# ON DUALS OF MULTIPLICATIVE DESIGNS

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## ABSTRACT

A multiplicative design is a square design (that is, a set S of n elements called varieties, and a collection of n subsets of S called blocks) in which each block may be assigned a positive number, called the block's weight, less than the size of the block in such a way that the size of the intersection of two distinct blocks is the geometric mean of their weights. A uniform design is a multiplicative design in which the difference between the weight and size of a block is independent of the choice of the block. A $\lambda$-design is a multiplicative design with identical weights in which not all of the block sizes are equal.

It is conjectured that if a multiplicative design has a multiplicative dual, and if neither design belongs to a specific class of designs, then both are uniform designs. Two cases of this conjecture are proved, one of which is this generalization of a result of K. N. Majumdar: a $\lambda$-design with a multiplicative dual has $\lambda = 1$. Degenerate multiplicative designs are investigated. A generalization to multiplicative designs of Henry B. Mann's upper bound on the multiplicity of a repeated variety is also proved.

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# SECTION 1

## INTRODUCTION

By a combinatorial design, or simply a design, we mean a finite set $S = \{a_1, \ldots, a_m\}$ and a finite indexed collection $S_1, \ldots, S_n$ of subsets of S. Traditionally the elements $a_i$ are called varieties and the subsets $S_j$, which need not be distinct, are called blocks. We define the following integers commonly associated with a design:

| | |
|---|---|
| $m$ | variety order |
| $r_i = \lvert\{j : a_i \in S_j\}\rvert$ <br> $1 \le i \le m$ | replication numbers |
| $\mu_{ij} = \lvert\{\ell : \{a_i, a_j\} \subseteq S_\ell\}\rvert$ <br> $1 \le i, j \le m,\ i \ne j$ | linkage numbers |
| $n$ | block order |
| $k_j = \lvert S_j\rvert,\ 1 \le j \le n$ | block sizes |
| $\lambda_{ij} = \lvert S_i \cap S_j\rvert$ <br> $1 \le i, j \le n,\ i \ne j$ | overlap numbers . |

To each design we associate the (0, 1)-matrix $A = [a_{ij}]$ of size m by n in which $a_{ij} = 0$ or $a_{ij} = 1$ according as $a_i \notin S_j$ or $a_i \in S_j$. This matrix representation is quite useful and we will often identify a design with its incidence matrix whenever no confusion could result. If A is the incidence matrix of a design, a dual of the design is a design whose incidence matrix is $A^T$, the transpose of A. We have

$$A^T A = \mathrm{diag}[k_1-\lambda_{11}, \ldots, k_n-\lambda_{nn}] + [\lambda_{ij}] ,$$
$$AA^T = \mathrm{diag}[r_1-\mu_{11}, \ldots, r_m-\mu_{mm}] + [\mu_{ij}] ,$$

where $\mu_{ii}$ for $1 \le i \le m$ and $\lambda_{jj}$ for $1 \le j \le n$ may be given arbitrary values.

We define some axiomatically restricted designs. Suppose the overlap numbers $\lambda_{ij}$ of a design have a constant value $\lambda$ for $i \neq j$ and that $k_j > \lambda$ for $1 \le j \le n$. It is known [9, 11] then that the block order n of the design cannot exceed the variety order m, that is

$$m \ge n . \tag{1.1}$$

This relation is known as <u>Fisher's inequality</u>. If in addition when $m = n$ and the block sizes $k_j$ of the design also have a common value k, the design is called a $(v, k, \lambda)$-<u>design</u> where $v = m = n$. Conditions imposed on a $(v, k, \lambda)$-design to exclude degeneracies are $v > k + 1$, $k > \lambda + 1$, and $\lambda > 0$. A useful necessary condition [8] on the parameters of a $(v, k, \lambda)$-design is

$$k^2 - \lambda v = k - \lambda . \tag{1.2}$$

If $m = n$, if the overlap numbers have a common value $\lambda$, and if the block sizes are <u>not</u> all equal, the design is called a $\lambda$-<u>design</u> [8]. The conditions imposed here to exclude uninteresting degeneracies are $k_j > \lambda > 0$ for $1 \le j \le n$.

Suppose that in a certain design we can associate with each block $S_j$ a positive number $\lambda_j$, called its <u>weight</u>, such that the (i, j)-overlap

$$\lambda_{ij} = |S_i \cap S_j| = \sqrt{\lambda_i}\sqrt{\lambda_j}$$

for all $i \neq j$. If in addition the block order $n$ and variety order $m$ are equal and $m = n \geq 3$, the design is called a multiplicative design. The conditions imposed here to exclude degeneracies are $k_j > \lambda_j > 0$ for $1 \leq j \leq n$. A multiplicative design in which $k_j - \lambda_j$ is constant for $1 \leq j \leq n$ is called a uniform design. Ryser [10] defined these designs and constructed several classes of them. Degenerate multiplicative designs are not without interest and these are discussed in some detail in Section 2.

Define a multiplicative design to be bordered if, through row and column permutations, the incidence matrix A of the design can be brought to the form

$$A = \left[\begin{array}{c|c} a_{11} & X^T \\ \hline Y & B \end{array}\right],$$

where X and Y are column matrices, each consisting entirely of zeroes or entirely of ones, and where B is a $(v, k, \lambda)$-design, possibly a degenerate one. Several classes of bordered multiplicative designs may be immediately constructed in terms of $(v, k, \lambda)$-designs.

It is known [8] that the dual of a $(v, k, \lambda)$-design is also a $(v, k, \lambda)$-design. Ryser [10] proved this generalization to uniform designs.

Theorem 1.1 [Ryser]. The dual of a uniform design is a uniform design.

A weak converse to Theorem 1.1 might be: If a multiplicative design has a multiplicative dual, then both the design and its dual are uniform. This is seen to fail for many bordered multiplicative designs, but there is some evidence that for all other designs the converse is true.

Conjecture 1.2. If a nonbordered multiplicative design has a multiplicative dual, then both the design and its dual are uniform designs.

Sections 3 and 4 prove certain cases of Conjecture 1.2. There is less hard evidence for the following conjecture.

Conjecture 1.3. A multiplicative design has at most two distinct weights.

But if Conjecture 1.3 holds, then the results of Sections 3 and 4 go a long way in establishing Conjecture 1.2.

If in the definition of a multiplicative design we do not necessarily require that $m = n$, then we call the design a partial multiplicative design. An inequality of Fisher type [10] implies $m \geq n$, however. If in addition $k_j - \lambda_j$ is constant for $1 \leq j \leq n$, it is called a partial uniform design. These are investigated in Section 5, and in particular we prove a generalization of an upper bound due to H. B. Mann [7, 5] on the multiplicity of the repeated variety.

