LEAST SQUARE POLYNOMIAL SPLINE APPROXIMATION

Thesis by

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In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1972

(Submitted May 16, 1972)

ACKNOWLEDGEMENT

I wish to thank Professor Robert P. Dilworth for his interest in my general well being as a graduate student, Professor Martin H. Schultz for suggesting my thesis topic and many fruitful ideas concerning its investigation, and Professor Herbert B. Keller for his guidance in the preparation of the thesis itself.

I greatly appreciate the financial support which I have received from California Institute of Technology, the U. S. Office of Education through a National Defense Education Act Fellowship, and the Naval Undersea Research and Development Center through an Advanced Graduate Fellowship.

Special thanks go to my wife and daughters, Suzanne, Rachel and Sarah, for their love, patience, and encouragement.

ABSTRACT

Bounds are derived for both the L^2- and L^∞-norms of the error in approximating sufficiently smooth functions by polynomial splines using an integral least square technique based on the theory of orthogonal projection in real Hilbert space. Quadrature schemes for the approximate solution of this least square problem are examined and bounds for the error due to the use of such schemes are derived. The question of the consistency of such quadrature schemes with the least square error is investigated and asymptotic results are presented. Numerical results are also included.

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INTRODUCTION

In this paper we consider polynomial spline approximation techniques based on the theory of orthogonal projection in real Hilbert space. Splines are used as the approximations since they have smoothness properties and have been used to interpolate to large classes of smooth functions with small errors. In addition, with the proper choice of basis functions, splines give rise to bounded well-conditioned matrices without orthonormalization. The motivation for the use of an integral least square technique is the hope that it might generate approximations which would smooth errors due to "noisy" data. Interpolation techniques should be avoided, if possible, when attempting to approximate such data sets.

We begin, in Section 1, with a discussion of a least square approximation theory for finite dimensional subspaces of real Hilbert space. In Section 2, we confine our attention to the Hilbert space $L^2[0,1]$ with inner product defined for $g, h \in L^2[0,1]$ by

$$(g,h) \equiv \int_0^1 g(x)\, h(x)\, dx.$$

We consider the finite dimensional subspaces to be spaces of polynomial spline functions. Standard spline interpolation results are given [14] and then used to bound the L^2-norms of the least square error and its derivatives. We then employ the L^2-bounds to derive L^∞-bounds for the same error functions using a Sobolev type inequality. We discuss the application of techniques based on this least square theory to empirical data, i.e., tabulated functions. In Sections 3 and 4, we discuss, in detail, the use of certain types of quadrature schemes for the approximate solution of the least square spline approximation problem for a tabulated function. We derive bounds for the error introduced into the least square spline approximation by the use of these quadrature schemes. We then investigate the consistency of the orders of accuracy of the discretized techniques with the order of accuracy of the true integral least square technique. Asympototic results are presented throughout Sections 2, 3, and 4 wherever appropriate. Finally, in Section 5, we present some numerical results based on programs coded in FORTRAN implementing some of the techniques discussed here as well as the standard discrete least squares technique. These techniques are compared in their effectiveness in approximating by polynomial splines the exponential function as well as several discrete data sets of particular interest.

1. THE LEAST SQUARE PROBLEM IN REAL HILBERT SPACE

In this section we formulate the least square problem in real Hilbert space. We demonstrate the equivalence of this problem to that of solving an appropriate linear system of equations. The matrix involved is shown to be positive definite and symmetric thereby guaranteeing that a unique solution to the system of equations exists. We then conclude that a unique solution to the least square problem always exists. Finally, we discuss the context in which these concepts are to be employed in this paper.

Let H be a real Hilbert space with the inner product of two elements $f, g \in H$ denoted by (f, g). This inner product satisfies the following properties for all $f, g, h \in H$ and any real number α.

$$\begin{aligned} &\text{i)} & (\alpha f, g) &= \alpha(f, g) \\ &\text{ii)} & (f + g, h) &= (f, h) + (g, h) \\ &\text{iii)} & (f, g) &= (g, f) \\ &\text{iv)} & (f, f) &> 0 \text{ for } f \neq 0 \\ &\text{v)} & (f, f) &= 0 \text{ for } f = 0. \end{aligned} \tag{1.1}$$

Then $||f|| \equiv (f, f)^{1/2}$ defines a norm on H and $d(f, g) \equiv ||f - g||$ defines a metric on H.

Let G be any finite dimensional subspace of H. Then, given any $f \epsilon H$, we wish to find an element $g \epsilon G$ which minimizes $d(f, g) = ||f - g||$. We call this problem the least square problem for $f \in H$ associated with the finite dimensional subspace G of H.

Suppose that the elements $g_1, g_2, \ldots, g_n$ form a basis for G. Let $\underline{\alpha} \equiv (\alpha_1, \ldots, \alpha_n)$ and define

$$F(\underline{\alpha}) = F(\alpha_1, \ldots, \alpha_n) \equiv \left\| f - \sum_{i=1}^{n} \alpha_i g_i \right\|^2 \tag{1.2}$$

Clearly, our original problem is equivalent to finding an n-tuple $\underline{\hat{\alpha}} = (\hat{\alpha}_1, \ldots, \hat{\alpha}_n)$ which minimizes $F(\underline{\alpha})$. Using the definition of the norm $||\cdot||$ and the properties of the inner product $(\cdot, \cdot)$, we find that

$$F(\underline{\alpha}) = F(\alpha_1, \ldots, \alpha_n) = \left(f - \sum_{i=1}^{n} \alpha_i g_i, \; f - \sum_{i=1}^{n} \alpha_i g_i \right) =$$

$$(f, f) - 2 \sum_{i=1}^{n} \alpha_i (f, g_i) + \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j (g_i, g_j)$$

is a quadratic function of the α_i, $1 \le i \le n$. Consequently, the partial derivatives of F with respect to the α_i, $1 \le i \le n$, evaluated at such a minimum must equal zero, i.e., for $1 \le i \le n$

$$\frac{\partial F}{\partial \alpha_i}(\underline{\hat{\alpha}}) = -2(f, g_i) + 2 \sum_{j=1}^{n} \hat{\alpha}_j (g_i, g_j) = 0. \tag{1.3}$$

We shall write this system of equations, known as the normal equations, as

$$A\underline{\alpha} - \underline{\hat{k}} = 0 \tag{1.4}$$

where the entries, a_{ij}, of A and the components, $\hat{k}_i$, of $\underline{\hat{k}}$ are defined by

$$a_{ij} \equiv (g_i, g_j), \ \hat{k}_i \equiv (f, g_i), \ \text{for } 1 \le i,j \le n. \tag{1.5}$$

Of course the matrix A is the well known Gram matrix or Gramian of the elements $g_1, \ldots, g_n$ of H.

The matrix A is symmetric since $(g_i, g_j) = (g_j, g_i)$ for all i, j by property (iii) of the inner product. Now given any n-tuple $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$

$$\begin{aligned} \underline{\alpha}^T A \underline{\alpha} &= \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j (g_i, g_j) \\ &= \left(\sum_{i=1}^{n} \alpha_i g_i, \sum_{j=1}^{n} \alpha_j g_j\right) \\ &= \left\|\sum_{i=1}^{n} \alpha_i g_i\right\|^2 \ge 0. \end{aligned} \tag{1.6}$$

But equality in (1.6) implies that

$$\left\|\sum_{i=1}^{n} \alpha_i g_i\right\| = 0.$$

Applying property (v) of the inner product yields

$$\sum_{i=1}^{n} \alpha_i g_i = 0$$

and, consequently, $\alpha_i = 0$ all i, since the g_i, $1 \le i \le n$, are linearly independent. So $\underline{\alpha}^T A \underline{\alpha} \ge 0$ with equality only if $\underline{\alpha} \equiv 0$. Therefore, A is also positive definite. But then A has a unique inverse, A^{-1}, and $\underline{\hat{\alpha}} = A^{-1}\hat{\underline{k}}$ is the unique solution to the system (1.4). Therefore,

$$\hat{g} = \sum_{i=1}^{n} \hat{\alpha}_i g_i$$

is the unique solution to the least square problem for $f \in H$ associated with the finite dimensional subspace $G = \text{span}\{g_1, \ldots g_n\}$ of H. This completes the proof of the following well known theorem which is true, in fact, for any closed subspace G of H and is known as the Projection Theorem.

Theorem 1.1 — Given any $f \in H$, the least square problem for f associated with G always has a unique solution.

Note that our proof of the theorem also provides a potential means by which such solutions may be obtained.

In later sections of this paper, least square problems in the real Hilbert space $L^2[0,1]$ with inner product

$$(g, h) \equiv \int_0^1 g(x)\, h(x)\, dx$$

will be discussed. These least square problems will be posed with respect to finite dimensional (sub)spaces of polynomial splines. In the following section we discuss such spaces of polynomial splines and the least square problem in this context.

2. THE LEAST SQUARE PROBLEM IN $L^2[0,1]$ WITH RESPECT TO POLYNOMIAL SPLINE SPACES

In this section we first define the concept of polynomial spline spaces and state some standard spline interpolation results. We then examine, in detail, the least square problem in $L^2[0,1]$ with respect to these finite dimensional spaces of polynomial splines. We use L^2-error bounds for polynomial spline interpolation to derive both L^2- and L^∞-error bounds for least square spline approximation. We conclude with a discussion of the implementation of this technique.

We begin with the following definitions. For each non-negative integer, N, let $\mathscr{P}_N[0,1]$ denote the set of all partitions, Δ, of the interval $[0,1]$ of the form

$$\Delta: 0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1. \tag{2.1}$$

Moreover, let $\mathscr{P}[0,1] = \bigcup_{N=0}^{\infty} \mathscr{P}_N[0,1]$.

If $\Delta \in \mathscr{P}_N[0,1]$, d is a positive integer and z is an integer such that $-1 \le z \le d-1$, the polynomial spline space, $Sp(d, \Delta, z)$, is defined to be the set of all real valued functions $s(x) \in C^z[0,1]$ such that, on each subinterval $[x_i, x_{i+1}]$, $0 \le i \le N$, of $[0,1]$ determined by Δ, $s(x)$ is a polynomial of degree d. Here, $C^{-1}[0,1]$ is defined to be the set of all piecewise continuous functions on $[0,1]$ with each discontinuity a simple jump discontinuity at one of the points x_i, $1 \le i \le N$. We note that $Sp(d, \Delta, z) \subseteq L^2[0,1]$ and, since $L^2[0,1]$ is a real Hilbert space with respect to the inner product $(\cdot,\cdot)$ defined for $g, h \in L^2[0,1]$ by

$$(g, h) \equiv \int_0^1 g(x)\, h(x)\, dx,$$

we may study the least square problem for any $f \in L^2[0, 1]$ with respect to the finite dimensional subspace $Sp(d, \Delta, z)$. Theorem 1.1 applies and, consequently, we know that a unique solution to this problem always exists.

Note also that, for $d = 2m - 1$, $m \geq 1$, and $m - 1 \leq z \leq 2m - 2 = d - 1$, the definition of polynomial splines given here agrees with that of deficient splines of [1]. For generalizations of the concept of spline space, the reader is invited to study [16]. In fact, the polynomial spline interpolation results to be stated in this section remain essentially unchanged if one allows the integer z to depend on the partition points x_i, $1 \leq i \leq N$, in such a way that $m - 1 \leq z(x_i) \leq 2m - 2$.

As in [1], we define the interpolation mapping $\mathcal{I}_m : C^{m-1}[0, 1] \to Sp(2m - 1, \Delta, z)$ by $\mathcal{I}_m f = s$ where

$$D^k s(x_i) \equiv D^k f(x_i) \begin{cases} 0 \leq k \leq 2m - 2 - z \quad , & 1 \leq i \leq N \\ 0 \leq k \leq m - 1 \quad , & i = 0, N + 1. \end{cases} \tag{2.2}$$

The preceding interpolation mapping corresponds to the Type I interpolation mapping of [16].

We shall soon need the following basic result on polynomial spline interpolation.

Theorem 2.1 — The interpolation mapping given by (2.2) is well defined for all $\Delta \epsilon \mathscr{P}[0,1]$, $1 \le m$, and $m - 1 \le z \le 2m - 2$.

Before we state the result which gives bounds for the error in polynomial spline interpolation, we must first make the following definitions. For each positive integer, p, let $K^p[0,1]$ denote the collection of all real valued functions, f(x), defined on [0,1] such that $f \epsilon C^{p-1}[0,1]$, $D^{p-1}f$ is absolutely continuous, and $D^p f \epsilon L^2[0,1]$ where $Df \equiv df/dx$ denotes the derivative of f. Also, given any $\Delta \epsilon \mathscr{P}_N[0,1]$ of the form (2.1), let $\overline{\Delta} \equiv \max_{0 \le i \le N}(x_{i+1} - x_i)$ and $\underline{\Delta} \equiv \min_{0 \le i \le N}(x_{i+1} - x_i)$. The following theorem is a composite of special cases of Theorems 3.5 and 4.1 of [14].

Theorem 2.2 — Let $1 \le m$, $0 \le N$, $\Delta \epsilon \mathscr{P}_N[0,1]$ and let $m - 1 \le z \le 2m - 2$. Then, for any $f \epsilon K^{2m}[0,1]$ and $0 \le j \le 2m$,

$$||D^j(f - \mathscr{I}_m f)||_{L^2[0,1]} \le K_{m,z,j}(\overline{\Delta})^{2m-j} \, ||D^{2m}f||_{L^2[0,1]} \tag{2.3}$$

where

$$K_{m,z,j} = \frac{[(z-2+m)!]^2}{\pi^{2m-j}}, \quad \text{if } 0 \le j \le 2m-2-z,$$

$$= \frac{[(z-2+m)!]^2}{j!\,\pi^{2m-j}}, \quad \text{if } 2m-2-z < j \le m-1,$$

$$= \frac{(z-2+m)!}{\pi^m}, \quad \text{if } j = m,$$

$$= \frac{2}{\pi^{2m-j}} + \frac{[(z-2+m)!+2]}{\pi^m}\left[\frac{(3m)!}{(4m-j)!}\right]^2 (\overline{\Delta}/\underline{\Delta})^{j-m}, \text{ if } m+1 \le j \le 2m-2,$$

$$= \frac{2}{(2m-1)!\,\pi} + \frac{[(z-2+m)!+2]}{\pi^m}\left[\frac{(3m)!}{(2m+1)!}\right]^2 (\overline{\Delta}/\underline{\Delta})^{m-1}, \quad \text{if } j = 2m-1,$$

$$= 1 \quad, \text{ if } j = 2m.$$

We note, as does the author of [14], that $\mathscr{I}_m f$ is not necessarily in $K^j[0,1]$ for $z+1 < j \le 2m$ and, in this case, we must define $||D^j(f - \mathscr{I}_m f)||_{L^2[0,1]}$ by

$$||D^j(f - \mathscr{I}_m f)||_{L^2[0,1]} \equiv \left(\sum_{i=0}^{N} ||D^j(f - \mathscr{I}_m f)||^2_{L^2[x_i, x_{i+1}]}\right)^{1/2}.$$

These interpolation results enable us to give bounds for the L^2-norms of the error and its derivatives in approximating elements of certain classes of functions by polynomial splines using the least square technique of Section 1 of this paper in the context described at the beginning of this section.

We have confined our attention to $L^2[0,1]$ with inner product of $g, h \in L^2[0,1]$ defined by

$$(g, h) \equiv \int_0^1 g(x)\, h(x)\, dx.$$

As we have noted, $L^2[0, 1]$ is a Hilbert space with respect to this inner product since the norm which is induced by this inner product is the L^2-norm. Consequently, the least square problem for any $f \in L^2[0, 1]$ with respect to any finite dimensional subspace of $L^2[0, 1]$ always has a unique solution. In particular, if the subspace is a polynomial spline space, $Sp(d, \Delta, z)$, $\Delta \in \mathscr{P}[0, 1]$ and $-1 \le z \le d - 1$, we shall denote the solution $\hat{s}$. By definition, $\hat{s}$ is that element of $Sp(d, \Delta, z)$ which minimizes $||f - s||_{L^2[0, 1]}$ over $Sp(d, \Delta, z)$, i.e., for any $s \in Sp(d, \Delta, z)$

$$||f - \hat{s}||_{L^2[0, 1]} \le ||f - s||_{L^2[0, 1]} \tag{2.5}$$

Now, if $d = 2m - 1$, $f \in K^{2m}[0, 1] \subseteq L^2[0, 1]$, and $m - 1 \le z \le 2m - 2$, then $\mathscr{I}_m f$ is a well-defined element of $Sp(2m - 1, \Delta, z)$ and, consequently,

$$||f - \hat{s}||_{L^2[0, 1]} \le ||f - \mathscr{I}_m f||_{L^2[0, 1]}$$

Finally, Theorem 2.2 with $j = 0$ applied to the right hand side of (2.6) gives us the following theorem in the case that $j = 0$.

Theorem 2.3 — Let $1 \le m$, $0 \le N$, $\Delta \in \mathscr{P}_N[0, 1]$ and $m - 1 \le z \le 2m - 2$. For any function $f \in K^{2m}[0, 1] \subseteq L^2[0, 1]$, if $\hat{s}$ is the least square approximation to f in $Sp(2m - 1, \Delta, z)$, then

$$||D^j(f - \hat{s})||_{L^2[0,1]} \le C_{m,z,j}(\overline{\Delta})^{2m-j}\, ||D^{2m}f||_{L^2[0,1]} \tag{2.7}$$

where

$$\begin{aligned} C_{m,z,j} &= K_{m,z,o}\,, \quad \text{if } j = 0, \\ &= K_{m,z,j} + 2\left[\frac{(2m-1)!}{(2m-j-1)!}\right]^2 K_{m,z,o}(\overline{\Delta}/\underline{\Delta})^j, \quad \text{if } 1 \le j \le 2m-1 \\ &= K_{m,z,2m}, \quad \text{if } j = 2m, \text{ and} \end{aligned} \tag{2.8}$$

the $K_{m,z,j}$ are given by (2.4).

<u>Proof</u>: We assume $1 \le j \le 2m - 1$ since we have already established the result of the theorem in the case that $j = 0$ and it is immediate in the case that $j = 2m$ since $D^{2m}\hat{s} \equiv 0$ on $[0,1]$. We shall need the following lemma which gives an inequality of E. Schmidt that relates the L^2-norm of the derivative of a polynomial to the L^2-norm of the polynomial itself. See Appendix A for a proof of this result.

<u>Lemma</u>: Let $p_m(x)$ be a polynomial of degree m on $[a,b]$. Then

$$||Dp_m||_{L^2[a,b]} \le \frac{(m+1)^2}{b-a}\, ||p_m||_{L^2[a,b]}\,. \tag{2.9}$$

We now proceed with the proof of the theorem. We first use the triangle inequality to obtain

$$||D^j(f - \hat{s})||_{L^2[0,1]} \le ||D^j(f - \mathscr{I}_m f)||_{L^2[0,1]} + ||D^j(\mathscr{I}_m f - \hat{s})||_{L^2[0,1]} \tag{2.10}$$

recalling that $\mathscr{I}_m f$ is a well defined element of the spline space Sp(2m - 1, Δ, z) to which $\hat{s}$ also belongs. Applying (2.9) j times to the second term on the right hand side of the inequality (2.10) we obtain

$$||D^j(\mathscr{I}_m f - \hat{s})||_{L^2[0,1]} \leq \frac{\prod_{i=1}^{j}(2m - i)^2}{(\underline{\Delta})^j} ||\mathscr{I}_m f - \hat{s}||_{L^2[0,1]} =$$

$$\left[\frac{(2m-1)!}{(2m-j-1)!}\right]^2 (\underline{\Delta})^{-j} ||\mathscr{I}_m f - \hat{s}||_{L^2[0,1]}$$

$$\leq \left[\frac{(2m-1)!}{(2m-j-1)!}\right]^2 (\underline{\Delta})^{-j} \left\{||f - \mathscr{I}_m f||_{L^2[0,1]} + ||f - \hat{s}||_{L^2[0,1]}\right\}$$

$$\leq 2\left[\frac{(2m-1)!}{(2m-j-1)!}\right]^2 (\underline{\Delta})^{-j} ||f - \mathscr{I}_m f||_{L^2[0,1]} . \tag{2.11}$$

Combining (2.10) and (2.11) we obtain

$$||D^j(f - \hat{s})||_{L^2[0,1]}$$
$$\leq ||D^j(f - \mathscr{I}_m f)||_{L^2[0,1]} + 2\left[\frac{(2m-1)!}{(2m-j-1)!}\right]^2 (\underline{\Delta})^{-j} ||f - \mathscr{I}_m f||_{L^2[0,1]} . \tag{2.12}$$

Finally, applying Theorem 2.2 to the terms on the right hand side of (2.12) gives

$$\|D^j(f - \hat{s})\|_{L^2[0,1]}$$

$$\leq K_{m,z,j}(\overline{\Delta})^{2m-j} \|D^{2m}f\|_{L^2[0,1]}$$

$$+ 2\left[\frac{(2m-1)!}{(2m-j-1)!}\right]^2 (\underline{\Delta})^{-j} K_{m,z,o}(\overline{\Delta})^{2m} \|D^{2m}f\|_{L^2[0,1]} =$$

$$\left\{K_{m,z,j} + 2\left[\frac{(2m-1)!}{(2m-j-1)!}\right]^2 (\overline{\Delta}/\underline{\Delta})^j K_{m,z,o}\right\}(\overline{\Delta})^{2m-j} \|D^{2m}f\|_{L^2[0,1]} \tag{2.13}$$

the result of the theorem in the case that $1 \leq j \leq 2m - 1$.

We immediately have the following corollary.

Corollary: Given a sequence of partitions $\{\Delta^j\}_{j=1}^{\infty}$ of $[0,1]$ such that $\lim_{j\to\infty} \overline{\Delta}^j = 0$, if $\hat{s}_j$, for each j, is the least square spline approximation in $Sp(2m-1, \Delta, z)$, $m - 1 \leq z \leq 2m - 2$, to $f \in K^{2m}[0,1]$, then

$$\lim_{j\to\infty} \|f - \hat{s}_j\|_{L^2[0,1]} = 0.$$

If, in addition, $(\overline{\Delta}^j/\underline{\Delta}^j) \leq M$ all j, then, for $1 \leq k \leq 2m - 1$,

$$\lim_{j\to\infty} \|D^k(f - \hat{s}_j)\|_{L^2[0,1]} = 0.$$

We now introduce a lemma which enables us to derive bounds for the L^∞-norm of the least square error in terms of our bounds for the L^2-norm of this same error.

