Economic Fluctuations and Capitalistic Production: A Case Study in Robustness Constraints

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ABSTRACT

A central pursuit of macroeconomic research is to understand the source of short run variations in aggregate economic variables. To this end, the branch of macroeconomics known as Real Business Cycle (RBC) theory emphasizes the role of disturbances to the real economy while abstracting from nominal variables (e.g. money). According to RBC theory, business cycles are the result of optimal responses to exogenous stochastic disturbances on technology in a structure of capitalistic production. In this report, we contend that the structure of capitalistic production *per se* constrains the ability of the economy to absorb shocks. That is, even if the feedback behavior in the model is designed to mitigate fluctuation (and is not necessarily optimal relative to some inter-temporal utility), the resulting sensitivity is nevertheless constrained by a lower bound. Moreover, we show that this lower bound is exacerbated with increasing steady state consumption, capital and investment. Concretely, we show that the Ramsey model, linearized about its steady state equilibrium, has a non-minimum phase structure and therefore its sensitivity is constrained by the control theoretic design limits. Moreover, the non-minimum phase zero is given by the inverse of the discount factor. As the discount factor approaches unity, steady state consumption approaches optimal steady state consumption, but the non-minimum phase zero approaches the closed unit circle exacerbating the sensitivity constraints.

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INTRODUCTION

1.1 Background and Motivation

One of the primary tasks of macroeconomic research is to elucidate the key mechanisms underlying business cycles. The term *business cycle* refers to fluctuations and co-movements of key aggregate economic variables around trend-line growth. Such variables include production, employment, investment, wages, interest rates, and consumption. Periods of rapid upswings in the aggregate variables are referred to as expansions, and general declines are referred to as recessions. Despite the use of the term "cycle," economists generally agree that aggregate fluctuations are aperiodic with varying duration and occurring at irregular intervals. The now widely held empirical definition for the business cycle was established by [1]. Figure 1.1 shows real gross domestic product (GDP) since 1947 and figure 1.2 shows the unemployment rate for the U.S. since 1948; periods of recession are denoted by shaded vertical bars.

There are a great number of theories regarding the causes of business cycles (not all of them being mutually exclusive). Austrian business cycle theory, in the tradition of Carl Menger and Eugen von Bohm-Bawerk, emphasizes misallocation within an economy's inter-temporal capital structure resulting from credit-induced booms [2]. Keynesian theory regards incomplete nominal adjustment (such as sluggish

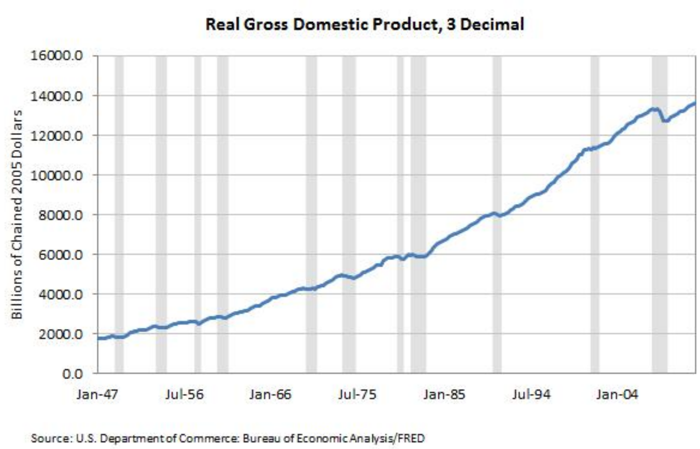

Figure 1.1: U.S. Real GDP

Figure 1.2: U.S. Unemployment Rate

price and wage adjustments) as a key transmission mechanism of fluctuations to output and employment [3]. Economists in the Monetarist tradition underscore the importance of the supply of money as a factor in recessions [4]. On the empirical side, [5] highlights the consistent empirical prominence of residential investment during business cycles and the fact that it is the best leading indicator of an oncoming recession. He suggest that housing should play a prominent role in the conduct of monetary policy. [6] shows, in an multi-sector equilibrium model, that fluctuations in the match between resources and wants creates large fluctuations in output and employment because moving resources from one sector to another is costly. Importantly, he concludes that a higher degree of specialization in the economy implies more severe fluctuations, along with a higher average level of output and growth.

Despite the variety of schools of thought regarding the causes of business cycles, the prevailing mainstream view in modern macroeconomics holds that economic fluctuations are the responses of a dynamic economic system subject to exogenous random shocks [7]. This view is motivated by the observation that relative movements of the key variables exhibit characteristic patterns during the business cycle, despite the fact that the fluctuations as a whole do not seem to follow a regular cyclical pattern [8]. The proximate causes (shocks) of expansions and contractions may vary from business cycle to business cycle and occur at irregular intervals, but the general behavior during the business cycle is somewhat consistent.

