Nonvanishing of L-functions for $\mathbf{GL}(n)$

Thesis by

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To my family.

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Abstract

In this thesis I study two different approaches towards proving average results on values of L-functions, with an interest toward establishing new results on automorphic L-functions, especially concerning the nonvanishing of L-functions of degree > 2 at the center of the critical strip (and at other points of the complex plane), and their applications, particularly to p-adic L-functions. In the first problem, I evaluate a twisted average of L-values using the approximate functional equation in order to prove a result on the determination of isobaric representations of $\mathrm{GL}(3, \mathbb{A}_{\mathbb{Q}})$ by certain L-values of p-power twists. I also provide an application to the adjoint p-adic L-function of an elliptic curve. More specifically, I show that if E is an elliptic curve over $\mathbb{Q}$ with semistable reduction at some fixed prime p, then the adjoint p-adic L-function of E evaluated at any infinite set of integers relatively prime to p completely determines up to a quadratic twist the isogeny class of E.

For the second problem, which is part of a long project, I present some results towards proving an average result for the degree 4 L-function on $\mathrm{GSp}(4)/\mathbb{Q}$ at the center using the relative trace formula. More specifically, I consider a suitable relative trace formula such that the spectral side is an average of central L-values of genus 2 holomorphic Siegel eigenforms of weight k and level N twisted by some fixed character. I then work towards computing the corresponding geometric side.

Published content and contributions

Chapter 2 of this thesis is based on paper [Nas15] (DOI: 10.1016/j.jnt.2014.12.012) of which I am the sole author. Apart from a modified introduction and minor changes, the results in that chapter come directly from this paper.

Contents

Chapter 1

Introduction

In many cases, the values of an L-function inside the critical strip can encode important arithmetic information. The study of the behavior of L-functions at the center of the critical strip is especially important in this sense, as suggested by its ties with several conjectures. For example, the Birch and Swinnerton-Dyer conjecture suggests that the behavior of the L-function associated to an elliptic curve E over $\mathbb{Q}$ at the center of the critical strip $0 \leq \mathrm{Re}(s) \leq 2$ determines the rank of $E(\mathbb{Q})$. There are also generalizations such as Deligne's conjecture on special values of L-functions and work due to Beilinson and Kato on the leading term in the Taylor series of the L-function at the center.

The construction and study of p-adic L-functions is also an important and related part of current research. It is conjectured that p-adic L-functions can be constructed in general settings, but have only been shown to exist in a limited number of cases. It is known that you can construct a p-adic L-function associated to a modular form by interpolating p-power twists of the associated complex L-function at special values. Work of Ash and Ginzburg shows that a p-adic L-function can be constructed for certain automorphic representations π of $\mathrm{GL}(2n, \mathbb{A}_{\mathbb{Q}})$ under some conditions, such as the nonvanishing of the twisted complex L-function $L(\pi \otimes \chi, 1/2)$ by some character

χ that is trivial at infinity.

One method of proving nonvanishing results for values of L-functions inside the critical strip is to consider suitable averages over families of L-functions. To evaluate such averages, one can use the traditional approach using an approximate functional equation or, among others, the more recent approach using the relative trace formula.

In Chapter 2, we use the approximate functional equation to compute a twisted average of L-functions which allows us to prove a result on the determination of isobaric representations π of $\mathrm{GL}(3, \mathbb{A}_{\mathbb{Q}})$ by twisted L-values $L(\pi\otimes\chi, \beta)$ with χ ranging over primitve p-power order characters and β a fixed point inside the critical strip but outside the central line. The method used also gives nonvanishing of infinitely many such twisted L-values for isobaric automorphic representations of $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$ for $n \geq 3$.

We also provide an application on the determination of elliptic curves over $\mathbb{Q}$ up to isogeny by the adjoint p-adic L-function. The main result of Chapter 2 (Theorem 2.4) can be summarized as follows:

Theorem. *Let p be an odd prime and E, E' be elliptic curves over $\mathbb{Q}$ with semistable reduction at p. Suppose*

$$L_p(Sym^2E, n) = CL_p(Sym^2E', n)$$

for all $(n, p) = 1$ elements of an arbitrary infinite set Y and $C \in \overline{\mathbb{Q}}$. Then E' is isogenous to a quadratic twist E_D of E. If E and E' have square free conductor, then E and E' are isogenous over $\mathbb{Q}$.