# SECTION 2

## DEGENERATE MULTIPLICATIVE DESIGNS

By a (nondegenerate) multiplicative design on the parameters $k_1, \ldots, k_n$ and $\lambda_1, \ldots, \lambda_n$ we mean a (0, 1)-matrix A of order n satisfying

$$A^T A = \operatorname{diag}[k_1 - \lambda_1, \ldots, k_n - \lambda_n] + [\sqrt{\lambda_i}\sqrt{\lambda_j}] \tag{2.1}$$

and

$$0 < \lambda_i < k_i \quad \text{for} \quad 1 \le i \le n \quad . \tag{2.2}$$

The intent of the conditions (2.2) is to exclude unimportant degenerate cases. Analogous conditions for $(v, k, \lambda)$-designs and $\lambda$-designs exclude only readily identifiable degenerate configurations. The instances of A in which (2.1) holds and (2.2) fails are not as clear, however, and the following discussion will describe many of them.

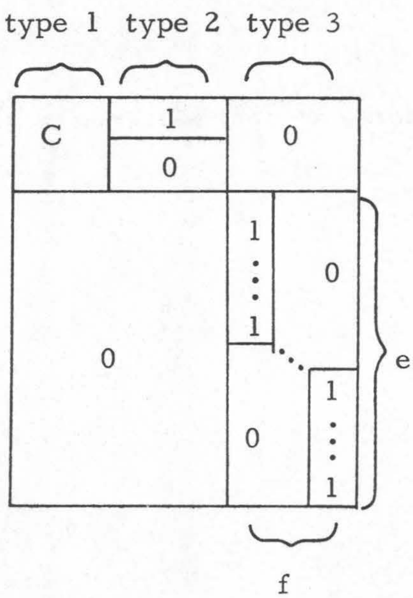
Condition (2.1) and the structure of A require that $0 \le \lambda_i$ and $0 \le k_i$ for $1 \le i \le n$. With no loss of generality we set $\lambda_1 \le \ldots \le \lambda_n$. Then $k_i \ge \sqrt{\lambda_i}\sqrt{\lambda_{i+1}} \ge \lambda_i$ for $1 \le i < n$, but $k_n \ge \lambda_n$ does not necessarily hold. Indeed if B is a $(v, k, \lambda)$-design and if A is formed in the following way from B

$$A = \left[\begin{array}{c|c} & 1 \\ B & \vdots \\ & 1 \\ \hline 1 \cdots 1 & 1 \end{array}\right]$$

then A satisfies (2.1) with $n = v+1$, $k_1 = \cdots = k_{n-1} = k+1$, $k_n = v+1$, $\lambda_1 = \cdots = \lambda_{n-1} = \lambda+1$ and $\lambda_n = (k+1)^2/(\lambda+1)$. But $k_n \geq \lambda_n$ for $k \geq 2\lambda+1$ and $k_n \leq \lambda_n$ for $k < 2\lambda+1$. If a $(v, k, \lambda)$-design satisfies one of the conditions $k \geq 2\lambda+1$, $k < 2\lambda+1$, its complement satisfies the other.

Assume now that $k_i \geq \lambda_i \geq 0$ for $1 \leq i \leq n$. If $k_i = \lambda_i = 0$ for some $i$, the $i^{th}$ column consists entirely of zeros and its presence does not affect the rest of the matrix. We hence take $k_i > 0$ for $1 \leq i \leq n$. Call the column $i$ of A to be type 1, 2 or 3 according as $k_i > \lambda_i > 0$, $k_i = \lambda_i > 0$, or $k_i > \lambda_i = 0$. If $k_i = \lambda_i > 0$ and $k_j = \lambda_j > 0$ with $\lambda_i \geq \lambda_j$ then the inner product $\sqrt{\lambda_i}\sqrt{\lambda_j}$ cannot exceed the column sum $\lambda_j$, whence $\lambda_i \leq \lambda_j$. Hence $\lambda_i = \lambda_j$ for columns of type 2.

We now see that A must have the form

A = 

(2.3)

Note that $e \geq f$. If $e = f$, it is possible that no columns of type 2 occur, in which case A has the form

$$\left[\begin{array}{c|c} I & 0 \\ \hline 0 & A' \end{array}\right]$$

where I is an identity matrix of arbitrary order and A' is a nondegenerate multiplicative design.

If $e > f$, however, columns of type 2 must occur. Otherwise C in (2. 3) would be a nondegenerate partial multiplicative design of a size that violates the Fisher inequality. When columns of type 2 occur, C cannot be square. If C were square, say with parameters $k'_1, \ldots, k'_g$ and $\lambda'_1, \ldots, \lambda'_g$ we could form a square D,

$$D = \left[\begin{array}{c|c} & 1 \\ & \vdots \\ & 1 \\ \cline{2-2} C & 0 \\ & \vdots \\ & 0 \\ \hline 0 \ldots 0 & 0 \end{array}\right] \quad \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} k'_{g+1} = \lambda'_{g+1} \quad .$$

A column of type 2 is adjoined and then a row of zeros. The determinant of $D^T D$ may be computed explicitly to be

$$\det(D^T D) = (k'_1 - \lambda'_1) \cdots (k'_g - \lambda'_g) k'_{g+1} > 0 \quad ,$$

whence $\det D \neq 0$. But D contains a row of zeros and hence is singular, a contradiction.

Constructions exist where columns of type 2 occur and C is of size $g$ by $g-1$. Replace the last column of a projective plane P of order $m$ by its complement. Then the first $g-1 = m^2 + m$ columns of the altered P denote C with $k_1' = \cdots = k_{g-1}' = m+1$ and $\lambda_1' = \cdots = \lambda_{g-1}' = 1$. The last column is of type 2 with $k_g' = \lambda_g' = m^2$.

A question on degenerate multiplicative designs remains unanswered. Under what conditions can a partial nondegenerate multiplicative design of size $n$ by $r$, with $n > r$, be augmented by a degenerate column of type 2?

## SECTION 3

## THE $\lambda$-DESIGNS WITH MULTIPLICATIVE DUALS

In [6] K. N. Majumdar (alias Majindar) showed essentially that a $\lambda$-design whose dual is a $\lambda'$-design satisfies $\lambda = \lambda' = 1$. It is well-known [3, 9] that there is exactly one 1-design for each order $n > 3$, apart from block and variety labeling, and that this may be represented in matrix form as

$$\left[\begin{array}{c|c} 0 & 1 \cdots 1 \\ \hline 1 & \\ \vdots & I_{n-1} \\ 1 & \end{array}\right] . \tag{3.1}$$

Now $I_{n-1}$ is a degenerate $(v, k, \lambda)$-design with $(v, k, \lambda) = (n-1, 1, 0)$, so (3.1) satisfies the conditions of a bordered multiplicative design given in Section 1. It is also immediate that (3.1) is the only bordered multiplicative design that is also a $\lambda$-design. A $\lambda$-design is a non-uniform multiplicative design, so that Conjecture 1.2 in the case of a $\lambda$-design becomes: If a $\lambda$-design A has a multiplicative design as its dual, then A is a bordered multiplicative design, and hence $\lambda = 1$. This is the special case of Conjecture 1.2 proved in Theorem 3.2 below.