<u>Lemma</u>: Let u by absolutely continuous on [a, b] such that $Du \in L^2[a,b]$. Then

$$||u||_{L^\infty[a,b]} \leq \sqrt{2}\,(b-a)^{-1/2}\,||u||_{L^2[a,b]} + \sqrt{2}\,(b-a)^{1/2}\,||Du||_{L^2[a,b]}. \tag{2.14}$$

<u>Proof</u>: For any $x, x_1 \in [a,b]$

$$u(x) = u(x_1) + \int_{x_1}^{x} Du(\xi)\, d\xi$$

and, consequently,

$$|u(x)| \leq |u(x_1)| + \int_{x_1}^{x} |Du(\xi)|\, d\xi$$

$$\leq |u(x_1)| + |x - x_1|^{1/2} \left\{ \int_{x_1}^{x} |Du(\xi)|^2\, d\xi \right\}^{1/2}$$

$$\leq |u(x_1)| + (b-a)^{1/2}\, ||Du||_{L^2[a,b]}.$$

Squaring the inequality, we obtain

$$|u(x)|^2 \leq \left\{ |u(x_1)| + (b-a)^{1/2}\, ||Du||_{L^2[a,b]} \right\}^2$$

$$\leq 2|u(x_1)|^2 + 2(b-a)\, ||Du||^2_{L^2[a,b]}$$

and integrating both sides with respect to x_1 yields

$$(b-a)|u(x)|^2 \leq 2\|u\|^2_{L^2[a,b]} + 2(b-a)^2\|Du\|^2_{L^2[a,b]}$$

Therefore

$$|u(x)|^2 \leq 2(b-a)^{-1}\|u\|^2_{L^2[a,b]} + 2(b-a)\|Du\|^2_{L^2[a,b]}$$

and so

$$|u(x)| \leq \sqrt{2}(b-a)^{-1/2}\|u\|_{L^2[a,b]} + \sqrt{2}(b-a)^{1/2}\|Du\|_{L^2[a,b]}$$

from which the result of the lemma follows immediately.

This brings us to the following theorem.

Theorem 2.4 — Let $1 \leq m$, $0 \leq N$, $\Delta \epsilon \mathscr{P}_N[0,1]$ and $m - 1 \leq z \leq 2m - 2$. For any function $f \epsilon K^{2m}[0,1] \subseteq L^2[0,1]$, if $\hat{s}$ is the least square approximation to f in $Sp(2m-1, \Delta, z)$, then, for $0 \leq j \leq 2m - 1$,

$$\|D^j(f-\hat{s})\|_{L^\infty[0,1]} \leq C^\infty_{m,z,j}(\overline{\Delta})^{2m-j-1/2}\|D^{2m}f\|_{L^2[0,1]} \tag{2.15}$$

where

$$C^\infty_{m,z,j} = \sqrt{2}\left[C_{m,z,j+1} + C_{m,z,j}(\overline{\Delta}/\underline{\Delta})^{1/2}\right]. \tag{2.16}$$

Proof: $f \epsilon K^{2m}[0,1]$ implies that $D^j f \epsilon K^1[x_i, x_{i+1}], 0 \leq i \leq N$, $0 \leq j \leq 2m - 1$. Also, $\hat{s} \epsilon Sp(2m-1, \Delta, z)$ is a polynomial of degree $2m - 1$ on each subinterval $[x_i, x_{i+1}]$ of Δ, and consequently, $D^j\hat{s} \epsilon K^1[x_i, x_{i+1}], 0 \leq i \leq N$, $0 \leq j \leq 2m - 1$. Applying our lemma to $D^j(f - \hat{s})$ on each subinterval $[x_i, x_{i+1}]$ of Δ we obtain

$$||D^j(f - \hat{s})||_{L^\infty[x_i, x_{i+1}]}$$

$$\leq \sqrt{2}\,(x_{i+1} - x_i)^{-1/2}\,||D^j(f - \hat{s})||_{L^2[x_i, x_{i+1}]}$$

$$+ \sqrt{2}\,(x_{i+1} - x_i)^{1/2}\,||D^{j+1}(f - \hat{s})||_{L^2[x_i, x_{i+1}]} \tag{2.17}$$

$$\leq \sqrt{2}\,(\underline{\Delta})^{-1/2}\,||D^j(f - \hat{s})||_{L^2[0,1]} + \sqrt{2}\,(\overline{\Delta})^{1/2}\,||D^{j+1}(f - \hat{s})||_{L^2[0,1]}.$$

But (2.17) immediately implies that

$$||D^j(f - \hat{s})||_{L^\infty[0,1]} \leq \max_{0 \leq i \leq N} ||D^j(f - \hat{s})||_{L^\infty[x_i, x_{i+1}]}.$$

$$\leq \sqrt{2}\,(\underline{\Delta})^{-1/2}\,||D^j(f - \hat{s})||_{L^2[0,1]} + \sqrt{2}\,(\overline{\Delta})^{1/2}\,||D^{j+1}(f - \hat{s})||_{L^2[0,1]} \tag{2.18}$$

We now apply the results of Theorem 2.3 to the terms on the right hand side of (2.18) to yield

$$||D^j(f - \hat{s})||_{L^\infty[0,1]}$$

$$\leq \sqrt{2}\,(\underline{\Delta})^{-1/2}\,C_{m,z,j}(\overline{\Delta})^{2m-j}\,||D^{2m}f\,||_{L^2[0,1]}$$

$$+ \sqrt{2}\,(\overline{\Delta})^{1/2}\,C_{m,z,j+1}(\overline{\Delta})^{2m-j-1}\,||D^{2m}f\,||_{L^2[0,1]} =$$

$$\sqrt{2}\left[C_{m,z,j+1} + C_{m,z,j}(\overline{\Delta}/\underline{\Delta})^{1/2}\right](\overline{\Delta})^{2m-j-1/2}\,||D^{2m}f\,||_{L^2[0,1]}$$

the result of the theorem. This completes the proof.

Again, the corollary is immediate.

Corollary: Given a sequence of partitions $\{\Delta^j\}_{j=1}^{\infty}$ of [0, 1] such that $\lim_{j\to\infty} \overline{\Delta}^j = 0$ and $(\overline{\Delta}^j/\underline{\Delta}^j) \leq M$ all j, if $\hat{s}_j$, for each j, is the least square spline approximation in Sp(2m - 1, Δ, z), $m - 1 \leq z \leq 2m - 2$, to $f \in K^{2m}[0,1]$, then, for $0 \leq k \leq 2m - 1$,

$$\lim_{j\to\infty} \|D^k(f - \hat{s}_j)\|_{L^\infty[0,1]} = 0.$$

Setting theoretical considerations aside, we now turn to the practical problem of actually obtaining such approximations. The proof of Theorem 1.1 immediately leads us to the question of basis functions for the finite dimensional space of approximating functions. Let $\{s_i\}_{i=1}^{NS}$ be a set of basis functions for the NS-dimensional spline space, Sp(d, Δ, z). We note that $NS = d + 1 + N(d - z)$. In fact, the total number of indeterminates required to define an arbitrary element of Sp(d, Δ, -1) is $(N + 1)(d + 1)$ since we must determine the coefficients of a polynomial of degree d on each of N + 1 subintervals of Δ and there are no continuity constraints. In the case that there are constraints, and here we consider the integer z to depend on the interior mesh points x_i, $1 \leq i \leq N$, continuity of degree $z(x_i)$ at x_i, $1 \leq i \leq N$, introduces $z(x_i) + 1$ constraints and consequently reduces the number of indeterminates by $z(x_i) + 1$. Therefore,

$$NS = (N + 1)(d + 1) - \sum_{i=1}^{N}(z(x_i) + 1)$$

$$= d + 1 + Nd - \sum_{i=1}^{N} z(x_i)$$

which reduces to $NS = d + 1 + N(d - z)$ in the case that $z(x_i) \equiv z$, $1 \leq i \leq N$. Now, for $f \in L^2[0,1]$, we have seen that the least square approximation to f in $Sp(d, \Delta, z)$ is

$$\hat{s} = \sum_{i=1}^{NS} \hat{\alpha}_i s_i$$

where $\underline{\hat{\alpha}} \equiv (\hat{\alpha}_1, \ldots, \hat{\alpha}_{NS})$ is the unique solution to the system

$$A\underline{\alpha} - \underline{\hat{k}} = 0 \tag{2.19}$$

where the entries, a_{ij}, of A and the components, $\hat{k}_i$, of $\underline{\hat{k}}$ are defined for $1 \leq i, j \leq NS$ by

$$a_{ij} \equiv \int_0^1 s_i(x)\, s_j(x)\, dx \qquad \text{and}$$

$$\hat{k}_i \equiv \int_0^1 f(x)\, s_i(x)\, dx. \tag{2.20}$$

Therefore, in order to actually obtain the least square approximation to f in $Sp(d, \Delta, z)$, i.e., calculate the NS-tuple $\underline{\hat{\alpha}}$, we must have numerical values for the entries of A and the components of $\underline{\hat{k}}$ as well as an effective technique with which to solve the system (2.19). Since A is positive definite and symmetric, the point successive over-relaxation iterative method is guaranteed to converge and can be used to determine $\underline{\hat{\alpha}}$ (cf. [17, p. 59]). Another possibility would be to use the method of Cholesky (cf. [7, p. 127]). However, if A is also a band matrix, which indeed is the case when $d = 2m - 1$, $m \geq 1$, and $m - 1 \leq z \leq 2m - 2$ for suitably chosen basis functions (cf. [13]), Gaussian elimination can be used to efficiently solve the system (2.19). In any case, the zero structure of A determines the technique to be employed and its rate of convergence. In fact, once the basis functions for $Sp(d, \Delta, z)$ are chosen, the entries of A may be calculated directly as they are just sums of definite integrals of polynomials over subintervals $[x_i, x_{i+1}]$, $0 \leq i \leq N$, of Δ. Of course, the zero structure of A will then be determined and the appropriate technique can be chosen. The possibility of calculating the entries of $\hat{\underline{k}}$ directly seems remote since we may not have a representation of f which would admit such a calculation. Indeed, in many, if not most, practical applications, f is a tabulated function, i.e., its value may be known at only a finite number of discrete points. In such a situation, a quadrature scheme of some sort must be used to obtain the NS-tuple $\tilde{\underline{k}}$, an approximation to $\hat{\underline{k}}$, and we solve the system

$$A\underline{\alpha} - \tilde{\underline{k}} = 0 \qquad (2.21)$$

instead of (2.19). Let

$$\tilde{s} = \sum_{i=1}^{NS} \tilde{\alpha}_i s_i$$

where $\tilde{\underline{\alpha}} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_{NS})$ is the unique solution to (2.21). Recalling that the least square approximation is denoted $\hat{s}$, we wish to bound $||\hat{s} - \tilde{s}||$ in terms of known quantities in order to consider convergence results as well as the concept of consistency in the cases that bounds for $||f - \hat{s}||$ exist. To this end, let L denote the integral over [0, 1] and let $\tilde{L}$ be the quadrature rule used to determine $\underline{\tilde{k}}$, both regarded as bounded linear functionals on C[0, 1]. Then

$$\hat{k}_i = \int_0^1 f(x)\, s_i(x)\, dx = L[fs_i],$$

$1 \le i \le NS$, and $\tilde{k}_i = \tilde{L}fs_i$, $1 \le i \le NS$, and beginning with (1.6), we find that

$$\begin{aligned}
||\hat{s} - \tilde{s}||^2_{L^2[a,b]} &= (\underline{\hat{\alpha}} - \underline{\tilde{\alpha}})^T A(\underline{\hat{\alpha}} - \underline{\tilde{\alpha}}) \\
&= (\underline{\hat{\alpha}} - \underline{\tilde{\alpha}})^T(\underline{\hat{k}} - \underline{\tilde{k}}) = \sum_{i=1}^{NS} (\hat{\alpha}_i - \tilde{\alpha}_i)(\hat{k}_i - \tilde{k}_i) \\
&= \sum_{i=1}^{NS} (\hat{\alpha}_i - \tilde{\alpha}_i)(L[fs_i] - \tilde{L}[fs_i]) \\
&= (L - \tilde{L})\left[f \sum_{i=1}^{NS} (\hat{\alpha}_i - \tilde{\alpha}_i)\, s_i \right] \\
&= (L - \tilde{L})\left[f(\hat{s} - \tilde{s}) \right].
\end{aligned}$$

We shall use this relation in the next sections in order to develop our main results, much as a similar relationship leads to similar results in [6].

3. QUADRATURE SCHEMES OF THE INTERPOLATORY TYPE

In this section we first consider the concept of interpolatory quadrature and we derive error bounds for such schemes when applied to members of certain classes of functions. We then describe composite quadrature schemes (based on interpolatory formulae) which we shall use to obtain approximate solutions to the least square problems which we discussed in preceding sections. We study the error introduced into the approximation by the use of such a composite scheme. We examine the question of convergence for sequences of such approximations. We then define the concept of consistent quadrature schemes and conclude this section with an application of the discussion to the case at hand.

As in [8, p. 303], let the $n + 1$ distinct points $\tau_0 < \tau_1 < \cdots < \tau_n$ be given so that $a \le \tau_j \le b$ for all j. Then, for any function $\sigma \epsilon C[a, b]$, we may construct the interpolation polynomial, $P_n(x)$, of degree at most n such that $\sigma(\tau_j) = P_n(\tau_j)$ for all j. We take

$$\tilde{L}^*\sigma = \int_a^b P_n(x)\, dx \tag{3.1}$$

as an approximation to

$$L^*\sigma = \int_a^b \sigma(x)\, dx. \tag{3.2}$$

By using the Lagrange form for the interpolation polynomial

$$P_n(x) = \sum_{j=0}^{n} \phi_{n,j}(x)\, \sigma(\tau_j)$$

where

$$\phi_{n,j}(x) = \frac{\omega_n(x)}{(x - \tau_j)\, \omega_n'(\tau_j)}, \text{ for } x \epsilon [a,b],\ 0 \le j \le n,$$

and

$$\omega_n(x) = (x - \tau_0)(x - \tau_1) \cdots (x - \tau_n), \text{ for } x \epsilon [a,b],$$

we obtain from (3.1) the quadrature formula

$$\widetilde{L}^*\sigma = \sum_{j=0}^{n} w_{n,j}\, \sigma(\tau_j) \tag{3.3}$$

with the coefficients $w_{n,j}$ given by

$$w_{n,j} = \int_a^b \phi_{n,j}(x)\, dx, \quad 0 \le j \le n. \tag{3.4}$$

Any quadrature formula of this form is called an interpolatory quadrature formula.

We intend to consider the quadrature formula $\widetilde{L}$ of Section 2 to be a composite rule based on quadrature formulae of the interpolatory type. The following theorem will enable us to bound the error in approximating an integral by such a composite scheme. This result is quite general and sharper bounds exist for special interpolatory schemes such as Gaussian and Newton-Cotes quadrature formulae.

<u>Theorem 3.1</u> — Let $\tilde{L}^*$ be defined as above. Then, the quadrature error for any $\sigma \epsilon C^{n+1}[a,b]$ satisfies

$$|(L^* - \tilde{L}^*)\,\sigma| = |L^*\sigma - \tilde{L}^*\sigma| = \left| \int_a^b \sigma(t)\,dt - \sum_{j=0}^{n} w_{n,j}\,\sigma(\tau_j) \right|$$

$$\leq Q(b-a)^{n+3/2}\,\|D^{n+1}\sigma\|_{L^2[a,b]} \qquad (3.5)$$

where Q is independent of the length of the interval [a,b].

<u>Proof</u>: Clearly $(L^* - \tilde{L}^*)\,p_n = 0$ for any polynomial $p_n(x)$ of degree at most n. Consequently, we may employ Peano's Theorem [5, p. 109] to obtain the following representation for the quadrature error

$$L^*\sigma - \tilde{L}^*\sigma = \int_a^b D^{n+1}\sigma(t)\,K(t)\,dt \qquad (3.6)$$

where

$$K(t) = \frac{1}{n!}(L^* - \tilde{L}^*)_x\left[(x-t)_+^n\right] \qquad (3.7)$$

and

$$(x - t)_+^n = \begin{cases} (x - t)^n , & x \geq t \\ 0 , & x < t. \end{cases} \tag{3.8}$$

The notation $(L^* - \tilde{L}^*)_x$ means the error functional $(L^* - \tilde{L}^*)$ applied to the x-variable of $(x - t)_+^n$. Now, applying the Cauchy-Schwarz inequality to (3.6) we immediately obtain

$$|L^*\sigma - \tilde{L}^*\sigma| \leq \left\{\int_a^b |D^{n+1}\sigma(t)|^2 \, dt\right\}^{1/2} \left\{\int_a^b |K(t)|^2 \, dt\right\}^{1/2} =$$

$$\left\{\int_a^b |K(t)|^2 \, dt\right\}^{1/2} ||D^{n+1}\sigma||_{L^2[a,b]}. \tag{3.9}$$

We shall complete the proof of the theorem by demonstrating that

$$\int_a^b K^2(t) \, dt = (b - a)^{2n+3} \int_0^1 \overline{K}^2(t) \, dt \tag{3.10}$$

where $\overline{K}(t)$ is the kernel associated with the interpolatory quadrature scheme $\overline{L}^*$ based on the interval $[0, 1]$ corresponding to $\tilde{L}^*$ under the change of variables defined by

$$s = \frac{t - a}{b - a} \quad \text{for} \quad t \epsilon [a, b] \tag{3.11}$$

i.e., $\overline{L}^*$ is based on polynomial interpolation at the points ρ_i defined by

$$\rho_i = \frac{\tau_i - a}{b - a}, \quad 0 \le i \le n, \tag{3.12}$$

over the interval [0,1].

Let us first examine the structure of the kernel K(t). (3.7) and (3.8) imply that

$$n!K(t) = \left\{ \int_a^b (x - t)_+^n \, dx - \sum_{j=0}^{n} w_{n,j}(\tau_j - t)_+^n \right\}$$

$$= \left\{ \int_t^b (x - t)^n \, dx - \sum_{j=0}^{n} w_{n,j}(\tau_j - t)_+^n \right\}$$

$$= \frac{(b - t)^{n+1}}{n + 1} - \sum_{j=0}^{n} w_{n,j}(\tau_j - t)_+^n$$

$$= (-1)^{n+1} \begin{cases} \dfrac{(t - b)^{n+1}}{n + 1} + \displaystyle\sum_{j=0}^{n} w_{n,j}(t - \tau_j)^n, & a \le t \le \tau_0 \\[2ex] \dfrac{(t - b)^{n+1}}{n + 1} + \displaystyle\sum_{j=k}^{n} w_{n,j}(t - \tau_j)^n, & \tau_{k-1} < t \le \tau_k, \ 1 \le k \le n, \\[2ex] \dfrac{(t - b)^{n+1}}{n + 1}, & \tau_n < t \le b. \end{cases} \tag{3.13}$$

Introducing the change of variables defined by (3.11) into the integral

$$\int_a^b K^2(t)\,dt,$$

we obtain

$$\int_a^b K^2(t)\,dt = \int_0^1 K^2[a + s(b-a)](b-a)\,ds$$

$$= (b-a)\int_0^1 K^2[a + s(b-a)]\,ds \qquad (3.14)$$

since dt = (b - a) ds. But (3.13) implies that

$$n!K[a+s(b-a)] = (-1)^{n+1}\begin{cases} \dfrac{(a+s(b-a)-b)^{n+1}}{n+1} + \displaystyle\sum_{j=0}^{n} w_{n,j}(a+s(b-a)-a-\rho_j(b-a))^n, \\ \qquad\qquad 0 \le s \le \rho_0, \\ \dfrac{(a+s(b-a)-b)^{n+1}}{n+1} + \displaystyle\sum_{j=k}^{n} w_{n,j}(a+s(b-a)-a-\rho_j(b-a))^n, \\ \qquad\qquad \rho_{k-1} < s \le \rho_k,\ 1 \le k \le n, \\ \dfrac{(a+s(b-a)-b)^{n+1}}{n+1}, \quad \rho_n < s \le 1 \end{cases}$$

$$= (-1)^{n+1} \begin{cases} \dfrac{(s-1)^{n+1}(b-a)^{n+1}}{n+1} + \displaystyle\sum_{j=0}^{n} w_{n,j}(s-\rho_j)^n (b-a)^n, \; 0 \le s \le \rho_0, \\[2ex] \dfrac{(s-1)^{n+1}(b-a)^{n+1}}{n+1} + \displaystyle\sum_{j=k}^{n} w_{n,j}(s-\rho_j)^n (b-a)^n, \; \rho_{k-1} < s \le \rho_k, \\ \qquad\qquad\qquad\qquad 1 \le k \le n, \\[2ex] \dfrac{(s-1)^{n+1}(b-a)^{n+1}}{n+1}, \; \rho_n < s \le 1. \end{cases}$$

(3.15)

$$= (-1)^{n+1}(b-a)^{n+1} \begin{cases} \dfrac{(s-1)^{n+1}}{n+1} + \displaystyle\sum_{j=0}^{n} \frac{w_{n,j}}{b-a}(s-\rho_j)^n, \; 0 \le s \le \rho_0, \\[2ex] \dfrac{(s-1)^{n+1}}{n+1} + \displaystyle\sum_{j=k}^{n} \frac{w_{n,j}}{b-a}(s-\rho_j)^n, \; \rho_{k-1} < s \le \rho_k, \\ \qquad\qquad\qquad\qquad 1 \le k \le n, \\[2ex] \dfrac{(s-1)^{n+1}}{n+1}, \; \rho_n < s \le 1. \end{cases}$$

However, for any $\sigma \epsilon C[0,1]$

$$\overline{L}^*\sigma = \sum_{j=0}^{n} \overline{w}_{n,j}\sigma(\rho_j)$$

where

$$\overline{w}_{n,j} = \int_0^1 \overline{\phi}_{n,j}(x)\, dx \,, \quad 0 \le j \le n, \tag{3.16}$$

$$\bar{\phi}_{n,j}(x) = \frac{\bar{\omega}_n(x)}{(x - \rho_j)\,\bar{\omega}_n'(\rho_j)}, \text{ for } x\epsilon[0,1],\ 0 \le j \le n, \quad (3.17)$$

and

$$\bar{\omega}_n(x) = (x - \rho_0)(x - \rho_1)\cdots(x - \rho_n), \text{ for } x\epsilon[0,1]. \quad (3.18)$$

Now (3.16) implies that

$$\bar{w}_{n,j} = \frac{1}{(b-a)} \int_a^b \bar{\phi}_{n,j}\left(\frac{t-a}{b-a}\right) dt\ ,\ 0 \le j \le n, \quad (3.19)$$

under the inverse of the change of variables defined in (3.11). But (3.17) implies that, for $t\epsilon[a,b]$,

$$\bar{\phi}_{n,j}\left(\frac{t-a}{b-a}\right) = \frac{\bar{\omega}_n\left(\frac{t-a}{b-a}\right)}{\left(\frac{t-\tau_j}{b-a}\right)\bar{\omega}_n'\left(\frac{\tau_j-a}{b-a}\right)}\ ,\ 0 \le j \le n. \quad (3.20)$$