The consensus view among modern economists is that models of this flavor must have dynamic general equilibrium foundations [9]. That is, models of economic fluctuations should consist of a dynamic optimization framework in which agents optimize some inter-temporal utility function subject to constraints that represent available technology and institutional structure. Real Business Cycle (RBC) theory and New Keynesian theory are the two most prominent branches of macroeconomics that subscribe to this paradigm. RBC theory emphasizes real variables, real disturbances, efficient markets, and connections to long run growth, while New Keynesian models emphasize nominal variables such as prices, wages and interest rates, market imperfections and connections to monetary policy [3].

1.2 Contributions

In this report we explore the role that the basic capitalistic structure of an economy plays in establishing the existence of economic fluctuations. To do so we use a framework from which RBC theory is built called the Ramsey model. The Ramsey model, at its core, is a difference equation that models the basic capitalistic production process of an economy. The notion that production outputs are required as inputs to maintain production is what we refer to as *capitalistic* production. The Ramsey model also captures the notion of dynamic equilibrium (equilibrium in the economic sense). This means that streams of consumption/investment/capital are optimal in the sense of some inter-temporal utility function. This differs from the basic RBC model in two ways. In the RBC model a labor decision is included and a stochastic disturbance to production technology is introduced. So rather than dynamic equilibrium, we have stochastic dynamic equilibrium; i.e. optimal consumption/investment/capital behavior in the sense of expected utility.

Our purpose for employing such a framework is two-fold. Firstly, differences from the basic RBC model are inconsequential in the context of our contributions allowing us to appeal to the mainstream literature. Secondly, it assumes only very basic structure. That is, it abstracts from markets, money, frictions, etc. Hence, any constraints imposed by this real structure must be essential. That said, we emphasize that the purpose here is not to quantitatively reproduce actual business cycle behavior, but rather to contend that minimal assumptions give rise to conditions for which to expect the existence of fluctuations.

We use design limits from control theory to demonstrate this. [10] use similar design limits to explore design trade-offs encountered in monetary policy in a New Keynesian model. Similarly, [11] uses design limits to study robustness of optimal monetary policy rules. [12] looks at frequency domain implications of measurement error for the design of monetary policy. Our analysis, in contrast, abstracts from nominal variables and uses similar control theoretic tools to explore constraints that arise from the real structure of capitalistic production.

Concretely, we start with a Ramsey model in long run equilibrium. That is, trajectories for capital, investment and consumption are optimal with respect to infinite horizon utility and are in steady state. We show that the steady state is stable. We then assume that a small perturbation in the technology parameter induces small perturbations in all the other variables. This allows us to study the linearized system. We then assume investment as a control variable and consumption as an output. This assumption along with the capitalistic nature of the production process gives rise to a non-minimum zero. Thus, the Poisson integral applies and constrains sensitivity reduction. Limits on sensitivity reduction imply amplification of perturbations and hence economic fluctuations. Moreover, we show that the constraints depend on the discount factor to future utility. Specifically, as the discount factor approaches one, optimal steady state consumption is maximized and the non-minimum phase zero approaches 1, thereby exacerbating the constraints on sensitivity reduction.

1.3 Overview

In Chapter 2 we derive the Ramsey model. We start by reviewing the basic Solow growth model and then extend it to capture dynamic optimizing behavior. Chapter 3 establishes the main result; i.e. constraints on sensitivity reduction and how they depend on the discount factor. We start by departing from the RBC model of stochastic optimization and establishing the linearization of the Ramsey model around its long run equilibrium. We then note that the linearization has a non-minimum phase zero equal to the inverse of the discount factor. We state the sensitivity constraints and discuss their implications for business cycles. In Chapter 4 we conclude with a discussion and possibilities for future research directions.

Chapter 2

NEOCLASSICAL GROWTH MODEL

2.1 The Solow Growth Model

The Solow growth model is a simple model of economic growth based on production and capital accumulation. Production is given by the standard neoclassical aggregate production function $F : \mathbb{R}^2 \to \mathbb{R}$ which relates physical output to physical inputs. It represents the maximum attainable output given inputs while abstracting from details on how that maximum is actually attained. Production output Y in the Solow model is thus given by

$$Y = \Lambda F(K, AN)$$

where Λ represents a shock to production technology, K represents capital, N represents the labor force and A represents the "effectiveness" of labor or "knowledge". This interpretation of A follows from the fact that $A > 1$ results in an increase from $A = 1$ of the marginal productivity of labor for given capital. The Solow model itself does not actually account for shocks, but is included here for completeness as it will be utilized later; for now Λ can be taken as constant. As is standard in macroeconomics, F is assumed to satisfy the following conditions:

1. It is homogeneous of degree one, i.e. it satisfies $cy = F(cx_1, cx_2)$ for any c,
2. It has continuous first and second order partial derivatives which satisfy $F_i > 0$ and $F_{ii} < 0$ [1],
3. Its first order partial derivatives also satisfy $\lim_{x_i \to 0} F_i(x_i) = \infty$ and $\lim_{x_i \to \infty} F_i(x_i) = 0$.