A few facts are needed first. Ryser [9] and Woodall [11] proved that a $\lambda$-design (speaking now in matrix-theoretic terms) has precisely two distinct row sums $r_1$ and $r_2$, and that these numbers satisfy

$$r_1 + r_2 = n + 1 \quad . \tag{3.2}$$

With no loss of generality we henceforth insist that $r_1 > r_2 > 1$, and that the first $e_1$ rows of the incidence matrix A of the $\lambda$-design have row sum $r_1$, and the last $e_2$ rows of A have row sum $r_2$, where $e_1 + e_2 = n$. Lastly for each j in $1 \leq j \leq n$ let $k_j'$ and $k_j^*$ respectively denote the sum of the first $e_1$ entries, and the sum of the last $e_2$ entries of column j. Following Ryser, the sum of the inner products of column j with all other columns may be computed in two ways to get

$$k_j'(r_1 - 1) + k_j^*(r_2 - 1) = \lambda(n-1) = \lambda((r_1 - 1) + (r_2 - 1)) \quad . \tag{3.3}$$

Relation (3.2) was used in the last equality. Division of (3.3) by $r_2 - 1$ then yields

$$\rho k_j' + k_j^* = \lambda(\rho + 1) \tag{3.4}$$

where $\rho = (r_1 - 1)/(r_2 - 1) > 1$.

W. G. Bridges [2] has constructed a $\lambda$-design in the following way from each matrix B which is the incidence matrix of a degenerate $(v, 1, 0)$-design $I_v$ or of a nondegenerate $(v, k, \lambda')$-design with

$$(v, k, \lambda') \neq (4t-1, 2t-1, t-1)$$

for all $t \geq 2$. The rows of B are permuted so that the k ones of the first column appear initially. The matrix B is then partitioned into quadrants as shown

$$B = \left[\begin{array}{c|c} 1 & \\ \vdots & * \\ 1 & \\ \hline 0 & \\ \vdots & * \\ 0 & \end{array}\right]$$

and all but the lower right quadrant of B is complemented. The resultant matrix A can directly be shown to be the incidence matrix of a $\lambda$-design with $\lambda = k - \lambda'$. The column sums of A are $k_1 = v - k$ with multiplicity 1, and $k_2 = 2(k-\lambda')$ of multiplicity $v - 1$. (The Hadamard $(v, k, \lambda')$-designs B are excluded above because, precisely in those instances, $k_1 = k_2$, and A is a $(v, k, \lambda)$-design and not a $\lambda$-design.) Furthermore,

$$AA^T = \left[\begin{array}{ccc|ccc} v-k & & v-2k+\lambda' & & & \\ & \ddots & & & k-\lambda' & \\ v-2k+\lambda' & & v-k & & & \\ \hline & & & k+1 & & \lambda'+1 \\ & k-\lambda' & & & \ddots & \\ & & & \lambda'+1 & & k+1 \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right\} k \\ \left.\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right\} v-k \end{array} . \qquad (3.5)$$

It has been conjectured that this remarkable kind of derived $\lambda$-design, called <u>type I</u> by Bridges, characterizes all $\lambda$-designs. He and others have shown this to be the case for $\lambda \le 9$.

A theorem of Kramer [4] and Woodall [11] is given next; it is needed for the proof of Theorem 3.2. It is given again here because this version is more direct and more suited to our purposes. In the proof of Theorem 3.2 we also employ some techniques of Majumdar [6].

Theorem 3.1 (Kramer and Woodall): A $\lambda$-design in which the intersection of two rows depends only on the row sums of those rows is of type I.

Proof: We know that the matrix form of the $\lambda$-design $A = [a_{ij}]$ has precisely two distinct row sums $r_1$ and $r_2$, and by assumption that there exist numbers $\lambda_{11}$, $\lambda_{12}$, $\lambda_{22}$ such that

$$AA^T = \left[\begin{array}{cc|cc} r_1 & \lambda_{11} & & \\ \lambda_{11} & r_1 & \lambda_{12} & \\ \hline & & r_2 & \lambda_{22} \\ \lambda_{12} & & \lambda_{22} & r_2 \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c} 1 \\ 1 \end{array}}\right\} e_1 \\ \left.\vphantom{\begin{array}{c} 1 \\ 1 \end{array}}\right\} e_2 \end{array} .$$

In addition we have assumed

$$A^T A = \mathrm{diag}[k_1 - \lambda, \ldots, k_n - \lambda] + \lambda J ,$$

where J denotes the matrix of ones of order n. According as $1 \le i \le e_1$ or $e_1 < i \le n$, the (i, j)-entry of the identity $A(A^T A) = (AA^T)A$ may be computed explicitly to be

$$a_{ij}(k_j - \lambda) + r_1 \lambda = a_{ij}(r_1 - \lambda_{11}) + k_j' \lambda_{11} + k_j^* \lambda_{12} \tag{3.6}$$

or

$$a_{ij}(k_j - \lambda) + r_2 \lambda = a_{ij}(r_2 - \lambda_{22}) + k_j' \lambda_{12} + k_j^* \lambda_{22} , \tag{3.7}$$

where $k_j'$ and $k_j^*$ are respectively the sum of the first $e_1$ entries, and the sum of the last $e_2$ entries, of column j. If $0 < k_j' < e_1$, that is, if there is both a zero and a one among the first $e_1$ entries of column j, then we conclude from (6) that $k_j - \lambda = r_1 - \lambda_{11}$. Similarly $0 < k_j^* < e_2$ implies $k_j - \lambda = r_2 - \lambda_{22}$ by (7). Hence for each j in $1 \leq j \leq n$, at least one of (a), (b), (c) holds, and at least one of (d), (e), (f) holds, where

(a) $k_j' = 0$ (b) $k_j' = e_1$ (c) $k_j - \lambda = r_1 - \lambda_{11}$

(d) $k_j^* = 0$ (e) $k_j^* = e_2$ (f) $k_j - \lambda = r_2 - \lambda_{22}$ .

Call a column j of A <u>of special type</u> if one of (a) or (b) holds, and if one of (d) or (e) holds, for j. First note that (a) and (d) cannot both hold for column j, since then $k_j = 0$. Also (b) and (e) cannot both hold for column j, since then $k_j = n$. Either (a) and (e), or else (b) and (d), hold for a column j of special type. Two columns of A, say columns 1 and 2, cannot both be of special type, for then either $\lambda = 0$ or $k_1 = k_2 = \lambda$ would result. Thus there is at most one column of special type in A.

Suppose some column not of special type, say column j = 1, satisfies (a), so that (f) must also hold for j = 1. If in addition (d) holds for j = 2, then $\lambda = 0$, a contradiction. Also (e) cannot hold for j = 2, for then $k_1 = \lambda$. Thus if (a) and (f) holds for j = 1, then (f) holds for all columns j not of special type. In general all columns not of special type have the same column sum if any one of them satisfies (a). The same is true if any one of them satisfies (d).

Suppose instead that (b) and (f) hold for $j = 1$, but that neither (a) nor (d) hold for any column not of special type. If (e) holds for column $j = 2$ not of special type, then A cannot contain a column of special type, for it would be contained in column 1 or else in column 2. Now $n \geq 3$ by definition, (b) holds for $j = 1$, (e) holds for $j = 2$, and column 3 is not of special type. We claim (e) cannot also hold for $j = 3$. For then $e_2 \leq \lambda$ by the inner product of columns 2 and 3. But from $k_1^* + k_2' = \lambda$ and $k_1^* > 0$ we have $k_2' < \lambda$ whence by (3.4)

$$e_2 = k_2^* = \lambda - \rho(k_2' - \lambda) > \lambda \quad ,$$

a contradiction. Similarly (b) does not hold for $j = 3$ under these conditions. Because now (c) holds for $j = 2$ and $j = 3$ and (f) holds for $j = 1$ and $j = 3$, all three columns have the same column sum.

