Quasiparabolic Subgroups of Coxeter Groups and Their Hecke Algebra Module Structures

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Abstract

It is well known that the R-polynomial can be defined for the Hecke algebra of Coxeter groups, and the Kazhdan-Lusztig theory can be developed to understand the representations of Hecke algebra. There is also a generalization for the existence of R-polynomial and Kazhdan-Lusztig theory for the Hecke algebra module of standard parabolic subgroups of Coxeter groups. In recent work of Rains and Vazirani, a generalization of standard parabolic subgroups, called quasiparabolic subgroups, are introduced, and the corresponding Hecke algebra module is well-defined. However, the existence of the analogous involution (Kazhdan-Lusztig bar operator) on the Hecke algebra module of quasiparabolic subgroups is unknown in general. Assuming the existence of the bar-operator, the corresponding R-polynomials and Kazhdan-Lusztig polynomials can be constructed. We prove the existence of the bar operator for the corresponding Hecke algebra modules of quasiparabolic subgroups in finite classical Coxeter groups with a case-by-case verification (Chapter 4). As preparation, we classify all quasiparabolic subgroups of finite classical Coxeter groups. The approach is to first find all rotation subgroups of finite classical Coxeter groups (Chapter 2). Then we exclude the non-quasiparabolic subgroups and confirm the quasiparabolic subgroups (Chapter 3).

Contents

Chapter 1

Introduction

1.1 Background

Coxeter groups are a class of abstract groups generated by reflections. They are important in several areas of algebra, geometry and combinatorics. The concept of quasiparabolic sets and subgroups of Coxeter groups was introduced in recent work by Rains and Vazirani [14]. The first motivation for considering quasiparabolic sets was to study certain conjectures of symmetric functions using Hecke algebra techniques [1][13]. A motivating example of a quasiparabolic set is the conjugacy class of fixed-point-free involutions in S_{2n}. Also, the standard parabolic subgroups are a class of typical examples of quasiparabolic subgroups, and they are the origin of the name *quasiparabolic* [14].

One problem is to classify the quasiparabolic subgroups in all finite Coxeter groups. We manage to solve the problem for all finite classical Coxeter groups, and the quasiparabolic subgroups are listed in Theorem 16 of Chapter 3. In the original paper of Rains and Vazirani [14], the authors prove that all quasiparabolic subgroups are generated by rotations. Heading this direction, we first classify the rotation subgroups of finite classical Coxeter groups. Compared with the classification of reflection subgroups [6], the classification of rotation subgroups turn out to be much more complicated, and the results are given by Theorem 3, 8, 9 and 10. Then we exclude the non-quasiparabolic rotation subgroups, and confirm the quasiparabolic subgroups within the rotation subgroups. In particular, we prove the quasiparabolicity of a previously conjectural class of subgroups which have index 4 in the centralizer of the minimal fixed-point-free involutions of D_{2n}.

The Hecke algebra is closely related to the study of Chevalley groups [8][9]. In order to study the representation of Hecke algebra, Kazhdan and Lusztig introduced the R-polynomial and Kazhdan-Lusztig polynomial [10], and the coefficients of Kazhdan-Lusztig polynomials are closely related to intersection cohomology of Schubert varieties [11]. Thanks to the work of Deodhar [4][5], the Kazhdan-Lusztig theory can also generalized to the Hecke algebra module of standard parabolic subgroups of Coxeter groups. Rains and Vazirani analogously defined the Hecke algebra module of quasiparabolic subgroups. However, there is an obstruction for the existence of the Kazhdan-Lusztig bar operator. Based on the absence of counterexamples, the existence of K-L bar operator is conjectured in [14]. Assuming the existence of the K-L bar operator, Marberg has calculated the form of R-polynomials and Kazhdan-Lusztig polynomials [12]. Marberg also proved the existence of K-L bar operators for twisted involutions [12], which are a class of motivating examples of quasiparabolic subgroups of Coxeter groups [14]. Based on the classification of finite classical Coxeter groups, we are able to prove the existence of K-L bar operators for quasiparabolic subgroups of finite classical Coxeter groups as in Theorem 18.

1.2 Outline of the thesis

In the following section of Chapter 1, we review the definitions and basic properties of Coxeter groups and their quasiparabolic subgroups. In Chapter 2, we classify the rotation subgroups of finite classical Coxeter groups (type A, B and D). Based on the results in Chapter 2, we give the classification for quasiparabolic subgroups of finite classical Coxeter groups in Chapter 3. In Chapter 4, we first review the previously known results of Hecke algebras of Coxeter groups, and the Hecke algebra modules of quasiparabolic subgroups of Coxeter groups. Then from a case-by-case discussion, we verify the existence of Kazhdan-Lusztig bar operator of Hecke algebra modules of quasiparabolic subgroups of finite classical Coxeter groups.

1.3 Review of Coxeter groups and their quasiparabolic subgroups

We first review the concepts and properties of Coxeter groups, following [7] and [14], and set up the notation.

Definition 1. *A **Coxeter system** is a pair (W, S) consisting of a group W and a set of generators $S \subset W$, subject only to relations*

$$(ss')^{m(s,s')} = 1,$$

*where $m(s,s) = 1$, $m(s,s') = m(s',s) \geq 2$ for $s \neq s'$ in S. The elements $s \in S$ are called **simple reflections** of W. In addition, the conjugates of simple reflections are called **reflections** in W, and we denote this set by $R(W)$. Also, the product of two reflections is called a **rotation** in W.*

For each element $w \in W$, we denote $l(w)$ to be **the length** of w, being the least r expressing w as product of simple reflections $w = s_i \dots s_r$. The **Bruhat order** of W is the weakest partial order of W generated by the relations $w' < w$ if $w = w't$ for some $t \in R(W)$ and $l(w') < l(w)$.

Definition 2. *As for a W-set X, we define a **height function** $ht : X \to \mathbb{Z}$, and call the pair (X, ht) a **scaled W-set**, if $|ht(sx) - ht(x)| \leq 1$ for all $s \in S$. The W-set (X, ht) is **even** if for any pair $w \in W$, $x \in X$ s.t. $wx = x$, one has $l(w)$ even. Otherwise, the W-set (X, ht) is **odd**. For a scaled W-set X, an element $x \in X$ is W**-minimal** if $ht(sx) \geq ht(x)$ for all $s \in S$, and we make a similar definition for W-maximal elements.*

For any subset $I \subset S$, the subgroup W_I generated by $s_i \in I$ is called a **standard parabolic subgroup** of W. In [14], the authors introduce a generalization of standard parabolic subgroups, called quasiparabolic subgroups.

Definition 3. *A **quasiparabolic** W-set is a scaled W-set X satisfying the following properties:*

- *For all $r \in R(W)$, $x \in X$, if $ht(rx) = ht(x)$, then $rx = x$.*
- *For all $r \in R(W)$, $s \in S$, $x \in X$, if $ht(rx) > ht(x)$ and $ht(srx) < ht(sx)$, then $rx = sx$.*

If H is a subgroup of W, consider the scaled W-set W/H with height function

$$ht(wH) = \min_{v \in wH} l(v).$$

*Then H is a **quasiparabolic subgroup** of W, if W/H is a quasiparabolic W-set. H is an **even (or odd) quasiparabolic subgroup** of W, if W/H is an even (or odd) quasiparabolic W-set. For an odd subgroup H, its **even subgroup** H° is its subgroup containing all elements with even lengths.*

In particular, if X is a quasiparabolic W-set, then there is at most one maximal (or minimal) element in each orbit [14]. In addition, if x_0 is the minimal element of an orbit $O \subset X$, then O can be identified with the left coset W/H, where H is the stabilizer of x_0 in W.

Some motivating examples of quasiparabolic subgroups are standard parabolic subgroups and fixed-point free involutions in S_{2n} [14]. Then we have the basic question: what are all the quasiparabolic subgroups in finite Coxeter groups?

Recall that the classification of indecomposable finite Coxeter group is given by: $A_n(n \geq 1)$; $B_n(n \geq 2)$; $D_n(n \geq 4)$; $E_n(n = 6, 7, 8)$; F_4; $H_n(n = 3, 4)$; $I_2(m)(m \geq 5)$ [3]. In particular, the product of Coxeter groups of type A, B and D are called **classical Coxeter groups**. We are able to give the classification of quasiparabolic subgroups in finite classical Coxeter groups, and our approach to attack the problem relies on case-by-case discussion.

Definition 4. *Suppose H_0, H are subgroups of W. If H is generated by one reflection $r \in R(W)$ and H_0, where $|H| = 2|H_0|$, then we call H is a* ***double cover*** *of H_0 with r, and r is a* ***double cover reflection*** *of H_0. In addition, if $r \in S$ is a simple reflection in W, then H is a* ***simple double cover*** *of H_0 with s, and s is a* ***simple double cover reflection*** *of H_0.*

By [14], for odd quasiparabolic subgroup $H \subset W$, H must contain a simple reflection.

Theorem 1. *[14] Suppose the quasiparabolic subgroup $H \subset W$ contains an element of odd length. Then it contains a simple reflection.*

If H is a simple double cover of its even subgroup H_0, then the quasiparabolicity of H is determined by H_0 by the following theorem.

Theorem 2. *[14] If the subgroup $H \subset W$ contains a simple reflection, then H is quasiparabolic if and only if its even subgroup $H \cap W^0$ is quasiparabolic.*

In addition, Rains and Vazirani proved that all even quasiparabolic subgroups are generated by rotations.

Proposition 1. *[14] All even quasiparabolic subgroups are generated by rotations.*

We will start by classifying the rotation subgroups of finite subgroups in finite Coxeter groups of type A.

Chapter 2

Classification of Rotation Subgroups in Finite Classical Coxeter Groups

2.1 Rotation subgroups in finite Coxeter group of type A

Suppose $W = A_{n-1} = S_n$, a symmetric subgroup acting on symbols $1, \ldots, n$. The simple reflections $s_i = (i\ i{+}1)$ where $i = 1, \ldots, n-1$. Then the set of reflections $R(W)$ is equal to $\{(i\ j) : 1 \leq i < j \leq n\}$. So there are two types of rotations in S_n: 3-cycles $(a_1\ a_2\ a_3)$ and 2-rotations $(a_1\ a_2)(a_3\ a_4)$. Note that the two 2-rotations $(a_1\ a_2)(a_3\ a_4)$ and $(a_1\ a_3)(a_2\ a_4)$ will generate a Klein-4-group on symbols a_1, a_2, a_3, a_4, we carry out the classification by whether or not the 3-cycles and Klein-4-groups appear.

Proposition 2. *The indecomposable double covers of subgroups generated by 3-cycles are symmetric group S_k on k symbols.*

Proof. Note that the alternating group Alt_k on k symbols and a 3-cycle with i common symbol(s) $(i = 1, 2)$ will generate the group Alt_{k+3-i}. Then by induction on the number k of symbols will give the result. □

Proposition 3. *The indecomposable components H of double covers of subgroups generated by Klein-4-groups without appearance of 3-cycles are one of the following:*

- *B_k ($k \geq 2$), generated by $(a_{2i-1}\ a_{2i})(a_{2j-1}\ a_{2j})$, $(a_{2i-1}\ a_{2j-1})(a_{2i}\ a_{2j})$ ($1 \leq i < j \leq k$), and the double cover reflection $(a_1\ a_2)$;*
- *PGL(3,2), generated by B_3° on symbols $\{a_1, \ldots, a_6\}$ and the 2-rotation $(a_1\ a_3)(a_5\ a_7)$;*
- *AGL(3,2), generated by B_4° on symbols $\{a_1, \ldots, a_8\}$ and the 2-rotation $(a_1\ a_3)(a_5\ a_7)$.*

Proof. Suppose H_1, H_2 are two Klein-4-groups acting on 4 symbols. If there are 1 or 3 common symbols, a 3-cycle will be generated. So all distinct Klein-4-groups can only act on exactly 0 or 2 common symbols.

Now we consider the maximal k for a subgroup B_k° in H.

If $k \geq 5$, then it is impossible to add a Klein-4-group acting some common symbols, and without generating 3-cycles or B_{k+1}°.

If $k = 4$, then the only expansion for the subgroup will be $AGL(3,2)$.

If $k = 3$, then the only expansion for the subgroup will be $PGL(3,2)$. □

Proposition 4. *The indecomposable components of double covers of subgroups H generated by 3-cycles and Klein-4-groups are given by Proposition 2 and 3.*

Proof. Consider the subgroup Alt_k ($k \geq 3$) and a 2-rotation acting on some common symbols. They will generate an alternating subgroup on the orbit containing the k symbols acted by Alt_k. Then for the double cover of the whole subgroup H, if an orbit has a 3-cycle action, the action on this orbit will be the whole symmetric group or its even subgroup, the alternating subgroup. Otherwise, the action does not contain 3-cycles and, generated by Klein-4-groups, can be given by Proposition 3. □

Denote ΔS_k as the diagonal symmetric subgroup acting on $2k$ distinct symbols $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$, generated by 2-rotations $(a_i\ a_j)(b_i\ b_j)$ ($1 \leq i < j \leq k$). The Dil_{10} **in** S_5 is the subgroup acting on 5 symbols $\{a_1, \ldots, a_5\}$, generated by 2-rotations $(a_1\ a_2)(a_3\ a_4)$ and $(a_1\ a_3)(a_2\ a_5)$. The **twisted** Alt_5 **in** S_6 is the subgroup acting on 6 symbols $\{a_1, \ldots, a_6\}$, generated by 2-rotations $(a_1\ a_2)(a_3\ a_4)$, $(a_1\ a_2)(a_5\ a_6)$ and $(a_1\ a_3)(a_2\ a_5)$. We now study the subgroups generated by 2-rotations without appearance of 3-cycles or Klein-4-groups.

Proposition 5. *The indecomposable component of subgroups H generated by 2-rotations without the appearance of 3-cycles and Klein-4-groups, is given as follows:*

- $((\mathbb{Z}/2\mathbb{Z})^{\times k})^{\circ}$ *($k \geq 1$);*
- $\Delta(S_k)$ *($k \geq 3$);*
- Dil_{10} *in* S_5*;*
- *Twisted* Alt_5 *in* S_6.

Proof. Consider two 2-rotations $(a_1\ a_2)(a_3\ a_4)$ and $(b_1\ b_2)(b_3\ b_4)$. Define the **intersection type** of these two 2-rotations are $(c_1, c_2)(c_3, c_4)$, where $c_1 = |\{a_1, a_2\} \cap \{b_1, b_2\}|$, $c_2 = |\{a_1, a_2\} \cap \{b_3, b_4\}|$, $c_3 = |\{a_3, a_4\} \cap \{b_1, b_2\}|$, $c_4 = |\{a_3, a_4\} \cap \{b_3, b_4\}|$. In order to avoid the appearance of 3-cycles or Klein-4-groups, the only legitimate intersection type of two distinct 2-rotations are $(2, 0)(0, 0)$, $(1, 1)(1, 0)$, $(1, 0)(0, 1)$.

For the intersection type of $(1, 1)(1, 0)$, it will generate a subgroup Dil_{10} in S_5. If adding some more 2-rotations, the only expansion without appearance of 3-cycles or Klein-4-groups is twisted Alt_5 in S_6.

For the intersection type of $(2, 0)(0, 0)$, it will generate a subgroup $((\mathbb{Z}/2\mathbb{Z})^{\times 3})^{\circ}$. In general, the subgroup $((\mathbb{Z}/2\mathbb{Z})^{\times k})^{\circ}$ $(k \geq 3)$ can be generated by repeatedly adding $(a_1\ a_2)(c_1\ c_2)$, where c_1, c_2 are two new symbols. However, it is impossible to expand the group $H_1 = ((\mathbb{Z}/2\mathbb{Z})^{\times k})^{\circ}$ in another way when $k \geq 4$, without the new 2-rotation having exactly 1 common symbol with some 2-rotation in H_1, hence generating 3-cycles. While for the case $k = 3$, the only other expansion is twisted Alt_5 in S_6.

For the intersection type of $(1, 0)(0, 1)$, it will generate a subgroup $\Delta(S_3)$. Similar to the discussion for $((\mathbb{Z}/2\mathbb{Z})^{\times k})^{\circ}$, the only way to expand $\Delta(S_3)$ is $\Delta(S_k)$ $(k \geq 3)$ or twisted Alt_5 in S_6. So Proposition 5 gives all subgroups generated by 2-rotations without appearance of 3-cycles or Klein-4-groups. □

Now based on Proposition 4 and 5, we are able to give a full classification of rotation subgroups and their double covers in symmetric groups.

Theorem 3. *The indecomposable subgroups H generated by rotations in symmetric groups, or the double covers of such H, are one of the following groups, or the even subgroup of the direct product of some of these groups:*

- S_k *on k symbols, where $k \geq 2$;*

- *B_k on $2k$ symbols, where $k \geq 2$;*
- *ΔS_k on $2k$ symbols, where $k \geq 3$;*
- *$PGL(3,2)$ on 7 symbols;*
- *$AGL(3,2)$ on 8 symbols;*
- *Dil_{10} on 5 symbols;*
- *Twisted Alt_5 on 6 symbols;*
- *$K_4 \rtimes (\Delta S_3)$ on 7 symbols;*
- *$K_4 \rtimes (\Delta S_4)$ on 8 symbols.*

Proof. Note that for one 2-rotation $w = (a_1\ a_2)(a_3\ a_4)$, and a 3-cycle or Klein-4-group H_0, the only way adding w that will not generate a larger group generated by 3-cycles and Klein-4-groups, is exactly when the two symbols in one 2-cycle of w fall in the symbols acted by H_0. Then the even subgroups of direct product of S_k's and B_k's can be generated by those 3-cycles and Klein-4-groups.

For 3-cycles, all the expansions will give the **local double cover** (the indecomposable component of double cover) as S_k. For Klein-4-groups, if the local double cover is forbidden to have 3-cycles, then its local double cover is B_k acting on $2k$ symbols as Proposition 3. If $k \geq 3$, there will be no other way to further enlarge the group. While when $k = 2$, the new 2-rotations may have distinct common 2-cycles with the original Klein-4-group, generating $K_4 \rtimes (\Delta S_3)$ on 7 symbols or $K_4 \rtimes (\Delta S_4)$ on 8 symbols, and there is no other possible expansion of these two groups without generating 3-cycles. □

2.2 Rotation subgroups in finite Coxeter group of type B and D

We view the Coxeter group $W = B_n$ or D_n acting as signed permutations. Suppose $W = B_n$ has simple reflections $S = \{(1)_-, (1\ 2), \ldots, (n-1\ n)\}$, and the group $W = D_n$ has simple reflections $S = \{(1\ \overline{2}), (1\ 2), \ldots, (n-1\ n)\}$. (This is a non-standard convention, normally the simple reflections are $(n)_-$ and $(n-1\ \bar{n})$. We use $(1)_-$ and $(1\ \overline{2})$ in order to simplify the description in induction method by double cosets in Section 3.2.2 and the length function (3.1) of elements in D_{2n} in Section 3.6.) Then the reflections of $W = D_n$ has the form $(a\ b)$ or $(a\ \bar{b})$, and $W = B_n$ has one additional form $(a)_-$ of reflection on one sign-change.

Then the rotations of B_n and D_n must have the following form.

Proposition 6. *The rotations in $W = D_n$ have the form*

- *3-cycle $(a\ b\ c)$, or $(a\ b\ \bar{c})$;*
- *2-rotations (two disjoint 2-cycles) $(a\ b)(c\ d)$, $(a\ b)(c\ \bar{d})$, or $(a\ \bar{b})(c\ \bar{d})$;*
- *2-sign-change $(a)_-(b)_-$.*

If $W = B_n$, along with the above forms, the rotations can also have the following forms.

- *2-cycle and 1-sign-change in the 2-cycle symbol set $(a\ b)_- = (a\ b)(b)_-$;*
- *2-cycle and 1-sign-change out of the 2-cycle symbol set $(a\ b)(c)_-$,*

Our approach to classify the rotation subgroups H of $W = B_n$ or $W = D_n$ is similar to that when $W = A_n$, and we will use some results about Coxeter groups of type A for the *A-image* of H defined below, when $W = B_n$ or $W = D_n$.

Definition 5. *Suppose (W, S) and (W', S') are two Coxeter systems. A **Coxeter homomorphism** $\phi : W \to W'$ is a group homomorphism such that $\phi(S) \subset S' \cup \{1\}$.*

Consider the Coxeter homomorphisms

$$B_n \to A_{n-1} \times A_1$$

(The simple reflection $(i\ i+1)$ is mapped to $(i\ i+1)$ in A_{n-1} and the simple reflection $(1)_-$ is mapped to the generator of A_1), and

$$D_n \to A_{n-1}$$

*(The simple reflection $(i\ i+1)$ is mapped to $(i\ i+1)$ in A_{n-1} and the simple reflection $(1)_-$ is mapped to the identity 1). The images in the group A_{n-1} of the two maps above are called the **A-image** of $W = B_n$ or $W = D_n$.*

We first need to find out subgroups generated by 3-cycles in $W = B_n$ or $W = D_n$. There may be more possible cases than in A_n, because the 3-cycles in B_n or D_n may cause sign changes on some symbols (i.e., some signed symbol a is mapped to $-a$ by the rotation subgroup H).

2.2.1 The indecomposable groups generated by 3-cycles

If there is no sign change of symbols, the argument is exactly the same as for type A. We will get the subgroups $Alt_k (k \geq 3)$ generated by 3-cycles in $S_n \subset W$. For the case when the rotation subgroup H causes sign changes on some symbols, there must exist some 3-cycle generators which share at least two common symbols, and the common symbols may enable these signed symbols to be in the same orbit of their negative signed symbols.

For the case when two 3-cycles have 3 common symbols, it can be reduced to the case of distinct sign on 1 symbol. In this case, we claim the following

Proposition 7. *Suppose H is an indecomposable group generated by 3-cycles, and there exist two 3-cycles in the generators, which have the same 3 symbols but with distinct signs on 1 symbol, for example, $(a\ b\ c)$ and $(a\ b\ \overline{c})$. Then H is $\mathbb{F}_2^{n-1} \rtimes Alt_n$, the subgroup of B_n with even permutations on the symbol set $\{\pm a_1, \ldots, \pm a_n\}$ and even number of sign changes, acting on the whole n symbols with signs.*

Proof. Without loss of generality, suppose H contains $(a\ b\ c)$ and $(a\ b\ \overline{c})$, generating the subgroup $\mathbb{F}_2^2 \rtimes Alt_3$ of B_3, with even permutation on the symbol set $\{\pm a_1, \ldots, \pm a_n\}$, and an even number of sign changes acting on signed symbols $\{a, b, c\}$. Then it will generate a 2-sign-change $(a)_-(b)_-$. Note that the A-image of action of H on these k symbols is the same as Alt_n, where $n \geq 3$. In addition, H can induce all even numbers of sign changes on the n symbols.

Thus, $H = \mathbb{F}_2^{n-1} \rtimes Alt_n$, the subgroup of B_n with even permutation on the symbol set $\{\pm a_1, \ldots, \pm a_n\}$ and even sign changes. In addition, H can not be expanded to larger subgroup of B_n by adding (positive) 3-cycles in B_n. □

For the case that no two 3-cycles have 3 common symbols, H can also induce sign changes. Apart from the cases when H is a subgroup of A_n, we only have the case that there are two 3-cycles with 2 common symbols, where there is exactly one distinct sign on the two common symbols. We first investigate the case of subgroups generated by only two 3-cycles, with 2 common symbols. Without loss of generality, we may assume these two 3-cycles to be $(a_1\ a_2\ a_3)$ and $(a_1\ \overline{a_2}\ a_4)$.

Example 1. *The 3-cycles* $(a_1\ a_2\ a_3)$ *and* $(a_1\ \overline{a_2}\ a_4)$ *generate a subgroup isomorphic to* $SL(2,3)$ *of order 24.*

If we add one more 3-cycle not belonging to the copy of $SL(2,3)$ above, we claim it must be extended to $\mathbb{F}_2^{n-1} \rtimes Alt_n$.

Lemma 1. *If there are* $n \geq 5$ *symbols involved in an indecomposable subgroup* H*, which is generated by 3-cycles, then* H *is* Alt_n *or* $\mathbb{F}_2^{n-1} \rtimes Alt_n$*.*

Proof. If H is not Alt_n, then there are two 3-cycles which will cause sign changes on some symbols. If they generate $\mathbb{F}_2^{k-1} \rtimes Alt_k$, then a 2-sign-change $(a)_-(b)_-$ is generated. The positive symbol 3-cycle is included in H, and all elements in the group $\mathbb{F}_2^{n-1} \rtimes Alt_n$ are included in H. In addition, H is the subgroup of $\mathbb{F}_2^{n-1} \rtimes Alt_n$. Thus H must be $\mathbb{F}_2^{n-1} \rtimes Alt_n$, if the subgroup $\mathbb{F}_2^{k-1} \rtimes Alt_k$ exists.

In fact, we will generate this group by getting the alternating subgroup Alt_k on k symbols, and a 2-sign-change on any two of the k symbols. From $SL(2,3)$ generated by 3-cycles $(a_1\ a_2\ a_3)$ and $(a_1\ \overline{a_2}\ a_4)$, we know that any 3 symbols will have their 3-cycle in positive or negative signs. If the new added 3-cycle does not have new symbols, it will be in the group $SL(2,3)$, or it will enlarge it to $\mathbb{F}_2^3 \rtimes Alt_4$. If the new added 3-cycle has some new symbols, then it will have exactly one common symbol with some 3-cycle in $SL(2,3)$. They will generate elements in Alt_5. By adding other 3-cycles in the original group $SL(2,3)$, we will generate the group $\mathbb{F}_2^{k-1} \rtimes Alt_k$, where $k \geq 5$. The lemma is proved. □

By Example 1 and Lemma 1, we know all the indecomposable subgroups generated by 3-cycles are listed as follows.

Theorem 4. *All the indecomposable subgroups in* B_n *and* D_n *generated by 3-cycles are*

- *Alternating subgroup* Alt_k $(k \geq 3)$ *on* k *symbols;*
- $\mathbb{F}_2^{k-1} \rtimes Alt_k$*;*
- $SL(2,3)$*.*

2.2.2 The indecomposable groups generated by K_4's, with no 3-cycles

Next we will study the indecomposable groups generated by K_4's, with no 3-cycles.

- **Subgroups generated by two K_4's**

First we study all possible groups generated by two K_4's with some common symbols. Similar to the argument in the group S_n, we have the following restrictions for two K_4's.

Proposition 8. *Two K_4's can only 0 or 2 or 4 common symbols, without generating 3-cycles.*

- **Indecomposable groups generated by K_4's**

By the argument in the group $W = S_n$, the A-image can only be B_k°, $PGL(3,2)$, $AGL(3,2)$.

For $H = B_k^\circ$, suppose the symbols are paired by $\{a_{2i-1}, a_{2i}\}$. By adding possible sign changes, it may become $\mathbb{F}_2^{k-1} \rtimes B_k^\circ$, where $\mathbb{F}_2^{k-1}$ is generated by all two pairs of symbol sign changes $(a_{2i-1})_-(a_{2i})_-(a_{2j-1})_-(a_{2j})_-$; $\mathbb{F}_2^k \rtimes B_k^\circ$, where $\mathbb{F}_2^k$ is generated by pairs of symbols in same block sign changes $(a_{2i-1})_-(a_{2i})_-$; or $\mathbb{F}_2^{2k-1} \rtimes B_k^\circ$, where $\mathbb{F}_2^{2k-1}$ is generated by arbitrary pairs of symbol sign changes $(a_{2i-1})_-(a_{2i})_-$.

For $PGL(3,2)$, adding possible sign changes, then $H = \mathbb{F}_2^3 \rtimes PGL(3,2)$, where $\mathbb{F}_2^3$ is generated by sign changes $(a_1)_-(a_2)_-(a_3)_-(a_4)_-$ which are generated by two K_4's on signed symbols $\{a_1, a_2, a_3, a_4\}$, and $\{a_1, \overline{a_2}, a_3, \overline{a_4}\}$. Or $H = \mathbb{F}_2^6 \rtimes PGL(3,2)$, where $\mathbb{F}_2^6$ is generated by all pairs of symbol sign changes $(a_1)_-(a_2)_-$.

