PERIODIC SUPERSONIC MOTIONS
OF A THIN WING OF FINITE SPAN

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SUMMARY

The present paper supplies some general theorems with which periodic supersonic motions of a thin wing of fairly general planform may be analyzed to yield valuable three-dimensional results. It is shown that the method developed by Evvard (Refs. 1,2) for treating the steady supersonic motion of a thin wing with subsonic leading or side edges is valid for an oscillating wing of similar planform. Illustrations of the application of these general theorems are furnished by a careful study of several types of periodic oscillations of a rectangular wing. The present report includes a complete analysis for the case of plunging oscillations. Important steps have also been taken towards solution of the cases of pitching and rolling oscillations. The essential results are presented in a number of vector diagrams giving the magnitudes and phase angles of the lift and moment. Computations are made for several aspect ratios at two Mach numbers ($M=10/7,2$) when the reduced frequency (k) ranges from 0 to 2.0. It is found that the lift and moment vectors acting on a rectangular wing with supersonic plunging oscillations have positive phase angles within certain ranges of Mach numbers and aspect ratios, while the corresponding vectors acting on a wing of infinite span with the same kind of motion have negative phase angles for every Mach number. This new discovery indicates strongly the necessity of revising present day wing flutter calculations.

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I. INTRODUCTION

The determination of the forces and moments acting on an oscillating wing in a main stream of uniform supersonic speed is important because of its vital role in the prediction of the flutter and aerodynamic instability characteristics of high speed aircraft and supersonic missiles. In the past, calculations concerning this problem were mostly based on an analysis of a two-dimensional wing, i.e. a wing of infinite span (see Refs. 3, 4, 5, 6 and 7). A satisfactory theory of the corresponding problem for a three-dimensional wing of finite span is not yet available. The objective of the present research is to provide some calculations for a three-dimensional oscillating wing moving at a supersonic speed.

For a thin oscillating wing at a small angle of attack, the basic equation for the disturbance velocity potential may be derived, on basis of small disturbances, as a second order linear differential equation. By applying the principle of superposition, the solution of a linear partial differential equation may be superimposed to yield more solutions. Physically, this principle opens the way to replace the wing by a distribution of an infinite number of elementary sources or acoustic radiators. In the present paper, explicit expressions for the velocity potentials satisfying prescribed boundary conditions in either purely or mixed supersonic regions on a three-dimensional oscillating wing at a supersonic speed are obtained by the source superposition method.

To determine the aerodynamic loading over a wing surface during its motion, it is necessary to compute the distribution of the pressure discontinuity across the wing. After the velocity potential has been found, the linear Bernoulli's equation may be applied to evaluate this pressure distribution. And, then, general expressions of the aerodynamic forces and moments can be derived. General formulae for the lift and moment (due to lift) are given in the present paper.

For the purpose of illustrating these general results, detailed considerations are given to the cases of periodic supersonic motions of a rectangular wing. Many interesting features of the supersonic oscillations of a wing with finite span are discovered. It is believed that the tip effects for a rectangular oscillating wing of ordinary aspect ratio can not be neglected. For a certain range of Mach numbers, there are significant indications of reversal of the sign of the phase angles of the lift and moment vectors, due to the presence of wing tips. Therefore, the flutter computations on the basis of two-dimensional results are erroneous and should be revised.

II. FORMULATION OF THE MATHEMATICAL PROBLEM

§ 2.1 Preliminary considerations

It is intended to study the periodic supersonic motions of a thin wing of a general planform as shown in Fig. 1. As in Ref. (7), it is convenient to designate the region ODO' as a purely supersonic region while the remainder part of the wing is regarded as a mixed supersonic region. For the present analysis, it is assumed that

(1) the subsonic leading edges OA and O'B are independent, i.e. any Mach wave emitted from a point on OA will not first meet O'B, and vice versa;

(2) the trailing edge AB is purely supersonic, consequently, any disturbance behind the wing will not affect the conditions on the wing and at the trailing edge the Kutta-Joukowski condition need not hold;

(3) the air flow is isentropic and irrotational, consequently, the presence of strong shock waves is excluded while the existence of a velocity potential function is assured;

(4) the viscosity effects are neglected, consequently, there exists a tangency condition such that the air flow past the wing will be tangential to the wing section at every instant; and

(5) the angle of attack and the motion of the wing are such that the disturbance velocities due to the wing are very small in comparison with the free stream sound speed

and their second order terms may be neglected.

§ 2.2 Basic differential equation and boundary conditions

The basic differential equation of the disturbance velocity potential due to the motion of a thin oscillating wing may be derived on the basis of assumptions (3) and (5). The equation reads (Fig. 2)

$$\frac{1}{a^2}\frac{\partial^2\phi}{\partial t^2} + \frac{2U}{a^2}\frac{\partial^2\phi}{\partial x\partial t} + \left(\frac{U^2}{a^2}-1\right)\frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \qquad (1)$$

where U is in the positive x-direction. For periodic supersonic motions, $U/a > 1$, the equation is of the hyperbolic type. Symbols are defined in appendix §8.1.

It is required to find particular solutions of this differential equation which satisfy some prescribed boundary conditions. To the approximation of the linear equation, it is permissible to replace the prescribed boundary conditions on the actual wing surface by the same boundary conditions referred to the x,y plane where z=0. (Ref. 8). Because of the nature of the prescribed periodic motions and by the assumptions (1), (2) and (4), it is easy to write down several boundary conditions as follows:

(1) The z-components of the disturbance velocities on the top and bottom surfaces of the wing are respectively

$$\lim_{z\to+0}\frac{\partial\phi}{\partial z}(x,y,z,t) = w_r(x,y,+0)\exp(i\nu t)$$
$$= -U\Lambda_r(x,y,+0)\exp(i\nu t) \qquad (2)$$

$$\lim_{z \to -0} \frac{\partial \phi}{\partial z}(x, y, z, t) = w_B\,(x, y, -0)\; exp(i\nu t)$$
$$= U \Lambda_B\,(x, y, -0)\; exp(i\nu t) \tag{2a}$$

where +0 and -0 designate the top and bottom surfaces respectively of the wing. A sign convention for Λ_T and Λ_B, i.e. the effective slopes of the streamlines on the top and bottom surfaces respectively, is adopted as shown in Fig.3. (compare with Ref. 1).

(2) In front of the wing, i.e. in the region vOO'G in Fig. 1, there exists a zone of silence in which $\phi \equiv 0$. And, the conditions behind the wing are of no interest.

Something must be said about the conditions in the regions in front of OA and O'B but behind the leading Mach waves Ov and O'G. (Fig. 1). In these regions, interactions between the top and bottom surfaces of the wing cannot be avoided; this is because of the subsonic nature of the boundaries OA and O'B which, instead of forming a complete barrier to isolate the top and bottom surfaces, permit the spilling of effects of the disturbances from the top to the bottom surfaces, and vice versa. The interaction effects appear as a z-component disturbance velocity which is not specified in advance. To overcome this uncertainty, it is necessary to make use of another physical law. Since the fluid medium can not sustain any strain, pressure equilibrium must be maintained off the wing. By means of the linearized Bernoulli's equation (Ref. 9) this pressure continuity condition may be stated analytically as

$$\frac{\partial \phi_T}{\partial t} + U \frac{\partial \phi_T}{\partial x} = \frac{\partial \phi_B}{\partial t} + U \frac{\partial \phi_B}{\partial x} \tag{3}$$

Eq. (3) serves as a complimentary boundary condition. It is valid everywhere off the wing, in particular, in the regions AOv and BO'G.

III. SOURCE-SUPERPOSITION METHOD OF SOLUTION

§ 3.1 Elementary oscillating source potential

As pointed out in Ref. (7), Eq. (1) also represents the partial differential equation for the velocity potential of a disturbance source which is moving in the negative x-direction at a uniform speed U and causing the propagation of sound wave of small amplitude in a fluid medium. Simplification of this equation may be achieved with the introduction of coordinates transformations as follows:

$$
\begin{aligned}
x' &= \frac{1}{\beta} x \\
y' &= i y \\
z' &= i z \\
t' &= \beta a t - \frac{U x}{\beta a}
\end{aligned}
\qquad (4)
$$

where $\beta = \left(\frac{U^2}{a^2} - 1\right)^{1/2}$

After this transformation, Eq.(1) becomes

$$\frac{\partial^2 \phi}{\partial x'^2} + \frac{\partial^2 \phi}{\partial y'^2} + \frac{\partial^2 \phi}{\partial z'^2} = \frac{\partial^2 \phi}{\partial t'^2} \qquad (5)$$

This equation is of general occurrence in investigations of undulatory disturbances propagated with unit velocity independent of the wave length. Extensive discussions on this equation may be found in any treatise of mathematical physics. (Ref. 10). It is convenient to introduce the spherical polar coordinates (r, θ, ω) through the following transformations:

$$y' = r\sin\theta\cos\omega \qquad r = \left[\left(\frac{x}{\beta}\right)^2 - y^2 - z^2\right]^{1/2}$$
$$z' = r\sin\theta\sin\omega \qquad \cos\theta = \left[1 - \frac{\beta^2}{x^2}(y^2+z^2)\right]^{-1/2}$$
$$x' = r\cos\theta \qquad \tan\omega = z/y \tag{6}$$

By Eqs. (5) and (6), the wave equation in spherical polar coordinates is obtained as follows :

$$\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\omega^2} = \frac{\partial^2\phi}{\partial t'^2} \tag{7}$$

By means of separation of variables, the harmonic solutions of Eq. (7) are found as follows (Ref. 11):

$$\phi = \sum_{\ell,m,n} A_{mn}\, r^{-1/2}\begin{Bmatrix} J_{n+\frac{1}{2}}(\ell r) \\ J_{-(n+\frac{1}{2})}(\ell r)\end{Bmatrix}\begin{Bmatrix} P_n^m(\cos\theta) \\ Q_n^m(\cos\theta)\end{Bmatrix}\begin{Bmatrix}\cos m\omega \\ \sin m\omega\end{Bmatrix}\exp(i\ell t') \tag{8}$$

where ℓ, m and n are the separation parameters and A_{mn} is an arbitrary constant. $J_{(n+\frac{1}{2})}(\ell r)$ and $J_{-(n+\frac{1}{2})}(\ell r)$ are the Bessel functions of the first kind of order $(n+\frac{1}{2})$ and $-(n+\frac{1}{2})$ respectively. $P_n^m(\cos\theta)$ and $Q_n^m(\cos\theta)$ are the Associated Legendre functions, of degree n and order m, of the first and second kinds respectively.

On taking m=n=0 in Eq. (8), a simple particular solution is found

$$\phi_1 = A r^{-1/2} J_{-1/2}(\ell r)\exp(i\ell t') \tag{9}$$

$$= A\left(\frac{2}{\pi\ell}\right)^{1/2}\frac{1}{r}\cos\ell r\,\exp(i\ell t') \tag{9a}$$

where A is an arbitrary constant. By converting back to x,y,z,t system, Eq. (9a) yields

$$\phi_1 = A_1 \frac{1}{r} \cos \frac{\nu}{\beta a} r \; exp\left[i\nu \left(t - \frac{Ux}{\beta^2 a^2} \right) \right] \tag{9b}$$

where
$$r = \frac{1}{\beta} \left[x^2 - \beta^2 (y^2 + z^2) \right]^{1/2} \tag{10}$$
$$\nu = l a \beta$$
$$A_1 = A \left(\frac{2\beta a}{\pi \nu} \right)^{1/2}$$

Eq. (9b) represents the disturbance velocity potential at a point (x,y,z), at an instant t, due to a simple harmonically oscillating source of strength A_1, located at the origin of the coordinate system. When the disturbance source is situated at an arbitrary point (ξ, η, ζ), the velocity potential at (x,y,z), at instant t is (see Refs. 12 and 13)

$$\phi_1 = A_1(\xi, \eta, \zeta) \frac{\beta}{r_1} \cos \frac{\nu r_1}{\beta^2 a} \; exp\left\{ i\nu \left[t - \frac{U}{\beta^2 a^2} (x - \xi) \right] \right\} \tag{9c}$$

where $r_1 = \left[(x-\xi)^2 - \beta^2 (y-\eta)^2 - \beta^2 (z-\zeta)^2 \right]^{1/2}$ (10a)

and $A_1(\xi, \eta, \zeta)$ defines the space variation of the source strength. Of course, ϕ_1 is a solution of Eq. (1). The exponential factor involving t indicates clearly the periodic nature of the motion. Therefore, as in Eq. (2) or (2a), ν is the frequency of the periodic motion.

At this point, it is interesting to consider a physical picture as follows. In Fig. 4, an oscillating source is located at the point (ξ, η, ζ) at time t. Since the disturbance source is moving in the negative x-direction, at successive earlier instants τ_n, the disturbance centers have to be located at the points ($\xi + U\tau_n$, η, ζ) in order that the disturbance source reaches the point (ξ, η, ζ) at time t. In particular, two sound waves originated at two distinct disturbance centers, at two earlier instants τ_1 and τ_2, will pass through a particular field point (x, y, z) at time t. This physical fact may be used to derive the source potential represented in Eq. (9c). (Ref. 7).

Von Karman's three rules of supersonic aerodynamics may be verified. (Ref. 8). At any instant t, the disturbance source at (ξ, η, ζ) is forbidden to send signals ahead of the enveloped region of the successive wave fronts from successive earlier instants. The enveloped region which is a circular cone in space with vertex at (ξ, η, ζ) at instant t, thus represents the zone of action for the disturbance source; while outside this circular cone, there exists a zone of silence. In fact, Eq.(9c) gives real values for ϕ_1 only in the interior of the cone. This downstream facing Mach cone represents the <u>region of influence</u> of the source at (ξ, η, ζ) at instant t. Conversely, the pressure, density and velocities at an arbitrary field point (x, y, z) at an instant t will depend on the disturbances issued at previous instants from the points that lie on or inside a similar Mach cone extending upstream from (x, y, z). This upstream facing Mach cone represents the <u>domain of dependence</u> for a field point (x, y, z) at instant t.

§ 3.2 <u>Disturbance velocity potential of an oscillating wing</u>

Elementary oscillating sources of the type expressed in Eq. (9c) will now be distributed to represent the top and bottom surfaces of a thin oscillating wing. In a linear theory, a wing surface may, indeed, be simulated by a distribution of elementary sources in the x,y plane. For such a distribution of elementary sources, the disturbance velocity potential at a point (x, y, z) at time t may be computed by a surface integral

$$\Phi(x, y, z, t) = \iint_S \phi_1(x, y, z, t; \xi, \eta)\, d\xi\, d\eta \tag{11}$$

where

$$\phi_1(x,y,z,t;\xi,\eta) = \beta A(\xi,\eta)\, \frac{\cos\left\{\frac{\nu}{\beta^2 a}\left[(x-\xi)^2-\beta^2(y-\eta)^2-\beta^2 z^2\right]^{1/2}\right\}}{\left[(x-\xi)^2-\beta^2(y-\eta)^2-\beta^2 z^2\right]^{1/2}}\, \exp\left\{i\nu\left[t-\frac{U}{\beta^2 a^2}(x-\xi)\right]\right\}$$

and S is the region of integration in the x,y plane. Hence, to compute the disturbance velocity potential due to a wing performing small oscillations in a supersonic stream is to determine the surface integral Φ. By the principle of superposition, this surface integral Φ, is again a solution of the linear partial differential equation, Eq. (1).

In order to proceed further, it is necessary to seek answers to the following questions : (1) what is the proper region of integration S ? (2) what is the appropriate strength of the distributed sources ? To answer these questions, it is natural to consider the given boundary conditions.

§ 3.3 Velocity potential at a point in the purely supersonic region at instant t

Since the top and bottom surfaces, in a purely supersonic region, are completely isolated; a field point (x, y, z), where $z \geqslant 0$, will be influenced only by the singularities or disturbances distributed over the top surface of the wing, included in the upstream facing Mach cone from (x, y, z). (Fig. 5). Then, with reference to Fig. 1, the region of integration S for the surface integral, Eq. (11), is bounded by the curves $\xi = \xi_1$, and $(x-\xi)^2 - \beta^2(y-\eta)^2 - \beta^2 z^2 = 0$; and the velocity potential at a point (x, y, z) in the vicinity of the wing at an instant t, may be written as

$$\Phi(x,y,z,t)$$

$$= \beta \exp(i\nu t) \int_{\xi_1}^{x-\beta z} \exp\left[-\frac{i\nu U}{\beta^2 a^2}(x-\xi)\right] d\xi \int_{y-\frac{1}{\beta}[(x-\xi)^2-\beta^2 z^2]^{1/2}}^{y+\frac{1}{\beta}[(x-\xi)^2-\beta^2 z^2]^{1/2}} \frac{A(\xi,\eta)\cos\left\{\frac{\nu}{\beta^2 a}[(x-\xi)^2-\beta^2(y-\eta)^2-\beta^2 z^2]^{1/2}\right\}}{[(x-\xi)^2-\beta^2(y-\eta)^2-\beta^2 z^2]^{1/2}} d\eta \tag{12}$$

Let $$\beta(y-\eta) = [(x-\xi)^2-\beta^2 z^2]^{1/2} \cos\theta_1 \tag{13}$$

and $$d\eta = \frac{1}{\beta}[(x-\xi)^2-\beta^2 z^2]^{1/2} \sin\theta_1 \, d\theta_1 \tag{13a}$$

then Eq. (12) becomes

$$\Phi(x,y,z,t) = \exp(i\nu t) \int_{\xi_1}^{x-\beta z} \exp\left[-\frac{i\nu U}{\beta^2 a^2}(x-\xi)\right] d\xi$$

$$\times \int_0^{\pi} A\left(\xi, y-\frac{1}{\beta}[(x-\xi)^2-\beta^2 z^2]^{1/2}\cos\theta_1\right) \cos\left\{\frac{\nu}{\beta^2 a}[(x-\xi)^2-\beta^2 z^2]^{1/2}\sin\theta_1\right\} d\theta_1 \tag{12a}$$

By differentiating Eq. (12a) with respect to z, it is found that

$$\frac{\partial \Phi}{\partial z}(x, y, z, t) = -\pi\beta \exp\left[i\nu\left(t - \frac{Uz}{\beta a^2}\right)\right] A(x-\beta z, y)$$
$$+ \exp(i\nu t) \int_{\xi_1}^{x-\beta z} \exp\left[-i\nu \frac{U}{\beta^2 a^2}(x-\xi)\right] d\xi$$
$$\times \int_0^{\pi} \frac{\partial}{\partial z}\left[A\left\{\xi, y - \frac{1}{\beta}[(x-\xi)^2-\beta^2 z^2]^{1/2} \cos\theta_1\right\} \cos\left\{\frac{\nu}{\beta^2 a}[(x-\xi)^2-\beta^2 z^2]^{1/2} \sin\theta_1\right\}\right] d\theta_1 \qquad (14)$$

It is easily seen that in the second term on the right hand side of Eq. (14), the differentiation of the integrand with respect to z under the integral signs will yield the following result :

$$\frac{\partial}{\partial z}\left[A(\xi, y - h_2\cos\theta_1) \cos\left(\frac{\nu h_2}{\beta a}\sin\theta_1\right)\right]$$
$$= -\frac{\beta z}{h_2} \frac{\partial}{\partial h_2}\left[A(\xi, y - h_2\cos\theta_1) \cos\left(\frac{\nu h_2}{\beta a}\sin\theta_1\right)\right] \qquad (15)$$

where, of course, $\quad h_2 = \frac{1}{\beta}[(x-\xi)^2 - \beta^2 z^2]^{1/2}$ (16)

By substituting Eq. (15) into Eq. (14), it is found that

$$\frac{\partial \Phi}{\partial z}(x, y, z, t) = -\pi\beta \exp\left[i\nu\left(t - \frac{Uz}{\beta a^2}\right)\right] A(x-\beta z, y)$$
$$-\beta z \exp(i\nu t) \int_{\xi_1}^{x-\beta z} \exp\left[-\frac{i\nu U}{\beta^2 a^2}(x-\xi)\right] \frac{d\xi}{h_2} \int_0^{\pi} \frac{\partial}{\partial h_2}\left[A(\xi, y - h_2\cos\theta_1) \cos\left(\frac{\nu h_2}{\beta a}\sin\theta_1\right)\right] d\theta_1 \qquad (14a)$$

On taking the limiting value of Eq. (14a) at z = +0, it is seen that

(1) the first term becomes $-\pi\beta A(x, y)\exp(i\nu t)$;

(2) the second term vanishes with z; because as z = +0,

the multiple integral which is finite is multiplied by $z=+0$.

Hence it is concluded that

$$\frac{\partial \Phi}{\partial z}(x, y, +0, t) = -\pi\beta A(x, y, +0) \exp(i\nu t) \tag{17}$$

By Eqs. (2) and (17), it is easily found that

$$w_T(x, y, +0) \exp(i\nu t) = -U\Lambda_T(x, y, +0) \exp(i\nu t) = -\pi\beta A(x, y, +0) \exp(i\nu t) \tag{18}$$

Similarly, by considerations of the conditions on the bottom surface, it may be shown that

$$w_B(x, y, -0) \exp(i\nu t) = U\Lambda_B(x, y, -0) \exp(i\nu t) = \pi\beta A(x, y, -0) \exp(i\nu t) \tag{19}$$

Eqs. (18) and (19) may be regarded as confirmation of the following theorem :

<u>Theorem 1</u> The strength of the source at any point at any instant on the surface of an oscillating wing is linearly dependent on the downwash at that point and at that instant and is independent of the downwash of the neighboring points.

Eq. (12) and theorem 1 provide definite answers as to the proper region of integration and appropriate source strength for the determination of the velocity potential at a point (x, y, z) in the purely supersonic region of an oscillating wing at supersonic speed, at an instant t. Therefore, it is simple to state the following theorem :

<u>Theorem 2</u> The velocity potential at an instant t, at

a point P in the purely supersonic region on the surface of a three-dimensional oscillating wing (Fig. 1), may be computed by

$$\Phi(x, y, \pm 0, t) = \mp \frac{1}{\pi} \exp(i\nu t) \int_{\xi_1}^{x} \exp\left[-\frac{i\nu U}{\beta^2 a^2}(x-\xi)\right] d\xi \int_{y-\frac{1}{\beta}(x-\xi)}^{y+\frac{1}{\beta}(x-\xi)} \begin{Bmatrix} w_T(\xi,\eta) \\ w_B(\xi,\eta) \end{Bmatrix} \frac{\cos\left\{\frac{\nu}{\beta^2 a}\left[(x-\xi)^2-\beta^2(y-\eta)^2\right]^{1/2}\right\}}{\left[(x-\xi)^2-\beta^2(y-\eta)^2\right]^{1/2}} d\eta \tag{12b}$$

where $z = \pm 0$ refers to the $\begin{Bmatrix} \text{top} \\ \text{bottom} \end{Bmatrix}$ surface of the wing.

§ 3.4 Velocity potential at a point in the mixed supersonic region at instant t

A mixed supersonic region may be converted into a "pseudo-purely supersonic region." This is accomplished by assuming that the region ahead of the wing but behind the leading Mach waves is occupied by a thin impermeable diaphragm (Fig. 1) which is an extension of the wing having the following properties (Ref. 1) :

(1) It will not alter the flow over the wing;

(2) It will sustain no pressure gradient (see Eq.(3)).

With the presence of this diaphragm, the top and bottom surfaces of the wing may again be considered as independent of each other. However, there is an unknown velocity component in the z-direction in the diaphragm region, which actually is due to the interaction effects between the top and bottom surfaces.

In order to compute the velocity potential at the point

Q, it is convenient to first consider a point N located on the trace of the upstream facing Mach cone from Q, in the diaphragm plane (Fig. 1). Let the unknown vertical velocity or downwash and the effective slope of the streamline on the top surface of the diaphragm be $w_{D_T}(x_N, y_N)$ and $\Lambda_{D_T}(x_N, y_N)$ respectively. Then, by Eq. (12b), it is found that

$$\Phi_{D_T}(x_N, y_N, +0, t) = -\frac{1}{\pi}\exp(i\nu t)\iint_{S_W} \frac{w_T(\xi,\eta)\cos\left\{\frac{\nu}{\beta^2 a}\left[(x_N-\xi)^2-\beta^2(y_N-\eta)^2\right]^{1/2}\right\}}{\left[(x_N-\xi)^2-\beta^2(y_N-\eta)^2\right]^{1/2}\exp\left[\frac{i\nu U}{\beta^2 a^2}(x_N-\xi)\right]}\,d\xi\,d\eta$$

$$-\frac{1}{\pi}\exp(i\nu t)\iint_{S_D} \frac{w_{D_T}(\xi,\eta)\cos\left\{\frac{\nu}{\beta^2 a}\left[(x_N-\xi)^2-\beta^2(y_N-\eta)^2\right]^{1/2}\right\}}{\left[(x_N-\xi)^2-\beta^2(y_N-\eta)^2\right]^{1/2}\exp\left[\frac{i\nu U}{\beta^2 a^2}(x_N-\xi)\right]}\,d\xi\,d\eta \qquad (20)$$

where S_W is the region of the wing and S_D is the region of the diaphragm included in the upstream facing Mach cone from $N(x_N, y_N, +0)$, at instant t.

The regions of integration S_W and S_D are most easily expressed in terms of the oblique u,v coordinates defined as follows :

$$u = \frac{M}{2\beta}(\xi - \beta\eta)$$

$$v = \frac{M}{2\beta}(\xi + \beta\eta) \qquad (21)$$

or

$$\xi = \frac{\beta}{M}(v + u)$$

$$\eta = \frac{1}{M}(v - u) \qquad (22)$$

With these coordinates transformations, the point (x_N, y_N) is transformed into (u_N, v_N), where

$$u_N = \frac{M}{2\beta}(x_N - \beta y_N)$$

$$v_N = \frac{M}{2\beta}(x_N + \beta y_N) \tag{21a}$$

or

$$x_N = \frac{\beta}{M}(v_N + u_N)$$

$$y_N = \frac{1}{M}(v_N - u_N) \tag{22a}$$

The surface integral Φ_{D_T} in Eq. (20) becomes

$$\Phi_{D_T}(u_N, v_N, +0, t)$$

$$= -\frac{1}{\pi M}\exp(i\nu t)\int_0^{u_N}\frac{du}{(u_N-u)^{1/2}\exp[\frac{i\nu}{\beta a}(u_N-u)]}\int_{v_1(u)}^{v_2(u)}\frac{w_T(u,v)\cos\{\frac{2\nu}{M\beta a}[(u_N-u)(v_N-v)]^{1/2}\}}{(v_N-v)^{1/2}\exp[\frac{i\nu}{\beta a}(v_N-v)]}dv$$

$$-\frac{1}{\pi M}\exp(i\nu t)\int_0^{u_N}\frac{du}{(u_N-u)^{1/2}\exp[\frac{i\nu}{\beta a}(u_N-u)]}\int_{v_2(u)}^{v_N}\frac{w_{D_T}(u,v)\cos\{\frac{2\nu}{M\beta a}[(u_N-u)(v_N-v)]^{1/2}\}}{(v_N-v)^{1/2}\exp[\frac{i\nu}{\beta a}(v_N-v)]}dv \tag{20a}$$

where $w_T(u,v)$ is the downwash on the top surface of the wing, $w_{D_T}(u,v)$ is the downwash on the top surface of the diaphragm, the area bounded by $0 \leq u \leq u_N$ and $v_1(u) \leq v \leq v_2(u)$ is S_W, and the area bounded by $0 \leq u \leq u_N$ and $v_2(u) \leq v \leq v_N$ is S_D. (Fig.1).

Similarly, for the corresponding point $N(u_N, v_N, -0, t)$ on the bottom surface of the diaphragm, it is obtained that

$$\Phi_{D_B}(u_N, v_N, -0, t)$$

$$= \frac{1}{\pi M}\exp(i\nu t)\int_0^{u_N}\frac{du}{(u_N-u)^{1/2}\exp[\frac{i\nu}{\beta a}(u_N-u)]}\int_{v_1(u)}^{v_2(u)}\frac{w_B(u,v)\cos\{\frac{2\nu}{M\beta a}[(u_N-u)(v_N-v)]^{1/2}\}}{(v_N-v)^{1/2}\exp[\frac{i\nu}{\beta a}(v_N-v)]}dv$$

$$+\frac{1}{\pi M}\exp(i\nu t)\int_0^{u_N}\frac{du}{(u_N-u)^{1/2}\exp[\frac{i\nu}{\beta a}(u_N-u)]}\int_{v_2(u)}^{v_N}\frac{w_{D_B}(u,v)\cos\{\frac{2\nu}{M\beta a}[(u_N-u)(v_N-v)]^{1/2}\}}{(v_N-v)^{1/2}\exp[\frac{i\nu}{\beta a}(v_N-v)]}dv \tag{20b}$$

Off the wing, the z-component of the disturbance velocity must be continuous. In terms of the effective slopes of the stream lines, this conditions is, with the sign convention of Fig. 3,

$$\Lambda_{D_T}(u,v) = -\Lambda_{D_B}(u,v) = \Lambda_D(u,v) \qquad (23)$$

From Eq. (3), it is found that, in the diaphragm region,

$$\Phi_{D_T}(x,y,+0,t) = \Phi_{D_B}(x,y,-0,t) + F(x-Ut,y) \qquad (24)$$

where F is an integration function. The leading Mach wave Ov (Fig. 1) from the origin, O, represents a line of infinitesimal disturbance along which F(x-Ut,y) can be set equal to zero at all times. F remains zero along y = constant lines for values of x not intercepted by the wing. (see Ref. 9). Therefore, in Eq. (24), F may be put to zero. Then, from Eqs. (20a), (20b), (23) and (24), it is obtained that

$$\frac{1}{2}\int_0^{u_N}\frac{du}{(u_N-u)^{1/2}\exp\left[\frac{i\nu}{\beta a}(u_N-u)\right]}\int_{v_1(u)}^{v_2(u)}\frac{[\Lambda_B(u,v)-\Lambda_T(u,v)]\cos\left\{\frac{2\nu}{M\beta a}[(u_N-u)(v_N-v)]^{1/2}\right\}}{(v_N-v)^{1/2}\exp\left[\frac{i\nu}{\beta a}(v_N-v)\right]}dv$$

$$=\int_0^{u_N}\frac{du}{(u_N-u)^{1/2}\exp\left[\frac{i\nu}{\beta a}(u_N-u)\right]}\int_{v_2(u)}^{v_N}\frac{\Lambda_D(u,v)\cos\left\{\frac{2\nu}{M\beta a}[(u_N-u)(v_N-v)]^{1/2}\right\}}{(v_N-v)^{1/2}\exp\left[\frac{i\nu}{\beta a}(v_N-v)\right]}dv \qquad (25)$$

When $\nu = 0$, this reduces to

$$\frac{1}{2}\int_0^{u_N}\frac{du}{(u_N-u)^{1/2}}\int_{v_1(u)}^{v_2(u)}\frac{\Lambda_B(u,v)-\Lambda_T(u,v)}{(v_N-v)^{1/2}}dv = \int_0^{u_N}\frac{du}{(u_N-u)^{1/2}}\int_{v_2(u)}^{v_N}\frac{\Lambda_D(u,v)\,dv}{(v_N-v)^{1/2}} \qquad (26)$$

Inasmuch as the limits of integration of the u-integrals are the same for <u>all values</u> of u_N and owing to the nature of the functions, the two integrals with respect to v may be equated along lines of <u>constant v</u> that extend across the wing and the diaphragm (Fig. 1). Therefore,

$$\int_{V_2(u)}^{V_N} \frac{\Lambda_D(u,v)\,dv}{(V_N - v)^{1/2}} = \int_{V_1(u)}^{V_2(u)} \frac{\Lambda_B(u,v) - \Lambda_T(u,v)}{2\,(V_N - v)^{1/2}}\,dv \tag{27}$$

This is the fundamental result of Ref. (1) and also is the basic equation of Ref. (2). The above argument is valid because the terms containing u_N do not appear in the v-integrals, and hence the equality Eq. (27) is true for all u_N's on the line $v=v_N$.

The parallel treatment of Eq. (25) would be possible if the terms containing $(u_N-u)(v_N-v)$ can be separated as in Eq. (26), under the integral signs. The present treatment represents a first attempt towards this end. The isolation of terms containing (u_N-u) from terms containing (v_N-v) such that the v-integrals are free of the (u_N-u) factor, may be accomplished by the following procedures.