In all remaining cases, either (c) holds for all columns $j$ not of special type, or (f) does, or both. In either case, all columns not of special type have the same column sum. The upshot of this and the preceding two paragraphs is that all columns not of special type have the same column sum.

Not all of the column sums of A are equal, for then by a standard theorem [9] a $(v, k, \lambda)$-design results for A, not a $\lambda$-design. So necessarily a unique column of special type occurs. We have either

$$A = \left[\begin{array}{c|c} 1 & \\ \vdots & A_1 \\ 1 & \\ \hline 0 & \\ \vdots & A_2 \\ 0 & \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\ \vdots\\ 1\end{array}}\right\} e_1 \\ \left.\vphantom{\begin{array}{c}0\\ \vdots\\ 0\end{array}}\right\} e_2 \end{array} \tag{3.8a}$$

or

$$A = \left[\begin{array}{c|c} 0 & \\ \vdots & A_1 \\ 0 & \\ \hline 1 & \\ \vdots & A_2 \\ 1 & \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\ \vdots\\ 1\end{array}}\right\} e_1 \\ \left.\vphantom{\begin{array}{c}0\\ \vdots\\ 0\end{array}}\right\} e_2 \end{array} \tag{3.8b}$$

where $k_2 = \cdots = k_n$.

Suppose that A has the form (3.8a). Columns 1 and j, where $1 < j \leq n$, have inner product $\lambda$, and so $k'_j = \lambda$ for $1 < j \leq n$. From (3.4) we conclude that

$$k^*_j = \lambda - \rho(k'_j - \lambda) = \lambda \quad .$$

Take $1 < j < \ell \leq n$, and let x be the number of solutions i in the interval $1 \leq i \leq e_1$ of $a_{ij} = a_{i\ell} = 1$. Using $k'_j = k^*_j = \lambda$ it is not hard to show that $a_{ij} = a_{i\ell} = 0$ has $e_2 - \lambda - x$ solutions i in the interval $e_1 < i \leq n$. If A is transformed into A'' as shown,

$$A'' = \left[\begin{array}{c|c} 0 & \\ \vdots & A_1 \\ 0 & \\ \hline 1 & \\ \vdots & J - A_2 \\ 1 & \end{array}\right] ,$$

it is clear that A'' is a $(v, k, \lambda)$-design with $(v, k, \lambda) = (n, e_2, e_2 - \lambda)$.

Similarly, if A has the form (3.8b), the matrix formed by complementing all elements in (3.8b) except those of $A_2$, is a $(v, k, \lambda)$-design with $(v, k, \lambda) = (n, e_1, e_1 - \lambda)$. With either form A is a $\lambda$-design of type I. The proof is finished.

<u>Theorem 3.2.</u> If a $\lambda$-design has a multiplicative design as its dual, then $\lambda = 1$.

<u>Proof:</u> Let $A = [a_{ij}]$ denote the incidence matrix of the $\lambda$-design. For $1 \leq i \leq n$ there are appropriate parameters $k_i$, $r_i$, $\lambda_i$ such that

$$A^T A = \operatorname{diag}[k_1 - \lambda, \ldots, k_n - \lambda] + \lambda J \quad ,$$

$$AA^T = \operatorname{diag}[r_1 - \lambda_1, \ldots, r_n - \lambda_n] + [\sqrt{\lambda_i}\sqrt{\lambda_j}] \quad ,$$

where J denotes the matrix of ones of order n. Let J' denote the matrix of ones of size n by 1. The equality $A^T(AJ') = (A^T A)J'$ yields

$$A^T \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} = \begin{bmatrix} \lambda(n-1) + k_1 \\ \vdots \\ \lambda(n-1) + k_n \end{bmatrix} . \tag{3.9}$$

Premultiplication of (3.9) by A then gives

$$\begin{bmatrix} r_1 & & \sqrt{\lambda_i}\sqrt{\lambda_j} \\ & \ddots & \\ \sqrt{\lambda_i}\sqrt{\lambda_j} & & r_n \end{bmatrix} \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} = \begin{bmatrix} \lambda(n-1)r_1 \\ \vdots \\ \lambda(n-1)r_n \end{bmatrix} + A \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix} . \tag{3.10}$$

Similarly the equality $A(A^T J') = (AA^T)J'$ yields

$$A \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix} = \begin{bmatrix} r_1 + s\sqrt{\lambda_1} - \lambda_1 \\ \vdots \\ r_n + s\sqrt{\lambda_n} - \lambda_n \end{bmatrix} , \qquad (3.11)$$

where $s = \sqrt{\lambda_1} + \cdots + \sqrt{\lambda_n}$. Now (3. 10) and (3. 11) together imply

$$\begin{bmatrix} r_1^2 + t\sqrt{\lambda_1} - r_1\lambda_1 \\ \vdots \\ r_n^2 + t\sqrt{\lambda_n} - r_n\lambda_1 \end{bmatrix} = \begin{bmatrix} \lambda(n-1)r_1 \\ \vdots \\ \lambda(n-1)r_n \end{bmatrix} + \begin{bmatrix} r_1 + s\sqrt{\lambda_1} - \lambda_1 \\ \vdots \\ r_n + s\sqrt{\lambda_n} - \lambda_n \end{bmatrix} , \qquad (3.12)$$

where $t = r_1\sqrt{\lambda_1} + \cdots + r_n\sqrt{\lambda_n}$. The $i^{th}$ column entry of (3. 12) may be rewritten as

$$(r_i - 1)\lambda_i + (s-t)\sqrt{\lambda_i} + r_i + \lambda(n-1)r_i - r_i^2 = 0 \qquad (3.13)$$

We have $r_i > 1$ by nondegeneracy assumptions, so that (3. 13) is a quadratic equation in $\sqrt{\lambda_i}$.

Permute the rows of A so that $r_i = s_1$ for $1 \le i \le e$, and $r_i = s_2$ for $e < i \le n$, where $s_1$ and $s_2$ are the two distinct row sums of A guaranteed in (3. 2). Then $\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_e}$ each satisfy the same quadratic equation (3. 13), and so have among themselves at most two distinct values. Suppose that both roots of (3. 13) are assumed, so that for some $i \neq j$,

$$\sqrt{\lambda_i} + \sqrt{\lambda_j} = \frac{t-s}{s_1 - 1} = \sqrt{\lambda_i} + \sqrt{\lambda_j} + d \ .$$

But we have $n \ge 3$ and $d > 0$. This is a contradiction. Hence $\lambda_1, \ldots, \lambda_e$ have a common value $\mu_1$, and similarly $\lambda_{e+1}, \ldots, \lambda_n$ have a common value $\mu_2$.