For $AGL(3,2)$, adding possible sign changes, H may become $\mathbb{F}_2^4 \rtimes AGL(3,2)$, where $\mathbb{F}_2^4$ is generated by sign changes $(a_1)_-(a_2)_-(a_3)_-(a_4)_-$ which are generated by two K_4's on signed symbols $\{a_1, a_2, a_3, a_4\}$, and $\{a_1, \overline{a_2}, a_3, \overline{a_4}\}$. Or $\mathbb{F}_2^7 \rtimes AGL(3,2)$, where $\mathbb{F}_2^6$ is generated by all pairs of symbol sign changes $(a_1)_-(a_2)_-$.

Summing up the results above, we have

Theorem 5. *The indecomposable groups H generated by Klein-4-groups K_4 in $W = B_n$ or $W = D_n$ are one of the following groups described as in the argument above:*

- *B_k° acting as permutations on $2k$ symbols;*
- *$\mathbb{F}_2^{k-1} \rtimes B_k^\circ$ acting as permutation on $2k$ symbols, where $\mathbb{F}_2^{k-1}$ is generated by sign changes on four symbols of any two pairs of symbols;*
- *$\mathbb{F}_2^k \rtimes B_k^\circ$ acting as permutation on $2k$ symbols, where $\mathbb{F}_2^k$ is generated by sign changes on any pair of symbols;*

- $PGL(3,2)$
- $\mathbb{F}_2^3 \rtimes PGL(3,2)$
- $\mathbb{F}_2^6 \rtimes PGL(3,2)$
- $AGL(3,2)$
- $\mathbb{F}_2^4 \rtimes AGL(3,2)$
- $\mathbb{F}_2^7 \rtimes AGL(3,2)$

2.2.3 The indecomposable groups generated by 3-cycles and K_4's

- The case of one 3-cycle and one K_4

Next we consider the indecomposable groups generated by 3-cycles and K_4's.

Theorem 6. *The indecomposable groups H generated by 3-cycles and K_4's are given by Theorem 4 and 5, along with*

- $\mathbb{F}_2 \times Alt_4$ *with alternating group acting as permutations on 4 symbols, and allowing sign changes $(a_1)_-(a_2)_-(a_3)_-(a_4)_-$ on all four symbols.*

Proof. We first study the cases when one 3-cycle and one K_4 have some common symbols.

Lemma 2. *If one 3-cycle and one K_4 have some common symbols, they will generate Alt_k ($k = 4,5,6$) or $\mathbb{F}_2^{k-1} \rtimes Alt_k$ ($k = 4,5$) or $\mathbb{F}_2 \times Alt_4$.*

Proof.
- If the 3-cycle and K_4 have three common symbols, they may have 0 or 1 distinct signs.

  If they have 0 distinct signs, by the argument when $W = S_n$, they will generate the alternating group Alt_4.

  If they have 1 distinct sign, say $(a_1\ a_2\ a_3)$ and $\{a_1, a_2, \overline{a_3}, a_4\}$, then the 3-cycle $(a_1\ a_2\ \overline{a_4})$ will be generated, and the whole group H is $\mathbb{F}_2 \times Alt_4$. Here, H contains the alternating group acting on 4 symbols, and allowing sign changes $(a_1)_-(a_2)_-(a_3)_-(a_4)_-$ on all four symbols. If we add one more 3-cycle or K_4 into H, it will generate $\mathbb{F}_2^{k-1} \rtimes Alt_k$ ($k = 4,5,6,7$).

- If the 3-cycle and K_4 have exactly two common symbols, they may have 0 or 1 distinct signs.

  If they have 0 distinct signs, by the argument when $W = S_n$, they will generate the alternating group Alt_5.

  If they have 1 distinct sign, say $(a_1\ a_2\ a_3)$ and $\{a_1, \overline{a_2}, a_4, a_5\}$, then the whole group H is $\mathbb{F}_2^4 \rtimes Alt_5$. Here, H contains the alternating group acting on 5 symbols, and allowing an even number of sign changes.

- If the 3-cycle and K_4 have exactly one common symbol, then they can be embedded into S_6 as when W has type A. By the argument when $W = S_n$, they will generate the alternating group Alt_6.

□

- Indecomposable groups generated by 3-cycles and K_4's

We know that if the indecomposable group contain both 3-cycles and K_4's, it must be Alt_k or $\mathbb{F}_2^{k-1} \rtimes Alt_k$. Which one it is depends on whether sign changes are generated. So no more kinds of groups can be generated other than those in Theorem 6.

□

2.2.4 Determining the indecomposable groups containing no 3-cycles or K_4's

- The case of two intersecting 2-rotations

Now we will figure out the indecomposable rotation subgroups H containing no 3-cycles or K_4's. We first investigate the possible cases for two distinct intersecting 2-rotations. Suppose there is an original 2-rotation $(a_1\ a_2)(a_3\ a_4)$, and denote by $(z_1, z_2)(z_3, z_4)$ the intersection type of the A-image of two 2-rotations.

After checking all possible intersection types, we know that only two 2-cycles with the intersecting types $(2,0)(0,2)$, $(1,1)(1,1)$, $(1,1)(1,0)$, $(2,0)(0,0)$, and $(1,0)(0,1)$ can generate neither 3-cycles nor K_4's. In addition, only types $(2,0)(0,2)$, $(1,1)(1,1)$, and $(2,0)(0,0)$ can cause sign changes on some symbols.

We next discuss all possible groups generated by two 2-rotations, without the appearance of 3-cycles or K_4's.

The classification goes according to the intersecting types of the first two given 2-rotations.

- **The intersection type** $(1,1)(1,1)$

In this case, in order to avoid the appearance of K_4, we can assume the two given 2-rotations are $(a_1\ a_2)(a_3\ a_4)$ and $(a_1\ \overline{a_3})(a_2\ a_4)$. We will prove that H can only be expanded by adding 2-rotations, without the appearance of 3-cycles or K_4's, as follows.

Proposition 9. *There is only one expansion of* Dil_8 *generated by* $(a_1\ a_2)(a_3\ a_4)$ *and* $(a_1\ \overline{a_3})(a_2\ a_4)$*, without the appearance of 3-cycles or* K_4*'s. This expansion is* $(\mathbb{Z}/4\mathbb{Z})^{\times(k-1)} \rtimes Sym_k$ *(*$k \geq 2$*).*

Proof. We try to add a third 2-rotation to the original group Dil_8. The group Dil_8 can not be expanded only on the original four symbols, without the appearance of K_4's.

So there must be new symbols involved in the third 2-rotation.

If there is only 1 new symbol involved, then the intersecting type of the third 2-rotation and the original two 2-rotations should be $(1,1)(1,0)$, generating two copies of Dil_{10}. However, some 3-cycle will be generated in this case, which is forbidden.

If there are exactly 2 new symbols involved, then the intersection types $(2,0)(0,0)$ and $(1,0)(0,1)$ are allowed. If the third 2-rotation has intersection type $(2,0)(0,0)$ with one of the given 2-rotation, then it will have intersection type $(1,1)(0,0)$ (or $(1,0)(1,0)$) with another 2-rotation, generating a copy of K_4.

Thus only the intersection type $(1,0)(0,1)$ is allowed. Without loss of generality, we may assume the new 2-rotation is $(a_1\ a_5)(a_4\ a_6)$. Then the group is generated by $(a_1\ a_4)_-(a_3\ a_2)_-$, $(a_1\ a_4)_-(a_5\ a_6)_-$, $(a_1\ a_2)(a_4\ a_3)$, $(a_1\ \overline{a_3})(a_4\ a_2)$, $(a_1\ a_5)(a_4\ a_6)$, and the whole subgroup is $((\mathbb{Z}/4\mathbb{Z})^{\times 2}) \rtimes Sym_3$.

We can go on adding 2-rotations, with only the intersection type $(1,0)(0,1)$, and this will generate the subgroup $((\mathbb{Z}/4\mathbb{Z})^{\times(k-1)}) \rtimes Sym_k$, for all $k \geq 3$. □

In the following arguments, we can forbid the appearance the type $(1,1)(1,1)$.

- **The intersection type** $(1,1)(1,0)$

The argument in two 2-rotations implies that if the intersection type is $(1,1)(1,0)$, they will generate group Dil_{10}.

We now analyze indecomposable subgroups including intersection type $(1,1)(1,0)$, without the appearance of 3-cycles, K_4's and the intersection type $(1,1)(1,1)$.

Proposition 10. *The indecomposable subgroups including intersection type* $(1,1)(1,0)$*, without the appearance of 3-cycles,* K_4*'s and the intersection type* $(1,1)(1,1)$*, are one of the following groups:*

1. *The dihedral group* Dil_{10} *acting on* 5 *symbols;*
2. *The semi-direct product* $\mathbb{F}_2^4 \rtimes Dil_{10}$*, where* Dil_{10} *acts on* 5 *symbols, and sign changes on all pairs of symbols are allowed;*
3. *Twisted* Alt_5 *in* S_6 *on* 6 *symbols;*
4. H_3 *acting on* 6 *symbols with signs;*
5. *The semi-direct product* $\mathbb{F}_2^5 \rtimes Alt_5$*, where* Alt_5 *is twisted* Alt_5 *in* S_6 *on* 6 *symbols, and sign changes on all pairs of symbols are allowed.*

Proof. By the argument when $W = S_n$, we know that the A-image can only be Dil_{10} or twisted Alt_5 in S_6.

The base case is that there are only two 2-rotations, with intersection type $(1,1)(1,0)$, say $(a_1\ a_2)(a_3\ a_4)$ and $(a_1\ a_3)(a_2\ a_5)$. They will generate Dil_{10}.

If we add the third generator and expand the group Dil_{10}, it may not expand the A-image permuting the 5 symbols, then sign changes on one or two 2-cycles in one 2-rotation in the Dil_{10} will be added. and we will get the subgroup $H = \mathbb{F}_2^4 \rtimes Dil_{10}$.

If the third generator expand the group Dil_{10} on the A-image, then by the argument in the group S_n, the A-image must be twisted Alt_5 in S_6.

If no sign changes are induced, say, adding $(a_1\ a_2)(a_5\ a_6)$, H will be twisted Alt_5 in S_6.

Otherwise, if there exist some sign changes, then we can assume that the new 2-rotation is $(a_1\ \overline{a_2})(a_5\ a_6)$. The A-image on 6 symbols will still be the twisted subgroup Alt_5 in S_6, and sign changes on all 6 symbols will be generated, so the group order is at least 120. In addition, this group can be generated by $(a_1\ a_2)(a_3\ a_4)$, $(a_1\ a_4)(a_3\ a_5)$, and $(a_1\ \overline{a_2})(a_5\ a_6)$, which are generators of H_3. Since H_3 has 120 elements, this group is actually H_3.

If some more 2-rotations are introduced, note that the A-image on A_n can not be expanded, so only 2-rotations causing sign changes can be introduced. Then sign changes on all pairs of symbols can be generated, and the group is $H = \mathbb{F}_2^5 \rtimes Alt_5$, where the A-image is twisted Alt_5 in S_6, and all pairs of sign changes are allowed. □

In the following argument of this section, we can forbid the appearance of intersection type $(1,1)(1,0)$ too. Then only the intersection types $(2,0)(0,0)$ and $(1,0)(0,1)$ are allowed.

- **The intersection type** $(2,0)(0,0)$

If only the intersection types $(2,0)(0,0)$ and $(1,0)(0,1)$ are allowed, by the argument when $W = S_n$, the A-image of H can only be $(S_2^{\times k})^\circ$ $(k \geq 3)$. Suppose the A-image of the k 2-cycles on the $2k$ unsigned-symbols are $(a_{2i-1}\ a_{2i})$, where $i = 1, \ldots, k$. If we also consider the signs, then only 2-cycles $(a_{2i-1}\ a_{2i})$ and $(a_{2i-1}\ \overline{a_{2i}})$ are allowed.

Since all these 2-cycles commute, the group will be a direct product of groups constructed by an even number of 2-cycles $(a_{2i-1}\ a_{2i})$ or $(a_{2j-1}\ \overline{a_{2j}})$. So the group is $((\mathbb{Z}/2\mathbb{Z})^{\times k_1})^\circ \times \cdots \times ((\mathbb{Z}/2\mathbb{Z})^{\times k_l})^\circ$, where the components $((\mathbb{Z}/2\mathbb{Z})^{\times k_m})^\circ$ have no common 2-cycles, and the union of all 2-cycles covers all $2k$ symbols.

- **The intersection type** $(1,0)(0,1)$

By the argument when $W = S_n$, for the type of $(1,0)(0,1)$, the A-image of the rotation subgroup H is ΔS_k.

Suppose the two orbits are $\{a_1, \ldots, a_k\}$ and $\{a_{k+1}, \ldots, a_{2k}\}$. On each orbit, the permutation on the symbols generate the group S_k. If some sign changes happen, it must be generated by 2-rotations on same symbols, with distinct signs. Without loss of generality, we may assume they are $(a_1\ a_2)(a_{k+1}\ a_{k+2})$ and $(a_1\ \overline{a_2})(a_{k+1}\ a_{k+2})$, or $(a_1\ a_2)(a_{k+1}\ a_{k+2})$ and $(a_1\ \overline{a_2})(a_{k+1}\ \overline{a_{k+2}})$.

For the case of $(a_1\ a_2)(a_{k+1}\ a_{k+2})$ and $(a_1\ \overline{a_2})(a_{k+1}\ a_{k+2})$, all even sign changes on symbols $\{a_1, a_2, \ldots, a_k\}$ are generated. The group H will be $\mathbb{F}_2^{k-1} \rtimes \Delta S_k$.

If there are additional two 2-rotations that can generate sign changes $(a_i)_-(a_j)_-$ $(k+1 \leq i < j \leq 2k)$ on symbols $\{a_{k+1}, \ldots, a_{2k}\}$, then H will be $\mathbb{F}_2^{2(k-1)} \rtimes \Delta S_k$, where even sign changes on both symbol sets $\{a_1, \ldots, a_k\}$ and $\{a_{k+1}, \ldots, a_{2k}\}$ are allowed.

For the case of adding $(a_1\ \overline{a_2})(a_{k+1}\ \overline{a_{k+2}})$, then all even sign changes that agree on the two symbol sets $\{a_1, \ldots, a_k\}$ and $\{a_{k+1}, \ldots, a_{2k}\}$ are generated. The group will be $\Delta(\mathbb{F}_2^{k-1} \rtimes S_k)$.

If some other sign changes $(a_i)_-(a_j)_-$ not in $\Delta(\mathbb{F}_2^{k-1} \rtimes S_k)$ can be generated, then the whole group will be $\mathbb{F}_2^{2(k-1)} \rtimes \Delta S_k$, where $\mathbb{F}_2^{k-1}$ acts on both symbol sets $\{a_1, \ldots, a_k\}$ and $\{a_{k+1}, \ldots, a_{2k}\}$ as even number of sign changes.

From the arguments above in this section, we can classify all subgroups generated by 2-rotations in B_n, without the appearance of 3-cycles or K_4's.

Theorem 7. *All subgroups not containing 3-cycles or K_4's in $W = B_n$ or $W = D_n$, generated by 2-rotations, are given as follows:*

- *The semi-direct product $(\mathbb{Z}/4\mathbb{Z})^{\times(k-1)} \rtimes Sym_k$ ($k \geq 2$), where $(\mathbb{Z}/4\mathbb{Z})^{\times(k-1)}$ is the zero summation subgroup of $(\mathbb{Z}/4\mathbb{Z})^{\times k}$, with each component corresponding to powers of $(2i-1\ 2i)_-$, and Sym_k corresponding to permutations among blocks $\{2i-1, 2i\}$ ($1 \leq i \leq k$);*
- *The dihedral group Dil_{10} acting on 5 symbols;*
- *The semi-direct product $\mathbb{F}_2^4 \rtimes Dil_{10}$, where Dil_{10} acting on 5 symbols, and sign changes on all pairs of symbols are allowed;*
- *Twisted Alt_5 in S_6 on 6 symbols;*
- *H_3 acting on 6 symbols with signs, where the A-image is twisted Alt_5 in S_6;*
- *The semi-direct product $\mathbb{F}_2^5 \rtimes Alt_5$, where Alt_5 is twisted Alt_5 in S_6 on 6 symbols, and $\mathbb{F}_2^5$ acts by an even number of sign changes on the 6 symbols;*
- *The direct product $(S_2^{\times k_1})^\circ \times \cdots \times (S_2^{\times k_l})^\circ$ where the components $(S_2^{\times k_m})^\circ$ contain 2-cycles in the form $(a_{2i-1}\ a_{2i})$ or $(a_{2i-1}\ \overline{a_{2i}})$, ($1 \leq i \leq k$) and the union of all 2-cycles covers all $2k$ symbols $\{a_1, \ldots, a_{2k}\}$;*
- *ΔS_k permuting $2k$ symbols;*
- *$\mathbb{F}_2^{k-1} \rtimes \Delta S_k$;*
- *$\Delta(\mathbb{F}_2^{k-1} \rtimes S_k)$;*
- *$\mathbb{F}_2^{2(k-1)} \rtimes \Delta S_k$.*

2.2.5 Determining the indecomposable groups having a given normal subgroup generated by 3-cycles and K_4's, and containing no additional 3-cycles or K_4's

- **The indecomposable groups with a normal subgroup with the appearance of 3-cycles**

Now we extend to determine the indecomposable groups having a given normal subgroup as in Theorem 4, and containing no additional 3-cycles or K_4's. We first study the groups with the appearance of 3-cycles.

Proposition 11. *Suppose a group G is generated by 3-cycles, and we add 2-rotations which share some common cycles with G, then they will generate the subgroup H of $(\prod_i G_i) \times G'$, where each component G_i is one of the following groups:*

- *S_k permuting k symbols;*
- *$\mathbb{F}_2^{k-1} \rtimes S_k$, where the A-image S_k permutes k symbols, and $\mathbb{F}_2^{k-1}$ is the subgroup of even number of sign changes;*
- *$\mathbb{F}_2 \times S_4$, where the A-image S_4 permutes 4 symbols $\{a_1, a_2, a_3, a_4\}$, and $\mathbb{F}_2$ is the subgroup of sign changes $(a_1)_-(a_2)_-(a_3)_-(a_4)_-$ on all the 4 symbols.*

and $G' = A_1^{\times j}$ is a group generated by 2-cycles, with j relations that the parity of G_i agrees with the appearance of one corresponding 2-cycle in G'.

Proof. We first consider the group generated by one 3-cycle and one 2-rotation, which share some common symbols. By the argument when $W = S_n$, if there is no sign change generated by the 3-cycle and 2-rotation, then the group will be Alt_k $(k = 4, 5, 6, 7)$ or $(S_l \times S_2)^\circ$ $(l = 3, 4)$. If there are some sign changes generated by the 3-cycle and 2-rotation, then there is a 2-cycle in the 2-rotation, which can generate sign changes along with the 3-cycle. Without loss of generality, we assume the 3-cycle is $(a_1\ a_2\ a_3)$, and the 2-cycle is $(a_1\ \overline{a_2})$. The other 2-cycle of the 2-rotation may have 0 or 1 common symbols with the 3-cycle $(a_1\ a_2\ a_3)$.

If there is 1 common symbol, say the 2-cycle is $(a_3\ a_4)$, then the 3-cycle $(a_1\ a_2\ \overline{a_4})$ will be generated. The whole group is the direct product of $\mathbb{F}_2$ and Alt_4, where Alt_4 is the alternating group on symbols $\{a_1, a_2, a_3, \overline{a_4}\}$, and $\mathbb{F}_2$ corresponds the sign changes $(a_1)_-(a_2)_-(a_3)_-(a_4)_-$ on all 4 symbols.

If there are no common symbols, say the 2-cycle is $(a_4\ a_5)$, then the sign changes on any two symbols of $\{a_1, a_2, a_3\}$ can be generated. The whole group H will be the even subgroup of $(\mathbb{F}_2^2 \rtimes S_3) \times S_2$.

Now, we can calculate the possible groups generated by more than 2 rotations. For the A-image as Alt_k on k symbols, where $k \geq 5$, the appearance of sign changes on 4 symbols will generate subgroup of F_2^{k-1} for all sign changes on even number of symbols.

In addition, all subgroups containing 3-cycles will have A-image as Alt_k acting on all the symbols, which are in the same orbit of the 3-cycle.

So we get our claim in the proposition. □

- **The indecomposable groups generated by 2-rotations without the appearance of 3-cycles**

We next study the groups generated by 2-rotations (including K_4's), without the appearance of 3-cycles. So for one K_4 and one 2-rotation, they can only have 0 or 2 or 4 common symbols.

In addition, we can consider the largest subgroup generated by K_4's. When we add 2-rotations, they will not generate a larger group which can be generated by only K_4's.

So if one K_4 and one 2-rotation have 4 common symbols, they may only generate 2 or 4 sign changes, which can be obtained by two K_4's, too.

If one K_4 and one 2-rotation have no common symbols, they will commute, and we will have similar properties as in the case when $W = S_n$.

So we only need to treat the case that one K_4 and one 2-rotation have 2 common symbols. Note that we require that there is no larger group generated by K_4's, after we add the 2-rotation. So the two common symbols must lie in the same 2-cycle in the 2-rotation.

Note that from the argument with W in type A, if the A-image is larger than B_k° with $k \geq 3$, then when we add some 2-cycles in the group, all these 2-cycles can be treated as adding one given 2-cycle.

So we will study the possibilities in adding 2-cycles to the groups generated by K_4's.

Proposition 12. *Consider adding at least one 2-cycle in a group H generated by K_4's, and suppose H permutes at least 4 symbols. If the appearance of 3-cycles or the group $K_4 \rtimes (\Delta S_3)$ on 7 symbols is forbidden, then the group G must be a subgroup of $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^{\times k} \rtimes Sym_k$, where Sym_k permutes k pairs $\{b_{2i-1}, b_{2i}\}$ $(1 \leq i \leq k)$ and keeps the A-image parity and signs of symbols, while each $(u_i, v_i) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ indicates the parity of 2-cycles $(b_{2i-1}\ b_{2i})$ and $(b_{2i-1}\ \overline{b_{2i}})$. All the appearances of $(b_{2i-1}\ b_{2i})$ $(1 \leq i \leq k)$ are equivalent, and so is $(b_{2i-1}\ \overline{b_{2i}})$. In addition, G is listed as follows:*

1. *B_k acting as permutation on $2k$ symbols, i.e., all v_i's are zero;*

2. *$(\mathbb{Z}/2\mathbb{Z})^{\times(2k-1)} \rtimes Sym_k$, where $\sum_i u_i = 0$ in $\mathbb{Z}/2\mathbb{Z}$, and $\sum_i v_i$ corresponds to the parity of $(b_1\ \overline{b_2})$; or where $\sum_i v_i = 0$ in $\mathbb{Z}/2\mathbb{Z}$, and $\sum_i u_i$ corresponds to the parity of $(b_1\ b_2)$;*

3. *$(\mathbb{Z}/2\mathbb{Z})^{\times(2k)} \rtimes Sym_k$, where $\sum_i u_i$ corresponds to the parity of $(b_1\ b_2)$, and $\sum_i v_i$ corresponds to the parity of $(b_1\ \overline{b_2})$;*

4. *$(\mathbb{Z}/2\mathbb{Z})^{\times(2k)} \rtimes Sym_k$, where $\sum_i(u_i + v_i)$ corresponds to the parity of the sum of $(b_1\ b_2)$ and $(b_1\ \overline{b_2})$.*

Proof. By Theorem 5, the groups generated by K_4's have A-image B_k°, $PGL(3,2)$, $AGL(3,2)$, with possible sign changes on all two pairs, or all pairs, or no symbols. For $AGL(3,2)$ and $PGL(3,2)$, if we add a 2-rotation, then 3-cycles can be generated, and the group action is transitive on all symbols, which follows as in Theorem 4. When the A-image of H is B_k° ($k \geq 3$), which is generated by $\{a_1, a_2, a_{2i-1}, a_{2i}\}$, $(2 \leq i \leq k)$, then the 2-rotation can only be $(2i-1\ 2i)$ or $(2i-1\ \overline{2i})$ in order to avoid the appearance of 3-cycles. In addition, adding $(a_{2i-1}\ a_{2i})$ is equivalent to adding $(a_1\ a_2)$, adding $(a_{2i-1}\ \overline{a_{2i}})$ is equivalent to adding $(a_1\ \overline{a_2})$.

When the A-image of H is B_2°, if the appearance of group $K_4 \rtimes (\Delta S_3)$ is forbidden, then no new 2-cycles with A-image having exactly 1 common symbol with some 2-cycle in H can appear.

So for both cases, we just need to consider adding $(a_1\ a_2)$ or $(a_1\ \overline{a_2})$.

- Suppose the original group is B_k°. If only $(a_1\ a_2)$ is added, then the new group is B_k, where all b_i's are zero.

  If only $(a_1\ \overline{a_2})$ is added, then the new group is $(\mathbb{Z}/2\mathbb{Z})^{\times(2k-1)} \rtimes Sym_k$, where $\sum_i a_i = 0$ in $\mathbb{Z}/2\mathbb{Z}$, and $\sum_i b_i$ corresponds to one A_1 component in G' in Proposition 11 in relation with $(a_1\ \overline{a_2})$.

  If both $(a_1\ a_2)$ and $(a_1\ \overline{a_2})$ are added, then the new group is $(\mathbb{Z}/2\mathbb{Z})^{\times(2k)} \rtimes Sym_k$, where $\sum_i a_i$ corresponds to one A_1 component in G' in relation with $(a_1\ a_2)$, and $\sum_i b_i$ corresponds one A_1 component in G' in relation with $(a_1\ \overline{a_2})$.

- Suppose the original group is $\mathbb{F}_2^{k-1} \rtimes B_k^\circ$, with sign changes on even pairs of symbols allowed. If only $(a_1\ a_2)$ is added, then the new group is $(\mathbb{Z}/2\mathbb{Z})^{\times(2k-1)} \rtimes Sym_k$, where $\sum_i b_i = 0$ in $\mathbb{Z}/2\mathbb{Z}$, and $\sum_i a_i$ corresponds to the parity of $(a_1\ a_2)$.

  If only $(a_1\ \overline{a_2})$ is added, then the new group is $(\mathbb{Z}/2\mathbb{Z})^{\times(2k-1)} \rtimes Sym_k$, where $\sum_i a_i = 0$ in $\mathbb{Z}/2\mathbb{Z}$, and $\sum_i b_i$ corresponds to the parity of $(a_1\ \overline{a_2})$.

If both $(a_1\ a_2)$ and $(a_1\ \overline{a_2})$ are added, then the new group is $(\mathbb{Z}/2\mathbb{Z})^{\times(2k)} \rtimes Sym_k$, where $\sum_i a_i$ corresponds to the parity of appearance $(a_1\ a_2)$, and $\sum_i b_i$ corresponds to the parity of $(a_1\ \overline{a_2})$.

- Suppose the original group is $\mathbb{F}_2^k \rtimes B_k^\circ$, with sign changes on any pairs of symbols allowed, then the new group is $(\mathbb{Z}/2\mathbb{Z})^{\times(2k-1)} \rtimes Sym_k$, where $\sum_i(a_i + b_i)$ corresponds to the parity of the sum of $(a_1\ a_2)$ and $(a_1\ \overline{a_2})$.

□

The cases for A-image as B_2° is more intricate. Suppose the A-image K_4 group acts on symbols $\{a_1, a_2, a_3, a_4\}$. Then we may add 2-cycles $(a_1\ a_2)$, $(a_1\ \overline{a_2})$, $(a_1\ a_3)$, $(a_1\ \overline{a_3})$, $(a_1\ a_4)$, or $(a_1\ \overline{a_4})$.