In Chapter 3, I present some results which are part of a long project towards establishing an average result for the degree 4 L-function on $\mathrm{GSp}(4)/\mathbb{Q}$ at the center. More precisely, the purpose of this project is to establish the following:

($\star$) Fix a Siegel weight $k \geq 3$. Then for a suitable fixed character χ_0, there exist infinitely many genus 2 holomorphic Siegel eigenforms π of trivial central character and weight k such that

$$L(\pi \otimes \chi_0, 1/2) \neq 0.$$

as we vary the level $N \to \infty$.

Since the nonvanishing statement in ($\star$) is only possible modulo the root number, we consider forms π with root number 1. In particular I am looking at self-dual forms so the root number can only be ± 1.

Alternatively, we can consider the slightly modified problem:

($\star\star$) Fix a level $N > 1$. Then for a suitable fixed character χ_0, there exist infinitely many genus 2 holomorphic Siegel eigenforms π of trivial central character and level N such that

$$L(\pi \otimes \chi_0, 1/2) \neq 0$$

as we vary the weight $k \to \infty$.

My approach to proving these problems is: ($\star$) would follow from the spectral side of a suitable relative trace formula on $\mathrm{GSp}(4)/\mathbb{Q}$ with respect to subgroups $\mathrm{GL}(2)/F$ with F an auxiliary imaginary quadratic field and U the unipotent radical of the Siegel parabolic subgroup, if the corresponding geometric side is nonvanishing as we vary $N \to \infty$. Similarily, ($\star\star$) would follow if the corresponding geometric side is nonvanishing as $k \to \infty$.

It should be noted that no nonvanishing results at the center are known for any family of L-functions of degree ≥ 4. By work of Bloch and Kato, nonvanishing at the center of the degree 4 L-function would imply that a certain associated Selmer group of the Siegel eigenform π is finite.

There are three steps to solving this problem: (1) Verify that the spectral side gives the desired average, (2) Find special leading terms on the geometric side that give a nonvanishing contribution, and (3) Show that these leading terms dominate the others.

I give a suitable test function and show that this allows the spectral side to be identified with a weighted average of central degree 4 L-values and identify the leading terms on the geometric side and an oscillating behavior in the remaining terms, which should allow me to show that the former terms dominate the latter.

Chapter 2

Determination of elliptic curves by their adjoint p-adic L-functions[1]

2.1 Introduction

There has been a lot of interest in the study of L-functions associated to symmetric powers of motives attached to modular forms, and in particular to the study of the Bloch-Kato conjecture for $L(Sym^2E, s)$, the L-function associated to the symmetric square of an elliptic curve at the critical value $s = 2$.

In [DD97], Dabrowski and Delbourgo define the p-adic L-function attached to the motive Sym^2E at the critical point $s = 2$ as the Mazur-Mellin transform of a p-adic distribution $\mu_p(Sym^2E)$ on $\mathbb{Z}_p^\times$. This distribution is defined by interpolating the values of the complex symmetric square L-function $L(Sym^2E, \chi, 2)$ at all twists by Dirichlet characters of p-power order. They also show that the distribution $\mu_p(Sym^2E)$ is in fact a bounded measure on $\mathbb{Z}_p^\times$ if E has good ordinary reduction at p or bad multiplicative reduction at p, and is an h-admissible measure with $h = 2$ if E has good supersingular reduction at p.

[1]This chapter is a modified version of the author's paper [Nas15]

The main result of this chapter is stated in Theorem 2.4. It is a result on the determination of elliptic curves over $\mathbb{Q}$ up to isogeny by the adjoint p-adic L-function. Using the theory of h-admissible measures developed by Visik [Vis76], we show (see Lemma 2.4) that Theorem 2.4 reduces to proving that the twisted L-values $\{L(Sym^2E, \chi, 2)\}$ with χ ranging over Dirichlet characters with p-power conductor determine the isogeny class of E up to quadratic twist. Note that just knowing nonvanishing of the complex L-values twisted by p-power characters gives the nonvanishing of the p-adic L-function, but not that the p-adic L-function determines the isogeny class of E, which requires a further argument.