We have shown that the intersection of two rows of A, say rows i and j with $i \neq j$, depends only on the row sums of those rows; this intersection is namely

$$\begin{array}{lll} \mu_1 & \text{if} & r_i = r_j = s_1 \ , \\ \mu_2 & \text{if} & r_i = r_j = s_2 \ , \\ \sqrt{\mu_1}\sqrt{\mu_2} & \text{if} & \{r_i, r_j\} = \{s_1, s_2\} \ . \end{array}$$

By Theorem 3.1, A is of type I. Suppose A is derived not from the degenerate $(n, 1, 0)$-design $I_n$, but from a nondegenerate $(v, k, \lambda')$-design B whose parameters satisfy $(v, k, \lambda) \neq (4t-1, 2t-1, t-1)$. From the description (5) of $AA^T$, and from the nondegeneracy conditions $k > 1$, $v - k > 1$ on B, we see that

$$\mu_1 = v - 2k + \lambda', \quad \sqrt{\mu_1}\sqrt{\mu_2} = k - \lambda', \quad \mu_2 = \lambda' + 1 \ ,$$

or essentially the same situation with $\mu_1$ and $\mu_2$ interchanged. We evaluate $\mu_1\mu_2$ in two ways to get

$$(k-\lambda')^2 = (v-2k+\lambda')(\lambda'+1) \quad . \tag{3.14}$$

If the usual necessary condition for $(v, k, \lambda)$-designs,

$$k(k-1) = \lambda'(v-1) \quad , \tag{3.15}$$

is subtracted from (3.14) we have

$$v = 3k - 2\lambda' \quad . \tag{3.16}$$

Together (3. 15) and (3. 16) show that

$$(k-\lambda')(k-2\lambda'-1) = 0 \quad ,$$

whence $k = 2\lambda' + 1$ because $k > \lambda'$ for nondegenerate $(v, k, \lambda)$-designs. This and (3. 16) imply that, for some $t \geq 2$,

$$(v, k, \lambda') = (4t-1,\ 2t-1,\ t-1) \quad ,$$

a contradiction.

Hence A is derived from the degenerate $(n, 1, 0)$-design $I_n$. It is routine to show that the construction gives an A of the form (3. 1), a form which characterizes 1-designs. Hence $\lambda = 1$, and the proof is finished.

## SECTION 4

## A SECOND CASE OF THE CONJECTURE

In Section 3 the case of the conjecture was proved in which the multiplicative design has exactly one distinct weight and its multiplicative dual has arbitrarily many distinct weights. The case which suggests itself next is that case in which both the design and its dual have precisely two distinct weights. Theorem 4.1 below proves this with an additional assumption on the block sizes of the design and its dual.

<u>Theorem 4.1.</u> Suppose the dual of a nonbordered multiplicative design is also a multiplicative design. Suppose also that each design has exactly two distinct weights, and that for each design, the size of a block depends only on the weight of that block. Then the design and its dual are both uniform designs.

<u>Proof:</u> We have assumed that there are invariants $n$, $e_1$, $e_2$, $f_1$, $f_2$, $k_1$, $k_2$, $\lambda_1$, $\lambda_2$, $r_1$, $r_2$, $\mu_1$, $\mu_2$ with $\lambda_1 \neq \lambda_2$ and $\mu_1 \neq \mu_2$, and a $(0, 1)$-matrix $A = [a_{ij}]$ of order $n$ satisfying

$$A^TA = \begin{array}{c} \left[\begin{array}{c|c} \begin{matrix} k_1 & & \lambda_1 \\ & \ddots & \\ \lambda_1 & & k_1 \end{matrix} & \sqrt{\lambda_1}\sqrt{\lambda_2} \\ \hline \sqrt{\lambda_1}\sqrt{\lambda_2} & \begin{matrix} k_2 & & \lambda_2 \\ & \ddots & \\ \lambda_2 & & k_2 \end{matrix} \end{array}\right] \\ \underbrace{\qquad\qquad}_{f_1}\underbrace{\qquad\qquad}_{f_2} \end{array}, \qquad AA^T = \begin{array}{c} \left[\begin{array}{c|c} \begin{matrix} r_1 & & \mu_1 \\ & \ddots & \\ \mu_1 & & r_1 \end{matrix} & \sqrt{\mu_1}\sqrt{\mu_2} \\ \hline \sqrt{\mu_1}\sqrt{\mu_2} & \begin{matrix} r_2 & & \mu_2 \\ & \ddots & \\ \mu_2 & & r_2 \end{matrix} \end{array}\right] \\ \underbrace{\qquad\qquad}_{e_1}\underbrace{\qquad\qquad}_{e_2} \end{array}$$

Partition the matrix A into quadrants as shown:

$$A = \begin{array}{c} \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \begin{array}{l} \}\, e_1 \\ \}\, e_2 \end{array} \\ \begin{array}{cc} \underbrace{\qquad}_{f_1} & \underbrace{\qquad}_{f_2} \end{array} \end{array} .$$

Let $s_i'$ and $s_i^*$ be respectively the sum of the first $f_1$ entries, and of the last $f_2$ entries, of row i of A. Likewise let $\ell_j'$ and $\ell_j^*$ be respectively the sum of the first $e_1$ entries, and of the last $e_2$ entries, of column j of A. Note that $s_i' + s_i^*$ is $r_1$ or $r_2$ according as $1 \le i \le e_1$ or $e_1 < i \le n$, and that $\ell_j' + \ell_j^*$ is $k_1$ or $k_2$ according as $1 \le j \le f_1$ or $f_1 < j \le n$.

In the matrix equality $A(A^TA) = (AA^T)A$, the upper left quadrant of size $e_1$ by $f_1$ may be written explicitly as

$$\begin{aligned} (k_1-\lambda_1)A_{11} + \lambda_1 \cdot \mathrm{diag}[s_1', \ldots, s_{e_1}'] \cdot J + \lambda_{12} \cdot \mathrm{diag}[s_1^*, \ldots, s_{e_1}^*] \cdot J \\ = (r_1-\mu_1)A_{11} + \mu_1 \cdot J \cdot \mathrm{diag}[\ell_1', \ldots, \ell_{f_1}'] \\ + \mu_{12} \cdot J \cdot \mathrm{diag}[\ell_1^*, \ldots, \ell_{f_1}^*] \end{aligned} \tag{4.1}$$

where J denotes here the matrix of ones of size $e_1$ by $f_1$, and where $\lambda_{12} = \sqrt{\lambda_1}\sqrt{\lambda_2}$ and $\mu_{12} = \sqrt{\mu_1}\sqrt{\mu_2}$ is written for brevity.

First suppose that $k_1 - \lambda_1 = r_1 - \mu_1$ in (4.1). With $s_i^* = r_1 - s_i'$ and $\ell_j^* = k_1 - \ell_j'$, this supposition allows us to write (4.1) as

$$\begin{aligned} (\lambda_1-\lambda_{12}) \cdot \mathrm{diag}[s_1', \ldots, s_{e_1}'] \cdot J + r_1\lambda_{12} J \\ = (\mu_1-\mu_{12}) \cdot J \cdot \mathrm{diag}[\ell_1', \ldots, \ell_{f_1}'] + k_1\mu_{12} J . \end{aligned} \tag{4.2}$$

Note that $\lambda_1 \neq \lambda_2$ and $\mu_1 \neq \mu_2$, respectively imply that $\lambda_1 - \lambda_{12}$ and $\mu_1 - \mu_{12}$ are nonzero. The entries of the matrix on the left side of (4. 2) are constant within the same row, and those of the matrix on the right side of (4. 2) are constant within the same column. Hence the entries are constant within each matrix, and $A_{11}$ has constant row sums and constant column sums.