Proposition 13. *Suppose we add 2-rotations to the group $H = K_4$ on symbols $\{a_1, a_2, a_3, a_4\}$, and the appearance of B_3° on 6 symbols is forbidden. If the group G is not generated by K_4's, and the A-image of G has subgroup $K_4 \rtimes (\Delta S_3)$, then it must be one of the following groups:*

- $K_4 \rtimes (\Delta S_3)$ *acting on 7 symbols as when W has type A, with two orbits $\{a_1, a_2, a_3, a_4\}$ and $\{a_5, a_6, a_7\}$;*
- $\mathbb{F}_2 \times (K_4 \rtimes (\Delta S_3))$*, where the subgroup $\mathbb{F}_2$ generated by the sign changes $(a_1)_-(a_2)_-(a_3)_-(a_4)_-$ on all 4 symbols $\{a_1, a_2, a_3, a_4\}$;*
- $\mathbb{F}_2^2 \rtimes (K_4 \rtimes (\Delta S_3))$*, where the subgroup $\mathbb{F}_2^2$ corresponds to even number of sign changes on symbols of $\{a_5, a_6, a_7\}$;*
- $\mathbb{F}_2^3 \rtimes (K_4 \rtimes (\Delta S_3))$*, where the subgroup $\mathbb{F}_2^3$ corresponds to even number of sign changes on symbols in $\{a_1, a_2, a_3, a_4\}$;*
- $\mathbb{F}_2^3 \rtimes (K_4 \rtimes (\Delta S_3))$*, where the subgroup $\mathbb{F}_2^3$ is generated by $(a_1)_-(a_2)_-(a_3)_-(a_4)_-$, $(a_1)_-(a_2)_-(a_5)_-(a_6)_-$, and $(a_2)_-(a_3)_-(a_6)_-(a_7)_-$;*
- $(\mathbb{F}_2 \times \mathbb{F}_2^2) \rtimes (K_4 \rtimes (\Delta S_3))$*, where the subgroup $\mathbb{F}_2$ corresponds to sign changes $(a_1)_-(a_2)_-(a_3)_-(a_4)_-$ on all 4 symbols $\{a_1, a_2, a_3, a_4\}$ and the subgroup $\mathbb{F}_2^2$ corresponds to even number of sign changes on symbols in $\{a_5, a_6, a_7\}$;*

- $(\mathbb{F}_2^3 \times \mathbb{F}_2^2) \rtimes (K_4 \rtimes (\Delta S_3))$, *where the subgroup* $\mathbb{F}_2^3$ *corresponds to even number of changes on symbols in* $\{1, 2, 3, 4\}$ *and the subgroup* $\mathbb{F}_2^2$ *corresponds to even number of changes on symbols in* $\{5, 6, 7\}$*;*
- $K_4 \rtimes (\Delta S_4)$ *acting on 8 symbols as when* W *has type* A*, with two orbits* $\{a_1, a_2, a_3, a_4\}$*, and* $\{a_5, a_6, a_7, a_8\}$*;*
- $\mathbb{F}_2 \times (K_4 \rtimes (\Delta S_4))$*, where the subgroup* $\mathbb{F}_2$ *is generated by the sign changes* $(a_1)_-(a_2)_-(a_3)_-(a_4)_-$ *on all 4 symbols* $\{a_1, a_2, a_3, a_4\}$*,*
- $(\mathbb{F}_2 \times \mathbb{F}_2) \rtimes (K_4 \rtimes (\Delta S_4))$*, where the first* $\mathbb{F}_2$ *is generated by the sign changes* $(a_1)_-(a_2)_-(a_3)_-(a_4)_-$ *on all 4 symbols* $\{a_1, a_2, a_3, a_4\}$*, and the second* $\mathbb{F}_2$ *is generated by the sign changes* $(a_5)_-(a_6)_-(a_7)_-(a_8)_-$ *on all 4 symbols* $\{a_5, a_6, a_7, a_8\}$*,*
- $(\mathbb{F}_2^3 \times \mathbb{F}_2) \rtimes (K_4 \rtimes (\Delta S_4))$*, where* $\mathbb{F}_2^3$ *corresponds to even number of sign changes on symbols* $\{a_1, a_2, a_3, a_4\}$*, and* $\mathbb{F}_2$ *is generated by the sign changes* $(a_5)_-(a_6)_-(a_7)_-(a_8)_-$ *on all 4 symbols* $\{a_5, a_6, a_7, a_8\}$
- $(\mathbb{F}_2^3 \times \mathbb{F}_2) \rtimes (K_4 \rtimes (\Delta S_4))$*, generated by* H *and* $(a_1\ a_2)(a_5\ a_6)$*,* $(a_1\ a_3)(a_5\ a_7)$*,* $(a_1\ \overline{a_2})(a_3\ \overline{a_4})$*,* $(a_5\ \overline{a_6})(a_7\ \overline{a_8})$*,* $(a_1\ \overline{a_2})(a_5\ \overline{a_6})$*;*
- $(\mathbb{F}_2^3 \times \mathbb{F}_2^3) \rtimes (K_4 \rtimes (\Delta S_4))$*, where the first* $\mathbb{F}_2^3$ *corresponds to even number of sign changes on symbols* $\{a_1, a_2, a_3, a_4\}$*, and the second* $\mathbb{F}_2^3$ *corresponds to even number of sign changes on symbols* $\{a_5, a_6, a_7, a_8\}$*;*
- $(\mathbb{F}_2 \times K_4) \rtimes S_4$*, by adding* $(a_1\ a_2)(a_5\ a_6)$*,* $(a_2\ a_3)(a_6\ a_7)$ *and* $(a_1\ \overline{a_2})(a_7\ a_8)$ *into* H*;*

Proof. We have K_4 on symbols $\{a_1, a_2, a_3, a_4\}$, and add two 2-rotations sharing exactly one common symbol in $\{a_1, a_2, a_3, a_4\}$, then each of the 2-rotations must have exactly 2 common symbols with K_4 on $\{1, 2, 3, 4\}$, and the 2 common symbols must be in the same 2-cycle of the 2-rotations. In addition, the remaining 2-cycles of the two 2-rotations also share exactly 1 common symbol. The common 2-cycles may have 0 or 1 distinct signs with the K_4 on $\{1, 2, 3, 4\}$, Then there are three cases.

- The two common 2-cycles with K_4 in the two 2-rotations have same signs as K_4. For example, the 2-rotations are $(a_1\ a_2)(a_5\ a_6)$ and $(a_1\ a_3)(a_5\ a_7)$. Then the two 2-rotations and the K_4 will generate the group $K_4 \rtimes (\Delta S_3)$.

- For the two common 2-cycles with K_4 in the two 2-rotations, one has the same signs as K_4, and the other has 1 distinct sign from K_4. For example, the 2-rotations are $(a_1\ a_2)(a_5\ a_6)$ and $(a_1\ \overline{a_3})(a_5\ a_7)$. They will generate $\mathbb{F}_2 \times (K_4 \rtimes (\Delta S_3))$, where K_4 acts on symbols $\{a_1, a_2, \overline{a_3}, \overline{a_4}\}$, and $\mathbb{F}_2$ is generated by sign changes $(a_1)_-(a_2)_-(a_3)_-(a_4)_-$.

- Both of the two common 2-cycles with K_4 in the two 2-rotations have 1 distinct sign from K_4. For example, the 2-rotations are $(a_1\ \overline{a_2})(a_5\ a_6)$ and $(a_1\ \overline{a_3})(a_5\ a_7)$. Then the 2-rotation $(a_2\ a_3)(a_6\ a_7)$ will be generated. It is equivalent to add $(a_1\ \overline{a_2})(a_5\ a_6)$ and $(a_2\ a_3)(a_6\ a_7)$ to K_4 on symbols $\{a_1, a_2, a_3, a_4\}$. We can treat it exactly same as the second case.

When we add more 2-rotations to the groups above, If there are no new symbols acted on by G, then only 2-rotations with all two 2-cycles having the same symbols as some 2-rotations in the original group can be added. It is equivalent to add sign changes of all symbols in 1 or 2 2-cycles. Then the sign changes on symbols $\{a_1, a_2, a_3, a_4\}$ may be

- no sign changes, corresponding to trivial group; or
- sign changes on all 4 symbols, corresponding to $\mathbb{F}_2$; or
- sign changes on an even number of symbols, corresponding to $\mathbb{F}_2^3$.

The sign changes on symbols $\{a_5, a_6, a_7\}$ may be

- no sign changes, corresponding to trivial group; or
- sign changes on an even number of symbols, corresponding to $\mathbb{F}_2^2$.

Also, the sign changes can be $\mathbb{F}_2^3$, generated by $(a_1)_-(a_2)_-(a_3)_-(a_4)_-$, $(a_1)_-(a_2)_-(a_5)_-(a_6)_-$, and $(a_2)_-(a_3)_-(a_6)_-(a_7)_-$. Thus we are able to get the groups with A-image of $K_4 \rtimes (\Delta S_3)$ in Proposition 13, and actually all these groups contain a subgroup $K_4 \rtimes (\Delta S_3)$ on 7 symbols.

If there are some new symbols introduced when we add new 2-rotations, we may assume we already have the group $K_4 \rtimes (\Delta S_3)$ on 7 symbols $\{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ as before. There will be only one new symbol a_8 can be introduced. Without loss of generality, we may assume the 2-rotation is $(a_1\ a_2)(a_7\ a_8)$ or $(a_1\ \overline{a_2})(a_7\ a_8)$.

If we add $(a_1\ a_2)(a_7\ a_8)$, the group $K_4 \rtimes (\Delta S_4)$ will be generated. By adding possible sign changes on symbols in 1 or 2 2-cycles, we will have the following groups:

- $K_4 \rtimes (\Delta S_4)$ acting on 8 symbols as when W has type A, with two orbits $\{a_1, a_2, a_3, a_4\}$, and $\{a_5, a_6, a_7, a_8\}$;
- $\mathbb{F}_2 \times (K_4 \rtimes (\Delta S_4))$, where the subgroup $\mathbb{F}_2$ is generated by the sign changes $(a_1)_-(a_2)_-(a_3)_-(a_4)_-$ on all 4 symbols $\{a_1, a_2, a_3, a_4\}$,
- $(\mathbb{F}_2 \times \mathbb{F}_2) \rtimes (K_4 \rtimes (\Delta S_4))$, where the first $\mathbb{F}_2$ is generated by the sign changes $(a_1)_-(a_2)_-(a_3)_-(a_4)_-$ on all 4 symbols $\{a_1, a_2, a_3, a_4\}$, and the second $\mathbb{F}_2$ is generated by the sign changes $(a_5)_-(a_6)_-(a_7)_-(a_8)_-$ on all 4 symbols $\{a_5, a_6, a_7, a_8\}$,
- $(\mathbb{F}_2^3 \times \mathbb{F}_2) \rtimes (K_4 \rtimes (\Delta S_4))$, where $\mathbb{F}_2^3$ corresponds to an even number of sign changes on the symbols $\{a_1, a_2, a_3, a_4\}$, and $\mathbb{F}_2$ is generated by the sign changes $(a_5)_-(a_6)_-(a_7)_-(a_8)_-$ on all 4 symbols $\{a_5, a_6, a_7, a_8\}$
- $(\mathbb{F}_2^3 \times \mathbb{F}_2) \rtimes (K_4 \rtimes (\Delta S_4))$, generated by H and $(a_1\ a_2)(a_5\ a_6)$, $(a_1\ a_3)(a_5\ a_7)$, $(a_1\ \overline{a_2})(a_3\ \overline{a_4})$, $(a_5\ \overline{a_6})(a_7\ \overline{a_8})$, $(a_1\ \overline{a_2})(a_5\ \overline{a_6})$;
- $(\mathbb{F}_2^3 \times \mathbb{F}_2^3) \rtimes (K_4 \rtimes (\Delta S_4))$, where the first $\mathbb{F}_2^3$ corresponds to an even number of sign changes on the symbols $\{a_1, a_2, a_3, a_4\}$, and the second $\mathbb{F}_2^3$ corresponds to an even number of sign changes on the symbols $\{a_5, a_6, a_7, a_8\}$;

If we add $(a_1\ \overline{a_2})(a_7\ a_8)$, the sign changes on all 4 symbols $(a_1)_-(a_2)_-(a_3)_-(a_4)_-$ in $\{a_1, a_2, a_3, a_4\}$ will be generated, and the group H is the semi-direct product $(\mathbb{F}_2 \times K_4) \rtimes S_4$.

If some more 2-rotations are added to H, they will generate $(\mathbb{F}_2^3 \times \mathbb{F}_2) \rtimes (K_4 \rtimes (\Delta S_4))$ or $(\mathbb{F}_2^3 \times \mathbb{F}_2^3) \rtimes (K_4 \rtimes (\Delta S_4))$, which are listed before when we add $(a_1\ a_2)(a_7\ a_8)$ into $K_4 \rtimes (\Delta S_3)$ on $\{a_1, \ldots, a_7\}$.

□

Now we are able to consider all the subgroups of $W = B_n$ or $W = D_n$, which are generated by A-rotations (2-rotations or 3-cycles). We will consider which groups are able to produce 2-cycles (A-reflections) to having non-trivial relation with other groups.

Definition 6. *Suppose W is a Coxeter group, and H_0 is a rotation subgroup of W. A reflection $r \in R(W)$ is called a **double cover reflection** of H_0, if the subgroup H, generated by H_0 and r, is a double cover of H, and there exists some other $r' \neq r \in R(W)$ such that $rr' \in H_0$. All such reflections $r' \in R(W)$ and r itself gives a **double cover reflection class**. (It is easy to see that* *double cover reflection class gives an equivalence relation.) If a double cover reflection class contains a simple reflection, it is a* ***simple double cover reflection class****. Suppose H_1 and H_2 are two rotation subgroups acting on disjoint symbol sets, or $H_1 = H_2$ in W, r_1 and r_2 are respectively double cover reflections of H_1 and H_2, and suppose $r_1 \neq r_2$ when $H_1 = H_2$. Then the subgroup H generated by H_1 and H_2 with rotation $(r_1\ r_2)$, is called a* ***double cover product*** *of H_1 and H_2 with rotation $(r_1\ r_2)$. If both r_1 and r_2 are in some simple double cover reflection classes, then the double cover of $H_1 \times H_2$ with r_1, is called a* ***simple double cover product*** *of H_1 and H_2 with rotation $(r_1\ r_2)$.*

Summing up all the results in this section, we will get all subgroups generated by A-rotations (3-cycles or 2-rotations) in $W = B_n$ or $W = D_n$, by adding $r_1 r_2$ where r_1 and r_2 are in distinct double cover reflection class.

Summarizing the previous A-rotation subgroups, all the possible double cover reflection classes come from the following groups:

Theorem 8. *All the indecomposable A-rotation groups, which can not be written as the double cover product of some smaller subgroups, are given as follows:*

- *Trivial group with one double cover reflection class $(i\ j)$, or with one double cover reflection class $(i)_-$, or with two double cover reflection classes $(i\ j)$ and $(i\ \overline{j})$;*
- *Alt_k with one double cover reflection class $(i\ j)$;*
- *$\mathbb{F}_2^{i-1} \rtimes Alt_i$ with two double cover reflection classes $(i\ j)$ and $(i)_-$;*
- *$\mathbb{F}_2 \times Alt_4$ with one double cover reflection class $(i\ j)$;*
- *B_i° in S_{2i} with one double cover reflection class $(i\ j)$ where $\{i, j\}$ are two symbols in one block;*
- *$(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^{\times(k-1)} \rtimes Sym_k$, $(k \geq 2)$ with two double cover reflection classes $(i\ j)$ and $(i\ \overline{j})$, where $\{i, j\}$ are two symbols in one block;*
- *$(\mathbb{Z}/4\mathbb{Z})^{\times(k-1)} \rtimes Sym_k$ $(k \geq 2)$, with no double cover reflection class;*
- *$SL(2,3)$ generated by $(a_1\ a_2\ a_3)$ and $(a_1\ \overline{a_2}\ a_4)$, with no double cover reflection class;*
- *$PGL(3,2)$, with no double cover reflection class;*

- $\mathbb{F}_2^3 \rtimes PGL(3,2)$, *with no double cover reflection class;*
- $\mathbb{F}_2^6 \rtimes PGL(3,2)$, *with one double cover reflection class* $(i)_-$*;*
- $AGL(3,2)$, *with no double cover reflection class;*
- $\mathbb{F}_2^4 \rtimes AGL(3,2)$, *with no double cover reflection class;*
- $\mathbb{F}_2^7 \rtimes AGL(3,2)$, *with one double cover reflection class* $(i)_-$*;*
- *The dihedral group* Dil_{10} *acting on* 5 *symbols, with no double cover reflection class;*
- $\mathbb{F}_2^4 \rtimes Dil_{10}$, *with one double cover reflection class* $(i)_-$*;*
- *Twisted* Alt_5 *in* Sym_6 *on* 6 *symbols, with no double cover reflection class;*
- H_3 *where the A-image is twisted* Alt_5 *in* Sym_6*, with no double cover reflection class;*
- $\mathbb{F}_2^5 \rtimes Alt_5$, *with one double cover reflection class* $(i)_-$*;*
- ΔS_k *permuting* $2k$ *symbols, with no double cover reflection class;*
- $\mathbb{F}_2^{k-1} \rtimes \Delta S_k$, *with one double cover reflection class* $(i)_-$*, where* i *is a symbol in the orbit where the sign changes of* $\mathbb{F}_2^{k-1}$ *act;*
- $\Delta(\mathbb{F}_2)^{\times k-1} \rtimes S_k$, *with no double cover reflection class;*
- $\mathbb{F}_2^{2(k-1)} \rtimes \Delta S_k$, *with two double cover reflection classes* $(i)_-$ *and* $(j)_-$*, where* i *and* j *are two symbols in the two orbits of* ΔS_k *respectively;*
- $K_4 \rtimes (\Delta S_3)$, *with no double cover reflection class;*
- $\mathbb{F}_2 \times (K_4 \rtimes (\Delta S_3))$, *where the subgroup* $\mathbb{F}_2$ *corresponds to the sign changes on all 4 symbols* $\{a_1, a_2, a_3, a_4\}$*, with no double cover reflection class;*
- $\mathbb{F}_2^2 \rtimes (K_4 \rtimes (\Delta S_3))$, *where the subgroup* $\mathbb{F}_2^2$ *acts on* $\{a_5, a_6, a_7\}$*, with one double cover reflection class* $(a_5)_-$*;*
- $\mathbb{F}_2^3 \rtimes (K_4 \rtimes (\Delta S_3))$, *where the subgroup* $\mathbb{F}_2^3$ *acts on* $\{a_1, a_2, a_3, a_4\}$*, with one double cover reflection class* $(a_1)_-$*;*

- $\mathbb{F}_2^3 \rtimes (K_4 \rtimes (\Delta S_3))$, *where the subgroup* $\mathbb{F}_2^3$ *is generated by* $(a_1)_-(a_2)_-(a_3)_-(a_4)_-$, $(a_1)_-(a_2)_-(a_5)_-(a_6)_-$, *and* $(a_2)_-(a_3)_-(a_6)_-(a_7)_-$, *with no double cover reflection class;*

- $(\mathbb{F}_2 \times \mathbb{F}_2^2) \rtimes (K_4 \rtimes (\Delta S_3))$, *where the subgroup* $\mathbb{F}_2$ *corresponds to sign changes* $(a_1)_-(a_2)_-(a_3)_-(a_4)_-$ *on all 4 symbols* $\{a_1, a_2, a_3, a_4\}$ *and the subgroup* $\mathbb{F}_2^2$ *corresponds to an even number of sign changes on symbols in* $\{a_5, a_6, a_7\}$, *with one double cover reflection class* $(a_5)_-$*;*

- $(\mathbb{F}_2^3 \times \mathbb{F}_2^2) \rtimes (K_4 \rtimes (\Delta S_3))$, *where the subgroup* $\mathbb{F}_2^3$ *acts on* $\{a_1, a_2, a_3, a_4\}$ *and* $\mathbb{F}_2^2$ *acts on* $\{a_5, a_6, a_7\}$, *with two double cover reflection classes* $(a_1)_-$ *and* $(a_5)_-$*;*

- $K_4 \rtimes (\Delta S_4)$, *with no double cover reflection class;*

- $\mathbb{F}_2 \times (K_4 \rtimes (\Delta S_4))$, *where the subgroup* $\mathbb{F}_2$ *is generated by the sign changes* $(a_1)_-(a_2)_-(a_3)_-(a_4)_-$ *on one orbit* $\{a_1, a_2, a_3, a_4\}$, *with no double cover reflection class;*

- $(\mathbb{F}_2 \times \mathbb{F}_2) \rtimes (K_4 \rtimes (\Delta S_4))$, *where the first* $\mathbb{F}_2$ *is generated by the sign changes* $(a_1)_-(a_2)_-(a_3)_-(a_4)_-$ *on one orbit* $\{a_1, a_2, a_3, a_4\}$, *and the second* $\mathbb{F}_2$ *is generated by the sign changes* $(a_5)_-(a_6)_-(a_7)_-(a_8)_-$ *on the other orbit* $\{a_5, a_6, a_7, a_8\}$, *with no double cover reflection class;*

- $(\mathbb{F}_2^3 \times \mathbb{F}_2) \rtimes (K_4 \rtimes (\Delta S_4))$, *where* $\mathbb{F}_2^3$ *acts on one orbit* $\{a_1, a_2, a_3, a_4\}$, *and* $\mathbb{F}_2$ *is generated by the sign changes* $(a_5)_-(a_6)_-(a_7)_-(a_8)_-$ *on the other orbit* $\{a_5, a_6, a_7, a_8\}$, *with one cover reflection class* $(a_1)_-$*;*

- $(\mathbb{F}_2^3 \times \mathbb{F}_2) \rtimes (K_4 \rtimes (\Delta S_4))$, *generated by* $K_4 \rtimes S_3$ *on two orbits* $\{a_1, a_2, a_3, a_4\}$ *and* $\{a_5, a_6, a_7\}$, *and rotations* $(a_1\ \overline{a_2})(a_3\ \overline{a_4})$, $(a_5\ \overline{a_6})(a_7\ \overline{a_8})$, $(a_1\ \overline{a_2})(a_5\ \overline{a_6})$, *with no double cover reflection class;*

- $(\mathbb{F}_2^3 \times \mathbb{F}_2^3) \rtimes (K_4 \rtimes (\Delta S_4))$, *where the first* $\mathbb{F}_2^3$ *acts on one orbit* $\{a_1, a_2, a_3, a_4\}$, *and the second* $\mathbb{F}_2^3$ *acts on the other orbit* $\{a_5, a_6, a_7, a_8\}$, *with two double cover reflection classes* $(a_1)_-$ *and* $(a_5)_-$*;*

- $(\mathbb{F}_2 \times K_4) \rtimes S_4$, *generated by* $K_4 \rtimes S_3$ *on two orbits* $\{a_1, a_2, a_3, a_4\}$ *and* $\{a_5, a_6, a_7\}$, *and rotations* $(a_1\ \overline{a_2})(a_7\ a_8)$.

2.2.6 The indecomposable groups generated by D-rotations

D-rotations are A-rotations together with the 2-sign-changes $(a)_-(b)_-$. we will see if any other types of rotation subgroup are generated in addition to the double cover products of groups listed in Theorem 8.

Theorem 9. *The indecomposable rotation subgroups H in $W = D_n$ can be written as the double cover product, with A-rotations or 2-sign-changes, of the following groups:*

- *The A-rotation groups listed in Theorem 8;*
- *The semi-direct product $((\mathbb{Z}/4\mathbb{Z})^{\times(k-1)} \times \mathbb{Z}/2\mathbb{Z}) \rtimes Sym_k$ $(k \geq 2)$, where $(\mathbb{Z}/4\mathbb{Z})^{\times(k-1)} \times \mathbb{Z}/2\mathbb{Z}$ is the subgroup $\{(a_1, \ldots, a_n) \mid 2(\sum_{1\leq i\leq n} a_i) = 0 \in \mathbb{Z}/4\mathbb{Z}\}$ of $(\mathbb{Z}/4\mathbb{Z})^{\times k}$, with each component corresponding to $(2i-1\ 2i)_-$, and Sym_k corresponds to permutations among blocks $\{2i-1, 2i\}$ $(1 \leq i \leq k)$. It has no double cover reflection class;*
- *$\Delta B_k = \Delta(\mathbb{F}_2^k \rtimes S_k)$, with no double cover reflection class.*

Proof. If the added 2-sign-changes have their 2 symbols in the same orbit of the A-image, and if the A-image is one of the groups A_k, $PGL(3,2)$, $AGL(3,2)$, Dil_{10}, ΔA_k, $K_4 \rtimes (\Delta S_3)$, or $K_4 \rtimes (\Delta S_4)$. Then all sign changes on an even number of symbols can be generated, and no new groups will be obtained, this is because the A-image orbit with at least 2 symbols are 2-transitive,

If the A-image is B_k with k pairs of symbols $\{a_{2i-1}, a_{2i}\}$, and the 2 symbols are in distinct pairs, then all sign changes on an even number of symbols can be generated, which is a group already listed in Theorem 8.

When the 2 symbols are in the same pair, if H has a subgroup B_k°, the obtained group is already listed in Theorem 8.

If $H = (\mathbb{Z}/4\mathbb{Z})^{\times(k-1)} \rtimes Sym_k$ $(k \geq 2)$, we will have a new group by adding 2-sign-change $(a_1)_-(a_2)_-$, where a_1, a_2 are in the same block of the A-image B_k°.

The new group is the semi-direct product $((\mathbb{Z}/4\mathbb{Z})^{\times(k-1)} \times \mathbb{Z}/2\mathbb{Z}) \rtimes Sym_k$ $(k \geq 2)$, where $(\mathbb{Z}/4\mathbb{Z})^{\times(k-1)} \times \mathbb{Z}/2\mathbb{Z}$ is the subgroup $\{(a_1, \ldots, a_n) \mid 2(\sum_{1\leq i\leq n} a_i) = 0 \in \mathbb{Z}/4\mathbb{Z}\}$ of $(\mathbb{Z}/4\mathbb{Z})^{\times k}$, with each component corresponding to $(2i-1\ 2i)_-$, and Sym_k corresponds to permutations of $\{2i-1, 2i\}$ $(1 \leq i \leq k)$.

If the added 2-sign-changes have their 2 symbols in distinct orbits, only when $W = \Delta A_k$ it can generate ΔB_k, which has no double cover reflection class.

Otherwise, the new groups will be obtained as the double cover products of groups listed in Theorem 9 with A-rotations and 2-sign-changes. □

2.2.7 The indecomposable groups generated by B-rotations

For B-rotations, apart from D-rotations (A-rotations and 2-sign-changes), we may also have the composition of one 2-cycle and 1 sign change: $(a\ b)(c)_-$, or $(a\ \bar{b})(c)_-$.

Theorem 10. *All the rotation subgroups in $W = B_n$ are obtained by double cover products of groups in Theorem 8 and 9 with B-rotations, or*

- *Dil_6 in B_3 generated by $(a_1\ a_2)(a_3)_-$ and $(a_1\ a_3)(a_2)_-$, with no double cover reflection class;*
- *Dil_{16} in B_4 generated by $(a_1\ a_2)(a_3\ a_4)$ and $(a_1\ a_3)(a_2)_-$, with no double cover reflection class;*
- *$Dil_8 = \Delta B_2$ in B_4 generated by $(a_1\ a_2)(a_3)_-$ and $(a_3\ a_4)(a_2)_-$, with no double cover reflection class.*

Proof. Suppose we add $(a_1\ a_2)(a_1)_-$ or a 2-cycle-1-sign-change $(a_1\ a_2)(a_3)_-$ into the rotation subgroup H. The element $(a_1\ a_2)(a_1)_-$ with 3-cycles or 2-rotations or 2-sign-changes will only generate subgroups as a double cover product of those in Theorem 9.

For the rotation $(a_1\ a_2)(a_3)_-$, if it intersects some 3-cycle, and only shares 1 or 2 common symbols, then only the double cover product of Alt_k ($k \geq 3$), $\mathbb{F}_2^{k-1} \rtimes Alt_k$, or the trivial group, can be generated.

If they share 3 common symbols, there is only one new type of group Dil_6 generated by $(a_1\ a_2\ a_3)$ and $(a_1\ \overline{a_2})(a_3)_-$. If H is expanded by some other rotations, then the subgroup $\mathbb{F}_2^{k-1} \rtimes Alt_k$ acting on a set of symbols containing a_1, a_2, and a_3, will appear.

If the rotation $(a_1\ a_2)(a_3)_-$ intersects with some 2-rotations, there is only one new type of group Dil_{16}, which is generated by $(a_1\ a_2)(a_3)_-$ and $(a_1\ a_3)(a_2\ a_4)$. If H is expanded by some other rotations, then $\mathbb{F}_2^{k-1} \rtimes Alt_k$ acting on the orbit containing $a_1, \ldots, a_4$ will be generated.