The first condition, referred to as constant returns to scale, asserts that the economy is at point where there are no more gains to be made from further specialization [3]. More importantly, it allows total output to be exhaustively distributed to the owner's of inputs according to their marginal products, i.e. $F(x_1, x_2) = F_1 x_1 + F_2 x_2$ [13]. The second condition states that the marginal

[1] Here F_i represents the partial derivative of F with respect to input i and x_j represents the j^{th} input to F

product of inputs are everywhere positive but are subject to diminishing marginal returns. Lastly, the third item is referred to as the Inada conditions [14]. They guarantee a nonzero stable stationary point in the model.

In the Solow model the labor force is the entire population and so full employment is assumed (this assumption is held throughout the report as it is not essential to the the results). The labor force and its effectiveness are assumed to grow exponentially as

$$\begin{aligned} N_{t+1} &= nN_t \\ A_{t+1} &= aA_t, \end{aligned} \tag{2.1}$$

where $n > 1$ and $a > 1$ are exogenous parameters. Thus the growth rates for the population and technology are constant and given by $n-1$ and $a-1$, respectively.

Capital is assumed to grow under a *capitalistic* process; i.e., a process by which some of the output produced is used as input in the production function:

$$\begin{aligned} K_{t+1} &= \Lambda F(I_t, A_t N_t) \\ C_t &= K_t - I_t. \end{aligned} \tag{2.2}$$

Here C is consumption and I is investment. In words, current consumption is given by current capital less capital used as input to the production of next period capital (investment). Next period capital consists solely of new units of output, and so 100% depreciation per period is assumed. It should be noted that there exist different variations of the basic process model in the economics literature. Most texts write the model as

$$\begin{aligned} K_{t+1} &= I_t \\ C_t &= \Lambda F(K_t, A_t N_t) - I_t. \end{aligned}$$

In this version, K_t is interpreted as capital available at the beginning of the period but C_t (resp. I_t) is interpreted as consumption (resp. investment) at the end of the period since production takes time. To avoid any confusion we use the model in (2.2), consistent with, for example, [15].

Defining $k := \frac{K}{AN}$, and using (2.1) and (2.2) we can write

$$\begin{aligned} k_{t+1} &= \frac{\Lambda F(I_t, A_t N_t)}{anA_t N_t} \\ &= \lambda F(i_t, 1) \end{aligned}$$

which follows from constant returns to scale and where $\lambda := \frac{\Lambda}{an}$ and $i := \frac{I}{AN}$. Further defining $c := \frac{C}{AN}$ and $f(\cdot) := F(\cdot, 1)$ we get the so called intensive form of the model,

$$\begin{aligned} k_{t+1} &= \lambda f(i_t) \\ c_t &= k_t - i_t. \end{aligned} \tag{2.3}$$

where $k_t, c_t, i_t, \lambda \geq 0$. Lower case variables thus correspond to upper case variables as measured relative to trend growth in effective labor. That is, subtracting the linear trend on a log plot of an non-intensive variable gives the log plot of the respective intensive variable. Lastly, the Solow growth model consists of (2.3) along with the assumption that investment is a constant fraction of the production output (a constant savings rate s); that is, $i_t = sk_t$ with $0 \leq s \leq 1$.

Balanced Growth Path and the Golden Rule

The balanced growth path refers to the stationary point of (2.3); it is given by the solution $k^* > 0$ to $k = \lambda f(sk)$. The solution exists and is stable by the conditions on the partial derivatives of the production function F (which also hold for the intensive form of the production function f) . It is referred to as the balanced growth since a constant intensive variable implies a constant growth rate of the respective non-intensive variable; a rate equal to the growth rate of effective labor. Thus, in steady state all non-intensive variables grow at the same rate.

In particular, steady state consumption is given by $c^*(s) = \lambda f\left(sk^*(s)\right) - sk^*(s)$, where the dependence on the savings rate of steady state capital and consumption is explicitly shown. A natural question to ask is whether there exists a savings rate that achieves optimal consumption in steady state. Such a savings rate does in fact exist since steady state consumption as a function of the savings rate is continuous over a compact set. Moreover, consider the partial of c^* with respect to s,

$$\frac{\partial c^*}{\partial s} = \left(\lambda f'(sk^*(s)) - 1\right)\left(k^*(s) + s\frac{\partial k^*}{\partial s}\right).$$

Since the second multiplicative term on the rhs is always positive, the savings rate that corresponds to optimal consumption in steady state is given by $\lambda f'(sk^*(s)) = 1$. The steady state level of capital k^* that corresponds to this savings is known as the golden-rule level of capital.