The term $(v_N-v)(u_N-u)$ vanishes at (u_N,v_N), therefore the equality Eq. (25) actually should be

$$\lim_{\varepsilon\to 0} \int_0^{u_N-\varepsilon} \frac{du}{(u_N-u)^{1/2}\exp[\frac{i\nu}{\beta a}(u_N-u)]} \int_{V_1(u)}^{V_2(u)} \frac{[\Lambda_B(u,v)-\Lambda_T(u,v)]\cos\{\frac{2\nu}{M\beta a}[(u_N-u)(V_N-v)]^{1/2}\}}{2\,(V_N-v)^{1/2}\exp[\frac{i\nu}{\beta a}(V_N-v)]}\,dv$$

$$= \lim_{\substack{\varepsilon\to 0\\ \varepsilon_1\to 0}} \int_0^{u_N-\varepsilon} \frac{du}{(u_N-u)^{1/2}\exp[\frac{i\nu}{\beta a}(u_N-u)]} \int_{V_2(u)}^{V_N-\varepsilon_1} \frac{\Lambda_D(u,v)\cos\{\frac{2\nu}{M\beta a}[(u_N-u)(V_N-v)]^{1/2}\}}{(V_N-v)^{1/2}\exp[\frac{i\nu}{\beta a}(V_N-v)]}\,dv \tag{25a}$$

The nature of the functions Λ_B, Λ_r and Λ_b must be such as to insure the existence of the improper integrals. Thus, except for the singularity (u_N, v_N), in the finite integration regions, the integrands are defined and bounded everywhere. Now, the circular functions are defined by power series; in particular, the power series expansion of the cosine function is

$$\cos z = \sum_{n=0}^{\infty} \frac{(-)^n}{(2n)!} z^{2n} \tag{28}$$

The series in Eq. (28) has the following properties : (Ref. 14)

(1) it converges absolutely for all values of z (real and complex),

(2) it converges uniformly in any bounded domain of values of z, and consequently,

(3) it is a continuous function of z for all values of z.

Because of the uniform continuity, the cosine function in Eq. (25a) may be expanded in an infinite series and the orders of integration and summation may be inverted. Thus

$$\sum_{n=0}^{\infty} \frac{(-)^n \left(\frac{\nu}{\beta a}\right)^{2n} \left(\frac{1}{M}\right)^{2n} \pi^{1/2}}{n!\, \Gamma(n+\frac{1}{2})} \lim_{\varepsilon \to 0} \int_0^{u_N-\varepsilon} \frac{(u_N-u)^{n-\frac{1}{2}} du}{\exp[\frac{i\nu}{\beta a}(u_N-u)]} \int_{v_1(u)}^{v_2(u)} \frac{[\Lambda_B(u,v)-\Lambda_r(u,v)](v_N-v)^{n-\frac{1}{2}}}{2\exp[\frac{i\nu}{\beta a}(v_N-v)]} dv$$

$$= \sum_{n=0}^{\infty} \frac{(-)^n \left(\frac{\nu}{\beta a}\right)^{2n} \left(\frac{1}{M}\right)^{2n} \pi^{1/2}}{n!\, \Gamma(n+\frac{1}{2})} \lim_{\substack{\varepsilon \to 0 \\ \varepsilon_1 \to 0}} \int_0^{u_N-\varepsilon} \frac{(u_N-u)^{n-\frac{1}{2}} du}{\exp[\frac{i\nu}{\beta a}(u_N-u)]} \int_{v_2(u)}^{v_N-\varepsilon_1} \frac{\Lambda_b(u,v)(v_N-v)^{n-\frac{1}{2}}}{\exp[\frac{i\nu}{\beta a}(v_N-v)]} dv \tag{25b}$$

where Γ represents the well-known Gamma-function. With the conviction that the improper integrals under question exist, the "lim" signs may be left out.

In Eq. (25b), unlike in Eq. (25a), the v-integrals do not contain u_N terms, and the problem has been reduced to one analogous to that of Eq. (26). Now, it may be pointed out that since Eq. (25a) is derived by equating the velocity potentials on the top and bottom surfaces of the diaphragm (Fig. 1), the two sides of Eq. (25b) may conveniently be considered as power series in $(\frac{1}{M})$ of a potential function Φ, satisfying the original linear differential equation, Eq. (1); consequently corresponding terms may be equated.

Therefore, for <u>constant</u> value of v_N, with n being any positive integer, it is seen that

$$\int_{v_2(u)}^{v_N} \frac{\Lambda_D(u,v)\,(v_N - v)^{n-\frac{1}{2}}\,dv}{\exp[\frac{i\nu}{\beta a}(v_N - v)]} = \int_{v_1(u)}^{v_2(u)} \frac{[\Lambda_B(u,v) - \Lambda_T(u,v)]\,(v_N - v)^{n-\frac{1}{2}}\,dv}{2\,\exp[\frac{i\nu}{\beta a}(v_N - v)]} \tag{29}$$

In this system of simultaneous integral equations $\Lambda_B(u,v)$ and $\Lambda_T(u,v)$ are known while $\Lambda_D(u,v)$ is unknown. Consider, say, (N+1) integral equations corresponding to n=0,1,2,.... ....N. (of course, in the limit, $N \to \infty$). In order that these (N+1) simultaneous equations may determine one unknown Λ_D, it is necessary that the (N+1) equations are not mutually independent, that is, the (N+1) equations are reducible to <u>one</u> equation. In fact, this is true for the system, Eq. (29). For instance, when n=1, it is obtained from Eq. (29) that

$$\int_{v_2(u)}^{v_N} \frac{\Lambda_D(u,v)\,(v_N-v)^{\frac{1}{2}}dv}{\exp[\frac{i\nu}{\beta a}(v_N-v)]} = \int_{v_1(u)}^{v_2(u)} \frac{[\Lambda_B(u,v)-\Lambda_T(u,v)]\,(v_N-v)^{\frac{1}{2}}dv}{2\exp[\frac{i\nu}{\beta a}(v_N-v)]} \tag{30}$$

Carry out a differentiation of Eq. (30) with respect to v_N. The result of this differentiation plus $(\frac{i\nu}{\beta a})$ times Eq. (30) yields

$$\int_{v_2(u)}^{v_N} \frac{\Lambda_D(u,v)\,(v_N-v)^{-\frac{1}{2}}dv}{\exp[\frac{i\nu}{\beta a}(v_N-v)]} = \int_{v_1(u)}^{v_2(u)} \frac{[\Lambda_B(u,v)-\Lambda_T(u,v)]\,(v_N-v)^{-\frac{1}{2}}dv}{2\exp[\frac{i\nu}{\beta a}(v_N-v)]} \tag{31}$$

which is Eq. (29) for n=0. Therefore, when Λ_D satisfies Eq. (30), it also satisfies Eq. (31). This argument can be carried on, by induction, to include the case for every n. Therefore, the system, Eq. (29), is consistent and determines an unique function Λ_D.

For the determination of the contribution of the diaphragm on the velocity potential at a point $Q(u_Q, v_Q, \pm 0)$ on the top and bottom surfaces of the wing, it is not necessary to solve the integral equation, Eq. (29), explicitly. Let this contribution be called $\Phi_{w_D}(u_Q, v_Q, \pm 0, t)$ (see Fig. 1). Then,

$$\begin{aligned}
&\Phi_{w_D}(u_Q, v_Q, \pm 0, t) \\
&= \pm\frac{U}{\pi M}\exp(i\nu t)\int_0^{u'} \frac{du}{(u_Q-u)^{\frac{1}{2}}\exp[\frac{i\nu}{\beta a}(u_Q-u)]}\int_{v_2(u)}^{v_Q} \frac{\Lambda_D(u,v)\cos\{\frac{2\nu}{M\beta a}[(u_Q-u)(v_Q-v)]^{\frac{1}{2}}\}\,dv}{(v_Q-v)^{\frac{1}{2}}\exp[\frac{i\nu}{\beta a}(v_Q-v)]} \\
&= \pm\frac{U}{\pi M}\exp(i\nu t)\sum_{n=0}^{\infty}\frac{(-)^n(\frac{\nu}{\beta a})^{2n}(\frac{1}{M})^{2n}\pi^{\frac{1}{2}}}{n!\,\Gamma(n+\frac{1}{2})}\int_0^{u'}\frac{(u_Q-u)^{n-\frac{1}{2}}du}{\exp[\frac{i\nu}{\beta a}(u_Q-u)]}\int_{v_2(u)}^{v_Q}\frac{\Lambda_D(u,v)(v_Q-v)^{n-\frac{1}{2}}dv}{\exp[\frac{i\nu}{\beta a}(v_Q-v)]}
\end{aligned} \tag{32}$$

where u' is the u-coordinate of the intersection point of the curves : $v=v_2(u)$ and $v=v_Q$, i.e. $v_2(u')=v_Q$.

By comparing

$$\int_{v_2(u)}^{v_Q} \frac{\Lambda_D(u,v)(v_Q-v)^{n-\frac{1}{2}}}{\exp[\frac{i\nu}{\beta a}(v_Q-v)]}dv \quad \text{with} \quad \int_{v_2(u)}^{v_N} \frac{\Lambda_D(u,v)(v_N-v)^{n-\frac{1}{2}}}{\exp[\frac{i\nu}{\beta a}(v_N-v)]}dv$$

it is seen that they are identical if every v_N in the latter is replaced by v_Q. But the value of v_N along the v=constant line passing through the point $(u_Q, v_Q, \pm 0)$ is v_Q (Fig. 1). Hence for every positive integer n, and on the line v=constant=v_Q on wing or v_N on diaphragm, Eq. (29) may be extended to be

$$\int_{v_2(u)}^{v_Q} \frac{\Lambda_D(u,v)(v_Q-v)^{n-\frac{1}{2}}dv}{\exp[\frac{i\nu}{\beta a}(v_Q-v)]} = \int_{v_2(u)}^{v_N} \frac{\Lambda_D(u,v)(v_N-v)^{n-\frac{1}{2}}dv}{\exp[\frac{i\nu}{\beta a}(v_N-v)]}$$
$$= \int_{v_1(u)}^{v_2(u)} \frac{[\Lambda_B(u,v)-\Lambda_T(u,v)](v_N-v)^{n-\frac{1}{2}}dv}{2\exp[\frac{i\nu}{\beta a}(v_N-v)]} = \int_{v_1(u)}^{v_2(u)} \frac{[\Lambda_B(u,v)-\Lambda_T(u,v)](v_Q-v)^{n-\frac{1}{2}}dv}{2\exp[\frac{i\nu}{\beta a}(v_Q-v)]} \tag{33}$$

On substituting Eq. (33) into Eq. (32), it is found that

$$\Phi_{w_D}(u_Q, v_Q, \pm 0, t)$$
$$= \pm \frac{U}{\pi M}\exp(i\nu t)\sum_{n=0}^{\infty}\frac{(-)^n(\frac{\nu}{\beta a})^{2n}(\frac{1}{M})^{2n}\pi^{\frac{1}{2}}}{n!\,\Gamma(n+\frac{1}{2})}\int_0^{u'}\frac{(u_Q-u)^{n-\frac{1}{2}}du}{\exp[\frac{i\nu}{\beta a}(u_Q-u)]}\int_{v_1(u)}^{v_2(u)}\frac{[\Lambda_B(u,v)-\Lambda_T(u,v)](v_Q-v)^{n-\frac{1}{2}}}{2\exp[\frac{i\nu}{\beta a}(v_Q-v)]}dv$$
$$= \pm \frac{U}{\pi M}\exp(i\nu t)\int_0^{u'}\frac{du}{(u_Q-u)^{\frac{1}{2}}\exp[\frac{i\nu}{\beta a}(u_Q-u)]}\int_{v_1(u)}^{v_2(u)}\frac{[\Lambda_B(u,v)-\Lambda_T(u,v)]\cos\{\frac{2\nu}{M\beta a}[(u_Q-u)(v_Q-v)]^{\frac{1}{2}}\}}{2(v_Q-v)^{\frac{1}{2}}\exp[\frac{i\nu}{\beta a}(v_Q-v)]}dv \tag{32a}$$

In Eq. (32a) an important theorem is established. The theorem may be stated as follows :

<u>Theorem 3</u> In computation of the velocity potential

at an instant t at a point Q (Fig. 1) in the mixed supersonic region of an oscillating wing at supersonic speed, the contribution of the diaphragm may be evaluated by Eq. (32a). In other words, the contribution of the diaphragm can be evaluated by an equivalent integration over a portion of the wing surface.

Now, the velocity potential Φ at the point Q on the top wing surface at instant t may be computed. It is

$$\begin{aligned}
&\Phi(u_Q, v_Q, +0, t) \\
&= \frac{U}{\pi M} \exp(i\nu t) \int_0^{u'} \frac{du}{(u_Q-u)^{1/2} \exp\left[\frac{i\nu}{\beta a}(u_Q-u)\right]} \int_{v_1(u)}^{v_2(u)} \frac{[\Lambda_B(u,v) - \Lambda_T(u,v)] \cos\left\{\frac{2\nu}{M\beta a}[(u_Q-u)(v_Q-v)]^{1/2}\right\}}{2(v_Q-v)^{1/2} \exp\left[\frac{i\nu}{\beta a}(v_Q-v)\right]} dv \\
&+ \frac{U}{\pi M} \exp(i\nu t) \int_0^{u'} \frac{du}{(u_Q-u)^{1/2} \exp\left[\frac{i\nu}{\beta a}(u_Q-u)\right]} \int_{v_1(u)}^{v_2(u)} \frac{\Lambda_T(u,v) \cos\left\{\frac{2\nu}{M\beta a}[(u_Q-u)(v_Q-v)]^{1/2}\right\}}{(v_Q-v)^{1/2} \exp\left[\frac{i\nu}{\beta a}(v_Q-v)\right]} dv \\
&+ \frac{U}{\pi M} \exp(i\nu t) \int_{u'}^{u_Q} \frac{du}{(u_Q-u)^{1/2} \exp\left[\frac{i\nu}{\beta a}(u_Q-u)\right]} \int_{v_1(u)}^{v_Q} \frac{\Lambda_T(u,v) \cos\left\{\frac{2\nu}{M\beta a}[(u_Q-u)(v_Q-v)]^{1/2}\right\}}{(v_Q-v)^{1/2} \exp\left[\frac{i\nu}{\beta a}(v_Q-v)\right]} dv
\end{aligned} \tag{34}$$

In Eq. (34), the first surface integral represents the contribution from the diaphragm, while the last two surface integrals are the contribution from the top of the wing. (Fig. 1). By combining the first and second surface integrals, it is seen that

$$\begin{aligned}
&\Phi(u_Q, v_Q, +0, t) \\
&= \frac{U}{\pi M} \exp(i\nu t) \int_0^{u'} \frac{du}{(u_Q-u)^{1/2} \exp\left[\frac{i\nu}{\beta a}(u_Q-u)\right]} \int_{v_1(u)}^{v_2(u)} \frac{[\Lambda_B(u,v) + \Lambda_T(u,v)] \cos\left\{\frac{2\nu}{M\beta a}[(u_Q-u)(v_Q-v)]^{1/2}\right\}}{2(v_Q-v)^{1/2} \exp\left[\frac{i\nu}{\beta a}(v_Q-v)\right]} dv \\
&+ \frac{U}{\pi M} \exp(i\nu t) \int_{u'}^{u_Q} \frac{du}{(u_Q-u)^{1/2} \exp\left[\frac{i\nu}{\beta a}(u_Q-u)\right]} \int_{v_1(u)}^{v_Q} \frac{\Lambda_T(u,v) \cos\left\{\frac{2\nu}{M\beta a}[(u_Q-u)(v_Q-v)]^{1/2}\right\}}{(v_Q-v)^{1/2} \exp\left[\frac{i\nu}{\beta a}(v_Q-v)\right]} dv
\end{aligned} \tag{34a}$$

By doing the same thing for the bottom surface of the wing, it is seen that

$$\Phi(u_Q, v_Q, -0, t)$$
$$= \frac{U}{\pi M} \exp(i\nu t) \int_0^{u'} \frac{du}{(u_Q-u)^{1/2} \exp\left[\frac{i\nu}{\beta a}(u_Q-u)\right]} \int_{v_1(u)}^{v_2(u)} \frac{\left[\Lambda_B(u,v)+\Lambda_T(u,v)\right] \cos\left\{\frac{2\nu}{M\beta a}\left[(u_Q-u)(v_Q-v)\right]^{1/2}\right\}}{2(v_Q-v)^{1/2} \exp\left[\frac{i\nu}{\beta a}(v_Q-v)\right]} dv$$
$$+ \frac{U}{\pi M} \exp(i\nu t) \int_{u'}^{u_Q} \frac{du}{(u_Q-u)^{1/2} \exp\left[\frac{i\nu}{\beta a}(u_Q-u)\right]} \int_{v_1(u)}^{v_Q} \frac{\Lambda_B(u,v) \cos\left\{\frac{2\nu}{M\beta a}\left[(u_Q-u)(v_Q-v)\right]^{1/2}\right\}}{(v_Q-v)^{1/2} \exp\left[\frac{i\nu}{\beta a}(v_Q-v)\right]} dv \tag{35}$$

Eqs. (34a) and (35) may be restated in the following theorem :

<u>Theorem 4</u> A. The velocity potential, in the mixed supersonic region on the top surface of a three-dimensional oscillating wing, may be computed by Eq. (34a); or in the x,y coordinates,

$$\Phi(x, y, +0, t)$$
$$= \frac{U}{\pi} \exp(i\nu t) \iint_{S_{w_1}} \frac{\left[\Lambda_B(\xi,\eta)+\Lambda_T(\xi,\eta)\right] \cos\left\{\frac{\nu}{\beta^2 a}\left[(x-\xi)^2-\beta^2(y-\eta)^2\right]^{1/2}\right\}}{2\left[(x-\xi)^2-\beta^2(y-\eta)^2\right]^{1/2} \exp\left[\frac{i\nu U}{\beta^2 a^2}(x-\xi)\right]} d\xi\, d\eta$$
$$+ \frac{U}{\pi} \exp(i\nu t) \iint_{S_{w_2}} \frac{\Lambda_T(\xi,\eta) \cos\left\{\frac{\nu}{\beta^2 a}\left[(x-\xi)^2-\beta^2(y-\eta)^2\right]^{1/2}\right\}}{\left[(x-\xi)^2-\beta^2(y-\eta)^2\right]^{1/2} \exp\left[\frac{i\nu U}{\beta^2 a^2}(x-\xi)\right]} d\xi\, d\eta \tag{34b}$$

where (x,y,+0) represents a point Q (Fig. 1), S_{w_1} is the area bounded by $0 \leq u \leq u'$ and $v_1(u) \leq v \leq v_2(u)$, and S_{w_2} is the area bounded by $u' \leq u \leq u_Q$ and $v_1(u) \leq v \leq v_Q$.

B. The velocity potential, at a corresponding point on the bottom surface of the same wing at the same instant t, is given by Eq. (35); or, in the x,y coordinates,

$$\Phi(x, y, -0, t)$$
$$= \frac{U}{\pi} \exp(i\nu t) \iint_{S_{w_1}} \frac{[\Lambda_B(\xi,\eta) + \Lambda_T(\xi,\eta)] \cos\left\{\frac{\nu}{\beta^2 a}\left[(x-\xi)^2 - \beta^2(y-\eta)^2\right]^{\frac{1}{2}}\right\}}{2\left[(x-\xi)^2 - \beta^2(y-\eta)^2\right]^{\frac{1}{2}} \exp\left[\frac{i\nu U}{\beta^2 a^2}(x-\xi)\right]} d\xi\, d\eta$$
$$+ \frac{U}{\pi} \exp(i\nu t) \iint_{S_{w_2}} \frac{\Lambda_B(\xi,\eta) \cos\left\{\frac{\nu}{\beta^2 a}\left[(x-\xi)^2 - \beta^2(y-\eta)^2\right]^{\frac{1}{2}}\right\}}{\left[(x-\xi)^2 - \beta^2(y-\eta)^2\right]^{\frac{1}{2}} \exp\left[\frac{i\nu U}{\beta^2 a^2}(x-\xi)\right]} d\xi\, d\eta \tag{35a}$$

§ 3.5 Symmetrical and antisymmetrical solutions

In a linear treatment, the effects of camber and thickness of the airfoil section may be considered separately. The calculations for an actual wing at an angle of attack different from zero may be performed in two parts : (1) the solution for a symmetrical wing of the given thickness distribution at zero angle of attack, essentially a drag problem and (2) the solution for a thin wing of zero thickness with the given camber distribution at the given angle of attack, essentially a lift problem.

For a drag problem, it is clear that in Eqs. (2) and (2a) the functions w_T and w_B are related in the following way :

$$w_T(x, y, +0) = -w_B(x, y, -0) \tag{36}$$

Or, in terms of the effective slopes of the streamlines, it is required that

$$\Lambda_T(x, y, +0) = \Lambda_B(x, y, -0) \tag{36a}$$

Thus, by Eqs. (12b), (34b) and (35a), the following result is easily obtained :

$$\Phi(x, y, +0, t) = \Phi(x, y, -0, t) \tag{37}$$

On the other hand, for a lift problem, the functions w_T and w_B are so related that

$$w_T(x, y, +0) = w_B(x, y, -0) \tag{38}$$

Or, in terms of Λ_T and Λ_B, this is

$$\Lambda_T(x, y, +0) = -\Lambda_B(x, y, -0) \tag{38a}$$

Thus, by Eqs. (12b), (34b) and (35a), it is easily verified that

$$\Phi(x, y, +0, t) = -\Phi(x, y, -0, t) \tag{39}$$

Eqs. (37) and (39), therefore, characterize respectively the symmetrical and antisymmetrical solutions for a general problem. It is worthy of mentioning that in the case of the symmetrical solutions, $\frac{1}{2}(\Lambda_T + \Lambda_B) = \Lambda_T = \Lambda_B$ by Eq. (36a), so, it is convenient to replace $\frac{1}{2}(\Lambda_T + \Lambda_B)$ in Eq. (34b) by Λ_T, and in Eq. (35a) by Λ_B. And, in the case of the antisymmetrical solutions, $\Lambda_T + \Lambda_B = 0$ by Eq. (38a), so, in Eqs. (34b) and (35a), the integrals over $S_{w,}$ vanish.

IV. GENERAL FORMULAE FOR THE LIFT AND MOMENT

§ 4.1 Antisymmetrical velocity potential functions

The previous discussions were devoted to the computations of particular solutions of Eq. (1), representing the disturbance velocity potential functions which satisfy a set of boundary conditions (Eqs. (2), (2a) and (3)) prescribed by the given periodic supersonic motions and the known geometry of a thin oscillating wing. These considerations are of fundamental importance in the formulation of a linear theory for the unsteady motions of a thin wing moving faster than sound speed. As a matter of fact, after the velocity potentials have been computed with the general theorems thus far obtained, it is possible to determine the aerodynamic loading over a general oscillating wing as shown in Fig. 1 without too much difficulty. It is intended now to derive the general expressions for the lift and the pitching moment due to the lift. For this purpose, it is, of course, necessary to consider only the antisymmetrical solutions mentioned in §3.5. On the basis of the general results obtained in § 3.3 and § 3.4, the following results are valid :

(1) For a typical point P(x,y,+0) in the purely supersonic region ODO' on the top surface of the oscillating wing shown in Fig. 1, the velocity potential at an instant t is given by Eq. (12b).

(2) For a typical point Q(x,y,+0) in the mixed super-

sonic region ODEA on the top surface of the oscillating wing shown in Fig. 1, the velocity potential at an instant t is given by

$$\Phi(x,y,+0,t) = \frac{U}{\pi}\exp(i\nu t)\iint_{S_{w_2}} \frac{\Lambda_T(\xi,\eta)\cos\left\{\frac{\nu}{\beta^2 a}\left[(x-\xi)^2-\beta^2(y-\eta)^2\right]^{\frac{1}{2}}\right\}}{\left[(x-\xi)^2-\beta^2(y-\eta)^2\right]^{\frac{1}{2}}\exp\left[\frac{i\nu U}{\beta^2 a^2}(x-\xi)\right]}\,d\eta\, d\xi \qquad (34c)$$

where S_{w_2} is the area $QQ_1Q_2Q_3$ bounded by the three Mach lines QQ_1, Q_1Q_2, QQ_3 and the leading edge Q_2Q_3. Eq. (34c) is obtained from Eq. (34b) simply by dropping the surface integral over S_{w_1}.

(3) Similarly, the velocity potential at an instant t, at a point K in the mixed supersonic region O'DFB, or at a point C in the mixed supersonic region DEF, on the top surface of the oscillating wing shown in Fig. 1 may also be computed by Eq. (34c). For the point K, S_{w_2} is the area $KK_1K_2K_3$ bounded by the three Mach lines KK_1, KK_3, K_2K_3 and the leading edge K_1K_2. For the point C, S_{w_2} is the area $CC_1C_2C_3C_4$ bounded by the four Mach lines CC_1, C_1C_2, C_3C_4, CC_4, and the leading edge C_2C_3.

§ 4.2 <u>General expressions of the lift and moment</u>

In a linear theory, the pressure disturbance due to the wing may be determined by the linearized Bernoulli's equation

$$p_1 = p - p_0 = -\rho\left(\frac{\partial}{\partial t}\phi + U\frac{\partial}{\partial x}\phi\right) \qquad (40)$$

where p_1 is the disturbance pressure expressed as the deviation of the local pressure p from the free stream pressure p_0. On the top surface of an oscillating wing, the disturbance pressure p_{1T} is, then,

$$p_{1T} = -\rho\left[\frac{\partial}{\partial t}\Phi(x,y,+0,t) + U\frac{\partial}{\partial x}\Phi(x,y,+0,t)\right] \tag{40a}$$

For an antisymmetrical or lift problem, by Eq. (39), it is clear that

$$p_{1T} = -p_{1B} \tag{41}$$

where p_{1B} is the corresponding disturbance pressure on the bottom surface of the wing. In an oscillating wing as shown in Fig. 1, it is necessary to substitute for Φ, in Eq. (40a), by the potential function given either in Eq. (12b) or in Eq. (34c), according to whether the purely or the mixed supersonic region is considered. Thus, it is convenient to treat these regions separately. The total lift force L exerted on a region of area S on the wing is therefore

$$L = \iint_S (p_{1B} - p_{1T})\, dS = -2\iint_S p_{1T}\, dS \tag{42}$$

L is positive if the lift acts upward. The average lift coefficient in the region under consideration is, as usual,

$$C_L = \frac{L}{\frac{1}{2}\rho U^2 S} \tag{42a}$$

The pitching moment due to L in Eq. (42), about the y-axis, i.e. the line x=0, may be evaluated as

$$M = -2\iint_S x\, p_{1T}\, dS \tag{43}$$

A counterclockwise moment is positive. The average moment coefficient in the area S is defined as

$$C_M = \frac{M}{\frac{1}{2}\rho U^2 S x_c} \tag{43a}$$

where x_c is a convenient characteristic length, for instance, the average chord of the wing.

V. PLUNGING OSCILLATIONS OF A RECTANGULAR WING

§ 5.1 Disturbance velocity potentials

A rectangular wing is shown in Fig. 6 (compare with Fig. 1). This rectangular wing is assumed to be an ideal flat plate of zero thickness. Its aspect ratio, $AR = b/x_c$, is assumed to be sufficiently large such that $AR \geqslant 1/\beta$ (the condition of independent subsonic edges). In Part V, it is assumed that the rectangular wing is undergoing a slight, periodic plunging motion (as in a bending oscillation) such that

$$w_r(\xi, \eta)\, exp(i\nu t) = w_o\, exp(i\nu t) \qquad \text{at every instant} \tag{44}$$

where w_o is a real constant which may also be expressed as

$$w_o = -U\Lambda_o \tag{45}$$

Λ_o, in Eq. (45), is the maximum angle of attack of the wing. The purely supersonic region ODO' and the mixed supersonic regions ODEA, O'DFB and DEF will be designated for simplicity as regions II, I, III and IV respectively (Fig.6). It will be tacitly assumed that the velocity potential functions in region II are continuous in the space and time variables; as are the velocity potential functions in region I or III or IV. In the following exposition the analytical expression for the velocity potential in the purely supersonic region will be found different from those for the velocity potentials in the mixed supersonic regions. In fact, the conditions in the purely supersonic region will be shown identical to those of a two-dimensional flow field, while the conditions in the mixed supersonic regions are quite different. The feasibility of joining together two different flow fields across a common boundary is explained by the nature of the hyperbolic differential equation, Eq. (1).

With reference to Fig. 6 and by Eq. (12b), it is immediately found for a point P(x,y,+0) in region II that

$$\Phi(x,y,+0,t)$$
$$= \frac{U\Lambda_0}{\pi}\exp(i\nu t)\int_0^x \exp\left[-\frac{i\nu U}{\beta^2 a^2}(x-\xi)\right]d\xi\int_{y-\frac{1}{\beta}(x-\xi)}^{y+\frac{1}{\beta}(x-\xi)}\frac{\cos\left\{\frac{\nu}{\beta^2 a}\left[(x-\xi)^2-\beta^2(y-\eta)^2\right]^{1/2}\right\}}{\left[(x-\xi)^2-\beta^2(y-\eta)^2\right]^{1/2}}d\eta \qquad (46)$$

By introducing a set of new integration variables θ_2 and σ defined by

$$\sigma = \frac{\nu U}{\beta^2 a^2}(x-\xi)$$
$$\sigma\cos\theta_2 = \frac{\nu U}{\beta a^2}(y-\eta) \qquad (47)$$

Eq. (46) may be integrated once with respect to θ_2 to yield

$$\Phi(x,y,+0,t) = \Lambda_0 a\frac{\beta a}{\nu}\exp(i\nu t)\int_0^k \exp(-i\sigma)J_0\left(\frac{\sigma}{M}\right)d\sigma \qquad (46a)$$

where $\qquad k = \frac{M^2}{\beta^2}\frac{\nu x}{U}\ . \qquad (48)$

is sometimes called the "reduced frequency".

It is obvious that in Eq. (46) identical results will be obtained with assignment of any value of y, for instance, $y = 0$. Thus, the final expression of Φ is independent of the y-variable. This expression for Φ in Eq. (46a) has been previously obtained by several authors in their researches on the periodic plunging motions of a two-dimensional wing moving at a supersonic speed (Refs. 3 to 7).

An alternative expression of this same Φ at the point P(x,y,+0) in the region II can be obtained in the u,v plane. With reference to Fig. 6 and by applying the coordinates transformations in Eq. (22) to Eq. (12b), it is obtained that

$$\Phi(u_p, v_p, +0, t) = \frac{U A_0}{\pi M} \exp(i\nu t) \int_{-v_p}^{u_p} \frac{du}{(u_p - u)^{1/2} \exp\left[\frac{i\nu}{\beta a}(u_p - u)\right]} \int_{-u}^{v_p} \frac{\cos\left\{\frac{2\nu}{M\beta a}\left[(u_p - u)(v_p - v)\right]^{1/2}\right\}}{(v_p - v)^{1/2} \exp\left[\frac{i\nu}{\beta a}(v_p - v)\right]} dv \tag{49}$$

By writing the cosine term in its exponential form and with the introduction of new integration variables p and q defined by

$$p = (v_p - v)^{1/2} - \frac{1}{M}(u_p - u)^{1/2}$$
$$q = (v_p - v)^{1/2} + \frac{1}{M}(u_p - u)^{1/2} \tag{50}$$

Eq. (49) can be rewritten as

$$\Phi(u_p, v_p, +0, t) = \frac{U A_0}{\pi M} \exp(i\nu t) \int_{-v_p}^{u_p} (u_p - u)^{-1/2} \exp\left[-\frac{i\nu}{\beta a}\frac{\beta^2}{M^2}(u_p - u)\right] du$$
$$\times \left\{ \int_{-\frac{1}{M}(u_p - u)^{1/2}}^{(v_p + u)^{1/2} - \frac{1}{M}(u_p - u)^{1/2}} \exp\left[-\frac{i\nu}{\beta a} p^2\right] dp + \int_{\frac{1}{M}(u_p - u)^{1/2}}^{(v_p + u)^{1/2} + \frac{1}{M}(u_p - u)^{1/2}} \exp\left[-\frac{i\nu}{\beta a} q^2\right] dq \right\}$$

$$= \frac{U\Lambda_0}{\pi M} \exp(i\nu t) \int_{-v_p}^{u_p} (u_p - u)^{-\frac{1}{2}} \exp\left[-\frac{i\nu\beta^2}{\beta a M^2}(u_p - u)\right] du \int_{-(v_p+u)^{\frac{1}{2}} + \frac{1}{M}(u_p-u)^{\frac{1}{2}}}^{(v_p+u)^{\frac{1}{2}} + \frac{1}{M}(u_p-u)^{\frac{1}{2}}} \exp\left[-\frac{i\nu}{\beta a} p^2\right] dp \tag{49a}$$

In Eq. (49a), introduce another set of new variables

$$\begin{aligned} P &= \left(\frac{\nu}{\beta a}\right)^{\frac{1}{2}} p \\ G &= \left[\frac{\nu}{\beta a}(u_p - u)\right]^{\frac{1}{2}} \end{aligned} \tag{51}$$

then it is easily found that

$$\begin{aligned} &\Phi(u_p, v_p, +0, t) \\ &= \frac{2U\Lambda_0}{\pi M} \frac{\beta a}{\nu} \exp(i\nu t) \int_0^{\left[\frac{\nu}{\beta a}(u_p+v_p)\right]^{\frac{1}{2}}} \exp\left[-i\frac{\beta^2}{M^2} G^2\right] dG \int_{\frac{G}{M} - \left[\frac{\nu}{\beta a}(u_p+v_p) - G^2\right]^{\frac{1}{2}}}^{\frac{G}{M} + \left[\frac{\nu}{\beta a}(u_p+v_p) - G^2\right]^{\frac{1}{2}}} \exp(-iP^2)\, dP \end{aligned} \tag{49b}$$

By Eqs. (22) and (48),

$$\frac{\nu}{\beta a}(u_p + v_p) = \frac{\nu}{\beta a} \frac{M}{\beta} x = k$$

(where x is, of course, the x-coordinate of the point P in Fig. 6) and therefore, Eq. (49b) becomes

$$\begin{aligned} &\Phi(u_p, v_p, +0, t) \\ &= \frac{2}{\pi} \Lambda_0 a \frac{\beta a}{\nu} \exp(i\nu t) \int_0^{k^{\frac{1}{2}}} \exp\left(-i\frac{\beta^2}{M^2} G^2\right) dG \int_{\frac{G}{M} - (k - G^2)^{\frac{1}{2}}}^{\frac{G}{M} + (k - G^2)^{\frac{1}{2}}} \exp(-iP^2)\, dP \end{aligned} \qquad (49c)^*$$

---

* Of course, here $\Phi(u_p, v_p, +0, t)$ merely indicates that the potential functions are computed at the point $P(u_p, v_p, +0, t)$.