For the intersection of two 2-cycle-1-sign-changes, H will generate one new type of group $\Delta(B_2)$, which is generated by $(a_1\ a_2)(a_3)_-$ and $(a_3\ a_4)(a_1)_-$. There are no other subgroups without the appearance of $\mathbb{F}_2^{k-1} \rtimes Alt_k$, or Dil_6 and Dil_{16} described above.

In summary, all the rotation subgroups in $W = B_n$ will be the double cover product of the groups in Theorem 8, 9 and 10. $\square$

Chapter 3

Classification of Quasiparabolic Subgroups in Finite Classical Coxeter Groups

3.1 Quasiparabolic subgroups in finite Coxeter group of type A

We consider the double cosets $W_I w H$, where W_I is a standard parabolic subgroup of W, and $H \leq W$ is a subgroup of W. Our philosophy is to study the candidate quasiparabolic subgroups in H among those in the standard parabolic subgroup W_I, and go by induction on the set I of simple reflections.

Proposition 14. *[14] Suppose the transitive scaled W-set (X, ht) has a unique W-minimal element, and the stabilizer of that element is a quasiparabolic subgroup. Then X is quasiparabolic. On the other hand, if X is quasiparabolic, then the stabilizer of a minimal element of X is a quasiparabolic subgroup of W.*

If w is a minimal representative of the double coset $W_I w H$, and H is a quasiparabolic subgroup of W, then the left cosets $W_I w H/H$ form a transitive quasiparabolic W_I-set, and by Proposition 14, the stabilizer $H_{w,I}$ of the minimal element wH is a quasiparabolic subgroup of W_I.

In the following examples, we may see a few standard parabolic subgroups W_I, and calculate some subgroups $H_{w,I}$ of W_I serving as stabilizers of minimal elements wH. (We regard W as

$A_{n-1} = S_n$ on n symbols $[n] = \{1, 2, \dots, n\}$, with simple reflections $s_i = (i\ i+1)$.)

Example 2. *Let $I = \{s_i : 1 \leq i \leq n-2\}$. The elements w satisfying that*

$$w(i) = \begin{cases} i, & 1 \leq i \leq j-1; \\ n, & i = j; \\ i-1, & j+1 \leq i \leq n, \end{cases}$$

where j is a maximal symbol of an orbit of H, are minimal representatives of the double coset $W_I wH$. In particular, w is a minimal representative of the coset wH under the W_I action. Then the stabilizer $H_{j,I} = wStab_{H,j}w^{-1} \leq W_I$ of wH is obtained by relabeling the symbols as follows,

- *Taking the stabilizer $Stab_{H,j}$ of symbol j in H;*
- *deleting the symbol j;*
- *moving the symbols from $j+1$ to n one smaller than before, so that the symbol lies in $[n-1]$.*

Since $W_I wH/H$ is a quasiparabolic W_I-set, the stabilizer $H_{j,I}$ of wH, is quasiparabolic in W_I.

Similarly, if j is a minimal symbol of an orbit of H, we can pick $I = \{s_i : 2 \leq i \leq n-1\}$, and obtain the stabilizer $H_{j,I}$ by,

- *taking the stabilizer $Stab_{H,j}$ of symbol j in H;*
- *deleting the symbol j;*
- *moving the symbols from 1 to $j-1$ one larger than before. (For convenience of induction, we may translate the new symbols between 2 and n, to 1 to $n-1$, so that the symbols lie in [n-1].)*

If H is a quasiparabolic subgroup of W, then $H_{j,I}$ is a quasiparabolic subgroup of W_I.

Now we will list a few subgroups H of S_n, which are not quasiparabolic, and they will be helpful in ruling out many subgroups which are not quasiparabolic.

Example 3. *In the group $W = S_4$, the subgroup $H_1 = Alt_3$ generated by the 3-cycle $(1\ 2\ 4)$, the subgroup $H_2 = \Delta(\mathbb{Z}/2\mathbb{Z})$ generated by the 2-rotation $(1\ 3)(2\ 4)$, and the subgroup $H_3 = \Delta(\mathbb{Z}/2\mathbb{Z})$ generated by the 2-rotation $(1\ 4)(2\ 3)$ are not quasiparabolic.*

In the group $W = S_5$*, the subgroup* $H_1 = \Delta(\mathbb{Z}/2\mathbb{Z})$ *generated by the 2-rotation* $(1\ 3)(4\ 5)$*, the subgroup* $H_2 = K_4$ *acting on symbols* $1, 3, 4, 5$*, and the subgroup* $H_3 = K_4$ *acting on symbols* $1, 2, 4, 5$*, are not quasiparabolic.*

The examples suggest that the subgroups H, whose orbits are not on consecutive symbols, are not quasiparabolic in $W = S_n$.

More precisely, for any two symbols $i, j \in [n]$ $(i < j)$, if $\exists h \in H$, s.t., $h(i) = j$, then $\forall k$ satisfying $i < k < j$, $\exists \tilde{h} \in H$, s.t., $h(i) = k$. In this case, we say H has **consecutive orbits**.

Theorem 11. *For any quasiparabolic subgroup* H *of* S_n*,* H *has consecutive orbits.*

Proof. Suppose H is a quasiparabolic subgroup and has an orbit on non-consecutive symbols. We will obtain a contradiction by the induction method in Example 2.

If H is an odd subgroup, the components will be given by Theorem 3. Now consider the component with a non-consecutive orbit.

If the component is S_k, then a copy of S_2 with two non-consecutive symbols will be obtained, but a quasiparabolic S_2 should contain a simple reflection, forcing the orbit to contain two neighboring symbols. So the original H is not quasiparabolic.

If the component is B_k, then a copy of B_2 with four non-consecutive symbols will be obtained. By taking their even subgroup K_4, by Example 3, the K_4 with non-consecutive four symbols will not be quasiparabolic. So the original H is not quasiparabolic.

If the component H_0 is the even subgroup of a direct products of S_k's and B_k's, then we are able to delete symbols as in Example 2 and claim H is not quasiparabolic. In particular, if some S_k° has non-consecutive orbits, we will have symbols such that $a < b < c$ and a, c is in the orbit, but b is not. We are able to reduce H_0 to be $\Delta(\mathbb{Z}/2\mathbb{Z})$ and keep the symbols a, b, c, so the $\Delta(\mathbb{Z}/2\mathbb{Z})$ has non-consecutive orbits, and hence is not quasiparabolic. On the other hand, if some B_k° has non-consecutive orbits, it can be reduced to K_4 on non-consecutive orbit, hence is not quasiparabolic. So H is not quasiparabolic in this case.

If the component is ΔS_k, Dil_{10}, or twisted Alt_5 in S_6, it can be reduced to $\Delta(\mathbb{Z}/2\mathbb{Z})$ with non-consecutive orbits. So the original H is not quasiparabolic.

If the component is $PGL(3, 2)$ or $AGL(3, 2)$, it can be reduced to B_3° with non-consecutive orbits. So the original H is not quasiparabolic.

If the component is $K_4 \rtimes (\Delta S_3)$ or $K_4 \rtimes (\Delta S_4)$, it can be reduced to $(B_2 \times S_2)^\circ$ or ΔS_3 on non-consecutive orbits. Hence the original H is not quasiparabolic.

From this discussion on all subgroups generated by rotations, or their double covers, H can be reduced to one of the cases in Example 3, which are not quasiparabolic. So a quasiparabolic subgroup H must have all its orbits acting on consecutive symbols. □

Example 4. *Let $I = \{s_i : i \neq n-2\}$. Consider the elements w satisfying that*

$$w(i) = \begin{cases} i, & 1 \leq i \leq j-1; \\ n-1, & i = j; \\ i-1, & j+1 \leq i \leq k-1; \\ n, & i = k; \\ i-2, & k+1 \leq i \leq n, \end{cases}$$

where $j < k$ are two symbols in $[n]$, such that for any $w' \in H$, $w'(j) + w'(k) \leq j + k$. Then these types of w are minimal representatives of the double coset $W_I wH$. In particular, w is a minimal representative of the coset wH under W_I action. Then the subgroup $H_{j,k,I} = wStab_{H,j,k}w^{-1} \leq W_I$ of wH, is obtained by,

- *taking the subgroup $Stab_{H,j,k} = \{w \in H : \{w(j), w(k)\} = \{j, k\}\}$ in H;*
- *moving symbols j, k correspondingly to $n-1$, n, and keeping the order of the other symbols, so that the element lies in W_I.*

Since $W_I wH/H$ is a quasiparabolic W_I-set, the subgroup $H_{j,k,I}$ is quasiparabolic in W_I.

A similar operation can be done for the two smallest symbols in one orbit, and it will be omitted here.

In the following examples, we will give a few quasiparabolic subgroups of S_n, besides the standard quasiparabolic subgroups.

Definition 7. *Suppose W is a Coxeter group. Denote W^+ the semidirect product of W by the group of permutations of S that induce Coxeter automorphisms of W. An involution $\iota \in W^+$ is* ***perfect*** *if for all $r \in R(W)$, $(r\iota)^4 = 1$.*

Example 5. *[14] The W-action by conjugation on the perfect involutions $\mathcal{I}$, with height function $(l(\iota)-n)/2$, makes $\mathcal{I}$ a quasiparabolic W-set. One orbit of $\mathcal{I}$ is the set of fixed-point-free involutions. Then the stabilizer of the minimal (or maximal) element, which is B_n with the n blocks as symbols $\{1,2\},\ldots,\{2n-1,2n\}$ (or $\{1,2n\},\ldots,\{n,n+1\}$), is a quasiparabolic subgroup.*

By [14], the image and preimage of quasiparabolic subgroups under Coxeter homomorphism are also quasiparaoblic.

Proposition 15. *[14] Let $\phi : W \to W'$ be a Coxeter homomorphism. If $H \subset W$ is quasiparabolic, then so is $\phi(H)$; if $H' \subset W'$ is quasiparabolic, then so is $\phi^{-1}(H')$.*

Example 6. *[14] The image of the Coxeter homomorphism*

$$S_n \xrightarrow{\Delta} S_n \times S_n \hookrightarrow S_m, \quad \textit{where } m \geq 2n,$$

with the symbol i mapped to a_1+i (or $a_1+(n+1-i)$) and a_2+i (or $a_2+(n+1-i)$), where $0 \leq a_1 \leq a_2 - n \leq m - 2n$, gives a subgroup $H = \Delta S_n$ in $W = S_m$. Since H is the image of a quasiparabolic subgroup of $S_n \times S_n$ under a Coxeter homomorphism, H is quasiparabolic in W.

Example 7. *For the group $AGL(3,2)$ in S_8, if the group $AGL(3,2)$ in S_8 is generated by B_4°'s with the two possible quasiparabolic case in Example 5, then it will be a quasiparabolic subgroup.*

For the group $PGL(3,2)$ in S_7, if the group $PGL(3,2)$ in S_7 is generated by B_3° with blocks $\{1,2\},\{3,4\},\{5,6\}$ and B_3 with blocks $\{2,7\},\{3,6\},\{4,5\}$, then it will be a quasiparabolic subgroup.

For the group $K_4 \rtimes (\Delta S_4)$ in S_8, if the group $K_4 \rtimes (\Delta S_4)$ in S_8 is generated by K_4 on symbols $\{1,2,3,4\}$ and $\{5,6,7,8\}$, and ΔS_4 in Example 6, then it is quasiparabolic. Also, if it is generated by K_4 on symbols $\{1,2,3,4\}$ and $\{5,6,7,8\}$, and 2-rotations $(1\ 2)(6\ 7)$ and $(2\ 3)(5\ 6)$, then it is also quasiparabolic. When we take the stabilizer of the symbol 8, we will get corresponding quasiparabolic subgroup $K_4 \rtimes (\Delta S_3)$ of S_7.

For the group Alt_5 in S_6, if the group Alt_5 in S_6 is generated by 2-rotations $(1\ 2)(3\ 4)$, $(1\ 2)(5\ 6)$ and $(2\ 3)(4\ 5)$, then it is quasiparabolic. When we take the stabilizer of the symbol 6, we will get a corresponding quasiparabolic subgroup Dil_{10} of S_5.

We will next show that there will not be more quasiparabolic subgroups than those in the examples.

Proposition 16. *For the subgroup $H = B_n^\circ$ in $W = S_{2n}$, H is quasiparabolic if and only if H is described in Example 5.*

Proof. For the case $n = 2$, $B_2^\circ = K_4$ on 4 symbols, which is trivial.

For the case $n \geq 3$, consider the subgroup H_1 obtained by modifying H in Example 2 for the symbol 1. Then we know the other symbol paired with symbol 1 in same block should be 2 or n, otherwise the stabilizer H_1 of the symbol 1 will have non-consecutive symbols in one orbit, and H_1 will not be quasiparabolic. This forces H non-quasiparabolic. Similarly, the symbol n should be paired with the symbol 1 or $n-1$. So we have two cases:

1. the symbol 1 is paired with the symbol 2, and the symbol n is paired with the symbol $n-1$;

2. the symbol 1 is paired with the symbol n.

The two cases will give the two possible quasiparabolic $H = B_n^\circ$ subgroups in $W = S_{2n}$.

- If the symbol 1 is paired with the symbol 2, and the symbol n is paired with the symbol $n-1$, we claim H must have its n blocks as $\{2i-1, 2i\}$ $(1 \leq i \leq n)$. Otherwise, suppose $2j-1$ is the smallest odd symbol that is not paired with $2j$, then by deleting all symbols $1, 2, \ldots, 2j-2$ with the operation from Example 2, we have $H' = B_{n-j+1}^\circ < W' = S_{2(n-j+1)}$ on symbols $2j-1, 2j, \ldots, 2n$, but the symbol $2j-1$ is not paired with the symbol $2j$ or symbol n, forcing H' to be non-quasiparabolic in W', so H is non-quasiparabolic, too.

- If the symbol 1 is paired with the symbol n, we claim H must have its n blocks as $\{i, 2n+1-i\}$ $(1 \leq i \leq n)$. Otherwise, suppose $1 < j < n$ is the smallest symbol that is not paired with $2n+1-j$. Then by deleting all symbols $1, 2, \ldots, j-2$ with the operations from Example 2, we have $H' = B_{n-j+1}^\circ < W' = S_{2(n-j+1)}$ on symbols $j-1, j, \ldots, 2n-j+2$, and the symbol $j-1$ is paired with $2n-j+2$, while j is not paired with $2n-j+1$. Now consider the subgroup $H'' = B_{n-j-1}^\circ \times (\Delta S_2)$ of $W'' = S_{2(n-j)} \times S_2$, obtained by modifying H' as in Example 4. Then H'' will have an orbit consisting of symbols j' and $2n-j+2$, while $j' < 2n-j+1$. So H'' has an orbit with non-consecutive symbols, forcing H'' to be non-quasiparabolic in W''. Then H is non-quasiparabolic in W, too.

In summary, all quasiparabolic subgroups $H = B_n^\circ$ are described in Example 5. $\square$

Proposition 17. *For the subgroup $H = \Delta S_n$ in $W = S_m$, where $m \geq 2n$, H is quasiparabolic in W if and only if H is described in Example 6.*

Proof. From Theorem 11, the two orbits of $H = \Delta S_n$ should be consecutive. We suppose the two orbits are $\{a_1+1, \dots, a_1+n\}$ and $\{a_2+1, \dots, a_2+n\}$, where $0 \leq a_1 \leq a_2 - n \leq m - 2n$. By the operation from Example 2, the symbol $a_1 + 1$ and the symbol $a_1 + n$ should be paired with the symbol $a_2 + 1$ or $a_2 + n$. By the operation from Example 4 inductively as in the case of B_n above, H is uniquely determined after we fix the symbol in the orbit $\{a_2+1, \dots, a_2+n\}$ pairing to $a_1 + 1$. □

Proposition 18. *The quasiparabolic subgroups isomorphic to $AGL(3,2)$, $PGL(3,2)$, $K_4 \rtimes (\Delta S_3)$, $K_4 \rtimes (\Delta S_4)$, Dil_{10}, and twisted Alt_5, are all given in Example 7 (up to an isomorphism of the Coxeter diagram of S_n).*

Proof. For the subgroup $PGL(3,2)$ in S_7, the stabilizer of the symbol 7 is B_3°, which should be paired by $\{1,2\}$, $\{3,4\}$, $\{5,6\}$ or $\{1,6\}$, $\{2,5\}$, $\{3,4\}$. Also the stabilizer of the symbol 1 is B_3°, which should be paired by $\{2,3\}$, $\{4,5\}$, $\{6,7\}$ or $\{2,7\}$, $\{3,6\}$, $\{4,5\}$. Since there are no two K_4's having exactly 3 common symbols in $PGL(3,2)$, the quasiparabolic subgroup $PGL(3,2)$ in S_7 should be described as in Example 7.

For the subgroup $AGL(3,2)$ in S_8, the stabilizer of the symbol 1 or the symbol 8 is $PGL(3,2)$, which should be quasiparabolic as in Example 7. Then the only possibility is described in Example 7.

Consider the subgroup $K_4 \rtimes (\Delta S_3)$ in S_7, with orbits $\{1,2,3,4\}$ and $\{5,6,7\}$, the stabilizer of the symbol 1 is ΔS_3. Then the simple reflections (2 3) and (3 4) should be paired with the simple reflections (5 6) or (6 7), giving the two possible quasiparabolic subgroups in Example 7.

For the subgroup $K_4 \rtimes (\Delta S_4)$ in S_8, if the stabilizer of symbol 8 is $K_4 \rtimes (\Delta S_3)$, then the simple reflections (1 2) and (2 3) should be paired with the simple reflection (5 6) or (6 7), giving the two possible quasiparabolic subgroups in Example 7.

For the subgroup Dil_{10} in S_5, the stabilizer of the symbol 5 is ΔS_2, which should be (1 2)(3 4). Similarly, the quasiparabolic subgroup Dil_{10} should also have (2 3)(4 5).

For the subgroup twisted Alt_5 in S_6, the stabilizer of symbol 1 or 6 is Dil_{10}, which has been determined above. So the quasiparabolic subgroup twisted Alt_5 should also be described as in Example 7. □

$(S_{i_1} \times \ldots \times S_{i_k} \times B_{j_1} \times \ldots \times B_{j_l})^\circ$ is the even subgroup of $S_{i_1} \times \ldots \times S_{i_k} \times B_{j_1} \times \ldots \times B_{j_l}$. So we only need to find all possible quasiparabolic subgroup of the form $S_{i_1} \times \ldots \times S_{i_k} \times B_{j_1} \times \ldots \times B_{j_l}$ in group S_n.

In fact, all S_i's and B_j's need to have consecutive symbols, and B_j's should have symbols paired as in Example 5. In particular, we need to rule out a case which is not treated before. For the quasiparabolic subgroup B_2, we can not have the pairing on the symbols $\{1, 3\}$ and $\{2, 4\}$, since the stabilizer of symbol 4 is S_2 on symbol 1 and 3, which is non-quasiparabolic.

When all the S_i's and B_j's are quasiparabolic, then their direct product is quasiparabolic. We list these quasiparabolic subgroups in the following theorem.

Theorem 12. *The subgroup $(S_{i_1} \times \ldots \times S_{i_k} \times B_{j_1} \times \ldots \times B_{j_l})^\circ$ in S_n is quasiparabolic if and only if*

1. *All S_i's and B_j's acts on consecutive symbols AND*
2. *All B_j's have symbols paired as in one of the two cases in Example 5.*

Combining Theorem 12 and Proposition 17 and 18, all even quasiparabolic subgroups of S_n are given by the examples in Theorem 12, Example 6 and 7. Note that only the case in Theorem 12 can serve as the even subgroup of some odd quasiparabolic subgroup, by adding one simple reflection as a generator. Then we state the classification of all quasiparabolic subgroups at the end of this part as a theorem.

Theorem 13. *All quasiparabolic subgroups of $W = S_n$ are the direct product of these groups acting on consecutive symbols:*

1. *S_i;*
2. *B_j on j pairs of symbols satisfying one of the cases in Example 5;*
3. *$(S_{i_1} \times \ldots \times S_{i_k} \times B_{j_1} \times \ldots \times B_{j_l})^\circ$ as in Theorem 12;*
4. *ΔS_i as in Example 6;*
5. *$K_4 \rtimes (\Delta S_3)$ acting on 7 symbols as in Example 7;*
6. *$K_4 \rtimes (\Delta S_4)$ acting on 8 symbols as in Example 7;*

7. *$PGL(3,2)$ acting on 7 symbols as in Example 7;*

8. *$AGL(3,2)$ acting on 8 symbols as in Example 7;*

9. *Dil_{10} acting on 5 symbols as in Example 7;*

10. *Twisted Alt_5 in S_6 as in Example 7.*

3.2 Preparation work for quasiparabolic subgroups in finite Coxeter groups of type B and D

3.2.1 Connection between quasiparabolic subgroups in B_n, D_n and those in A_n

In this part, we are going to use Coxeter homomorphism to get information about quasiparabolic subgroups in finite Coxeter groups of type B and D from those in finite Coxeter groups of type A. We present the Coxeter group $W = B_n$ or D_n acting as signed permutations. Suppose $W = B_n$ has simple reflections $S = \{(1)_-, (1\ 2), \ldots, (n-1\ n)\}$. Denote $s_i = (n-i\ n+1-i)$ where $1 \leq i \leq n-1$, and $s_n = (1)_-$. The group $W = D_n$ has simple reflections $S = \{(1\ \overline{2}), (1\ 2), \ldots, (n-1\ n)\}$. Denote $s_i = (n-i\ n+1-i)$ where $1 \leq i \leq n-1$, and $s_n = (1\ \overline{2})$.

For the Coxeter homomorphisms

$$B_n \to A_{n-1} \times A_1,$$

and

$$D_n \to A_{n-1},$$

the A-image of quasiparabolic subgroup H of B_n or D_n should have their orbits on consecutive symbols, similar to the case in A_n.

3.2.2 Induction method by double cosets

Similar to the method of classifying quasiparabolic subgroups in A_n, we may go through some standard parabolic subgroups, and see the operation on subgroups H in B_n and D_n.

Example 8. *Suppose $W = B_n$ with the given simple reflections at the beginning of Section 2.2. Let $I = \{(1)_-, (1\ 2), \ldots, (n-2\ n-1)\} \subset S$, and let W_I be the corresponding standard parabolic subgroup. Suppose H is quasiparabolic in W, and j is a maximal symbol (including the sign) in an orbit of H. If $j > 0$, we consider the element $w \in W$ satisfying that*

$$w(i) = \begin{cases} i, & 1 \le i \le j-1; \\ n, & i = j; \\ i-1, & j+1 \le i \le n, \end{cases}$$

while if $j < 0$, let $w \in W$ satisfy that

$$w(i) = \begin{cases} i, & 1 \le i \le -j-1; \\ -n, & i = -j; \\ i-1, & -j+1 \le i \le n. \end{cases}$$

Then w is a minimal representative of the double coset $W_I w H$. In particular, w is a minimal representative of the coset wH under the W_I action. Denote $Stab_{H,j}$ as the stabilizer of symbol j in H. Then the stabilizer $H_{j,I} = wStab_{H,j}w^{-1} \le W_I$ of wH, is obtained by,

- *taking the stabilizer $Stab_{H,j}$ of the symbol j in H;*
- *deleting the symbol j (and $-j$);*
- *moving the symbols from $j+1$ to n one smaller than before (and symmetrically for $-(j+1)$ to $-n$), so that the elements are acted on by W_I.*

Since $W_I wH/H$ is a quasiparabolic W_I-set, the stabilizer $H_{j,I}$ of wH, is quasiparabolic in W_I.

When $W = D_n$, still letting $I = \{(1\ \overline{2}), (1\ 2), \ldots, (n-2\ n-1)\} \subset S$, let W_I be the corresponding standard parabolic subgroup. Suppose H is quasiparabolic in W, and j is a maximal symbol (including the sign) in an orbit of H. If $j > 0$, we consider the element $w \in W$ satisfying that

$$w(i) = \begin{cases} i, & 1 \le i \le j-1; \\ n, & i = j; \\ i-1, & j+1 \le i \le n, \end{cases}$$

while if $j < 0$, let $w \in W$ satisfy that

$$w(i) = \begin{cases} -1, & i = 1; \\ i, & 2 \leq i \leq -j-1; \\ -n, & i = -j; \\ i-1, & -j+1 \leq i \leq n. \end{cases}$$

Similar to the case of $W = B_n$, we will obtain the stabilizer $H_{j,I}$, quasiparabolic in W_I by

- *taking the stabilizer $Stab_{H,j}$ of symbol j in H;*
- *deleting the symbol j (and $-j$);*
- *moving the symbols from $j+1$ to n one smaller than before (and symmetrically for $-(j+1)$ to $-n$). In addition, if $j < 0$, we need to add a negative sign for symbol 1, so that the element lies in W_I.*

Example 9. *Suppose $W = B_n$ with the given simple reflections at the beginning of Section 2.2. Let $I = \{(1)_-, (1\ 2), (2\ 3), \ldots, (n-3\ n-2), (n-1\ n)\} \subset S$, and let W_I be the corresponding standard parabolic subgroup. Let H be quasiparabolic in W, and let $j < k$ be two signed symbols in $\pm[n] = \{\pm 1, \ldots, \pm n\}$. Suppose $l \in \pm[n]$, define*

$$w_l = \begin{cases} s_1 \ldots s_{n-l}, & l > 0; \\ s_1 \ldots s_{n-1} s_n s_{n-1} \ldots s_{n+1-l}, & l < 0. \end{cases}$$

If for all $w' \in H$ such that $\{w'(j), w'(k)\} = \{j, k\}$, $w'(j) + w'(k) \leq j + k$, then the following $w = w_j w_k \in W$ is a minimal representative element of the double coset $W_I w H$.

In particular, w is a minimal representative of the coset wH under the W_I action. Denote by $Stab_{H,j}$ the stabilizer of symbol j in H. Then the stabilizer $H_{j,I} = wStab_{H,j}w^{-1} \leq W_I$ of wH, is obtained by,

- *taking the stabilizer $Stab_{H,j}$ of symbol j in H;*
- *moving the symbols j, k to $n-1, n$ and keeping the signs and order of other symbols, so that the element are acted on by W_I.*

Since $W_I wH/H$ is a quasiparabolic W_I-set, the stabilizer $H_{j,k,I}$ of wH, is quasiparabolic in W_I.

When $W = D_n$, still letting $I = \{(1\ \overline{2}), (1\ 2), (2\ 3), \ldots, (n-3\ n-2), (n-1\ n)\} \subset S$, W_I be the corresponding standard parabolic subgroup, H be quasiparabolic in W, and $j < k$ are two signed symbols in $\pm[n]$. Suppose $l \in \pm[n]$, define

$$w_l = \begin{cases} s_1 \ldots s_{n-l}, & l > 0; \\ s_1 \ldots s_{n-2} s_n s_{n-1} \ldots s_{n+1-l}, & l < 0. \end{cases}$$

If for all $w' \in H$ such that $\{w'(j), w'(k)\} = \{j, k\}$, $w'(j) + w'(k) \leq j + k$, then the following $w = w_j w_k \in W$ is a minimal representative element of the double coset $W_I w H$.

Similar to the case of $W = B_n$, we will obtain the stabilizer $H_{j,I}$, quasiparabolic in W_I by

- *taking the stabilizer $Stab_{H,j}$ of symbol j in H;*
- *moving the symbols from j, k to $n-1, n$ and keeping the signs and order of other symbols (possibly changing the sign of the new symbol 1), so that the element lies in W_I.*

3.2.3 Orbits not including its negative symbols

Suppose H is quasiparabolic in $W = B_n$ or $W = D_n$. In this part, we will show that those orbits of H not including its negative symbols will include symbols with same signs for the case of $W = B_n$, and either itself or its dual (by the nontrivial diagram automorphism switching 1 and -1) will have the same signs as the case of $W = D_n$.

Similar to the method for the case when W has type A, we will list a few non-quasiparabolic cases in small B_n or D_n, and use the induction operation to rule out those non-quasiparabolic subgroups in general B_n or D_n.