2.2 The Ramsey-Cass-Koopmans Model

The Ramsey model extends the Solow model by removing the restriction that investment is a constant fraction of output (determined as an exogenous parameter) and introduces preferences over consumption at different time periods. Preferences are captured by a utility function, and investment is, therefore, determined as the strategy maximizing utility over an infinite horizon subject to the constraints of the capitalistic process. Formally, the problem is stated as

$$\begin{aligned} &\max_{i_0, i_1, \ldots} U = \sum_{t=0}^{\infty} \beta^t u(c_t) \\ &s.t. \\ &\qquad \begin{array}{rcl} k_{t+1} &=& \lambda f(i_t) \\ c_t &=& k_t - i_t \\ c_t &\geq& 0 \\ i_t &\geq& 0 \\ k_{t+1} &\geq& 0 \end{array} \end{aligned} \tag{2.4}$$

for $t = 0, 1, \ldots$, where k_0 is given, $\beta \in (0, 1)$, and $u : [0, \infty) \to \mathbb{R}$ satisfies the same conditions on the first and second order derivatives as the production function. The function u is the single period utility function and represents the utility derived from current consumption of production output. The parameter β is the discount factor and represents the degree to which current consumption is valued over future consumption; as β approaches 1, valuation of future consumption approaches that of current consumption. The utility function U, therefore, serves to rank consumption streams, and the Ramsey model posits that given initial capital, investment behaves so that consumption follows a utility maximizing trajectory.

This basic framework of inter-temporal utility optimization subject to constraints imposed by production possibilities and possibly other elements (e.g. market frictions, monetary constraints, stochastic uncertainty etc..) is referred to as a dynamic general equilibrium (DGE) model, and serves as the basic workhorse for modern macroeconomic study [9]. The reason for the popularity of this paradigm is two-fold. Firstly, DGE represents a unified modeling framework that is apt to address both questions of long run growth and short run fluctuations. Secondly, such an aggregate model, under certain conditions, can be decomposed as a decentralized model where many firms and households optimize their own objective functions subject to given prices, wages and interest rates consistent with competitive equilibrium [16]. Since, we are interested in the basic determinant of short run fluctuations (and not necessarily elements that can exacerbate fluctuations) the simplest of such DGE models, as given by (2.4), is the preferred model here.

First Order Conditions

The derivation here is analogous to that by [17]. To characterize the optimal trajectories of capital, consumption and investment, we use dynamic programming to derive first order conditions . The Bellman equation for (2.4) is given by

$$v(k) = \max_{0 \leq i \leq k} \left[u(k-i) + \beta v\left(\lambda f(i)\right) \right]. \tag{2.5}$$

The conditions on the production function and single period utility function guarantee that the Bellman principle applies so that the solution to the Bellman equation gives the solution to (2.4). Moreover, these conditions guarantee that solution v exists, is differentiable, strictly increasing, and strictly concave, the optimal policy function h is increasing and differentiable, and v is the fixed point of a suitably defined contraction mapping.

Substituting the optimal policy function into (2.5) gives

$$v(k) = u\left(k - h(k)\right) + \beta v\left(\lambda f\left(h(k)\right)\right).$$

Differentiating, we get

$$v'(k) = u'\left(k - h(k)\right)\left(1 - h'(k)\right) + \beta\left[v'\left(\lambda f\left(h(k)\right)\right)\lambda f'\left(h(k)\right)h'(k)\right]. \tag{2.6}$$

To simplify this expression, consider the first order condition that corresponds to maximizing the rhs of (2.5),

$$-u'(k-i) + \beta v'\left(\lambda f(i)\right)\lambda f'(i) = 0. \tag{2.7}$$

We use this expression to simplify the second term on the rhs of (2.6) to get

$$\begin{aligned} v'(k) &= u'\left(k - h(k)\right)\left(1 - h'(k)\right) + u'\left(k - h(k)\right)h'(k) \\ &= u'(k-i). \end{aligned}$$

Here (and through the reminder of the derivation) i refers to optimal investment $i = h(k)$. Since this expression holds for arbitrary values of k we can write, $v'\left(k_{t+1}\right) = u'\left(k_{t+1} - i_{t+1}\right)$. Substituting this into (2.7) allows us to write first order conditions, independent of the value function, as

$$1 = \beta \frac{u'\left(k_{t+1} - i_{t+1}\right)}{u'\left(k_t - i_t\right)} \lambda f'\left(i_t\right).$$

This expression along with the expression $k_{t+1} = \lambda f\left(i_t\right)$ is a system that governs the optimal trajectories of capital, investment, and consumption.

The Saddle Path, Balanced Growth, and the Modified Golden Rule

The first order conditions along with the expression for next period capital comprise a two dimensional non-linear system in capital and investment. The stationary point of this system is given by

$$\begin{aligned} k^* &= \lambda f\left(i^*\right) \\ 1 &= \beta\lambda f'\left(i^*\right). \end{aligned} \tag{2.8}$$

Using the implicit function theorem, the Jacobian of the system evaluated at the stationary point can be written as

$$A = \begin{bmatrix} 0 & \beta^{-1} \\ -1 & 1+\beta^{-1}+\frac{\beta u'(k^*-i^*)\lambda f''(i^*)}{u''(k^*-i^*)} \end{bmatrix}.$$

Since $\det(A) > 1$ and $tr(A) > 1 + \det(A)$, the eigenvalues of A, λ_1 and λ_2, are real and satisfy $0 < \lambda_1 < 1 < \lambda_2$. The stationary point, therefore, is a saddle point. It can thus be shown, as in e.g. [17, p. 16], that the one dimensional stable manifold is the graph of the optimal policy function that uniquely solves (2.4); it is referred to as the saddle path.