If f is the newfom of weight 2 associated to E by Wiles' modularity theorem, and π the unitary cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$ generated by f, then

$$L(Sym^2E, s) = L(Sym^2\pi, s-1),$$

where $Sym^2\pi$ is the automorphic representation of $\mathrm{GL}(3, \mathbb{A}_{\mathbb{Q}})$ associated to π by Gelbart and Jacuqet [GJ78]. It is well-known that $Sym^2\pi$ is cuspidal only if E is non-CM, otherwise it is an isobaric sum of unitary cuspidal automorphic representations. Theorem 2.4 is then a consequence of a result on the determination of isobaric automorphic representations of GL(3) which is summarized in Theorem 2.2.

One of the main ingredients in the proof of Theorem 2.2 is the computation of a twisted average of the form

$$\sum_{\chi \bmod p^a} \overline{\chi}(s)\chi(r)L(\pi \otimes \chi, \beta),$$

where the sum if over primitive p-power order characters of conductor p^a. This sum is computed in Theorem 2.1 and it uses an approximate equation similar to that used in [Luo05]. A proof of the approximate functional equation is provided in Section 2.2.2. Note that special care is required when π or π' above are not cuspidal, and in such cases I require $L(\pi \otimes \chi, s)$ and $L(\pi' \otimes \chi, s)$ to be entire for all p-power twists.

A consequence of the computation of this twisted average is given by a nonvanishing result on p-power twists of isobaric representations for $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$ for $n \geq 3$. This result is summarized in Corollary 2.1. Note, however, that proving nonvanishing of $L(\pi \otimes \chi, \beta)$ for infinitely many p-power twists and a fixed β will not imply the determination result of Theorem 2.2. Proving determination as in Theorem 2.2 is a stronger result which I am only able to show for $n = 3$ and not for $n > 4$.

To give some context to the nonvanishing of twisted L-values, note that nonvanishing results have been proved for many families of twisted L-values. In particular, building on work of Rohrlich [Roh84] and Ramakrishnan and Barthel [BR94], Luo [Luo05] showed that nonvanishing of infinitely many twisted L-values $\{L(\pi \otimes \chi, \beta)\}$ with χ ranging over all Dirichlet characters, holds for $\beta \notin \left[\frac{2}{n}, 1 - \frac{2}{n}\right]$ and π a cuspidal automorphic representation of $\mathrm{GL}(3, \mathbb{A}_{\mathbb{Q}})$. In particular, Luo is able to obtain nonvanishing at the center for $n = 3$. However, in my case the set of characters considered is much sparser.

We now give an outline of the rest of the chapter. In Section 2.2 we present the basic properties of the standard L-function associated to an isobaric representation of $\mathrm{GL}(n)$ as well as give a proof of the approximate functional equation. In Section 2.3 we give proofs for Theorem 2.1 and Theorem 2.2, as well as give an application on the determination of π by certain twisted L-values of the isobaric automorphic representation $Ad(\pi)$ of $\mathrm{GL}(3, \mathbb{A}_{\mathbb{Q}})$ when π is a cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$. This result is summarized in Theorem 2.3. Finally, in Section 2.4 we give a proof of Theorem 2.4.

2.2 Preliminaries

2.2.1 The standard L-function of GL(n)

Let π be an irreducible automorphic representation of $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$ and $L(\pi, s)$ its associated standard L-function. Write $\pi = \otimes'_v \pi_v$ as a restricted direct product with π_v admissible irreducible representations of the local groups $\mathrm{GL}(n, \mathbb{Q}_v)$. The Euler product

$$L(\pi, s) = \prod_v L(\pi_v, s) \tag{2.1}$$

converges for $\mathrm{Re}(s)$ large. There exist conjugacy classes of matrices $A_v(\pi) \in \mathrm{GL}(n, \mathbb{C})$ such that the local L-functions at finite places v with π_v unramified are

$$L(\pi_v, s) = \det(1 - A_v(\pi)v^{-s})^{-1}. \tag{2.2}$$

Let $A_v(\pi) = [\alpha_{1,v}(\pi), \cdots, \alpha_{n,v}(\pi)]$ be the diagonal representatives of the conjugacy classes.

For S a set of places of $\mathbb{Q}$ we can define

$$L^S(\pi, s) = \prod_{v \notin S} L_v(\pi, s) \tag{2.3}$$

called the incomplete L-function associated to set S.