Example 10. *Suppose $W = B_2$, then the subgroup $H_1 = \mathbb{Z}/2\mathbb{Z}$ generated by $(1\ \overline{2})$ is not a quasiparabolic subgroup of W.*

Suppose $W = B_2 \times B_2$, then the subgroups $H_1 = \Delta(\mathbb{Z}/2\mathbb{Z})$ generated by $(1\ 2)(1'\ \overline{2'})$ and $H_2 = \Delta(\mathbb{Z}/2\mathbb{Z})$ generated by $(1\ \overline{2})(1'\ \overline{2'})$, are not quasiparabolic in W.

Suppose $W = B_3$, then the subgroup $H_1 = Alt_3$ generated by $(1\ 2\ \overline{3})$ or $(1\ \overline{2}\ 3)$ or $(1\ \overline{2}\ \overline{3})$ is not a quasiparabolic subgroup of W.

Suppose $W = B_4$, then the subgroup $H_1 = \Delta(\mathbb{Z}/2\mathbb{Z})$ generated by $(1\ \overline{2})(3\ 4)$, the subgroup $H_2 = \Delta(\mathbb{Z}/2\mathbb{Z})$ generated by $(1\ 2)(3\ \overline{4})$, the subgroup $H_3 = \Delta(\mathbb{Z}/2\mathbb{Z})$ generated by $(1\ \overline{2})(3\ \overline{4})$, *are not quasiparabolic subgroups of* W*. The subgroup* $H_4 = K_4$ *with an orbit on 4 signed-symbols* $\{1, (\pm)2, (\pm)3, (\pm)4\}$*, except when these 4 symbols have the same signs (i.e., at least one of symbols 2,3,4 has negative sign), is also not a quasiparabolic subgroup of* W*. From the non-quasiparabolicity of* H_2*, the group* B_2*, whose even subgroup is* $B_2^\circ = H_2 = K_4$ *on 4 signed-symbols not in same signs, is also non-quasiparabolic.*

Example 11. *Suppose* $W = D_3$ *(also the same as* A_3*, but we let* W *act on 3 signed-symbols), the subgroup* $H_1 = \mathbb{Z}/2\mathbb{Z}$ *generated by* $(2\ \overline{3})$*, and the subgroup* $H_2 = Alt_3$ *generated by* $(1\ 2\ \overline{3})$ *(or the dual generated by* $(\overline{1}\ 2\ \overline{3})$*,) are not quasiparabolic subgroups of* W*.*

Suppose $W = D_3 \times D_3$*, the subgroup* $H_1 = \Delta(\mathbb{Z}/2\mathbb{Z})$ *generated by* $(2\ \overline{3})(2'\ \overline{3'})$*, the subgroup* $H_2 = \Delta(S_3)$ *generated by* $(1\ 2\ 3)(1'\ 2'\ \overline{3'})$*, the subgroup* $H_3 = \Delta(S_3)$ *generated by* $(1\ 2\ \overline{3})(1'\ 2'\ \overline{3'})$*, are not quasiparabolic subgroups of* W*.*

Suppose $W = D_4$*, the subgroup* $H_1 = \Delta(\mathbb{Z}/2\mathbb{Z})$ *generated by* $(1\ 2)(3\ \overline{4})$*, the subgroup* $H_2 = K_4$ *with an orbit of 4 signed-symbols* $\{1, 2, (\pm)3, (\pm)4\}$*, when these 4 symbols do not have the same signs (i.e., at least one of symbols 3,4 has negative sign), and the subgroup* $H_3 = Alt_3$ *generated by 3-cycle on* $\{2, (\pm)3, (\pm)4\}$*, when these 3 symbols do not have the same signs (i.e., at least one of symbols 3,4 has negative sign), are not quasiparabolic subgroups of* W*. From the non-quasiparabolicity of* H_2*, the group* B_2*, whose even subgroup is* $B_2^\circ = H_2 = K_4$ *on 4 signed-symbols, where symbols 2,3,4 are not in the same signs, is also non-quasiparabolic in* W*.*

Suppose $W = D_5$*, then the subgroup* $H_1 = \Delta(\mathbb{Z}/2\mathbb{Z})$ *generated by* $(2\ \overline{3})(4\ 5)$*, and the subgroup* $H_2 = K_4$ *acting on 4 signed-symbols* $\{2, -3, 4, 5\}$*, are not quasiparabolic subgroups of* W*. By the non-quasiparabolicity of* H_2*, the group* B_2 *whose even subgroup is* $B_2^\circ = H_2 = K_4$ *on signed-symbols* $\{2, -3, 4, 5\}$ *is also non-quasiparabolic in* W*.*

Definition 8. *A subgroup* H *of finite classical Coxeter group* W *has* ***A form****, if*

- $H = H_A$*, where* H_A *is the A-image of* H*, and* H_A *is quasiparabolic in the A-image of* W*.*
- *Each orbit of* H *should have symbols with same signs (excluding the symbol 1 when the direct product component of* W *has type D).*

With these small non-quasiparabolic cases, we are able to prove the following theorem that all symbols in an orbit, which does not include a symbol and its negative, should have A form.

Theorem 14. *Suppose W is a finite classical Coxeter group, and H is a quasiparabolic subgroup of W. If i is a symbol that is not in the same orbit as $-i$ under the action of H, then H should have A form.*

Proof. Recall the classification of quasiparabolic subgroups in S_n on consecutive symbols in Theorem 13,

- S_i;
- B_j;
- $(S_{i_1} \times \ldots \times S_{i_k} \times B_{j_1} \times \ldots \times B_{j_l})^\circ$;
- ΔS_i;
- $K_4 \rtimes S_3$ acting on 7 symbols;
- $K_4 \rtimes S_4$ acting on 8 symbols;
- $PGL(3,2)$ acting on 7 symbols;
- $AGL(3,2)$ acting on 8 symbols;
- Dil_{10} acting on 5 symbols;
- Twisted Alt_5 in Sym_6.

Each orbit $\mathcal{O}$ of the H-action not including a symbols and its negative must become one of the orbits in the above subgroups. Suppose $\mathcal{O}$ has two symbols (not including the symbol 1 in the case of W has type D) with distinct signs. We list the set of positive symbols $\mathcal{O}_+ = \{i_1, \ldots, i_k\}$, and the set of negative symbols $\mathcal{O}_- = \{-j_1, \ldots, -j_l\}$, where $i_1 < \ldots < i_k$ and $j_1 < \ldots < j_l$. By the operation in Example 8 and 9, we can finally get the following subgroups on symbols of distinct signs, which are non-quasiparabolic, forcing H to be non-quasiparabolic.

- When $\mathcal{O}$ is the orbit of S_i, by the operation in Example 8, we may inductively delete the symbols $i_k, i_{k-1}, \ldots, i_2$ and $-j_l, -j_{l-1}, \ldots, -j_2$. We will get $\mathbb{Z}/2\mathbb{Z}$ on symbols (not including symbol 1 when W has type D) with distinct signs, which is non-quasiparabolic.

- When $\mathcal{O}$ is the orbit of B_i, we first conduct the operation in Example 8, so that $\mathcal{O}_+$ and $\mathcal{O}_-$ are non-empty, and (at least) one of $\mathcal{O}_+$ or $\mathcal{O}_-$ (we may suppose it's $\mathcal{O}_+$) has 1 or 2 remaining symbols. If both $\mathcal{O}_+$ and $\mathcal{O}_-$ have 1 or 2 remaining symbols, then $\mathbb{Z}/2\mathbb{Z}$ or B_2 on symbols (not including symbol 1 when W has type D) on distinct signs will be obtained, which are non-quasiparabolic.

  Otherwise, when $\mathcal{O}_+$ has 2 symbols i_1, i_2 remaining, if they are in the same pair, we are able to delete all but two symbols in $\mathcal{O}_-$ to get $B_2 \subset S_4$ in $W = B_4$ or $W = D_4$ where the symbols $2, 3, 4$ does not have same signs, which are non-quasiparabolic by Example 10 and 11. If the 2 symbols remaining in $\mathcal{O}_+$ are in distinct pairs, we are able to conduct the operation in Example 8 to eliminate the symbol i_2 in $\mathcal{O}_+$, and leave $\mathcal{O}_+$ with exactly 1 symbol i_1.

  When $\mathcal{O}_+$ has 1 symbol i_1, suppose i_1 is paired with some $-j_i \in \mathcal{O}_-$. We conduct the operation in Example 8, until $-j_{i+1}$, or we have $H = B_2$ with 4 remaining symbols not with the same signs. In this case we are able to show H is non-quasiparabolic by Example 10 and 11. Now the remaining symbols are $i_1, -j_1, -j_2, \ldots, -j_i$ with $i \geq 5$, where i_1 and $-j_i$ are in the same pair. We operate as in Example 9 for the symbols j_{i-1}, j_i. Then we will have a component K_4 including the symbols $i_1, -j_{j-1}, -j_i$ not in same signs, which is non-quasiparablic.

- When $\mathcal{O}$ is the orbit of the S_i's component or the B_j's component of $(S_{i_1} \times \ldots \times S_{i_k} \times B_{j_1} \times \ldots \times B_{j_l})^\circ$, we follow a similar argument as when $\mathcal{O}$ is the orbit of the S_i or B_j, to prove $\mathcal{O}$ should have same signs.

- When $\mathcal{O}$ is the orbit of ΔA_i, by the operation in Example 8, we may inductively delete the symbols $i_k, i_{k-1}, \ldots, i_2$ and $-j_l, -j_{l-1}, \ldots, -j_2$. We will get $\Delta(\mathbb{Z}/2\mathbb{Z})$ with at least one orbit on symbols (not including symbol 1 when W has type D) with distinct signs, which is non-quasiparabolic.

- When $\mathcal{O}$ is the orbit of the 3-symbol orbit of $K_4 \rtimes S_3$ in $W = W_1 \times W_2$, where $W_1 = B_3$ or D_3 (when $W = D_3$, we require symbols $2, 3$ have distinct signs) or D_4 (when $W = D_4$, we require $\mathcal{O}$ on symbols $2, 3, 4$) and $W_2 = S_4$ or B_4 or D_4. we are able to reduce it to Alt_3 on 3 symbols not with same signs (not including symbol 1), which is non-quasiparabolic.

- When $\mathcal{O}$ is the orbit of the 4-symbol orbit of $K_4 \rtimes S_3$, or $K_4 \rtimes S_4$, or $PGL(3, 2)$, or $AGL(3, 2)$

we are able to reduce it to K_4 on 4 symbols not with same signs (not including symbol 1), which is non-quasiparabolic.

- When $\mathcal{O}$ is the orbit of Dil_{10}, or twisted Alt_5 in S_6, we will get $\Delta(\mathbb{Z}/2\mathbb{Z})$ with at least one orbit on symbols (not including symbol 1 when W has type D) with distinct signs, which is non-quasiparabolic.

In summary, if $\mathcal{O}$ is an orbit of a quasiparabolic subgroup H of W, and $\mathcal{O}$ does not contain a symbol and its negative, then all symbols in $\mathcal{O}$ (excluding symbol 1 for the orbit acted by a type D component of W) should have the same signs. □

3.3 Restriction on sign changes for quasiparabolic subgroups

We claim the sign changes on some orbits are forbidden if some smaller orbits do not have 'enough' sign changes.

Example 12. *For $W = B_3$ or $W = D_3$, the subgroup $H_1 = F_2$ generated by $(1)_-(3)_-$ or $(2)_-(3)_-$, and the subgroup $H_2 = B_2^\circ$ are not quasiparabolic. For $W = B_3$, the subgroup $H = \mathbb{Z}/2\mathbb{Z}$ generated by $(1\ 2)(3)_-$ or $(1\ \overline{2})(3)_-$ or $(1)_-(2\ \overline{3})$ is also not quasiparabolic.*

From Example 12, if the subgroup H has some orbit with sign changes, and the sign changes are independent with the sign changes on k smaller symbols (i.e., symbols closer to 1), and those smaller k symbols do not have $\mathbb{F}_2^k$ of arbitrary sign changes on these k symbols as a subgroup, then H can be reduced to Example 12 by repeating the operations in Example 8. So H will not be quasiparabolic in W.

Example 13. *For $W = B_4$ or $W = D_4$, the subgroup $H = \Delta(\mathbb{F}_2 \times \mathbb{Z}/2\mathbb{Z})$ generated by $\{(1\ 2)(3\ 4), (1\ \overline{2})(3\ \overline{4})\}$ or $\{(1\ \overline{2})(3\ 4), (1\ 2)(3\ \overline{4})\}$, is not quasiparabolic in W.*

From Example 13, if a subgroup H can be reduced to $\Delta(\mathbb{F}_2 \times \mathbb{Z}/2\mathbb{Z})$ as in the example, it will not be quasiparabolic. Next we will give some eligible sign changes for quasiparabolic subgroups when W has components of type B or D.

Definition 9. *A subgroup H of $W = B_n$ has **B form**, if*

- *$H = \mathbb{F}_2^m \rtimes H_A$, where $\mathbb{F}_2^m$ is the subgroup of arbitrary sign changes on symbols $1, \ldots, m$, and H_A is the A-image of H.*

- *H_A is quasiparabolic in S_n.*

- *For orbits of H without sign changes, they should have symbols with the same signs.*

*When W has multiple components, a component W_i of type B has **B form** under H if*

- *the subgroup $\mathbb{F}_2^m$ on arbitrary sign changes on symbols $1, \dots, m$ is included in H, and H does not generate sign changes on other symbols $m+1, \dots, n$.*

- *$H \cup W_i$ has quasiparabolic A-image in S_n.*

- *The orbits of H in symbols $m+1, \dots, n$ have symbols with the same signs.*

Definition 10. *A subgroup H of $W = B_n$ or $W = D_n$ has **D form** if*

- *$H = \mathbb{F}_2^{m-1} \rtimes H_A$, where $\mathbb{F}_2^{m-1}$ is the subgroup of an even number of sign changes on the symbols $1, \dots, m$, and H_A is the A-image of H. Or when $W = B_n$, $H = (\mathbb{F}_2^m \rtimes H_A)^\circ$, where $\mathbb{F}_2^m$ is the subgroup of arbitrary sign changes on symbols $1, \dots, m$, and H_A is the A-image of H.*

- *H_A is quasiparabolic in S_n.*

- *The orbits of H without sign changes have symbols with the same signs.*

*When W has multiple components, a component W_i of type B or type D has **D form** under H if*

- *the subgroup $\mathbb{F}_2^{m-1}$ on an even number of sign changes on the symbols $1, \dots, m$ are included in H, and H does not generate sign changes on the other symbols $m+1, \dots, n$.*

- *$H \cup W_i$ has quasiparabolic A-image in S_n.*

- *The orbits of H in symbols $m+1, \dots, n$ have symbols with the same signs.*

Definition 11. *A subgroup H of $W = D_n$ has **D2 form** if*

- *$H = D_2 \times (H_{A,1})^\circ \times (H_{A,2})^\circ$. Here $H_{A,1}$ and $H_{A,2}$ are odd quasiparabolic subgroups of S_{n-2} on disjoint symbols sets in $3, \dots, n$, without sign changes. D_2 acts on the symbols $1, 2$, and the simple reflections $(1\ 2)$ and $(1\ \overline{2})$ correspond to the parity of $H_{A,1}$ and $H_{A,2}$, respectively.*

For the case $H_A = B_n$ or $H_A = B_n^\circ$, H may also have special-D form or even-special-D form as follows.

Definition 12. *A subgroup H of $W = D_n$ has* ***special-D form*** *if*

- $H = \mathbb{F}_2^m \rtimes H_A$, *where* $\mathbb{F}_2^m$ *is the subgroup of an even number of sign changes on the m blocks* $\{1,2\},\ldots,\{2m-1,2m\}$, *or when $m = 2$, it can also have blocks* $\{1,4\},\{2,3\}$. *In addition, H_A is the A-image of H, equal to B_m°, or B_m in S_{2m}, generated by* $(2i-1\ 2i)(2j-1\ 2j)$, $(2i-1\ \overline{2i})(2j-1\ \overline{2j})$ *and* $(2i-1\ 2j-1)(2i\ 2j)$ ($1 \le i < j \le m$), *or its image under the Coxeter automorphism.*
- *H_A is quasiparabolic in S_n.*
- *The orbits of H without sign changes have symbols with the same signs.*

Definition 13. *A subgroup H of $W = D_n$ has* ***even-special-D form*** *if*

- $H = \mathbb{F}_2^{m-1} \rtimes H_A$, *where* $\mathbb{F}_2^{m-1}$ *is the subgroup consisting of an even number of sign changes on the m blocks* $\{1,2\},\ldots,\{2m-1,2m\}$, *or when $m = 2$, it can also have blocks* $\{1,4\},\{2,3\}$. *In addition, H_A is the A-image of H, equal to B_m° or B_m in S_{2m}, generated by* $(2i-1\ 2i)(2j-1\ 2j)$, $(2i-1\ \overline{2i})(2j-1\ \overline{2j})$ *and* $(2i-1\ 2j-1)(2i\ 2j)$ ($1 \le i < j \le m$), *or its image under the Coxeter automorphism.*

  Also, $H = \mathbb{F}_2^3 \rtimes PGL(3,2)$ *by adding* $(1\ 3)(5\ 7)$ *into the copy of* $\mathbb{F}_2^2 \rtimes B_3^\circ$ *above, or* $H = \mathbb{F}_2^4 \rtimes AGL(3,2)$ *by adding* $(2\ 4)(6\ 8)$ *into the copy of* $\mathbb{F}_2^3 \rtimes PGL(3,2)$ *above.*
- *H_A is quasiparabolic in S_n.*
- *The orbits of H without sign changes have symbols with the same signs.*

There are also a few examples of even quasiparabolic subgroups, which can generate sign changes on their orbits.

Example 14. *For $W = B_3$, the subgroup $H = Dil_6$ generated by* $(1)_-(2\ 3)$ *and* $(2)_-(1\ 3)$ *is quasiparabolic.*

For $W = B_4$, the subgroup $H = Dil_{16}$ generated by $(1\ 2)(3\ 4)$ *and* $(2\ 3)(1)_-$ *is quasiparabolic.*

For $W = D_4$, the subgroup $H = \mathbb{F}_2 \times Alt_4$ generated by $(1\ 2\ 3)$, $(1\ 2\ 4)$ *and* $(1\ \overline{2})(3\ \overline{4})$ *is quasiparabolic.*

For $W = D_6$, the subgroup $H = H_3$ generated by $(1\ 2)(3\ 4)$, $(2\ 3)(4\ 5)$ *and* $(1\ \overline{2})(5\ 6)$ *is quasiparabolic.*

3.4 Ruling out non-quasiparabolic subgroups of B_n and D_n

We will list out some examples of non-quasiparabolic subgroups generated by rotations.

Example 15. *The subgroups generated by 3-cycles, when the orbit $\mathcal{O}$ is closed under negation, may be $\mathbb{F}_2^{k-1} \rtimes Alt_k$, $\mathbb{F}_2 \times Alt_4$ or $SL(2,3)$, but the $SL(2,3)$ can not be quasiparabolic, by checking all possibilities.*

The group $H = \mathbb{F}_2 \times Alt_4$ generated by $(1\ 2\ 3)$, $(2\ 3\ 4)$ and $(1\ \overline{2})(3\ \overline{4})$ is quasiparabolic in $W = D_4$, but no other $\mathbb{F}_2 \times Alt_4$ is quasiparabolic in either $W = B_4$ or $W = D_4$. (Except for changing the sign of the symbol 1 in all the two 3-cycles).

Example 16. *For subgroups $H = Dil_8 \leq W$ generated by 2-rotations $\{(a_1\ a_2)(a_3\ a_4), (a_1\ a_3)(a_2\ \overline{a_4})\}$ on $\{\pm1, \pm2, \pm3, \pm4\}$, where $W = B_4$ or $W = D_4$, the element $w = (1)_-(2)_-(3)_-(4)_-$ with sign changes on all 4 symbols is in H. We may assume that the generators have the 2-cycle involving the symbol 4 having 2 symbols with the same signs. There are 6 cases for $W = B_4$ and 3 cases for $W = D_4$, which are all non-quasiparabolic. (These cases have generators $\{(1\ 2)(3\ 4), (1\ \overline{3})(2\ 4)\}$, or $\{(1\ 3)(2\ 4), (1\ \overline{2})(3\ 4)\}$, or $\{(1\ 2)(3\ 4), (1\ 4)(2\ \overline{3})\}$, or $\{(1\ 4)(2\ 3), (1\ \overline{2})(3\ 4)\}$, or $\{(1\ 3)(2\ 4), (1\ \overline{4})(2\ 3)\}$, or $\{(1\ 4)(2\ 3), (1\ \overline{3})(2\ 4)\}$).*

The subgroup $H = (\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \rtimes S_2$, which is obtained by adding $(a_3)_-(a_4)_-$ to the group Dil_8 above, is also non-quasiparabolic in either $W = B_4$ or $W = D_4$.

So the subgroups $(\mathbb{Z}/4\mathbb{Z})^{\times(k-1)} \rtimes S_k$ $(k \geq 2)$, and $((\mathbb{Z}/4\mathbb{Z})^{\times(k-1)} \times \mathbb{Z}/2\mathbb{Z}) \rtimes S_k$ $(k \geq 2)$ are non-quasiparabolic.

Example 17. *For subgroups $H = \mathbb{F}_2 \times K_4 \leq W$, where $W = B_4$ or $W = D_4$, and $\mathbb{F}_2$ corresponds to sign changes on all 4 symbols. Then the subgroup must contain a subgroup K_4, and all K_4's by changing two signs of the 4 symbols of the original K_4. So we may assume the original K_4 having 0 or 1 symbols with negative signs. If the K_4 has 1 negative symbol, then we may assume it is symbol 1, because we can freely change any two symbols' signs simultaneously in the 2-cycle presentation of 2-rotations. There are 2 possibilities of H generated by $\{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ \overline{2})(3\ \overline{4})\}$, or $\{(\overline{1}\ 2)(3\ 4), (\overline{1}\ 3)(2\ 4), (1\ 2)(3\ \overline{4})\}$. For the case $W = B_4$, all these 2 cases are not quasiparabolic.*

While for the case $W = D_4$, these 2 cases are dual to each other, and they are quasiparabolic subgroups of $W = D_4$. In fact, these two quasiparabolic subgroups are small cases of the conjectural quasiparabolic subgroups in [14]. We will prove the quasiparabolicity of the general case in Section

3.6. However, when we consider $W = D_5$ and the subgroup $H = \mathbb{F}_2 \times K_4$ acts on symbols 2,3,4,5 (suppose $(1\ \overline{2}) \in S$), then H is not a quasiparabolic subgroup of W.

Example 18. *Let $H = \mathbb{F}_2^2 \rtimes K_4 \leq W$, where $W = B_4$ or $W = D_4$. We take the stabilizer of the symbol 4 under the operation in Example 8, then the stabilizer is $\mathbb{F}_2$ on sign changes of 2 symbols. From Example 12, the groups $\mathbb{F}_2$ of sign changes on symbols $\{2,3\}$ or $\{1,3\}$, are not quasiparabolic. So the symbol 4 should be paired with symbol 3 on the sign changes.*

Then there are 2 cases of $H = \mathbb{F}_2^2 \rtimes K_4$, generated by $\{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ \overline{2})(3\ 4)\}$, or $\{(\overline{1}\ 2)(3\ 4), (\overline{1}\ 3)(2\ 4), (1\ 2)(3\ 4)\}$,

When $W = B_4$, then the H generated by $\{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ \overline{2})(3\ 4)\}$ is quasiparabolic. In fact, it is the $n = 2$ case of the even subgroup of the centralizer of the minimal fixed-point-free involutions of B_{2n}, which is quasiparabolic by [14]. In contrast, the remaining case is not quasiparabolic. When $W = D_4$, then the two cases of H are dual to each other, and both are quasiparabolic. They are also the $n = 2$ case of the even subgroup of the centralizer of the minimal fixed-point-free involutions of D_{2n}, which is quasiparabolic.

Example 19. *From the classification of rotation subgroups, there are 3 possibilities for the subgroups H of W with A-image of quasiparabolic $PGL(3,2)$. These are $PGL(3,2)$, $\mathbb{F}_2^6 \rtimes PGL(3,2)$ with an even number of sign changes, and $\mathbb{F}_2^3 \rtimes PGL(3,2)$ generated by sign-changes on 4 symbols a, b, c, d with $(ab)(cd) \in H$. From Example 17, when we fix the symbols 5,6,7, then the stabilizer is $\mathbb{F}_2 \times K_4$, which is not quasiparabolic in B_4. So $\mathbb{F}_2^3 \rtimes PGL(3,2)$ is not quasiparabolic in B_7. However, if $W = D_7$, then the subgroup $\mathbb{F}_2^3 \rtimes PGL(3,2)$ is quasiparabolic, if it is generated by $(1\ 2)(3\ 4)$, $(1\ 2)(5\ 6)$, $(1\ 3)(2\ 4)$, $(3\ 5)(4\ 6)$, $(1\ 3)(5\ 7)$ and $(1\ \overline{2})(3\ \overline{4})$.*

From the classification of rotation subgroups, there are 3 possibilities for the subgroup H of W with A-image of quasiparabolic $AGL(3,2)$. These are $AGL(3,2)$, $\mathbb{F}_2^7 \rtimes AGL(3,2)$ with an even number of sign changes, and $\mathbb{F}_2^4 \rtimes AGL(3,2)$ generated by sign-changes on 4 symbols a, b, c, d with $(ab)(cd) \in H$. From Example 17, when we fix the symbol 8, we will obtain $\mathbb{F}_2^3 \rtimes PGL(3,2)$, which is not quasiparabolic in B_7. So $\mathbb{F}_2^4 \rtimes AGL(3,2)$ is not quasiparabolic in B_8. However, if $W = D_8$, then the subgroup $\mathbb{F}_2^4 \rtimes AGL(3,2)$ is quasiparabolic, if it is generated by the above quasiparabolic $\mathbb{F}_2^3 \rtimes PGL(3,2)$ in $D_7 \subset D_8$, and the 2-rotation $(5\ 6)(7\ 8)$.

3.5 Quasiparabolic subgroups with A-image as indecomposable quasiparabolic subgroups

Example 20. *For the case that the A-image of H is S_i, when $W = D_n$, the only possibilities are $H = S_i$, or $H = \mathbb{F}_2^{i-1} \rtimes S_i$ with an even number of sign changes, or $H = \mathbb{F}_2 \times S_4$ with its even subgroup $\mathbb{F}_2 \times Alt_4$ in Example 14. While $W = B_n$, there is one more possibility of $H = \mathbb{F}_2^i \rtimes S_i$ with arbitrary sign changes, but $H = \mathbb{F}_2 \times S_4$ is no longer quasiparabolic in $W = B_n$. When $H = S_i$, $H = \mathbb{F}_2^{i-1} \rtimes S_i$ or $H = \mathbb{F}_2^i \rtimes S_i$, H is quasiparabolic in W if and only if W has B form or D form.*

When $H = \mathbb{F}_2 \times S_4$ in $W = D_n$, H can only act on the symbols $1, 2, 3, 4$. (The $\mathbb{F}_2 \times S_4$ acting on $2, 3, 4, 5$ is non-quasiparabolic, as is $\mathbb{F}_2 \times S_4$ acting on $1, 2, 3, 4$ except for the group with the even subgroup $\mathbb{F}_2 \times Alt_4$ in Example 14 or its image under the Coxeter homomorphism.)

When $H = Dil_6$ or $H = Dil_{16}$ in Theorem 10, the only possible quasiparabolic groups are given by Example 14. While if $H = Dil_8$ in Theorem 10, H is non-quasiparabolic in B_4.

Example 21. *For the case that the A-image of H is B_j, when $W = D_n$, the subgroups $H = B_j$ in A form, or $H = \mathbb{F}_2^{2j-1} \rtimes B_j$ in D form, or $H = \mathbb{F}_2^j \rtimes B_j$ in special-D form, or $H = \mathbb{F}_2^{j-1} \rtimes B_j$ in even-special-D form, are quasiparabolic. While if $W = B_n$, the subgroup $H = \mathbb{F}_2^{j-1} \rtimes B_j$ will not be quasiparabolic, and there is one more quasiparabolic subgroup which is $H = \mathbb{F}_2^{2j} \rtimes B_j$ in B form.*

For $H = \mathbb{F}_2^j \rtimes B_j$ (or $H = \mathbb{F}_2^{j-1} \rtimes B_j$), only the subgroup satisfying special-D form (or even-special-D form) can be quasiparabolic. When $j = 2$, only the groups in special-D form (or even-special-D form) can be quasiparabolic. So H must act on the symbols $\pm 1, \ldots, \pm 2j$.