In Eq. (49c), let $G/k^{1/2} = \mu$ (52)

so, finally, it is found that

$$\Phi(u_p, v_p, +0, t) = \frac{2}{\pi} k^{1/2} A_o a \frac{\beta a}{\nu} \exp(i\nu t) \int_0^1 f(\mu, k, M)\, d\mu \tag{49d}$$

where

$$f(\mu, k, M) = \exp\left(-i\frac{\beta^2}{M^2} k \mu^2\right) \int_{k^{1/2}\left[\frac{\mu}{M} - (1-\mu^2)^{1/2}\right]}^{k^{1/2}\left[\frac{\mu}{M} + (1-\mu^2)^{1/2}\right]} \exp(-iP^2)\, dP \tag{53}$$

In Eq. (49d), the final expression of Φ again does not contain the y-variable. And, because (1) the regions of integration (Fig. 6) in the x,y plane and the u,v plane are identical for Eqs. (46) and (49), and (2) these regions are distributed with elementary oscillating sources of identical strength, the two different ways of integration should yield the <u>same</u> result. Therefore, on comparing Φ from Eqs. (46a) and (49d), it is seen that the integral functions

$$\int_0^k \exp(-i\sigma) J_o\left(\frac{\sigma}{M}\right) d\sigma$$

and $$\frac{2}{\pi} k^{1/2} \int_0^1 f(\mu, k, M)\, d\mu$$

must be equivalent. In the present paper, these integral functions will be designated as the T-functions, viz.

$$T = T_r - iT_i = \int_0^k (\cos\sigma - i\sin\sigma)\, J_o\left(\frac{\sigma}{M}\right) d\sigma \tag{54a}$$

$$= \frac{2}{\pi} k^{1/2} \int_0^1 f(\mu, k, M)\, d\mu \tag{54b}$$

where the real and imaginary parts of the complex T-function are T_r and $(-T_i)$ respectively. T_r and T_i (and so T) are evidently functions of two variables k and M. In Ref. (15), Schwarz expanded the T-functions in Eq. (54a) into infinite series by which he computed the numerical values of T_r and T_i to six decimal places for $0 \leq k \leq 2$ and $1 \leq M \leq 10$ at conveniently small intervals of k. In appendix § 8.3, the function $f(\mu, k, M)$ in Eq. (53) is expanded into a uniformly convergent series (Ref. 16) which can be integrated term by term to yield from Eq. (54b) new infinite series for the T-functions. On writing out the first five terms of the new infinite series, it is obtained that

$$
\begin{aligned}
T_r = {} & k\,(a_0 J_0 \cos k + b_0 J_1 \sin k) \\
& + k^2 [a_1 J_1 \cos k + (b_1 J_2 + b_2 J_0) \sin k] \\
& + k^3 [(a_2 J_2 + a_3 J_0) \cos k + (b_3 J_3 + b_4 J_1) \sin k] \\
& + k^4 [(a_4 J_3 + a_5 J_1) \cos k + (b_5 J_4 + b_6 J_2 + b_7 J_0) \sin k] \\
& + k^5 [(a_6 J_4 + a_7 J_2 + a_8 J_0) \cos k + (b_8 J_5 + b_9 J_3 + b_{10} J_1) \sin k] \\
& + \cdots\cdots
\end{aligned}
\qquad (55)
$$

$$
\begin{aligned}
-T_i = {} & k\,(b_0 J_1 \cos k - a_0 J_0 \sin k) \\
& + k^2 [(b_1 J_2 + b_2 J_0) \cos k - a_1 J_1 \sin k] \\
& + k^3 [(b_3 J_3 + b_4 J_1) \cos k - (a_2 J_2 + a_3 J_0) \sin k] \\
& + k^4 [(b_5 J_4 + b_6 J_2 + b_7 J_0) \cos k - (a_4 J_3 + a_5 J_1) \sin k] \\
& + k^5 [(b_8 J_5 + b_9 J_3 + b_{10} J_1) \cos k - (a_6 J_4 + a_7 J_2 + a_8 J_0) \sin k] \\
& + \cdots\cdots
\end{aligned}
\qquad (56)
$$

where

$$a_0 = 1$$

$$a_1 = \frac{1}{3M^3} + \frac{1}{M}$$

$$
\begin{aligned}
a_2 &= \frac{1}{6M^4} + \frac{1}{3M^2} - \frac{1}{10} \\
a_3 &= -\frac{1}{6M^4} - \frac{1}{3M^2} - \frac{1}{6} \\
a_4 &= -\frac{1}{140M^7} + \frac{1}{20M^5} + \frac{1}{12M^3} - \frac{1}{20M} \\
a_5 &= -\frac{1}{60M^7} - \frac{3}{20M^5} - \frac{1}{4M^3} - \frac{7}{60M} \\
a_6 &= -\frac{1}{280M^8} + \frac{1}{90M^6} + \frac{1}{60M^4} - \frac{1}{70M^2} + \frac{1}{1512} \\
a_7 &= -\frac{1}{210M^8} - \frac{2}{45M^6} - \frac{1}{15M^4} - \frac{2}{105M^2} + \frac{1}{126} \\
a_8 &= \frac{1}{120M^8} + \frac{1}{30M^6} + \frac{1}{20M^4} + \frac{1}{30M^2} + \frac{1}{120} \\
b_0 &= -\frac{1}{M} \\
b_1 &= -\frac{1}{2M^2} + \frac{1}{6} \\
b_2 &= \frac{1}{2M^2} + \frac{1}{2} \\
b_3 &= \frac{1}{60M^5} - \frac{1}{6M^3} + \frac{1}{12M} \\
b_4 &= \frac{1}{12M^5} + \frac{1}{2M^3} + \frac{5}{12M} \\
b_5 &= \frac{1}{120M^6} - \frac{1}{20M^4} + \frac{1}{40M^2} - \frac{1}{840} \\
b_6 &= \frac{1}{30M^6} + \frac{1}{6M^4} + \frac{1}{10M^2} - \frac{1}{30} \\
b_7 &= -\frac{1}{24M^6} - \frac{1}{8M^4} - \frac{1}{8M^2} - \frac{1}{24} \\
b_8 &= -\frac{1}{15120M^9} - \frac{1}{420M^7} + \frac{1}{120M^5} - \frac{1}{180M^3} + \frac{1}{1680M} \\
b_9 &= -\frac{1}{560M^9} + \frac{1}{140M^7} + \frac{1}{24M^5} + \frac{1}{60M^3} - \frac{27}{1680M} \\
b_{10} &= -\frac{1}{360M^9} - \frac{1}{30M^7} - \frac{1}{12M^5} - \frac{7}{20M^3} - \frac{1}{40M}
\end{aligned}
\tag{57}
$$

and all the Bessel functions (J's) are of the argument k/M. In Table 1, the values of T_r and T_i computed by Eqs. (55) and (56) for $0 \leq k \leq 1.0$ at $M=2$ are compared with those given by Schwarz in Ref. (15). Therefore, for small values of k, very good accuracy for T_r and T_i is obtained by using the formulae given in Eqs. (55) and (56).

However, attention will now be directed to the evaluation of the velocity potential function at an instant t at a point $Q(u_Q, v_Q, +0)$ in the mixed supersonic region I on the top surface of the rectangular oscillating wing (Fig. 6). By Eqs. (34a) and (34c), it is immediately obtained that

$$\Phi(u_Q, v_Q, +0, t)$$
$$= \frac{U\Lambda_0}{\pi M} \exp(i\nu t) \int_{v_Q}^{u_Q} \frac{du}{(u_Q-u)^{1/2} \exp\left[\frac{i\nu}{\beta a}(u_Q-u)\right]} \int_{-u}^{v_Q} \frac{\cos\left\{\frac{2\nu}{M\beta a}\left[(u_Q-u)(v_Q-v)\right]^{1/2}\right\} dv}{(v_Q-v)^{1/2} \exp\left[\frac{i\nu}{\beta a}(v_Q-v)\right]} \tag{58}$$

From previous computations (Eq. (49a)), this may be re-written as

$$\Phi(u_Q, v_Q, +0, t)$$
$$= \frac{U\Lambda_0}{\pi M} \exp(i\nu t) \int_{v_Q}^{u_Q} (u_Q-u)^{-1/2} \exp\left[-\frac{i\nu}{\beta a}\frac{\beta^2}{M^2}(u_Q-u)\right] du \int_{-(v_Q+u)^{1/2}+\frac{1}{M}(u_Q-u)^{1/2}}^{(v_Q+u)^{1/2}+\frac{1}{M}(u_Q-u)^{1/2}} \exp\left(-\frac{i\nu}{\beta a}p^2\right) dp \tag{58a}$$

By introducing the variables P and G defined in Eq. (51a)*, Eq. (58a) becomes

$$\Phi(u_Q, v_Q, +0, t)$$
$$= \frac{2U\Lambda_0}{\pi M} \frac{\beta a}{\nu} \exp(i\nu t) \int_0^{\left[\frac{\nu}{\beta a}(u_Q-v_Q)\right]^{1/2}} \exp\left(-i\frac{\beta^2}{M^2}G^2\right) dG \int_{\frac{G}{M}-\left[\frac{\nu}{\beta a}(u_Q+v_Q)-G^2\right]^{1/2}}^{\frac{G}{M}+\left[\frac{\nu}{\beta a}(u_Q+v_Q)-G^2\right]^{1/2}} \exp(-iP^2) dP \tag{58b}$$

By Eqs. (22) and (48),

$$\frac{\nu}{\beta a}(u_Q + v_Q) = \frac{\nu}{\beta a}\frac{M}{\beta}x = k$$
$$\frac{\nu}{\beta a}(u_Q - v_Q) = -\frac{\nu}{\beta a}My = k\frac{\beta|y|}{x} \quad **$$

---

* $P = \left(\frac{\nu}{\beta a}\right)^{1/2} p \qquad G = \left[\frac{\nu}{\beta a}(u_Q-u)\right]^{1/2}$ (51a)

** With the coordinate system as shown in Fig. 6, all y values on the wing are negative. Thus, $y = -|y|$.

(where x,y are, of course, the Cartesian coordinates of the point Q in Fig. 6) and therefore, Eq. (58b) may be rewritten as

$$\Phi(x, y, +0, t)$$
$$= \frac{2}{\pi}\Lambda_0 a \frac{\beta a}{\nu} \exp(i\nu t) \int_0^{\left(k\frac{\beta|y|}{x}\right)^{1/2}} \exp\left(-i\frac{\beta^2}{M^2}G^2\right) dG \int_{\frac{G}{M}-(k-G^2)^{1/2}}^{\frac{G}{M}+(k-G^2)^{1/2}} \exp(-iP^2)\, dP \tag{58c}$$

By substitution of Eqs. (52) and (53), Eq. (58c) finally becomes

$$\Phi(x, y, +0, t) = \frac{2}{\pi} k^{1/2} \Lambda_0 a \frac{\beta a}{\nu} \exp(i\nu t) \int_0^{\left(\beta\frac{|y|}{x}\right)^{1/2}} f(\mu, k, M)\, d\mu \tag{58d}$$

By comparison of Eqs. (49d) and (58d), it is concluded that

(1) The velocity potential functions in the purely and mixed supersonic regions on a rectangular wing with plunging oscillations have the same inner P-integral, viz. the Fresnel integral (see Appendix §8.2); but, then, they differ in the integration limits in the outer μ-integrals. For the mixed supersonic case, the integration limits for the μ-integral contain the conical flow variable, $\beta|y|/x$, that characterizes Busemann's steady rectangular wing tip theory (Ref. 17). Thus, while only three variables, viz. k(x), M and t, enter the velocity potential function in the purely supersonic region, one additional variable, $\beta|y|/x$, enters the velocity potential function in the mixed supersonic region. Therefore, the purely supersonic region II (Fig. 6)

is a two-dimensional flow field but the mixed supersonic region I is a genuine three-dimensional flow field. The two types of flow fields are patched together along a Mach line OD (Fig. 6). This is in conformity with a very important principle in the theory of hyperbolic differential equations : Whenever the flow in two adjacent regions is described by expressions which are analytically different then the two regions are necessarily separated by a Mach line. (Ref. 18).

(2) In a linear theory, strong shock waves must be excluded. The Mach line OD is an infinitesimal disturbance wave across which the flow variables such as density, pressure and velocities, and consequently, the velocity potential and its derivatives (to the second order) must be continuous.* Therefore, along the Mach line from the wing tip, O, in the present theory, the velocity potential function in the two-dimensional flow region will have to change continuously into the velocity potential function in the three-dimensional flow region. This is, indeed, verified in the above calculations because the conical flow variable $\beta|y|/x$ assumes the value unity on the Mach line OD, (Fig. 6), and so, the velocity potential function Φ given in Eq. (58d)

* In mathematics, one speaks of "continuity along a path", which means that the conditions for continuity (Ref. 19) are fulfilled for all points lying on the path in question, irrespective of the values of the function for other points.

changes over to the form of Φ given in Eq. (49d).

Furthermore, the above computations in the u,v plane illustrate well the parallelism in the evaluation of the velocity potentials in the wing tip region and the two-dimensional flow region of a rectangular oscillating wing. This particular advantage is lost when the computations are carried out in the x,y plane.

Now, return to Fig. 6. It is easily seen that for plunging oscillations, the regions III and I of the rectangular wing should have identical flow conditions because of symmetry. Therefore, the above deductions for region I are equally true for region III of the rectangular wing.

From the discussions in §4.1, it is possible to determine the velocity potential Φ at an instant t at a point C (x,y,+0) in the region IV of the rectangular wing (Fig. 6) by the following superposition formula :

$$\Phi(x, y, +0, t) = \Phi_1 + \Phi_3 - \Phi_2 \tag{59}$$

where Φ_1 is the velocity potential for a right hand wing tip, Φ_3 is the corresponding velocity potential for a left hand wing tip and Φ_2 is the two-dimensional velocity potential.

§ 5.2 <u>Lift and moment expressed as definite integrals</u>

For the present discussion, it is most convenient to consider a rectangular wing as shown in Fig. 7. Thus, the total lift and moment acting on the wing consist of three parts, viz., (1) lift L_o and moment M_o due to the contribution of region II, (2) lift L_1 and moment M_1 due to the contribution of region I and (3) lift L_2 and moment M_2 due to the contribution of region II'. Of course, the velocity potentials in the regions II and II' which are purely supersonic, are given by Eqs. (46a) and (49d); and the velocity potential in the region I which is mixed supersonic, is given by Eq. (58d).

Formula of C_{L_0} in region II

The lift contribution due to a portion of region II (Fig. 7) of unity span may be expressed by Eqs. (49d), (40a) and (42) as

$$\begin{aligned} L_0 &= -2\int_0^1 dy \int_0^{x_c} p_{1T}\, dx \\ &= 2\rho \int_0^{x_c} \left[U \frac{\partial \Phi}{\partial x}(x,y,+0,t) + i\nu \Phi(x,y,+0,t) \right] dx \end{aligned} \tag{60}$$

where by Eqs. (49d) and (54b),

$$\Phi(x,y,+0,t) = \Lambda_0 a \frac{\beta a}{\nu} \exp(i\nu t)\, T(k,M) \tag{61}$$

With $S=x_c$, it is easily seen that by Eqs. (42a), (60) and (61)

$$\begin{aligned} C_{L_0} &= \frac{4}{Ux_c}\int_0^{x_c} \frac{\partial \Phi}{\partial x}(x,y,+0,t)\,dx + \frac{4\nu i}{U^2 x_c}\int_0^{x_c} \Phi(x,y,+0,t)\,dx \\ &= \frac{4}{Ux_c}\Phi(x_c,y,+0,t) + \frac{4\nu i}{U^2 x_c}\int_0^{x_c}\Phi(x,y,+0,t)\,dx \\ &= \Lambda_0 a \frac{\beta a}{\nu}\exp(i\nu t)\,\frac{4}{Ux_c}\left[(T)_{k_c} + \frac{i\nu}{U}\int_0^{x_c} T\,dx \right] \end{aligned} \tag{60a}$$

where $(T)_{k_c}$ means that the T-functions are evaluated for $k_c = \frac{\nu x_c}{U}\frac{M^2}{\beta^2}$. Replacing x_c in Eq. (60a) by k_c defined as above yields

$$\frac{\beta\, C_{L_0}}{4A_0 \exp(i\nu t)} = \frac{1}{k_c}\left[(T)_{k_c} + i\frac{\beta^2}{M^2}\int_0^{k_c} T\, dk \right] \tag{60b}$$

It is interesting to consider the limiting case when $k_c = 0$ or $\nu = 0$ for Eq. (60b). Thus, it is easily found that

$$\lim_{\nu=0}\left(\frac{\beta\, C_{L_0}}{4A_0 \exp(i\nu t)}\right) = \frac{\beta (C_{L_0})_{\nu=0}}{4A_0} = 1$$

which verifies the well-known Ackeret formula (Ref. 20)

$$(C_{L_0})_{\nu=0} = \frac{4A_0}{\beta} \tag{62}$$

Then, Eq. (60b) may be rewritten as

$$\frac{C_{L_0}}{C^*_{L_0}} = \frac{1}{k_c}\left[(T_r - iT_i) + i\frac{\beta^2}{M^2}(A_r - iA_i) \right]_{k_c} \tag{60c}$$

where $C^*_{L_0} = (C_{L_0})_{\nu=0}\ \exp(i\nu t)$ (63)

and $(A_r - iA_i)_{k_c} = \int_0^{k_c} (T_r - iT_i)\, dk$ (64)

$C^*_{L_0}$ is the "quasi-steady" lift coefficient in region II (Fig. 7), i.e. the lift coefficient in region II computed by the steady state theory from the instantaneous angle of attack.

Formula of C_{M_0} in region II

The moment contribution due to the lift acting on a portion of region II (Fig. 7) of unity span may be expressed by Eqs. (49d), (40a) and (43) as

$$M_o = -2\int_0^1 dy \int_0^{x_c} x\, p_{1T}\, dx = 2\rho \int_0^{x_c} x\left[U\frac{\partial\Phi}{\partial x}(x,y,+0,t) + i\nu\Phi(x,y,+0,t)\right]dx \tag{65}$$

where Φ is the function defined in Eq. (61). Then, by Eq. (43a), it is seen that

$$C_{M_o} = \frac{4}{Ux_c^2}\int_0^{x_c} x\,\frac{\partial\Phi}{\partial x}(x,y,+0,t)\,dx + \frac{4i\nu}{U^2x_c^2}\int_0^{x_c} x\,\Phi(x,y,+0,t)\,dx \tag{65a}$$

Integrating the first integral on the right-hand side of Eq.(65a) by parts yields

$$\begin{aligned} C_{M_o} &= \frac{4}{Ux_c^2}\left[x_c\Phi(x_c,y,+0,t) - \int_0^{x_c}\Phi(x,y,+0,t)\,dx + \frac{i\nu}{U}\int_0^{x_c} x\,\Phi(x,y,+0,t)\,dx\right] \\ &= \Lambda_o a\,\frac{\beta a}{\nu}\exp(i\nu t)\frac{4}{Ux_c}\left[(T)_{k_c} - \frac{1}{x_c}\int_0^{x_c} T\,dx + \frac{i\nu}{Ux_c}\int_0^{x_c} x\,T\,dx\right] \\ &= \frac{4\Lambda_o}{\beta k_c}\exp(i\nu t)\left[(T)_{k_c} - \frac{1}{k_c}\int_0^{k_c} T\,dk + \frac{i\beta^2}{k_cM^2}\int_0^{k_c} k\,T\,dk\right] \\ &= \frac{4\Lambda_o}{\beta k_c}\exp(i\nu t)\left[(T_r - iT_i) - \frac{1}{k}(A_r - iA_i) + \frac{i\beta^2}{kM^2}(B_r - iB_i)\right]_{k_c} \end{aligned} \tag{65b}$$

where $(B_r - iB_i)_{k_c} = \int_0^{k_c} k\,(T_r - iT_i)\,dk$ (66)

From Eq. (65b) it is easily found that

$$\lim_{\nu=0}\left(\frac{\beta C_{M_o}}{4\Lambda_o\exp(i\nu t)}\right) = \frac{(C_{M_o})_{\nu=0}}{(C_{L_o})_{\nu=0}} = \frac{1}{2}$$

or $\qquad (C_{M_o})_{\nu=0} = \frac{1}{2}(C_{L_o})_{\nu=0}$ (67)

that is, on a steady wing, the center of lift in the purely supersonic region is at the mid-chord point from the leading edge. Now, on introducing

$C_{M_o}^* = (C_{M_o})_{\nu=0}\exp(i\nu t)$ (68)

which is the "quasi-steady" moment coefficient in region II (Fig. 7), Eq. (65b) becomes

$$\frac{C_{M_0}}{C^*_{M_0}} = \frac{2}{k_c}\left[(T_r - iT_i) - \frac{1}{k}(A_r - iA_i) + i\frac{\beta^2}{kM^2}(B_r - iB_i)\right]_{k_c} \tag{65c}$$

Eq. (65c) is the desired formula.

<u>Formula of C_{L_1} in region I</u>

The two wing tips in Fig. 7 contribute the same amount of lift because of symmetry. Thus, the lift contribution of the region I which has an area of $S = x_c^2/\beta$ may be expressed by Eqs. (58d), (40a) and (42) as

$$\begin{aligned} L_1 &= -4\int_0^{x_c} dx \int_{-x/\beta}^{0} p_{1T}\, dy \\ &= 4\rho \int_0^{x_c} dx \int_{-x/\beta}^{0} \left[U\frac{\partial \Phi}{\partial x}(x,y,+0,t) + i\nu\Phi(x,y,+0,t)\right] dy \end{aligned} \tag{69}$$

where Φ is the function defined by Eq. (58d). And so, the lift coefficient C_{L_1} in region I is by Eq. (42a),

$$C_{L_1} = \frac{8\beta}{Ux_c^2}\int_0^{x_c} dx \int_{-x/\beta}^{0} \frac{\partial \Phi}{\partial x}(x,y,+0,t)\,dy + \frac{8i\nu\beta}{U^2x_c^2}\int_0^{x_c} dx\int_{-x/\beta}^{0}\Phi(x,y,+0,t)\,dy \tag{69a}$$

The first double integral in Eq. (69a) may be treated as follows :

$$\begin{aligned} \int_0^{x_c} dx \int_{-x/\beta}^{0} \frac{\partial \Phi}{\partial x}(x,y,+0,t)\,dy &= \int_{-x_c/\beta}^{0} dy \int_{-\beta y}^{x_c} \frac{\partial \Phi}{\partial x}(x,y,+0,t)\,dx \\ &= \int_{-x_c/\beta}^{0}\left[\Phi(x_c,y,+0,t) - \Phi(-\beta y, y, +0, t)\right] dy \\ &= \frac{2}{\pi}A_0 a\frac{\beta a}{\nu}\exp(i\nu t)\Big[k_c^{1/2}\int_{-x_c/\beta}^{0} dy\int_0^{\left(\frac{\beta|y|}{x_c}\right)^{1/2}} f(\mu, k_c, M)\,d\mu \\ &\qquad - \int_{-x_c/\beta}^{0}(k)^{1/2}_{-\beta y}\,dy\int_0^1 f(\mu,\{k\}_{-\beta y}, M)\,d\mu\Big] \end{aligned} \tag{70}$$

$$= \frac{2}{\pi} \Lambda_o a \frac{\beta a}{\nu} \exp(i\nu t) \Big[\frac{x_c}{\beta} k_c^{1/2} \int_0^1 (1-\mu^2) f(\mu, k_c, M)\, d\mu - \frac{1}{\beta} \int_0^{x_c} k^{1/2} dx \int_0^1 f(\mu, k, M)\, d\mu \Big] \tag{70a}$$

where the last operation consists of changing the order of integrations with respect to y and to μ in the first double integral of Eq. (70) and replacing $(-\beta y)$ by x in the second double integral of Eq. (70).

Similarly, the second double integral in Eq. (69a) may be, by a change in the order of integrations, converted into

$$\int_0^{x_c} dx \int_{-x/\beta}^{0} \Phi(x, y, +0, t)\, dy = \frac{2}{\pi} \Lambda_o a \frac{\beta a}{\nu} \exp(i\nu t) \int_0^{x_c} k^{1/2} dx \int_{-x/\beta}^{0} dy \int_0^{\left(\frac{\beta|y|}{x}\right)^{1/2}} f(\mu, k, M)\, d\mu$$

$$= \frac{2}{\pi} \Lambda_o a \frac{\beta a}{\nu} \exp(i\nu t) \frac{1}{\beta} \int_0^{x_c} x k^{1/2} dx \int_0^1 (1-\mu^2) f(\mu, k, M)\, d\mu \tag{71}$$

Before summing up these results, it is convenient to define the H-functions of two variables k(x) and M as follows :

$$H = H_r - i H_i = \frac{2}{\pi} k^{1/2} \int_0^1 (1-\mu^2) f(\mu, k, M)\, d\mu \tag{72}$$

where H_r and $(-H_i)$ denote the real and imaginary parts respectively of the complex H-function.

Now, with the definitions of the T- and H-functions as given in Eqs. (54b) and (72) respectively, and by substitution of Eqs. (70a) and (71) into Eq. (69a), it is found that

$$C_{L_1} = \frac{8}{U x_c} \Lambda_o a \frac{\beta a}{\nu} \exp(i\nu t) \Big[(H_r - iH_i)_{k_c} - \frac{1}{x_c} \int_0^{x_c} T dx + \frac{i\nu}{U x_c} \int_0^{x_c} x H dx \Big]$$

$$= \frac{8\Lambda_o}{\beta k_c} \exp(i\nu t) \Big[(H_r - iH_i)_{k_c} - \frac{1}{k_c} \int_0^{k_c} (T_r - iT_i) dk + \frac{i\beta^2}{k_c M^2} \int_0^{k_c} k (H_r - iH_i) dk \Big]$$

$$= \frac{8\Lambda_o}{\beta k_c} \exp(i\nu t) \Big[(H_r - iH_i) - \frac{1}{k}(A_r - iA_i) + \frac{i\beta^2}{k M^2} (C_r - iC_i) \Big]_{k_c} \tag{69b}$$

where $(C_r - iC_i)_{k_c} = \int_0^{k_c} k (H_r - iH_i)\, dk$ (73)

It can be easily verified that

$$\lim_{k_c = 0} \Big[\frac{1}{k_c^2} A_r(k_c) \Big] = \frac{1}{2}$$

$$\lim_{k_c = 0} \Big[\frac{1}{k_c} H_r(k_c) \Big] = \frac{3}{4}$$

and that all the other terms on the right-hand side of Eq. (69b) vanish in the limit when $k_c = 0$. Therefore, it is easily seen that

$$(C_{L_1})_{\nu=0} = 2 (C_{L_0})_{\nu=0} \left(\frac{3}{4} - \frac{1}{2}\right) = \frac{1}{2} (C_{L_0})_{\nu=0} \tag{74}$$

which is, of course, the same result as found by Busemann (Ref. 17) in his conical flow field theory of the rectangular wing tip with steady motion. Introduce the "quasi-steady" lift coefficient in region I (Fig. 7) defined as

$$C_{L_1}^* = (C_{L_1})_{\nu=0} \exp(i\nu t) \tag{75}$$

then, Eq. (69b) may be rewritten as

$$\frac{C_{L_1}}{C_{L_1}^*} = \frac{4}{k_c} \Big[(H_r - iH_i) - \frac{1}{k} (A_r - iA_i) + \frac{i\beta^2}{k M^2} (C_r - iC_i) \Big]_{k_c} \tag{69c}$$

which is the required formula.

In a later section, methods of calculating the H-functions defined in Eq. (72) will be discussed. The H-functions are of primary importance for studying the flow conditions in the rectangular oscillating wing tips region. Their appearance in the moment coefficient C_{M_1} will be shown presently.