Let $\boxplus$ be the isobaric sum introduced in [JS81]. We can define an irreducible automorphic representation, called an isobaric representation, $\pi_1 \boxplus \cdots \boxplus \pi_m$ of $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$, $n = \sum_{i=1}^m n_i$, for m cuspidal automorphic representations $\pi_i \in \mathrm{GL}(n_i, \mathbb{A}_{\mathbb{Q}})$. Such a representation satisfies

$$L^S(\boxplus_{j=1}^m \pi_j, s) = \prod_{j=1}^m L^S(\pi_j, s)$$

with S any finite set of places.

The following is a generalization of the Strong Multiplicity One Theorem for isobaric representations due to Jacquet and Shalika [JS81]:

Theorem (Generalized Strong Multiplicity One)**.** *Consider two isobaric representations* π_1 *and* π_2 *of* $GL(n, \mathbb{A}_{\mathbb{Q}})$ *and* S *a finite set of places of* $\mathbb{Q}$ *that contains* ∞*. Then* $\pi_{1,v} \cong \pi_{2,v}$ *for all* $v \notin S$ *implies* $\pi_1 \cong \pi_2$*.*

We call an isobaric representation tempered if each π_i in the isobaric sum $\pi = \pi_1 \boxplus \cdots \boxplus \pi_m$ is a tempered cuspidal automorphic representation, or more specifically if each local factor $\pi_{i,v}$ is tempered.

We will consider a subset of the set of isobaric representations of $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$, more specifically those given by an isobaric sum of unitary cuspidal automorphic representations. We denote this subset by $\mathcal{A}_u(n)$. We will also consider the case when the unitary cuspidal automorphic representations in the isobaric sum are tempered, which is expected to always hold given the generalized Ramanujan conjecture.

Let $n \geq 3$ and let $\pi \in \mathcal{A}_u(n)$ be an isobaric sum of unitary cuspidal automorphic representations of $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$ with (unitary) central character ω_π and contragradient representation $\tilde{\pi}$. We have

$$L(\pi_\infty, s) = \prod_{j=1}^{n} \pi^{-\frac{s-\mu_j}{2}} \Gamma\left(\frac{s-\mu_j}{2}\right), \; L(\tilde{\pi}_\infty, s) = \prod_{j=1}^{n} \pi^{-\frac{s-\overline{\mu_j}}{2}} \Gamma\left(\frac{s-\overline{\mu_j}}{2}\right) \tag{2.4}$$

for some $\mu_j \in \mathbb{C}$, with π in this context denoting the transcendental number.

The L-function is defined for $\mathrm{Re}(s) > 1$ by the absolutely convergent Dirichlet series

$$L(\pi, s) = \sum_{m=1}^{\infty} \frac{a_\pi(m)}{m^s} \tag{2.5}$$

with $a_\pi(1) = 1$. This extends to a meromorphic function on $\mathbb{C}$ with a finite number of poles.

It is known that the coefficients $a_\pi(m)$ of the Dirichlet series satisfy

$$\sum_{m \leq M} |a_\pi(m)|^2 \ll_\epsilon M^{1+\epsilon} \tag{2.6}$$

for $M \geq 1$ (cf. Theorem 4 in [Mol02], [JPSS83, JS81, Sha81, Sha88]). For this property to hold, it is necessary that π be an isobaric sum of unitary cuspidal automorphic representations, rather than any unitary isobaric representation.

If π is in fact an isobaric sum of tempered cuspidal automorphic representations, then we have that the coefficients $a_\pi(m)$ satisfy

$$|a_\pi(m)| \ll_\epsilon m^\epsilon.$$

The completed L-function $\Lambda(\pi, s) = L(\pi_\infty, s)L(\pi, s)$ obeys the functional equation

$$\Lambda(\pi, s) = \epsilon(\pi, s)\Lambda(\tilde{\pi}, 1 - s), \tag{2.7}$$

where the ϵ-factor is given by

$$\epsilon(\pi, s) = f_\pi^{1/2-s} W(\pi) \tag{2.8}$$

and f_π and $W(\pi)$ are the conductor and the root number of π.