In addition, H should have pairing $\{2i-1, 2i\}$ $(1 \leq i \leq j)$. The other pairing $\{i, 2j+1-i\}$ can not generate a quasiparabolic subgroup, because after the operation in Example 8 for the symbol $2j$, the symbol 1 will be stabilized, and then we will have $\mathbb{F}_2^{j-1} \rtimes B_{j-1}$ (or $\mathbb{F}_2^{j-2} \rtimes B_{j-1}$) acting on symbols $\{2, 3 \ldots, 2j-1\}$, which is non-quasiparabolic.

In addition, H should contain the elements $(2i-1\ 2i+1)(2i\ 2i+2)$ ($i \geq 2$ for W in type D). Otherwise, since $(2i-1)_-(2i)_-(2i+1)_-(2i+2)_- \in H$, and H contains K_4 acting on symbols $\{\pm(2i-1), \ldots, \pm(2i+2)\}$, then $(2i-1\ 2i)(2i+1\ \overline{2i+2}) \in H$. After the operations in Example 8 for symbols larger than $2i+2$, we will reduce the A-image of the group to B_{i+1} on symbols $1, \ldots, 2i+2$. Then by applying the operation in Example 9 for symbols $2i, 2i+2$, and we will have an orbit $\{2i-1, -2i\}$, with symbols with distinct signs ($i \geq 2$ when W has type D). So H is non-quasiparabolic. Thus the only quasiparabolic $H = \mathbb{F}_2^j \rtimes B_j$ (or $H = \mathbb{F}_2^{j-1} \rtimes B_j$) should have *special-D form (or even-special-D form) as in Definition 12 (or 13).*

Example 22. *For the case that the A-image of H is* $(A_{i_1} \times \ldots \times A_{i_k} \times B_{j_1} \times \ldots \times B_{j_l})^\circ$, *by Example 12, the quasiparabolic subgroups should be in the form of*

- $(A_{i_1} \times \ldots \times A_{i_k} \times B_{j_1} \times \ldots \times B_{j_l})^\circ$ *in A form without sign changes, or*
- $\mathbb{F}_2^{m-1} \rtimes (A_{i_1} \times \ldots \times A_{i_k} \times B_{j_1} \times \ldots \times B_{j_l})^\circ$ *in D form with an even number sign changes on symbols* $1, \ldots, m$, *or in B form with arbitrary sign changes on symbols* $1, \ldots, m-1$.
- $D_2 \times H_{A,1}^\circ \times H_{A,2}^\circ$ *in D2 form with* D_2 *having simple reflections* $(1\ 2)$ *and* $(1\ \overline{2})$ *corresponding to the parity of* $H_{A,1}$ *and* $H_{A,2}$.

If H is not in the form above, then we may find

- *some orbits without sign changes containing symbols with distinct signs, which forces H to become non-quasiparabolic by Theorem 14, or*
- *H does not have sign changes in B form or D form, which is not quasiparabolic.*

Example 23. *For the case that the A-image* H_A *of H is* ΔS_k, *a quasiparabolic subgroup should have quasiparabolic A-image. In addition, suppose H generates sign changes on some orbits, then one orbit should be* $1, 2, \ldots, k$, *and the other orbit is* $l, l+1, \ldots, l+k-1$. *In addition, the symbol i should be paired with* $l+i$, *otherwise, we will finally have* $\mathbb{F}_2^j \rtimes (\Delta\mathbb{Z}/2\mathbb{Z})$ $(j = 1, 2, 3, 4)$ *on symbols* $2, 3, 4, 5$, *which is non-quasiparabolic.*

The quasiparabolic subgroups can be

- $H = \Delta S_k$,
- $H = \mathbb{F}_2^{k-1} \rtimes \Delta S_k$ *in D form,*
- $H = \mathbb{F}_2^{k} \rtimes \Delta S_i$ *in B form,*
- $H = \mathbb{F}_2^{2k-1} \rtimes \Delta S_i$ *in D form,*
- $H = \mathbb{F}_2^{2k} \rtimes \Delta S_i$ *in B form.*

There are no other subgroups generated by rotations with A-image $H_A = \Delta S_i$ *when W is indecomposable.*

If $W = W_1 \times W_2$ has two direct product components W_1, W_2, and the two orbits of H are acted separately by W_1 and W_2, except for the cases which H having A form, B form, or D form on W_1 and W_2, we still have the possibilities $\Delta(\mathbb{F}_2^k \rtimes S_k) = \Delta(B_k)$ or $\Delta(\mathbb{F}_2^{k-1} \rtimes S_k) = \Delta(D_k)$. $\Delta(B_k)$ is quasiparabolic in $B_{n_1} \times B_{n_2}$, and $\Delta(D_k)$ is quasiparabolic in $D_{n_1} \times D_{n_2}$, if $\Delta(D_k)$ is generated by $(i\ i+1)(i'\ (i+1)')$ and $(i\ \overline{i+1})(i'\ \overline{(i+1)'})$, and $\Delta(B_k)$ is generated by $(i\ i+1)(i'\ (i+1)')$ and $(i)_-(i')_-$, for $1 \leq i < k$, and $i' \in \{1', \ldots, (k-1)'\}$, where W_1 acts on symbols $1, \ldots, n_1$, and W_2 acts on symbols $1', \ldots, n_2'$. However, $\Delta(D_k)$ is non-quasiparabolic in W when W_1 or W_2 has type B.

Other copies of $\Delta(D_k)$ or $\Delta(B_k)$ may contain some elements $(i\ i+1)(i'\ \overline{(i+1)'})$ ($i \geq 2$ if W_i has type D). First apply the operation in Example 8 for symbols $i+2, \ldots, k$, and get $\Delta(D_j)$ or $\Delta(B_j)$. Then by applying the operation in Example 9 for the symbols $i, i+1$, then we will have a component $\Delta(\mathbb{Z}/2\mathbb{Z})$ generated by $(i\ i+1)(i'\ \overline{(i+1)'})$, containing an orbit of symbols with distinct signs. Thus only the copies of $\Delta(D_k)$ or $\Delta(B_k)$ containing $(i\ i+1)(i'\ (i+1)')$ as in the paragraph above can be quasiparabolic in $W = W_1 \times W_2$.

Example 24. *For the case that the A-image H_A of H is $K_4 \rtimes (\Delta S_3)$, if $W = B_n$, then the quasiparabolic subgroups are*

- $H = K_4 \rtimes (\Delta S_3)$ *in A form,*
- $H = \mathbb{F}_2^2 \rtimes (K_4 \rtimes (\Delta S_3))$ *in D form,*
- $H = \mathbb{F}_2^3 \rtimes (K_4 \rtimes (\Delta S_3))$ *in B form,*
- $H = \mathbb{F}_2^3 \rtimes (K_4 \rtimes (\Delta S_3))$ *in D form,*
- $H = \mathbb{F}_2^4 \rtimes (K_4 \rtimes (\Delta S_3))$ *in B form,*
- $H = \mathbb{F}_2^6 \rtimes (K_4 \rtimes (\Delta S_3))$ *in D form,*
- $H = \mathbb{F}_2^7 \rtimes (K_4 \rtimes (\Delta S_3))$ *in B form.*

When $W = D_n$, only the above subgroups in A form or D form can remain as quasiparabolic subgroups of W, also $H = \mathbb{F}_2 \times (K_4 \rtimes (\Delta S_3))$ in even-special-D form with sign changes on all symbols $\{1, 2, 3, 4\}$ is quasiparabolic. Other subgroups generated by rotations are not quasiparabolic in W.

Assume $W = W_1 \times W_2$ has two direct product components. There are some other possible quasiparabolic subgroups besides the previous ones.

- *For $H = (\mathbb{F}_2 \times \mathbb{F}_2^2) \rtimes (K_4 \rtimes (\Delta S_3))$ generated by $(1\ 2)(3\ 4)$, $(1\ 3)(2\ 4)$, $(1\ \overline{2})(3\ \overline{4})$, $(1\ 2)(1'\ 2')$, $(2\ 3)(2'\ 3')$ and $(1\ 2)(1'\ \overline{2'})$, H is quasiparabolic in $W = D_4 \times B_3$ or $W = D_4 \times D_3$.*
- *For $H = (\mathbb{F}_2^3 \times \mathbb{F}_2^2) \rtimes (K_4 \rtimes (\Delta S_3))$, where the subgroup $\mathbb{F}_2^3$ acts on the symbols $1, 2, 3, 4$ and $\mathbb{F}_2^2$ acts on symbols $1', 2', 3'$, H is quasiparabolic in W if W_1 acting on the symbols $1, 2, 3, 4$ and W_2 acting on $1', 2', 3'$ has type B or D, with two possible double cover reflection classes $(1)_-$ when W_1 has type B, and $(1')_-$ when W_2 has type B.*
- *For $H = \mathbb{F}_2^3 \rtimes (K_4 \rtimes (\Delta S_3))$, which is generated by $K_4 \rtimes (\Delta S_3)$ acting on $1, 2, 3, 4$ and $1', 2', 3'$, and the 2-rotation $(1\ \overline{2})(1'\ \overline{2'})$, H is quasiparabolic in $W = D_4 \times D_3$.*

Example 25. *For the case that the A-image H_A of H is $K_4 \rtimes (\Delta S_4)$, if $W = B_n$, then the quasiparabolic subgroups are*

- *$H = K_4 \rtimes (\Delta S_4)$ in A form,*
- *$H = \mathbb{F}_2^3 \rtimes (K_4 \rtimes (\Delta S_4))$ in D form,*
- *$H = \mathbb{F}_2^4 \rtimes (K_4 \rtimes (\Delta S_4))$ in B form,*
- *$H = \mathbb{F}_2^7 \rtimes (K_4 \rtimes (\Delta S_4))$ in D form,*
- *$H = \mathbb{F}_2^8 \rtimes (K_4 \rtimes (\Delta S_4))$ in B form.*

When $W = D_n$, only the above subgroups in A form or D form are quasiparabolic subgroups of W. In addition the subgroup $H = \mathbb{F}_2 \times (K_4 \rtimes (\Delta S_4))$ in even-special-D form with sign changes on all symbols $\{1, 2, 3, 4\}$ is quasiparabolic in D_8. The copy of $H = (\mathbb{F}_2 \times K_4) \rtimes S_4$ generated by $K_4 \rtimes S_3$ on $1, 2, 3, 4$ and $5, 6, 7$, and the 2-rotation $(1\ \overline{2})(7\ 8)$, is quasiparabolic in D_8. (Actually this $H = (\mathbb{F}_2 \times K_4) \rtimes S_4$ is the image of the subgroup $\mathbb{F}_2^3 \rtimes (K_4 \rtimes (\Delta S_3))$ in $D_4 \times D_3 = D_4 \times A_3$ in Example 24 under the injective Coxeter homomorphism $D_4 \times A_3 \to D_8$.) Other subgroups generated by rotations are not quasiparabolic in W.

Assume $W = W_1 \times W_2$ has two direct product components. There are some other possible quasiparabolic subgroups besides the previous ones.

- For $H = (\mathbb{F}_2 \times \mathbb{F}_2^3) \rtimes (K_4 \rtimes (\Delta S_4))$ generated by $(1\ 2)(3\ 4)$, $(1\ 3)(2\ 4)$, $(1\ \overline{2})(3\ \overline{4})$, $(1\ 2)(1'\ 2')$, $(2\ 3)(2'\ 3')$, $(3\ 4)(3'\ 4')$ and $(1\ 2)(1'\ \overline{2'})$, H is quasiparabolic in $W = D_4 \times B_4$ or $W = D_4 \times D_4$.

- For $H = (\mathbb{F}_2 \times \mathbb{F}_2) \rtimes (K_4 \rtimes (\Delta S_4))$ generated by $K_4 \rtimes (\Delta S_4)$ acting on $1, 2, 3, 4$ and $1', 2', 3', 4'$, and the 2-rotations $(1\ \overline{2})(3\ \overline{4})$ and $(1'\ \overline{2'})(3'\ \overline{4'})$, H is quasiparabolic in $W = D_4 \times D_4$.

- For $H = (\mathbb{F}_2^3 \times \mathbb{F}_2^3) \rtimes (K_4 \rtimes (\Delta S_4))$, where the subgroup $\mathbb{F}_2^3$ acts on symbols $1, 2, 3, 4$ and $\mathbb{F}_2^3$ acts on symbols $1', 2', 3', 4'$, H is quasiparabolic in W if W_1 acting on symbols $1, 2, 3, 4$ and W_2 acting on $1', 2', 3', 4'$ has type B or D, with two possible double cover reflection classes $(1)_-$ when W_1 has type B, and $(1')_-$ when W_2 has type B.

- For $H = \mathbb{F}_2^4 \rtimes (K_4 \rtimes (\Delta S_4))$, which is generated by $K_4 \rtimes (\Delta S_4)$ acting on $1, 2, 3, 4$ and $1', 2', 3', 4'$, and the 2-rotation $(1\ \overline{2})(1'\ \overline{2'})$, H is quasiparabolic in $W = D_4 \times D_4$.

Example 26. *Assume that the A-image H_A of H is $PGL(3, 2)$. If $W = B_n$, the quasiparabolic subgroups are $H = PGL(3, 2)$ in A form, or $H = \mathbb{F}_2^6 \rtimes PGL(3, 2)$ in D form, or $H = \mathbb{F}_2^7 \rtimes PGL(3, 2)$ in B form. When $W = D_n$, $H = \mathbb{F}_2^7 \rtimes PGL(3, 2)$ will not be a subgroup of W, while $H = \mathbb{F}_2^3 \rtimes PGL(3, 2)$ in even-special-D form with sign changes on the symbols $\{1, \ldots, 7\}$, is a quasiparabolic subgroup of W.*

Example 27. *Assume that the A-image H_A of H is $AGL(3, 2)$. If $W = B_n$, the quasiparabolic subgroups are $H = AGL(3, 2)$ in A form, or $H = \mathbb{F}_2^7 \rtimes AGL(3, 2)$ in D form, or $H = \mathbb{F}_2^8 \rtimes AGL(3, 2)$ in B form. When $W = D_n$, $H = \mathbb{F}_2^8 \rtimes AGL(3, 2)$ will not be a subgroup of W, while $H = \mathbb{F}_2^4 \rtimes AGL(3, 2)$ in even-special-D form with sign changes on the symbols $\{1, \ldots, 8\}$, is a quasiparabolic subgroup of W.*

Example 28. *Assume that the A-image H_A of H is Dil_{10}. When $W = B_n$ or $W = D_n$, the quasiparabolic subgroups are $H = Dil_{10}$ in A form, or $H = \mathbb{F}_2^4 \rtimes Dil_{10}$ in D form, or $H = \mathbb{F}_2^5 \rtimes Dil_{10}$ in B form.*

Example 29. *Assume that the A-image H_A of H is Alt_5 in Sym_6. When $W = B_n$ or $W = D_n$, the quasiparabolic subgroups are $H = Alt_5$ in A form, or $H = \mathbb{F}_2^5 \rtimes Alt_5$ in D form, or $H = \mathbb{F}_2^6 \rtimes Alt_5$ in B form.*

3.6 A class of quasiparabolic subgroups which have index 4 in the centralizer of the minimal fixed-point-free involutions of D_{2n}

We denote by $\tilde{H} = \mathbb{F}_2^{n-1} \rtimes B_n$ the centralizer of the minimal fixed-point-free involutions of D_{2n}, and by $H = \mathbb{F}_2^{n-2} \rtimes B_n^\circ$ an index 4 subgroup of $\tilde{H}$, generated by the B_n° on the n pairs of symbols, and $\mathbb{F}_2^{n-2}$ by allowing sign changes on even pairs of symbols acted on by B_n°.

From Theorem 4.3 in [14], $\tilde{H}$ is a quasiparabolic subgroup of $W = D_{2n}$, i.e., the left cosets of $\tilde{H}$ in W form a quasiparabolic W-set.

We will directly write down the standard minimal representative w of the left coset $w\tilde{H}$, and wH, and find the relationship between the two W-sets $W/\tilde{H}$ and W/H.

Recall that for any $w \in W = D_{2n}$, the length of w is given by [2]

$$l(w) = |\{(i,j) : 1 \leq i < j \leq 2n, w(i) > w(j)\}| + |\{(i,j) : 1 \leq i < j \leq 2n, -w(i) > w(j)\}|. \quad (3.1)$$

Note that given an element w in the left coset $w\tilde{H}$, w has n blocks $(w(1), w(2)), \ldots, (w(2n-1), w(2n))$. The following $w' \in W$ still gives $w'\tilde{H} = w\tilde{H}$.

- $w'(i) = \begin{cases} w(i) & \text{if } i \neq 2j-1, 2j \\ w(2j) & \text{if } i = 2j-1 \\ w(2j-1) & \text{if } i = 2j \end{cases}$ by switching the two symbols within one block;
- $w'(i) = \begin{cases} w(i) & \text{if } i \neq 2j-1, 2j, 2k-1, 2k \\ w(2k-1) & \text{if } i = 2j-1 \\ w(2k) & \text{if } i = 2j \\ w(2j-1) & \text{if } i = 2k-1 \\ w(2j) & \text{if } i = 2k \end{cases}$ by switching two blocks;
- $w'(i) = \begin{cases} w(i) & \text{if } i \neq 2j-1, 2j \\ -w(i) & \text{if } i = 2j-1, 2j \end{cases}$ by changing the signs of the two symbols in one block.

Now we may describe the standard form of a minimal representative w in a left coset $w\tilde{H}$.

Definition 14. *For $w \in W = D_{2n}$, a* block $(w(2i-1), w(2i))$ *is in* unit-form *if* $w(2i) > |w(2i -$

$1)| > 0$. *If a block is not in unit-form, then their* unit-form $(Uw(2i-1), Uw(2i))$ *is the image* $(wa(2i-1), wa(2i))$*, with the composition action under* $a \in K_4$*, where* K_4 *is a group generated by reflections* $(2i-1\ 2i)$ *and* $(2i-1\ \overline{2i})$*, and the two signed symbols* $wa(2i-1), wa(2i)$ *satisfying* $wa(2i) > |wa(2i-1)| > 0$*. A block* $(w(2i-1), w(2i))$ *is* smaller than $(w(2j-1), w(2j))$ *if and only if the smaller signed symbol in the unit form of* $(w(2i-1), w(2i))$ *is smaller than that of* $(w(2j-1), w(2j))$*, i.e.,* $\min\{Uw(2i-1), Uw(2i)\} < \min\{Uw(2j-1), Uw(2j)\}$.

Proposition 19. *Any left coset* $w\tilde{H}$ *will contain a minimal representative in the following form:*

- *In each block,* $w(2i-1), w(2i)$ *satisfies* $w(2i) > |w(2i-1)| > 0$*, i.e.,* $(w(2i-1), w(2i))$ *is in unit-form.*
- *The blocks are arranged so that* $w(1) < w(3) < \ldots < w(2n-1)$*.*

Proof. By (possibly) switching the two symbols within one block and changing the signs of the two symbols within one block, we are able to assume $w(2i) > |w(2i-1)| > 0$. By sorting the blocks, we are able to assume $w(1) < w(3) < \ldots < w(2n-1)$. So there exists an element w as described in Proposition 19 in every left coset $w\tilde{H}$.

The remaining task is to show the elements in Proposition 19 are minimal representatives of $w\tilde{H}$. In fact, from (3.1) for the length of w in $W = D_{2n}$, $w(i) > w(i+1) \iff w > ws_{2n-i}$, where $s_{2n-i} = (i\ i+1)$. So the minimal representative element w must satisfy $w(2i) > w(2i-1)$ for each block. In addition, by allowing sign changes of the two symbols in one block (and possibly switching the two elements within the block) we may force $w(2i) > |w(2i-1)| > 0$ without increasing the length. Also, for any two neighboring blocks $w(2i-1), w(2i)$ and $w(2i+1), w(2i+2)$, if $w(2i+2) > |w(2i+1)| > 0$ and $w(2i+1) > w(2i-1)$, then switching these two blocks without changing the signs and the order of two symbols within each block, will not decrease the length. Thus, for any representative $w' \in w\tilde{H}$, if we arrange the n blocks so that $w(1) < w(3) < \ldots < w(2n-1)$, and $w(2i) > w(2i-1)$ as in the form of Proposition 19, we will not increase the length of the representative. Thus, $w\tilde{H}$ contains such a minimal representative w. ☐

Next, we will describe the minimal representative w_0 in a left coset wH, which is obtained similarly to that in $w\tilde{H}$.

Proposition 20. *Any left coset* wH *will contain a minimal representative* w_0 *in the following form:*

- *In each block, when $i \geq 2$, $w_0(2i-1), w_0(2i)$ satisfies $w_0(2i) > |w_0(2i-1)| > 0$, i.e., $(w_0(2i-1), w_0(2i))$ is in unit-form.*
- *The blocks are arranged so that* $\max\{\min\{w_0(1), w_0(2)\}, \min\{-w_0(1), -w_0(2)\}\} < w_0(3) < \ldots < w_0(2n-1)$.

Proof. The subgroups H allow all operations in $\tilde{H}$ with some more restrictions:

- The number of operations that switch the two symbols within one block should be even.
- The number of operations that change the signs of the two symbols in one block should be even.

For any element w in wH, when $i > 1$, we may apply the operations as follows:

- If $w(2i-1) > |w(2i)| > 0$, then switch $w(2i-1), w(2i)$ and also switch $w(1), w(2)$;
- If $-w(2i-1) > |w(2i)| > 0$, then switch $w(2i-1), w(2i)$ and also switch $w(1), w(2)$, and change the signs of the four symbols;
- If $-w(2i) > |w(2i-1)| > 0$, then change the signs of the four symbols $w(2i-1), w(2i), w(1), w(2)$.

From (3.1), all these three operations will not increase the length of w. Similar to the case of $w\tilde{H}$, we will be able to arrange the blocks with $w(2i) > |w(2i-1)| > 0$, for all $i \geq 2$.

Similar to the case in Proposion 19, switching two neighboring blocks $w(2i-1), w(2i)$ and $w(2i+1), w(2i+2)$ will not increase the length, if each block has unit-form and $w(2i-1) > w(2i+1)$. Also, we can switch the first block and second block and set the second block in unit form, without increasing the length, so that $\max\{\min\{w(1), w(2)\}, \min\{-w(1), -w(2)\}\} < w(3)$. Thus any element w in wH can be switched to the given form in the proposition under the operations above, without increasing its length. □

Now we are able to prove the quasiparabolicity of the subgroup H in W.

Theorem 15. *The subgroup H is quasiparabolic in $W = D_{2n}$.*

Proof. From the form of the minimal representatives of w_1 in wH, w_1 will have exactly one of the four forms related to minimal representative w_0 in $w\tilde{H}$: w_0, $w_0(1\ 2) = w_0 s_{2n-1}$, $w_0(1\ \overline{2}) = w_0 s_{2n}$, or $w_0(1)_-(2)_- = w_0 s_{2n-1} s_{2n}$. Denote the four forms above as the K_4**-forms** of w_1. Which one of the four forms is determined by the relation of $w(1), w(2)$ in the first block:

- If $w_1(2) > |w_1(1)| > 0$, then $w_1 = w_0$;
- If $w_1(1) > |w_1(2)| > 0$, then $w_1 = w_0 s_{2n-1}$;
- If $-w_1(1) > |w_1(2)| > 0$, then $w_1 = w_0 s_{2n}$;
- If $-w_1(2) > |w_1(1)| > 0$, then $w_1 = w_0 s_{2n-1} s_{2n}$.

We will be able to write $W/H = W/\tilde{H} \times K_4$. For $wH = (w\tilde{H}, a)$, where $a \in K_4 = \{1, (1\ 2) = s_{2n-1}, (1\ \overline{2}) = s_{2n}, (1)_-(2)_- = s_{2n-1}s_{2n}\}$, the height is given by $\mathrm{ht}(wH) = \mathrm{ht}((w\tilde{H}, a)) = \mathrm{ht}(w\tilde{H}) + l(a)$. In addition, the action of the simple reflections $s_i \in S$ is given as follows:

- When $1 \le i \le n-1$, $s_i = (2n-i\ 2n-i+1)$, if there exists some j s.t., $\{w(2j-1), w(2j)\} \subset \{\pm(2n-i), \pm(2n-i+1)\}$, i.e., $s_i w\tilde{H} = w\tilde{H}$, then

$$s_i wH = s_i(w\tilde{H}, a) = \begin{cases} (w\tilde{H}, (1\ 2)a) & \text{if } w(2j-1) \text{ and } w(2j) \text{ have the same sign;} \\ (w\tilde{H}, (1\ \overline{2})a) & \text{if } w(2j-1) \text{ and } w(2j) \text{ have distinct signs.} \end{cases}$$

  While if $s_i w\tilde{H} \neq w\tilde{H}$, then we may possibly switch two neighboring blocks, if their minimal symbols are $\pm(2n-i)$ and $\pm(2n-i+1)$ respectively. Then $s_i wH = s_i(w\tilde{H}, a) = (s_i w\tilde{H}, a)$.

- For the simple reflection $s_{2n} = (1\ \overline{2})$, if $s_{2n} w\tilde{H} = w\tilde{H}$, then

$$s_{2n} wH = s_{2n}(w\tilde{H}, a) = \begin{cases} (w\tilde{H}, (1\ \overline{2})a) & \text{if } w(2j-1) \text{ and } w(2j) \text{ have the same sign;} \\ (w\tilde{H}, (1\ 2)a) & \text{if } w(2j-1) \text{ and } w(2j) \text{ have distinct signs.} \end{cases}$$

  While if $s_{2n} w\tilde{H} \neq w\tilde{H}$, then the two symbols 1,2 must be minimal symbols in their blocks (in unit-form), and these two blocks are neighbor to each other in the minimal representative form. So we only need to possibly switch these two blocks and keep all but the smallest block in unit-form, without changing the K_4-form of w. Thus $s_{2n} wH = s_{2n}(w\tilde{H}, a) = (s_{2n} w\tilde{H}, a)$.

From the action of the simple reflections, we will be able to write down the action of general reflection $t = (i\ j)$ on left cosets $W/H = W/\tilde{H} \times K_4$. If $tw\tilde{H} = w\tilde{H}$, then $twH = t(w\tilde{H}, a) = (w\tilde{H}, ta)$, where

$$ta = \begin{cases} (1\ 2)a & \text{if } (i, j \text{ has same signs) xor } (w^{-1}(i), w^{-1}(j) \text{ has distinct signs);} \\ (1\ \overline{2})a & \text{if } (i, j \text{ has distinct signs) xor } (w^{-1}(i), w^{-1}(j) \text{ has distinct signs);} \end{cases}$$

If $tw\tilde{H} \neq w\tilde{H}$, when i, j have the same signs ($i < j$), then

$$t = (i\ j) = s_{2n-j-1}s_{2n-j-2} \ldots s_{2n-i+1}s_{2n-i}s_{2n-i+1} \ldots s_{2n-j-1}.$$

Suppose b_i is the signed symbol so that for some $1 \leq k_i \leq n$, $\{w(2k_i - 1), w(2k_i)\} = \{b_i, i\}$ or $\{w(2k - 1), w(2k)\} = \{b_i, -i\}$. (Similar definition for b_j.) If $|i| < |b_i| < |j|$, then if $w(2k_i - 1), w(2k_i)$ have the same signs, then the block $\{w(2k_i - 1), w(2k_i)\}$ contributes a multiplication of $(1\ 2)$ in the K_4 part; if $w(2k_i - 1), w(2k_i)$ have distinct signs, then the block $\{w(2k_i - 1), w(2k_i)\}$ contributes a multiplication of $(1\ \overline{2})$ in the K_4 part. A similar contribution is made by the block $\{w(2k_j - 1), w(2k_j)\}$ in the K_4 part.