Introducing the same change of variables into (3.18), we obtain, for $t\epsilon[a,b]$,

$$\bar{\omega}_n\left(\frac{t-a}{b-a}\right) = \prod_{j=0}^{n}\left(\frac{t-a}{b-a} - \frac{\tau_j - a}{b-a}\right)$$

$$= \frac{1}{(b-a)^{n+1}} \prod_{j=0}^{n} (t - \tau_j) = \frac{\omega_n(t)}{(b-a)^{n+1}}\ . \quad (3.21)$$

Consequently

$$\overline{\omega}_n'\left(\frac{\tau_j - a}{b - a}\right) = \prod_{\substack{k=0 \\ k\neq j}}^{n} \left(\frac{\tau_j - a}{b - a} - \frac{\tau_k - a}{b - a}\right)$$

$$= \frac{1}{(b - a)^n} \prod_{\substack{k=0 \\ k\neq j}}^{n} (\tau_j - \tau_k) = \frac{\omega_n'(\tau_j)}{(b - a)^n}. \tag{3.22}$$

Substituting (3.21) and (3.22) into (3.20), we obtain, for $t \epsilon [a, b]$,

$$\overline{\phi}_{n,j}\left(\frac{t - a}{b - a}\right) = \frac{\omega_n(t)}{(t - \tau_j)\,\omega_n'(\tau_j)} = \phi_{n,j}(t)\ ,\quad 0 \le j \le n, \tag{3.23}$$

and combining (3.4), (3.19), and (3.23), we find that

$$\overline{\omega}_{n,j} = \frac{\omega_{n,j}}{b - a}\ . \tag{3.24}$$

Repeating the derivation of (3.13) for $\overline{K}(s)$ instead of $K(t)$, we obtain

$$n!\overline{K}(s) = (-1)^{n+1} \begin{cases} \dfrac{(s-1)^{n+1}}{n+1} + \displaystyle\sum_{j=0}^{n} \frac{w_{n,j}}{b-a}(s-\rho_j)^n\ , & 0 \le s \le \rho_0, \\[2ex] \dfrac{(s-1)^{n+1}}{n+1} + \displaystyle\sum_{j=k}^{n} \frac{w_{n,j}}{b-a}(s-\rho_j)^n, & \rho_{k-1} < s \le \rho_k, \\ & 1 \le k \le n, \\[2ex] \dfrac{(s-1)^{n+1}}{n+1}\ , & \rho_n < s \le 1\ . \end{cases} \tag{3.25}$$

A comparison of (3.15) and (3.25) implies, upon cancellation of n!, that

$$K(a + s(b - a)) = (b - a)^{n+1}\,\overline{K}(s) \tag{3.26}$$

which, when substituted into (3.14), yields

$$\int_a^b K^2(t)\,dt = (b - a)^{2n+3} \int_0^1 \overline{K}^2(s)\,ds.$$

Combining this with (3.9) gives (3.5), the result of the theorem. Clearly,

$$Q = \left\{ \int_0^1 \overline{K}^2(s)\,ds \right\}^{1/2}. \tag{3.27}$$

Given partitions Δ_i^* of the form

$$\Delta_i^* : x_i \le \tau_{i,0} < \tau_{i,1} < \cdots < \tau_{i,n} \le x_{i+1}$$

of the subintervals $[x_i, x_{i+1}]$, $0 \le i \le N$, determined by Δ, we define the composite rule $\tilde{L}$ by

$$\tilde{L}\sigma = \sum_{i=0}^{N} \sum_{j=0}^{n} w_{n,j}^{i} \cdot \sigma(\tau_{i,j}) \tag{3.28}$$

in terms of the weights $w_{n,j}^{i}$, $0 \le j \le n$, of the interpolatory quadratures L_i^*, $0 \le i \le N$, defined over the partitions Δ_i^*. This brings us to the following theorem, which can be improved in those cases that sharper bounds exist for the interpolatory schemes employed in the composite rule.

Theorem 3.2 — Let $\Delta \epsilon \mathscr{P}_n[0,1]$, $N \geq 0$, and let the partitions Δ_i^*, $0 \leq i \leq N$, of the intervals $[x_i, x_{i+1}]$ be given. Let $Sp(d, \Delta, z)$, $d \leq n$, be a polynomial spline space, i.e., $-1 \leq z \leq d-1$. For $f \epsilon C^{n+1}[0,1] \subseteq K^{n+1}[0,1] \subseteq L^2[0,1]$, let $\hat{s}$ be the least square approximation to f in $Sp(d, \Delta, z)$, i.e., if $\{s_i\}_{i=1}^{NS}$ form a basis for $Sp(d, \Delta, z)$ then

$$\hat{s} \equiv \sum_{i=1}^{NS} \hat{\alpha}_i s_i$$

where $\underline{\hat{\alpha}} \equiv (\hat{\alpha}_1, \ldots, \hat{\alpha}_{NS})$ is the unique solution to the system (2.19). Finally, let $\tilde{s}$ be the discretized least square approximation to f in $Sp(d, \Delta, z)$ defined by

$$\tilde{s} \equiv \sum_{i=1}^{NS} \tilde{\alpha}_i s_i$$

where $\underline{\tilde{\alpha}} \equiv (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_{NS})$ is the unique solution to the system (2.21) with $\tilde{k}$ determined by the composite scheme $\tilde{L}$ defined in (3.28). Then

$$||\hat{s} - \tilde{s}||_{L^2[0,1]} \leq K(\overline{\Delta})^{n-d+1/2} \tag{3.29}$$

where K is a positive constant not necessarily independent of Δ and, again, $\overline{\Delta} \equiv \max_{0 \leq i \leq N} (x_{i+1} - x_i)$.

Proof: Using (2.22) and applying (3.5) to the corresponding interpolatory quadrature scheme $\tilde{L}_i^*$ in each subinterval determined by Δ with the appropriate normalized constant denoted Q_i, we obtain

$$||\hat{s} - \tilde{s}||^2_{L^2[0,1]} = |(L - \tilde{L})[f(\hat{s} - \tilde{s})]|$$

$$\leq \max_{0 \leq i \leq N} Q_i \sum_{j=0}^{N} (h_j)^{n+3/2} ||D^{n+1}[f(\hat{s} - \tilde{s})]||_{L^2[x_j, x_{j+1}]}$$

$$= Q \sum_{j=0}^{N} (h_j)^{n+3/2} ||D^{n+1}[f(\hat{s} - \tilde{s})]||_{L^2[x_j, x_{j+1}]} \tag{3.30}$$

where $Q \equiv \max_{0 \leq i \leq N} Q_i$ and where $h_j = x_{j+1} - x_j$, $0 \leq j \leq N$. However

$$||D^{n+1}[f(\hat{s} - \tilde{s})]||_{L^2[x_j, x_{j+1}]}$$

$$\leq \sum_{k=0}^{n+1} \binom{n+1}{k} ||D^{n-k+1}f||_{L^\infty[x_j, x_{j+1}]} ||D^k(\hat{s} - \tilde{s})||_{L^2[x_j, x_{j+1}]}$$

$$\leq C_f \sum_{k=0}^{d} \binom{n+1}{k} ||D^k(\hat{s} - \tilde{s})||_{L^2[x_j, x_{j+1}]} \tag{3.31}$$

since $f \epsilon C^{n+1}[0,1]$ implies the existence of the positive constant $C_f \equiv \max_{0 \leq k \leq n+1} ||D^k f||_{L^\infty[0,1]}$ and $\hat{s}, \tilde{s} \epsilon Sp(d, \Delta, z)$ implies that $\hat{s} - \tilde{s}$ is a polynomial of degree $\leq d$ on each subinterval $[x_j, x_{j+1}]$ and so $||D^k(\hat{s} - \tilde{s})||_{L^2[x_j, x_{j+1}]} = 0$, $0 \leq j \leq N$, $d + 1 \leq k \leq n + 1$. Combining (3.30) and (3.31) and applying Schmidt's inequality, (2.9), k times to $||D^k(\hat{s} - \tilde{s})||_{L^2[x_j, x_{j+1}]}$, $0 \leq j \leq N$, $1 \leq k \leq d$, we find that

$$\|\hat{s}-\tilde{s}\|^2_{L^2[0,1]} \le Q\cdot C_\rho \cdot \sum_{j=0}^{N}\left\{(h_j)^{n+3/2}\cdot \sum_{k=0}^{d}\binom{n+1}{k}\|D^k(\hat{s}-\tilde{s})\|_{L^2[x_j,x_{j+1}]}\right\}$$

$$\le Q\cdot C_\rho\cdot \sum_{j=0}^{N}\left\{(h_j)^{n+3/2}\cdot \sum_{k=0}^{d}\binom{n+1}{k}\left[\frac{d!}{(d-k)!}\right]^2 (h_j)^{-k}\|\hat{s}-\tilde{s}\|_{L^2[x_j,x_{j+1}]}\right\}$$

$$\le Q\cdot C_\rho\cdot \sum_{j=0}^{N}\left\{(h_j)^{n-d+3/2}.\|\hat{s}-\tilde{s}\|_{L^2[0,1]}\cdot \sum_{k=0}^{d}\binom{n+1}{k}\left[\frac{d!}{(d-k)!}\right]^2 (h_j)^{d-k}\right\}$$

$$\le Q\cdot C_\rho\cdot \left\{\sum_{k=0}^{d}\binom{n+1}{k}\left[\frac{d!}{(d-k)!}\right]^2\right\}\cdot \|\hat{s}-\tilde{s}\|_{L^2[0,1]}\cdot\sum_{j=0}^{N}(h_j)^{n-d+3/2}$$

$$\le Q\cdot C_\rho\cdot \left\{\sum_{k=0}^{d}\binom{n+1}{k}\left[\frac{d!}{(d-k)!}\right]^2\right\}\cdot (\overline{\Delta})^{n-d+1/2}\cdot \|\hat{s}-\tilde{s}\|_{L^2[0,1]} \tag{3.32}$$

since $h_j \le 1$, $0 \le j \le N$, and

$$\sum_{j=0}^{N} h_j = 1.$$

Cancelling the factor $\|\hat{s}-\tilde{s}\|_{L^2[0,1]}$ from both sides of (3.32), we obtain

$$\|\hat{s}-\tilde{s}\|_{L^2[0,1]} \le K(\overline{\Delta})^{n-d+1/2}$$

where

$$K \equiv Q\cdot C_\rho\cdot\left\{\sum_{k=0}^{d}\binom{n+1}{k}\left[\frac{d!}{(d-k)!}\right]^2\right\}$$

is not necessarily independent of Δ. This completes the proof.

The following corollary is immediate.

Corollary: Given a sequence of partitions $\{\Delta^j\}_{j=1}^{\infty}$ of [0, 1] such that $\lim_{j\to\infty} \overline{\Delta}^j = 0$, let $\hat{s}_j$, for each j, be the least square approximation in $Sp(d, \Delta^j, z)$, $-1 \leq z \leq d - 1$, to $f \in C^{n+1}[0,1]$. Let $\mathscr{F} \subseteq \mathscr{P}_{n-1}[0,1]$ be finite. For each j, let $\tilde{s}_j$ be a discretized least square approximation in $Sp(d, \Delta^j, z)$ to f obtained by using a composite quadrature rule $\tilde{L}_j$ of the form given by (3.28) where all partitions of the subintervals determined by Δ^j over which the interpolatory formulae are defined, when scaled to [0, 1], are members of the finite set $\mathscr{F}$. Then, if $d \leq n$

$$\lim_{j\to\infty} \|\hat{s}_j - \tilde{s}_j\|_{L^2[0,1]} = 0.$$

This means, of course, that the errors introduced into the approximation by the use of composite schemes of this type tend to zero with $\overline{\Delta}^j$. These errors may or may not be small compared to $\|f - \hat{s}_j\|_{L^2[0,1]}$. Nevertheless, combining the corollary to Theorem 2.3 with this last corollary, we obtain the following result.

Corollary: If, in addition to the hypothesis of the corollary just given, we also assume that $d = 2m - 1$ and $m - 1 \leq z \leq 2m - 2 = d - 1$, then

$$\lim_{j\to\infty} \|f - \tilde{s}_j\|_{L^2[0,1]} = 0.$$

We proceed to define the concept of the consistency of quadrature schemes for the approximate solution of the least square problem (cf. [6]). Let d be any fixed positive integer and let $\mathscr{C}$ be a collection of partitions, Δ, of [0, 1]. For each $\Delta \epsilon \mathscr{C}$, let $Sp(d, \Delta, z)$ be a space of polynomial splines and let $\hat{s}_\Delta$, the least square approximation to $f \epsilon L^2[0,1]$ in $Sp(d, \Delta, z)$, satisfy

$$||f - \hat{s}_\Delta||_N \leq \mathscr{K}(\overline{\Delta})^{\ell} \tag{3.33}$$

where $||\cdot||_N$ is some norm on $L^2[0,1]$ and $\mathscr{K}$ and ℓ are positive constants independent of Δ. In addition, for each $\Delta \epsilon \mathscr{C}$, let $\tilde{s}_\Delta$, that element of $Sp(d, \Delta, z)$ obtained as an approximation to $\hat{s}_\Delta$ using some bounded linear functional $\tilde{L}_\Delta$, satisfy

$$||\hat{s}_\Delta - \tilde{s}_\Delta||_N \leq \mathscr{K}'(\overline{\Delta})^{\ell'} \tag{3.34}$$

where $\mathscr{K}'$ and ℓ' are positive constants independent of Δ. Then, the triangle inequality, (3.33) and (3.34) imply that

$$\begin{aligned} ||f - \tilde{s}_\Delta||_N &\leq ||f - \hat{s}_\Delta||_N + ||\hat{s}_\Delta - \tilde{s}_\Delta||_N \\ &\leq \mathscr{K}(\overline{\Delta})^{\ell} + \mathscr{K}'(\overline{\Delta})^{\ell'} \\ &\leq (\mathscr{K} + \mathscr{K}')(\overline{\Delta})^{\min(\ell, \ell')} \end{aligned}$$

for all $\Delta \epsilon \mathscr{C}$ since $\overline{\Delta} \leq 1$. Consequently, if $\min(\ell, \ell') \geq \ell$, i.e., $\ell' \geq \ell$, the order of accuracy of the splines $\tilde{s}_\Delta, \Delta \epsilon \mathscr{C}$, as approximations to f is no worse than the order of accuracy of the spline approximations $\hat{s}_\Delta$. In this case, we say that the choice of functionals, $\tilde{L}_\Delta$, is consistent in the norm $||\cdot||_N$ with the bounds given by (3.33).

The results of Theorems 2.2 and 3.2 immediately give us the following theorem.

Theorem 3.3 — Let $\mathscr{C} = \mathscr{P}[0,1]$, $d = 2m - 1$, and $m - 1 \leq z \leq 2m - 2$. Let $\mathscr{F} \subseteq \mathscr{P}_{n-1}[0,1]$ be finite. For each $\Delta \epsilon \mathscr{C}$, let the linear functional in (3.28) be defined in terms of interpolation over partitions of subintervals of Δ all of which, when scaled to $[0,1]$, are members of $\mathscr{F}$. Then, for $f \epsilon C^{n+1}[0,1]$ and $4(2m-1) \leq 2n-1$, i.e., $n \geq \frac{8m-3}{2}$ or $m \leq \frac{2n+3}{8}$, this choice of linear functionals is consistent in the L^2-norm with the bounds for the least square error given by (2.7).

We note that this result can be improved in those cases in which special bounds exist for the interpolatory formulae which are employed in the composite rules.

4. QUADRATURE SCHEMES OF THE FILON TYPE

In this section we investigate the use of quadrature schemes of the Filon type for the approximate solution of the least square problem in $L^2[0,1]$. Beginning with a definition of Filon type quadrature, we note its dependence on interpolation. We derive bounds for the error in approximating the solution of the least square problem by such a technique in terms of the error in the interpolation used to define the quadrature. This leads us to the derivation of bounds for the error in piecewise Lagrange interpolation. We discuss the question of convergence for sequences of approximations based on Filon type quadrature schemes using this type of interpolation and we conclude with a theorem on the consistency of such Filon type schemes with the least square error.

Just as in the preceding section, we are faced with the problem of approximating the components of $\underline{\hat{k}}$, i.e., the integrals

$$\hat{k}_i \equiv \int_0^1 f(x)\, s_i(x)\, dx\ , \quad 1 \le i \le NS,$$

where the splines $\{s_i\}_{i=1}^{NS}$ form a basis for the polynomial spline space $Sp(d, \Delta, z)$. In the last section we considered interpolating the integrands by polynomials and integrating the interpolates as approximations to the integrals. This method of approximating the components of $\underline{\hat{k}}$ depends only on point evaluations of the basis functions, s_i, $1 \le i \le NS$, when, in fact, we have explicit piecewise polynomial representations for them. In this section we consider quadrature rules based on interpolating the function, f, by a piecewise polynomial, denoted $\tilde{f}$, and using the representations of the basis functions directly in calculating the approximations to the integrals in question, i.e., we define the vector $\underline{\tilde{k}}$, as an approximation to $\underline{\hat{k}}$, by

$$\tilde{k}_i \equiv \tilde{L}[fs_i] = \int_0^1 \tilde{f}(x)\, s_i(x)\, dx\ , \quad 1 \le i \le NS. \tag{4.1}$$

Quadrature schemes for integrals of product integrands in which only one of the factors requires approximation are said to be of the Filon type, (cf. [5, p. 62]). Since $\tilde{f}$ and all the basis functions, s_i, $1 \le i \le NS$, are piecewise polynomials, each component of $\underline{\tilde{k}}$ is just the sum of definite integrals of polynomials and can be calculated directly. Here, again, we let

$$\tilde{s} = \sum_{i=1}^{NS} \tilde{\alpha}_i s_i$$

where $\underline{\tilde{\alpha}} \equiv (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_{NS})$ is the unique solution of the linear system of (2.21) when $\underline{\tilde{k}}$ is defined by (4.1). We state and then prove the following theorem which give bounds for the L^2-norm of the error in approximating the least square spline approximation to f by $\tilde{s}$ in terms of the L^q-norm, $2 \le q \le \infty$, of the error in approximating f by the interpolate $\tilde{f}$.

Theorem 4.1 — Let $\Delta \epsilon \mathscr{P}_N[0,1]$, $N \geq 0$, be of the form (2.1) and let $Sp(d, \Delta, z)$ be a polynomial spline space, i.e., $-1 \leq z \leq d-1$. For $f \epsilon C[0,1]$, let $\hat{s}$ be the least square spline approximation to f in $Sp(d, \Delta, z)$, i.e., if the splines $\{s_i\}_{i=1}^{NS}$ form a basis for $Sp(d, \Delta, z)$, then

$$\hat{s} \equiv \sum_{i=1}^{NS} \hat{\alpha}_i s_i$$

where $\underline{\hat{\alpha}} \equiv (\hat{\alpha}_1, \ldots, \hat{\alpha}_{NS})$ is the unique solution to the linear system of equations defined by (2.19). Finally let

$$\tilde{s} \equiv \sum_{i=1}^{NS} \tilde{\alpha}_i s_i$$

be the discretized least square approximation to $f \epsilon Sp(d, \Delta, z)$ where $\underline{\tilde{\alpha}} \equiv (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_{NS})$ is the unique solution to the system (2.21) with $\underline{\tilde{k}}$ determined by the functional $\tilde{L}$ defined in (4.1). Then, for $2 \leq q \leq \infty$,

$$||\hat{s} - \tilde{s}||_{L^2[0,1]} \leq ||f - \tilde{f}||_{L^q[0,1]} . \tag{4.2}$$

Proof: Beginning with (2.22), just as in the proof of Theorem 3.2, we obtain for $2 \leq q \leq \infty$,

$$\|\hat{s} - \tilde{s}\|^2_{L^2[0,1]} = (L - \tilde{L})[f(\hat{s} - \tilde{s})]$$

$$= \int_0^1 [f(x) - \tilde{f}(x)] \cdot [\hat{s}(x) - \tilde{s}(x)]\, dx$$

$$\leq \|f - \tilde{f}\|_{L^2[0,1]} \cdot \|\hat{s} - \tilde{s}\|_{L^2[0,1]}$$

$$\leq \|f - \tilde{f}\|_{L^q[0,1]} \cdot \|\hat{s} - \tilde{s}\|_{L^2[0,1]}$$

Cancelling $\|\hat{s} - \tilde{s}\|_{L^2[0,1]}$ from each side yields (4.2).

If bounds for the error in the interpolation can be derived, they can be coupled with the results of Theorem 4.1 to investigate convergence and consistency results analogous to those given at the end of the preceding section. To be more specific, we could discuss the convergence of sequences of discretized least square spline approximations to a function, f, obtained using a Filon type quadrature scheme based on the interpolate to f. And, for collections of such quadrature schemes, we could investigate the question of their consistency with our bounds for the L^2-norm of the least square error.

Error bounds for piecewise Hermite (osculatory) interpolation can be derived using the Peano Kernel Theorem (cf. [4]). In order to obtain such interpolates to a tabulated function, tables of values of certain derivatives of the function are required in all but the linear case. In order to avoid this difficulty, we employ piecewise Lagrange interpolation for our results based on Filon type quadrature schemes. We note, however, that this type of interpolation may coincide with piecewise Hermite interpolation in the linear case. Using rather general error bounds for Lagrange interpolation over the interval [a,b] (cf. [12, p. 105]), we are able to derive global error bounds for piecewise Lagrange interpolation.

We begin with the following definition. For any positive integer s, let $P_s[a,b]$ be the set of all polynomials of degree at most s defined on [a,b]. Given any function $f \in C[a,b]$ and a partition Δ^* of [a,b] of the form

$$\Delta^* : a \le \tau_0 < \tau_1 < \cdots < \tau_{s-1} < \tau_s \le b,$$

let its unique Δ^*-interpolate be the element $f^* \in P_s[a,b]$ such that

$$f^*(\tau_j) = f(\tau_j) , \quad 0 \le j \le s.$$

This, of course, is the standard definition of Lagrange interpolation.