Let χ denote an even primitive Dirichlet character that is unramified at ∞ and with odd conductor q coprime to f_π. The twisted L-function obeys the functional equation (see for example [JPSS83])

$$\Lambda(\pi \otimes \chi, s) = \epsilon(\pi \otimes \chi, s)\Lambda(\tilde{\pi} \otimes \overline{\chi}, 1-s), \tag{2.9}$$

where $\Lambda(\pi \otimes \chi, s) = L(\pi_\infty, s)L(\pi \otimes \chi, s)$. The ϵ-factor is given by (cf. Proposition 4.1 in [BR94])

$$\epsilon(\pi \otimes \chi, s) = \epsilon(\pi, s)\omega_\pi(q)\chi(f_\pi)q^{-ns}\tau(\chi)^n \tag{2.10}$$

with $\tau(\chi)$ the Gauss sum of the character χ, and ω_π the central character of π.

Since $L(\pi \otimes \chi, s)$ does not vanish in the half-plane $\mathrm{Re}(s) > 1$, it is enough to consider $1/2 \leq \mathrm{Re}(s) \leq 1$. Twisting π by a unitary character $|\cdot|^{it}$ if needed, we can take $s \in \mathbb{R}$. Thus, from now on,

$$\frac{1}{2} \leq s \leq 1. \tag{2.11}$$

2.2.2 Approximate functional equation

We present a construction introduced in [Luo05, LR97]. For a smooth function g with compact support on $(0,\infty)$, normalized such that $\int_0^\infty g(u)\frac{du}{u} = 1$, we can introduce an entire function k given by

$$k(s) = \int_0^\infty g(u)u^{s-1}du$$

such that $k(0) = 1$ by normalization and k decreases rapidly in vertical strips. We then consider two functions for $y > 0$,

$$F_1(y) = \frac{1}{2\pi i}\int_{(2)} k(s)y^{-s}\frac{ds}{s}, \tag{2.12}$$

$$F_2(y) = \frac{1}{2\pi i}\int_{(2)} k(-s)G(-s+\beta)y^{-s}\frac{ds}{s}, \tag{2.13}$$

with $G(s)=\frac{L(\tilde{\pi}_\infty,1-s)}{L(\pi_\infty,s)}$ and the integrals above over $\mathrm{Re}(s)=2$. The functions $F_1(y)$ and $F_2(y)$ obey the following relations (see [Luo05]):

1. $F_{1,2}(y) \ll C_m y^{-m}$ for all $m \geq 1$, as $y \to \infty$.

2. $F_1(y) = 1 + O(y^m)$ for all $m \geq 1$ for y small enough.

3. $F_2(y) \ll_\epsilon 1 + y^{1-\eta-\mathrm{Re}(\beta)-\epsilon}$ for any $\epsilon > 0$, where $\eta = \max_{1\leq j\leq n} \mathrm{Re}(\mu_j)$ and μ_j as in (2.4). If π is tempered then $\eta = 0$ and in general the following inequality holds (see [LRS99]):
$$0 \leq \eta \leq \frac{1}{2} - \frac{1}{n^2+1}. \tag{2.14}$$

The following approximate functional equation was first used in [LR97] for cuspidal automorphic representations of $\mathrm{GL}(n)$ over $\mathbb{Q}$. A similar approximate functional equation was proved in [BH12] for slightly different rapidly decreasing functions.

Proposition If $\pi \in \mathcal{A}_u(n)$ and χ is a primitive Dirichlet character of conductor q such that $L(\pi\otimes\chi,s)$ is entire, then for any $\frac{1}{2} \leq \beta \leq 1$

$$\begin{aligned} L(\pi\otimes\chi,\beta) &= \sum_{m=1}^{\infty} \frac{a_\pi(m)\chi(m)}{m^\beta} F_1\left(\frac{my}{f_\pi q^n}\right) + \omega_\pi(q)\epsilon(0,\pi)\tau(\chi)^n (f_\pi q^n)^{-\beta} \times \\ &\times \sum_{m=1}^{\infty} \frac{a_{\tilde{\pi}}(m)\overline{\chi}(mf'_\pi)}{m^{1-\beta}} F_2\left(\frac{m}{y}\right), \end{aligned}$$

where f'_π is the multiplicative inverse of f_π modulo q.