Suppose instead that $k_1-\lambda_1 \neq r_1-\mu_1$. Then (4. 1) may be solved for $A_{11}$ with the result that

$$A_{11} = [a_i + b_j] \quad , \tag{4. 3}$$

for some real numbers $a_1, \ldots, a_{e_1}$ and $b_1, \ldots, b_{f_1}$. If the rank of the (0, 1)-matrix $A_{11}$ is greater than 1, $A_{11}$ must contain a nonsingular (0, 1)-submatrix of order 2. Apart from row and column permutations, these submatrices are characterized by

$$\begin{bmatrix} 0 & 1 \\ 1 & * \end{bmatrix} \quad , \tag{4. 4}$$

where the asterisk * denotes either 0 or 1. Because (4. 4) is a submatrix of $A_{11}$, (4. 3) implies that the lower right entry * is 2. This contradiction implies that the rank of $A_{11}$ is 0 or 1. By similar reasoning (4. 3) excludes

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{4. 5}$$

as a possible submatrix of $A_{11}$. It is straightforward to show that a

(0, 1)-matrix of rank 0 or 1, containing no submatrix which by row and column permutations could be brought to (4.5), has one of the forms

$$\text{(a) } 0 \qquad \text{(b) } J \qquad \text{(c) } [0\,|\,J] \qquad \text{(d) } \left[\frac{0}{J}\right] \, . \qquad (4.6)$$

This argument for $A_{11}$ can be similarly applied to each quadrant $A_{ij}$ with $1 \leq i \leq 2$ and $1 \leq j \leq 2$. Then either (1) $r_i - \mu_i = k_j - \lambda_j$ and $A_{ij}$ has constant row sums and constant column sums, or (2) $A_{ij}$ has one of the forms (4.6a, b, c, d). If (1) holds for any pair of adjacent quadrants (that is, for $A_{11}$ and $A_{12}$, $A_{12}$ and $A_{22}$, $A_{22}$ and $A_{21}$, or $A_{21}$ and $A_{11}$) then $r_1 - \mu_1 = r_2 - \mu_2$ or $k_1 - \lambda_1 = k_2 - \lambda_2$ may be deduced. These are respectively the conditions for the uniformity of $A^T$ and of $A$. By Theorem 1.1, if one of $A$ and $A^T$ is uniform, they both are. In this instance we are done.

Apart from row and column permutations the remaining instances fall into three cases,

$$\text{(I)} \begin{bmatrix} \text{no} & \text{no} \\ \text{no} & \text{no} \end{bmatrix} \qquad \text{(II)} \begin{bmatrix} \text{no} & \text{no} \\ \text{no} & \text{yes} \end{bmatrix} \qquad \text{(III)} \begin{bmatrix} \text{yes} & \text{no} \\ \text{no} & \text{yes} \end{bmatrix} , \qquad (4.7)$$

where a quadrant is marked yes or no according as condition (1) above does or does not hold for that quadrant.

In cases (I) and (II), the row sums of $A_{11}$ are constant, or the column sums are constant, or both. Say at least the first possibility holds. Then each of $A_{11}$ and $A_{12}$ has the form (a), (b) or (c). Since $A^T$ is a nondegenerate design, $A$ does not have repeated rows. Hence $e_1 = 1$ here. Similarly if $A_{21}$ has constant row sums, then $e_2 = 1$.

But $n = e_1 + e_2 = 2$ contradicts $n \geq 3$, a condition assumed at the outstart. Then $A_{21}$ has constant column sums, so that $A_{11}$ does as well. Hence $f_1 = 1$ for the same reason that $e_1 = 1$. In case (I) alone the entire argument can be applied to quadrant $A_{22}$ instead of $A_{11}$, with the conclusion that $e_2 = f_2 = 1$. Again the contradiction $n = e_1 + e_2 = 2$ results.

In case (II) alone we still have $e_1 = f_1 = 1$. Because $A_{22}$ has constant row sums and column sums, $A_{12}$ has constant column sums and $A_{21}$ has constant row sums. Hence $A$ may by row and column permutations be brought to the form

$$A = \left[\begin{array}{c|c} a_{11} & X^T \\ \hline Y & A_{22} \end{array}\right]$$

where $a_{11}$ is of course either 0 or 1, X and Y are column matrices, each consisting of either n-1 zeros or else of n-1 ones, and $A_{22}$ is a $(v, k, \lambda)$-design with $v = n - 1$. This describes the form of the bordered multiplicative designs discussed earlier and excluded by hypotheses of the theorem.

In case (III) we gather that $A_{11}$ and $A_{22}$ each have constant row sums and constant column sums, and hence that $A_{12}$ and $A_{21}$ each have constant row sums and constant column sums. Each of $A_{12}$ and $A_{21}$ then has form (a) or form (b). Note that $A_{11}$ then has constant inner products between distinct rows and constant inner products between distinct columns. Fisher's inequality (1.1) may now be applied twice, to $A_{11}$ and to $A_{11}{}^{T}$, to get respectively $e_1 \geq f_1$ and $e_1 \leq f_1$. Hence $A_{11}$ is a $(v', k'\lambda')$-design with $v' = e_1 = f_1$. Likewise $A_{22}$ is a $(v^*, k^*, \lambda^*)$-design with $v^* = e_2 = f_2$.

If $A_{12}$ and $A_{21}$ are both of the form (a), clearly $\sqrt{\lambda_1}\sqrt{\lambda_2} = 0$, whence either $\lambda_1 = 0$ or $\lambda_2 = 0$. This contradicts the nondegeneracy of A. If $A_{12}$ and $A_{21}$ are both of the form (b) we may calculate explicitly

$$\lambda_1 = v^* + \lambda' \ , \qquad \lambda_2 = v' + \lambda^* \ , \qquad \sqrt{\lambda_1}\sqrt{\lambda_2} = k' + k^* \ ,$$

whence $\lambda_1\lambda_2$ may be evaluated in two ways to get

$$[v^*(v'-k') - k^*(k'-\lambda') - (k'^2-\lambda'v')] + [k'(v^*-k^*) - \lambda'(k^*-\lambda^*) - (k^{*2}-\lambda^*v^*)] = 0 \quad . \tag{4.8}$$

The condition (1.2) for $(v, k, \lambda)$-designs, applied respectively to $A_{11}$ and $A_{22}$, yields

$$k'^2 - \lambda'v' = k' - \lambda' \quad ,$$
$$k^{*2} - \lambda^*v^* = k^* - \lambda^* \quad .$$

With these equalities, (4.8) may be rewritten as

$$[v^*(v'-k') - (k^*+1)(k'-\lambda')] + [k'(v^*-k^*) - (\lambda'+1)(k^*-\lambda^*)] = 0 \tag{4.9}$$