When i, j have distinct signs ($|i| < |j|$), then

$$t = (i\ j) = s_{2n-i-1} \ldots s_{2n-1}s_{2n-j-1} \ldots s_{2n-2}s_{2n}s_{2n-2} \ldots s_{2n-j-1}s_{2n-1} \ldots s_{2n-i-1}.$$

Also denote b_i and k_i as before. If $|b_i| < |i| < |j|$, then the block $\{w(2k_i - 1), w(2k_i)\}$ contributes a multiplication of $(1\ 2)(1\ \overline{2})$ in the K_4 part. A similar contribution is made by $w(2k_j - 1), w(2k_j)$ in the K_4 part.

If $|i| < |b_i| < |j|$, then the block $\{w(2k_i - 1), w(2k_i)\}$ contributes a multiplication of $(1\ 2)$ (or $(1\ \overline{2})$) in the K_4 part, if $w(2k_i - 1), w(2k_i)$ have distinct signs (or the same sign). However, the block $\{w(2k_j - 1), w(2k_j)\}$ contributes a multiplication of $(1\ 2)$ (or $(1\ \overline{2})$) in the K_4 part, if $w(2k_j - 1), w(2k_j)$ have the same sign (or distinct signs).

Now we are able to verify the quasiparabolic property of H, with the aid of the quasiparabolicity of $\tilde{H}$.

First we prove a lemma

Lemma 3. *Suppose $t \in T(W)$ is a reflection, $\tilde{H}$ and H are as before. If $twH > wH$, then $tw\tilde{H} \geq w\tilde{H}$.*

Proof. Let $wH = (w\tilde{H}, a) \in W/\tilde{H} \times K_4$. Suppose $twH > wH$ and $tw\tilde{H} < w\tilde{H}$. Then the difference of the height of twH and wH from the K_4 part must be at least 2. Then the K_4 part of wH must be the identity, and the K_4 part of twH must be $(1\ 2)(1\ \overline{2})$. From the argument above for the action of t on the K_4 part, there are only the following possible cases:

- $t = (i\ j)$, where i, j have the same sign. $|i| < b_i, b_j < |j|$, and exactly one block $\{w(2k_i - 1), w(2k_i)\}$ and $\{w(2k_j - 1), w(2k_j)\}$ have the same sign.

  If the block $\{w(2k_i - 1), w(2k_i)\}$ have the same sign, then by Proposition 19, since wH has unit-form in the K_4 part, we may assume $b_j < 0$, $0 < i < b_i < j$, $k_j < k_i$. However, $twH < wH$ in this case, which is impossible.

  If the block $\{w(2k_j - 1), w(2k_j)\}$ have the same sign, then by Proposition 19, since wH has unit-form in the K_4 part, we may assume $0 < b_i, b_j < j$, $k_i < k_j$, but $\text{ht}(twH) - \text{ht}(wH) \geq 3$, forcing $tw\tilde{H} > w\tilde{H}$. A contradiction is achieved.

- When $t = (i\ \bar{j})$, $|i| < |b_i|, |b_j| < |j|$, and 0 or 2 blocks $\{w(2k_i - 1), w(2k_i)\}$ and $\{w(2k_j - 1), w(2k_j)\}$ have the same sign.

  If 2 blocks have the same sign, by Proposition 19, we may have $0 < i < b_i, b_j < j$ and $k_i < k_j$, then $\text{ht}(twH) - \text{ht}(wH) \geq 3$, forcing $tw\tilde{H} > w\tilde{H}$, which is impossible.

  If no blocks have the same sign, we may have $k_j < i < 0 < k_i < j$, and $k_j < k_i$. Then $\text{ht}(twH) < \text{ht}(wH)$, which is impossible.

- Assume $t = (i\ \bar{j})$, and one of $|b_i|, |b_j|$ is smaller than $|i|$, and the other is larger than $|j|$. Then $twH = t'wH$, where $t' = (b_i\ b_j)$ or $t' = (b_i\ \overline{b_j})$, which we have discussed in the previous two cases.

In sum, all the cases above are impossible. So when $twH > wH$, we must have $tw\tilde{H} \geq w\tilde{H}$. □

In fact, $W/H = W/\tilde{H} \times K_4$ is an even W-set, so we only need to verify QP2: If $\text{ht}(rwH) > \text{ht}(wH)$ and $\text{ht}(srwH) < \text{ht}(swH)$, then $rw = sw$.

From the lemma, we are able to see that $\text{ht}(rw\tilde{H}) \geq \text{ht}(w\tilde{H})$ and $\text{ht}(srw\tilde{H}) \leq \text{ht}(sw\tilde{H})$.

- If $\text{ht}(rw\tilde{H}) > \text{ht}(w\tilde{H})$, by the quasiparabolic property of $\tilde{H}$, we have $\text{ht}(srw\tilde{H}) < \text{ht}(sw\tilde{H})$, and $rw\tilde{H} = sw\tilde{H}$. In addition, the K_4 part of swH is equal to that of wH. Suppose $s = (i\ i+1)$, then r switches the two symbols b_i, b_{i+1}.

  If b_i, b_{i+1} have the same sign, then $r = (b_i\ b_{i+1})$, and we will not change the K_4 part of wH.

  If b_i, b_{i+1} have distinct signs, then $r = (b_i\ \overline{b_{i+1}})$, and we will not change the K_4 part of wH, too.

- If $\mathrm{ht}(rw\tilde{H}) = \mathrm{ht}(w\tilde{H})$, then $rw\tilde{H} = w\tilde{H}$ and $srw\tilde{H} = sw\tilde{H}$. Note that the actions of r and s on the K_4 part commute, and the action of s on the K_4 part for $rw\tilde{H}$ and $w\tilde{H}$ are the same. So by $rwH > wH$, $srwH < swH$, we have $rwH = swH$.

The quasiparabolic property of H can thus be derived from the quasiparabolic property of $\tilde{H}$ in $W = D_{2n}$. □

3.7 Quasiparabolic subgroups of products of finite Coxeter groups

We have studied all the indecomposable quasiparabolic subgroups H in finite classical Coxeter groups W. Since all quasiparabolic subgroups H are rotation subgroups or their simple double covers, and by operations in Example 2 and 8, all components in the double cover product H should be quasiparabolic, we see that any quasiparabolic subgroups will be double cover products of these smaller quasiparabolic subgroups.

In addition, we will prove H will be a simple double cover product of the smaller quasiparabolic subgroups.

Proposition 21. *Suppose H_1 and H_2 are two quasiparabolic subgroups in a finite classical Coxeter group W, and H_1 and H_2 act on disjoint symbol sets. If r_1 and r_2 are representatives of double cover reflection classes of H_1 and H_2 respectively, and at least one of them is in a simple double cover reflection class, then the double cover product H of H_1 and H_2 with reflection r_1r_2 is not quasiparabolic.*

Proof. If exactly one of r_1 and r_2 is in a simple double cover reflection class, then if H is quasiparabolic, the double cover $\tilde{H} = \tilde{H}_1 \times \tilde{H}_2$ of H with r_1 (or identically with r_2), is also quasiparabolic. Then the double cover $\tilde{H}_1$ and $\tilde{H}_2$ of H_1 and H_2 with reflections r_1 and r_2 are quasiparabolic. However, one of $\tilde{H}_1$ and $\tilde{H}_2$ does not have simple reflections, so it is not quasiparabolic by Lemma 2.12 in [14]. By this contradiction, H is not quasiparabolic if exactly one of r_1 and r_2 is in a simple double cover reflection class.

If none of r_1 and r_2 is in a simple double cover reflection class, then if r_1 and r_2 are in distinct direct product components of W, we are able to apply the operations in Example 2 and 8 to delete the symbols not acted by r_1, and reduce r_1 to a simple reflection. This will be $(1\ 2)$ if r_1 is in component of type A; $(1\ 2)$ or $(1\ \overline{2})$ if r_1 is in component of type D; $(1\ 2)$ or $(1)_-$ or $(1\ \overline{2})$ if r_1 is in component of type B. If r_1 is operated to $(1\ \overline{2})$, we may assume $(1\ \overline{2})$ is in component B_2, then $(1\ \overline{2})$ is equivalent to $(2)_-$ in B_2, which can be operated to $(1)_-$ by Example 8. As long as r_1 is operated to a simple reflection, then by the same argument as in the paragraph above, we are able to show the group obtained from H is non-quasiparabolic, thus H is also non-quasiparabolic.

If r_1 and r_2 are in the same direct product component of W, first by the consecutive property of the orbit, we may assume that the two symbols a_1, b_1 acted by r_1 are both smaller than the two symbols a_2, b_2 acted by r_2. Then we may also apply the operations in Example 2 and 8 to delete the symbols less than a_2, b_2 other than a_1, b_1,, and transform r_1 to simple reflection, or delete all symbols other than a_1, b_1, a_2, b_2, and get a non-quasiparabolic $\Delta(\mathbb{Z}/2\mathbb{Z})$ generated by $(1\ \overline{2})(3\ 4)$ or $(1\ \overline{2})(3\ \overline{4})$ in B_4. So H can not be quasiparabolic.

In summary, if a double cover product of two quasiparabolic subgroup is still quasiparabolic, it should be a simple double cover product. □

Now we list all indecomposable quasiparabolic subgroups in finite classical Coxeter groups W.

Theorem 16. *All quasiparabolic subgroups H are the simple double cover products of the following quasiparabolic subgroups, their images of the following inductive Coxeter homomorphisms and the even subgroups of their pre-images of following projective Coxeter homomorphisms.*

The quasiparabolic subgroups are listed as follows:

- *Trivial group in A_1, B_1 or D_2;*
- *Alt_k in A_{k-1};*
- *B_k° in A_{2k-1} generated by $(2i-1\ 2i)(2j-1\ 2j)$ and $(2i-1\ 2j-1)(2i\ 2j)$, or generated by $(i\ 2k+1-i)(j\ 2k+1-j)$ and $(i\ j)(2k+1-i\ 2k+1-j)$;*
- *$\mathbb{F}_2^{k-1} \rtimes B_k^\circ$ in D_{2k} generated by $(2i-1\ 2i)(2j-1\ 2j)$, $(2i-1\ 2j-1)(2i\ 2j)$, and $(2i-1\ \overline{2i})(2j-1\ \overline{2j})$;*
- *$\mathbb{F}_2^{k} \rtimes B_k^\circ$ in B_{2k} generated by $(2i-1\ 2i)(2j-1\ 2j)$, $(2i-1\ 2j-1)(2i\ 2j)$, and $(2i-1\ 2i)(2j-1\ \overline{2j})$;*
- *$\mathbb{F}_2 \times Alt_4$ in D_4 generated by $(1\ 2\ 3)$, $(2\ 3\ 4)$ and $(1\ \overline{2})(3\ \overline{4})$;*

- *$PGL(3,2)$ in A_6 generated by quasiparabolic B_3° in S_6 generated by $(2i-1\ 2i)(2j-1\ 2j)$ and $(2i-1\ 2j-1)(2i\ 2j)$, and 2-rotation $(1\ 3)(5\ 7)$;*
- *$\mathbb{F}_2^3 \rtimes PGL(3,2)$ in D_7 generated by quasiparabolic $PGL(3,2)$ in S_7, and 2-rotation $(1\ \overline{2})(3\ \overline{4})$;*
- *$AGL(3,2)$ in A_7 generated by quasiparabolic B_4° in S_8 generated by $(2i-1\ 2i)(2j-1\ 2j)$ and $(2i-1\ 2j-1)(2i\ 2j)$, and 2-rotation $(1\ 3)(5\ 7)$;*
- *$\mathbb{F}_2^4 \rtimes AGL(3,2)$ in D_8 generated by quasiparabolic $AGL(3,2)$ in S_8, and 2-rotation $(1\ \overline{2})(3\ \overline{4})$;*
- *Dil_{10} in A_4 generated by $(1\ 2)(3\ 4)$ and $(2\ 3)(4\ 5)$;*
- *Twisted Alt_5 in A_5 generated by $(1\ 2)(3\ 4)$, $(2\ 3)(4\ 5)$ and $(3\ 4)(5\ 6)$;*
- *H_3 in D_6 generated by $(1\ 2)(3\ 4)$, $(2\ 3)(4\ 5)$ and $(\overline{1}\ 2)(5\ 6)$;*
- *Dil_6 in B_3 generated by $(1)_-(2\ 3)$ and $(2)_-(1\ 3)$;*
- *Dil_{16} in B_4 generated by $(1\ 2)(3\ 4)$ and $(2\ 3)(1)_-$;*
- *ΔS_k in $A_{k-1} \times A_{k-1}$ generated by $(i\ j)(i'\ j')$;*
- *ΔD_k in $D_k \times D_k$ generated by $(i\ j)(i'\ j')$ and $(\overline{1}\ 2)(\overline{1'}\ 2')$;*
- *ΔB_k in $B_k \times B_k$ generated by $(i\ j)(i'\ j')$ and $(1)_-(1')_-$;*
- *$K_4 \rtimes (\Delta S_3)$ in $A_3 \times A_2$ generated by quasiparabolic ΔS_3 in $A_2 \times A_2$, and 2-rotation $(1\ 2)(3\ 4)$;*
- *$\mathbb{F}_2 \times (K_4 \rtimes (\Delta S_3))$ in $D_4 \times A_2$ generated by quasiparabolic $K_4 \rtimes (\Delta S_3)$ in $A_3 \times A_2$, and 2-rotation $(1\ \overline{2})(3\ \overline{4})$;*
- *$\mathbb{F}_2^3 \rtimes (K_4 \rtimes (\Delta S_3))$ in $D_4 \times D_3$ generated by quasiparabolic $\mathbb{F}_2 \times (K_4 \rtimes (\Delta S_3))$ in $D_4 \times A_2$ and 2-rotation $(1\ \overline{2})(1'\ \overline{2'})$;*
- *$K_4 \rtimes (\Delta S_4)$ in $A_3 \times A_3$ generated by quasiparabolic ΔS_4 in $A_3 \times A_3$, and 2-rotation $(1\ 2)(3\ 4)$;*
- *$\mathbb{F}_2 \times (K_4 \rtimes (\Delta S_4))$ in $D_4 \times A_3$ generated by quasiparabolic $K_4 \rtimes (\Delta S_4)$ in $A_3 \times A_3$, and 2-rotation $(1\ \overline{2})(3\ \overline{4})$;*
- *$(\mathbb{F}_2^2) \rtimes (K_4 \rtimes (\Delta S_4))$ in $D_4 \times D_4$ generated by quasiparabolic $\mathbb{F}_2 \times (K_4 \rtimes (\Delta S_4))$ in $D_4 \times A_3$, and 2-rotation $(1'\ \overline{2'})(3'\ \overline{4'})$;*

- $(\mathbb{F}_2^4) \rtimes (K_4 \rtimes (\Delta S_4))$ *in* $D_4 \times D_4$ *generated by quasiparabolic* $(\mathbb{F}_2^2) \times (K_4 \rtimes (\Delta S_4))$ *in* $D_4 \times D_4$*, and 2-rotation* $(1\ \overline{2})(1'\ \overline{2'})$*;*

The inductive Coxeter homomorphisms are listed as follows:

- $A_n \times W' \to A_{n+1} \times W'$*;*
- $B_n \times W' \to B_{n+1} \times W'$*;*
- $D_n \times W' \to D_{n+1} \times W'$*;*
- $A_{n_1} \times A_{n_2} \times W' \to A_{n_1+n_2+1} \times W'$*;*
- $B_{n_1} \times A_{n_2} \times W' \to B_{n_1+n_2+1} \times W'$*;*
- $D_{n_1} \times A_{n_2} \times W' \to D_{n_1+n_2+1} \times W'$*.*

The projective Coxeter homomorphisms are listed as follows:

- $B_n \times W' \to A_{n-1} \times W'$*;*
- $D_n \times W' \to A_{n-1} \times W'$*.*

Chapter 4

The Kazhdan-Lusztig Theory of Quasiparabolic Subgroups

4.1 Hecke algebra modules of quasiparabolic sets

In this section, we recall some basic results and notation about Hecke algebra modules of quasiparabolic sets. The Hecke algebra $H_W(q)$ of a Coxeter system (W, S) is the $\mathbb{Z}[q]$-algebra with generators $T(s)$ for $s \in S$ with relations [10]

$$\begin{aligned} T(s)T(w) =& T(sw), \text{ if } l(sw) > l(w); \\ T(s)^2 =& (q-1)T(s) + qT(1). \end{aligned}$$

The Hecke algebra $H_W(q)$ has a basis $T(w)$ for $w \in W$. We are able to define the **Kazhdan-Lusztig bar operator** $\overline{T(w)} = (T(w^{-1}))^{-1}$, and expand $\overline{T(w)}$ as the linear combination of $T(x)$ with $x \leq w$ over $\mathbb{Z}[q, q^{-1}]$ with $\bar{q} = q^{-1}$ [10].

$$\overline{T(w)} = \epsilon_w q_w^{-1} \sum_{x \leq w} \epsilon_x R_{x,w}(q) T(x), \tag{4.1}$$

where $R_{x,w} \in \mathbb{Z}[q]$ is a polynomial of degree $l(w) - l(x)$ in q, and $R_{w,w}(q) = 1$, $\epsilon_w = (-1)^{l(w)}$ and $q_w = q^{l(w)}$. From the existence of the R-polynomial of $H_W(q)$, we are able to build Kazhdan-Lusztig theory. See [7].

For a quasiparabolic W-set X, we are able to define $H_W(q)$-modules $H_X^{\pm}(q)$ with a basis $x \in X$

as follows:

Definition 15. *[14] Let $T(X)$ be the free $\mathbb{Z}[q]$-module with basis $T(x)$ for $x \in X$. For $s \in S$, define endomorphisms $T_{\pm}(s)$ of $T(X)$ by*

$$T_{\pm}(s)T(x) = \begin{cases} T(sx) & ht(sx) > ht(x); \\ \epsilon_{\pm} T(x) & ht(sx) = ht(x); \\ (q-1)T(x) + qT(sx) & ht(sx) < ht(x). \end{cases} \tag{4.2}$$

Here $\epsilon_+ = q$ and $\epsilon_- = -1$ are the action of the generators $T(s)$ of $H_W(q)$ under the trivial representation 1_+ and the sign representation 1_-. Then the map $T(s) \to T_{\pm}(s)$ gives $T(X)$ a $H_W(q)$-module structure, denoted by $H_X^{\pm}(q)$.

We are also curious about building the Kazhdan-Lusztig theory on $H_X^{\pm}(q)$, however, the existence of the R-polynomial is not known in general. Assuming the existence of the R-polynomial, an analogue Kazhdan-Lusztig theory can be derived. See [12].

In fact, the existence of the R-polynomial is equivalent to the following conjecture.

Conjecture 1. *[14] Let $H \subset W$ be a quasiparabolic subgroup, and $(W/H)^-$ denote the scaled W-set $(W/H, -ht)$. Then there is an isomorphism with coefficients in $\mathbb{Z}[q, q^{-1}]$*

$$H_{W/H}^{\pm}(q) \simeq H_{(W/H)^-}^{\pm}(q) \tag{4.3}$$

of $H_W(q)$-modules in which $T^{\pm}(H)$ maps to $T^{\pm}(H)$.

Equivalently, this isomorphism exists iff the annihilator of $T^{\pm}(H)$ in $H_{(W/H)^{\pm}}(H)$ is mapped into the annihilator of $T^{\pm}(H)$ in $H_{(W/H)^{\mp}}(H)$ by the Hecke algebra bar operator in $H_W(q)$.

In other words, suppose w_1 and w_2 are both minimal representative of the left coset wH, then $\overline{T(w_1)}T_+(H) = \overline{T(w_2)}T_+(H)$, and $\overline{T(w_1)}T_-(H) = \overline{T(w_2)}T_-(H)$.

In [4] and [5], Deodhar studied the Hecke algebra module for standard parabolic subgroups W_I in W, and developed the analogues of R-polynomials and Kazhdan-Lusztig theory. Note that for any $w \in W$, there is a unique decomposition $w = w^I w_I$, where $w_I \in W_I$, and w^I is the unique minimal representative of the left coset wW_I. So the existence of the bar operator of the Hecke algebra module for a standard parabolic subgroup W_I is trivial. Another motivating example of a quasiparabolic set is the set $\mathcal{I}$ of perfect involutions in W^+, where W^+ is the semidirect product of W by the Coxeter automorphisms of W. In this case, the K-L bar operator for its Hecke algebra module also exists [12].

Now we consider other quasiparabolic sets.

4.2 The existence of K-L bar operator for Hecke algebra modules of quasiparabolic subgroups in finite classical Coxeter groups

By Proposition 7.4 in [14], if a quasiparabolic W-set X and a W'-set X' have K-L bar operators on their Hecke algebra modules, then so does the quasiparabolic $W \times W'$-set $X \times X'$.

By Theorem 7.6 in [14], if H is a quasiparabolic subgroup of W_I, and the Hecke algebra module $H_{W_I/H}$ has a K-L bar operator, then so does the Hecke algebra module $H_{W/H}$, by the isomorphism of the induced representation

$$H_{(W/H)^+}(q) \simeq \mathrm{Ind}_{H_I(q)}^{H_W(q)} H_{(W_I/H)^+}(q) \simeq \mathrm{Ind}_{H_I(q)}^{H_W(q)} H_{(W_I/H)^-}(q) \simeq H_{(W/H)^-}(q)$$

from

$$H_{(W_I/H)^+}(q) \simeq H_{(W_I/H)^-}(q).$$

Also, by Lemma 7.9 in [14], the annihilator $I_{W/H}$ of $T(H)$ in $H_{W/H}(q)$ is generated by $T(w) - T(w')$, where w and w' are distinct minimal representatives of the left coset wH. In addition, $I_{W/H}$ can be generated by the elements $T((st)^{k/2}w)$ and $T((ts)^{k/2}w)$, where $(st)^{k/2}w$ and $(ts)^{k/2}w$ are two distinct minimal representatives of $(st)^{k/2}wH$. Here $(st)^{k/2}wH$ is the maximal point of the $\langle s,t\rangle$-orbit of wH, and wH is the minimal point of the $\langle s,t\rangle$-orbit of wH.

Now we are able to prove that the quasiparabolic $\Delta(W)$ in $W \times W$ has a Kazhdan-Lusztig bar operator in its Hecke algebra module.

Proposition 22. *The Hecke algebra module $H_{(W\times W)/(\Delta W)}$ has a Kazhdan-Lusztig bar operator.*

Proof. Suppose the first $W = W_1$ has simple reflections $S = \{s_1, \ldots, s_n\}$, and the second $W = W_2$ has simple reflections $S' = \{s'_1, \ldots, s'_n\}$. Note that all left cosets $w\Delta(W)$ has minimal representatives $w \in W_1$, and $(W \times W)/\Delta(W) \simeq W(= W_1)$ as a $W(= W_1)$-set. For the generators

$T((st)^{k/2}w) - T((ts)^{k/2}w)$ in the annihilator of $T(H)$, $(H = \Delta(W))$ we may assume $w \in W_1$. If s or t is in W_2, then they commute with all $w \in W_1$, and thus we may move s or t to the right of $(st)^{k/2}w$ and get s_0 or t_0 in W_1 such that $s_0\Delta(W) = s\Delta(W)$, $t_0\Delta(W) = t\Delta(W)$.

Then $\overline{T((st)^{k/2}wH)} - \overline{T((ts)^{k/2}wH)} = \overline{T(w_1H)} - \overline{T(w_2H)}$, where $w_1, w_2 \in W_1$, and $(st)^{k/2}wH = w_1H$, $(ts)^{k/2}wH = w_2H$. From the existence of the K-L bar operator in the W-set W, we know $\overline{T(w_1H)} - \overline{T(w_2H)} = 0$, thus $\overline{T((st)^{k/2}w)} - \overline{T((ts)^{k/2}w)}$ also lies in the annihilator of $T(H)$. So the Hecke algebra module $H_{(W\times W)/(\Delta W)}$ has the Kazhdan-Lusztig bar operator. □

Besides, the following lemma may also be useful to reduce the number of generators in the annihilator of $T(H)$ we need to check, when we try to prove the existence of the K-L bar operator.

Lemma 4. *Consider a Coxeter group W, and a subgroup H is quasiparabolic in W. Suppose W_I is a standard parabolic subgroup of W, and $H_I = W_I \cap H$. If $|W/H| = |W_I/H_I| < \infty$, then for any $w \in W_I$, if w is a minimal representative of wH_I in W_I, then w is also a minimal representative of wH in W. Thus the Hecke algebra modules $H_{W_I/H_I}(q)$ and $H_{W/H}(q)$ are isomorphic as H_{W_I}-modules.*

Proof. This follows directly from the deletion criterion of quasiparabolic subgroups (Corollary 2.8 in [14]). All minimal representatives w of wH_I in W_I will be automatically become minimal representatives of wH in W. The condition $|W/H| = |W_I/H_I| < \infty$ ensures that no more left cosets exist in W/H other than those in W_I/H_I. □

Now we are able to prove if the Hecke algebra module of a quasiparabolic subgroup H in W has a K-L bar operator, then so does the Hecke algebra module for the pre-image $\phi^{-1}(H)$ of H under a projective Coxeter homomorphism ϕ, if W is a subgroup of its pre-image $\phi^{-1}(W)$.

Proposition 23. *Suppose $\phi : W_1 \to W_2$ is a projective Coxeter homomorphism, and the short exact sequence*

$$1 \to \ker(\phi) \to W_1 \to W_2 \to 1$$

splits. Let H_2 be quasiparabolic in W_2, and let $H_1 = \phi^{-1}(H_2)$ be the pre-image of H_2. Suppose the $H_{W_2}(q)$-Hecke algebra module $H_{W_2/H_2}(q)$ has a K-L bar operator. Then the $H_{W_1}(q)$-Hecke algebra module $H_{W_1/H_1}(q)$ has a K-L bar operator.

Proof. The W_2-sets W_2/H_2 and W_1/H_1 are isomorphic. Since W_2 can be viewed as a subgroup of W_1, then for any generators $T((st)^{k/2}wH_1)-T((ts)^{k/2}wH_1)$ of the annihilator of $T(H_1)$ is H_{W_1/H_1}. The elements s,t,w can be projected to W_2 and

$$\overline{T[(st)^{k/2}wH_1]-T[(ts)^{k/2}wH_1]} = \overline{T[(\phi(s)\phi(t))^{k/2}\phi(w)H_2]-T[(\phi(t)\phi(s))^{k/2}\phi(w)H_2]},$$

where $\phi(s),\phi(t),\phi(w)\in W_2$, and $(\phi(s)\phi(t))^{k/2}\phi(w)H_2 = (\phi(t)\phi(s))^{k/2}\phi(w)H_2$. By the existence of the K-L bar operator of H_{W_2/H_2},

$$\overline{T[(\phi(s)\phi(t))^{k/2}\phi(w)H_2]-T[(\phi(t)\phi(s))^{k/2}\phi(w)H_2]} = 0.$$

So H_{W_1/H_1} also has a K-L bar operator. □

In [12], Marberg proved that a quasiparabolic (X,ht) admits a K-L bar operator if and only if its even double cover $(\tilde{X},\tilde{\mathrm{ht}})$ admits a bar operator. Using a similar idea, we will show that a quasiparabolic $(W/H,\mathrm{ht})$ admits a K-L bar operator if and only if its double simple cover $(W/\tilde{H},\mathrm{ht})$ admits a K-L bar operator.

Theorem 17. *Suppose H is quasiparabolic in W and $\tilde{H}$ is a simple double cover of H with simple reflection t. Then a quasiparabolic $(W/H,ht)$ admits a K-L bar operator if and only if $(W/\tilde{H},ht)$ admits a K-L bar operator.*

Proof. We follow the proof of Theorem 3.3 in [12].

Define a $W\times A_1$-set $(W/\tilde{H}\times\mathbb{F}_2,\mathrm{ht}')$ as follows:

$$\mathrm{ht}'(w\tilde{H},0)=\mathrm{ht}(w\tilde{H}),$$

and

$$\mathrm{ht}'(w\tilde{H},1)=\mathrm{ht}(w\tilde{H})+1.$$

The $W\times A_1$-action on $(W/\tilde{H}\times\mathbb{F}_2,\mathrm{ht}')$ is as follows: For $s\in S(W)$, if $sw\tilde{H}\neq w\tilde{H}$, or $swH=wH$, then $s(w\tilde{H},a)=(sw\tilde{H},a)$.