Formula of C_{M_1} in region I

The moment due to the lift force L_1 in Eq. (69) may be expressed by Eqs. (58d), (40a) and (43) as

$$\begin{aligned} M_1 &= -4\int_0^{x_c} x\,dx \int_{-x/\beta}^{0} p_{1T}\,dy \\ &= 4\rho\int_0^{x_c} x\,dx \int_{-x/\beta}^{0} \left[U\frac{\partial\Phi}{\partial x}(x,y,+0,t) + i\nu\Phi(x,y,+0,t)\right]dy \end{aligned} \tag{76}$$

where Φ is the function defined by Eq. (58d). And so, the moment coefficient C_{M_1} in region I is by Eq. (43a)

$$C_{M_1} = \frac{8\beta}{Ux_c^3}\int_0^{x_c} x\,dx\int_{-x/\beta}^{0}\frac{\partial\Phi}{\partial x}(x,y,+0,t)\,dy + \frac{8i\beta\nu}{U^2x_c^3}\int_0^{x_c} x\,dx\int_{-x/\beta}^{0}\Phi(x,y,+0,t)\,dy \tag{76a}$$

Consider the first double integral. By integration by parts, it is obtained that

$$\begin{aligned} \int_0^{x_c} x\,dx\int_{-x/\beta}^{0}\frac{\partial\Phi}{\partial x}(x,y,+0,t)\,dy &= \int_{-x_c/\beta}^{0} dy\int_{-\beta y}^{x_c} x\frac{\partial\Phi}{\partial x}(x,y,+0,t)\,dx \\ &= \int_{-x_c/\beta}^{0}\left[x_c\Phi(x_c,y,+0,t) + \beta y\Phi(-\beta y,y,+0,t) - \int_{-\beta y}^{x_c}\Phi(x,y,+0,t)\,dx\right] \end{aligned} \tag{77}$$

in which the first integral on the right is by previous computations,

$$\int_{-x_c/\beta}^{0} x_c\,\Phi(x_c, y, +0, t)\,dy = \Lambda_0 a \frac{\beta a}{\nu} \exp(i\nu t) \frac{x_c^2}{\beta}(H)_{k_c} \tag{77a}$$

the second integral on the right is

$$\int_{-x_c/\beta}^{0} \beta y\,\Phi(-\beta y, y, +0, t)\,dy = \frac{2}{\pi}\Lambda_0 a \frac{\beta a}{\nu}\exp(i\nu t)\int_{-x_c/\beta}^{0}(\beta y)(k)_{-\beta y}^{1/2}dy\int_0^1 f[\mu, (k)_{-\beta y}, M]\,d\mu$$

$$= -\frac{2}{\pi}\Lambda_0 a \frac{\beta a}{\nu}\exp(i\nu t)\frac{1}{\beta}\int_0^{x_c} x\,k^{1/2}dx\int_0^1 f(\mu, k, M)\,d\mu$$

$$= -\Lambda_0 a \frac{\beta a}{\nu}\exp(i\nu t)\frac{1}{\beta}\int_0^{x_c} x\,T\,dx \tag{77b}$$

and the third integral on the right is by Eq. (71),

$$\int_{-x_c/\beta}^{0} dy\int_{-\beta y}^{x_c}\Phi(x, y, +0, t)\,dx = \int_0^{x_c} dx\int_{-x/\beta}^{0}\Phi(x, y, +0, t)\,dy = \Lambda_0 a \frac{\beta a}{\nu}\exp(i\nu t)\frac{1}{\beta}\int_0^{x_c} xH\,dx \tag{77c}$$

Consider next the second double integral. By substitution of Eq. (58d) for Φ and then interchanging the order of integrations with respect to y and μ, it is found that

$$\int_0^{x_c} x\,dx\int_{-x/\beta}^{0}\Phi(x, y, +0, t)\,dy = \frac{2}{\pi}\Lambda_0 a \frac{\beta a}{\nu}\exp(i\nu t)\int_0^{x_c} x\,k^{1/2}dx\int_{-x/\beta}^{0}dy\int_0^{(\beta|y|/x)^{1/2}} f(\mu, k, M)\,d\mu$$

$$= \frac{2}{\pi}\Lambda_0 a \frac{\beta a}{\nu}\exp(i\nu t)\frac{1}{\beta}\int_0^{x_c} x^2 k^{1/2}dx\int_0^1 (1-\mu^2) f(\mu, k, M)\,d\mu = \Lambda_0 a \frac{\beta a}{\nu}\frac{1}{\beta}\exp(i\nu t)\int_0^{x_c} x^2 H\,dx \tag{78}$$

Hence by using Eqs. (76a), (77), (77a), (77b), (77c) and (78), the following result is readily obtained :

$$C_{M_1} = \Lambda_0 a \frac{\beta a}{\nu}\exp(i\nu t)\frac{8}{U x_c}\left[(H)_{k_c} - \frac{1}{x_c^2}\int_0^{x_c} xT\,dx - \frac{1}{x_c^2}\int_0^{x_c} xH\,dx + \frac{i\nu}{U x_c^2}\int_0^{x_c} x^2 H\,dx\right]$$

$$= \frac{8\Lambda_0}{\beta k_c}\exp(i\nu t)\left[(H)_{k_c} - \frac{1}{k_c^2}\int_0^{k_c} kT\,dk - \frac{1}{k_c^2}\int_0^{k_c} kH\,dk + \frac{i\beta^2}{k_c^2 M^2}\int_0^{k_c} k^2 H\,dk\right]$$

$$= \frac{8\Lambda_0}{\beta k_c}\exp(i\nu t)\left[(H_r - iH_i) - \frac{1}{k^2}(B_r - iB_i) - \frac{1}{k^2}(C_r - iC_i) + \frac{i\beta^2}{k^2 M^2}(D_r - iD_i)\right]_{k_c} \tag{76b}$$

where
$$(D_r - iD_i)_{k_c} = \int_0^{k_c} k^2 (H_r - iH_i)\, dk \tag{79}$$

It is important to observe that

$$\lim_{k_c=0} \left[\frac{1}{k_c} H_r(k_c)\right] = \frac{3}{4}$$

$$\lim_{k_c=0} \left[\frac{1}{k_c^3} B_r(k_c)\right] = \frac{1}{3}$$

$$\lim_{k_c=0} \left[\frac{1}{k_c^3} C_r(k_c)\right] = \frac{1}{4}$$

and that the limiting values of the remaining terms on the right-hand side of the Eq. (76b) vanish when $k_c = 0$. Therefore, it is seen that

$$(C_{M_1})_{\nu=0} = 4(C_{L_1})_{\nu=0}\left(\frac{3}{4} - \frac{1}{3} - \frac{1}{4}\right) = \frac{2}{3}(C_{L_1})_{\nu=0} \tag{80}$$

Hence, the center of lift in the steady rectangular wing tips region is located at two thirds of the chord from the leading edge.

Finally, define the "quasi-steady" moment coefficient in the region I (Fig. 7) as

$$C_{M_1}{}^* = (C_{M_1})_{\nu=0} \exp(i\nu t) \tag{81}$$

Then, Eq. (76b) may be rewritten as

$$\frac{C_{M_1}}{C_{M_1}{}^*} = \frac{6}{k_c}\left[(H_r - iH_i) - \frac{1}{k^2}(B_r - iB_i) - \frac{1}{k^2}(C_r - iC_i) + \frac{i\beta^2}{k^2M^2}(D_r - iD_i)\right]_{k_c} \tag{76c}$$

This is the desired formula.

<u>Formula of C_{L_2} in region II'</u>

The lift contribution due to the region II' (Fig. 7) may be expressed by Eq. (42) as

$$L_2 = -4\int_0^{x_c} dx \int_{-x_c/\beta}^{-x/\beta} p_{1T}\, dy \tag{82}$$

Since the region II' is purely supersonic, the potential function Φ and the pressure disturbance p_{1T} are both independent of the y-variable (see Eqs. (49d) and (40a)). Therefore, Eq. (82) may be integrated once to yield

$$L_2 = -4\int_0^{x_c} \left(\frac{x_c}{\beta} - \frac{x}{\beta}\right) p_{1T}\, dx \tag{82a}$$

And so, the lift coefficient C_{L_2} in region II' is by Eq. (42a)

$$C_{L_2} = -\frac{8}{\rho U^2 x_c}\int_0^{x_c} p_{1T}\, dx + \frac{8}{\rho U^2 x_c^2}\int_0^{x_c} x\, p_{1T}\, dx \tag{83}$$

It is recognized that the first factor on the right-hand side of Eq. (83) is simply $2C_{L_0}$, and the second factor on the right-hand side of Eq. (83) is simply $-2C_{M_0}$. Hence, it is obtained that

$$C_{L_2} = 2\,(C_{L_0} - C_{M_0}) \tag{83a}$$

Formula of C_{M_2} in region II'

The moment due to the lift L_2 given in Eq. (82) may be expressed by Eq. (43) as

$$M_2 = -4\int_0^{x_c} dx \int_{-x_c/\beta}^{-x/\beta} x\, p_{1T}\, dy \tag{84}$$

Again, on account of the fact that the disturbance pressure p_{1T} in the region II' is independent of the y-variable, Eq. (84) may easily be reduced to

$$M_2 = -4\int_0^{x_c}\left(\frac{x_c}{\beta}-\frac{x}{\beta}\right)x\,p_{1T}\,dx \tag{84a}$$

And, by Eq. (43a), it is found that

$$C_{M_2} = -\frac{8}{\rho U^2 x_c^2}\int_0^{x_c} x\,p_{1T}\,dx + \frac{8}{\rho U^2 x_c^3}\int_0^{x_c} x^2 p_{1T}\,dx$$
$$= 2C_{M_0} + \frac{8}{\rho U^2 x_c^3}\int_0^{x_c} x^2 p_{1T}\,dx \tag{85}$$

The factor

$$\int_0^{x_c} x^2 p_{1T}\,dx = -\rho\int_0^{x_c}\left[U\frac{\partial\Phi}{\partial x}(x,y,+0,t)+i\nu\Phi(x,y,+0,t)\right]x^2dx$$

may be, by integrating the first term in the bracket [....] by parts, reduced to

$$\int_0^{x_c} x^2 p_{1T}\,dx = -\rho\left\{U\left[x_c^2\Phi(x_c,y,+0,t)-2\int_0^{x_c}x\Phi(x,y,+0,t)dx\right]+i\nu\int_0^{x_c}x^2\Phi(x,y,+0,t)\,dx\right\} \tag{86}$$

In Eq. (86), substitution of Eq. (61) for Φ yields

$$\int_0^{x_c} x^2 p_{1T}\,dx = -\Lambda_0 a\frac{\beta a}{\nu}\exp(i\nu t)\,\rho\left\{U\left[x_c^2(T)_{k_c}-2\int_0^{x_c}xT\,dx\right]+i\nu\int_0^{x_c}x^2T\,dx\right\}$$
$$= -\Lambda_0 a\frac{\beta a}{\nu}\exp(i\nu t)\,\rho\left\{U\left[x_c^2(T)_{k_c}-2\left(\frac{U\beta^2}{\nu M^2}\right)^2\int_0^{k_c}kT\,dk\right]+i\nu\left(\frac{U\beta^2}{\nu M^2}\right)^3\int_0^{k_c}k^2T\,dk\right\} \tag{86a}$$

By combination of Eqs. (85) and (86a), it is found that

$$C_{M_2} = 2C_{M_0} + \bar{C}_M \tag{85a}$$

where

$$\bar{C}_M = -\frac{8\Lambda_0}{\beta k_c}\exp(i\nu t)\left[(T)_{k_c}-\frac{2}{k_c^2}\int_0^{k_c}kT\,dk+\frac{i\beta^2}{k_c^2M^2}\int_0^{k_c}k^2T\,dk\right]$$
$$= -\frac{8\Lambda_0}{\beta k_c}\exp(i\nu t)\left[(T_r-iT_i)-\frac{2}{k^2}(B_r-iB_i)+\frac{i\beta^2}{k^2M^2}(E_r-iE_i)\right]_{k_c} \tag{87}$$

and $$(E_r - iE_i)_{k_c} = \int_0^{k_c} k^2(T_r - iT_i)\,dk \tag{88}$$

Lift and moment of the entire wing

The above calculations give the expressions of the average lift and moment coefficients in the various regions of the rectangular wing as shown in Fig. 7. For the average lift and moment coefficients of the entire wing, it is proposed to introduce the symbols C_{L_w} and C_{M_w} respectively. Then, it is obtained that

$$C_{L_w} = \frac{1}{b x_c}\left[C_{L_o} x_c \left(b - 2\frac{x_c}{\beta}\right) + (C_{L_1} + C_{L_2})\frac{x_c^2}{\beta}\right]$$

$$= C_{L_o}\left(1 - \frac{2}{\beta A\!R}\right) + (C_{L_1} + C_{L_2})\frac{1}{\beta A\!R} \qquad (89)$$

which by Eq. (83a) may be rewritten as

$$C_{L_w} = C_{L_o} + (C_{L_1} - 2 C_{M_o})\frac{1}{\beta A\!R} \qquad (89a)$$

From Fig.7, this formula evidently holds when $\infty \geqslant \beta A\!R \geqslant 2$. But, it also holds when $1 \leqslant \beta AR < 2$ (Fig. 6). This is because of the fact mentioned in Eq. (59). Indeed, if the lift coefficient in region IV (Fig. 6) were actually calculated on basis of this potential function, Eq. (59), it can be used in conjunction with the lift coefficients in regions I, II, III (Fig. 6) as derived above to yield the same C_{L_w} as given in Eq. (89a). Hence, this equation is valid for $1 \leqslant \beta AR \leqslant \infty$.

A ratio C_{L_w}/C_{L_o} is interesting because it represents the ratio of the lift coefficient for a finite wing to that for an infinite wing; in other words, it represents the aerodynamic efficiency of a rectangular wing. This ratio is easily obtained as

$$\frac{C_{L_W}}{C_{L_0}} = 1 + \left(\frac{C_{L_1}}{C_{L_0}} - 2\frac{C_{M_0}}{C_{L_0}} \right) \frac{1}{\beta AR} \qquad 1 \leq \beta\, AR \leq \infty \tag{90}$$

By a similar procedure, the C_{M_W} can be derived as follows :

$$C_{M_W} = \frac{1}{b x_c^2}\left[C_{M_0} x_c^2 \left(b - 2\frac{x_c}{\beta}\right) + \left(C_{M_1} + 2C_{M_0} + \bar{C}_M\right)\frac{x_c^3}{\beta} \right]$$

$$= C_{M_0} + \left(C_{M_1} + \bar{C}_M\right)\frac{1}{\beta AR} \qquad 1 \leq \beta\, AR \leq \infty \tag{91}$$

And the ratio C_{M_W}/C_{M_0} can be then expressed as

$$\frac{C_{M_W}}{C_{M_0}} = 1 + \left(\frac{C_{M_1}}{C_{M_0}} + \frac{\bar{C}_M}{C_{M_0}} \right) \frac{1}{\beta AR} \qquad 1 \leq \beta\, AR \leq \infty \tag{92}$$

§ 5.3 <u>Dual correlation relations between T and H-functions</u>

Differentiation of the function $\frac{\pi}{2}k^{-\frac{1}{2}}T$ (Eq. (54b)) with respect to k yields after some calculations that

$$\frac{\pi}{2}\frac{\partial}{\partial k}\left(\frac{T}{k^{1/2}}\right) = -i\frac{\beta^2}{M^2}\int_0^1 \mu^2 f(\mu, k, M)\, d\mu + \frac{\pi}{4k^{1/2}} \exp(-ik)\left[J_0\left(\frac{k}{M}\right) - \frac{i}{M}J_1\left(\frac{k}{M}\right)\right] \tag{93}$$

By Eqs. (72) and (93), the following formula can be easily obtained :

$$H = T - i\frac{M^2}{\beta^2}\frac{1}{k}\left\{ k\frac{\partial T}{\partial k} - \frac{1}{2}T - \frac{1}{2}k\exp(-ik)\left[J_0\left(\frac{k}{M}\right) - \frac{i}{M}J_1\left(\frac{k}{M}\right)\right]\right\} \tag{93a}$$

But by Eq. (54a), it is seen that

$$\frac{\partial T}{\partial k} = \exp(-ik)\, J_0\left(\frac{k}{M}\right) \quad * \tag{94}$$

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* This can also be obtained from Eq. (54b).

and therefore, Eq. (93a) may be rewritten as

$$H = T - \frac{M^2}{2\beta^2}\frac{1}{k}\left\{ k\exp(-ik)\left[iJ_0(\tfrac{k}{M}) - \tfrac{1}{M}J_1(\tfrac{k}{M})\right] - iT\right\} \tag{93b}$$

By separation of the real and imaginary parts, a set of dual correlation relations between T and H is found as follows :

$$H_r = T_r + \frac{M^2}{2\beta^2}\left\{\left[\frac{1}{M}J_1(\tfrac{k}{M})\cos k - J_0(\tfrac{k}{M})\sin k\right] + \frac{T_i}{k}\right\} \tag{95}$$

$$H_i = T_i + \frac{M^2}{2\beta^2}\left\{\left[\frac{1}{M}J_1(\tfrac{k}{M})\sin k + J_0(\tfrac{k}{M})\cos k\right] - \frac{T_r}{k}\right\} \tag{96}$$

Eqs. (95) and (96) may be used to calculate the values of H_r and H_i on basis of the values of T_r and T_i given in Ref. (15). In Table 2, the values of T_r, T_i, H_r and H_i are given for $0 \leq k \leq 2.0$ and $M = 10/7, 2$, where the values of T_r and T_i are taken from Ref. (15) while the values of H_r and H_i are computed by Eqs. (95) and (96).

It may noted that Eqs. (95) and (96) are not entirely satisfactory for the evaluation of H_r and H_i at very small k, because of the presence of the terms with denominator k.* Means of relieving this inconvenience are provided as follows :

(1) The T-series given in Eqs. (55) and (56) may be used to compute the values of T_r and T_i to seven or eight digits for very small k. These values of T_r and T_i may be used in Eqs. (95) and (96) for the evaluation of H_r and H_i

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* Indeed, it is more convenient to compute $H_r k$ and $H_i k$ because the definite integrals C_r, C_i, D_r and D_i contain only $H_r k$, $H_i k$, $H_r k^2$ and $H_i k^2$ respectively.

to five or six significant figures.

(2) Or alternatively, the H_r and H_i functions can be expanded into infinite series similar to Eqs. (55) and (56), as in appendix § 8.4. These H-series may then be used for accurate determination of the values of H_r and H_i for small k, as illustrated in Table 3.

§ 5.4 Evaluation of the definite integrals and numerical results (see also § 8.5)

The most efficient and simplest method of numerical evaluation of the definite integrals in Eqs. (64), (66), (73), (79) and (88) is by application of the Euler-Maclaurin formula (Ref. 14), viz.

$$\int_a^{a+rw} F(x)\,dx = w\left[\tfrac{1}{2}F(a) + F(a+w) + F(a+2w) + \cdots\cdots + \tfrac{1}{2}F(a+rw)\right]$$
$$+ \sum_{m=1}^{n-1} \frac{(-)^m B_m w^{2m}}{(2m)!}\left[F^{(2m-1)}(a+rw) - F^{(2m-1)}(a)\right] + R_n \qquad (97)$$

where

$$R_n = \frac{w^{2n+1}}{(2n)!}\int_0^1 \phi_{2n}(t)\left[\sum_{m=0}^{r-1} F^{(2n)}(a+mw+wt)\right]dt\,, \qquad (98)$$

B_m is the mth Bernoullian number,

and $\phi_n(t)$ is the nth order Bernoullian polynomial.

For the sake of securing reasonable accuracy without too much computational labor, it suffices to employ a cut-off version of Eq. (97) as follows :

$$\int_a^{a+rw} F(x)\,dx = w\left[\tfrac{1}{2}F(a) + F(a+w) + F(a+2w) + \cdots\cdots + \tfrac{1}{2}F(a+rw)\right]$$
$$+ (-)\frac{w^2}{2!\,6}\left[F'(a+rw) - F'(a)\right] + (-)^2\frac{w^4}{4!\,30}\left[F^{(3)}(a+rw) - F^{(3)}(a)\right] \qquad (97a)$$

Accordingly, values of $A_r, A_i, B_r, B_i, C_r, C_i, D_r, D_i, E_r$ and

E_i are determined for k_c=0,.4,.8,1.2,1.6,2.0, and M=10/7, 2.00, by application of Eq. (97a).* Results are tabulated in Tables 4,5,6,7 and 8. The formulae Eqs. (60c),(65c), (69c),(76c),(87),(90) and (92) are then used to compute the respective aerodynamic coefficients. Results of these computations are summarized in Tables 9,10,11 and 12. These data are plotted as the vector diagrams in Figs.8,9,10,11, 12,13,14 and 15. The analysis of the plunging oscillations of a rectangular wing moving at a supersonic speed therefore comes to an end. Discussions of these results are postponed to Part VII.

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* Mathematical tables such as Refs. (21) and (22) are useful for this task.

VI. OTHER TYPES OF OSCILLATIONS OF A RECTANGULAR WING

§ 6.1 Pitching oscillations of a rectangular wing

Suppose that the rectangular flat plate (Fig.6) is undergoing a periodic pitching motion (as in a torsion oscillations) about the y-axis such that

$$w_r(\xi,\eta)\exp(i\nu t) = -\Lambda_o U\xi\exp(i\nu t) \quad \text{at every instant} \tag{99}$$

Then, with reference to Fig. 6 and by substitution of Eq. (99) into Eq. (12b), it is immediately found for a point P (x,y,+0) in region II that

$$\Phi(x,y,+o,t) = \frac{U\Lambda_o}{\pi}\exp(i\nu t)\int_o^x \xi\exp\left[\frac{-i\nu U}{\beta^2a^2}(x-\xi)\right]d\xi\int_{y-\frac{1}{\beta}(x-\xi)}^{y+\frac{1}{\beta}(x-\xi)}\frac{\cos\left\{\frac{\nu}{\beta^2a}\left[(x-\xi)^2-\beta^2(y-\eta)^2\right]^{1/2}\right\}}{\left[(x-\xi)^2-\beta^2(y-\eta)^2\right]^{1/2}}d\eta \tag{100}$$

which as in Eq. (46a) may be integrated once to yield

$$\Phi(x,y,+o,t) = \Lambda_o a\frac{\beta a}{\nu}\exp(i\nu t)\left[x\int_o^k J_o\left(\frac{\sigma}{M}\right)\exp(-i\sigma)d\sigma - \frac{\beta^2a^2}{\nu U}\int_o^k \sigma J_o\left(\frac{\sigma}{M}\right)\exp(-i\sigma)\,d\sigma\right] \tag{100a}$$

By integration by parts twice it is found that (see the recurrence relation in § 8.5)

$$\int_o^k \sigma J_o\left(\frac{\sigma}{M}\right)\exp(-i\sigma)\,d\sigma = \frac{M^2}{\beta^2}\left\{k\exp(-ik)\left[iJ_o\left(\frac{k}{M}\right)-\frac{1}{M}J_1\left(\frac{k}{M}\right)\right]-iT\right\} \tag{101}$$

And, combination of Eqs. (54a),(93b),(100a) and (101) yields

$$\Phi(x,y,+o,t) = \Lambda_o a\left(\frac{\beta a}{\nu}\right)^2\frac{\beta}{M}\exp(i\nu t)\,k\,(2H-T) \tag{100b}$$

This equation of the velocity potential at a point P on the rectangular wing (Fig. 6) may again be verified by an integration in the u,v plane. Thus, Eq. (99) may be rewritten as

$$W_T(u,v)\exp(i\nu t) = -U\Lambda_o\frac{\beta}{M}(u+v)\exp(i\nu t) \tag{99a}$$

and Eq. (100) is transformed into

$$\begin{aligned}\Phi(u_p,v_p,+0,t) &= \frac{U\Lambda_o}{\pi M}\frac{\beta}{M}\exp(i\nu t)\Big\{\int_{-v_p}^{u_p}\frac{u\,du}{(u_p-u)^{1/2}\exp[\frac{i\nu}{\beta a}(u_p-u)]}\int_{-u}^{v_p}\frac{\cos\{\frac{2\nu}{M\beta a}[(u_p-u)(v_p-v)]^{1/2}\}dv}{(v_p-v)^{1/2}\exp[\frac{i\nu}{\beta a}(v_p-v)]}\\ &\quad+\int_{-v_p}^{u_p}\frac{du}{(u_p-u)^{1/2}\exp[\frac{i\nu}{\beta a}(u_p-u)]}\int_{-u}^{v_p}\frac{v\cos\{\frac{2\nu}{M\beta a}[(u_p-u)(v_p-v)]^{1/2}\}dv}{(v_p-v)^{1/2}\exp[\frac{i\nu}{\beta a}(v_p-v)]}\Big\}\end{aligned} \tag{102}$$

After some calculations, Eq. (102) may be rewritten as

$$\begin{aligned}\Phi(u_p,v_p,+0,t) &= \frac{a\Lambda_o}{\pi}\frac{\beta}{M}\exp(i\nu t)\Big\{-\left(1+\frac{1}{M^2}\right)\int_{-v_p}^{u_p}\frac{(u_p-u)^{1/2}du}{\exp[\frac{i\nu}{\beta a}\frac{\beta^2}{M^2}(u_p-u)]}\int_{q_1}^{q_2}\exp\left(-\frac{i\nu}{\beta a}q^2\right)dq\\ &\quad+\left(i\frac{\beta a}{2\nu}+\frac{Mx}{\beta}\right)\int_{-v_p}^{u_p}\frac{du}{(u_p-u)^{1/2}\exp[\frac{i\nu}{\beta a}\frac{\beta^2}{M^2}(u_p-u)]}\int_{q_1}^{q_2}\exp\left(-\frac{i\nu}{\beta a}q^2\right)dq\\ &\quad+i\frac{\beta a}{2\nu}\int_{-v_p}^{u_p}\frac{du}{\exp[\frac{i\nu}{\beta a}\frac{\beta^2}{M^2}(u_p-u)]}\Big[\left(-\frac{q_2}{(u_p-u)^{1/2}}+\frac{2}{M}\right)\exp\left(-\frac{i\nu}{\beta a}q_2^2\right)\\ &\quad+\left(\frac{q_1}{(u_p-u)^{1/2}}-\frac{2}{M}\right)\exp\left(-\frac{i\nu}{\beta a}q_1^2\right)\Big]\Big\}\end{aligned} \tag{102a}$$

where $q_1 = -(v_p+u)^{1/2}+\frac{1}{M}(u_p-u)^{1/2}$ and $q_2 = (v_p+u)^{1/2}+\frac{1}{M}(u_p-u)^{1/2}$ (103)

It can be shown without much trouble that

$$\int_{-v_p}^{u_p} \frac{du}{\exp\left[\frac{i\nu}{\beta a}\frac{\beta^2}{M^2}(u_p-u)\right]}\left[\left(-\frac{q_2}{(u_p-u)^{1/2}}+\frac{2}{M}\right)\exp\left(-\frac{i\nu}{\beta a}q_2^2\right)+\left(\frac{q_1}{(u_p-u)^{1/2}}-\frac{2}{M}\right)\exp\left(-\frac{i\nu}{\beta a}q_1^2\right)\right]$$

$$= -\frac{\pi}{2}\left(\frac{\beta a}{\nu}\right)^2 k \exp(-ik)\left[i J_0\left(\frac{k}{M}\right)-\frac{1}{M}J_1\left(\frac{k}{M}\right)\right] \tag{104}$$

And, therefore, combination of Eqs. (102a), (93b), (54b) and (104) yields

$$\Phi(u_p, v_p, +0, t) = A_0 a\left(\frac{\beta a}{\nu}\right)^2\frac{\beta}{M}\exp(i\nu t)\, k(2H-T)$$

which is Eq. (100b).

For a point $Q(u_a, v_a, +0)$ in region I of a rectangular wing (Fig. 6) with pitching oscillations, the velocity potential Φ at an instant t is by Eqs. (34a),(38a) and (99a)

$$\Phi(u_a, v_a, +0, t)$$

$$= \frac{UA_0}{\pi M}\frac{\beta}{M}\exp(i\nu t)\left\{\int_{v_a}^{u_a}\frac{u\,du}{(u_a-u)^{1/2}\exp\left[\frac{i\nu}{\beta a}(u_a-u)\right]}\int_{-u}^{v_a}\frac{\cos\left\{\frac{2\nu}{M\beta a}\left[(u_a-u)(v_a-v)\right]^{1/2}\right\}dv}{(v_a-v)^{1/2}\exp\left[\frac{i\nu}{\beta a}(v_a-v)\right]}\right.$$

$$\left.+\int_{v_a}^{u_a}\frac{du}{(u_a-u)^{1/2}\exp\left[\frac{i\nu}{\beta a}(u_a-u)\right]}\int_{-u}^{v_a}\frac{v\cos\left\{\frac{2\nu}{M\beta a}\left[(u_a-u)(v_a-v)\right]^{1/2}\right\}dv}{(v_a-v)^{1/2}\exp\left[\frac{i\nu}{\beta a}(v_a-v)\right]}\right\} \tag{105}$$

As in Eq. (102a), this may be reduced to

$$\Phi(u_a, v_a, +0, t)$$

$$= \frac{aA_0}{\pi}\frac{\beta}{M}\exp(i\nu t)\left\{-\left(1+\frac{1}{M^2}\right)\int_{v_a}^{u_a}\frac{(u_a-u)^{1/2}du}{\exp\left[\frac{i\nu}{\beta a}\frac{\beta^2}{M^2}(u_a-u)\right]}\int_{q_1}^{q_2}\exp\left(-\frac{i\nu}{\beta a}q^2\right)dq\right.$$

$$+i\frac{\beta a}{2\nu}\int_{v_a}^{u_a}\frac{du}{\exp\left[\frac{i\nu}{\beta a}\frac{\beta^2}{M^2}(u_a-u)\right]}\left[\left(-\frac{q_2}{(u_a-u)^{1/2}}+\frac{2}{M}\right)\exp\left(-\frac{i\nu}{\beta a}q_2^2\right)+\left(\frac{q_1}{(u_a-u)^{1/2}}-\frac{2}{M}\right)\exp\left(-\frac{i\nu}{\beta a}q_1^2\right)\right]$$

$$\left.+\left(i\frac{\beta a}{2\nu}+\frac{M\chi}{\beta}\right)\int_{v_a}^{u_a}\frac{du}{(u_a-u)^{1/2}\exp\left[\frac{i\nu}{\beta a}\frac{\beta^2}{M^2}(u_a-u)\right]}\int_{q_1}^{q_2}\exp\left(-\frac{i\nu}{\beta a}q^2\right)dq\right\} \tag{105a}$$

where $q_1 = -(v_a + u)^{\frac{1}{2}} + \frac{1}{M}(u_a - u)^{\frac{1}{2}}$, $q_2 = (v_a + u)^{\frac{1}{2}} + \frac{1}{M}(u_a - u)^{\frac{1}{2}}$. (106)

It is then possible to derive the following formula

$$\begin{aligned} &\Phi(u_a, v_a, +0, t) \\ &= 2\frac{a\Lambda_0}{\pi}\frac{\beta}{M}\exp(i\nu t)\left(\frac{\beta a}{\nu}\right)^2\Big\{-\left(1+\frac{1}{M^2}\right)k^{3/2}\int_0^{\left(\frac{\beta|y|}{x}\right)^{1/2}} f(\mu,k,M)\,\mu^2 d\mu \\ &\qquad + \left(\frac{i}{2}+k\right)k^{1/2}\int_0^{\left(\frac{\beta|y|}{x}\right)^{1/2}} f(\mu,k,M)\,d\mu \\ &\qquad - ik\exp(-ik)\Big[\int_0^{\left(\frac{\beta|y|}{x}\right)^{1/2}} (1-\mu^2)^{1/2}\cos\left[\frac{2\mu k}{M}(1-\mu^2)^{1/2}\right]d\mu \\ &\qquad + \frac{i}{M}\int_0^{\left(\frac{\beta|y|}{x}\right)^{1/2}} \mu\sin\left[\frac{2\mu k}{M}(1-\mu^2)^{1/2}\right]d\mu\Big]\Big\} \end{aligned} \qquad (105b)$$

The essence of the discussions in § 5.1 may apply equally well to these new velocity potentials represented by Eqs. (100b) and (105b). The principle of patching up of the two-dimensional and the three-dimensional flow fields along a Mach line is again verified because when $\frac{\beta|y|}{x} = 1$, Eq. (105b) reduces to Eq. (100b). Calculations of the lift and moment in the different regions on the wing can be carried out on basis of Eqs. (100b) and (105b) in the same manner as in § 5.2. It is anticipated that the detail computations may be complicated but no essential difficulty is involved.

§ 6.2 Rolling oscillations of a rectangular wing

In this case, the downwash condition on the top surface of the wing is specified by

$$w_T(\xi,\eta)\exp(i\nu t) = -\Lambda_0 U\eta\exp(i\nu t) \quad \text{at every instant} \qquad (107)$$

or in the u,v coordinates

$$w_T(u,v)\exp(i\nu t) = -\frac{U\Lambda_0}{M}(v-u)\exp(i\nu t) \tag{107a}$$

With reference to Fig. 6 and by substitution of Eq. (107) in Eq. (12b) it is found for a point P(x,y,+0) in region II that

$$\Phi(x,y,+0,t)$$
$$= \frac{U\Lambda_0}{\pi}\exp(i\nu t)\int_0^x \exp\left[-\frac{i\nu U}{\beta^2 a^2}(x-\xi)\right]d\xi \int_{y-\frac{1}{\beta}(x-\xi)}^{y+\frac{1}{\beta}(x-\xi)} \frac{\eta\cos\left\{\frac{\nu}{\beta^2 a}\left[(x-\xi)^2-\beta^2(y-\eta)^2\right]^{1/2}\right\}d\eta}{\left[(x-\xi)^2-\beta^2(y-\eta)^2\right]^{1/2}} \tag{108}$$

which may be integrated once to yield

$$\Phi(x,y,+0,t) = \Lambda_0 a\frac{\beta a}{\nu}\exp(i\nu t)\,T_y = -\Lambda_0 a\left(\frac{\beta a}{\nu}\right)^2\frac{1}{M}\,k\,\frac{\beta|y|}{x}\,T\exp(i\nu t) \tag{108a}$$

This same expression can be obtained by integration in the u,v plane. Thus

$$\Phi(u_p,v_p,+0,t)$$
$$= \frac{U\Lambda_0}{\pi M}\frac{1}{M}\exp(i\nu t)\left\{-\int_{-v_p}^{u_p}\frac{u\,du}{(u_p-u)^{1/2}\exp\left[\frac{i\nu}{\beta a}(u_p-u)\right]}\int_{-u}^{v_p}\frac{\cos\left\{\frac{2\nu}{M\beta a}\left[(u_p-u)(v_p-v)\right]^{1/2}\right\}dv}{(v_p-v)^{1/2}\exp\left[\frac{i\nu}{\beta a}(v_p-v)\right]}\right.$$
$$\left.+\int_{-v_p}^{u_p}\frac{du}{(u_p-u)^{1/2}\exp\left[\frac{i\nu}{\beta a}(u_p-u)\right]}\int_{-u}^{v_p}\frac{v\cos\left\{\frac{2\nu}{M\beta a}\left[(u_p-u)(v_p-v)\right]^{1/2}\right\}dv}{(v_p-v)^{1/2}\exp\left[\frac{i\nu}{\beta a}(v_p-v)\right]}\right\} \tag{109}$$

which as in Eq. (102a) may be rewritten as

$$\Phi(u_p,v_p,+0,t)$$
$$= \frac{a\Lambda_0}{\pi}\frac{1}{M}\exp(i\nu t)\left\{\left(1-\frac{1}{M^2}\right)\int_{-v_p}^{u_p}\frac{(u_p-u)^{1/2}du}{\exp\left[\frac{i\nu}{\beta a}\frac{\beta^2}{M^2}(u_p-u)\right]}\int_{q_1}^{q_2}\exp\left(-\frac{i\nu}{\beta a}q^2\right)dq\right.$$

$$+i\frac{\beta a}{2\nu}\int_{-v_p}^{u_p}\frac{du}{\exp\left[\frac{i\nu}{\beta a}\frac{\beta^2}{M^2}(u_p-u)\right]}\left[\left(-\frac{q_2}{(u_p-u)^{1/2}}+\frac{2}{M}\right)\exp\left(-\frac{i\nu}{\beta a}q_2^2\right)+\left(\frac{q_1}{(u_p-u)^{1/2}}-\frac{2}{M}\right)\exp\left(-\frac{i\nu}{\beta a}q_1^2\right)\right]$$

$$+\left(i\frac{\beta a}{2\nu}+My\right)\int_{-v_p}^{u_p}\frac{du}{(u_p-u)^{1/2}\exp\left[\frac{i\nu}{\beta a}\frac{\beta^2}{M^2}(u_p-u)\right]}\int_{q_1}^{q_2}\exp\left(-\frac{i\nu}{\beta a}q^2\right)dq\ \Big\} \tag{109a}$$

where q_1 and q_2 are as defined in Eq. (103). And, combination of Eqs. (109a), (93b), (54b) and (104) yields Eq. (108a).