The number of (0, 0) entries in two distinct columns of a $(v, k, \lambda)$-design is $v - 2k + \lambda$. Hence $v - k \geq k - \lambda$, and it can be shown that the inequality is strict for nondegenerate designs. If both $A_{11}$

and $A_{22}$ are nondegenerate $(v, k, \lambda)$-designs, observe that

$$v^* > k^* + 1 , \qquad v' - k' > k' - \lambda'$$
$$k' > \lambda' + 1 , \qquad v^* - k^* > k^* - \lambda^*$$

imply that the left side of (4.9) is positive, a contradiction. At least one of $A_{11}$ and $A_{22}$ is then degenerate, say $A_{11}$. Here $A_{11}$ must have the form 0, I, J, or J-I, apart from row and column permutations. It has neither the form 0 nor the form J, for then case (II) applies instead, and $v' > 1$ for the same reason. If $A_{11}$ has the form I, so that $(v', k', \lambda') = (v', 1, 0)$, equation (4.9) becomes

$$[v^*(v'-1) - (k^*+1)] + [v^* - 2k^* + \lambda^*] = 0 \quad .$$

In any case $v^* - 2k^* + \lambda^* \geq 0$, so that $v' = 2$ and $v^* = k^* + 1$. Hence $A_{22}$ is degenerate. In fact $A_{22}$ necessarily has the form J - I. The total design A then has the form J - I and is inadmissible because $\lambda_1 = \lambda_2$. If $A_{11}$ instead has the form J - I, (4.9) gives

$$[v^* - (k^*+1)] + [(v'-1)(v^*-2k^*+\lambda^*)] = 0 \quad .$$

We again conclude that $v^* = k^* + 1$, and that A has the form J - I.

Finally we consider the case when $A_{12}$ is of the form 0 and $A_{21}$ is of the form J. The case with 0 and J interchanged is essentially the same. Still with $A_{11}$ a $(v', k', \lambda')$-design and with $A_{22}$ a $(v^*, k^*, \lambda^*)$-design, we compute

$$\lambda_1 = v^* + \lambda' , \qquad \lambda_2 = \lambda^* , \qquad \sqrt{\lambda_1}\sqrt{\lambda_2} = k^* \quad .$$

Equate the two evaluations for $\lambda_1\lambda_2$ to get $\lambda'\lambda^* = k^{*2} - \lambda^* v^*$, whence by the usual necessary condition (1.2) for $(v, k, \lambda)$-designs, $\lambda'\lambda^* = k^* - \lambda^*$. Multiplicativity for $A^T$ gives the dual statement $\lambda'\lambda^* = k' - \lambda'$. Hence we have

$$k' - \lambda' = k^* - \lambda^* \quad . \tag{4.10}$$

From (4.10) we see that

$$k_1 - \lambda_1 = (v^* + k') - (v^* + \lambda') = k' - \lambda' = k^* - \lambda^* = k_2 - \lambda_2 \ ,$$

$$r_1 - \mu_1 = k' - \lambda' = k^* - \lambda^* = (v' + k^*) - (v' + \lambda^*) = r_2 - \mu_2 \quad .$$

These are respectively the conditions for the uniformity of the designs A and $A^T$. This completes the proof.

This argument is by no means delicate. In fact it seems likely that Theorem 4.1 can be proved without the condition that the size of a block depends only on the weight of the block in both the design and its dual. If this could be done, it would follow from Theorem 3.2 and the strengthened Theorem 4.1 that Conjecture 1.3 implies Conjecture 1.2.

## SECTION 5

## A GENERALIZATION OF MANN'S INEQUALITY

In the definition of a multiplicative or of a uniform design, the requirement that the variety order $m$ and the block order $n$ are equal is highly restrictive. An inequality [10] of Fisher type implies $m \geq n$ in any case. If we replace $m = n$ by the requirement $m \geq n$, we call the design a partial multiplicative design. If in addition $k_j - \lambda_j$ is constant for $1 \leq j \leq n$, it is called a partial uniform design. We can also reasonably require that the row sums of the design are at least two, since a row with no ones or with a single one does not contribute to column intersections.

Partial multiplicative designs can of course be created by simply deleting columns of a multiplicative design. To illustrate how easily other partial multiplicative designs can be constructed if $m$ is allowed to grow much larger than $n$, we let $\lambda_1, \ldots, \lambda_n$ be any $n$ positive numbers such that $\sqrt{\lambda_i}\sqrt{\lambda_j}$ is integral for $1 \leq i < j \leq n$. Let $R_{ij}$ be the row vector of length $n$ whose $i^{th}$ and $j^{th}$ entries are one and whose other entries are zero. Let $A$ be a matrix containing as its rows precisely $\sqrt{\lambda_i}\sqrt{\lambda_j}$ copies of $R_{ij}$ for each pair $i, j$ satisfying $1 \leq i < j \leq n$. If we set $s = \sqrt{\lambda_1} + \cdots + \sqrt{\lambda_n}$, then $m = (s^2 - \lambda_1 - \cdots - \lambda_n)/2$ and $k_j = s\sqrt{\lambda_j} - \lambda_j$ for $1 \leq j \leq n$. This partial multiplicative design $A$ is uniform only when

$$\left.\begin{array}{ll}\text{(a)} & \lambda_1 = \cdots = \lambda_n\,, \\ \text{(b)} & n = 4 \quad \text{and} \quad \lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4, \quad \text{or else,} \\ \text{(c)} & \lambda_1 \neq \lambda_2 = \cdots = \lambda_n \quad \text{and} \quad \lambda_1 = (n-3)^2\lambda_2\,, \quad (n > 4)\end{array}\right\} \tag{5.1}$$

This example suggests that the study of partial multiplicative designs might profitably be restricted to partial multiplicative designs in which $m$ is not much larger than $n$, or in which the design is a partial uniform design.

A block design is a design in which the overlap numbers $\lambda_j$ are all equal, and the replication numbers $r_i$ are all equal. Of course $m \geq n$. H. B. Mann [7] and van Lint and Ryser [5] have shown that a variety (row) of a block design is repeated in the design at most $m/n$ times. (A variety of a design is said to be repeated $s$ times in the design if there are precisely $s - 1$ other varieties in the design which belong to precisely the same blocks as the given variety.) This upper bound $m/n$ can actually be assumed when a block design is formed by repeating each row of a $(v, k, \lambda)$-design, with $v = n$, a constant number $u$ of times. Then $u = m/n$.