Because of the local character of piecewise polynomial interpolation, we may focus our attention on the interval [0,1]. For fixed $x_0 \in [0,1]$, the error in this interpolation, denoted by F and defined for $f \in C[0,1]$ by $F(f) \equiv f(x_0) - f^*(x_0)$, is a linear functional on C[0,1]. We note that the definition of F depends on x_0 and Δ^*. Following [12, p. 85] in using Lagrange's interpolation formula, we see that this error functional is an elementary Stieltjes integral, i.e., there exists a function $\mu(x, x_0)$ of bounded variation with respect to $x \in [0,1]$ for each $x_0 \in [0,1]$ such that for any $f \in C[0,1]$

$$F(f) = \int_0^1 f(x)\, d\mu(x, x_0) \tag{4.3}$$

In order to give an explicit representation for $\mu(x, x_0)$, let $\ell_i(x) \in P_s[0,1]$, $0 \le i \le s$, be defined by

$$\ell_i(\tau_j) = \delta_{i,j}, \quad 0 \le j \le s,$$

where $\delta_{i,j}$ is the Kronecker delta function. Then Lagrange's formula for the Δ^*-interpolate of $f \in C[0,1]$ is given by

$$f^*(x) = \sum_{i=0}^{s} f(\tau_i)\, \ell_i(x). \tag{4.4}$$

Consequently, defining $\mu(x, x_0)$, for $0 \le \tau_j < x_0 < \tau_{j+1} \le 1$, $0 \le j \le s-1$, by

$$\mu(x, x_0) = \begin{cases} 0, & 0 \le x \le \tau_0, \\[2ex] -\sum_{i=0}^{k} \ell_i(x_0), & \tau_k < x \le \tau_{k+1}, \quad 0 \le k \le j-1, \\[2ex] -\sum_{i=0}^{j} \ell_i(x_0), & \tau_j < x < x_0, \\[2ex] 1 - \sum_{i=0}^{j} \ell_i(x_0), & x_0 \le x < \tau_{j+1}, \\[2ex] 1 - \sum_{i=0}^{k} \ell_i(x_0), & \tau_k < x \le \tau_{k+1}, \quad j+1 \le k \le s-1, \\[2ex] 1 - \sum_{i=0}^{s} \ell_i(x_0), & \tau_s < x \le 1, \end{cases} \tag{4.5}$$

so that $\mu(x, x_0)$ is a step function with simple jump discontinuities of magnitude 1 at x_0 and $-\ell_j(x_0)$ at τ_j, $0 \le j \le s$, we immediately obtain

$$\int_0^1 f(x)\, d\mu(x, x_0) = f(x_0) - \sum_{j=0}^{s} f(\tau_j)\, \ell_j(x_0)$$

$$= f(x_0) - f^*(x_0) = F(f).$$

This representation implies that F is a bounded linear functional on C[0, 1]. However, we also have, for any $g \epsilon P_s[0, 1]$,

$$F(g) = g(x_0) - g^*(x_0) = 0$$

since g certainly interpolates itself over Δ^* and interpolation over Δ^* is unique. We are now in the position to apply the Peano Kernel Theorem (cf. [12, p. 25]) to the functional F.

We must first generalize the spaces $K^p[0, 1]$ defined in Section 2. For any positive integer p and any extended real number r, $1 \le r \le \infty$, let $K^{p,r}[a, b]$ be the collection of all real valued functions, f(x), defined on [a, b] such that $f \epsilon C^{p-1}[a, b]$, $D^{p-1}f$ is absolutely continuous, and $D^p f \epsilon L^r[a, b]$. Note that, for all positive integers p, $K^{p,2}[0, 1] = K^p[0, 1]$ as defined in Section 2.

<u>Theorem 4.2</u> — For $1 \le p \le s + 1$, given any $f \epsilon K^{p,r}[0, 1]$, then, for any fixed $x_0 \epsilon [0, 1]$, the functional of (4.3) can be expressed as

$$F(f) = f(x_0) - f^*(x_0) = \int_0^1 D^p f(t)\, K_{\Delta^*, p}(t, x_0)\, dt \qquad (4.6)$$

where

$$K_{\Delta^*,p}(t,x_0) \equiv F_x\left\{\frac{(x-t)^{p-1}}{(p-1)!}\right\} = \int_t^1 \frac{(x-t)^{p-1}}{(p-1)!}\, d\mu(x,x_0). \qquad (4.7)$$

We remark that F_x in (4.7) means the application of F to $\{(x-t)_+^{p-1}/(p-1)!\}$ considered, for fixed t, as a function of x, and, as usual,

$$(x-t)_+^{p-1} = \begin{cases} (x-t)^{p-1}, & x \ge t, \\ 0, & x < t. \end{cases}$$

The explicit representations (4.4) and (4.5) allow us to determine the kernels $K_{\Delta^*,p}(t,x_0)$ although they are by no means uncomplicated in all but the linear case. Formula (4.5) implies that $\mu(x,x_0)$ is of bounded variation on [0,1], uniformly with respect to $x_0 \epsilon [0,1]$, i.e., there exists a constant K dependent on Δ^* but independent of x_0 such that Var $\mu(x,x_0) \le K$ all $x, x_0 \epsilon [0,1]$. Thus, as $|(x-t)^{p-1}|$ is bounded on [0,1] x [0,1], it follows from (4.7) that the kernel, $K_{\Delta^*,p}(t,x_0)$, is uniformly bounded on [0,1] x [0,1]. Consequently, if $1/r + 1/r' = 1$, then the function

$$g_{\Delta^*,p,r}(x_0) \equiv \left\{\int_0^1 |K_{\Delta^*,p}(t,x_0)|^{r'}\, dt\right\}^{1/r'}$$

is an element of $L^q[0,1]$, $1 \le q \le \infty$, and we can define the constant $c_{\Delta^*,p,r,q}$ by

$$c_{\Delta^*,p,r,q} \equiv \left\{\int_0^1 |g_{\Delta^*,p,r}(x_0)|^q\, dx_0\right\}^{1/q}.$$

Then, applying Holder's inequality to (4.6) gives

$$|f(x_0) - f^*(x_0)| \le ||D^p f||_{L^r[0,1]} \, g_{\Delta^*,p,r}(x_0) \tag{4.8}$$

and integrating the q-th power of both sides of (4.8) with respect to x_0 gives, with the definition of $c_{\Delta^*,p,r,q}$, the following corollary to Theorem 4.2 (cf. [12, p. 105]).

Corollary: For $1 \le p \le s + 1$, given any $f \epsilon K^{p,r}[0,1]$, then

$$||f - f^*||_{L^q[0,1]} \le c_{\Delta^*,p,r,q} \, ||D^p f||_{L^r[0,1]}$$

for $1 \le q, r \le \infty$.

We now obtain the analogous result for the interval $[a,b]$. For any $f \epsilon K^{p,r}[a,b]$, $1 \le p \le s + 1$, (4.6) can be written as

$$f(a + x_0[b-a]) - f^*(a + x_0[b-a]) = (b-a)^p \int_0^1 D^p f(a + t[b-a]) \, K_{\Delta^*,p}(t, x_0) \, dt$$

where $0 \le x_0 \le 1$. Consequently, we have a second corollary to Theorem 4.2.

Corollary: For $1 \le p \le s + 1$, given any $f \epsilon K^{p,r}[a,b]$, then

$$||f - f^*||_{L^q[a,b]} \le (b-a)^{p - 1/r + 1/q} \cdot c_{\Delta^*,p,r,q} \cdot ||D^p f||_{L^r[a,b]}$$

for $1 \le q, r \le \infty$.

If Δ^* is given as a partition over $[a,b]$, $c_{\Delta^*,p,r,q}$ will be interpreted to mean the normalized constant defined over $[0,1]$.

We are now in the position to estimate the global error in piecewise Lagrange interpolation. Given a partition Δ_T of the interval [a, b] of the form

$$\Delta_T : a = x_0 < x_1 < \cdots < x_N < x_{N+1} = b$$

and partitions Δ_i^* of the subintervals $[x_i, x_{i+1}]$, $0 \le i \le N$, of the form

$$\Delta_i^* : x_i \le \tau_{i,0} < \tau_{i,1} < \cdots < \tau_{i,s} \le x_{i+1} ,$$

we define the $(\Delta_T)^*$-interpolate to $f \epsilon C[a,b]$ by

$$\tilde{f}(x) \equiv f_i^*(x) , \quad x \epsilon [x_i, x_{i+1}] , \quad 0 \le i \le N,$$

where f_i^* is the Δ_i^*-interpolate to f as defined earlier in this section. Note that $\tilde{f}$ need not be continuous at the points x_i, $1 \le i \le N$, although continuity at x_i is guaranteed by $\tau_{i-1,s} = x_i = \tau_{i,0}$.

In the following theorem, we give bounds for the global error in $(\Delta_T)^*$-interpolation.

<u>Theorem 4.3</u> — For $1 \le p \le s + 1$, given any $f \epsilon K^{p,r}[a,b]$, if $\tilde{f}$ is the $(\Delta_T)^*$-interpolate to f as defined above, then

$$||f - \tilde{f}||_{L^q[a,b]} \le (\overline{\Delta}_T)^{p-1/r+1/q} \cdot \max_{0 \le i \le N} c_{\Delta_i^*,p,r,q} \cdot ||D^p f||_{L^r[a,b]} , \quad (4.9)$$

for any $q \ge r$, and, if $1 \le q \le r$,

$$||f - \tilde{f}||_{L^q[a,b]} \le (\overline{\Delta}_T)^{p-1/r+1/q} \cdot (b-a)^{(r-q)/rq} \cdot \max_{0 \le i \le N} c_{\Delta_i^*,p,r,r} \cdot ||D^p f||_{L^r[a,b]} .$$

(4.10)

Proof: With the definition of $K^{p,r}[a,b]$ and the hypothesis of the theorem, it is clear that $D^p f \in L^r[a,b]$ and $f - \tilde{f} \in L^q[a,b]$ for $1 \le q \le \infty$. For $0 \le i \le N$, let

$$\upsilon_i \equiv \left\{ \int_{x_i}^{x_{i+1}} |f(t) - \tilde{f}(t)|^q \, dt \right\}^{1/q}$$

and

$$\omega_i \equiv \left\{ \int_{x_i}^{x_{i+1}} |D^p f(t)|^r \, dt \right\}^{1/r} .$$

Then, from the second corollary to Theorem 4.2, we have, for $0 \le i \le N$,

$$\upsilon_i = \left\{ \int_{x_i}^{x_{i+1}} |f(t) - f_i^*(t)|^q \, dt \right\}^{1/q}$$

$$\le (x_{i+1} - x_i)^{p-1/r-1/q} \cdot c_{\Delta_i^*,p,r,q} \cdot \omega_i .$$

Here, the constants $c_{\Delta_i^*,p,r,q}$ are interpreted to be the normalized constants defined over the interval $[0,1]$ in terms of the appropriately scaled partition. But then

$$||f-\tilde{f}||_{L^q[a,b]} = \left\{\int_a^b |f(t) - \tilde{f}(t)|^q \, dt\right\}^{1/q}$$

$$= \left\{\sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} |f(t) - \tilde{f}(t)|^q \, dt\right\}^{1/q}$$

$$= \left\{\sum_{i=0}^{N} \upsilon_i^q\right\}^{1/q} \le \left\{\sum_{i=0}^{N} \left[(x_{i+1}-x_i)^{p-1/r+1/q} \cdot c_{\Delta_i^*,p,r,q} \cdot \omega_i\right]^q\right\}^{1/q}$$

$$\le (\overline{\Delta}_T)^{p-1/r+1/q} \cdot \max_{0\le i\le N} c_{\Delta_i^*,p,r,q} \cdot \left\{\sum_{i=0}^{N} \omega_i^q\right\}^{1/q} . \qquad (4.11)$$

For $q \ge r$, Jensen's inequality [2 , p. 18] gives

$$\left\{\sum_{i=0}^{N} \omega_i^q\right\}^{1/q} \le \left\{\sum_{i=0}^{N} \omega_i^r\right\}^{1/r} = \left\{\sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} |D^p f(t)|^r \, dt\right\}^{1/r}$$

$$(4.12)$$

$$= \left\{\int_a^b |D^p f(t)|^r \, dt\right\}^{1/r} = ||D^p f||_{L^r[a,b]} .$$

Combining (4.11) and (4.12) gives (4.9), the first result of the theorem. Namely, for $q \ge r$,

$$||f-\tilde{f}||_{L^q[a,b]} \le (\overline{\Delta}_T)^{p-1/r+1/q} \cdot \max_{0\le i\le N} c_{\Delta_i^*,p,r,q} \cdot ||D^p f||_{L^r[a,b]} . \qquad (4.9)$$

Now, for $1 \le q \le r$, the integral Holder's inequality gives

$$||f - \tilde{f}||_{L^q[a,b]} \le (b - a)^{(r-q)/rq} \, ||f - \tilde{f}||_{L^r[a,b]}$$

which, when combined with (4.9) in the case $q = r$, gives (4.10), the second result of the theorem. That is, for $1 \le q \le r$,

$$\|f - \tilde{f}\|_{L^q[a,b]} \le (\overline{\Delta}_T)^p (b - a)^{(r-q)/rq} \cdot \max_{0 \le i \le N} c_{\Delta_i^*, p, r, q} \cdot \|D^p f\|_{L^r[a,b]} .$$

This completes the proof of the theorem.

As a corollary to Theorems 4.1 and 4.3, we have the following result. We do not employ these theorems in their greatest generality. We assume $q = 2$ in the first theorem and $q = r = 2$ in the second.

<u>Corollary</u>: Let $\Delta \epsilon \mathscr{P}[0,1]$ and let $\hat{s}$ be the least square spline approximation in $Sp(d, \Delta, z)$ to $f \epsilon K^p[0,1]$. Given a finite subset $\mathscr{F} \subseteq \mathscr{P}_{s-1}[0,1]$ and a sequence of partitions $\{\Delta_T^j\}_{j=1}^{\infty}$ of $[0,1]$ such that $\lim_{j \to \infty} \overline{\Delta}_T^j = 0$, let $\tilde{s}_j$, for each j, be a discretized least square approximation in $Sp(d, \Delta, z)$ to f obtained using a Filon type quadrature scheme based on $(\Delta_T^j)^*$-interpolation where the partitions of the subintervals of Δ_T^j, scaled to the interval $[0,1]$, are all elements of the finite set $\mathscr{F}$. Then, if $p \le s + 1$,

$$\lim_{j \to \infty} \|\hat{s} - \tilde{s}_j\|_{L^2[0,1]} = 0.$$

This result tells us that the L^2-errors introduced into the approximation by the use of these Filon type schemes tend to zero with $\overline{\Delta}_T^j$. These errors may or may not be small compared to $\|f - \hat{s}\|_{L^2[0,1]}$. By combining the corollary to Theorem 2.3 with this last result, we obtain the following corollary.

Corollary: Let $\{\Delta^j\}_{j=1}^{\infty}$ be a sequence of partitions of [0,1] such that $\lim_{j\to\infty} \overline{\Delta}^j = 0$ and let $\hat{s}_j$, for each j, be the least square spline approximation in $Sp(2m-1,\Delta^j,z)$, $m-1 \le z \le 2m-2$, to $f \in K^{2m}[0,1]$. Given a finite subset $\mathscr{F} \subseteq \mathscr{P}_{s-1}[0,1]$ and a sequence of partitions $\{\Delta_T^j\}_{j=1}^{\infty}$ of [0,1] such that $\lim_{j\to\infty} \overline{\Delta}_T^j = 0$, let $\tilde{s}_j$, for each j, be a discretized least square approximation in $Sp(2m-1,\Delta^j,z)$ to f obtained using a Filon type quadrature scheme based on $(\Delta_T^j)^*$-interpolation where the partitions of the subintervals of Δ_T^j, scaled to the interval [0,1], are all elements of the finite set $\mathscr{F}$. Then, if $2m \le s+1$,

$$\lim_{j\to\infty} \|f - \tilde{s}_j\|_{L^2[0,1]} = 0.$$

Our final result of this section deals with the concept of the consistency of collections of such schemes as defined at the end of the preceding section and follows from Theorems 2.2, 4.1, and 4.3.

Theorem 4.4 — Let $\mathscr{C} = \mathscr{P}[0,1]$, $\mathscr{F} \subseteq \mathscr{P}_{2m-2}[0,1]$, $\mathscr{F}$ finite, $m-1 \le z \le 2m-2$ and, for $\Delta \in \mathscr{C}$, consider approximating the least square approximation in $Sp(2m-1,\Delta,z)$ to $f \in K^{2m}[0,1]$ using a linear functional of the form (4.1) based on $(\Delta_T)^*$-interpolation with $\overline{\Delta}_T \le \overline{\Delta}$ and the partitions of the subintervals of Δ_T, scaled to the interval [0,1], all in $\mathscr{F}$. Then this choice of linear functionals is consistent in the L^2-norm with the bounds for the least square error given by (2.7).

5. NUMERICAL RESULTS

In this section we present our numerical results based on FORTRAN codes of the techniques which we have considered in this paper. Listings of some of these codes and descriptions of their uses are included in Appendix B. We begin with a documentation of experiments designed to test the validity of some of the theoretical results. We follow with examples of least square spline approximations to data sets which are generally considered to be difficult to approximate with polynomials. We conclude with least square spline approximations to empirically determined data sets which are of practical interest. Wherever it seems appropriate, we include comments of computational interest. It seems appropriate now to mention that all numerical results were computed on a UNIVAC 1108.

Let Δ_N be the uniform partition of [0, 1] with mesh length $h_N = 1/(N + 1)$. Fix $m = 1$ or 2 and let $m - 1 \le z \le 2m - 1$. We begin with an examination of the errors in approximating the exponential function, $\exp(x) = e^x$, over [0, 1] by elements of the spline space $Sp(2m - 1, \Delta_N, z)$ using four different techniques. We define the splines $\hat{s}_N$, $\tilde{s}_N^1$, $\tilde{s}_N^2$, and $\tilde{s}_N^3 \in Sp(2m - 1, \Delta_N, z)$ as follows:

$\hat{s}_N$ $\equiv$ Least square approximation to exp as defined in Section 2,

$\tilde{s}_N^1$ $\equiv$ Discretized least square approximation to exp based on a composite interpolatory quadrature scheme as defined in Section 3,

$\tilde{s}_N^2$ $\equiv$ Discretized least square approximation to exp based on a Filon type quadrature scheme using piecewise Lagrange interpolation as defined in Section 4, and

$\tilde{s}_N^3 \equiv$ Least squares approximation to exp based on the standard discrete technique.

We note that $\hat{s}_N$ can be obtained since, for the exponential function, we can compute numerical values for the components of the vector $\hat{\underline{k}}$ of the system (2.19). The discretized least square approximations, $\tilde{s}_N^1$ and $\tilde{s}_N^2$, are obtained by solving the system (2.21) where $\tilde{\underline{k}}$, an approximation to $\hat{\underline{k}}$, in each case is determined by the appropriate quadrature scheme. The standard discrete least squares technique, which is used to obtain $\tilde{s}_N^3$, can be discussed in the context of Section 1 with only slight modifications. A (discrete) semi-inner product is employed instead of an inner product, i.e., property (iv) of (1.1), the defining relations for an inner product, is not satisfied, and the only loss the theory suffers is that the matrix involved cannot be guaranteed to be positive definite. Of course, the potential instability in solving the corresponding system must be considered when employing this purely discrete technique.

Theorems 2.3, 2.4, 3.2, 4.1, and 4.3 are employed to obtain the following appraisals where K, K^∞, K_1, and K_2 are all positive constants independent of h_N.

$$||\exp - \hat{s}_N||_{L^2[0,1]} \le K(h_N)^{2m}, \tag{5.1}$$

$$||\exp - \hat{s}_N||_{L^\infty[0,1]} \le K^\infty(h_N)^{2m-1/2}, \tag{5.2}$$

$$||\hat{s}_N - \tilde{s}_N^1||_{L^2[0,1]} \le K_1(h_N)^{n-2m+3/2}, \tag{5.3}$$

where n is the order of interpolatory quadrature in terms of which $\tilde{s}_N^1$ is defined, and

$$||\hat{s}_N - \tilde{s}_N^2||_{L^2[0,1]} \le K_2(h_T)^{s+1}, \tag{5.4}$$

where s is the degree of piecewise Lagrange interpolation employed in the Filon quadrature in terms of which $\tilde{s}_N^2$ is defined and h_T is the mesh width for the distribution of data for this interpolation technique. Combining (5.1) with (5.3) yields

$$||\exp - \tilde{s}_N^1||_{L^2[0,1]} \le [K + K_1](h_N)^{\min(2m, n-2m+3/2)} \tag{5.5}$$

and (5.1) with (5.4) yields

$$||\exp - \tilde{s}_N^2||_{L^2[0,1]} \le [K + K_2]\{\max(h_N, h_T)\}^{\min(2m, s+1)}. \tag{5.6}$$

We have no bounds for the error in the fourth approximation. However, for certain weighted discrete techniques, the results of Section 3 are valid. Explanatory remarks are in order. We observe, for example, that the interpolatory schemes of order n employed in Section 3 are exact for polynomials of degree $\le n$. Consequently, if $n \ge 4m - 1$, composite interpolatory schemes of order n are exact for products of splines in $Sp(2m - 1, \Delta, z)$ and, in particular, for the entries of the least square matrix defined in (2.20). Then, for the discrete technique with weights from the composite interpolatory scheme, Theorem 3.2 holds and we have appraisals in these special cases.