We have hence shown that the steady state of the one dimensional model of the basic capitalistic process (2.3) is stable when investment is chosen optimally with respect to the infinite horizon utility. The stationary point is given by (2.8), and, as in the Solow model, is referred to as the balanced growth path.

Recall from the exposition of the Solow model that the level of steady state investment corresponding to optimal steady state consumption is given by $1 = \lambda f'(i^*)$. Comparing this with (2.8) we see that steady state investment, and hence consumption and capital, in the Ramsey model fall below investment, consumption and capital corresponding to the golden rule. For this reason, the level of capital stock given by (2.8) is referred to as the modified golden rule. Importantly, note that as $\beta \to 1$ the steady state of the Ramsey model approaches the golden rule and hence approaches optimal steady state consumption. As we will see, this has important implications regarding how the Ramsey model behaves to disturbances from the steady state.

Chapter 3

THE LINEARIZED MODEL AND ROBUSTNESS CONSTRAINTS

The branch of macroeconomics known as Real Business Cycle (RBC) theory extends the Ramsey model by including hours worked as an additional set of decision variables and interpreting the shock parameter Λ as a stochastic process thereby turning the deterministic infinite horizon problem into a stochastic control problem. Models of this flavor have been shown to successfully replicate actual historical data on U.S. business cycles with simulated data from calibrated models [18]. This success has been largely responsible for establishing RBC theory as a core component of modern mainstream macroeconomics. It, however, has been met with some skepticism. In particular, the assumption that unemployment is the result of an optimal decision to allocate time toward leisure over labor has been called into question [19]. Secondly, the calibration process typically results in attributing a relatively large variance to the shock parameter Λ. Since this parameter is traditionally interpreted as shocks to technology, [20] has argued that such technology shocks are difficult to reconcile with actual data.

Despite its potential shortcomings, RBC theory has established dynamical general equilibrium as the central methodological paradigm for macroeconomic research. Moreover, recent work has addressed some of the criticisms leveled against RBC theory [21].

Given the structure of an RBC model, the theory conjectures that a tendency for economic fluctuations results from 1. optimizing behavior relative to 2. preferences subject to constraints imposed by 3. the capitalistic structure of production in a 4. dynamic and 5. uncertain environment. The RBC literature hypothesizes, as supported by simulation, that all these factors taken together are sufficient to generate business cycle phenomena. A natural question to ask is then whether each of those factors also represent quintessential elements.

Taking 4. and 5. as given, in this section we use control theory to underscore the role that the 3., the capitalistic structure of production, plays as a cause of short run fluctuations by abstracting from 1., optimizing behavior. Concretely, we adopt the same source of disturbances as that in RBC models (namely, perturbations in the parameter λ) and derive constraints that apply to any linear control laws, not just utility maximizing ones.

3.1 Linearized Model

In this section we view short run fluctuations in terms of small deviations from the balanced growth path of the Ramsey model. Assume there is an unanticipated perturbation (or shock) in the technology parameter so that the new value of the parameter is given by $\lambda(1+\Delta\lambda)$[1]. This induces a corresponding perturbation in the modified golden rule level of capital, investment and consumption; i.e. the new values are given by $k^*(1+\Delta k)$, $i^*(1+\Delta i)$, and $c^*(1+\Delta c)$, respectively. Assuming the perturbations vary with time, we have

$$\begin{aligned} k^*\left(1+\Delta k_{t+1}\right) &= \lambda\left(1+\Delta\lambda_t\right) f\left(i^*\left(1+\Delta i_t\right)\right) \\ c^*\left(1+\Delta c_t\right) &= k^*\left(1+\Delta k_t\right) - i^*\left(1+\Delta i_t\right). \end{aligned} \tag{3.1}$$

A first order approximation for the non-linear term is given by

$$\lambda\left(1+\Delta\lambda_t\right) f\left(i^*\left(1+\Delta i_t\right)\right) \approx \lambda f\left(i^*\right) + \lambda f'\left(i^*\right) i^*\Delta i_t + f\left(i^*\right)\lambda\Delta\lambda_t.$$

Substituting this linear approximation into (3.1) and recalling that the steady state satisfies (2.8) we get

$$\begin{aligned} k^*\Delta k_{t+1} &= \beta^{-1} i^*\Delta i_t + k^*\Delta\lambda_t \\ c^*\Delta c_t &= k^*\Delta k_t - i^*\Delta i_t. \end{aligned}$$

Defining $x_t := \Delta k_t$, $u_t := \Delta i_t$, $y_t := \Delta c_t$, and $d_t := \Delta\lambda_t$, for convenience, we can therefore write the linearization as

$$\begin{aligned} x_{t+1} &= \beta^{-1}\frac{i^*}{k^*}u_t + d_t \\ y_t &= \frac{k^*}{c^*}x_t - \frac{i^*}{c^*}u_t. \end{aligned}$$

Taking the z-transform of this linear system gives

$$Y = \frac{i^*}{c^*}\frac{\beta^{-1}-z}{z}U + \frac{k^*}{c^*}\frac{1}{z}D, \tag{3.2}$$

where the upper case variables refer to the z-transform of the respective lower case variables.