There are partial uniform designs which do not satisfy this bound $m/n$, such as the design given by (5.1b) when $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = \lambda_4 = 4$, for instance. Another, more interesting, incidence of failure for partial multiplicative designs is sketched in Figure 1. Here $A'$ and $A^*$, respectively denote a $(v', k', \lambda')$-design and a $(v^*, k^*, \lambda^*)$-design, and J and O, respectively denote appropriate matrices of ones and of zeros. Choose the parameters of $A^*$ so that $\lambda^* | k^*$ and $x | y\lambda^*$. If the parameters of $A'$ are given by

$$
A = \begin{array}{|c|c|}
\hline
A' & J \\
\hline
\multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{\vdots} \\
\hline
A' & J \\
\hline
0 & A^* \\
\hline
\multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{\vdots} \\
\hline
0 & A^* \\
\hline
\end{array}
\qquad
\begin{array}{l}
\\
x \text{ repetitions} \\
\\
\\
y \text{ repetitions} \\
\\
\end{array}
$$

Figure 1

$$\lambda' = \frac{k^*}{\lambda^*} - 1 \ , \qquad k' = \lambda'\left(\frac{y\lambda^*}{x} + 1\right) \ , \qquad v' = \frac{k'(k'-1)}{\lambda'} + 1 \ ,$$

then A is a partial multiplicative design. Each row is repeated either x or y times, and

$$m/n = (xv' + yv^*)/(v' + v^*) \quad .$$

Clearly $\min(x, y) \le m/n$, but $\max(x, y) \le m/n$ does not always hold. For instance, take

$$(v^*, k^*, \lambda^*) = (3, 2, 1) \ , \quad (v', k', \lambda') = (7, 3, 1) \ , \quad x = 1, \ y = 2, \qquad \text{or}$$
$$(v^*, k^*, \lambda^*) = (7, 3, 1) \ , \quad (v', k', \lambda') = (16, 6, 2) \ , \quad x = 1, \ y = 2 \quad .$$

In either case $y \nleq m/n$. The above examples are in fact partial uniform designs.

Mann's proof, however, suggests how a correct analogue to his inequality can be constructed.

<u>Theorem 5.1.</u> In a (nondegenerate) partial multiplicative design of variety order $m$ and block order $n$, let a given variety $z$ with replication number $e$ be repeated $s$ times. Let $z$ belong to the blocks $S_1, \ldots, S_e$, and set

$$K = k_1 + \cdots + k_e ,$$
$$L = (\sqrt{\lambda_1} + \cdots + \sqrt{\lambda_e})^2 - (\lambda_1 + \cdots + \lambda_e) .$$

Then

$$s \leq \frac{m(K+L) - K^2}{me^2 + K + L - 2Ke} \tag{5.2}$$

<u>Proof:</u> Consider the incidence matrix of size $m$ by $n$ of this partial multiplicative design. Permute the rows of A so that the $s$ copies of the row corresponding to $z$ occur initially. Also permute the columns of A so that the $e$ ones of the first $s$ rows occur initially. The matrix A then has the form

$$A = \begin{array}{r} s\left\{\vphantom{J}\right. \\ \vphantom{A_{21}} \end{array} \left.\underbrace{\begin{array}{|c|c|} \hline \overbrace{J}^{e} & 0 \\ \hline A_{21} & \\ \hline \end{array}}_{n}\right\} m \quad ,$$

where J and 0 denote appropriate matrices of ones and of zeros. Let $\Sigma$ denote here summation over $j$ in the range $1 \leq j \leq e$, and $\Sigma'$ denote summation over $i$ in the range $s+1 \leq i \leq m$. Let $a_{s+1}, \ldots, a_m$ be the row sums of $A_{21}$. The number of ones in $A_{21}$ can be computed in two ways as

$$\Sigma' a_i = \Sigma k_j - se = K - se \quad . \tag{5.3}$$

The number of ordered pairs of ones on the same row of $A_{21}$ can also be computed in two ways as

$$\begin{aligned} \Sigma' a_i(a_i - 1) &= (\Sigma \sqrt{\lambda_j})^2 - \Sigma \lambda_j - se(e-1) \\ &= L - se(e-1) \quad . \end{aligned} \tag{5.4}$$

The sum of (5. 3) and (5. 4) is

$$\Sigma' a_i^2 = K + L - se^2 \quad .$$

We see there are $K - se$ ones distributed among the $m - s$ rows of $A_{21}$. The Cauchy-Schwartz inequality implies that $\Sigma' a_i^2$ is minimized when the $a_i$ are all equal. Hence

$$K + L - se^2 \geq (K - se)^2/(m - s) \quad .$$

The quadratic term in $s$ vanishes upon multiplication of this inequality by $m - s$, so we conclude that (5. 2) holds. This completes the proof.

From the Cauchy-Schwartz inequality we also gather that equality holds in (5. 2) when and only when the $a_i$ are all equal. This condition is met, for example, when A consists of $s$ copies of each row of a bordered design

$$A' = \left[\begin{array}{c|c} a_{11} & X^T \\ \hline Y & B \end{array}\right] ,$$

and when the distinguished variety z corresponds to the first row of the matrix $A'$ shown above.

The inequality (5.2) does not have the same great usefulness of Mann's inequality $s \leq m/n$ because the numbers e, K, and L depend on the choice of the variety z in the design, where of course, m and n do not. For completeness we deduce Mann's result from Theorem 5.1.

Corollary 5.2 (Mann). If, in Theorem 5.1, the overlap numbers $\lambda_j$ of the partial multiplicative design have a common value $\lambda$, and its replication numbers $r_i$ have a common value k, then $s \leq m/n$.

Proof: A partial multiplicative design so defined is in fact a block design. It is known [8] that the block sizes $k_i$ have a common value r, and that

$$bk = rv , \qquad r - \lambda = rk - \lambda v , \tag{5.5}$$

where $b = m$ and $v = n$. With $k_i = r$, $e = k$, $K = rk$ and $L = \lambda k(k-1)$, inequality (5.2) becomes

$$s \leq \frac{k}{k} \cdot \frac{br + b\lambda k - b\lambda - r^2 k}{bk + r + \lambda k - \lambda - 2rk} ,$$

whence

$$- s(bk + r - \lambda - 2rk) + r(bk + \lambda k + r - \lambda - 2rk)$$
$$\geq - (br + b\lambda k - b\lambda - r^2 k) + r(bk + \lambda k + r - \lambda - 2rk) \quad ,$$

whence

$$(r-s)[(bk-rk) - (r-\lambda)k + (r-\lambda)]$$
$$\geq (b-r)(r-\lambda)(k-1) \quad . \tag{5.6}$$

The identities (5.5) may be applied to the left side of (5.6) to obtain

$$(r-s)[(rv-rk) - (r-\lambda)k + (rk-\lambda v)]$$
$$\geq (b-r)(r-\lambda)(k-1) \quad ,$$

that is,

$$(r-s)[r-\lambda)(v-k)] \geq (b-r)(r-\lambda)(k-1) \quad . \tag{5.7}$$

The partial multiplicative design of Theorem 5.1 is nondegenerate, so that $r > \lambda$. Hence (5.7) becomes

$$(r-s)(v-k) \geq (b-r)(k-1) \quad ,$$

whence

$$r-s \geq \frac{b-r}{v-k}(k-1) = \left[\frac{k}{r} \cdot \frac{b-r}{v-k}\right]\left[\frac{r}{k}(k-1)\right] = \frac{r}{k}(k-1) \quad .$$

We finally have

$$s \leq r - \frac{r}{k}(k-1) = \frac{r}{k} = \frac{b}{v} \quad .$$

The first of the identities (5.5) is invoked again at the last equality. Hence $s \leq b/v = m/n$, and we are finished.

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