Our first numerical results are presented in Tables 1, 2, and 3. We give approximate numerical values for the quantities $||\exp - \hat{s}_N||_{L^2[0,1]}$ and $||\exp - \hat{s}_N||_{L^\infty[0,1]}$ for the spline spaces $Sp(1, \Delta_N, 0)$, $Sp(3, \Delta_N, 2)$, and $Sp(3, \Delta_N, 1)$

TABLE 1. Least Square Linear Spline Approximation of the Exponential Function

h_N	$\|\|exp - \hat{s}_N\|\|_{L^2[0,1]}$	α	$\|\|exp - \hat{s}_N\|\|_{L^\infty[0,1]}$	α
1/2	$1.68 \cdot 10^{-2}$	--	$5.00 \cdot 10^{-2}$	--
1/3	$7.44 \cdot 10^{-3}$	2.01	$2.31 \cdot 10^{-2}$	1.90
1/4	$4.18 \cdot 10^{-3}$	2.00	$1.33 \cdot 10^{-2}$	1.92
1/5	$2.68 \cdot 10^{-3}$	2.00	$8.63 \cdot 10^{-3}$	1.94
1/6	$1.86 \cdot 10^{-3}$	2.00	$6.04 \cdot 10^{-3}$	1.95
1/7	$1.36 \cdot 10^{-3}$	2.00	$4.47 \cdot 10^{-3}$	1.96
1/8	$1.04 \cdot 10^{-3}$	2.00	$3.44 \cdot 10^{-3}$	1.97

TABLE 2. Least Square Cubic Spline Approximation of the Exponential Function

h_N	$\|\|exp - \hat{s}_N\|\|_{L^2[0,1]}$	α	$\|\|exp - \hat{s}_N\|\|_{L^\infty[0,1]}$	α
1/2	$4.53 \cdot 10^{-5}$	--	$1.82 \cdot 10^{-4}$	--
1/3	$1.63 \cdot 10^{-5}$	2.52	$3.11 \cdot 10^{-5}$	4.36
1/4	$5.30 \cdot 10^{-6}$	3.90	$1.09 \cdot 10^{-5}$	3.66
1/5	$2.30 \cdot 10^{-6}$	3.73	$4.81 \cdot 10^{-6}$	3.65
1/6	$1.13 \cdot 10^{-6}$	3.91	$2.40 \cdot 10^{-6}$	3.82
1/7	$6.21 \cdot 10^{-7}$	3.87	$1.35 \cdot 10^{-6}$	3.74
1/8	$3.68 \cdot 10^{-7}$	3.92	$8.06 \cdot 10^{-7}$	3.85

TABLE 3. Least Square Cubic Hermite Spline Approximation of the Exponential Function

h_N	$\|\|\exp - \hat{s}_N\|\|_{L^2[0,1]}$	α	$\|\|\exp - \hat{s}_N\|\|_{L^\infty[0,1]}$	α
1/2	$4.25 \cdot 10^{-5}$	--	$1.48 \cdot 10^{-4}$	--
1/3	$1.16 \cdot 10^{-5}$	3.20	$3.74 \cdot 10^{-5}$	3.38
1/4	$4.32 \cdot 10^{-6}$	3.44	$1.31 \cdot 10^{-5}$	3.65
1/5	$1.94 \cdot 10^{-6}$	3.60	$5.65 \cdot 10^{-6}$	3.77
1/6	$9.87 \cdot 10^{-7}$	3.69	$2.81 \cdot 10^{-6}$	3.83
1/7	$5.53 \cdot 10^{-7}$	3.75	$1.55 \cdot 10^{-6}$	3.86
1/8	$3.33 \cdot 10^{-7}$	3.79	$9.24 \cdot 10^{-7}$	3.88

for N = 1,2,...,7. Following [6], for each pair of consecutive entries, we have included the quantity

$$\alpha \equiv \log\left(\|\exp - \hat{s}_{n_1}\| / \|\exp - \hat{s}_{n_2}\|\right) / \log(h_{n_1}/h_{n_2}) \qquad (5.7)$$

defined in terms of successive values of the mesh spacing, $h_{n_1} > h_{n_2}$. The motivation for the definition (5.7) is the fact that as $h_n \to 0$ we have

$$\|\exp - \hat{s}_N\| \sim \mathscr{K}(h_N)^{\alpha}$$

for some constants α and $\mathscr{K}$ depending on the norm $\|\cdot\|$, but not on h_N. Then for two successive values of h, $h_{n_1} > h_{n_2}$,

$$\|\exp - \hat{s}_{n_1}\| / \|\exp - \hat{s}_{n_2}\| \sim (h_{n_1}/h_{n_2})^{\alpha}$$

from which the definition of α follows. In the tables enough values of h are given to see that the computed exponents of (5.7) in the L^2-norm are converging to the asymptotic values given by (5.1), i.e., $\alpha \sim 2m$. The loss predicted by (5.2) of 1/2 of an order of accuracy in moving from the L^2-norm to the L^{∞}-norm is apparently not realized in this case.

Tables 4, 5, and 6 include, for several values of n, approximate numerical values for the quantities $\|\hat{s}_N - \tilde{s}_N^1\|_{L^2[0,1]}$ where $\tilde{s}_N^1$ is an approximation in $Sp(2m-1, \Delta_N, z)$, $m = 1,2, m-1 \le z \le 2m-2$, to $\hat{s}_N$ determined by a composite interpolatory formula based on n+1-point open Newton-Cotes quadrature formulae. Again, we include the quantity α. The order of accuracy predicted by (5.3) as a function of n and m is $n - 2m + 3/2$ or $(2n - 4m + 3)/2$. We observe the following

TABLE 4. Interpolatory Quadrature and Linear Spline Spaces

h_N	n=1 $\|\|\hat{s} - \tilde{s}_N^1\|\|_{L^2[0,1]}$	α	n=2 $\|\|\hat{s} - \tilde{s}_N^1\|\|_{L^2[0,1]}$	α	n=3 $\|\|\hat{s} - \tilde{s}_N^1\|\|_{L^2[0,1]}$	α
1/2	$3.59 \cdot 10^{-2}$	--	$2.26 \cdot 10^{-4}$	--	$1.17 \cdot 10^{-4}$	--
1/3	$2.12 \cdot 10^{-2}$	1.30	$6.01 \cdot 10^{-5}$	3.27	$3.11 \cdot 10^{-5}$	3.27
1/4	$1.37 \cdot 10^{-2}$	1.53	$2.19 \cdot 10^{-5}$	3.51	$1.13 \cdot 10^{-5}$	3.51
1/5	$9.88 \cdot 10^{-3}$	1.45	$1.02 \cdot 10^{-5}$	3.44	$5.26 \cdot 10^{-6}$	3.44
1/6	$7.54 \cdot 10^{-3}$	1.48	$5.40 \cdot 10^{-6}$	3.48	$2.79 \cdot 10^{-6}$	3.47
1/7	$6.01 \cdot 10^{-3}$	1.48	$3.16 \cdot 10^{-6}$	3.46	$1.63 \cdot 10^{-6}$	3.48
1/8	$4.93 \cdot 10^{-3}$	1.48	$1.99 \cdot 10^{-6}$	3.48	$1.02 \cdot 10^{-6}$	3.50

TABLE 5. Interpolatory Quadrature and Cubic Spline Spaces

h_N	n=3 $\|\hat{s}_N - \tilde{s}_N^1\|_{L^2[0,1]}$	α	n=4 $\|\hat{s}_N - \tilde{s}_N^1\|_{L^2[0,1]}$	α	n=5 $\|\hat{s}_N - \tilde{s}_N^1\|_{L^2[0,1]}$	α	n=6 $\|\hat{s}_N - \tilde{s}_N^1\|_{L^2[0,1]}$	α	n=7 $\|\hat{s}_N - \tilde{s}_N^1\|_{L^2[0,1]}$	α
1/2	$1.59 \cdot 10^{-2}$	--	$9.13 \cdot 10^{-5}$	--	$5.49 \cdot 10^{-5}$	--	$1.06 \cdot 10^{-7}$	--	$6.17 \cdot 10^{-8}$	--
1/3	$9.88 \cdot 10^{-3}$	1.17	$2.77 \cdot 10^{-5}$	2.94	$1.66 \cdot 10^{-5}$	2.94	$1.48 \cdot 10^{-8}$	4.86	$9.44 \cdot 10^{-9}$	4.86
1/4	$5.13 \cdot 10^{-3}$	2.28	$8.02 \cdot 10^{-6}$	4.30	$4.82 \cdot 10^{-6}$	4.30	$2.40 \cdot 10^{-9}$	6.31	$1.54 \cdot 10^{-9}$	6.31
1/5	$3.98 \cdot 10^{-3}$	1.14	$4.06 \cdot 10^{-6}$	3.05	$2.44 \cdot 10^{-6}$	3.05	$7.82 \cdot 10^{-10}$	5.03	$5.00 \cdot 10^{-10}$	5.03
1/6	$2.86 \cdot 10^{-3}$	1.81	$2.03 \cdot 10^{-6}$	3.80	$1.22 \cdot 10^{-6}$	3.80	$2.72 \cdot 10^{-10}$	5.80	$1.74 \cdot 10^{-10}$	5.80
1/7	$2.33 \cdot 10^{-3}$	1.33	$1.22 \cdot 10^{-6}$	3.29	$7.34 \cdot 10^{-7}$	3.29	$1.20 \cdot 10^{-10}$	5.28	$7.70 \cdot 10^{-11}$	5.27
1/8	$1.88 \cdot 10^{-3}$	1.62	$7.55 \cdot 10^{-7}$	3.60	$4.54 \cdot 10^{-7}$	3.60	$5.68 \cdot 10^{-11}$	5.63	$3.62 \cdot 10^{-11}$	5.64

TABLE 6. Interpolatory Quadrature and Cubic Hermite Spline Spaces

h_N	n=3 $\|\|\hat{s}_N - \tilde{s}_N^1\|\|_{L^2[0,1]}$	α	n=4 $\|\|\hat{s}_N - \tilde{s}_N^1\|\|_{L^2[0,1]}$	α	n=5 $\|\|\hat{s}_N - \tilde{s}_N^1\|\|_{L^2[0,1]}$	α	n=6 $\|\|\hat{s}_N - \tilde{s}_N^1\|\|_{L^2[0,1]}$	α	n=7 $\|\|\hat{s}_N - \tilde{s}_N^1\|\|_{L^2[0,1]}$	α
1/2	$5.90 \cdot 10^{-2}$	--	$3.82 \cdot 10^{-4}$	--	$2.30 \cdot 10^{-4}$	--	$4.63 \cdot 10^{-7}$	--	$2.96 \cdot 10^{-7}$	--
1/3	$3.62 \cdot 10^{-2}$	1.21	$1.05 \cdot 10^{-4}$	3.19	$6.30 \cdot 10^{-5}$	3.19	$5.66 \cdot 10^{-8}$	5.19	$3.61 \cdot 10^{-8}$	5.19
1/4	$2.60 \cdot 10^{-2}$	1.15	$4.26 \cdot 10^{-5}$	3.13	$2.56 \cdot 10^{-5}$	3.13	$1.29 \cdot 10^{-8}$	5.13	$8.26 \cdot 10^{-9}$	5.13
1/5	$2.03 \cdot 10^{-2}$	1.12	$2.13 \cdot 10^{-5}$	3.11	$1.28 \cdot 10^{-5}$	3.11	$4.14 \cdot 10^{-9}$	5.11	$2.64 \cdot 10^{-9}$	5.11
1/6	$1.66 \cdot 10^{-2}$	1.11	$1.21 \cdot 10^{-5}$	3.10	$7.26 \cdot 10^{-6}$	3.10	$1.63 \cdot 10^{-9}$	5.10	$1.04 \cdot 10^{-9}$	5.10
1/7	$1.40 \cdot 10^{-2}$	1.09	$7.51 \cdot 10^{-6}$	3.09	$4.51 \cdot 10^{-6}$	3.09	$7.46 \cdot 10^{-10}$	5.09	$4.76 \cdot 10^{-10}$	5.09
1/8	$1.21 \cdot 10^{-2}$	1.09	$4.98 \cdot 10^{-6}$	3.08	$2.99 \cdot 10^{-6}$	3.08	$3.78 \cdot 10^{-10}$	5.09	$2.41 \cdot 10^{-10}$	5.09

discrepancies between the observed and predicted values of the order of accuracy of this technique in the L^2-norm. For the linear spline spaces, $Sp(1, \Delta_N, 0)$, we observe the values of 1.5, 3.5, and 3.5 for the limiting values of α when $n = 1, 2$, and 3, respectively, and yet the values predicted by the theoretical results are 0.5, 1.5, and 2.5. We observe, however, that special error bounds can be derived for odd point (even values of n) Newton-Cotes formulae which yield an additional order of accuracy. Consequently, we observe a constant discrepancy of one between the predicted and observed orders of accuracy in approximating the least square linear spline approximation to the exponential function using this type of discretized technique. We note that the corresponding table in the L^∞-norm reflects the loss of a half of an order of accuracy predicted by theoretical considerations. However, we have not included this table in this presentation of our numerical results. In the cubic case, i.e., $m = 2$ and $z = 2$, we observe the numbers of 1.5, 3.5, 3.5, 5.5, and 5.5 for the limiting values of α when $n = 3, 4, 5, 6$, and 7 and again find discrepancies with the predicted values of 0.5, 1.5, 2.5, 3.5, and 4.5. Considering the additional order of accuracy for odd point Newton-Cotes formulae, we again observe a constant discrepancy of one order of accuracy between the computed and predicted values of α. We also note that a predicted loss of a half an order of accuracy can be observed when the corresponding table in the L^∞-norm is computed. Finally, for the cubic Hermite spline spaces, i.e., $m = 2$ and $z = 1$, the observed values of α are tending to 1.0, 3.0, 3.0, 5.0, and 5.0 for $n = 3,4,5,6,7$ and the predicted values based on the special bounds for the odd point Newton-Cotes formulae are 1.5, 3.5, 3.5, 5.5, and 5.5 just as in the cubic case. However, here we observe the constant discrepancy of one half of an order of accuracy between the predicted and observed values for α. In this case, the loss of a half order of accuracy predicted by theoretical considerations when using L^2-error bounds and Sobolev type inequalities to generate L^∞ error bounds is not observed. These same discrepancies were observed in least square approximation of the function sin (2x), $0 \le x \le 1$. The analogous tables for the Filon type quadrature schemes show no discrepancies between the predicted and observed values of α. Consequently, we omit them from this presentation of numerical results.

Corresponding to the spline spaces $Sp(1, \Delta_N, 0)$, $Sp(3, \Delta_N, 2)$, and $Sp(3, \Delta_N, 1)$, respectively, in Tables 7, 8, and 9, we present approximate numerical values for the quantities $\|\exp - \tilde{s}_N^1\|_{L^2[0,1]}$, $\|\exp - \tilde{s}_N^2\|_{L^2[0,1]}$, and $\|\exp - \tilde{s}_N^3\|_{L^2[0,1]}$ where the quadrature schemes used to determine the discretized spline approximations are chosen to be consistent with the L^2-bounds for the least square error as given by (5.1). Specifically, the composite interpolatory formula employed to determine $\tilde{s}_N^1 \in Sp(2m - 1, \Delta_N, z)$ is based on $(4m - 1)$ - pt open ended Newton-Cotes formulae and the Filon scheme used to determine $\tilde{s}_N^2$ is based on piecewise Lagrange interpolation of degree $2m - 1$. For any fixed value of N, the data points used to determine the approximations are the same for each technique. Again the quantity, α, as defined in previous tables is included. We note that both discretized approximations, $\tilde{s}_N^1$ and $\tilde{s}_N^2$, exhibit the consistent behavior predicted by our theoretical considerations. We also note that the standard discrete least square technique generates spline approximations, $\tilde{s}_N^3$, which also exhibit this consistent behavior.

TABLE 7. Consistent Quadrature Schemes for Linear Spline Spaces

h_N	$\|\|\exp - \tilde{s}_N^1\|\|_{L^2[0,1]}$	α	$\|\|\exp - \tilde{s}_N^2\|\|_{L^2[0,1]}$	α	$\|\|\exp - \tilde{s}_N^3\|\|_{L^2[0,1]}$	α
1/2	$1.68 \cdot 10^{-2}$	--	$1.72 \cdot 10^{-2}$	--	$1.69 \cdot 10^{-2}$	--
1/3	$7.48 \cdot 10^{-3}$	2.01	$7.62 \cdot 10^{-3}$	2.00	$7.51 \cdot 10^{-3}$	2.00
1/4	$4.18 \cdot 10^{-3}$	2.00	$4.29 \cdot 10^{-3}$	2.00	$4.23 \cdot 10^{-3}$	2.00
1/5	$2.68 \cdot 10^{-3}$	2.00	$2.75 \cdot 10^{-3}$	2.00	$2.70 \cdot 10^{-3}$	2.00
1/6	$1.86 \cdot 10^{-3}$	2.00	$1.91 \cdot 10^{-3}$	2.00	$1.87 \cdot 10^{-3}$	2.00
1/7	$1.36 \cdot 10^{-3}$	2.00	$1.40 \cdot 10^{-3}$	2.00	$1.38 \cdot 10^{-3}$	2.00
1/8	$1.04 \cdot 10^{-3}$	2.00	$1.07 \cdot 10^{-3}$	2.00	$1.05 \cdot 10^{-3}$	2.00

TABLE 8. Consistent Quadrature Schemes for Cubic Spline Spaces

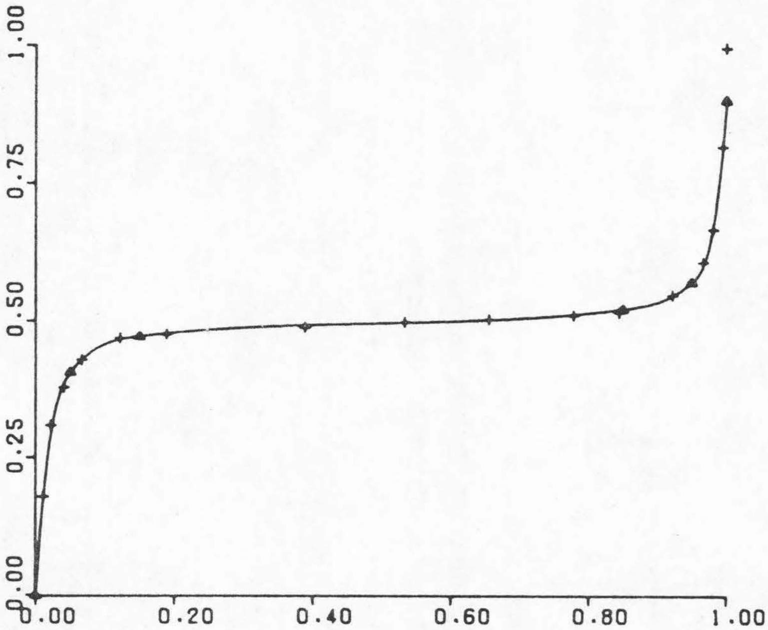
h_N	$\|\|\exp - \tilde{s}_N^1\|\|_{L^2[0,1]}$	α	$\|\|\exp - \tilde{s}_N^2\|\|_{L^2[0,1]}$	α	$\|\|\exp - \tilde{s}_N^3\|\|_{L^2[0,1]}$	α
1/2	$4.54 \cdot 10^{-5}$	--	$4.53 \cdot 10^{-5}$	--	$4.64 \cdot 10^{-5}$	--
1/3	$1.63 \cdot 10^{-5}$	2.52	$1.63 \cdot 10^{-5}$	2.52	$1.63 \cdot 10^{-5}$	2.58
1/4	$5.30 \cdot 10^{-6}$	3.90	$5.30 \cdot 10^{-6}$	3.90	$5.31 \cdot 10^{-6}$	3.90
1/5	$2.30 \cdot 10^{-6}$	3.73	$2.30 \cdot 10^{-6}$	3.73	$2.30 \cdot 10^{-6}$	3.74
1/6	$1.13 \cdot 10^{-6}$	3.91	$1.13 \cdot 10^{-6}$	3.91	$1.13 \cdot 10^{-6}$	3.91
1/7	$6.22 \cdot 10^{-7}$	3.87	$6.22 \cdot 10^{-7}$	3.87	$6.22 \cdot 10^{-7}$	3.87
1/8	$3.68 \cdot 10^{-7}$	3.92	$3.68 \cdot 10^{-7}$	3.92	$3.68 \cdot 10^{-7}$	3.92

TABLE 9. Consistent Quadrature Schemes for Cubic Hermite Spline Spaces

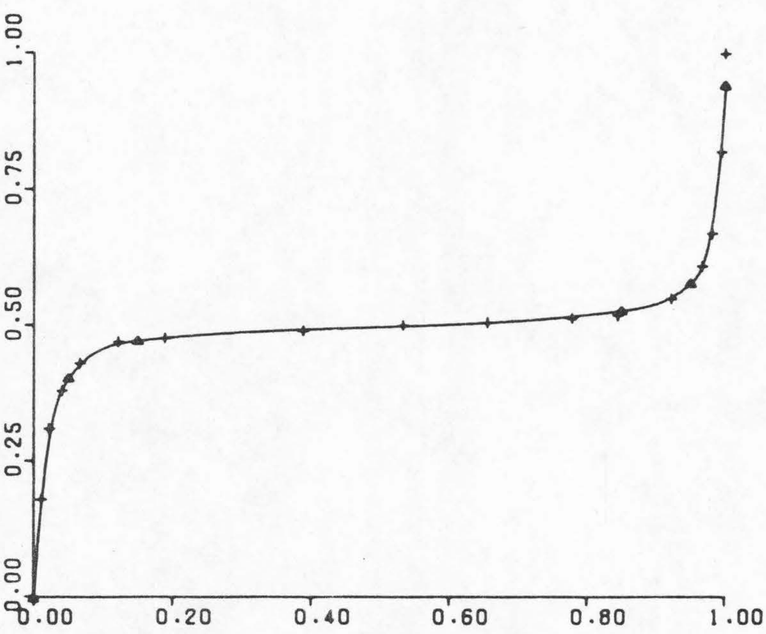
h_N	$\|\|\exp - \tilde{s}_N^1\|\|_{L^2[0,1]}$	α	$\|\|\exp - \tilde{s}_N^2\|\|_{L^2[0,1]}$	α	$\|\|\exp - \tilde{s}_N^3\|\|_{L^2[0,1]}$	α
1/2	$4.26 \cdot 10^{-5}$	--	$4.26 \cdot 10^{-5}$	--	$4.36 \cdot 10^{-5}$	--
1/3	$1.16 \cdot 10^{-5}$	3.20	$1.16 \cdot 10^{-5}$	3.20	$1.18 \cdot 10^{-5}$	3.22
1/4	$4.32 \cdot 10^{-6}$	3.45	$4.32 \cdot 10^{-6}$	3.44	$4.36 \cdot 10^{-6}$	3.46
1/5	$1.94 \cdot 10^{-6}$	3.60	$1.94 \cdot 10^{-6}$	3.60	$1.95 \cdot 10^{-6}$	3.61
1/6	$9.87 \cdot 10^{-7}$	3.69	$9.87 \cdot 10^{-7}$	3.69	$9.93 \cdot 10^{-7}$	3.70
1/7	$5.53 \cdot 10^{-7}$	3.75	$5.53 \cdot 10^{-7}$	3.75	$5.56 \cdot 10^{-7}$	3.76
1/8	$3.33 \cdot 10^{-7}$	3.79	$3.33 \cdot 10^{-7}$	3.79	$3.35 \cdot 10^{-7}$	3.80

We now turn to the approximation of data sets of special interest. The first two of these data sets are considered difficult to approximate using polynomials because of "their inability to take sharp bends followed by relatively flat behavior" (cf. [11, p. 15]). In Fig. 1 we present plots of two different spline approximations in $Sp(3,\Delta,2)$ with $\Delta \equiv \{0.0, 0.05, 0.15, 0.85, 0.95, 1.0\}$ to the data points (cf. [15]). On these plots, the knots are denoted Δ and the data points +. The upper plot represents a spline whose coefficients were determined using the standard discrete least square technique. The lower plot corresponds to a discretized least square approximation determined by Filon quadrature based on piecewise linear interpolation. Plots of the analogous approximations for another data set of similar interest (cf. [9]) are given in Fig. 2.