Proof. For $\sigma > 0,\ y > 0$ consider the integral:

$$\frac{1}{2\pi i}\int_{(\sigma)} k(s) L(\pi\otimes\chi, s+\beta)\left(\frac{y}{f_\pi q^n}\right)^{-s} \frac{\mathrm{d}s}{s}.$$

Since $k(s)$ and $L(\pi \otimes \chi, s+\beta)$ are entire functions, the only pole of the function

$$k(s)L(\pi \otimes \chi, s+\beta)\left(\frac{y}{f_\pi q^n}\right)^{-s} s^{-1}$$

is a simple pole at $s = 0$ with residue equal to

$$\lim_{s\to 0} k(s)L(\pi \otimes \chi, s+\beta)\left(\frac{y}{f_\pi q^n}\right)^{-s} = L(\pi \otimes \chi, \beta).$$

Then by the residue theorem

$$\begin{aligned} L(\pi \otimes \chi, \beta) &= \frac{1}{2\pi i}\int_{(\sigma)} k(s)L(\pi \otimes \chi, s+\beta)\left(\frac{y}{f_\pi q^n}\right)^{-s}\frac{\mathrm{d}s}{s} \\ &- \frac{1}{2\pi i}\int_{(-\sigma)} k(s)L(\pi \otimes \chi, s+\beta)\left(\frac{y}{f_\pi q^n}\right)^{-s}\frac{\mathrm{d}s}{s}. \end{aligned}$$

Taking $s \to -s$ in the second integral gives

$$\begin{aligned} L(\pi \otimes \chi, \beta) &= \frac{1}{2\pi i}\int_{(\sigma)} k(s)L(\pi \otimes \chi, s+\beta)\left(\frac{y}{f_\pi q^n}\right)^{-s}\frac{\mathrm{d}s}{s} \\ &+ \frac{1}{2\pi i}\int_{(\sigma)} k(-s)L(\pi \otimes \chi, -s+\beta)\left(\frac{y}{f_\pi q^n}\right)^{s}\frac{\mathrm{d}s}{s}. \end{aligned} \tag{2.15}$$

The functional equation is

$$L(\pi_\infty, s)L(\pi \otimes \chi, s) = \epsilon(\pi \otimes \chi, s)L(\tilde{\pi}_\infty, 1-s)L(\tilde{\pi} \otimes \overline{\chi}, 1-s),$$

which implies that

$$L(\pi \otimes \chi, s) = \epsilon(\pi \otimes \chi, s)G(s)L(\tilde{\pi} \otimes \overline{\chi}, 1-s).$$

Substituting this identity in the second integral gives

$$L(\pi \otimes \chi, \beta) = I_1 + I_2 \tag{2.16}$$

with

$$I_1 = \frac{1}{2\pi i} \int_{(\sigma)} k(s) L(\pi \otimes \chi, s + \beta) \left(\frac{y}{f_\pi q^n} \right)^{-s} \frac{\mathrm{d}s}{s}$$

and

$$I_2 = \frac{1}{2\pi i} \int_{(\sigma)} \epsilon(\beta - s, \pi \otimes \chi) G(\beta - s) k(-s) L(\tilde{\pi} \otimes \overline{\chi}, 1 + s - \beta) \left(\frac{y}{f_\pi q^n} \right)^{s} \frac{\mathrm{d}s}{s}.$$

Taking $\sigma = 2$ and substituting with $L(\pi \otimes \chi, s) = \sum_{m=1}^{\infty} a_\pi(m) \chi(m) m^{-s}$ in the region of absolute convergence gives

$$I_1 = \sum_{m=1}^{\infty} a_\pi(m) \chi(m) m^{-\beta} \cdot \frac{1}{2\pi i} \int_{(2)} k(s) \left(\frac{my}{f_\pi q^n} \right)^{-s} \frac{\mathrm{d}s}{s},$$

and by the definition of F_1,

$$I_1 = \sum_{m=1}^{\infty} a_\pi(m) \chi(m) m^{-\beta} F_1 \left(\frac{my}{f_\pi q^n} \right). \tag{2.17}$$

Similarly,

$$I_2 = \frac{1}{2\pi i} \int_{(2)} \epsilon(\beta - s, \pi \otimes \chi) G(\beta - s) k(-s) \sum_{m=1}^{\infty} a_{\tilde{\pi}}(m) \overline{\chi}(m) m^{-1-s+\beta} \left(\frac{y}{f_\pi q^n} \right)^{s} \frac{\mathrm{d}s}{s}$$

with $\epsilon(\beta - s, \pi \otimes \chi) = \epsilon(\beta - s, \pi) \omega_\pi(q) \chi(f_\pi) q^{-n(\beta - s)} \tau(\chi)^n$ and $\epsilon(\beta - s, \pi) = f_\pi^{1/2 - \beta + s} W(\pi)$.