[1] Δx represents the percentage deviation from the unperturbed value x.

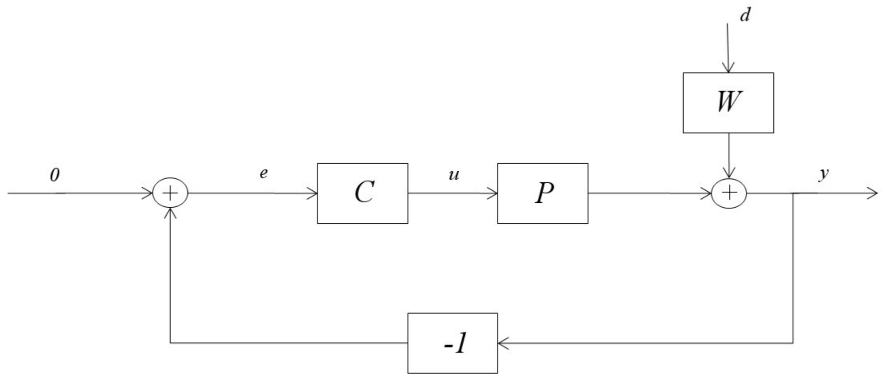

Figure 3.1: Negative Feedback System

3.2 Robustness Analysis

(3.2) gives the open loop frequency response of consumption in terms of two transfer functions: one corresponding to the open loop response of consumption to investment and the other corresponding to the open loop response of consumption to a disturbance. We posit that the system is structured as a negative feedback system with consumption as output and investment as the control variable. Thus, if we define

$$\begin{aligned} P(z) &:= \frac{i^*}{c^*}\frac{\beta^{-1}-z}{z} \\ W(z) &:= \frac{k^*}{c^*}\frac{1}{z} \end{aligned}$$

and we let $C(z)$ be any linear controller, we get the canonical negative feedback system shown in figure 3.1. As can be deduced from the figure, the transfer function from disturbance to output is given by WS, where S is the sensitivity function defined as,

$$S := \frac{1}{1+PC}.$$

The sensitivity function is well known to characterize the sensitivity of the closed loop transfer function to an infinitesimal perturbation in the nominal plant P [22]. In our context, we see that in order to mitigate fluctuations resulting from disturbances, i.e. $|WS| < 1$, the control law should be designed so as to make S as small as possible across a broad range of frequencies. The ability to design such a control law, however, is fundamentally constrained because P is non-minimum phase. With the assumption that consumption is output and investment is input, (3.2) shows the structure defining the basic capitalistic process necessarily induces a zero, $q = \beta^{-1}$, located outside the closed unit disk . The following theorems show that this non-minimum phase zero necessarily implies constraints on sensitivity reduction.

Theorem 1. *If the closed loop system of figure 3.1 is stable, then*

$$\|WS\|_\infty \geq \frac{k^*}{c^*}\beta.$$

Proof. Since the closed loop system is stable, WS is analytic in the complement of the closed unit disk, and so the maximum modulus theorem holds. Hence, $\frac{k^*}{c^*}\beta \leq |WS(\beta^{-1})| \leq \sup_{|z|>1} |WS(z)| = \|WS\|_\infty$. □

This theorem states that the non-minimum phase zero allows us to lower bound the worst case amplification of the consumption output due to a disturbance. Moreover, we show that for at least one type of production function, this lower bound gets worse as β approaches 1.

Lemma 2. *If* $\beta k^* - i^* > 0$*, then* $\frac{\partial}{\partial\beta}\left(\frac{k^*}{c^*}\right) > 0$.

Proof. Recall that the steady state conditions for capital, investment and consumption in the Ramsey model are given by

$$\begin{aligned} 1 &= \beta\lambda f'(i^*) \\ k^* &= \lambda f(i^*) \\ c^* &= \lambda f(i^*) - i^*. \end{aligned} \tag{3.3}$$

Implicit differentiation of these conditions gives

$$\begin{aligned} \frac{\partial i^*}{\partial\beta} &= -\beta^{-1}\frac{f'(i^*)}{f''(i^*)} \\ \frac{\partial k^*}{\partial\beta} &= \lambda f'(i^*)\frac{\partial i^*}{\partial\beta} \\ \frac{\partial c^*}{\partial\beta} &= (\lambda f'(i^*) - 1)\frac{\partial i^*}{\partial\beta} \end{aligned} \tag{3.4}$$

all of which are greater than zero by the properties of the production function. From the product rule we see that $\frac{\partial}{\partial\beta}\left(\frac{k^*}{c^*}\right) > 0$ is implied by

$$\frac{\partial k^*}{\partial\beta}c^* - \frac{\partial c^*}{\partial\beta}k^* > 0.$$