We conclude this section with the approximation of seven sets of data representing the velocity of sound in water versus depth. Of course, such data depend on many things including longitude, latitude, and the meteorological conditions where and when these velocities were determined. The use of cubic spline interpolates in ray tracing algorithms as approximations to sound velocity profiles has recently been investigated (cf. [10]). In Figs. 3 through 9 we present plots of two cubic spline approximations to the appropriate data sets. The techniques used to determine these approximations as well as our notation in the plots are identical to those of Figs. 1 and 2. We employed the partition $\Delta \equiv \{0.0, 400.0, 800.0, 1{,}600.0, 3{,}200.0, 6{,}400.0, 18{,}000.0\}$ for all splines in Figs. 3 through 9. We remark that the sparseness of the data for the deeper

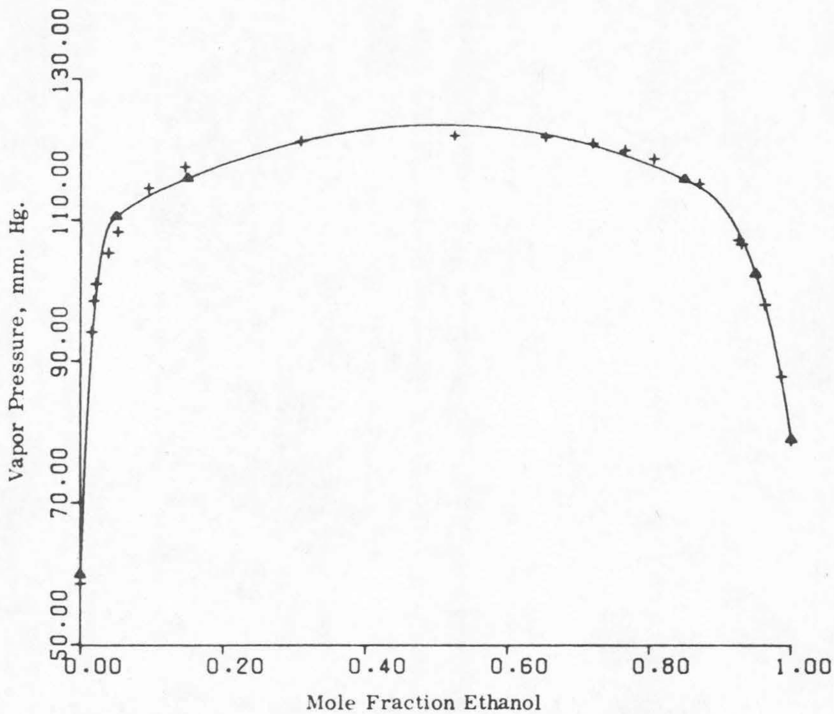


Discrete Least Squares Polynomial Spline Approximation

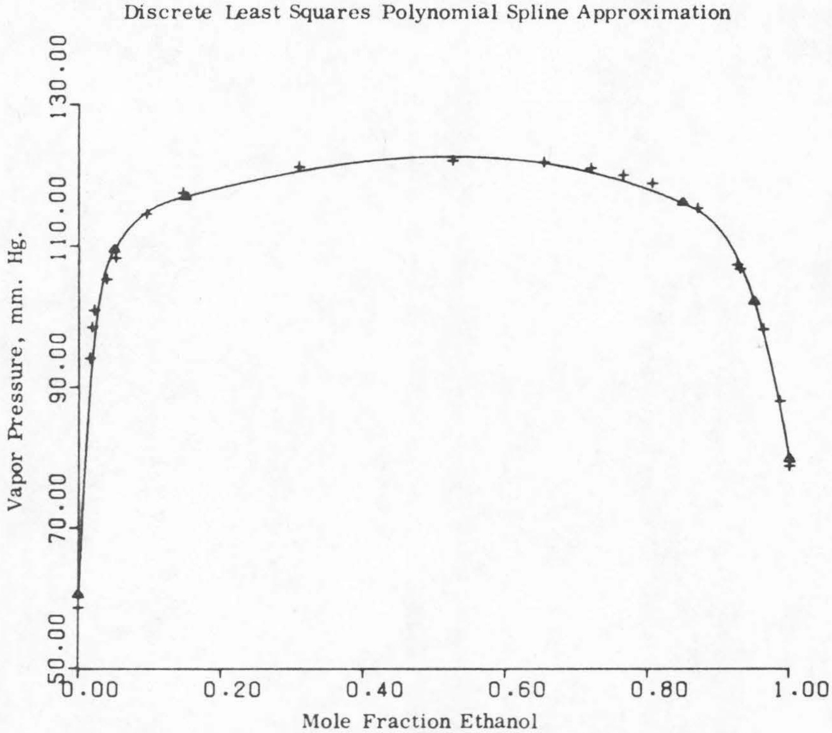


Discretized Integral Least Square Polynomial Spline Approximation

FIG. 1. A "Difficult" Data Set

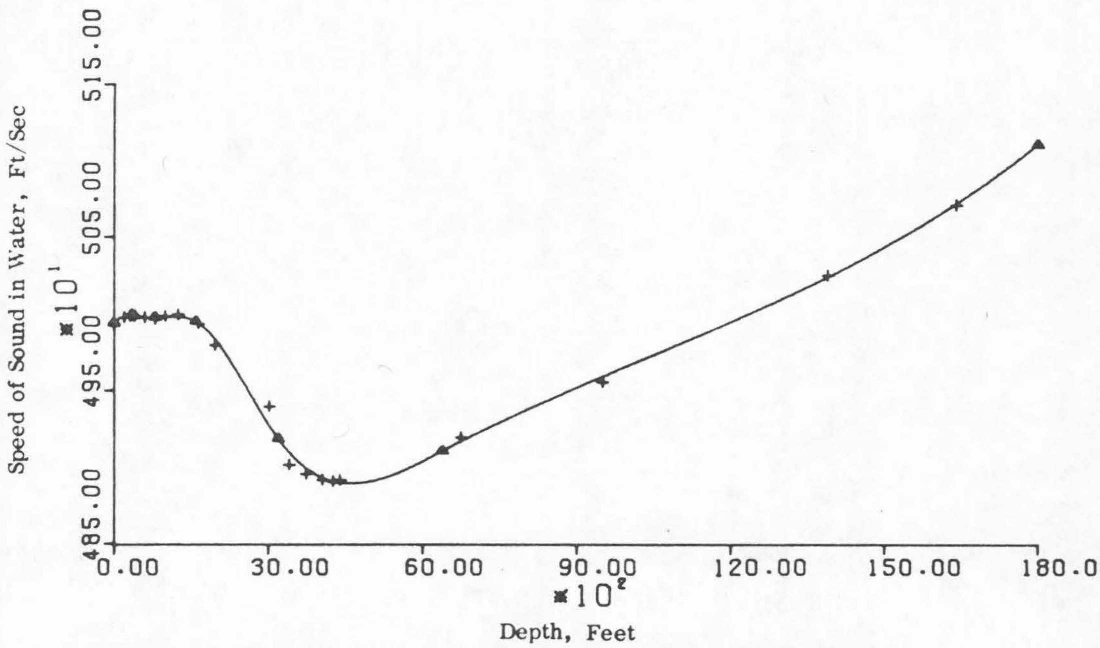


Discrete Least Squares Polynomial Spline Approximation

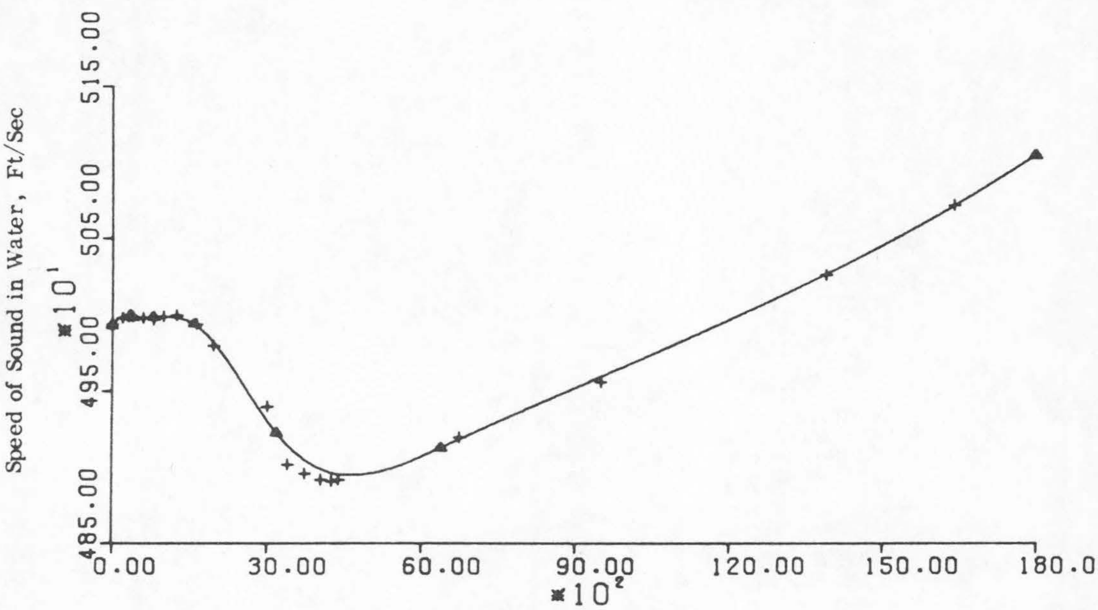


Discretized Integral Least Square Polynomial Spline Approximation

FIG. 2. Data for Ethanol — n-heptane Vapor Pressure.

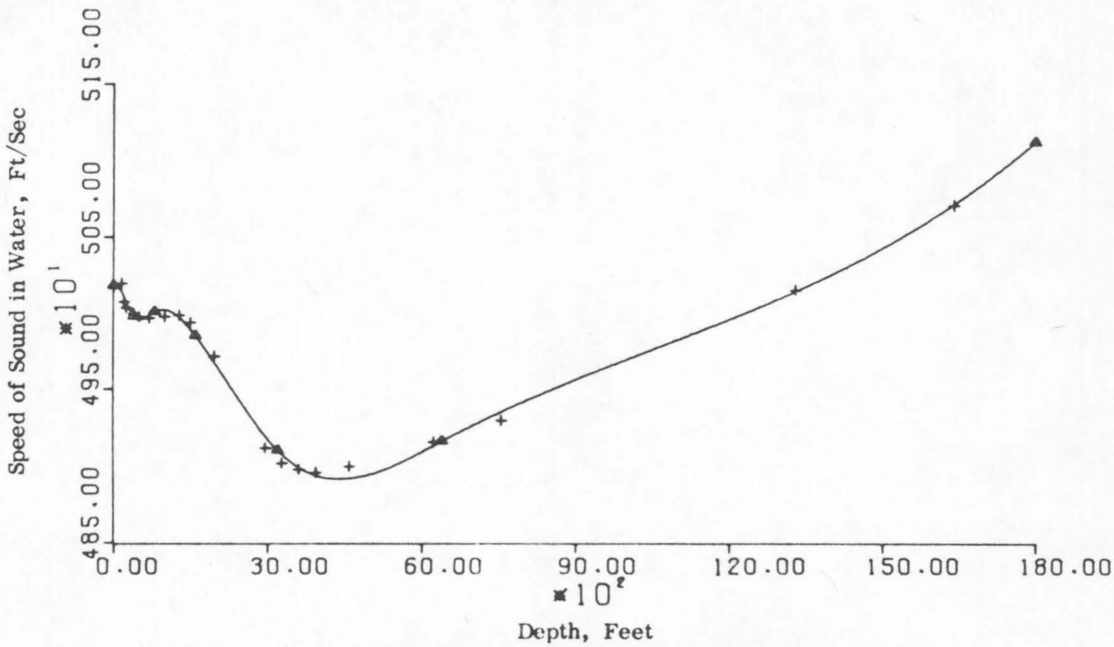


Discrete Least Squares Polynomial Spline Approximation

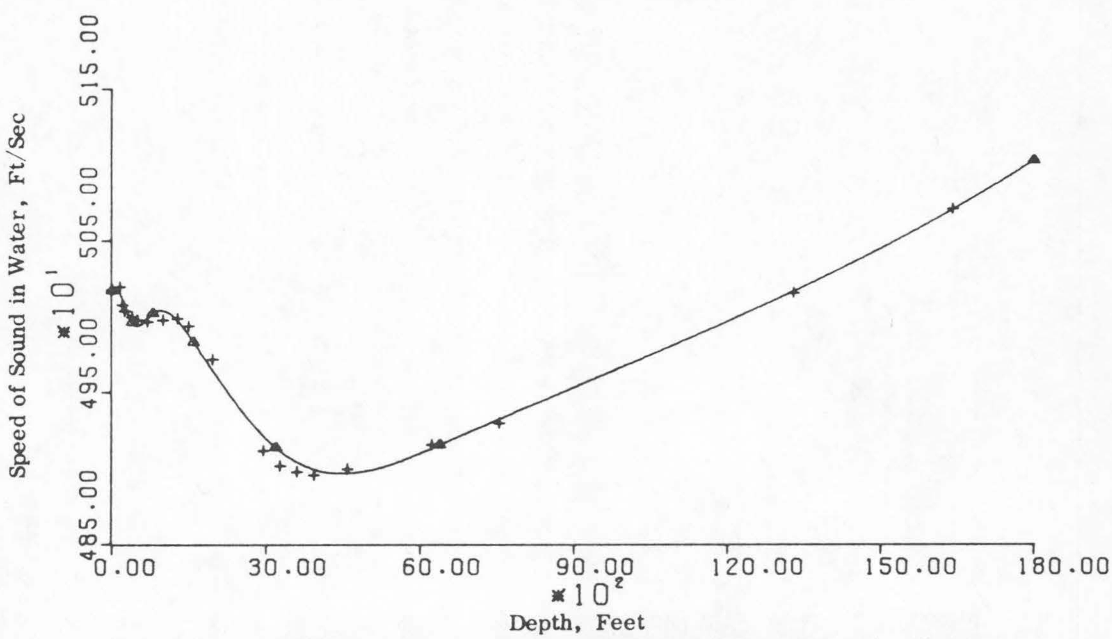


Discretized Integral Least Square Polynomial Spline Approximation

FIG. 3. Data for Ocean Area Alpha — Winter.

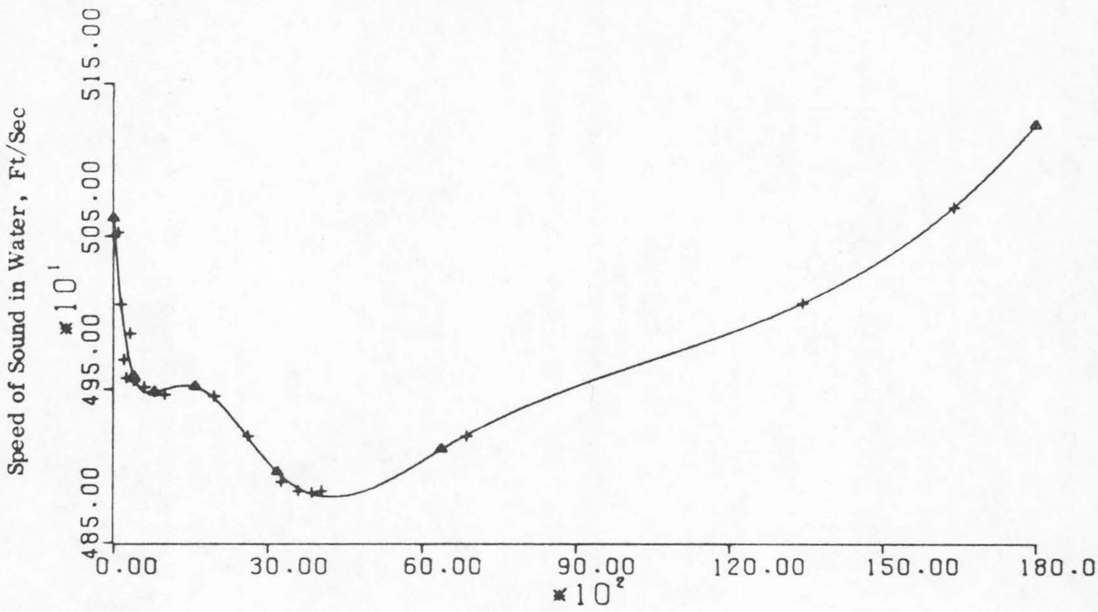


Discrete Least Squares Polynomial Spline Approximation

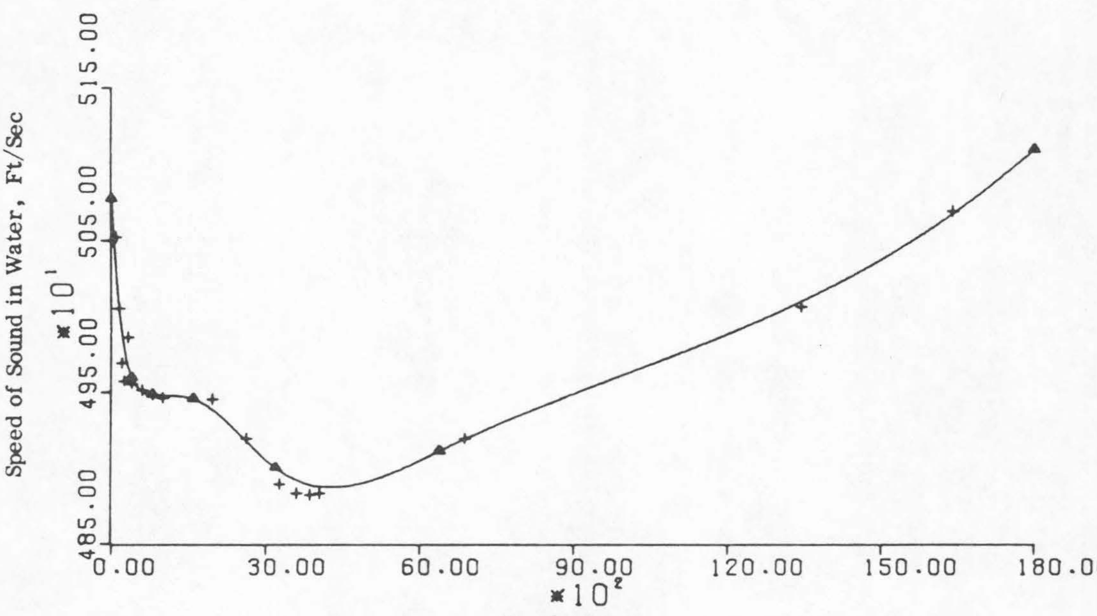


Discretized Integral Least Square Polynomial Spline Approximation

FIG. 4. Data for Ocean Area Alpha — Spring.

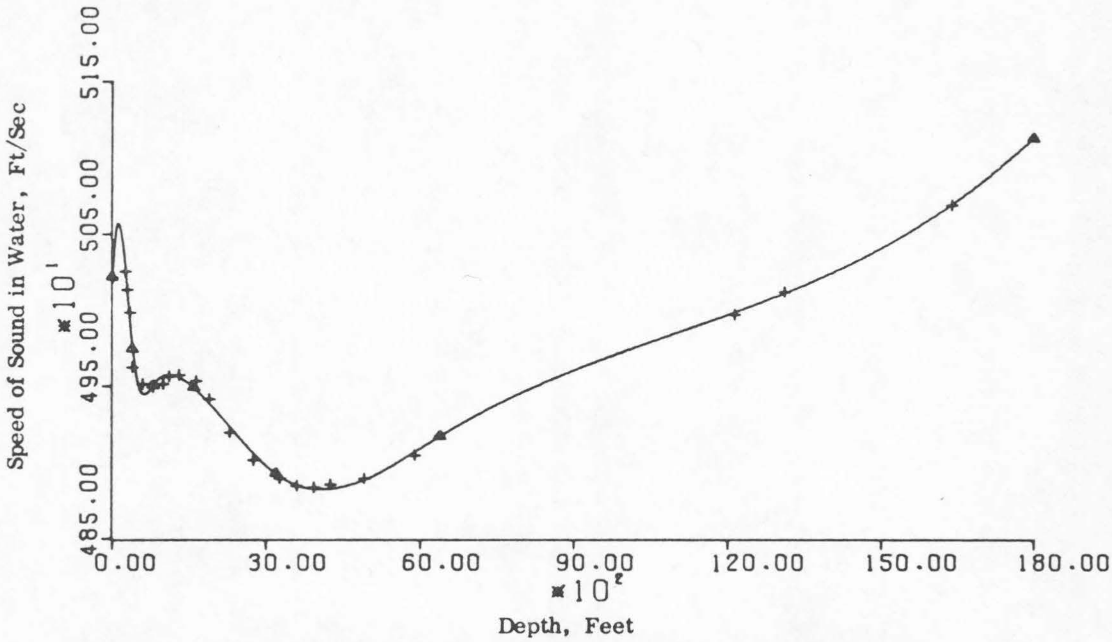


Discrete Least Squares Polynomial Spline Approximation

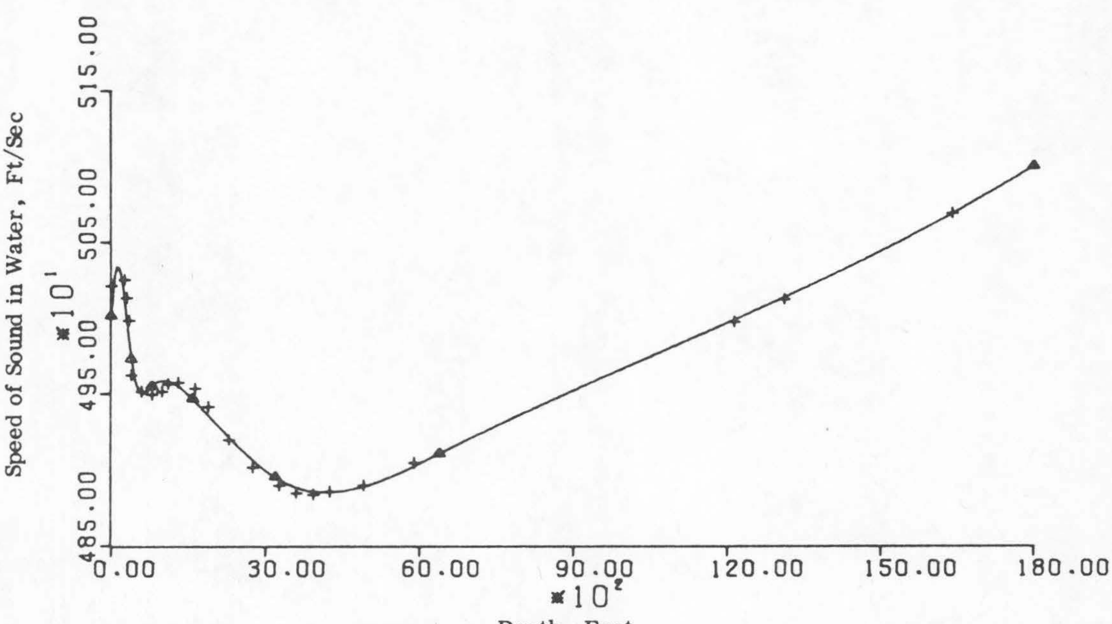


Discretized Integral Least Square Polynomial Spline Approximation

FIG. 5. Data for Ocean Area Alpha — Summer.