For a point $Q(u_Q, v_Q, +0)$ in region I of a rectangular wing (Fig. 6) with rolling oscillations, the velocity potential Φ at an instant t is by Eqs. (34a), (38a) and (107a)

$$\Phi(u_Q, v_Q, +0, t)$$

$$= \frac{UA_0}{\pi M}\frac{1}{M}\exp(i\nu t)\Big\{-\int_{v_Q}^{u_Q}\frac{u\,du}{(u_Q-u)^{1/2}\exp\left[\frac{i\nu}{\beta a}(u_Q-u)\right]}\int_{-u}^{v_Q}\frac{\cos\left\{\frac{2\nu}{M\beta a}\left[(u_Q-u)(v_Q-v)\right]^{1/2}\right\}dv}{(v_Q-v)^{1/2}\exp\left[\frac{i\nu}{\beta a}(v_Q-v)\right]}$$

$$+\int_{v_Q}^{u_Q}\frac{du}{(u_Q-u)^{1/2}\exp\left[\frac{i\nu}{\beta a}(u_Q-u)\right]}\int_{-u}^{v_Q}\frac{v\cos\left\{\frac{2\nu}{M\beta a}\left[(u_Q-u)(v_Q-v)\right]^{1/2}\right\}dv}{(v_Q-v)^{1/2}\exp\left[\frac{i\nu}{\beta a}(v_Q-v)\right]}\Big\} \tag{110}$$

which after some calculations may be written as

$$\Phi(u_Q, v_Q, +0, t) = 2\frac{aA_0}{\pi}\frac{1}{M}\left(\frac{\beta a}{\nu}\right)^2\exp(i\nu t)$$

$$\times\Big\{\left(1-\frac{1}{M^2}\right)k^{3/2}\int_0^{\left(\frac{\beta|y|}{x}\right)^{1/2}}f(\mu,k,M)\,\mu^2\,d\mu$$

$$+\left(\frac{i}{2}-\frac{\beta|y|}{x}k\right)k^{1/2}\int_0^{\left(\frac{\beta|y|}{x}\right)^{1/2}}f(\mu,k,M)\,d\mu$$

$$-ik\exp(-ik)\left[\int_0^{\left(\frac{\beta|y|}{x}\right)^{1/2}}(1-\mu^2)^{1/2}\cos\left[\frac{2\mu k}{M}(1-\mu^2)^{1/2}\right]d\mu+\frac{i}{M}\int_0^{\left(\frac{\beta|y|}{x}\right)^{1/2}}\mu\sin\left[\frac{2\mu k}{M}(1-\mu^2)^{1/2}\right]d\mu\right]\Big\} \tag{110a}$$

It can easily be shown that Eq. (110a) reduces to Eq. (108a) when $\frac{\beta|y|}{x} = 1$. And, calculations of the lift and moment on basis of Eqs. (108a) and (110a) may be carried out as in § 5.2.

VII. CONCLUSIONS

(1) In Ref. (7), the boundary value problem for the determination of the velocity potential in the purely supersonic region of a wing with unsteady motions at supersonic speed was treated by source-superposition method in a quite general manner. In fact, Theorems 1 and 2 mentioned above are included in Garrick and Rubinow's results. On specializing to considerations of periodic supersonic motions, the derivation of Eq. (17) becomes very simple. The derivation of the same equation in Ref. (7) is more complicated.

(2) In Ref. (9), the boundary value problem for the determination of the velocity potential in the mixed supersonic region of an unsteady wing was analyzed. Evvard's general results are expressed in an extraordinarily complicated form. The present results, as given in Theorems 3 and 4, however, are quite simple. The "equivalent integration area" idea (Theorem 3) was first discovered by Evvard in another paper on source-superposition method for steady supersonic wings (Ref. 1). The essential results, in Theorems 3 and 4, are to extend the validity of this idea to the cases of periodic motions.

(3) Since an arbitrary downwash function can usually be expanded as a Fourier series, the periodic motions may be considered as the basis for building up more general motions for an unsteady wing. This paves the way to extend

the validity of the theorems 1,2,3 and 4 to much more general unsteady motions.

(4) However, the "equivalent area" idea, Theorem 3, is not applicable to arbitrary unsteady supersonic motions. A particular type of motion which is of both theoretical and practical interest and which provides an example of the fallacy of the Theorems 3 and 4, is the so called "unit step" motion, in which a wing at rest starts abruptly at a certain instant and then maintains a steady motion. For composition of the velocity potential for a wing with motion of this nature, the "unit step" source will be useful. The "unit step" source can be derived from an oscillating source by a contour integration in the ν-plane,

$$\phi_2(x,y,z,t) = \frac{1}{2\pi i}\int_C \phi_1 \frac{d\nu}{\nu} = \frac{\beta}{2\pi i h_1}\int_C \cos\frac{\nu h_1}{\beta^2 a}\exp\left\{i\nu\left[t-\frac{U(x-\xi)}{\beta^2 a^2}\right]\right\}\frac{d\nu}{\nu} \qquad (111)$$

where C is the contour shown in Fig. 16. By writing the cosine term in exponential form Eq. (111) can be shown to yield

$$\phi_2(x,y,z,t) = \frac{\beta}{2h_1}\left\{ H\left(t-\frac{Ux}{\beta^2 a^2}+\frac{h_1}{\beta^2 a}\right) + H\left(t-\frac{Ux}{\beta^2 a^2}-\frac{h_1}{\beta^2 a}\right)\right\} \qquad (111a)$$

where $H(\mu)$ is the "unit step" function having the property that

$$H(\mu) = \begin{cases} 1 & \mu > 0 \\ 0 & \mu < 0 \end{cases} \qquad (112)$$

Now, draw a sphere of radius (at) enclosed in the circular cone from the "unit step" source at (ξ, η, ζ), with the center of the sphere located at a distance (Ut) from (ξ, η, ζ) (Fig. 17). Then the region of influence of the source is divided into <u>three</u> regions (by Eq. (112)):

(A) In region I, the influence is equivalent to that of a steady source.

(B) In region II, the influence is equivalent to that of a steady source of half strength.

(C) In region III, no influence of the source will be felt.

The region of dependence for a point (x,y,z) will consist of three similar regions.

Consider the lift problem of a rectangular flat plate wing performing "unit step" motion. Suppose that the velocity potential at a point S in the mixed supersonic region near the wing tip is to be computed at an instant t_1, such that $at_1 < |y|$ (Fig. 18). According to the argument above, the condition at S will depend on both regions A and B and the wing tip will have no influence. But in accordance with the Theorems 3 and 4, the domain of dependence at S would exclude the shaded region in Fig. 18 in the computation of the velocity potential at S, at instant t_1.

Therefore, Theorems 3 and 4 are <u>not</u> applicable to "unit step" wings. This fact is indicated (but not proved) by Eq. (111) because the operation of the contour integration will carry the cosine function to infinity such that the argument of Eq. (25b) breaks down in the proof of Theorem 3.

(5) In Part V, the general theorems developed in Part III are used to compute the supersonic plunging oscillations of a rectangular flat plate. This analysis provides a rather simple example of the remarkable tip effects of a three-dimensional oscillating wing. This is most easily seen by examining some of the interesting phase relations revealed in the lift and moment vector diagrams (Fig. 8,9,12 and 13). It is convenient to introduce the following complex representations of the lift and moment vectors given in Eqs. (60c),(65c),(69c) and (76c):

$$C_{L_0} = \frac{4\Lambda_0}{\beta}\left|\frac{C_{L_0}}{C_{L_0}^*}\right| \exp[i(\nu t + \epsilon_0)] \tag{60d}$$

$$C_{M_0} = \frac{2\Lambda_0}{\beta}\left|\frac{C_{M_0}}{C_{M_0}^*}\right| \exp[i(\nu t + \delta_0)] \tag{65d}$$

$$C_{L_1} = \frac{2\Lambda_0}{\beta}\left|\frac{C_{L_1}}{C_{L_1}^*}\right| \exp[i(\nu t + \epsilon_1)] \tag{69d}$$

$$C_{M_1} = \frac{4\Lambda_0}{3\beta}\left|\frac{C_{M_1}}{C_{M_1}^*}\right| \exp[i(\nu t + \delta_1)] \tag{76d}$$

where $\left|\frac{C_{L_0}}{C_{L_0}^*}\right| = \left[\left(\frac{C_{L_0}}{C_{L_0}^*}\right)_r^2 + \left(\frac{C_{L_0}}{C_{L_0}^*}\right)_i^2\right]^{1/2}$ is the modulus,

and $\epsilon_0 = \tan^{-1}\dfrac{(C_{L_0}/C_{L_0}^*)_i}{(C_{L_0}/C_{L_0}^*)_r}$ is the phase angle of the

vector $\frac{C_{L_0}}{C_{L_0}^*}$; and similarly, $\left|\frac{C_{M_0}}{C_{M_0}^*}\right|, \left|\frac{C_{L_1}}{C_{L_1}^*}\right|$ and $\left|\frac{C_{M_1}}{C_{M_1}^*}\right|$ are the moduli, and δ_0, ϵ_1 and δ_1 are the phase angles, respectively, of the vectors $\frac{C_{M_0}}{C_{M_0}^*}$, $\frac{C_{L_1}}{C_{L_1}^*}$ and $\frac{C_{M_1}}{C_{M_1}^*}$.

It is then observed that

(A) In Figs. 8 and 9, at $M = 10/7$, ϵ_0, δ_0, ϵ_1, δ_1 are all <u>negative</u> for $0 < k_c \leq 1.6$; they are all zero for $k_c = 0$. ϵ_1 and δ_1 become <u>positive</u> while ϵ_0 and δ_0 remain <u>negative</u> for values of k_c close to 2.0.

(B) In Figs. 12 and 13, at $M = 2$, ϵ_0, δ_0 are <u>negative</u> while ϵ_1, δ_1 are <u>positive</u> for $0 < k_c \leq 2.0$; they are all zero for $k_c = 0$.

The occurrence of this alternation in the phase relation is an interesting feature worthy of particular mentioning. The effect of this phase shift in the wing tips is undoubtedly a factor that requires further study.

(6) The following considerations will help to explain these phase shifts. Thus, from Eqs. (60c) and (65c), it is deduced that

$$(C_{L_0}/C^*_{L_0}) = \left\{ 1 + O(k_c^2) + O(k_c^4) + \cdots\cdots \right\} + i\left\{ -\frac{1}{2M^2} k_c + O(k_c^3) + O(k_c^5) + \cdots \right\} \tag{113}$$

$$(C_{M_0}/C^*_{M_0}) = \left\{ 1 + O(k_c^2) + O(k_c^4) + \cdots\cdots \right\} + i\left\{ -\frac{2}{3M^2} k_c + O(k_c^3) + O(k_c^5) + \cdots \right\} \tag{114}$$

From Eqs. (69c) and (76c), it is deduced that

$$(C_{L_1}/C^*_{L_1}) = \left\{ 1 + O(k_c^2) + O(k_c^4) + \cdots\cdots \right\} + i\left\{ \frac{M^2-3}{3M^2} k_c + O(k_c^3) + O(k_c^5) + \cdots \right\} \tag{115}$$

$$(C_{M_1}/C^*_{M_1}) = \left\{ 1 + O(k_c^2) + O(k_c^4) + \cdots\cdots \right\} + i\left\{ \frac{3(M^2-3)}{8M^2} k_c + O(k_c^3) + O(k_c^5) + \cdots \right\} \tag{116}$$

And, from Eqs. (89a) and (91), it is found that

$$C_{Lw}/C_{L_o}^* = \left\{\left(1-\frac{1}{2\beta AR}\right) + O(k_c^2) + O(k_c^4) + \cdots\cdots\right\} + i\left\{\frac{-3\beta AR + (M^2+1)}{6M^2\beta AR}k_c + O(k_c^3) + O(k_c^5) + \cdots\right\} \tag{117}$$

$$C_{M_w}/C_{M_o}^* = \left\{\left(1-\frac{2}{3\beta AR}\right) + O(k_c^2) + O(k_c^4) + \cdots\right\} + i\left\{\frac{-8/3\,\beta AR + (M^2+1)}{4M^2\beta AR}k_c + O(k_c^3) + O(k_c^5) + \cdots\right\} \tag{118}$$

From these equations, the following interesting results are easily seen:

(A) only the terms with <u>even</u> powers of k_c appear in the real components of the **vectors**;

(B) only the terms with <u>odd</u> powers of k_c appear in the imaginary components of the vectors;

(C) in the vicinity of $k_c = 0$, the first power term in k_c dominates the series for the imaginary components, therefore,

(a) by Eqs. (113) and (114), $(C_{L_o}/C_{L_o}^*)_i$ and $(C_{M_o}/C_{M_o}^*)_i$ are always <u>negative</u>, i.e., ϵ_o and δ_o remain <u>negative</u> for <u>all</u> Mach numbers (of course, $M>1$);

(b) by Eqs. (115) and (116), when $M^2>3$, $(C_{L_1}/C_{L_1}^*)_i$ and $(C_{M_1}/C_{M_1}^*)_i$ are <u>positive</u>, i.e., $\epsilon_1>0$, $\delta_1>0$; when $M^2<3$, $(C_{L_1}/C_{L_1}^*)_i$ and $(C_{M_1}/C_{M_1}^*)_i$ are <u>negative</u>, i.e., $\epsilon_1<0$, $\delta_1<0$; when $M^2=3$, $(C_{L_1}/C_{L_1}^*)_i$ and $(C_{M_1}/C_{M_1}^*)_i$ start with a term **at** most of the order of k_c^3.

(c) by Eq. (117), the vector $(C_{Lw}/C_{L_o}^*)$ will have <u>positive</u> phase angle when $1<M^2<1+(1/4)(3AR-[9AR^2-8]^{1/2})^2$ or

$M^2 > 1+(1/4)(3AR+[9AR^2-8]^{1/2})^2$; it will have negative phase angle when $1+(1/4)(3AR-[9AR^2-8]^{1/2})^2 < M^2 < 1+(1/4)(3AR+[9AR^2-8]^{1/2})^2$; its imaginary component will start with a term at most of the order of k_c^3 when $M^2 = 1+(1/4)(3AR \pm [9AR^2-8]^{1/2})^2$.

(d) by Eq. (118), the vector $C_{M_w}/C_{M_o}{}^*$ will have positive phase angle when $1<M^2<1+(1/4)(\frac{8}{3}AR-[(\frac{8}{3}AR)^2-8]^{1/2})^2$ or $M^2>1+(1/4)(\frac{8}{3}AR+[(\frac{8}{3}AR)^2-8]^{1/2})^2$; it will have negative phase angle when $1+(1/4)(\frac{8}{3}AR-[(\frac{8}{3}AR)^2-8]^{1/2})^2 < M^2 < 1+\frac{1}{4}(\frac{8}{3}AR+[(\frac{8}{3}AR)^2-8]^{1/2})^2$; its imaginary component will start with a term at most of the order of k_c^3 when $M^2 = 1+(1/4)(\frac{8}{3}AR \pm [(\frac{8}{3}AR)^2-8]^{1/2})^2$.

Hence, the phase relations calculated on basis of a two dimensional analysis are likely to be erroneous. And, aerodynamic instability and flutter computations that did not take into account the wing tip effects, are not reliable. Revision of these calculations are, consequently, necessary.

(7) It is interesting to study the effect of this phase shift on the one-dimensional flexural instability of a rectangular wing. For this purpose, it is necessary to compute the work done by the wing system. It is clear that the work done per cycle of the plunging oscillation is

$$W = \int_0^{2\pi/\nu} L \cdot \frac{dy}{dt}\, dt \tag{119}$$

where L is the lift force on the wing which can be expressed as follows (see Eq. (60d)):

$$L = |L| \exp[i(\nu t + \epsilon)] \tag{60e}$$

and dy/dt is the downwash velocity due to the wing as defined in Eqs. (44) and (45). Thus, it is seen that

$$\begin{aligned} W &= -\int_0^{2\pi/\nu} |L|\, U\Lambda_0 \cos(\nu t+\epsilon)\cos\nu t\; dt \\ &= -|L|\, U\Lambda_0 \int_0^{2\pi/\nu} [(\cos\nu t\cos\epsilon - \sin\nu t\sin\epsilon)\cos\nu t]\, dt \\ &= -|L|\, U\Lambda_0 \frac{\pi}{\nu}\cos\epsilon \end{aligned} \tag{119a}$$

And, therefore, for $\epsilon < \frac{\pi}{2}$, air absorbs energy from the wing system, i.e. the system is aerodynamically stable. Only when $\epsilon > \frac{\pi}{2}$, air supplies energy to the system, and consequently, there is danger of aerodynamic instability.

(8) In Figs. 10, 11, 14 and 15, it is seen that the higher is the aspect ratio, the closer is the behavior of a finite oscillating wing to the ideal infinite oscillating wing. Indeed, as AR $\to\infty$, it is easily obtained from Eqs. (90) and (92) respectively that

$$C_{L_W} = C_{L_0}\,, \quad C_{M_W} = C_{M_0} \qquad \text{as AR} \to \infty \tag{120}$$

Also, Eqs. (117) and (118) become Eqs. (113) and (114) respectively when AR$\to\infty$. Therefore, the tip effects are growing less conspicuous as the aspect ratio of a rectangular wing increases. This is, of course, not unexpected.

VIII. APPENDICES

§ 8.1. Symbols and notations

x, y, z	Cartesian coordinates
ξ, η, ζ	running Cartesian coordinates
t	time variable
ϕ	disturbance velocity potential
ϕ_1	elementary oscillating source potential
Φ	disturbance velocity potential on the wing surface
a	free stream sound speed
U	free stream speed in the positive x-direction
$U/a=M$	free stream Mach number
ν	frequency of the periodic motions of the wing
$w_T(\xi,\eta)\exp(i\nu t)$	z-component of the disturbance velocities at a point $(\xi,\eta,+0)$ on the top surface of the wing at an instant t.
$w_B(\xi,\eta)\exp(i\nu t)$	z-component of the disturbance velocities at a point $(\xi,\eta,-0)$ on the bottom surface of the wing at an instant t.
$w_{D_T}(\xi,\eta)\exp(i\nu t)$	z-component of the disturbance velocities at a point $(\xi,\eta,+0)$ on the top surface of the diaphragm at an instant t.
$w_{D_B}(\xi,\eta)\exp(i\nu t)$	z-component of the disturbance velocities at a point $(\xi,\eta,-0)$ on the bottom surface of the diaphragm at an instant t.
$\Lambda_T(\xi,\eta)$	effective slope of the top surface of the wing, at a point $(\xi,\eta,+0)$
$\Lambda_B(\xi,\eta)$	effective slope of the bottom surface of the wing, at a point $(\xi,\eta,-0)$
$\Lambda_{D_T}(\xi,\eta)$	effective slope of the top surface of the diaphragm, at a point $(\xi,\eta,+0)$

$\Lambda_{D_B}(\xi,\eta)$	effective slope of the bottom surface of the diaphragm, at a point $(\xi,\eta,-0)$
u,v	oblique (characteristic) coordinates
x_c	rectangular wing chord
b	rectangular wing span
$AR=b/x_c$	rectangular wing aspect ratio
Λ_o	rectangular wing maximum angle of attack
$k=\frac{M^2}{\beta^2}\frac{\nu x}{U}$	reduced frequency
p_1	disturbance pressure
p	local pressure
p_o	free stream pressure
ρ	free stream density
C_{L_o} , C_{M_o}	lift, moment coefficients in the region II of the rectangular wing (Fig.7)
$C_{L_o}^*$, $C_{M_o}^*$	"quasi-steady" lift, moment coefficients in the region II of the rectangular wing (Fig.7)
C_{L_1} , C_{M_1}	lift, moment coefficients in the region I of the rectangular wing (Fig.7)
$C_{L_1}^*$, $C_{M_1}^*$	"quasi-steady" lift, moment coefficients in the region I of the rectangular wing (Fig.7)
C_{L_2} , C_{M_2}	lift, moment coefficients in the region II' of the rectangular wing (Fig.7)
C_{L_W}	lift coefficient of the wing
C_{M_W}	moment coefficient of the wing
Φ_2	elementary "unit step" source potential
ϵ	phase angle of the lift vector
δ	phase angle of the moment vector
W	work done per cycle of oscillation

§ 8.2 Fresnel's integrals and Lommel's functions of two variables.

In two-dimensional problems of diffraction of light produced by a wedge or half plane, Sommerfield introduced the ingenious idea of many-valued wave functions (Refs. 23, 24 and 25). The Sommerfield's two-valued wave function may be expressed in a form involving the function

$$F(P_n) = \int_0^{P_n} exp(-iP^2)\, dP \tag{121}$$

This function has real and imaginary parts

$$C(P_n) = \int_0^{P_n} \cos P^2\, dP \tag{122}$$

$$S(P_n) = \int_0^{P_n} \sin P^2\, dP \tag{123}$$

which are known as Fresnel's integrals (Refs. 16 and 26). These integrals can be expanded in various forms, among which the series expansions due to Lommel will be discussed now and used later (§ 8.3 and § 8.4).

Let a new variable of integration $b = P^2$ (124) be introduced, then Eqs. (122) and (123) become

$$C(b_n) = \frac{1}{2}\int_0^{b_n} \cos b \frac{db}{b^{1/2}} = \frac{1}{2}\left(\frac{\pi}{2}\right)^{1/2} \int_0^{b_n} J_{-1/2}(b)\, db \tag{122a}$$

$$S(b_n) = \frac{1}{2}\int_0^{b_n} \sin b \frac{db}{b^{1/2}} = \frac{1}{2}\left(\frac{\pi}{2}\right)^{1/2} \int_0^{b_n} J_{+1/2}(b)\, db \tag{123a}$$

In accordance with Lommel, these integrals may be computed by (Ref 16)

$$C(b_n) = \frac{\pi^{1/2}}{2}\left[L_{1/2}(2b_n, 0) \cos b_n + L_{3/2}(2b_n, 0) \sin b_n \right] \tag{122b}$$

$$S(b_n) = \frac{\pi^{1/2}}{2}\left[L_{1/2}(2b_n, 0)\sin b_n - L_{3/2}(2b_n, 0)\cos b_n \right] \tag{123b}$$

where $L_{1/2}(2b_n, 0)$ and $L_{3/2}(2b_n, 0)$ are particular cases of Lommel's functions of two variables of ℓ-th order, $L_\ell(b,a)$, when $b=2b_n$, $a=0$ and $\ell=1/2$, $\ell=3/2$ respectively. In general, when ℓ is a non-integer, Lommel's functions $L_\ell(b,0)$ are defined by (Ref. 16)

$$L_\ell(b,0) = \sum_{m=0}^{\infty} \frac{(-)^m (\frac{1}{2}b)^{\ell+2m}}{\Gamma(\ell+2m+1)} \tag{125}$$

thus, for $\ell = 1/2$, $b = 2b_n$,

$$L_{1/2}(2b_n, 0) = \sum_{m=0}^{\infty} \frac{(-)^m (b_n)^{\frac{1}{2}+2m}}{\Gamma(\frac{1}{2}+2m+1)} \tag{125a}$$

and for $\ell = 3/2$, $b = 2b_n$,

$$L_{3/2}(2b_n, 0) = \sum_{m=0}^{\infty} \frac{(-)^m (b_n)^{\frac{3}{2}+2m}}{\Gamma(\frac{1}{2}+2m+2)} \tag{125b}$$

Besides, it is not difficult to verify the following formulae correlating $L_{1/2}(2b_n, 0)$ and $L_{3/2}(2b_n, 0)$:

$$\frac{d}{db_n} L_{3/2}(2b_n, 0) = L_{1/2}(2b_n, 0) \tag{126}$$

$$\frac{d}{db_n} L_{1/2}(2b_n, 0) = \frac{1}{(\pi b_n)^{1/2}} - L_{3/2}(2b_n, 0) \tag{127}$$

By simple elimination process, the following differential equation for $L_{1/2}(2b_n, 0)$ can be established

$$L_{1/2}(2b_n, 0) + \frac{d}{db_n}\left[\frac{d}{db_n} L_{1/2}(2b_n, 0) - \frac{1}{(\pi b_n)^{1/2}}\right] = 0 \tag{128}$$

Similarly, it is found that

$$\frac{d^2}{db_n^2} L_{3/2}(2b_n, 0) - \left[\frac{1}{(\pi b_n)^{1/2}} - L_{3/2}(2b_n, 0)\right] = 0 \tag{129}$$

Hence, Eqs. (125a) and (125b) may be considered as the series solutions of the differential equations, Eqs. (128) and (129) respectively. It is, therefore, concluded that the real and imaginary parts of Eq. (121) are Fresnel integrals, $C(b_n)$ and $S(b_n)$, which may be computed by Eqs. (122b) and (123b), where $L_{1/2}(2b_n, 0)$ is a solution of Eq. (128) in the form given by Eq. (125a), and $L_{3/2}(2b_n, 0)$ is a solution of Eq. (129) in the form given by Eq. (125b).

§ 8.3 Series representations of the T-functions

The function $f(\mu, k, M)$ defined in Eq. (53), can be rewritten as

$$f(\mu,k,M) = \exp\left(-\frac{i\beta^2}{M^2}k\mu^2\right)\left\{\int_0^{k^{1/2}\left[\frac{\mu}{M}+(1-\mu^2)^{1/2}\right]}(\cos P^2 - i\sin P^2)\,dP + \int_0^{+k^{1/2}\left[\frac{-\mu}{M}+(1-\mu^2)^{1/2}\right]}(\cos P^2 - i\sin P^2)\,dP\right\} \tag{53a}$$

In Eq. (53a), Fresnel's integrals may be replaced by Lommel's functions as discussed in § 8.2. Thus, it is found that

$$f(\mu,k,M) = \frac{(\pi)^{1/2}}{2}\exp(-ik)\left\{\exp\left[-i\frac{2\mu k}{M}(1-\mu^2)^{1/2}\right]\left[L_{1/2}(2b_1,0) + i L_{3/2}(2b_1,0)\right]\right.$$
$$\left. + \exp\left[i\frac{2\mu k}{M}(1-\mu^2)^{1/2}\right]\left[L_{1/2}(2b_2,0) + i L_{3/2}(2b_2,0)\right]\right\} \tag{53b}$$

where $$\begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} = k\left[(1-\mu^2)^{1/2} \pm \frac{\mu}{M}\right]^2 \tag{130}$$

Substitution of Eq. (53b) into Eq. (54b) yields

$$T = \left(\frac{k}{\pi}\right)^{1/2}\exp(-ik)\int_0^{\pi/2}\left\{\exp\left(-\frac{ik}{M}\sin 2\theta_2\right)\left[L_{1/2}(2b_1,0) + i L_{3/2}(2b_1,0)\right]\right.$$
$$\left. + \exp\left(\frac{ik}{M}\sin 2\theta_2\right)\left[L_{1/2}(2b_2,0) + i L_{3/2}(2b_2,0)\right]\right\}\cos\theta_2\, d\theta_2 \tag{131}$$

where θ_2, b_1 and b_2 are defined as follows:

$$\theta_2 = \sin^{-1}\mu \tag{132}$$

$$\begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} = k\left(\cos\theta_2 \pm \frac{\sin\theta_2}{M}\right)^2 \tag{130a}$$

On separating the real and imaginary parts of the integrand in Eq. (131), it is found that

$$T = \left(\frac{k}{\pi}\right)^{\frac{1}{2}} \exp(-ik)\left[(T_1 + T_6 + T_3 - T_8) + i(T_2 - T_5 + T_4 + T_7)\right] \tag{131a}$$

where
$$\int_0^{\pi/2} \cos\left(\frac{k}{M}\sin 2\theta_2\right)\begin{Bmatrix} L_{\frac{1}{2}}(2b_1, 0) \\ L_{\frac{3}{2}}(2b_1, 0)\end{Bmatrix}\cos\theta_2\, d\theta_2 = \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} \qquad (133),\ (134)$$

$$\int_0^{\pi/2} \cos\left(\frac{k}{M}\sin 2\theta_2\right)\begin{Bmatrix} L_{\frac{1}{2}}(2b_2, 0) \\ L_{\frac{3}{2}}(2b_2, 0)\end{Bmatrix}\cos\theta_2\, d\theta_2 = \begin{Bmatrix} T_3 \\ T_4 \end{Bmatrix} \qquad (135),\ (136)$$

$$\int_0^{\pi/2} \sin\left(\frac{k}{M}\sin 2\theta_2\right)\begin{Bmatrix} L_{\frac{1}{2}}(2b_1, 0) \\ L_{\frac{3}{2}}(2b_1, 0)\end{Bmatrix}\cos\theta_2\, d\theta_2 = \begin{Bmatrix} T_5 \\ T_6 \end{Bmatrix} \qquad (137),\ (138)$$

$$\int_0^{\pi/2} \sin\left(\frac{k}{M}\sin 2\theta_2\right)\begin{Bmatrix} L_{\frac{1}{2}}(2b_2, 0) \\ L_{\frac{3}{2}}(2b_2, 0)\end{Bmatrix}\cos\theta_2\, d\theta_2 = \begin{Bmatrix} T_7 \\ T_8 \end{Bmatrix} \qquad (139),\ (140)$$

To evaluate $T_1, T_2, \ldots\ldots T_8$, it is convenient to introduce Lommel's series for $L_{\frac{1}{2}}$ and $L_{\frac{3}{2}}$. Eqs. (125a) and (125b) can be rewritten as

$$L_{\frac{1}{2}}(2b_n, 0) = 2\left(\frac{b_n}{\pi}\right)^{\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(-)^m (2b_n)^{2m}}{\underbrace{1\cdot 3\cdot 5\cdots(4m+1)}_{(2m+1)\ \text{terms}}} \tag{125c}$$

$$L_{\frac{3}{2}}(2b_n, 0) = 2\left(\frac{b_n}{\pi}\right)^{\frac{1}{2}} \sum_{m=1}^{\infty} \frac{(-)^{m-1} (2b_n)^{2m-1}}{\underbrace{1\cdot 3\cdot 5\cdots(4m-1)}_{2m\ \text{terms}}} \tag{125d}$$

whence it follows that

$$\cos\theta_2 \begin{Bmatrix} L_{\frac{1}{2}}(2b_1, 0) \\ L_{\frac{1}{2}}(2b_2, 0)\end{Bmatrix} = \frac{1}{\pi^{\frac{1}{2}}} \sum_{m=0,2,4\ldots}^{\infty} \frac{(-)^{\frac{m}{2}}\, 2^{m+1}\left[k^{\frac{1}{2}}\left(\cos\theta_2 \pm \frac{\sin\theta_2}{M}\right)\right]^{2m+1} \cos\theta_2}{\underbrace{1\cdot 3\cdot 5\cdots\cdots(2m+1)}_{(m+1)\ \text{terms}}} \qquad (141),\ (142)$$

$$\cos\theta_2 \begin{Bmatrix} L_{3/2}(2b_1,0) \\ L_{3/2}(2b_2,0) \end{Bmatrix} = \frac{1}{\pi^{1/2}} \sum_{m=1,3,5\cdots}^{\infty} \frac{(-)^{\frac{m-1}{2}} 2^{m+1} \left[k^{1/2} \left(\cos\theta_2 \pm \frac{\sin\theta_2}{M} \right) \right]^{2m+1} \cos\theta_2}{\underbrace{1\cdot 3\cdot 5\cdots\cdots(2m+1)}_{(m+1)\ \text{terms}}} \qquad \begin{matrix}(143)\\(144)\end{matrix}$$