Otherwise if $sw\tilde{H}=w\tilde{H}$ and $swH\neq wH$, then $s(w\tilde{H},a)=(sw\tilde{H},a+1)$.

For $s\in A_1$, $s(w\tilde{H},a)=(w\tilde{H},a+1)$.

So for any $(w_1, a_1) \in W \times A_1$, its action on $W/\tilde{H} \times \mathbb{F}_2$ is given by $(w_1, a_1)(w\tilde{H}, a) = (w_1 w\tilde{H}, a + a_1 + \delta)$, where

$$\delta = \begin{cases} 0 & \text{if } \mathrm{ht}(w_1 w\tilde{H}) - \mathrm{ht}(w\tilde{H}) \equiv \mathrm{ht}(w_1 wH) - \mathrm{ht}(wH) \pmod 2, \\ 1 & \text{if } \mathrm{ht}(w_1 w\tilde{H}) - \mathrm{ht}(w\tilde{H}) \equiv \mathrm{ht}(w_1 wH) - \mathrm{ht}(wH) + 1 \pmod 2. \end{cases}$$

We will next check that $(W/\tilde{H} \times \mathbb{F}_2, \mathrm{ht}')$ is a quasiparabolic $W \times A_1$-set.

Lemma 5. *The $W \times A_1$-set $(W/\tilde{H} \times \mathbb{F}_2, ht')$ is quasiparabolic.*

Proof. We check the two conditions in the definition of quasiparabolic subgroups.

- If $r \in R(W \times A_1) = R(W) \cup R(A_1)$, and $\mathrm{ht}'[r(w\tilde{H}, a)] = \mathrm{ht}'[(w\tilde{H}, a)]$, we will verify $r(w\tilde{H}, a) = (w\tilde{H}, a)$.

  In fact, if $\mathrm{ht}'[r(w\tilde{H}, a)] = \mathrm{ht}'[(w\tilde{H}, a)]$, we must have $r \in R(W)$. We will prove $rw\tilde{H} = w\tilde{H}$. Otherwise, without loss of generality, we may suppose $rw\tilde{H} > w\tilde{H}$, then from $\mathrm{ht}'[r(w\tilde{H}, a)] = \mathrm{ht}'[(w\tilde{H}, a)]$, we will have $a = 1$, $r(w\tilde{H}, a) = r(w\tilde{H}, 1) = (rw\tilde{H}, 0)$, $\mathrm{ht}(rw\tilde{H}) = \mathrm{ht}(w\tilde{H}) + 1$, and $\mathrm{ht}(rwH) \equiv \mathrm{ht}(wH) \pmod 2$. Since $\mathrm{ht}(rwH) - \mathrm{ht}(rw\tilde{H})$ and $\mathrm{ht}(wH) - \mathrm{ht}(w\tilde{H})$ are equal to 0 or 1. So $\mathrm{ht}(rwH) - \mathrm{ht}(wH)$ is equal to 0 or 2.

  If $\mathrm{ht}(rwH) = \mathrm{ht}(wH)$, then since H is quasiparabolic in W, we have $rwH = wH$, and then $rw\tilde{H} = w\tilde{H}$.

  Otherwise, $\mathrm{ht}(rwH) = \mathrm{ht}(wH) + 2$, and then $\mathrm{ht}(rw\tilde{H}) = \mathrm{ht}(rwH) - 1$ and $\mathrm{ht}(w\tilde{H}) = \mathrm{ht}(wH)$. So $\mathrm{ht}(rwt\tilde{H}) = \mathrm{ht}(rwtH)$ and $\mathrm{ht}(wt\tilde{H}) = \mathrm{ht}(wtH) - 1$. We will get $\mathrm{ht}(rwtH) = \mathrm{ht}(wtH)$, then $rwtH = wtH$, forcing $rw\tilde{H} = rwt\tilde{H} = wt\tilde{H} = w\tilde{H}$.

  So QP1 holds for the $W \times A_1$-set $W/\tilde{H} \times \mathbb{F}_2$.

- If $r \in R(W \times A_1) = R(W) \cup R(A_1)$, $s \in S(W \times A_1)$, $\mathrm{ht}'[r(w\tilde{H}, a)] > \mathrm{ht}'(w\tilde{H}, a)$ and $\mathrm{ht}'[sr(w\tilde{H}, a)] < \mathrm{ht}'[s(w\tilde{H}, a)]$, we will verify $r(w\tilde{H}, a) = s(w\tilde{H}, a)$.

  If $r \in R(W)$ and $s \in S(W)$, from $\mathrm{ht}'[r(w\tilde{H}, a)] > \mathrm{ht}'(w\tilde{H}, a)$ and $\mathrm{ht}'[sr(w\tilde{H}, a)] < \mathrm{ht}'[s(w\tilde{H}, a)]$, we know that $rw\tilde{H} \geq w\tilde{H}$ and $srw\tilde{H} \leq sw\tilde{H}$.

  - ⋆ If one of these equalities holds, then the other also holds. In this case, $a = 0$, $r(w\tilde{H}, a) = r(w\tilde{H}, 0) = (w\tilde{H}, 1)$, and $sr(w\tilde{H}, a) = sr(w\tilde{H}, 0) = s(w\tilde{H}, 1) < s(w\tilde{H}, 0)$. Thus $s(w\tilde{H}, a) = s(w\tilde{H}, 0) = (w\tilde{H}, 1) = r(w\tilde{H}, a)$.

- ⋆ If none of the equalities hold, then $rw\tilde{H} > w\tilde{H}$ and $srw\tilde{H} < sw\tilde{H}$. Since H is quasiparabolic in W, we have $rw\tilde{H} = sw\tilde{H}$. Note that $\mathrm{ht}'[r(w\tilde{H}, a)] - \mathrm{ht}'(w\tilde{H}, a) = \mathrm{ht}(rw\tilde{H}, a) - \mathrm{ht}(w\tilde{H}, a) = 1$ and $\mathrm{ht}'[s(w\tilde{H}, a)] - \mathrm{ht}'[sr(w\tilde{H}, a)] = sw\tilde{H} - srw\tilde{H} = 1$, so the $\mathbb{F}_2$ part of $r(w\tilde{H}, a)$ and $s(w\tilde{H}, a)$ are all equal to a. Then $r(w\tilde{H}, a) = s(w\tilde{H}, a)$.

- If $r \in R(W)$ and $s \in S(A_1)$, then $\mathrm{ht}'[r(w\tilde{H}, a)] > \mathrm{ht}'[(w\tilde{H}, a)]$ and $\mathrm{ht}'[r(w\tilde{H}, a+1)] < \mathrm{ht}'[(w\tilde{H}, a+1)]$. So we must have $r(w\tilde{H}, a) = (w\tilde{H}, a+1) = s(w\tilde{H}, a)$.

- If $r \in S(A_1)$ and $s \in S(W)$, then $\mathrm{ht}'[(w\tilde{H}, a+1)] > \mathrm{ht}'[(w\tilde{H}, a)]$ and $\mathrm{ht}'[s(w\tilde{H}, a+1)] < \mathrm{ht}'[s(w\tilde{H}, a)]$. So $s(w\tilde{H}, a) = (w\tilde{H}, a+1) = r(w\tilde{H}, a)$.

- If $r \in S(A_1)$ and $s \in S(A_1)$, then $r(w\tilde{H}, a) = (w\tilde{H}, a+1) = s(w\tilde{H}, a)$.

□

We now go back to the proof of Theorem 17. For any $w \in W$ where w is a minimal representative in wH, if w is also a minimal representative in $w\tilde{H}$, then $wH = (w\tilde{H}, 0)$. Otherwise, wt is a minimal representative in $w\tilde{H}$, then $wH = (w\tilde{H}, 1)$.

So $(W/H, \mathrm{ht})$ is isomorphic to $(W/\tilde{H} \times \mathbb{F}_2, \mathrm{ht}')$ as W-sets.

Now it is obvious how to construct Hecke algebra modules $H_{(W/\tilde{H}\times\mathbb{F}_2)^\pm}(q)$ over $H_{W\times A_1}(q)$. Note that since $s_0 \in A_1$ commutes with all elements in $w \in W$, $H_{(W/\tilde{H}\times\mathbb{F}_2)^\pm}(q)$ admits a K-L bar operator if and only if $H_{(W/H)^\pm}(q)$ admits a K-L bar operator.

In addition, we have injective homomorphisms of $H_W(q)$-modules [12] and [14], from $H_{(W/\tilde{H})^\pm}(q)$ to $H_{(W/\tilde{H}\times\mathbb{F}_2)^+}(q)$ with

$$T_{(W/\tilde{H})^+}(w\tilde{H}) \mapsto (T(s_0) - \epsilon_-)T_{(W/\tilde{H}\times\mathbb{F}_2)^+}((w\tilde{H}, 0));$$

$$T_{(W/\tilde{H})^-}(w\tilde{H}) \mapsto (T(s_0) - \epsilon_+)T_{(W/\tilde{H}\times\mathbb{F}_2)^+}((w\tilde{H}, 0)).$$

Note that $T(s_0) - \epsilon_-$ and $T(s_0) - \epsilon_+$ are bar invariant in $H_{W\times A_1}(q)$, and commute with all elements in the subalgebra of $H_W(q)$. So if $H_{(W/\tilde{H}\times\mathbb{F}_2)^+}(q)$ has a K-L bar operator, then $H_{(W/\tilde{H})^\pm}(q)$ also has a K-L bar operator.

For the reverse direction, suppose $H_{(W/\tilde{H})^\pm}(q)$ admits a K-L bar operator, and note that

$$T_{(W/\tilde{H})^+}(\tilde{H}) - T_{(W/\tilde{H})^-}(\tilde{H}) = (\epsilon_+ - \epsilon_-)T_{(W/\tilde{H}\times\mathbb{F}_2)^+}((\tilde{H}, 0)) = (q+1)T_{(W/\tilde{H}\times\mathbb{F}_2)^+}((\tilde{H}, 0)).$$

If $H_{(W/\tilde{H})^{\pm}}(q)$ admits a K-L bar operator, note that for any $(w_1\tilde{H}, a) = (w_2\tilde{H}, a) \in W/\tilde{H} \times \mathbb{F}_2$,

$$\begin{aligned}
&(q+1)\overline{T((w_1,a))}T_{(W/\tilde{H}\times\mathbb{F}_2)^+}((\tilde{H},0)) \\
=&\overline{T(a)T(w_1)}T_{(W/\tilde{H})^+}(\tilde{H}) - \overline{T(a)T(w_1)}T_{(W/\tilde{H})^-}(\tilde{H}) \\
=&\overline{T(a)T(w_2)}T_{(W/\tilde{H})^+}(\tilde{H}) - \overline{T(a)T(w_2)}T_{(W/\tilde{H})^-}(\tilde{H}) \\
=&(q+1)\overline{T((w_2,a))}T_{(W/\tilde{H}\times\mathbb{F}_2)^+}((\tilde{H},0)).
\end{aligned}$$

Thus

$$\overline{T((w_1,a))}T_{(W/\tilde{H}\times\mathbb{F}_2)^+}((\tilde{H},0)) = \overline{T((w_2,a))}T_{(W/\tilde{H}\times\mathbb{F}_2)^+}((\tilde{H},0)),$$

and $H_{(W/\tilde{H}\times\mathbb{F}_2)^+}(q)$ also admits a K-L bar operator. □

By Theorem 17, since the centralizer $\mathbb{F}_2^k \rtimes B_k$ of the minimal perfect involution in D_{2n} is obtained by applying the simple double cover twice with reflections (1 2) and (1 $\overline{2}$) on the subgroup $\mathbb{F}_2^{k-1} \rtimes B_k^\circ$, and the Hecke algebra module of $\mathbb{F}_2^k \rtimes B_k$ in D_{2n} admits a K-L bar operator [12], we are able to conclude that the quasiparabolic subgroup $\mathbb{F}_2^{k-1} \rtimes B_k^\circ$ also admits a K-L bar operator.

Recall that the stabilizer of the minimal perfect involution (and its even subgroup), has a K-L bar operator for its Hecke algebra module [12]. The quasiparabolic subgroups B_k° in A_{2k-1} and $\mathbb{F}_2^k \rtimes B_k^\circ$ in B_{2k} will both have bar operators for their Hecke algebra modules.

In addition, the operations of taking simple double covers and double cover products also preserve the property that the corresponding Hecke algebra module admits a K-L bar operator.

Now all the indecomposable components listed in Theorem 16, which can be arbitrary large, including

- Alt_k in A_{k-1};
- B_k° in A_{2k-1};
- $\mathbb{F}_2^{k-1} \rtimes B_k^\circ$ in D_{2k};
- $\mathbb{F}_2^k \rtimes B_k^\circ$ in B_{2k};
- ΔS_k in $A_{k-1} \times A_{k-1}$;
- ΔD_k in $D_k \times D_k$;

- ΔB_k in $B_k \times B_k$,

admit K-L bar operators for their Hecke algebra modules.

For the remaining cases, we only need to check $\overline{T((st)^{k/2}wH)} = \overline{T((ts)^{k/2}wH)}$, for all the generators $T((st)^{k/2}w) - T((ts)^{k/2}w)$ of the annihilator $I_{W/H}$ of $T(H)$.

We will next prove a lemma that will help us to reduce lots of discussion for the remaining cases.

Lemma 6. *Suppose W, H, W_I, H_I satisfy the same conditions as in Lemma 4. In addition, suppose $w \in W_I$ is a minimal representative of wH, $s, t \in S$, $swH = twH$, $m(s,t) \in \{2,3\}$, and t commutes with all $w \in W_I$. If the H_{W_I}-modules $H_{W_I/H_I}(q) \simeq H_{W/H}(q)$ admit a K-L bar operator, and t is in the same simple double cover reflection class as some $s_0 \in W_I$, then $\overline{T(swH)} = \overline{T(twH)}$.*

Proof. We prove the lemma by induction on the length of w. When the length $l(w) = 0$, it is trivial to see that $\overline{T(sH)} = q^{-1}(T(sH) - (q-1)) = q^{-1}(T(tH) - (q-1)) = \overline{T(tH)}$.

Suppose the lemma holds for all $w \in W$ with $l(w) < l$, then for $l(w) = l$, we will prove $\overline{T(swH)} = \overline{T(twH)}$. Actually, since t commutes with w, we have $\overline{T(twH)} = \overline{T(wtH)} = \overline{T(ws_0H)}$. We write $w = s_1 \dots s_n$, then $swH = ws_0H$ is a $\langle s, s_1 \rangle$-maximal element in W/H. and the length of the $\langle s, s_1 \rangle$-orbit is greater than 1. Since $m(s,t) \in \{2,3\}$, the length of the $\langle s, s_1 \rangle$-orbit is equal to $m(s,t)$, and from the induction hypothesis, the K-L bar operator exists for elements with length less than l, so we have $\overline{T(swH)} = \overline{T(ws_0H)} = \overline{T(twH)}$. □

Now we are able to check the remaining cases in Theorem 16.

- $H = \mathbb{F}_2 \times Alt_4$ in D_4 generated by (1 2 3), (2 3 4) and $(1\ \overline{2})(3\ \overline{4})$:

  For $I = \{(1\ 2), (2\ 3), (1\ \overline{2})\}$ or $I = \{(1\ \overline{2}), (2\ 3)(3\ 4)\}$, we have $W/H = W_I/H_I$, and $H_I = Alt_3$ in $W_I = D_3$ which admits a K-L bar operator. By Lemma 4, we only need to check the case $swH = twH$ when $t = (1\ 2)$, $s = (3\ 4)$, which admits $\overline{T(swH)} = \overline{T(twH)}$.

- $H = Dil_6$ in B_3 generated by $(1)_-(2\ 3)$ and $(2)_-(1\ 3)$:

  For $I = \{(1\ 2), (1)_-\}$, we have $W/H = W_I/H_I$, and H_I is the trivial group in $W_I = B_2$ which admits a K-L bar operator. By Lemma 4, we only need to check the case $swH = twH$ when $t = (2\ 3)$, $s = (1\ 2)$ or $t = (1)_-$. They satisfy $\overline{T(swH)} = \overline{T(twH)}$.

- $H = Dil_{16}$ in B_4 generated by $(1\ 2)(3\ 4)$ and $(1)_-(2\ 3)$:

For $I = \{(2\ 3), (1\ 2), (1)_-\}$, we have $W/H = W_I/H_I$, and $H_I = (W_{\{(2\ 3),(1)_-\}})^\circ = \Delta(\mathbb{Z}/2\mathbb{Z})$ in $W_I = B_3$ which admits a K-L bar operator. By Lemma 4, we only need to check the case $swH = twH$ when $t = (3\ 4)$, $s \in I$. They satisfy $\overline{T(swH)} = \overline{T(twH)}$.

- Dil_{10} in S_5 generated by $(1\ 2)(3\ 4)$ and $(2\ 3)(4\ 5)$:

  For $I = S - \{(1\ 2)\}$ or $I = S - \{(4\ 5)\}$, we have $W/H = W_I/H_I$, and $H_I = \Delta(\mathbb{Z}/2\mathbb{Z})$ in $\Delta S_2 < W_I = S_4$ which admits a K-L bar operator. By Lemma 4, we only need to check the case $swH = twH$ when $s = (1\ 2)$, $t = (4\ 5)$. They satisfy $\overline{T(swH)} = \overline{T(twH)}$.

- $H = H_3$ in D_6 generated by $(1\ 2)(3\ 4)$, $(2\ 3)(4\ 5)$ and $(1\ \overline{2})(5\ 6)$:

  For $I = S - \{(5\ 6)\}$, we have $W/H = W_I/H_I$, and $H_I = Dil_{10}$ in $S_5 < W_I = D_5$ which admits a K-L bar operator. By Lemma 4, we only need to check the case $swH = twH$ when $t = (5\ 6)$, $s \in I$. They satisfy $\overline{T(swH)} = \overline{T(twH)}$.

- When H is twisted Alt_5 in S_6 generated by $(1\ 2)(3\ 4)$, $(2\ 3)(4\ 5)$ and $(1\ 2)(5\ 6)$:

  For $I = S - \{(4\ 5), (5\ 6)\}$, we have $W/H = W_I/H_I$. Let $J = I \cup \{(4\ 5)\} = S - \{(5\ 6)\}$, then $W/H = W_J/H_J$ and $H_J = Dil_{10}$ in S_5 admit a K-L bar operator. In addition, $t = (5\ 6)$ commutes with all $w \in W_I$, and by Lemma 4 and 6, we know H in W admits a K-L bar operator.

- $H = PGL(3, 2)$ in S_7 generated by a quasiparabolic copy of B_3° in S_6 generated by $(2i - 1\ 2i)(2j - 1\ 2j)$ and $(2i - 1\ 2j - 1)(2i\ 2j)$, and the 2-rotation $(1\ 3)(5\ 7)$:

  For $I = S - \{(5\ 6), (6\ 7)\}$, we have $W/H = W_I/H_I$. Let $J = I \cup \{(5\ 6)\} = S - \{(6\ 7)\}$, then $W/H = W_J/H_J$ and $H_J = B_3^\circ$ in S_6 admits a K-L bar operator. In addition, $t = (6\ 7)$ commutes with all $w \in W_I$, and by Lemma 4 and 6, we know H in W admits a K-L bar operator.

- $H = \mathbb{F}_2^3 \rtimes PGL(3, 2)$ in D_7 generated by a quasiparabolic copy of $PGL(3, 2)$ in S_7, and the 2-rotation $(1\ \overline{2})(3\ \overline{4})$:

  For $I = S - \{(5\ 6), (6\ 7)\}$, we have $W/H = W_I/H_I$. Let $J = I \cup \{(5\ 6)\} = S - \{(6\ 7)\}$, then $W/H = W_J/H_J$ and $H_J = \mathbb{F}_2^2 \rtimes B_3^\circ$ in D_6 admits a K-L bar operator. In addition, $t = (6\ 7)$ commutes with all $w \in W_I$, and by Lemma 4 and 6, we know H in W admits a K-L bar operator.

- $H = AGL(3,2)$ in S_8 generated by a quasiparabolic copy of B_4° in S_8 generated by $(2i-1\ 2i)(2j-1\ 2j)$ and $(2i-1\ 2j-1)(2i\ 2j)$, and the 2-rotation $(1\ 3)(5\ 7)$:

  For $I = S - \{(6\ 7), (7\ 8)\}$, we have $W/H = W_I/H_I$. Let $J = I \cup \{(6\ 7)\} = S - \{(7\ 8)\}$, then $W/H = W_J/H_J$ and $H_J = PGL(3,2)$ in S_7 admits a K-L bar operator. In addition, $t = (7\ 8)$ commutes with all $w \in W_I$, and by Lemma 4 and 6, we know H in W admits a K-L bar operator.

- $H = \mathbb{F}_2^4 \rtimes AGL(3,2)$ in D_8 generated by a quasiparabolic copy of $AGL(3,2)$ in S_8, and the 2-rotation $(1\ \overline{2})(3\ \overline{4})$:

  For $I = S - \{(6\ 7), (7\ 8)\}$, we have $W/H = W_I/H_I$. Let $J = I \cup \{(6\ 7)\} = S - \{(7\ 8)\}$, then $W/H = W_J/H_J$ and $H_J = \mathbb{F}_2^3 \rtimes PGL(3,2)$ in D_7 admits a K-L bar operator. In addition, $t = (6\ 7)$ commutes with all $w \in W_I$, and by Lemma 4 and 6, we know H in W admits a K-L bar operator.

- $H = K_4 \rtimes (\Delta S_3)$ in $S_4 \times S_3$ generated by a quasiparabolic copy of ΔS_3 in $S_3 \times S_3$, and the 2-rotation $(1\ 2)(3\ 4)$:

  For $I = S - \{(1\ 2)\}$ or $I = S - \{(3\ 4)\}$, we have that $H_I = \Delta S_3$ in $W_I = S_3 \times S_3$ admits a K-L bar operator. For $I = S - \{(1'\ 2')\}$ or $I = S - \{(2'\ 3')\}$, we have that $H_I = (K_4 \times S_2)^\circ$ in $W_I = S_4 \times S_2$ admits a K-L bar operator. In both cases, $W/H = W_I/H_I$, then for any $s, t \in S$, there exists at least one of the above four I's such that $s, t \in I$. By Lemma 4, we know H in W admits a K-L bar operator.

- $H = \mathbb{F}_2 \times (K_4 \rtimes (\Delta S_3))$ in $D_4 \times S_3$ generated by a quasiparabolic copy of $K_4 \rtimes (\Delta S_3)$ in $S_4 \times S_3$, and the 2-rotation $(1\ \overline{2})(3\ \overline{4})$:

  For $I = S - \{(1\ 2)\}$ or $I = S - \{(3\ 4)\}$, we have that $H_I = \Delta S_3$ in $S_3 \times S_3 < W_I = D_3 \times S_3$ admits a K-L bar operator. For $I = S - \{(1'\ 2')\}$ or $I = S - \{(2'\ 3')\}$, we have that $H_I = \mathbb{F}_2 \times (B_2 \times S_2)^\circ$ in $W_I = D_4 \times S_2$ admits a K-L bar operator. In both cases, $W/H = W_I/H_I$, then for any $s, t \in S$, there exists at least one of the above four I's such that $s, t \in I$. By Lemma 4, we know H in W admits a K-L bar operator.

- $H = \mathbb{F}_2^3 \rtimes (K_4 \rtimes (\Delta S_3))$ in $D_4 \times D_3$ generated by a quasiparabolic copy of $\mathbb{F}_2 \times (K_4 \rtimes (\Delta S_3))$ in $D_4 \times S_3$, and the 2-rotation $(1\ \overline{2})(1'\ \overline{2'})$:

For $I = S - \{(1\ 2)\}$ or $I = S - \{(3\ 4)\}$, we have that $H_I = \Delta(\mathbb{F}_2 \times S_3)$ in $W_I = D_3 \times D_3$ admits a K-L bar operator. For $I = S - \{(1'\ 2')\}$, we have that $H_I = \mathbb{F}_2 \times (B_2 \times S_2)^\circ$ in $W_I = D_4 \times A_2$ admits a K-L bar operator. In both cases, $W/H = W_I/H_I$, then for any $s, t \in S$, there exists at least one of the above three I's such that $s, t \in I$. By Lemma 4, we know H in W admits a K-L bar operator.

- $H = K_4 \rtimes (\Delta S_4)$ in $S_4 \times S_4$ generated by a quasiparabolic copy of ΔS_4 in $S_4 \times S_4$, and the 2-rotation $(1\ 2)(3\ 4)$:

  For $I = S - \{(1\ 2)\}$, $I = S - \{(3\ 4)\}$, $I = S - \{(1'\ 2')\}$, or $I = S - \{(3'\ 4')\}$ we have that $H_I = K_4 \rtimes (\Delta S_3)$ in $W_I = S_4 \times S_3$ admits a K-L bar operator. In addition, $W/H = W_I/H_I$, then for any $s, t \in S$, there exists at least one of the above four I's such that $s, t \in I$. By Lemma 4, we know H in W admits a K-L bar operator.

- $H = \mathbb{F}_2 \times (K_4 \rtimes (\Delta S_4))$ in $D_4 \times S_4$ generated by a quasiparabolic copy of $K_4 \rtimes (\Delta S_4)$ in $S_4 \times S_4$, and the 2-rotation $(1\ \overline{2})(3\ \overline{4})$:

  For $I = S - \{(1\ 2)\}$ or $I = S - \{(3\ 4)\}$, we have that $H_I = K_4 \rtimes (\Delta S_3)$ in $S_3 \times S_4 < W_I = D_3 \times S_4$ admits a K-L bar operator. For $I = S - \{(1'\ 2')\}$ or $I = S - \{(3'\ 4')\}$, we have that $H_I = \mathbb{F}_2 \times (K_4 \times (\Delta S_3))$ in $W_I = D_4 \times S_3$ admits a K-L bar operator. In both cases, $W/H = W_I/H_I$, then for any $s, t \in S$, there exists at least one of the above four I's such that $s, t \in I$. By Lemma 4, we know H in W admits a K-L bar operator.

- $H = (\mathbb{F}_2^2) \rtimes (K_4 \rtimes (\Delta S_4))$ in $D_4 \times D_4$ generated by a quasiparabolic copy of $\mathbb{F}_2 \times (K_4 \rtimes (\Delta S_4))$ in $D_4 \times S_4$, and the 2-rotation $(1'\ \overline{2'})(3'\ \overline{4'})$:

  For $I = S - \{(1\ 2)\}$, $I = S - \{(3\ 4)\}$, $I = S - \{(1'\ 2')\}$, or $I = S - \{(3'\ 4')\}$, we have that $H_I = \mathbb{F}_2 \times (K_4 \rtimes (\Delta S_3))$ in $D_4 \times S_3 < W_I = D_4 \times D_3$ admits a K-L bar operator. In addition, $W/H = W_I/H_I$, then for any $s, t \in S$, there exists at least one of the above four I's such that $s, t \in I$. By Lemma 4, we know H in W admits a K-L bar operator.

- $H = \mathbb{F}_2^4 \rtimes (K_4 \rtimes (\Delta S_4))$ in $D_4 \times D_4$ generated by a quasiparabolic copy of $(\mathbb{F}_2^2) \rtimes (K_4 \rtimes (\Delta S_4))$ in $D_4 \times D_4$, and the 2-rotation $(1\ \overline{2})(1'\ \overline{2'})$:

  For $I = S - \{(1\ 2)\}$, $I = S - \{(3\ 4)\}$, $I = S - \{(1'\ 2')\}$, or $I = S - \{(3'\ 4')\}$, we have that $H_I = \mathbb{F}_2^3 \rtimes (K_4 \rtimes (\Delta S_3))$ in $W_I = D_4 \times D_3$ admits a K-L bar operator. In addition,

$W/H = W_I/H_I$, then for any $s, t \in S$, there exists at least one of the above four I's such that $s, t \in I$. By Lemma 4, we know H in W admits a K-L bar operator.

Since taking double cover products, images of inductive Coxeter homomorphisms, and the two projective Coxeter homomorphisms in Theorem 16 also preserve the existence of the K-L bar operator, we are able to claim the following theorem.

Theorem 18. *If W is a finite classical Coxeter group, then Conjecture 1 holds.*

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