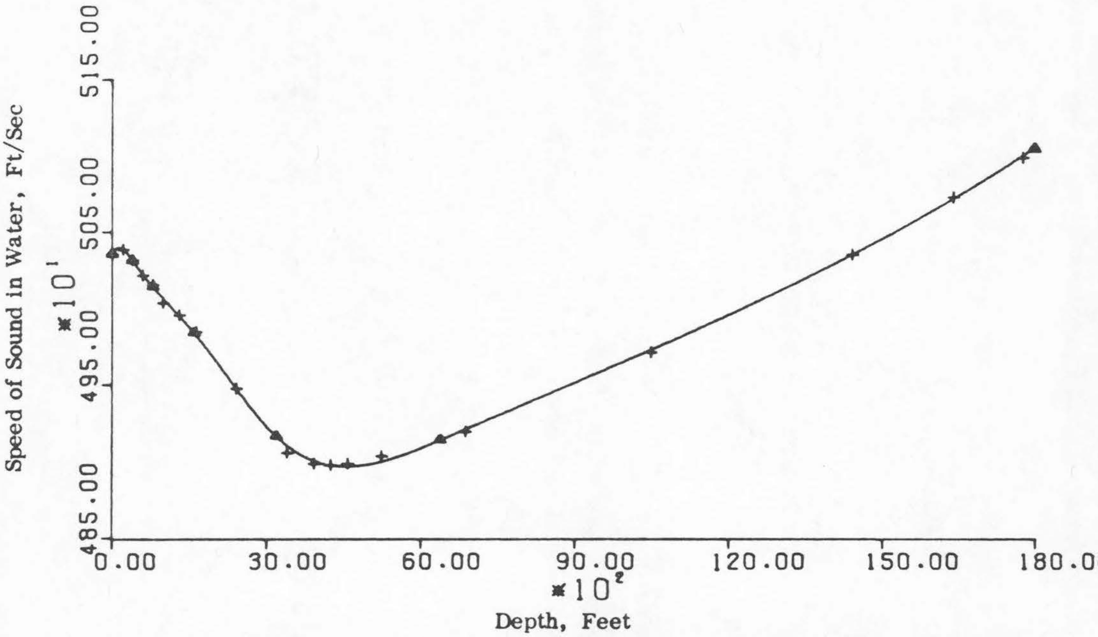


Discrete Least Squares Polynomial Spline Approximation

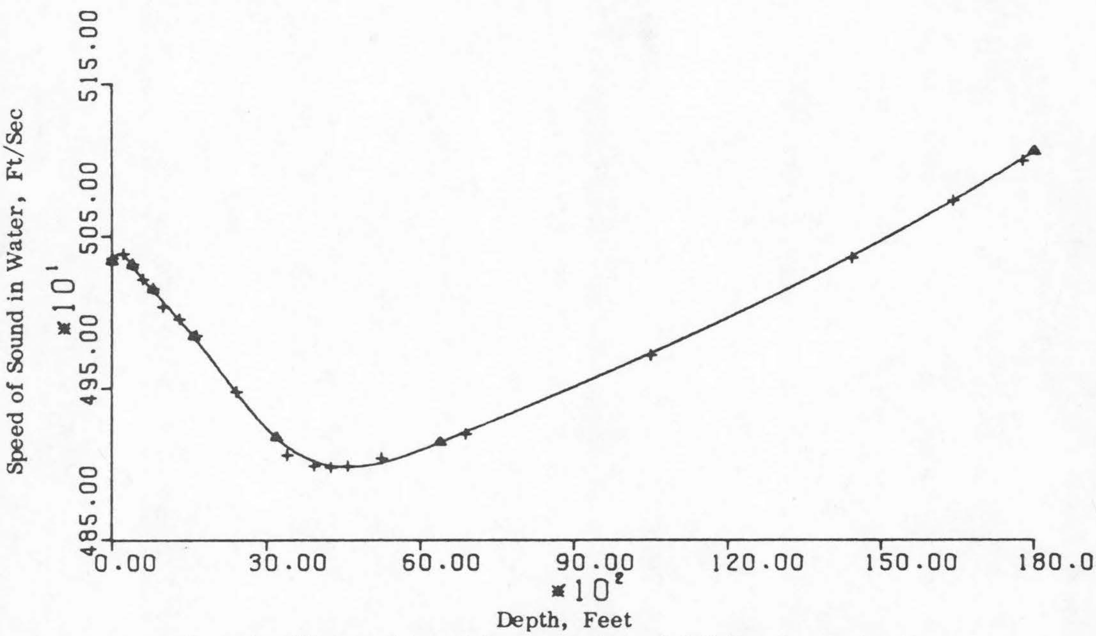


Discretized Integral Least Square Polynomial Spline Approximation

FIG. 6. Data for Ocean Area Alpha — Autumn.

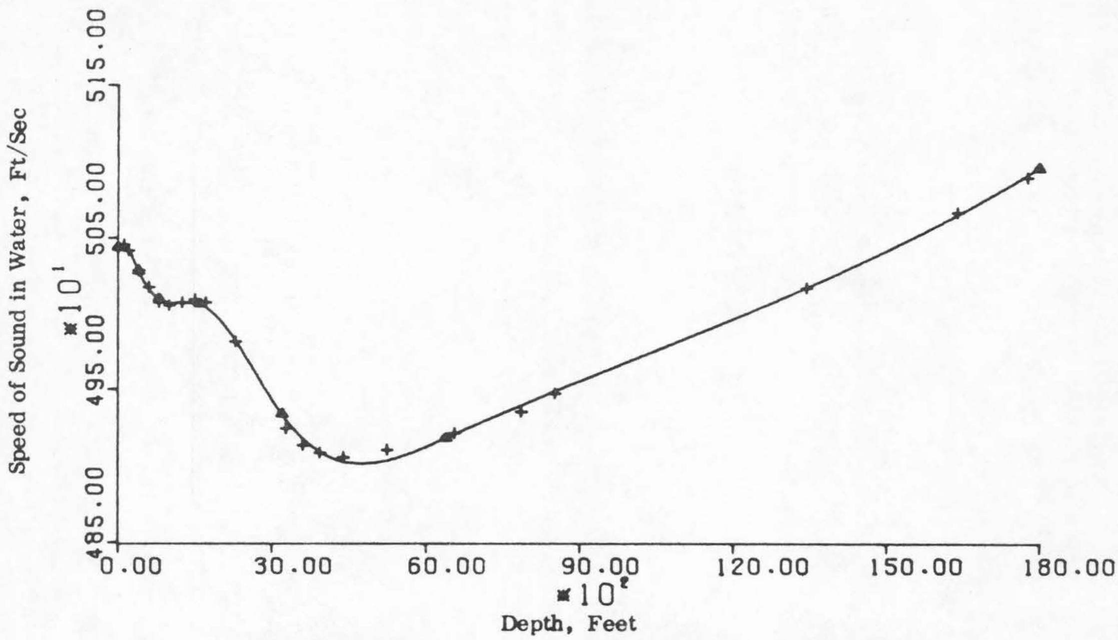


Discrete Least Squares Polynomial Spline Approximation

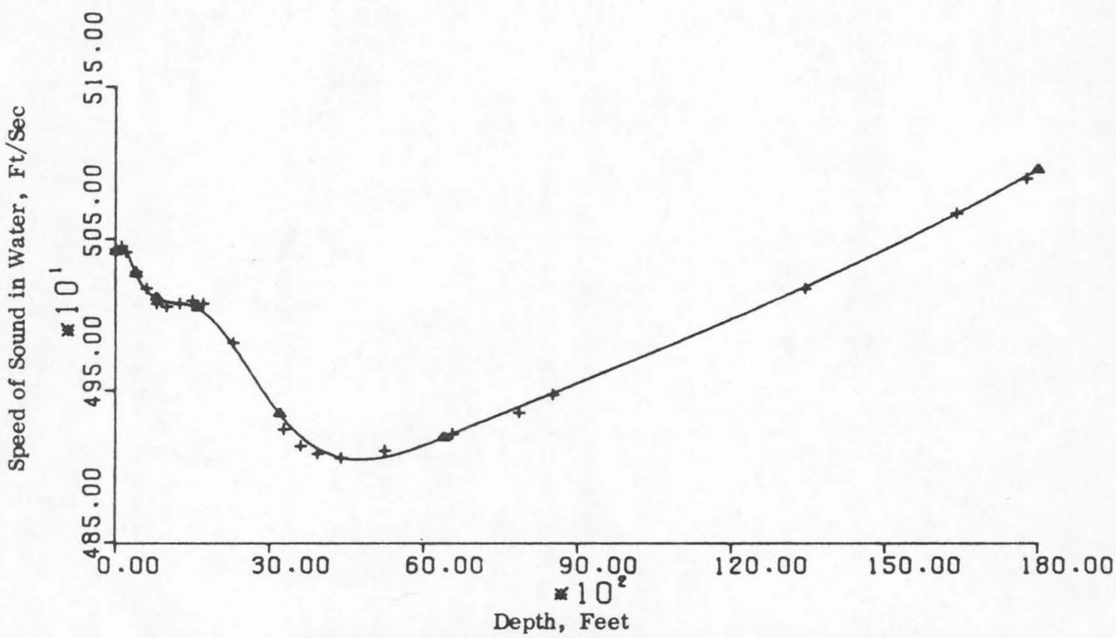


Discretized Integral Least Square Polynomial Spline Approximation

FIG. 7. Data for Ocean Area Bravo — Winter.

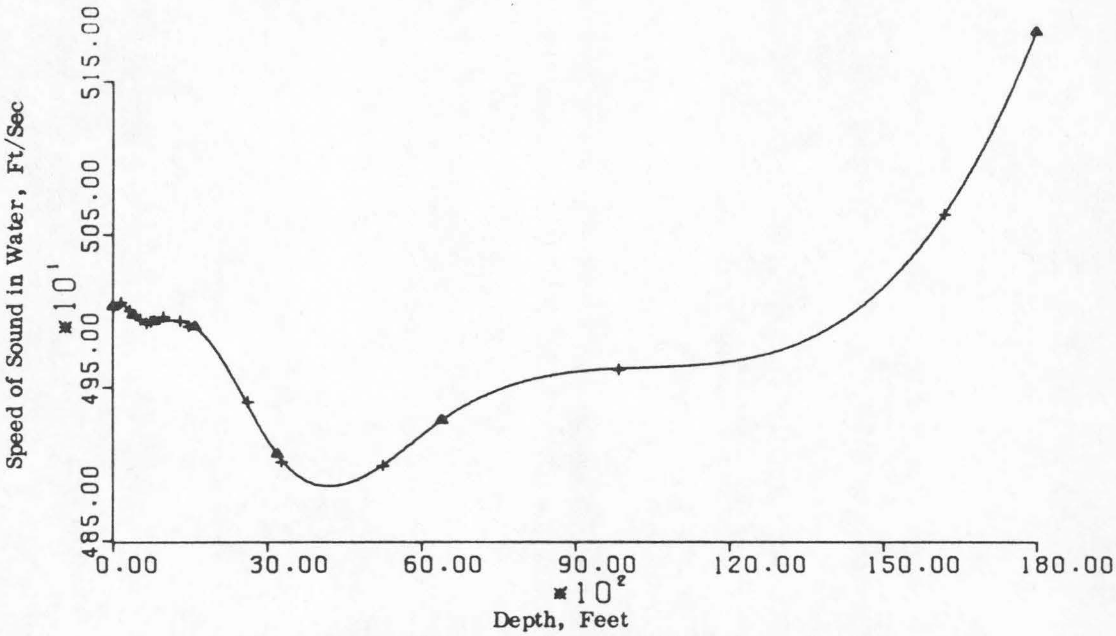


Discrete Least Squares Polynomial Spline Approximation

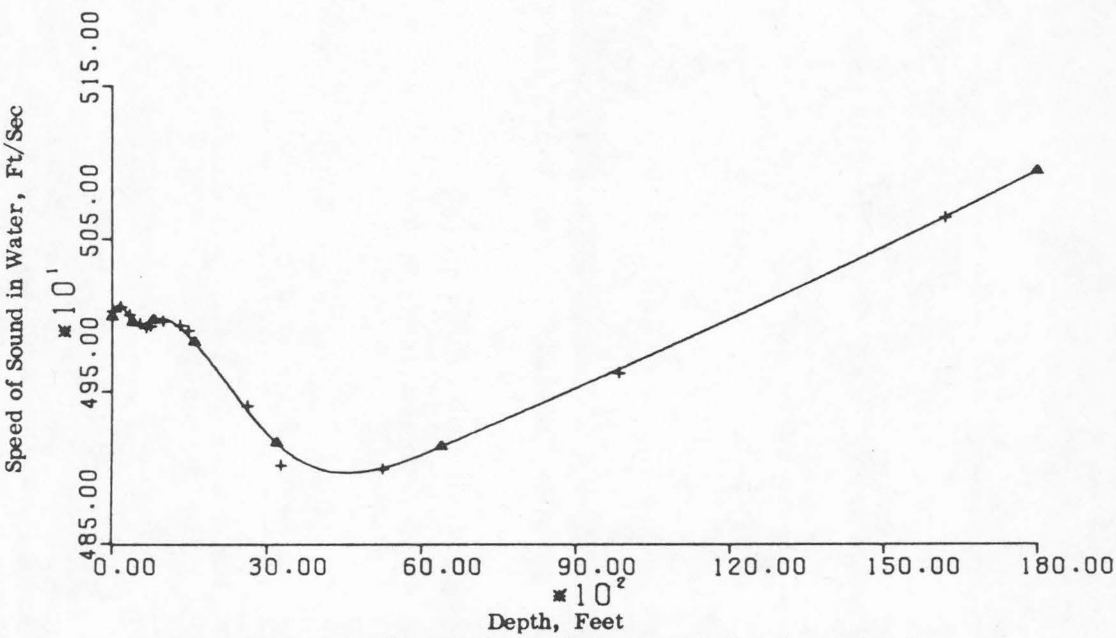


Discretized Integral Least Square Polynomial Spline Approximation

FIG. 8. Data for Ocean Area Bravo — Spring.



Discrete Least Squares Polynomial Spline Approximation



Discretized Integral Least Square Polynomial Spline Approximation

FIG. 9. Data for Ocean Area Gamma — Winter.

portions of the profiles makes the standard discrete technique unreliable because of its tendency to interpolate the data when it can. Indeed, the distribution of the data is the main reason that piecewise linear interpolation is employed in the Filon quadrature used to discretize the least square technique.

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Appendix A

ON AN INEQUALITY OF E. SCHMIDT

In 1932, E. Schmidt stated without proof the following inequality which relates the L^2-norm of a polynomial $p_M(x)$ of degree M on $[-1, 1]$ to the L^2-norm of its derivative:

$$\|Dp_M\|_{L^2[-1,1]} \leq k_M \cdot M^2 \cdot \|p_M\|_{L^2[-1,1]}.$$

In [3], R. Bellman gave a proof that $k_M \leq (1/\sqrt{2})(M + 1)/M)^2$ (and so $\limsup k_M \leq 1/\sqrt{2}$) based on the Cauchy-Schwarz inequality and properties of the Legendre polynomials. Employing essentially the same techniques, we are able to obtain the improved bound, $k_M \leq (1/2)((M + 1)/M)^2$, as well as an improved asymptotic result, $\limsup k_M \leq 1/2\sqrt{2}$.

Beginning with the following recurrence relation for Legendre polynomials (cf. [8, p. 206]), i.e.,

$$D\mathscr{P}_{n+1}(x) - D\mathscr{P}_{n-1}(x) = (2n + 1)\,\mathscr{P}_n(x)$$

where $\mathscr{P}_n$ is the n-th Legendre polynomial, we immediately have

$$D\mathscr{P}_{2n} = \sum_{k=0}^{n-1} (4k + 3)\,\mathscr{P}_{2k+1},$$

and (A.1)

$$D\mathscr{P}_{2n+1} = \sum_{k=0}^{n} (4k + 1)\,\mathscr{P}_{2k}.$$

Expressing $p_M(x)$ in terms of the Legendre polynomials

$$p_M(x) = \sum_{k=0}^{M} a_k \mathscr{P}_k(x), \tag{A.2}$$

and using the orthogonality relations of the $\mathscr{P}_n$, we find that

$$\int_{-1}^{1} [p_M(x)]^2\, dx = \sum_{k=0}^{M} a_k^2 \int_{-1}^{1} \mathscr{P}_k^2(x)\, dx$$

$$= 2 \sum_{k=0}^{M} \frac{a_k^2}{2k+1} \tag{A.3}$$

Beginning with (A.2), employing (A.1), and denoting the greatest integer less than or equal to x by [x], we find that

$$Dp_M(x) = \sum_{k=0}^{M} a_k\, D\mathscr{P}_k(x)$$

$$= \sum_{r=0}^{[M/2]} a_{2r}\, D\mathscr{P}_{2r}(x) + \sum_{r=0}^{[(M-1)/2]} a_{2r+1}\, D\mathscr{P}_{2r+1}(x)$$

$$= \sum_{0 \le 2k \le M-2} (4k+3)\, \mathscr{P}_{2k+1}(x) \sum_{r \ge k+1} a_{2r} + \sum_{0 \le 2k \le M-1} (4k+1)\, \mathscr{P}_{2k}(x) \sum_{r \ge k} a_{2r+1}.$$

Consequently, applying (A.3) to $Dp_M(x)$, we find that

$$\int_{-1}^{1} [Dp_M(x)]^2\, dx = 2 \sum_{0 \le 2k \le M-2} (4k+3) \left\{ \sum_{r \ge k+1} a_{2r} \right\}^2$$

$$+ 2 \sum_{0 \le 2k \le M-1} (4k+1) \left\{ \sum_{r \ge k} a_{2r+1} \right\}^2 . \qquad \text{(A.4)}$$

We now bound each of the sums on the right hand side of (A.4).

Setting $a_r = ((2r+1)/2)^{1/2}\, b_r$, we bound the first sum as follows:

$$2 \sum_{0 \le 2k \le M-2} (4k+3) \left\{ \sum_{r \ge k+1} a_{2r} \right\}^2 =$$

$$2 \sum_{0 \le 2k \le M-2} (4k+3) \left\{ \sum_{r \ge k+1} \left(\frac{4r+1}{2} \right)^{1/2} b_{2r} \right\}^2$$

$$\le 2 \sum_{0 \le 2k \le M-2} (4k+3) \left\{ \sum_{r \ge k+1} \frac{4r+1}{2} \right\} \cdot \left\{ \sum_{r \ge k+1} b_{2r}^2 \right\}$$

$$\le \sum_{0 \le 2k \le M-2} (4k+3) \left\{ \sum_{r \ge k+1} (4r+1) \right\} \cdot \sum_{r \ge 0} b_{2r}^2$$

$$\le \begin{cases} \dfrac{M(M+1)(M+2)(M+3)}{8} \displaystyle\sum_{r \ge 0} b_{2r}^2 , & \text{if } M \text{ is even,} \\[2ex] \dfrac{(M-1)\, M(M+1)(M+2)}{8} \displaystyle\sum_{r \ge 0} b_{2r}^2 , & \text{if } M \text{ is odd.} \end{cases} \qquad \text{(A.5)}$$

A similar calculation yields a similar bound for the second sum:

$$2 \sum_{0 \le 2k \le M-1} (4k+1) \left\{ \sum_{r \ge k} a_{2r+1} \right\}^2$$

$$\le \begin{cases} \dfrac{(M-1)\, M(M+1(M+2)}{8} \displaystyle\sum_{r \ge 0} b_{2r+1}^2 , & \text{if M is even,} \\ \\ \dfrac{M(M+1)(M+2)(M+3)}{8} \displaystyle\sum_{r \ge 0} b_{2r+1}^2 , & \text{if M is odd.} \end{cases} \tag{A.6}$$

We note that these bounds are due to H. Cheng. We immediately observe that

$$\int_{-1}^{1} [Dp_M(x)]^2 \, dx \le \frac{M(M+1)(M+2)(M+3)}{8} \sum_{k=0}^{M} b_k^2 = \frac{M(M+1)(M+2)(M+3)}{8} \sum_{k=0}^{M} \frac{2a_k^2}{2k+1}$$

$$= \frac{M(M+1)(M+2)(M+3)}{8} \int_{-1}^{1} [p_M(x)]^2 \, dx$$

by (A.3). Consequently,

$$k_M \le \left[\frac{(M+1)(M+2)(M+3)}{8M^3} \right]^{1/2}$$

and these bounds yield lim sup $k_M = 1/(2\sqrt{2})$. In fact, a result of Hille, Szego, and Tamarkin states that lim $k_M = 1/\pi$.

However, we may combine the bounds (A.5) and (A.6) in order to obtain a constant of the same form as that given by R. Bellman.

We have

$$\int_{-1}^{1} [Dp_M(x)]^2\, dx$$

$$\leq \left\{\frac{(M-1)\ M(M+1)(M+2)}{8} + \frac{M(M+1)(M+2)(M+3)}{8}\right\} \sum_{k=0}^{M} b_k^2 =$$

$$\frac{M(M+1)(M+2)(2M+2)}{8} \sum_{k=0}^{M} \frac{2a_k^2}{2k+1} \leq \frac{(M+1)^4}{4} \int_{-1}^{1} [p_M(s)]^2\, dx$$

and so

$$k_M \leq \frac{1}{2}\left(\frac{M+1}{M}\right)^2 .$$

We immediately have the following generalization for polynomials defined over the interval [a, b].

$$\|Dp_M\|_{L^2[a,b]} \leq \frac{(M+1)^2}{b-a} \|p_M\|_{L^2[a,b]} .$$

Appendix B

A CUBIC SPLINE APPROXIMATION PROGRAM — CSPLIT

We shall briefly describe the FORTRAN program CSPLIT and the subroutines on which it depends and remark that this is one program of many which we have used to investigate the techniques discussed in the main body of this paper. Note that all special purpose subroutines pertaining to least square approximation in cubic spline spaces have FORTRAN names beginning with the letter C. The names of the analogous routines in the linear and cubic Hermite cases begin with the letters L and H, respectively. All these special purpose codes are actually quite general allowing us great flexibility in the numerical work which we wish to pursue. However, there are definite limits on the size of the problem which we can handle with these programs as they are presently coded.