In Eqs. (141) and (143), the expression $(\cos\theta_2 + \frac{\sin\theta_2}{M})^{2m+1}\cos\theta_2$ can be expanded by binomial theorem into

$$\cos\theta_2\left(\cos\theta_2 + \frac{\sin\theta_2}{M}\right)^{2m+1} = \sum_{r=0}^{2m+1} \frac{(2m+1)!}{(2m+1-r)!\,r!} \frac{1}{M^{2m+1-r}} \sin^{2m+1-r}\theta_2 \cos^{r+1}\theta_2 \tag{145}$$

By method of induction, it can be easily proved that

$$\sin^{2n}A = \frac{1}{2^{2n-1}}\left\{ \sum_{r=0}^{n-1} \frac{(-)^{n-r}(2n)!}{(2n-r)!\,r!}\cos 2(n-r)A + \frac{(2n)!}{2(n)!(n)!} \right\} \tag{146}$$

$$\sin^{2n+1}A = \frac{1}{2^{2n}}\left\{ \sum_{r=0}^{n} \frac{(-)^{n-r}(2n+1)!}{(2n+1-r)!\,r!}\sin 2(n-r+\tfrac{1}{2})A \right\} \tag{147}$$

$$\cos^{2n}A = \frac{1}{2^{2n-1}}\left\{ \sum_{r=0}^{n-1} \frac{(2n)!}{(2n-r)!\,r!}\cos 2(n-r)A + \frac{(2n)!}{2(n)!(n)!} \right\} \tag{148}$$

$$\cos^{2n+1}A = \frac{1}{2^{2n}}\left\{ \sum_{r=0}^{n} \frac{(2n+1)!}{(2n+1-r)!\,r!}\cos 2(n-r+\tfrac{1}{2})A \right\} \tag{149}$$

In Eq. (145), if r is <u>odd</u>, (r+1) and (2m+1-r) are both <u>even</u>; by Eqs. (146) and (148), it is obtained that

$$\sin^{2m+1-r}\theta_2 \cos^{r+1}\theta_2 = \frac{(2m-r+1)!\,(r+1)!}{2^{2m+1}} W_{m,r}(\theta_2) \tag{150}$$

where

$$\begin{aligned} W_{m,r}(\theta_2) = {} & \sum_{j=0}^{m-\frac{r+1}{2}} \sum_{d=0}^{\frac{r-1}{2}} \frac{(-)^{m-j-\frac{r-1}{2}}\left[\cos 2(m-j-d+1)\theta_2 + \cos 2(m-r+d-j)\theta_2\right]}{(2m-r+1-j)!\,(r+1-d)!\,j!\,d!} \\ & + \frac{1}{\left[\left(\frac{2m-r+1}{2}\right)!\right]^2} \sum_{d=0}^{\frac{r-1}{2}} \frac{\cos 2\left(\frac{r+1}{2}-d\right)\theta_2}{(r+1-d)!\,d!} \\ & + \frac{1}{\left[\left(\frac{r+1}{2}\right)!\right]^2} \sum_{j=0}^{m-\frac{r+1}{2}} \frac{(-)^{m-j-\frac{r-1}{2}}\cos 2\left(m-j-\frac{r-1}{2}\right)\theta_2}{(2m-r+1-j)!\,j!} \\ & + \frac{1}{2\left[\left(\frac{r+1}{2}\right)!\left(\frac{2m-r+1}{2}\right)!\right]^2} \end{aligned} \tag{151}$$

Also, in Eq.(145), if r is even, (r+1) and (2m-r+1) are both odd; by Eqs. (147) and (149), it is obtained that

$$\sin^{2m+1-r}\theta_2 \cos^{r+1}\theta_2 = \frac{(2m-r+1)!\,(r+1)!}{2^{2m+1}} U_{m,r}(\theta_2) \tag{152}$$

where
$$U_{m,r}(\theta_2) = \sum_{g=0}^{m-\frac{r}{2}} \sum_{h=0}^{\frac{r}{2}} \frac{(-)^{m-g-\frac{r}{2}}\left[\sin 2(m-g-h+1)\theta_2 + \sin 2(m-r-g+h)\theta_2\right]}{(2m-r+1-g)!\,(r+1-h)!\; g!\; h!} \tag{153}$$

Combination of Eqs.(145),(150) and (152) yields

$$\cos\theta_2 \left(\cos\theta_2 + \frac{\sin\theta_2}{M}\right)^{2m+1} = \sum_{r=0,2,\cdots}^{2m} \frac{(2m+1)!\,(r+1)}{2^{2m+1} M^{2m+1-r}} U_{m,r}(\theta_2) + \sum_{r=1,3,\cdots}^{2m+1} \frac{(2m+1)!\,(r+1)}{2^{2m+1} M^{2m+1-r}} W_{m,r}(\theta_2) \tag{145a}$$

Substitution of Eq.(145a) into Eq.(141) gives

$$\cos\theta_2\, L_{\frac{1}{2}}(2b_1, 0) = \frac{1}{\pi^{\frac{1}{2}}} \sum_{m=0,2,\cdots}^{\infty} (-)^{\frac{m}{2}} k^{m+\frac{1}{2}} m! \left\{ \sum_{r=0,2,\cdots}^{2m} \frac{r+1}{M^{2m+1-r}} U_{m,r}(\theta_2) + \sum_{r=1,3,\cdots}^{2m+1} \frac{r+1}{M^{2m+1-r}} W_{m,r}(\theta_2) \right\} \tag{141a}$$

Substitution of Eq.(145a) into Eq.(143) gives

$$\cos\theta_2\, L_{\frac{3}{2}}(2b_1, 0) = \frac{1}{\pi^{\frac{1}{2}}} \sum_{m=1,3,\cdots}^{\infty} (-)^{\frac{m-1}{2}} k^{m+\frac{1}{2}} m! \left\{ \sum_{r=0,2,\cdots}^{2m} \frac{r+1}{M^{2m+1-r}} U_{m,r}(\theta_2) + \sum_{r=1,3,\cdots}^{2m+1} \frac{r+1}{M^{2m+1-r}} W_{m,r}(\theta_2) \right\} \tag{143a}$$

It is clear that in Eqs.(142) and (144), $\cos\theta_2 L_{\frac{1}{2}}(2b_2, 0)$ and $\cos\theta_2 L_{\frac{3}{2}}(2b_2, 0)$ can be computed by Eqs.(141a) and (143a) respectively, with θ_2 replaced by $(-\theta_2)$. (154)

Because of the analytic properties of Lommel's series (Ref. 16), the infinite series in Eqs. (141a) and (143a) are uniformly convergent and therefore the definite integrals $T_1, T_2, \ldots T_8$ can be evaluated by integration term by term. By this procedure, from Eqs.(133),(134),....(140) and (131a), it is finally obtained that

$$T=\frac{1}{\pi}\exp(-ik)\Big\{\Big[\sum_{m=0,2,\cdots}^{\infty}(-)^{\frac{m}{2}}k^{m+1}\,m!\sum_{r=1,3,\cdots}^{2m+1}\frac{r+1}{M^{2m+1-r}}\int_0^{\pi}W_{m,r}(\psi)\cos\left(\frac{k}{M}\sin\psi\right)d\psi$$

$$+\sum_{m=1,3,\cdots}^{\infty}(-)^{\frac{m-1}{2}}k^{m+1}\,m!\sum_{r=0,2,\cdots}^{2m}\frac{r+1}{M^{2m+1-r}}\int_0^{\pi}U_{m,r}(\psi)\sin\left(\frac{k}{M}\sin\psi\right)d\psi\Big]$$

$$+i\Big[\sum_{m=1,3,\cdots}^{\infty}(-)^{\frac{m-1}{2}}k^{m+1}\,m!\sum_{r=1,3,\cdots}^{2m+1}\frac{r+1}{M^{2m+1-r}}\int_0^{\pi}W_{m,r}(\psi)\cos\left(\frac{k}{M}\sin\psi\right)d\psi$$

$$-\sum_{m=0,2,\cdots}^{\infty}(-)^{\frac{m}{2}}k^{m+1}\,m!\sum_{r=0,2,\cdots}^{2m}\frac{r+1}{M^{2m+1-r}}\int_0^{\pi}U_{m,r}(\psi)\sin\left(\frac{k}{M}\sin\psi\right)d\psi\Big]\Big\} \qquad (131b)$$

where $\psi=2\theta_2$. Eq. (131b) is the desired series representation of the T-functions. Presumably, the velocity potential function Φ defined in Eq.(61) is a continuous function of k. Therefore, the T-series must be uniformly convergent in k. So, Eq.(131b) may be used to determine the values of T_r and T_i to any degree of accuracy desired. Evaluation of the T-series can easily be accomplished by using the following formulae:

$$\int_0^{\pi}\cos\left(\frac{k}{M}\sin\psi\right)\cos n\psi\,d\psi$$

$$=\begin{cases}\pi J_n\left(\frac{k}{M}\right), & \text{if n is zero or an even integer} \quad (156)\\ 0, & \text{if n is an odd integer} \quad (157)\end{cases}$$

$$\int_0^\pi \sin\left(\frac{k}{M}\sin\psi\right)\sin n\psi \, d\psi$$

$$= \begin{cases} \pi J_n\left(\frac{k}{M}\right) & \text{, if n is an odd integer} \quad (158) \\ 0 & \text{, if n is zero or an even integer} \quad (159) \end{cases}$$

The computations for the first few terms of the series are fairly straightforward. The results are given in Eqs. (55), (56) and (57) (§ 5.1).

§ 8.4 Series representations of the H-functions

The H-functions are defined in Eq.(72). In Eq.(93a), a correlation relation between T and H is established. Series representations of the H-functions, therefore, can be derived from Eqs.(131b) and (93a). On the other hand, an analysis similar to that just described in § 8.3 may be carried out as follows :

Substitution of Eq.(53b) into Eq.(72) yields

$$H = \left(\frac{k}{\pi}\right)^{\frac{1}{2}} \exp(-ik) \int_0^{\pi/2} \Big\{ \exp\left(-\frac{ik}{M}\sin 2\theta_2\right)\left[L_{\frac{1}{2}}(2b_1,0) + i L_{\frac{3}{2}}(2b_1,0)\right] + \exp\left(\frac{ik}{M}\sin 2\theta_2\right)\left[L_{\frac{1}{2}}(2b_2,0) + i L_{\frac{3}{2}}(2b_2,0)\right]\Big\} \cos^3\theta_2 \, d\theta_2 \tag{160}$$

where θ_2, b_1 and b_2 are as defined in Eqs.(132) and (130a). On separating the real and imaginary parts of the integrand in Eq.(160), it is obtained that

$$H = \left(\frac{k}{\pi}\right)^{\frac{1}{2}} \exp(-ik)\left[(H_1 + H_6 + H_3 - H_8) + i(H_2 - H_5 + H_4 + H_7)\right] \tag{160a}$$

where
$$\int_0^{\pi/2} \cos\left(\frac{k}{M}\sin 2\theta_2\right)\begin{Bmatrix} L_{1/2}(2b_1,0) \\ L_{3/2}(2b_1,0)\end{Bmatrix}\cos^3\theta_2\, d\theta_2 = \begin{Bmatrix} H_1 \\ H_2 \end{Bmatrix} \qquad \begin{matrix}(161)\\(162)\end{matrix}$$

$$\int_0^{\pi/2} \cos\left(\frac{k}{M}\sin 2\theta_2\right)\begin{Bmatrix} L_{1/2}(2b_2,0) \\ L_{3/2}(2b_2,0)\end{Bmatrix}\cos^3\theta_2\, d\theta_2 = \begin{Bmatrix} H_3 \\ H_4 \end{Bmatrix} \qquad \begin{matrix}(163)\\(164)\end{matrix}$$

$$\int_0^{\pi/2} \sin\left(\frac{k}{M}\sin 2\theta_2\right)\begin{Bmatrix} L_{1/2}(2b_1,0) \\ L_{3/2}(2b_1,0)\end{Bmatrix}\cos^3\theta_2\, d\theta_2 = \begin{Bmatrix} H_5 \\ H_6 \end{Bmatrix} \qquad \begin{matrix}(165)\\(166)\end{matrix}$$

$$\int_0^{\pi/2} \sin\left(\frac{k}{M}\sin 2\theta_2\right)\begin{Bmatrix} L_{1/2}(2b_2,0) \\ L_{3/2}(2b_2,0)\end{Bmatrix}\cos^3\theta_2\, d\theta_2 = \begin{Bmatrix} H_7 \\ H_8 \end{Bmatrix} \qquad \begin{matrix}(167)\\(168)\end{matrix}$$

By binomial theorem, it is easily seen that

$$\cos^3\theta_2\left(\cos\theta_2 + \frac{\sin\theta_2}{M}\right)^{2m+1} = \sum_{r=0}^{2m+1}\left\{\frac{(2m+1)!}{(2m+1-r)!\,r!}\,\frac{1}{M^{2m+1-r}}\sin^{2m+1-r}\theta_2\cos^{r+3}\theta_2\right\} \qquad (169)$$

If r is <u>even</u>, (r+3) and (2m+1-r) are both <u>odd</u>; by Eqs. (147) and (149), it is found that

$$\sin^{2m+1-r}\theta_2\cos^{r+3}\theta_2 = \frac{(2m-r+1)!\,(r+3)!}{2^{2m+3}}F_{m,r}(\theta_2) \qquad (170)$$

where
$$F_{m,r}(\theta_2) = \sum_{g=0}^{m-\frac{r}{2}}\sum_{h=0}^{\frac{r}{2}+1}\frac{(-)^{m-g-\frac{r}{2}}\left[\sin 2(m-g-h+2)\theta_2 + \sin 2(m-g-r+h-1)\theta_2\right]}{(2m+1-r-g)!\,g!\,(r+3-h)!\,h!} \qquad (171)$$

If r is <u>odd</u>, (r+3) and (2m+1-r) are both <u>even</u>; by Eqs. (146) and (148), it is found that

$$\sin^{2m+1-r}\theta_2\cos^{r+3}\theta_2 = \frac{(2m-r+1)!\,(r+3)!}{2^{2m+3}}E_{m,r}(\theta_2) \qquad (172)$$

where
$$E_{m,r}(\theta_2) = \sum_{j=0}^{m-\frac{r+1}{2}}\sum_{d=0}^{\frac{r+1}{2}}\frac{(-)^{m-j-\frac{r-1}{2}}\left[\cos 2(m-j-d+2)\theta_2 + \cos 2(m-j-r+d-1)\theta_2\right]}{(2m+1-r-j)!\,j!\,(r+3-d)!\,d!}$$

$$+\frac{1}{\left[\left(\frac{2m-r+1}{2}\right)!\right]^2}\sum_{d=0}^{\frac{r+1}{2}}\frac{\cos 2\left(\frac{r+3}{2}-d\right)\theta_2}{(r+3-d)!\,d!}$$

$$+\frac{1}{\left[\left(\frac{r+3}{2}\right)!\right]^2}\sum_{j=0}^{m-\frac{r+1}{2}}\frac{(-)^{m-j-\frac{r-1}{2}}\cos 2\left(m-j-\frac{r-1}{2}\right)\theta_2}{(2m+1-r-j)!\,j!}$$

$$+\frac{1}{2\left[\left(\frac{r+3}{2}\right)!\left(\frac{2m-r+1}{2}\right)!\right]^2} \tag{173}$$

Combination of Eqs. (169), (170) and (172) yields

$$\cos^3\theta_2\left(\cos\theta_2+\frac{\sin\theta_2}{M}\right)^{2m+1}$$

$$=\sum_{r=0,2,\cdots}^{2m}\frac{(2m+1)!\,(r+3)!}{2^{2m+3}\,r!\,M^{2m+1-r}}F_{m,r}(\theta_2)+\sum_{r=1,3,\cdots}^{2m+1}\frac{(2m+1)!\,(r+3)!}{2^{2m+3}\,r!\,M^{2m+1-r}}E_{m,r}(\theta_2) \tag{169a}$$

Thus, it is easily seen that

$$\cos^3\theta_2\, L_{1/2}(2b_1,0)=\frac{1}{\pi^{1/2}}\sum_{m=0,2,\cdots}^{\infty}\frac{(-)^{\frac{m}{2}}\,2^{m+1}\cos^3\theta_2\left[k^{1/2}\left(\cos\theta_2+\frac{\sin\theta_2}{M}\right)\right]^{2m+1}}{\underbrace{1\cdot3\cdot5\cdot\cdots\cdot(2m+1)}_{(m+1)\ \text{terms}}}$$

$$=\frac{1}{4\pi^{1/2}}\sum_{m=0,2,\cdots}^{\infty}(-)^{\frac{m}{2}}k^{m+1/2}\,m!\left\{\sum_{r=0,2,\cdots}^{2m}\frac{(r+3)!}{r!\,M^{2m+1-r}}F_{m,r}(\theta_2)+\sum_{r=1,3,\cdots}^{2m+1}\frac{(r+3)!}{r!\,M^{2m+1-r}}E_{m,r}(\theta_2)\right\} \tag{174}$$

and that

$$\cos^3\theta_2\, L_{3/2}(2b_1,0)=\frac{1}{\pi^{1/2}}\sum_{m=1,3,\cdots}^{\infty}\frac{(-)^{\frac{m-1}{2}}\,2^{m+1}\cos^3\theta_2\left[k^{1/2}\left(\cos\theta_2+\frac{\sin\theta_2}{M}\right)\right]^{2m+1}}{\underbrace{1\cdot3\cdot5\cdot\cdots\cdot(2m+1)}_{(m+1)\ \text{terms}}}$$

$$=\frac{1}{4\pi^{1/2}}\sum_{m=1,3,\cdots}^{\infty}(-)^{\frac{m-1}{2}}k^{m+1/2}\,m!\left\{\sum_{r=0,2,\cdots}^{2m}\frac{(r+3)!}{r!\,M^{2m+1-r}}F_{m,r}(\theta_2)+\sum_{r=1,3,\cdots}^{2m+1}\frac{(r+3)!}{r!\,M^{2m+1-r}}E_{m,r}(\theta_2)\right\} \tag{175}$$

The computations of $\cos^3\theta_2 L_{1/2}(2b_2, 0)$ can easily be accomplished by Eq.(174) with θ_2 replaced by $-\theta_2$. Similar procedure yields $\cos^3\theta_2 L_{3/2}(2b_2, 0)$ from Eq.(175). Therefore, by Eqs. (161),(162) ....(168) and (160a), it is obtained that

$$H = \frac{1}{4\pi} \exp(-ik)\Big\{\Big[\sum_{m=0,2,\cdots}^{\infty} (-)^{\frac{m}{2}} k^{m+1} m! \sum_{r=1,3,\cdots}^{2m+1} \frac{(r+3)!}{r!\,M^{2m+1-r}} \int_0^{\pi} \cos\left(\frac{k}{M}\sin\psi\right) E_{m,r}(\psi)\, d\psi$$

$$+ \sum_{m=1,3,\cdots}^{\infty} (-)^{\frac{m-1}{2}} k^{m+1} m! \sum_{r=0,2,\cdots}^{2m} \frac{(r+3)!}{r!\,M^{2m+1-r}} \int_0^{\pi} \sin\left(\frac{k}{M}\sin\psi\right) F_{m,r}(\psi)\, d\psi\Big]$$

$$+ i\Big[\sum_{m=1,3,\cdots}^{\infty} (-)^{\frac{m-1}{2}} k^{m+1} m! \sum_{r=1,3,\cdots}^{2m+1} \frac{(r+3)!}{r!\,M^{2m+1-r}} \int_0^{\pi} \cos\left(\frac{k}{M}\sin\psi\right) E_{m,r}(\psi)\, d\psi$$

$$- \sum_{m=0,2,\cdots}^{\infty} (-)^{\frac{m}{2}} k^{m+1} m! \sum_{r=0,2,\cdots}^{2m} \frac{(r+3)!}{r!\,M^{2m+1-r}} \int_0^{\pi} \sin\left(\frac{k}{M}\sin\psi\right) F_{m,r}(\psi)\, d\psi\Big]\Big\} \qquad (160b)$$

where $\psi = 2\theta_2$. Eq.(160b) represents the desired series representations of the H-functions. The first few terms of the H-series can be given as follows :

$$\begin{aligned} H_k = {} & k\left[(\alpha_1 J_0 + \alpha_2 J_2)\cos k + \beta_1 J_1 \sin k\right] \\ & + k^2\left[(\alpha_3 J_1 + \alpha_4 J_3)\cos k + (\beta_2 J_0 + \beta_3 J_2)\sin k\right] \\ & + k^3\left[(\alpha_5 J_0 + \alpha_6 J_2 + \alpha_7 J_4)\cos k + (\beta_4 J_1 + \beta_5 J_3)\sin k\right] \\ & + k^4\left[(\alpha_8 J_1 + \alpha_9 J_3 + \alpha_{10} J_5)\cos k + (\beta_6 J_0 + \beta_7 J_2 + \beta_8 J_4)\sin k\right] \\ & + k^5\left[(\alpha_{11} J_0 + \alpha_{12} J_2 + \alpha_{13} J_4 + \alpha_{14} J_6)\cos k + (\beta_9 J_1 + \beta_{10} J_3 + \beta_{11} J_5)\sin k\right] \\ & + \cdots\cdots\cdots \end{aligned} \qquad (176)$$

$$\begin{aligned} -H_i = {} & k\left[\beta_1 J_1 \cos k - (\alpha_1 J_0 + \alpha_2 J_2)\sin k\right] \\ & + k^2\left[(\beta_2 J_0 + \beta_3 J_2)\cos k - (\alpha_3 J_1 + \alpha_4 J_3)\sin k\right] \\ & + k^3\left[(\beta_4 J_1 + \beta_5 J_3)\cos k - (\alpha_5 J_0 + \alpha_6 J_2 + \alpha_7 J_4)\sin k\right] \end{aligned}$$

$$+ k^4 \left[(\beta_6 J_0 + \beta_7 J_2 + \beta_8 J_4) \cos k - (\alpha_8 J_1 + \alpha_9 J_3 + \alpha_{10} J_5) \sin k \right]$$

$$+ k^5 \left[(\beta_9 J_1 + \beta_{10} J_3 + \beta_{11} J_5) \cos k - (\alpha_{11} J_0 + \alpha_{12} J_2 + \alpha_{13} J_4 + \alpha_{14} J_6) \sin k \right]$$

$$+ \cdots\cdots\cdots\cdots \tag{177}$$

where

$$\alpha_1 = \frac{3}{4}$$

$$\alpha_2 = \frac{1}{4}$$

$$\alpha_3 = \frac{1}{4}\left(\frac{1}{2M^3} + \frac{5}{2M}\right)$$

$$\alpha_4 = \frac{1}{4}\left(-\frac{1}{6M^3} + \frac{1}{2M}\right)$$

$$\alpha_5 = -\frac{1}{2}\left(\frac{1}{8M^4} + \frac{5}{12M^2} + \frac{7}{24}\right)$$

$$\alpha_6 = -\frac{1}{2}\left(-\frac{1}{6M^4} - \frac{1}{3M^2} + \frac{28}{5!}\right)$$

$$\alpha_7 = -\frac{1}{2}\left(\frac{1}{4!M^4} - \frac{1}{2!3!M^2} + \frac{1}{5!}\right)$$

$$\alpha_8 = -\frac{3}{2}\left(\frac{2}{6!M^7} + \frac{1}{24M^5} + \frac{14}{3!4!M^3} + \frac{42}{6!M}\right)$$

$$\alpha_9 = -\frac{3}{2}\left(-\frac{3}{7!M^7} - \frac{1}{2!4!M^5} - \frac{1}{2!4!M^3} + \frac{27}{6!M}\right)$$

$$\alpha_{10} = -\frac{3}{2}\left(-\frac{1}{7!M^7} + \frac{1}{2!5!M^5} - \frac{1}{4!3!M^3} + \frac{1}{6!M}\right)$$

$$\alpha_{11} = 3!\left(\frac{1}{2!2!6!M^8} + \frac{1}{2!3!3!3!M^6} + \frac{7}{2!3!5!M^4} + \frac{1}{2!5!M^2} + \frac{11}{6!\,12}\right)$$

$$\alpha_{12} = 3!\left(-\frac{17}{8!M^8} - \frac{1}{2!3!4!M^6} - \frac{17}{4!5!M^4} - \frac{15}{2!7!M^2} + \frac{55}{8!}\right)$$

$$\alpha_{13} = 3!\left(\frac{1}{7!4M^8} + \frac{1}{6!M^6} + \frac{1}{2!3!5!M^4} - \frac{13}{7!M^2} + \frac{66}{9!}\right)$$

$$\alpha_{14} = 3!\left(\frac{1}{8!M^8} - \frac{1}{3!6!M^6} + \frac{1}{4!5!M^4} - \frac{1}{7!2!M^2} + \frac{1}{9!}\right)$$

$$\beta_1 = -\frac{1}{2M}$$

$$\beta_2 = \frac{1}{4}\left(\frac{1}{M^2} + \frac{5}{3}\right)$$

$$\beta_3 = \frac{1}{4}\left(-\frac{1}{M^2} + 1\right)$$

$$\beta_4 = \frac{1}{2}\left(\frac{1}{20M^5} + \frac{1}{2M^3} + \frac{7}{12M}\right)$$

$$\beta_5 = \frac{1}{2}\left(-\frac{1}{60M^5} - \frac{1}{6M^3} + \frac{1}{4M}\right)$$

$$\beta_6 = -\frac{3}{2}\left(\frac{1}{5!M^6} + \frac{1}{4!M^4} + \frac{7}{5!M^2} + \frac{1}{40}\right)$$

$$\beta_7 = -\frac{3}{2}\left(-\frac{8}{6!M^6} - \frac{2}{3!3!M^4} - \frac{4}{5!M^2} + \frac{1}{3!\,7}\right)$$

$$\beta_8 = -\frac{3}{2}\left(\frac{2}{6!M^6} + \frac{2}{3!4!M^4} - \frac{3}{5!M^2} + \frac{10}{7!}\right)$$

$$\beta_9 = -3!\left(\frac{36}{9!M^9}+\frac{10}{7!M^7}+\frac{20}{4!5!M^5}+\frac{1}{5!M^3}+\frac{132}{8!M}\right)$$
$$\beta_{10} = -3!\left(-\frac{2}{9!M^9}-\frac{5}{7!M^7}-\frac{10}{4!5!M^5}-\frac{2}{3!6!M^3}+\frac{110}{8!M}\right)$$
$$\beta_{11} = -3!\left(-\frac{6}{9!M^9}+\frac{1}{7!M^7}+\frac{2}{4!5!M^5}-\frac{6}{3!6!M^3}+\frac{10}{8!M}\right) \qquad (178)$$

and all the Bessel functions (J's) are of the argument k/M. In Table 3, Eqs.(176) and (177) are used to compute the values of H_k and H_i for M=2 and $0 \leq k \leq 1.0$. The good agreement of the data in Tables 2 and 3 verifies the statement that the H-series give accurate values of H_k and H_i for small k.

§ 8.5 Some reduction formulae and derivatives

(1) Because of the analytic properties of the T- and H-series, it is permissible to substitute the results in Eqs.(55),(56),(176) and (177) into Eqs.(64),(66),(73),(79) and (88), and then to integrate term by term. For such calculations, it is necessary, in general, to evaluate a definite integral of the following form :

$$\int_0^k \sigma^n \exp(-i\sigma) J_m\left(\frac{\sigma}{M}\right) d\sigma = M^{n+1}\int_0^{\bar{k}} \sigma^n \exp(-iM\sigma) J_m(\sigma)\, d\sigma \qquad (179)$$

where $\bar{k} = k/M$, and $n \geq m \geq 1$. On integrating the second integral in Eq.(179) by parts, it is found that

$$\int_0^{\bar{k}} \sigma^n \exp(-iM\sigma) J_m(\sigma)\, d\sigma$$
$$= \exp(-iM\bar{k})\, J_m(\bar{k}) \left\{ \frac{\bar{k}^n}{-iM} - \frac{(n-m)\bar{k}^{n-1}}{(-iM)^2} + \frac{(n-m)(n-m-1)\bar{k}^{n-2}}{(-iM)^3} + \cdots \right.$$
$$\left. + (-)^{n-m-1}\frac{(n-m)!\,\bar{k}^{m+1}}{(-iM)^{n-m}} + (-)^{n-m}\frac{(n-m)!\,\bar{k}^m}{(-iM)^{n-m+1}} \right\}$$

$$- \int_0^{\bar{k}} \exp(-iM\sigma) J_{m-1}(\sigma) \left\{ \frac{\sigma^n}{-iM} - \frac{(n-m)\sigma^{n-1}}{(-iM)^2} + \frac{(n-m)(n-m-1)\sigma^{n-2}}{(-iM)^3} + \cdots \right.$$
$$\left. + (-)^{n-m-1} \frac{(n-m)!\,\sigma^{m+1}}{(-iM)^{n-m}} + (-)^{n-m} \frac{(n-m)!\,\sigma^m}{(-iM)^{n-m+1}} \right\} d\sigma \tag{179a}$$

Introduce the new notations

$$k^n J_m(k) \exp(-iMk) = I_m^n(M,k) \tag{180}$$

$$\int_0^k I_m^n(M,\sigma)\, d\sigma = L_m^n(M,k) \tag{181}$$

Then, Eq.(179a) becomes

$$L_m^n(M,\bar{k}) = -\frac{1}{iM} \left[I_m^n(M,\bar{k}) - L_{m-1}^n(M,\bar{k}) \right]$$
$$- \frac{(n-m)}{(-iM)^2} \left[I_m^{n-1}(M,\bar{k}) - L_{m-1}^{n-1}(M,\bar{k}) \right]$$
$$+ \frac{(n-m)(n-m-1)}{(-iM)^3} \left[I_m^{n-2}(M,\bar{k}) - L_{m-1}^{n-2}(M,\bar{k}) \right]$$
$$+ \cdots\cdots\cdots$$
$$+ (-)^{n-m-1} \frac{(n-m)!}{(-iM)^{n-m}} \left[I_m^{m+1}(M,\bar{k}) - L_{m-1}^{m+1}(M,\bar{k}) \right]$$
$$+ (-)^{n-m} \frac{(n-m)!}{(-iM)^{n-m+1}} \left[I_m^m(M,\bar{k}) - L_{m-1}^m(M,\bar{k}) \right] \tag{182}$$

Eq.(182) is a reduction formula for $L_m^n(M,\bar{k})$; by applying this reduction formula repeatedly, it is possible to express $L_m^n(M,\bar{k})$ in terms of $L_0^n(M,\bar{k})$, $L_0^{n-1}(M,\bar{k})$,....., $L_0^1(M,\bar{k})$. In Ref.(4), a recurrence relation which may be written as follows in the present notations, is given

$$\frac{\beta^2}{M} L_o^n(M,\bar{k}) = \left[i J_o(\bar{k}) - \frac{J_1(\bar{k})}{M} + (1-n)\frac{J_o(\bar{k})}{M\bar{k}} \right] \bar{k}^n \exp(-iM\bar{k})$$

$$+ i(1-2n) L_o^{n-1}(M,\bar{k}) + (1-n)^2 \frac{1}{M} L_o^{n-2}(M,\bar{k}) \qquad (183)$$

By means of Eq.(183), therefore, $L_o^n(M,\bar{k})$ can be reduced to $L_o^o(M,\bar{k})$. From Eqs.(54a) and (181), it is easily seen that

$$L_o^o(M,\bar{k}) = \frac{1}{M}\int_0^k \exp(-i\sigma) J_o\left(\frac{\sigma}{M}\right) d\sigma = \frac{1}{M} T(M,k) \qquad (184)$$

Therefore, by using Eqs.(182) and (183) repeatedly, it is possible to express $L_m^n(M,\bar{k})$ in terms of the T-functions. And, the definite integrals A,B,C,D and E can be evaluated analytically in terms of the T-functions.