This gives

$$\begin{aligned} I_2 &= \sum_{m=1}^{\infty} a_{\tilde{\pi}}(m)\overline{\chi}(mf'_\pi)m^{-1+\beta}f_\pi^{1/2-\beta}W(\pi)\omega_\pi(q)q^{-n\beta}\tau(\chi)^n \times \\ &\times \frac{1}{2\pi i}\int_{(2)} G(\beta-s)k(-s)y^s m^{-s}\frac{\mathrm{d}s}{s}. \end{aligned}$$

By the definition of F_2,

$$I_2 = \omega_\pi(q)\epsilon(0,\pi)\tau(\chi)^n(f_\pi q^n)^{-\beta}\sum_{m=1}^{\infty}\frac{a_{\tilde{\pi}}(m)\overline{\chi}(mf'_\pi)}{m^{1-\beta}}F_2\left(\frac{m}{y}\right). \tag{2.18}$$

Here $W(\pi)f_\pi^{1/2} = \epsilon(0,\pi)$. Applying equations (2.16), (2.17), and (2.18) gives the desired approximate functional equation. □

2.2.3 Dihedral representations

Let π be a cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A}_\mathbb{Q})$ with conductor f_π. We have the symmetric square L-function $L(\pi,s,Sym^2)$ given by an Euler product with local factors

$$L_v(\pi,s,Sym^2) = (1-\alpha_v^2 v^{-s})^{-1}(1-\alpha_v\beta_v v^{-s})^{-1}(1-\beta_v^2 v^{-s})^{-1}$$

for primes v with $v \nmid f_\pi$ and $A_v(\pi) = \{\alpha_v,\beta_v\}$ the diagonal representatives of the conjugacy classes attached to π_v.

By [GJ78], there exists an isobaric automorphic representation $Sym^2(\pi)$ of $\mathrm{GL}(3,\mathbb{A}_\mathbb{Q})$ whose standard L-function agrees with $L(\pi,s,Sym^2)$ at least at primes v with $v \nmid f_\pi$. We have that $Sym^2(\pi)$ is cuspidal if and only if π is dihedral. A dihedral representation is a representation induced by the idele class character η of a quadratic extension

K of $\mathbb{Q}$. If $\pi = I_K^{\mathbb{Q}}(\eta)$ is a dihedral representation then

$$L(I_K^{\mathbb{Q}}(\eta), s) = L(\eta, s).$$

Let π be a (unitary) cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$. Suppose π is dihedral, of the form $I_K^{\mathbb{Q}}(\eta)$ for a (unitary) character η of C_K. We can express $Sym^2\pi$ as follows (see also [Kri12]). Let τ be the non-trivial automorphism of the degree 2 extension $K/\mathbb{Q}$. Note that

$$\eta\eta^{\tau} = \eta_0 \circ N_{K/\mathbb{Q}}, \tag{2.19}$$

where η_0 is the restriction of η to $C_{\mathbb{Q}}$. We have

$$I_K^{\mathbb{Q}}(\eta\eta^{\tau}) \cong \eta_0 \boxplus \eta_0\delta, \tag{2.20}$$

where δ is the quadratic character of $\mathbb{Q}$ associated to $K/\mathbb{Q}$.

If λ, μ are characters of C_K, then by applying Mackey:

$$I_K^{\mathbb{Q}}(\lambda) \boxtimes I_K^{\mathbb{Q}}(\mu) \cong I_K^{\mathbb{Q}}(\lambda\mu) \boxplus I_K^{\mathbb{Q}}(\lambda\mu^{\tau}). \tag{2.21}$$

Taking $\lambda = \mu = \eta$ in (2.21) and using (2.19) and (2.20),

$$\pi \boxtimes \pi \cong I_K^{\mathbb{Q}}(\eta^2) \boxplus \eta_0 \boxplus \eta_0\delta.$$

Since $\pi \boxtimes \pi = Sym^2(\pi) \boxplus \omega$ with $\omega = \eta_0\delta$,

$$Sym^2(\pi) \cong I_K^{\mathbb{Q}}(\eta^2) \boxplus \eta_0. \tag{2.22}$$

2.3 Determination of $\mathbf{GL}(3)$ cusp forms by p-power twists

2.3.1 A simple lemma involving Gauss sums

For an odd prime p, define the sets (following the notations in [LR97])

$$X_{(p)} = \{\chi \text{ a Dirichlet character of conductor } p^a \text{ for some } a\},$$

$$X^w_{(p)} = \{\chi \in X_{(p)} | \chi \text{ has } p\text{-power order}\}.$$

The characters of $X^w_{(p)}$ are called wild characters.