Using the expressions in (3.3) and (3.4), we note that the LHS is equivalent to

$$\left(k^* - \beta^{-1}i^*\right)\frac{\partial i^*}{\partial\beta}.$$

The lemma follows since $\frac{\partial i^*}{\partial\beta} > 0$. □

Theorem 3. *The Cobb-Douglas production function* [2] $f(i) = i^{\alpha}$ *where* $\alpha \in (0,1)$ *meets the condition of lemma 2,* $\forall \beta \in (0,1)$.

Proof. Applying (3.3) to $f(i) = i^{\alpha}$ we get that $i^* = (\alpha\beta\lambda)^{1/(1-\alpha)}$ and $k^* = \lambda(\alpha\beta\lambda)^{\alpha/(1-\alpha)}$. Thus,

$$\begin{aligned} \beta k^* - i^* &= \beta\lambda(\alpha\beta\lambda)^{\alpha/(1-\alpha)} - (\alpha\beta\lambda)^{1/(1-\alpha)} \\ &= (\beta\lambda - \alpha\beta\lambda)(\alpha\beta\lambda)^{\alpha/(1-\alpha)} \\ &= \beta\lambda(1-\alpha)(\alpha\beta\lambda)^{\alpha/(1-\alpha)} \\ &> 0. \end{aligned}$$

□

This theorem shows that for at least one class of production function, namely the Cobb-Douglas production function, the lower bound on $\|WS\|_\infty$ is made worse when β is increased toward 1. We next show that the non-minimum phase zero also imposes constraints directly on the sensitivity function; constraints that represent explicit trade-offs in sensitivity reduction over a range of frequencies.

Theorem 4. *(Poisson Integral for S). If the closed loop system is stable, then*

$$\int_{-\pi}^{\pi} log\left|S\left(e^{j\omega}\right)\right| \frac{1-\beta^2}{\beta^2 - 2\beta cos(\omega) + 1} d\omega = 0. \tag{3.5}$$

Proof. The proof is given in [24] with $q = \beta^{-1}$. □

Since the multiplicative term next to the log sensitivity is always positive, the Poisson integral tells us that weighted sensitivity reduction must be fully compensated by weighted sensitivity amplification. Moreover, note that the multiplicative term acts like a low pass filter amplifying low frequency log sensitivity and attenuating high frequency log sensitivity. Thus, low frequency sensitivity reduction must be accompanied by even larger sensitivity amplification at higher frequencies. This effect is exacerbated as $\beta \to 1$ since the cutoff frequency of the low pass filter gets smaller as β approaches 1 and the maximum magnitude gets larger.

[2] [23]

To make this more concrete, suppose that the economy is structured so that disturbances are sufficiently attenuated within some frequency band. Specifically assume that

$$\left|S\left(e^{j\omega}\right)\right| < \alpha, \quad \forall\omega \in [-\omega_1, \omega_1] \tag{3.6}$$

where $\omega_1 < \pi$. The following corollary (which follows from the Poisson Integral) states that the maximum sensitivity is lower bounded.

Corollary. *(The Water Bed Effect). If the closed loop system is stable and (3.6) holds, then*

$$\|S\|_\infty \geq \left(\frac{1}{\alpha}\right)^{\frac{\Omega(\omega_1)}{\pi - \Omega(\omega_1)}} \tag{3.7}$$

where Ω *is given by*

$$\Omega\left(\omega_1\right) = -\angle\left(\frac{\beta e^{j\omega_1} - 1}{\beta - e^{j\omega_1}}\right).$$

Proof. This proof is also given in [24] with $q = \beta^{-1}$. □

This corollary states that making the sensitivity arbitrarily small over some band, necessarily implies arbitrarily large maximum sensitivity. Importantly, note that $\Omega\left(\omega_1\right) \to \pi$ as $\beta \to 1$. This implies that the exponent in (3.7) becomes large as β approaches 1, thereby aggravating the constraint.

We have thus shown that in the Ramsey model, attenuation of fluctuations resulting from disturbances is necessarily limited. Since the sensitivity constraints apply for any linear controller, fluctuations are not *necessarily* the result of short run maximization of the utility function. Rather, they may be interpreted as necessary trade-offs that result from constraints given by the non-minimum phase structure of the basic capitalistic process. Moreover, this structure implies an explicit trade-off between sensitivity attenuation in different frequency bands.