Given numerical values for the points of the partition Δ(Z(i), i = 1, NM) and for the abscissa and ordinate of each data point (X(i,j), i = 1, NP, j = 1,2), CSPLIT is programmed to compute the coefficients of both a discrete and a discretized least square approximation to the data in the spline space Sp(3, Δ, 2) and to generate a CALCOMP plot tape with which graphs of these approximations are obtained. We note the use of COMMON statements in all but the general purpose polynomial manipulation routines (POLEX, POLINT, POLVAL, and LGRNGE) in order to yield access to the main variables to all subroutines needing it. This reduces the number of arguments required for the special purpose routines. We also note that most of the real variables in the programs are stored in double precision in order to avoid, as much as possible, rounding errors in the accumulation of the many inner products which must be calculated as well as in the solution of systems themselves.

The main program, CSPLIT, coordinates the use of the general and special purpose subroutines needed to generate and plot the indicated spline approximations. Listings of CSPLIT and its subroutines are given in Figs. 10 through 21. CINPUT is programmed to compute and store in the SC-array numerical values for the coefficients of the polynomial representations of the basis functions for the spline space $Sp(3, \Delta, 2)$. These basis functions have been chosen so that each has its support confined to at most four adjacent subintervals of the partition Δ (cf. [13]). Consequently, the matrices involved in both the systems which we must solve are band matrices.

A call to the subroutine CDLS fills the A-array and the first column of the B-array with the numerical values corresponding to the normal systems of equations for the discrete least squares approximation. CDLS depends on the cubic spline evaluation subroutine CEVAL. MATINV is called to obtain a solution to our system using Gaussian elimination. The coefficients of the discrete least square approximation to the data are found in the first column of the B-array and are then stored in the second column of this same array.

A call to the subroutine CMTRX fills the A-array with the entries of the least square matrix. We chose to base the discretized technique on a Filon type quadrature scheme which employs piecewise linear interpolation to the data. A call to the subroutine CFILON fills the first column of the B-array with numerical values based on our chosen type of quadrature. If we wished to employ a composite interpolatory type scheme, we would have used the subroutine CPLATE. Subsequent to a second call to the subroutine MATINV, the first two columns of the B-array contain the coefficient of the discretized and the discrete least square approximations to the data in the spline space $Sp(3, \Delta, 2)$. Finally, a call to the subroutine CSPLOT produces the CALCOMP plot tape used to generate graphs of these approximations. The graphs of the approximations presented in Figs. 1 through 9 of Section 5 of this paper were generated with the program CSPLIT (with slight modifications demanded by the different data formats).

```
      COMMON Z(20),SC(7,22,4),X(99,2),B(22,4),A(22,22)
      DOUBLE PRECISION Z,SC,X,B,A
      NDATA = 0
      READ(5,10) NM
   10 FORMAT(I10)
   40 FORMAT(8D10.4)
      READ(5,40) (Z(I),I=1,NM)
   30 READ(5,10) NP
      READ(5,20) ((X(I,J),J=1,2),I=1,NP)
   20 FORMAT(8F8.0)
      READ(5,10) MDATA
      NDATA = NDATA + 1
      CALL CINPUT(NM,0.0)
      CALL CDLS(NM,NP)
      NS = NM+2
      CALL MATINV(NS,1,DETERM,22,4)
      DO 50 I=1,NS
   50 B(I,2) = B(I,1)
      CALL CMTRX(NM)
      CALL CFILON(NM,NP,2)
      CALL MATINV(NS,1,DETERM,22,4)
      DO 60 I=1,NS
   60 WRITE(6,70) B(I,1),B(I,2)
   70 FORMAT(/,2D25.16)
      WRITE(6,80)
   80 FORMAT(/////)
      CALL CSPLOT(NM,NDATA,MDATA,NP)
      IF(MDATA.EQ.1) GO TO 30
      STOP
      END
```

FIG. 10. Program Listing for CSPLIT.

```
      SUBROUTINE CINPUT(NM,H)
      COMMON Z(20),SC(7,22,4),X(99,2),B(22,4),A(22,22)
      DOUBLE PRECISION Z,SC,X,B,A,C,U,F,T,P
      DIMENSION C(9),NF(9),U(9,9),K(9,9),F(9),T(40)
      DO 10 I=1,7
      DO 10 J=1,22
      DO 10 L=1,4
   10 SC(I,J,L) = 0.0D0
      IF(H.EQ.0.0) H = Z(2)-Z(1)
      T(1) = Z(1)-3.*H
      T(2) = Z(1)-2.*H
      T(3) = Z(1)-H
      DO 20 I=1,NM
   20 T(I+3) = Z(I)
      IF(H.EQ.0.0) H = Z(NM)-Z(NM-1)
      T(NM+4) = Z(NM)+H
      T(NM+5) = Z(NM)+2.*H
      T(NM+6) = Z(NM)+3.*H
      NF(1) = 1
      K(1,1) =  3
      NP5 = NM+5
      DO 40 I=4,NP5
      P = (T(I+1)-T(I))*(T(I+1)-T(I-1))*(T(I+1)-T(I-2))*(T(I+1)-T(I-3))
      C(1) = (-4.0D0)/P
      U(1,1) = T(I+1)
      CALL POLEX(1,C,NF,U,K,N,F)
      DO 30 J=1,4
   30 SC(J,I-3,4) = F(J)
   40 CONTINUE
      DO 60 I=5,NP5
      P = (T(I)-T(I+1))*(T(I)-T(I-1))*(T(I)-T(I-2))*(T(I)-T(I-3))
      C(1) = (-4.0D0)/P
      U(1,1) = T(I)
      CALL POLEX(1,C,NF,U,K,N,F)
      DO 50 J=1,4
   50 SC(J,I-3,3) = F(J) + SC(J,I-3,4)
   60 CONTINUE
      DO 80 I=6,NP5
      P = (T(I-1)-T(I+1))*(T(I-1)-T(I))*(T(I-1)-T(I-2))*(T(I-1)-T(I-3))
      C(1) = (-4.0D0)/P
      U(1,1) = T(I-1)
      CALL POLEX(1,C,NF,U,K,N,F)
      DO 70 J=1,4
   70 SC(J,I-3,2) = F(J) + SC(J,I-3,3)
   80 CONTINUE
      DO 100 I=7,NP5
      P = (T(I-2)-T(I+1))*(T(I-2)-T(I))*(T(I-2)-T(I-1))*(T(I-2)-T(I-3))
      C(1) = (-4.0D0)/P
      U(1,1) = T(I-2)
      CALL POLEX(1,C,NF,U,K,N,F)
      DO 90 J=1,4
   90 SC(J,I-3,1) = F(J) + SC(J,I-3,2)
  100 CONTINUE
      RETURN
      END
```

FIG. 11. Program Listing for Subroutine CINPUT.

```
      SUBROUTINE POLEX(NT,C,NF,U,K,N,F)
      DIMENSION C(9),NF(9),U(9,9),K(9,9),P(9,9),NN(9),F(9)
      DOUBLE PRECISION C,U,P,F
      DO 5 I=1,9
      DO 5 J=1,9
    5 P(I,J) = 0.0D0
      DO 60 J=1,NT
      P(J,1) = C(J)
      N = 1
      IF(NF(J).EQ.0) GO TO 50
      IF = NF(J)
      DO 40 I=1,IF
      LF = K(J,I)
      DO 30 L=1,LF
      P(J,N+1) = P(J,N)
      IF(N .LT.2) GO TO 20
      DO 10 KK=N ,2,-1
   10 P(J,KK) = P(J,KK-1)-U(J,I)*P(J,KK)
   20 P(J,1) = -U(J,I)*P(J,1)
      N = N+1
   30 CONTINUE
   40 CONTINUE
   50 NN(J) = N
   60 CONTINUE
      N = 0
      DO 70 J=1,NT
   70 N = MAX0(N,NN(J))
      DO 90 I=1,9
      F(I) = 0.0D0
      DO 80 J=1,NT
   80 F(I) = F(I) + P(J,I)
   90 CONTINUE
      RETURN
      END
```

FIG. 12. Program Listing for Subroutine POLEX.

```
      SUBROUTINE CDLS(NM,NP)
      COMMON Z(20),SC(7,22,4),X(99,2),B(22,4),A(22,22)
      DOUBLE PRECISION Z,SC,X,B,A,U,V
      NS = NM+2
      DO 20 I=1,NS
      B(I,1) = 0.0
      DO 10 J=1,NP
      CALL CEVAL(NM,I,1,X(J,1),V)
   10 B(I,1) = B(I,1) + X(J,2)*V
   20 CONTINUE
      DO 40 I=1,NS
      DO 40 J=1,NS
      A(I,J) = 0.0
      DO 30 K=1,NP
      CALL CEVAL(NM,J,1,X(K,1),U)
      CALL CEVAL(NM,I,1,X(K,1),V)
   30 A(I,J) = A(I,J) + U*V
   40 CONTINUE
      RETURN
      END
```

FIG. 13. Program Listing for Subroutine CDLS.

```
      SUBROUTINE CEVAL(NM,J,ND,S,V)
      COMMON Z(20),SC(7,22,4),X(99,2),B(22,4),A(22,22)
      DOUBLE PRECISION Z,SC,X,B,A,V,S
      IF(J.EQ.1) GO TO 40
      IF(J.EQ.2) GO TO 30
      IF(J.EQ.3) GO TO 20
      IF(S-Z(J-3))110,110,
      IF(S-Z(J-2))50,50,
   20 IF(S-Z(J-1))60,60,
   30 IF(S-Z(J))70,70,
   40 IF(S-Z(J+1))80,80,110
   50 K = 1
      GO TO (90,100),ND
   60 K = 2
      GO TO (90,100),ND
   70 K = 3
      GO TO (90,100),ND
   80 K = 4
      GO TO (90,100),ND
   90 V = SC(1,J,K) + SC(2,J,K)*S + SC(3,J,K)*S**2 + SC(4,J,K)*S**3
      GO TO 120
  100 V = SC(2,J,K) + 2.0*SC(3,J,K)*S + 3.0*SC(4,J,K)*S**2
      GO TO 120
  110 V = 0.0
  120 RETURN
      END
```

FIG. 14. Program Listing for Subroutine CEVAL.

```
      SUBROUTINE MATINV(N,M,DETERM,ND,MD)
      COMMON Z(20),SC(7,22,4),X(99,2),B(22,4),A(22,22)
      DIMENSION IPIVOT(50), INDEX(50,2), PIVOT(50)
      DOUBLE PRECISION Z,SC,X,B,A,PIVOT,AMAX,T,SWAP
      EQUIVALENCE (IROW,JROW), (ICOLUM,JCOLUM), (AMAX, T, SWAP)
   10 DETERM=1.0
   15 DO 20 J=1,N
   20 IPIVOT(J)=0
   30 DO 550 I=1,N
   40 AMAX=0.0
   45 DO 105 J=1,N
   50 IF (IPIVOT(J)-1) 60, 105, 60
   60 DO 100 K=1,N
   70 IF (IPIVOT(K)-1) 80, 100, 740
   80 IF (DABS(AMAX)-DABS(A(J,K))) 85, 100, 100
   85 IROW=J
   90 ICOLUM=K
   95 AMAX=A(J,K)
  100 CONTINUE
  105 CONTINUE
  110 IPIVOT(ICOLUM)=IPIVOT(ICOLUM)+1
  130 IF (IROW-ICOLUM) 140, 260, 140
  140 DETERM=-DETERM
  150 DO 200 L=1,N
  160 SWAP=A(IROW,L)
  170 A(IROW,L)=A(ICOLUM,L)
  200 A(ICOLUM,L)=SWAP
  205 IF(M) 260, 260, 210
  210 DO 250 L=1, M
  220 SWAP=B(IROW,L)
  230 B(IROW,L)=B(ICOLUM,L)
  250 B(ICOLUM,L)=SWAP
  260 INDEX(I,1)=IROW
  270 INDEX(I,2)=ICOLUM
  310 PIVOT(I)=A(ICOLUM,ICOLUM)
  320 DETERM=DETERM*PIVOT(I)
  330 A(ICOLUM,ICOLUM)=1.0
  340 DO 350 L=1,N
  350 A(ICOLUM,L)=A(ICOLUM,L)/PIVOT(I)
  355 IF(M) 380, 380, 360
  360 DO 370 L=1,M
  370 B(ICOLUM,L)=B(ICOLUM,L)/PIVOT(I)
  380 DO 550 L1=1,N
  390 IF(L1-ICOLUM) 400, 550, 400
  400 T=A(L1,ICOLUM)
  420 A(L1,ICOLUM)=0.0
  430 DO 450 L=1,N
  450 A(L1,L)=A(L1,L)-A(ICOLUM,L)*T
  455 IF(M) 550, 550, 460
  460 DO 500 L=1,M
  500 B(L1,L)=B(L1,L)-B(ICOLUM,L)*T
  550 CONTINUE
  600 DO 710 I=1,N
  610 L=N+1-I
  620 IF (INDEX(L,1)-INDEX(L,2)) 630, 710, 630
  630 JROW=INDEX(L,1)
  640 JCOLUM=INDEX(L,2)
  650 DO 705 K=1,N
  660 SWAP=A(K,JROW)
  670 A(K,JROW)=A(K,JCOLUM)
  700 A(K,JCOLUM)=SWAP
  705 CONTINUE
  710 CONTINUE
  740 RETURN
  750 END
```

FIG. 15. Program Listing for Subroutine MATINV.

```
      SUBROUTINE CFILON(NM,NP,MI)
      DIMENSION T(20),E(9),F(9),G(9)
      COMMON Z(20),SC(7,22,4),X(99,2),B(22,4),A(22,22)
      DOUBLE PRECISION Z,SC,X,B,A,T,E,F,G,V
      DO 10 I=1,22
   10 B(I,1) = 0.0D0
      MJ = MI + 3
      LL = 1
      KK = 1
      KI = 1
      KF = 2
   20 IF(Z(KF).GE.X(KK+MI-1,1)) GO TO 30
      KF = KF+1
      IF(KF.GE.NM) GO TO 30
      GO TO 20
   30 DO 40 I=1,MI
      E(I) = X(KK+I-1,1)
   40 F(I) = X(KK+I-1,2)
      CALL LGRNGE(MI,E,F,ND,G)
      T(KI) = X(LL,1)
      IF(LL,EQ.1) T(KI) = Z(1)
      II = KI+1
      IF = KF-1
      IF(II.GT.IF) GO TO 80
      DO 70 I=II,IF
   70 T(I) = Z(I)
   80 T(KF) = X(KK+MI-1,1)
      IF(NP.LE.KK+MI-1) T(KF) = Z(NM)
      DO 140 I=KI,IF
      IP3 = I+3
      DO 130 IS=I,IP3
      DO 120 J=1,MJ
      F(J) = 0.0D0
      DO 110 L=1,J
  110 F(J) = F(J)+SC(L,IS,I-IS+4)*G(J-L+1)
  120 CONTINUE
      CALL POLINT(MJ,F,T(I+1),T(I),V)
      B(IS,1) = B(IS,1)+V
  130 CONTINUE
  140 CONTINUE
      IF(T(KF).GE.Z(NM)) GO TO 170
      IF(KK+MI-1.GE.NP) GO TO 170
      KI = KF
      IF(Z(KF).GT.X(KK+MI-1,1)) KI = KF-1
      KF = KI+1
      LL = KK+MI-1
      KK = LL
      IF(NP.LT.KK+MI-1) KK = NP-MI+1
      GO TO 20
  170 RETURN
      END
```

FIG. 16. Program Listing for Subroutine CMTRX.

```
      SUBROUTINE POLINT(N,C,EB,EA,V)
      DIMENSION C(10)
      DOUBLE PRECISION C,EB,EA,V,U
      DO 10 J=N,1,-1
   10 C(J+1) = C(J)/DFLOAT(J)
      C(1) = 0.0D0
      NORDER = N+1
      CALL POLVAL(NORDER,C,EB,V)
      CALL POLVAL(NORDER,C,EA,U)
      V = V-U
      RETURN
      END
```

FIG. 17. Program Listing for Subroutine POLINT.

```
      SUBROUTINE POLVAL(NORDER,P,S,V)
      DIMENSION P(9)
      DOUBLE PRECISION P,S,V
      V = P(NORDER)
      IF(NORDER.EQ.1) RETURN
      DO 10 I=NORDER,2,-1
   10 V = S*V + P(I-1)
      RETURN
      END
```

FIG. 18. Program Listing for Subroutine POLVAL.

```
      SUBROUTINE CMTRX(NM)
      COMMON Z(20),SC(7,22,4),X(99,2),B(22,4),A(22,22)
      DOUBLE PRECISION Z,SC,X,B,A,F,U
      DIMENSION F(10),U(20,10)
      NM1 = NM-1
      DO 120 I=1,NM1
      DO 30 M=1,4
      DO 20 J=1,7
      F(J) = 0.0D0
      DO 10 K=1,J
   10 F(J) = F(J) + SC(K,I,4)*SC(J-K+1,I-M+4,M)
   20 CONTINUE
   30 CALL POLINT(7,F,Z(I+1),Z(I),U(I,M))
      DO 60 M=1,3
      DO 50 J=1,7
      F(J) = 0.0D0
      DO 40 K=1,J
   40 F(J) = F(J) + SC(K,I+1,3)*SC(J-K+1,I-M+4,M)
   50 CONTINUE
   60 CALL POLINT(7,F,Z(I+1),Z(I),U(I,M+4))
      DO 90 M=1,2
      DO 80 J=1,7
      F(J) = 0.0D0
      DO 70 K=1,J
   70 F(J) = F(J) + SC(K,I+2,2)*SC(J-K+1,I-M+4,M)
   80 CONTINUE
   90 CALL POLINT(7,F,Z(I+1),Z(I),U(I,M+7))
      DO 110 J=1,7
      F(J) = 0.0D0
      DO 100 K=1,J
  100 F(J) = F(J) + SC(K,I+3,1)*SC(J-K+1,I+3,1)
  110 CONTINUE
      CALL POLINT(7,F,Z(I+1),Z(I),U(I,10))
  120 CONTINUE
      DO 130 I=1,22
      DO 130 J=1,22
  130 A(I,J) = 0.0D0
      DO 140 I=1,NM1
      A(I,I) = A(I,I) + U(I,4)
      A(I,I+1) = A(I,I+1) + U(I,3)
      A(I,I+2) = A(I,I+2) + U(I,2)
      A(I,I+3) = A(I,I+3) + U(I,1)
      A(I+1,I+1) = A(I+1,I+1) + U(I,7)
      A(I+1,I+2) = A(I+1,I+2) + U(I,6)
      A(I+1,I+3) = A(I+1,I+3) + U(I,5)
      A(I+2,I+2) = A(I+2,I+2) + U(I,9)
      A(I+2,I+3) = A(I+2,I+3) + U(I,8)
      A(I+3,I+3) = A(I+3,I+3) + U(I,10)
  140 CONTINUE
      NM2 = NM+2
      DO 170 I=2,NM2
      I1 = I-1
      DO 160 J=1,I1
  160 A(I,J) = A(J,I)
  170 CONTINUE
      RETURN
      END
```

FIG. 19. Program Listing for Subroutine CFILON.

```
      SUBROUTINE LGRNGE(NPNTS,X,Y,NORDER,F)
      DIMENSION X(10),Y(10),C(10),NF(9),U(9,9),K(9,9),F(9)
      DOUBLE PRECISION X,Y,C,U,F
      DO 10 I=1,NPNTS
      NF(I) = NPNTS-1
      DO 10 J=1,NPNTS
   10 K(I,J) = 1
      DO 60 I=1,NPNTS
      C(I) = 1.0D0
      DO 20 J=1,NPNTS
      IF(J.EQ.I) GO TO 20
      C(I) = C(I)*(X(I)-X(J))
   20 CONTINUE
      C(I) = Y(I)/C(I)
      DO 50 J=1,NPNTS
      IF(J-I)30,50,40
   30 U(I,J) = X(J)
      GO TO 50
   40 U(I,J-1) = X(J)
   50 CONTINUE
   60 CONTINUE
      CALL POLEX(NPNTS,C,NF,U,K,NORDER,F)
      RETURN
      END
```

FIG. 20. Program Listing for Subroutine LGRNGE.

```
      SUBROUTINE CSPLOT(NM,N,M,NP)
      COMMON Z(20),SC(7,22,4),X(99,2),B(22,4),A(22,22)
      DIMENSION WORK(1024),XARRAY(503),YARRAY(503),Y(501)
      DOUBLE PRECISION Z,SC,X,B,A,T,V,Y
      IF(N.GT.1) GO TO 10
      CALL PLOTS(WORK(1),1024,8)
      CALL PLOT(5.0,1.75,-3)
   10 NPS = 501
      NM1 = NPS - 1
      NS = NM + 2
      DO 70 K=1,2
      DO 20 I=1,NP
      XARRAY(I) = X(I,1)
   20 YARRAY(I) = X(I,2)
      XARRAY(NP+1) = 0.0
      XARRAY(NP+2) = 3000.0
      YARRAY(NP+1) = 4850.0
      YARRAY(NP+2) = 100.0
      CALL AXIS(0.,0.,1H ,1,3.,90.,4850.,100.)
      CALL AXIS(0.,0.,1H ,-1,6.,0.,0.,3000.)
      CALL LINE(XARRAY(1),YARRAY(1),NP,1,-1,3)
      DO 40 I=1,NPS
      T = DFLOAT(I-1)*(Z(NM)-Z(1))/DFLOAT(NM1)
      Y(I) = 0.0
      DO 30 J=1,NS
      CALL CEVAL(NM,J,1,T,V)
   30 Y(I) = Y(I) + B(J,K)*V
      XARRAY(I) = T
   40 YARRAY(I) = Y(I)
      XARRAY(NPS+1) = 0.0
      XARRAY(NPS+2) = 3000.0
      YARRAY(NPS+1) = 4850.0
      YARRAY(NPS+2) = 100.0
      CALL LINE(XARRAY(1),YARRAY(1),NPS,1,0,2)
      XARRAY(NM+1) = 0.0
      XARRAY(NM+2) = 3000.0
      YARRAY(NM+1) = 4850.0
      YARRAY(NM+2) = 100.0
      DO 60 I=1,NM
      Y(I) = 0.0
      DO 50 J=1,NS
      CALL CEVAL(NM,J,1,Z(I),V)
   50 Y(I) = Y(I) + B(J,K)*V
      YARRAY(I) = Y(I)
   60 XARRAY(I) = Z(I)
      CALL LINE(XARRAY(1),YARRAY(1),NM,1,-1,2)
      IF(K.EQ.1) CALL PLOT(0.,4.5,-3)
      IF(K.EQ.2) CALL PLOT(8.5,-4.5,-3)
   70 CONTINUE
      IF(M.EQ.0) GO TO 90
      GO TO 110
   90 CALL PLOT(0.,0.,999)
  110 RETURN
      END
```

FIG. 21. Program Listing for Subroutine CSPLOT.