(2) For numerical evaluation of the definite integrals by Euler-Maclaurin method, the following formulae are useful :

$$T_r(k) = \int_0^k \cos\sigma \, J_o\left(\frac{\sigma}{M}\right) d\sigma \qquad (185)$$

$$T_r'(k) = \frac{\partial}{\partial k} T_r(k) = \cos k \, J_o\left(\frac{k}{M}\right) \qquad (186)$$

$$T_r''(k) = \frac{\partial^2}{\partial k^2} T_r(k) = -\sin k \, J_o\left(\frac{k}{M}\right) - \frac{1}{M}\cos k \, J_1\left(\frac{k}{M}\right) \qquad (187)$$

$$T_r'''(k) = \frac{\partial^3}{\partial k^3} T_r(k)$$

$$= \cos k \left\{ -\left(1 + \frac{1}{2M^2}\right) J_o\left(\frac{k}{M}\right) + \frac{1}{2M^2} J_2\left(\frac{k}{M}\right) \right\} + \sin k \, \frac{2}{M} J_1\left(\frac{k}{M}\right) \qquad (188)$$

$$T_i(k) = \int_0^k \sin\sigma \, J_o\left(\frac{\sigma}{M}\right) d\sigma \qquad (189)$$

$$T_i'(k) = \frac{\partial}{\partial k} T_i(k) = \sin k \, J_o\left(\frac{k}{M}\right) \qquad (190)$$

$$T_i''(k) = \frac{\partial^2}{\partial k^2} T_i(k) = \cos k\, J_0\left(\frac{k}{M}\right) - \frac{1}{M} J_1\left(\frac{k}{M}\right) \sin k \tag{191}$$

$$T_i'''(k) = \frac{\partial^3}{\partial k^3} T_i(k)$$

$$= \sin k \left[-\left(1 + \frac{1}{2M^2}\right) J_0\left(\frac{k}{M}\right) + \frac{1}{2M^2} J_2\left(\frac{k}{M}\right)\right] - \cos k \frac{2}{M} J_1\left(\frac{k}{M}\right) \tag{192}$$

$$H_r k = T_r k + \frac{M^2}{2\beta^2}\left[k\left(\frac{1}{M} J_1\left(\frac{k}{M}\right)\cos k - J_0\left(\frac{k}{M}\right)\sin k\right) + T_i\right] \tag{193}$$

$$(H_r k)' = \frac{\partial}{\partial k}(H_r k)$$

$$= (T_r k)' + \frac{M^2}{2\beta^2}\left\{k \cos k\left[\left(\frac{1}{2M^2} - 1\right) J_0\left(\frac{k}{M}\right) - \frac{1}{2M^2} J_2\left(\frac{k}{M}\right)\right] + \frac{1}{M}\cos k\, J_1\left(\frac{k}{M}\right)\right\} \tag{194}$$

$$(H_r k)'' = \frac{\partial^2}{\partial k^2}(H_r k)$$

$$\begin{aligned} &= (T_r k)'' + \frac{M^2}{2\beta^2}\Big\{-k \sin k\left[\left(\frac{1}{2M^2} - 1\right) J_0\left(\frac{k}{M}\right) - \frac{1}{2M^2} J_2\left(\frac{k}{M}\right)\right] \\ &\qquad + k \cos k\left[\left(-\frac{3}{4M^3} + \frac{1}{M}\right) J_1\left(\frac{k}{M}\right) + \frac{1}{4M^3} J_3\left(\frac{k}{M}\right)\right] \\ &\qquad + \cos k\left[\left(\frac{1}{M^2} - 2\right) J_0\left(\frac{k}{M}\right) - \frac{1}{M^2} J_2\left(\frac{k}{M}\right)\right] + T_i''\Big\} \end{aligned} \tag{195}$$

$$(H_r k)''' = \frac{\partial^3}{\partial k^3}(H_r k)$$

$$\begin{aligned} &= (T_r k)''' + \frac{M^2}{2\beta^2}\Big\{k \cos k\left[\left(-\frac{3}{8M^4} + 1\right) J_0\left(\frac{k}{M}\right) + \frac{1}{2M^4} J_2\left(\frac{k}{M}\right) - \frac{1}{8M^4} J_4\left(\frac{k}{M}\right)\right] \\ &\qquad - k \sin k\left[\left(-\frac{3}{2M^3} + \frac{2}{M}\right) J_1\left(\frac{k}{M}\right) + \frac{1}{2M^3} J_3\left(\frac{k}{M}\right)\right] \\ &\qquad + \cos k\left[\left(-\frac{9}{4M^3} + \frac{3}{M}\right) J_1\left(\frac{k}{M}\right) + \frac{3}{4M^3} J_3\left(\frac{k}{M}\right)\right] \\ &\qquad + \sin k\left[\left(-\frac{3}{2M^2} + 3\right) J_0\left(\frac{k}{M}\right) + \frac{3}{2M^2} J_2\left(\frac{k}{M}\right)\right] + T_i'''\Big\} \end{aligned} \tag{196}$$

$$H_i k = T_i k + \frac{M^2}{2\beta^2}\left[k\left(\cos k\, J_0\left(\frac{k}{M}\right) + \frac{1}{M} J_1\left(\frac{k}{M}\right)\sin k\right) - T_r\right] \tag{197}$$

$$(H_i k)' = \frac{\partial}{\partial k}(H_i k)$$

$$= (T_i k)' + \frac{M^2}{2\beta^2}\left\{k \sin k\left[\left(\frac{1}{2M^2} - 1\right) J_0\left(\frac{k}{M}\right) - \frac{1}{2M^2} J_2\left(\frac{k}{M}\right)\right] + \frac{1}{M} J_1\left(\frac{k}{M}\right)\sin k\right\} \tag{198}$$

$$(H_i k)'' = \frac{\partial^2}{\partial k^2}(H_i k)$$

$$= (T_i k)'' + \frac{M^2}{2\beta^2}\Big\{ k \cos k \Big[\Big(\frac{1}{2M^2} - 1\Big) J_0\Big(\frac{k}{M}\Big) - \frac{1}{2M^2} J_2\Big(\frac{k}{M}\Big)\Big]$$

$$+ k \sin k \Big[\Big(-\frac{3}{4M^3} + \frac{1}{M}\Big) J_1\Big(\frac{k}{M}\Big) + \frac{1}{4M^3} J_3\Big(\frac{k}{M}\Big)\Big]$$

$$+ \sin k \Big[\Big(\frac{1}{M^2} - 2\Big) J_0\Big(\frac{k}{M}\Big) - \frac{1}{M^2} J_2\Big(\frac{k}{M}\Big)\Big] - T_r'' \Big\} \tag{199}$$

$$(H_i k)''' = \frac{\partial^3}{\partial k^3}(H_i k)$$

$$= (T_i k)''' + \frac{M^2}{2\beta^2}\Big\{ k \cos k \Big[\Big(-\frac{3}{2M^3} + \frac{2}{M}\Big) J_1\Big(\frac{k}{M}\Big) + \frac{1}{2M^3} J_3\Big(\frac{k}{M}\Big)\Big]$$

$$+ k \sin k \Big[\Big(-\frac{3}{8M^4} + 1\Big) J_0\Big(\frac{k}{M}\Big) + \frac{1}{2M^4} J_2\Big(\frac{k}{M}\Big) - \frac{1}{8M^4} J_4\Big(\frac{k}{M}\Big)\Big]$$

$$+ \cos k \Big[\Big(\frac{3}{2M^2} - 3\Big) J_0\Big(\frac{k}{M}\Big) - \frac{3}{2M^2} J_2\Big(\frac{k}{M}\Big)\Big]$$

$$+ \sin k \Big[\Big(-\frac{9}{4M^3} + \frac{3}{M}\Big) J_1\Big(\frac{k}{M}\Big) + \frac{3}{4M^3} J_3\Big(\frac{k}{M}\Big)\Big] - T_r''' \Big\} \tag{200}$$

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TABLE ONE

Comparison of the values of T_r and T_i computed from Eqs. (55) and (56) with those given by Schwarz $M = 2.00$

Present theory .....with [ ];

Schwarz's results...without [ ].

k	$[T_r]$	T_r	$[T_i]$	T_i	$\frac{[T_r]-T_r}{T_r} \times 100$	$\frac{[T_i]-T_i}{T_i} \times 100$
0	0	0	0	0	0	0
.02	.019999	.019999	.000200	.000200	0	0
.06	.059960	.059960	.001799	.001799	0	0
.10	.099813	.099813	.004994	.004994	0	0
.14	.139486	.139486	.009778	.009778	0	0
.20	.198505	.198505	.019909	.019909	0	0
.40	.388143	.388150	.078562	.078547	-.0018%	.019%
.60	.560507	.560631	.172878	.172727	-.022%	.080%
.80	.707857	.708710	.298057	.297377	-.120%	.229%
1.00	.822892	.826654	.447598	.445887	-.455%	.384%

TABLE TWO

Values of T_r, T_i, H_r and H_i $M = 10/7$

k	T_r	T_i	H_r	H_i
0	0	0	0	0
.2	.198347	.019885	.148969	.013263
.4	.386936	.078171	.291826	.052236
.6	.556806	.170880	.422936	.114520
.8	.700504	.291770	.537539	.196356
1.0	.812672	.432895	.632103	.292925
1.2	.890424	.585292	.704556	.398756
1.4	.933505	.739762	.754390	.508171
1.6	.944205	.887636	.782624	.615732
1.8	.927046	1.021494	.792617	.717270
2.0	.888280	1.135727	.784901	.807216

TABLE TWO (Cont'd)

Values of T_r, T_i, H_r and H_i $M=2.00$

k	T_r	T_i	H_r	H_i
0	0	0	0	0
.2	.198505	.019909	.149067	.013278
.4	.388150	.078547	.292588	.052462
.6	.560631	.172727	.425348	.115633
.8	.708710	.297377	.542750	.199736
1.0	.826654	.445887	.641081	.300776
1.2	.910564	.610554	.717708	.414073
1.4	.958569	.783093	.771190	.534552
1.6	.970889	.955180	.801340	.657064
1.8	.949737	1.118987	.809162	.776697
2.0	.899092	1.267665	.796744	.889057

TABLE THREE

Values of H_r and H_i computed from Eqs.(176) and (177)
$M=2.00$

k	H_r	H_i	k	H_r	H_i
0	0	0	.20	.149067	.013278
.02	.0149991	.0001333	.40	.292588	.052462
.06	.0449747	.0011996	.60	.425348	.115633
.10	.0748803	.0033292	.80	.542750	.199736
.14	.1046795	.0065204	1.00	.641081	.300776

TABLE FOUR

Values of A_r and A_i
$M=10/7$

k	A_r	A_i	k	A_r	A_i
0	0	0	1.2	.622427	.254666
.4	.078686	.010520	1.6	.993665	.550127
.8	.299658	.080755	2.0	1.363040	.957411

TABLE FOUR (Cont'd)

Values of A_r and A_i M=2.00

k	A_r	A_i	k	A_r	A_i
0	0	0	1.2	.629820	.261114
.4	.078810	.010551	1.6	1.010864	.574316
.8	.301432	.081676	2.0	1.388790	1.020898

TABLE FIVE

Values of B_r and B_i

M=10/7 M=2.00

k	B_r	B_i	B_r	B_i
0	0	0	0	0
.4	.020913	.003151	.020954	.003161
.8	.157690	.048153	.158813	.048766
1.2	.482995	.225995	.489897	.232394
1.6	1.003439	.643687	1.024169	.675488
2.0	1.667556	1.380115	1.703460	1.483513

TABLE SIX

Values of C_r and C_i

M=10/7 M=2.00

k	C_r	C_i	C_r	C_i
0	0	0	0	0
.4	.015738	.002104	.015762	.002110
.8	.119864	.032302	.120573	.032671
1.2	.373458	.152800	.377896	.156673
1.6	.794933	.440103	.808676	.454124
2.0	1.363395	.957628	1.387520	1.023606

TABLE SEVEN

Values of D_r and D_i

	M=10/7		M=2.00	
k	D_r	D_i	D_r	D_i
0	0	0	0	0
.4	.004713	.000672	.004719	.000675
.8	.071389	.020590	.071855	.020843
1.2	.330549	.145374	.334901	.149337
1.6	.926057	.554368	.943622	.572106
2.0	1.953527	1.494327	1.989454	1.606362

TABLE EIGHT

Values of E_r and E_i

	M=10/7		M=2.00	
k	E_r	E_i	E_r	E_i
0	0	0	0	0
.4	.006261	.001007	.006261	.001010
.8	.093770	.030679	.094513	.031099
1.2	.425890	.214779	.432650	.221322
1.6	1.160435	.809149	1.186811	.852285
2.0	2.359391	2.146106	2.412818	2.320199

TABLE NINE

Values of the lift coefficients C_{L_0} and C_{L_1}
M=10/7

k	$(C_{L_0}/C_{L_0}^*)_r$	$(C_{L_0}/C_{L_0}^*)_i$	$(C_{L_1}/C_{L_1}^*)_r$	$(C_{L_1}/C_{L_1}^*)_i$
0	1.000000	0	1.000000	0
.4	.980753	-.095103	.977930	-.058710
.8	.927111	-.173680	.917796	-.094990
1.2	.850253	-.223212	.836024	-.092714
1.6	.765481	-.238042	.754666	-.046296
2.0	.688280	-.220289	.695152	.038310

TABLE NINE (Cont'd)

Values of the lift coefficients C_{L_0} and C_{L_1}
M=2.00

k	$(C_{L_0}/C_{L_0}^*)_r$	$(C_{L_0}/C_{L_0}^*)_i$	$(C_{L_1}/C_{L_1}^*)_r$	$(C_{L_1}/C_{L_1}^*)_i$
0	1.000000	0	1.000000	0
.4	.990158	-.048598	.995200	.034710
.8	.962459	-.089129	.982946	.076986
1.2	.922000	-.115157	.969264	.132356
1.6	.876016	-.123145	.956052	.202376
2.0	.832383	-.113036	.972402	.283424

TABLE TEN

Values of the moment coefficients C_{M_0} and C_{M_1}
M=10/7

k	$(C_{M_0}/C_{M_0}^*)_r$	$(C_{M_0}/C_{M_0}^*)_i$	$(C_{M_1}/C_{M_1}^*)_r$	$(C_{M_1}/C_{M_1}^*)_i$
0	1.000000	0	1.00	0
.4	.971190	-.126030	.973515	-.065505
.8	.891572	-.225748	.902019	-.103185
1.2	.779638	-.279664	.806421	-.093174
1.6	.660424	-.279950	.714660	-.029583
2.0	.558689	-.231794	.653070	+.078885

M=2.00

k	$(C_{M_0}/C_{M_0}^*)_r$	$(C_{M_0}/C_{M_0}^*)_i$	$(C_{M_1}/C_{M_1}^*)_r$	$(C_{M_1}/C_{M_1}^*)_i$
0	1.000000	0	1.000000	0
.4	.985260	-.064396	.994125	.039015
.8	.944098	-.115986	.979755	.087855
1.2	.884934	-.144622	.964269	.152700
1.6	.819668	-.145192	.948729	.227412
2.0	.761014	-.118418	.975576	.332238

TABLE ELEVEN

Lift coefficient of the rectangular wing, C_{Lw}

M=10/7

AR=1/β | AR=1

k	$(C_{Lw}/C_{Lo})_r$	$(C_{Lw}/C_{Lo})_i$	$(C_{Lw}/C_{Lo})_r$	$(C_{Lw}/C_{Lo})_i$
0	.500000	0	.509902	0
.4	.503422	.050418	.513256	.049420
.8	.514336	.101284	.523954	.099278
1.2	.534715	.152248	.543930	.149233
1.6	.567663	.201032	.576225	.197051
2.0	.615924	.241677	.623530	.236891

AR = 3 | AR = 5

k	$(C_{Lw}/C_{Lo})_r$	$(C_{Lw}/C_{Lo})_i$	$(C_{Lw}/C_{Lo})_r$	$(C_{Lw}/C_{Lo})_i$
0	.836634	0	.901980	0
.4	.837752	.016473	.902651	.009884
.8	.841318	.033093	.904791	.019856
1.2	.847977	.049744	.908786	.029847
1.6	.858742	.065684	.915245	.039410
2.0	.874510	.078964	.924706	.047378

M = 2.00

AR = 1/β | AR = 1

k	$(C_{Lw}/C_{Lo})_r$	$(C_{Lw}/C_{Lo})_i$	$(C_{Lw}/C_{Lo})_r$	$(C_{Lw}/C_{Lo})_i$
0	.500000	0	.711325	0
.4	.504634	.058250	.714000	.033631
.8	.518981	.115958	.722284	.066948
1.2	.544384	.171726	.736950	.099146
1.6	.578792	.222041	.756816	.128195
2.0	.634156	.262832	.788780	.151746

TABLE ELEVEN (Cont'd)

Lift coefficient of the rectangular wing, C_{Lw}

M = 2.00

k	AR = 3 $(C_{Lw}/C_{Lo})_r$	AR = 3 $(C_{Lw}/C_{Lo})_i$	AR = 5 $(C_{Lw}/C_{Lo})_r$	AR = 5 $(C_{Lw}/C_{Lo})_i$
0	.903775	0	.942265	0
.4	.904667	.011210	.942800	.006726
.8	.907428	.022316	.944457	.013390
1.2	.912317	.033049	.947390	.019829
1.6	.918939	.042732	.951363	.025639
2.0	.929593	.050582	.957756	.030349

TABLE TWELVE

Moment coefficient of the rectangular wing, C_{Mw}

M = 10/7

k	AR = 1/β $(C_{Mw}/C_{Mo})_r$	AR = 1/β $(C_{Mw}/C_{Mo})_i$	AR = 1 $(C_{Mw}/C_{Mo})_r$	AR = 1 $(C_{Mw}/C_{Mo})_i$
0	.333333	0	.346536	0
.4	.334598	.062541	.347776	.061302
.8	.339791	.128996	.352863	.126441
1.2	.341073	.199478	.354122	.283717
1.6	.381829	.289449	.394071	.283717
2.0	.445849	.382746	.456823	.375166

k	AR = 3 $(C_{Mw}/C_{Mo})_r$	AR = 3 $(C_{Mw}/C_{Mo})_i$	AR = 5 $(C_{Mw}/C_{Mo})_r$	AR = 5 $(C_{Mw}/C_{Mo})_i$
0	.782179	0	.869307	0
.4	.782592	.020434	.869555	.012260
.8	.784288	.042147	.870573	.025288
1.2	.784707	.065176	.870824	.039106
1.6	.798024	.094572	.878814	.056743
2.0	.818941	.125055	.891365	.075033

TABLE TWELVE (Cont'd)

Moment coefficient of the rectangular wing, C_{M_w}

$M = 2.00$

	$AR = 1/\beta$		$AR = 1$	
k	$(C_{M_w}/C_{M_o})_r$	$(C_{M_w}/C_{M_o})_i$	$(C_{M_w}/C_{M_o})_r$	$(C_{M_w}/C_{M_o})_i$
0	.333333	0	.615100	0
.4	.338132	.081429	.617871	.047013
.8	.353718	.163898	.626869	.094627
1.2	.384502	.249667	.644642	.144145
1.6	.431251	.327477	.671633	.189069
2.0	.523049	.410011	.724632	.236720

	$AR = 3$		$AR = 5$	
k	$(C_{M_w}/C_{M_o})_r$	$(C_{M_w}/C_{M_o})_i$	$(C_{M_w}/C_{M_o})_r$	$(C_{M_w}/C_{M_o})_i$
0	.871700	0	.923020	0
.4	.872624	.015671	.923574	.009403
.8	.875623	.031542	.925374	.018925
1.2	.881547	.048048	.928928	.028829
1.6	.890544	.063023	.934327	.037814
2.0	.908211	.078907	.944926	.047344

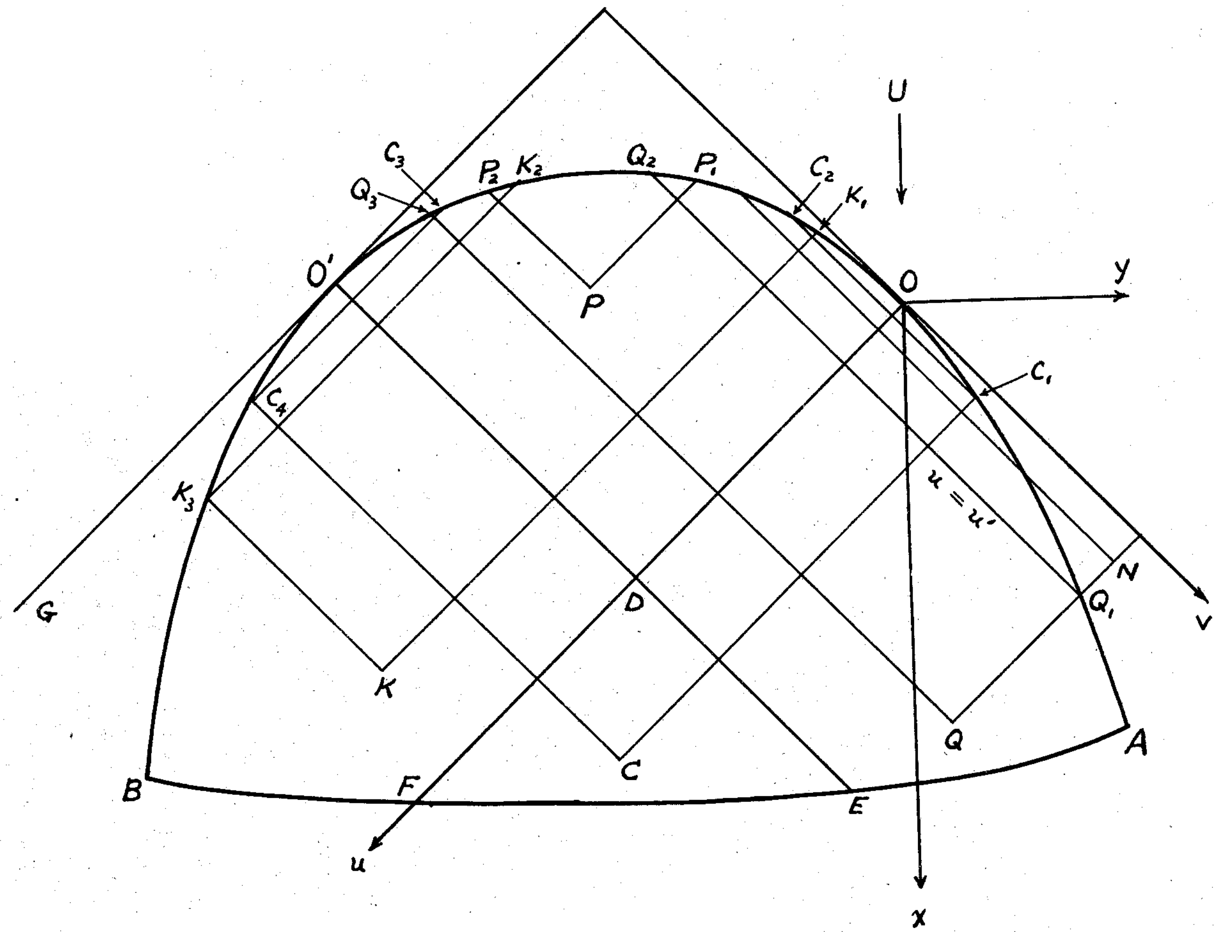


FIGURE 1

PLANFORM OF A GENERAL WING

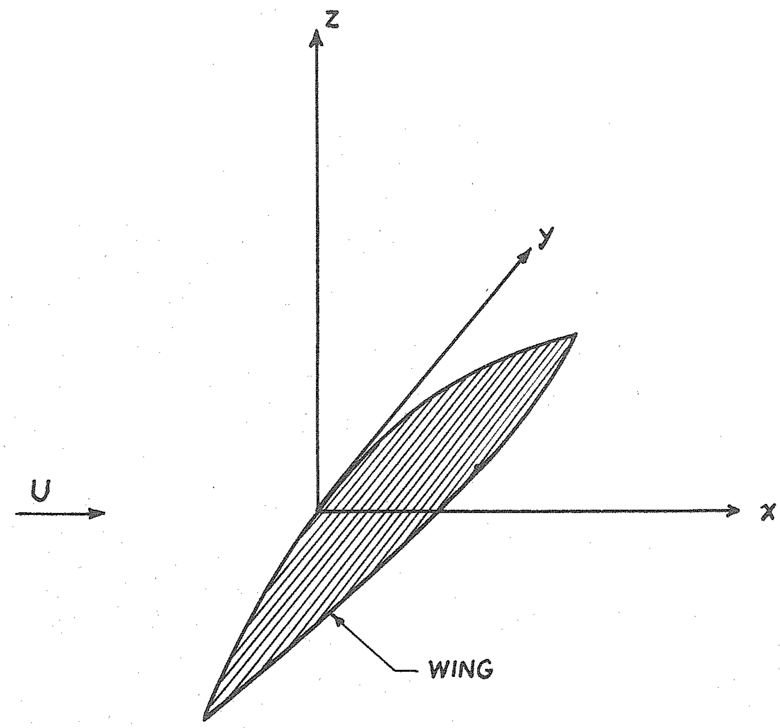


FIGURE 2

COORDINATE SYSTEM

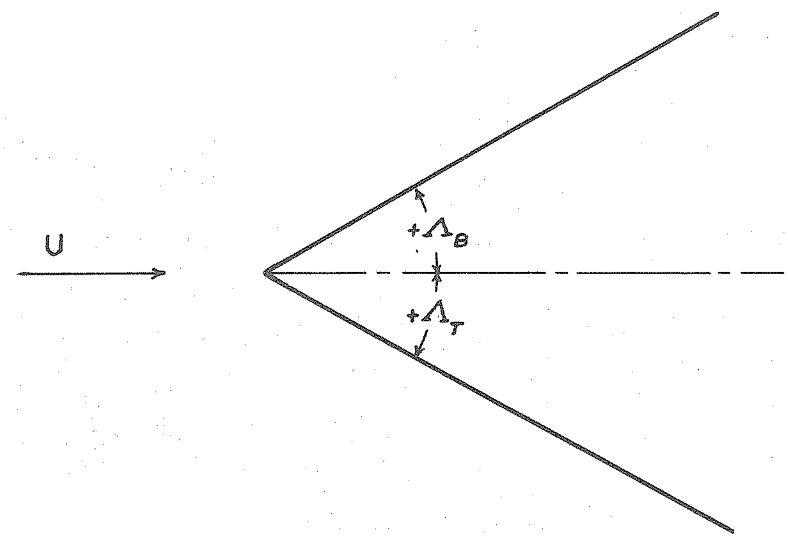
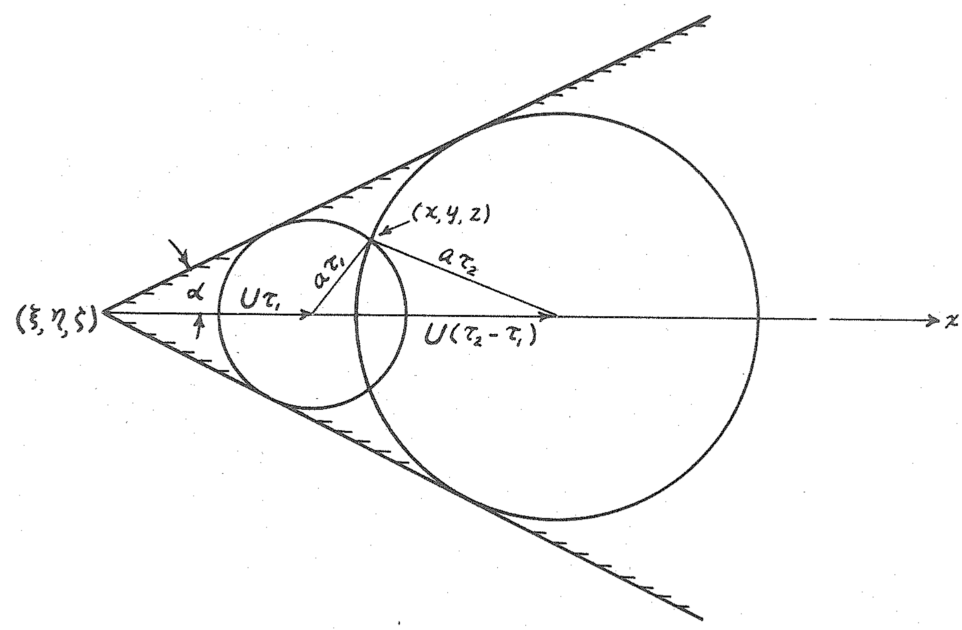