We have also shown that these sensitivity constraints are made worse when β approaches 1. From the analysis of the balanced growth path of the Ramsey model, recall that the steady state approaches the golden rule as $\beta \to 1$. Thus, assuming long run equilibrium, this implies another, perhaps surprising, trade-off. As preferences adjust so that optimal steady state consumption is approached, the constraints on sensitivity are made worse. The intuition behind this is that as $\beta \to 1$ future consumption becomes as much valued as current consumption. This puts pressure on the system to be more "capitalistic," hence exacerbating the non-minimum phase nature of the steady state.

Chapter 4

CONCLUSION

4.1 Discussion

In a model of economic growth called the Ramsey model, we demonstrated that fluctuations from steady state growth can be viewed in terms of robustness (sensitivity) constraints that are a necessary byproduct of the basic capitalistic process underlying the economy. We also showed that these constraints are aggravated as the economy becomes more capitalistic .

The model used consists of optimization of infinite horizon preferences subject to constraints that represent the capitalistic economy. One of the core branches of macroeconomics, Real Business Cycle theory, extends this model by adding a stochastic component that perturbs the production process of the economy. The stochastic control problem is then solved, and aggregate fluctuations are therefore viewed as optimal responses to stochastic perturbations. In our approach, we instead focus on general perturbations to the steady state of the linearized Ramsey model. This allows us to consider controls that are not necessarily optimal relative to a consumption utility function, and to apply tools from control theory.

We find that the ability to mitigate disturbances, as viewed through the sensitivity function, is limited. Specifically, the capitalistic nature of the basic process induces a non-minimum phase zero in the linearized plant. We showed that this zero establishes a lower bound on the norm of the consumption frequency response to disturbances. Moreover, through the Poisson Integral for the sensitivity function, this non-minimum phase zero establishes trade-offs in sensitivity across different frequency bands. Therefore, controls that attenuate low frequency disturbances, necessarily amplify higher frequency disturbances. Moreover, this trade-off is exacerbated with changing preferences. When preferences change toward valuing future consumption more, the capitalistic requirements on the basic system increase and this moves the non-minimum phase zero toward the unit disk, thereby exacerbating the trade-offs.

Thus, if the economy is structured so as to "smooth" the business cycle in the presence of (low to mid frequency) disturbances, it will necessarily be fragile to higher frequency disturbances. Moreover, the better it smooths in one band, the more fragile it becomes in another. In this sense, the capitalistic nature of the basic economic process itself may be a fundamental reason for the existence of the business cycle. Therefore, optimizing behavior in the presence of stochastic disturbances, as modeled by RBC theory, can be viewed as a particular manifestation of the trade-offs imposed by the non-minimum phase capitalistic structure.

4.2 Future Research Directions

It is well known that investment is more volatile than consumption in the business cycle [25]. This work focused on constraints related to mitigating fluctuations in consumption. Future work should be directed at understanding the implications of the given analysis on investment behavior and constraints that pertain to investment (such as the Bode integral for the complementary sensitivity function).

Secondly, the assumption that consumption be treated as output in the section on the linearized model is somewhat ad hoc. There is no apriori reason to restrict setting investment as output and consumption as the control input. Doing so would replace the non-minimum phase zero with an unstable pole. Since complementary Poisson and Bode integrals exist for the case of unstable poles, the analysis should follow similarly. This is needs to be looked at carefully.

Though the aim of this report was to establish sensitivity constraints on a model that contains only the most basic features of a capitalistic economy, possible future work includes demonstrating how the sensitivity constraints are incorporated into models with more realistic structure. For example, in our analysis we assumed 100% depreciation per period of output, i.e. completely perishable output. It seems straight forward to add a depreciation term to study durable output and its effects on the given constraints.

Another direction is to explore the structure studied in the seminal paper by [18]. In their model, capital is constructed in stages before it can be used in production. Each stage takes time and resources. It seems that each stage would possibly act as a delay in the formulation of the Poisson/Bode integral, thereby exacerbating the constraints.

The results could be extended further by considering the multi-input multi-output case. [15] study a multi-sector model that includes a labor decision. Specifically, they assume a multi-input multi-output production function given by

$$K_{i,t+1} = \lambda_i L_{it}^{b_i} \prod_{j=1}^{N} I_{ijt}^{a_{ij}}, \quad i = 1, 2, \ldots, N$$

with

$$
\begin{aligned}
Z_t &= H - \sum_{i=1}^{N} L_{it} \\
C_{jt} &= K_{jt} - \sum_{i=1}^{N} I_{ijt} \quad j = 1, 2, \ldots, N
\end{aligned}
$$

where multiple commodities are now assumed, labor enters into the production function, and there is a labor/leisure decision (Z is leisure, L is labor, and H is total time available per period). Moreover, the utility function now depends on multiple commodities and leisure. This model is analogous to the model used in this report. In fact, if the labor decision is ignored and one commodity is assumed, the core of the model becomes a specific instance of the Ramsey model without underlying effective labor and population growth. Thus, it seems that analogous constraints from MIMO control would apply to this case.

A subsequent direction would be to explore nonlinear versions of constraints arising from unstable zeros. This could help illuminate on what the constraints are in the full nonlinear system.

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