FIGURE 3

SIGN CONVENTION OF Λ's



$$\alpha = \sin^{-1} \frac{a}{U}$$

FIGURE 4

REGION OF INFLUENCE

OF SOURCE AT (ξ, η, ζ) AT INSTANT t

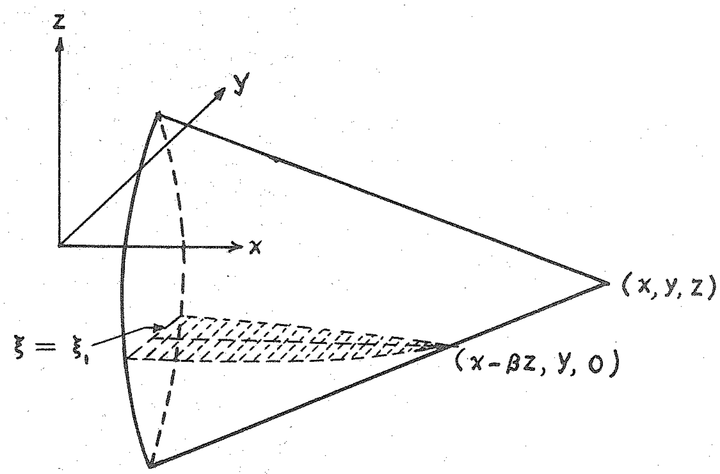


FIGURE 5

SINGULARITIES OR SOURCES IN x,y PLANE

THAT AFFECT CONDITIONS AT (x,y,z) AT INSTANT t

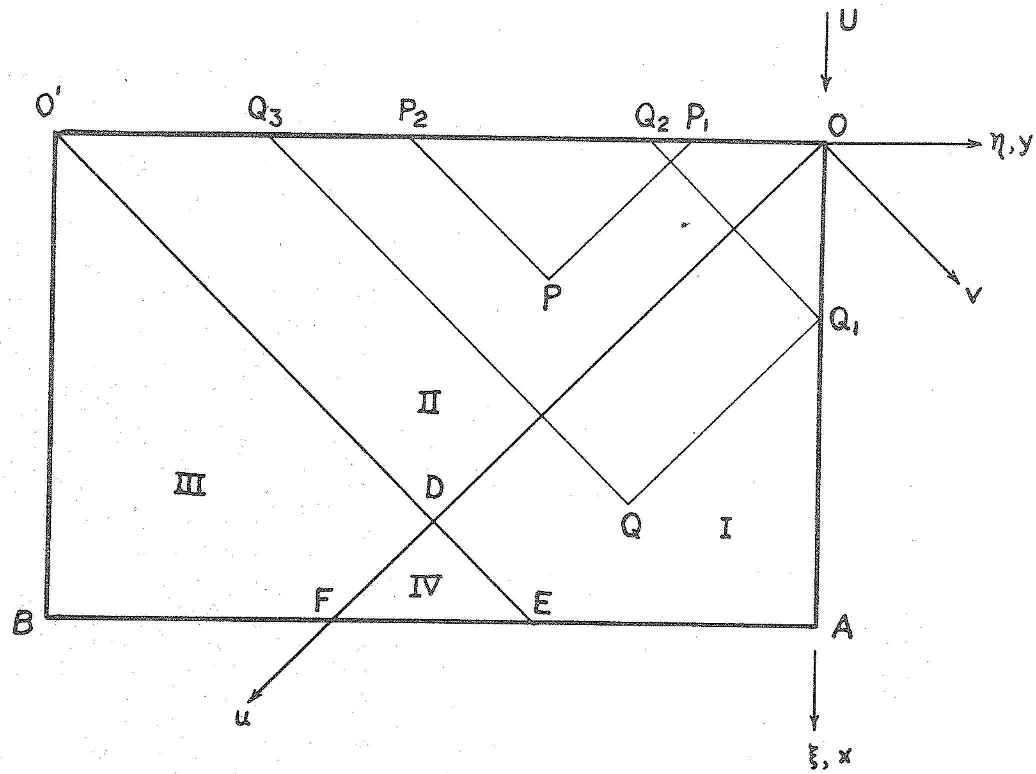


FIGURE 6

PLANFORM OF A RECTANGULAR WING, $1 \leq \beta AR < 2$

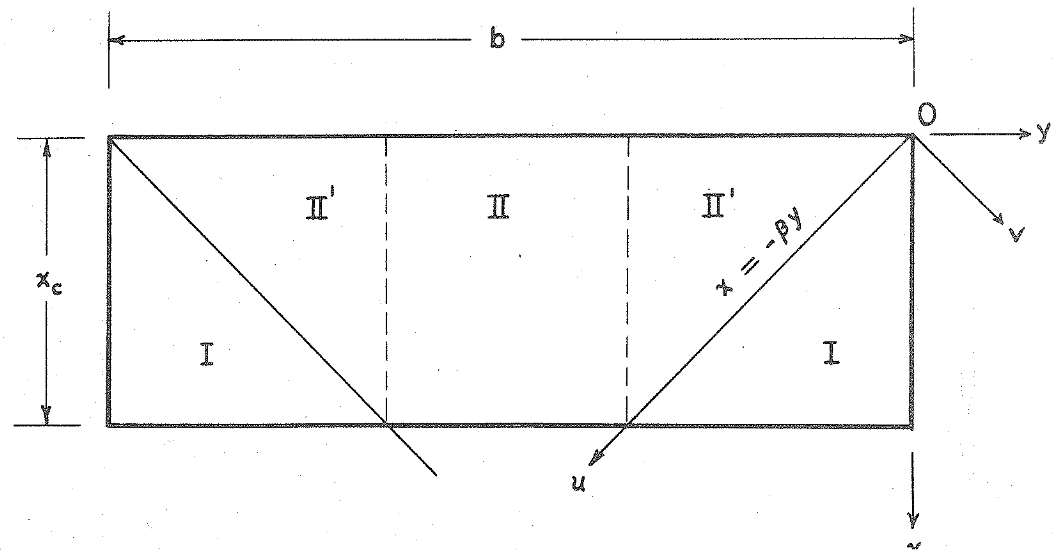


FIGURE 7

PLANFORM OF A RECTANGULAR WING, $2 < \beta AR < \infty$

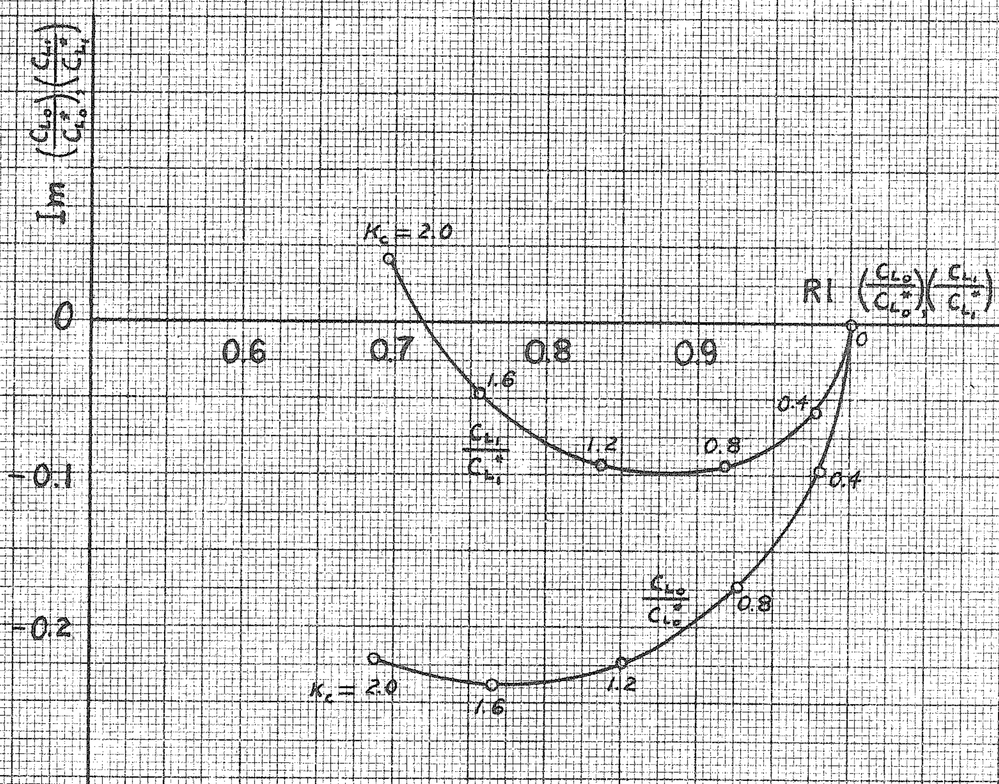


FIGURE 8

VECTOR DIAGRAM REPRESENTING C_{L_0} AND C_{L_1}

OF A RECTANGULAR OSCILLATING WING AT M=10/7

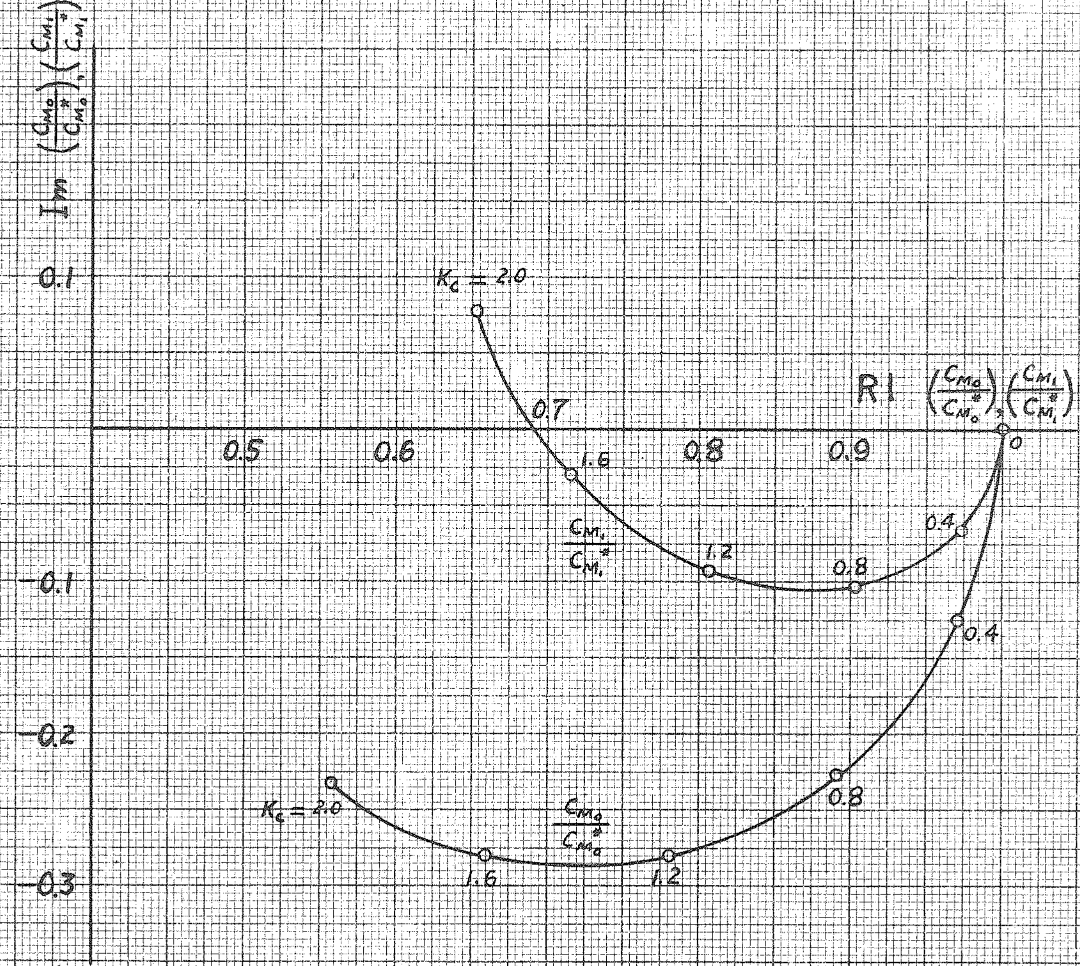


FIGURE 9

VECTOR DIAGRAM REPRESENTING C_{M_0} AND C_{M_1}

OF A RECTANGULAR OSCILLATING WING AT M=10/7

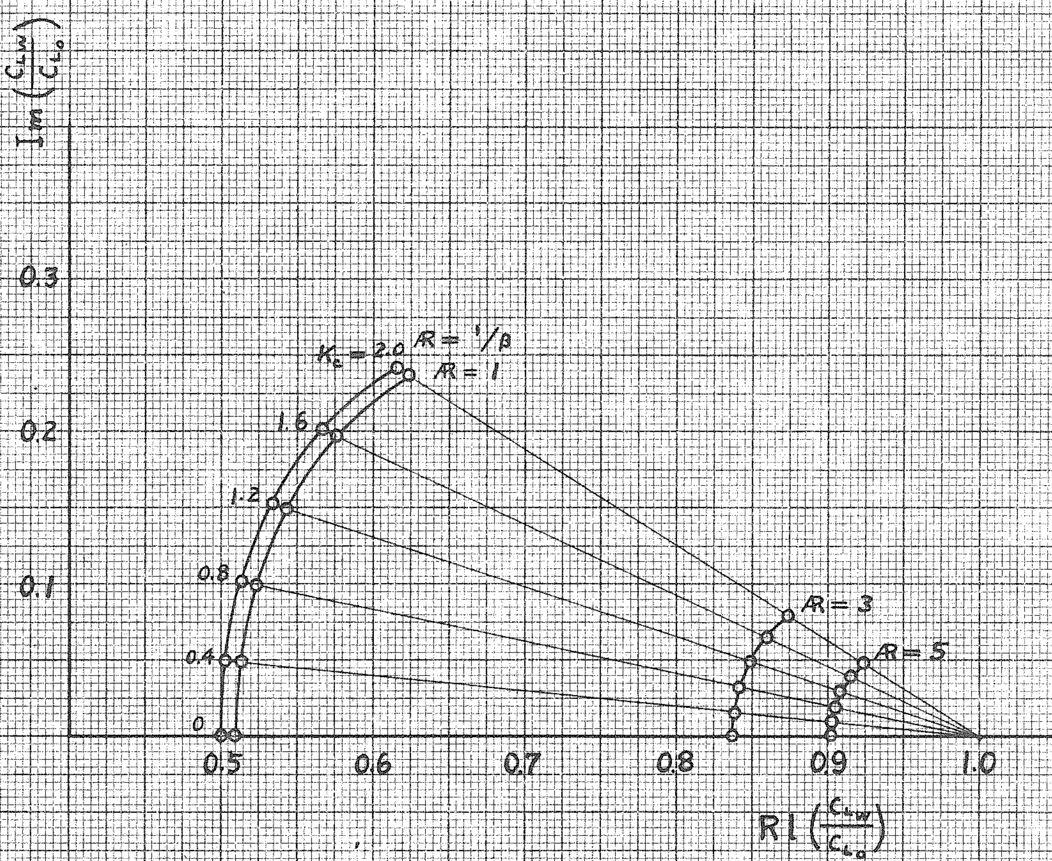


FIGURE 10

AERODYNAMIC EFFICIENCY OF A RECTANGULAR OSCILLATING WING AT M=10/7, (A) C_{L_W}/C_{L_0}

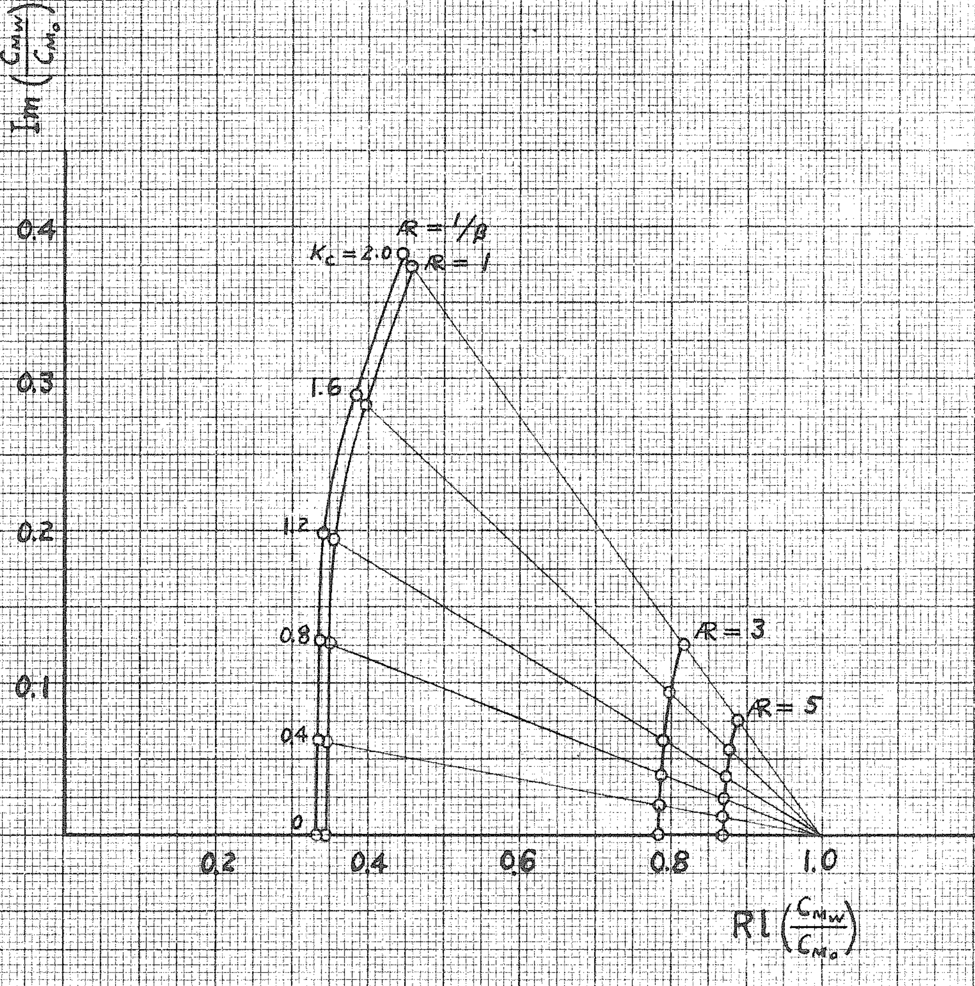


FIGURE 11

AERODYNAMIC EFFICIENCY OF A RECTANGULAR OSCILLATING WING AT M=10/7, (B) C_{M_W}/C_{M_o}

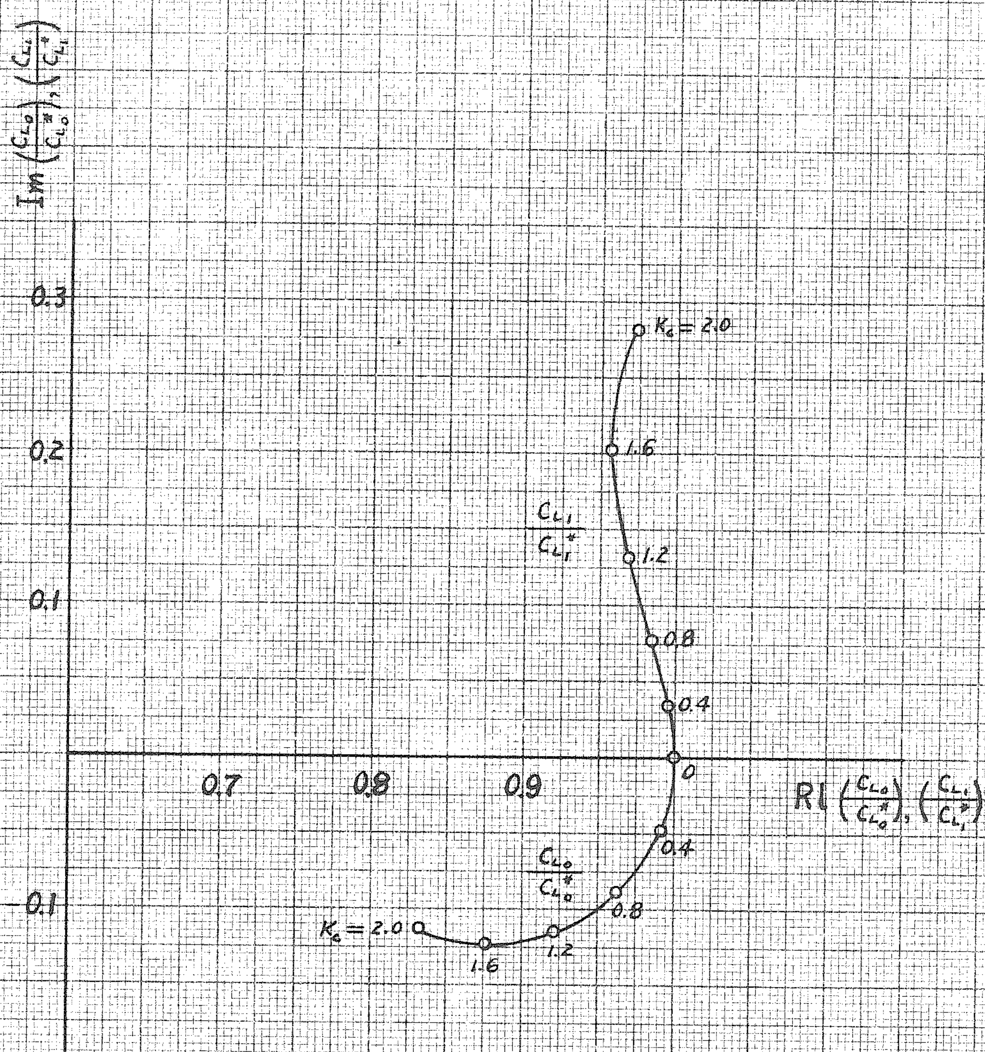


FIGURE 12

VECTOR DIAGRAM REPRESENTING C_{L_0} AND C_{L_1}

OF A RECTANGULAR OSCILLATING WING AT M=2.00

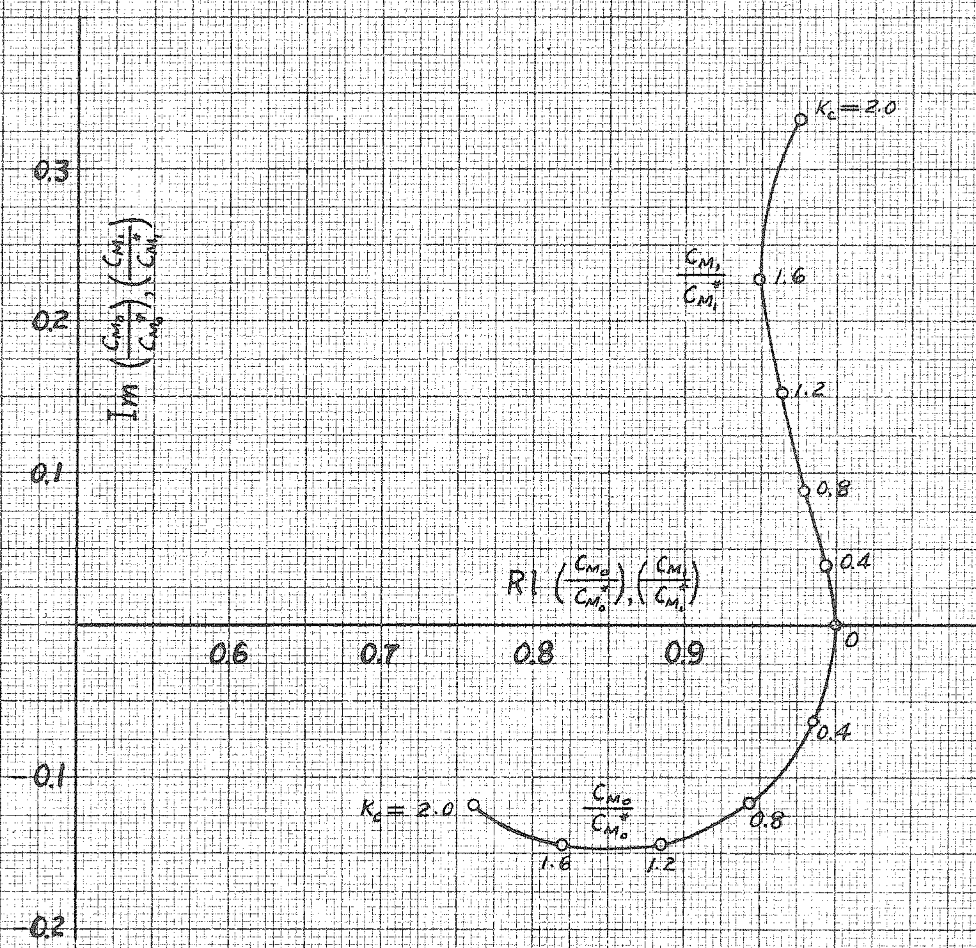


FIGURE 13

VECTOR DIAGRAM REPRESENTING C_{M_0} AND C_{M_1}

OF A RECTANGULAR OSCILLATING WING AT M=2.00

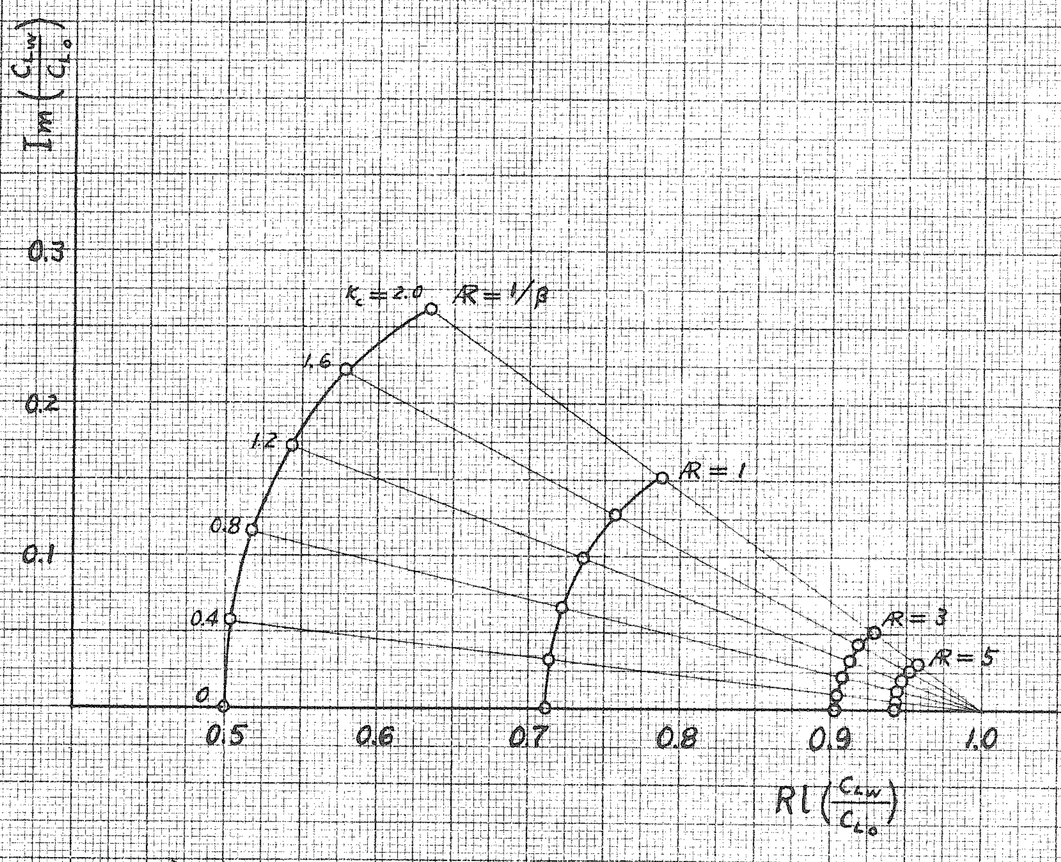


FIGURE 14

AERODYNAMIC EFFICIENCY OF A RECTANGULAR OSCILLATING WING AT M=2.00, (A) C_{L_W}/C_{L_0}

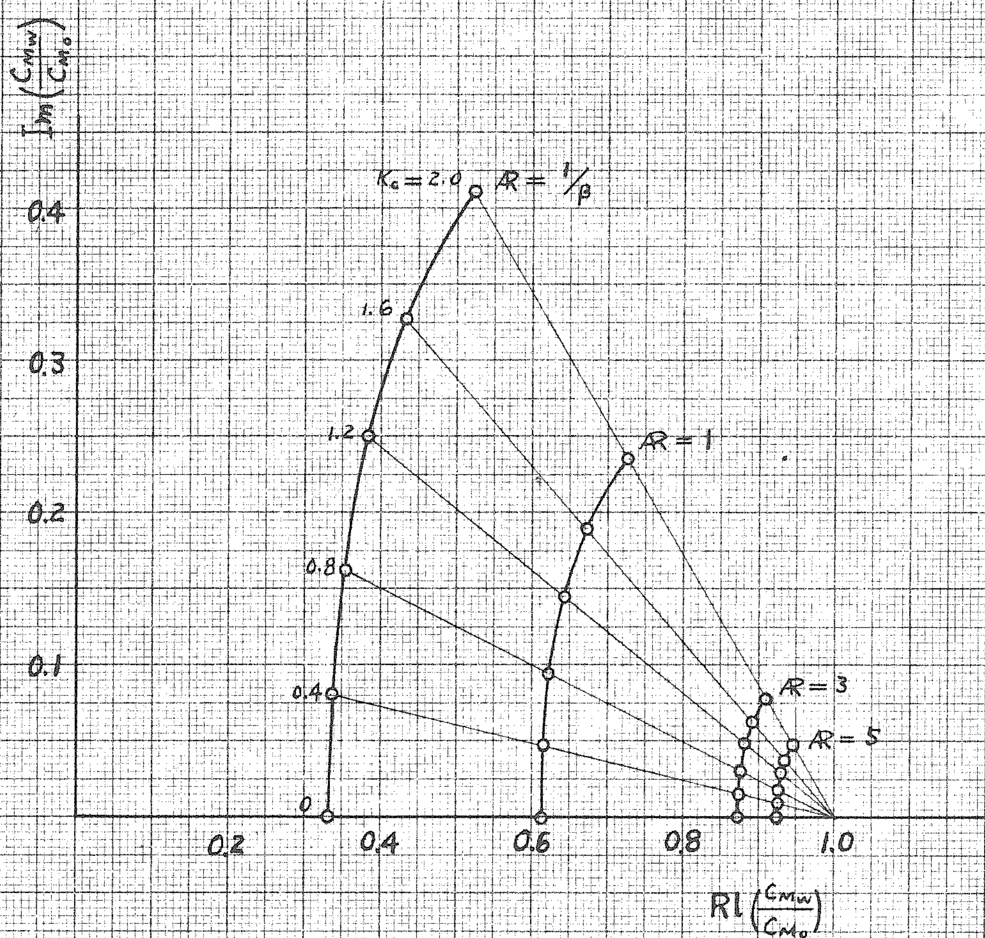


FIGURE 15

AERODYNAMIC EFFICIENCY OF A RECTANGULAR OSCILLATING WING AT M=2.00, (B) C_{M_W}/C_{M_0}

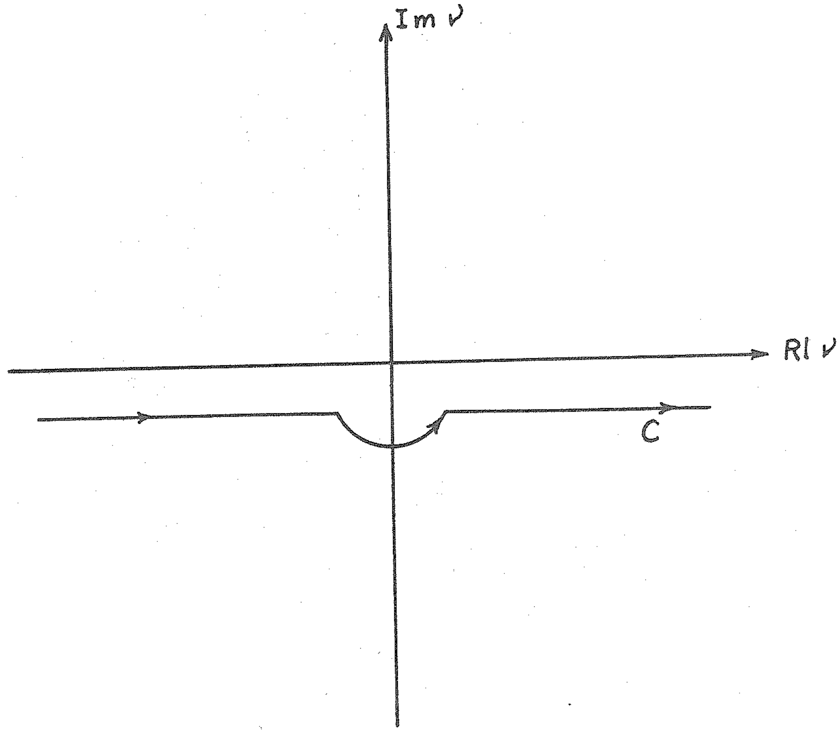


FIGURE 16

CONTOUR OF INTEGRATION IN THE ν-PLANE

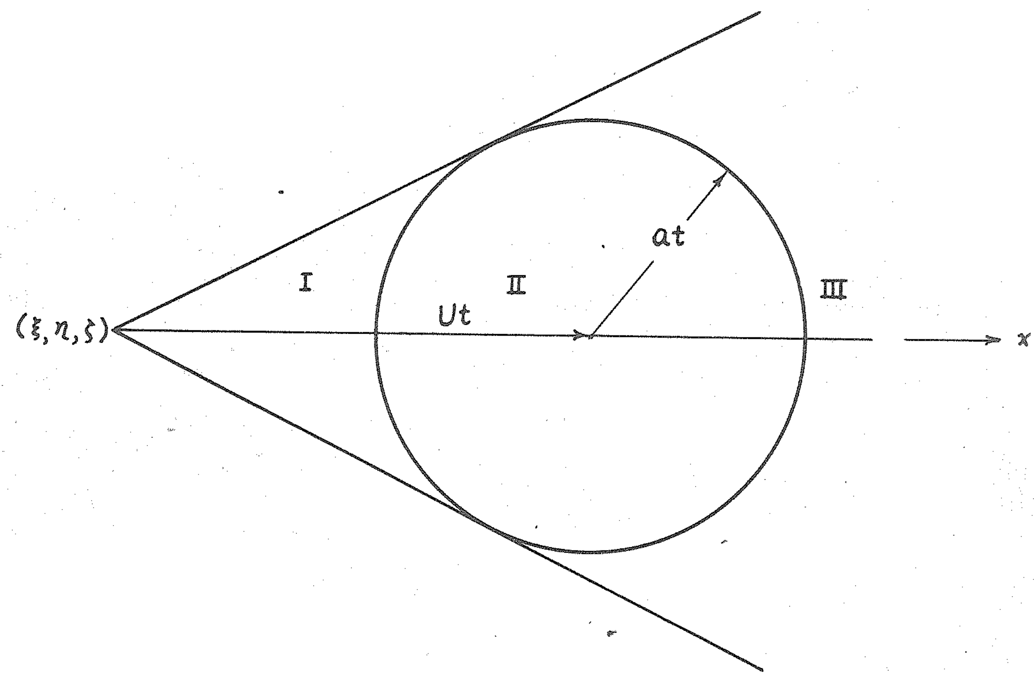


FIGURE 17

REGION OF INFLUENCE

OF AN "UNIT-STEP" SOURCE AT (ξ, η, ζ) AT AN INSTANT t

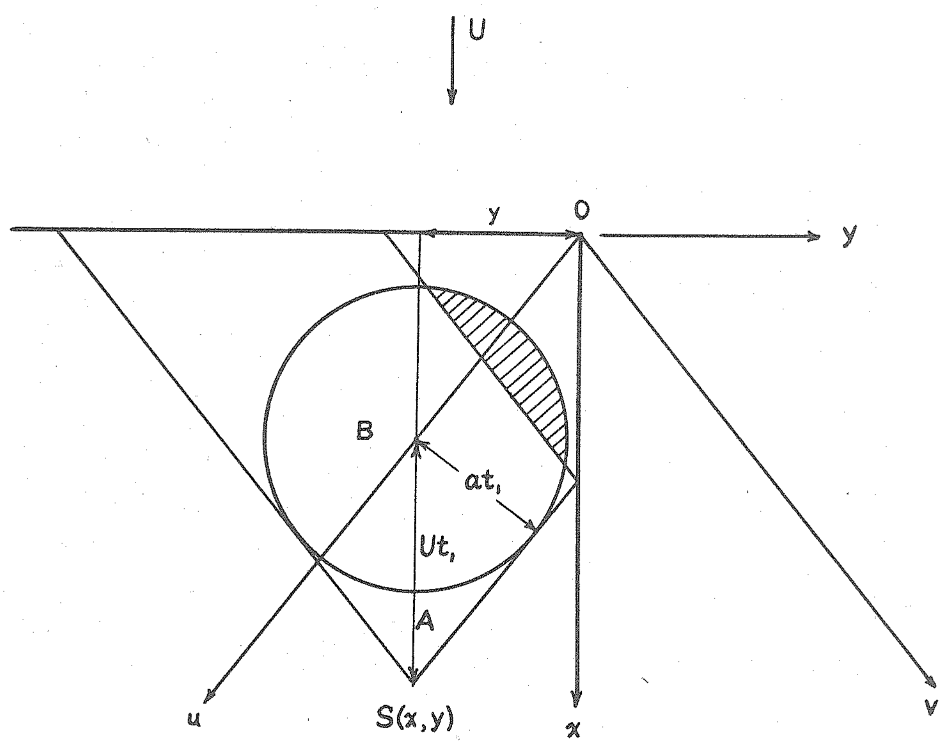


FIGURE 18

THE WING TIP REGION OF A RECTANGULAR FLAT

PLATE PERFORMING "UNIT-STEP" MOTION AT INSTANT t_1