If $\chi \in X_{(p)}$, then $\chi : (\mathbb{Z}/p^a\mathbb{Z})^\times \to \mathbb{C}^\times$ for some a. Note that $(\mathbb{Z}/p^a\mathbb{Z})^\times \cong \mathbb{Z}/p^{a-1}\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z}$. A character in $X_{(p)}$ is an element in $X^w_{(p)}$ if and only if it is trivial on the elements of exponent $p-1$.

We denote the integers mod p^a of exponent $p-1$ by S_a and the sum over all primitive wild characters of conductor p^a by $\sum^*_{\chi \bmod p^a}$.

Consider the set

$$G(p^a) := \ker((\mathbb{Z}/p^a\mathbb{Z})^\times \to (\mathbb{Z}/p)^\times) \cong \mathbb{Z}/p^{a-1}\mathbb{Z}. \tag{2.23}$$

Using the orthogonality of characters we get that summing over the primitive wild characters of conductor p^a gives (see [LR97])

$$\sum\nolimits^*_{\chi \bmod p^a} \chi = |G(p^a)|\delta_{S_a} - |G(p^{a-1})|\delta_{S_{a-1}}, \tag{2.24}$$

with $|G(p^a)| = p^{a-1}$ from (2.23) and δ_{S_a} the characteristic function of S_a.

The following result for hyper-Kloosterman sums was proved in [Yan98]:

Lemma 2.1. *Let p be a prime number, $1 < n < p$ and $q = p^a$ with $a > 1$. Let x' denote the inverse of x mod q and let $e(x) := e^{2\pi i x}$. Then for any integer z coprime to p the hyper-Kloosterman sum*

$$\Big| \sum_{\substack{x_1,\cdots,x_n (mod\ q)\\ (x_i,p)=1}} e\left(\frac{x_1 + \cdots + x_n + z x_1' \cdots x_n'}{q}\right)\Big|$$

is bounded by

$$\begin{cases} \leq (n+1)q^{n/2} & \text{if } 1 < n < p-1, a > 1 \\ \leq p^{1/2} q^{n/2} & \text{if } n = p-1, a \geq 5 \\ \leq p q^{n/2} & \text{if } n = p-1, a = 4 \\ \leq p^{1/2} q^{n/2} & \text{if } n = p-1, a = 3 \\ \leq q^{n/2} & \text{if } n = p-1, a = 2. \end{cases} \tag{2.25}$$

As a consequence of Lemma 2.1 we prove the following result:

Lemma 2.2. *Let $\tau(\chi)$ denote the Gauss sum of the character χ. If $(r,p) = 1$, then the following bound holds:*

$$\Big| \sum\nolimits^*_{\chi\ mod\ p^a} \overline{\chi}(r) \tau^n(\chi) \Big| \ll p^{1/2 + a(n+1)/2} \tag{2.26}$$

for $2 < n \leq p$.

Proof. If χ is a primitive character of conductor p^a, then

$$\tau(\chi) = \sum_{m=0}^{p^a - 1} \chi(m) e^{2\pi i m/p^a}.$$

Let

$$A := \sum\nolimits^*_{\chi \bmod p^a} \overline{\chi}(r)\tau^n(\chi),$$

then

$$A = \sum\nolimits^*_{\chi \bmod p^a} \left[\overline{\chi}(r) \left(\sum_{m=0}^{p^a-1} \chi(m) e^{2\pi i m/p^a} \right)^n \right].$$

We rewrite the above sum as

$$A = \sum\nolimits^*_{\chi \bmod p^a} \left[\overline{\chi}(r) \left(\sum_{x_1=0}^{p^a-1} \chi(x_1) e^{2\pi i x_1/p^a} \right) \cdots \left(\sum_{x_n=0}^{p^a-1} \chi(x_n) e^{2\pi i x_n/p^a} \right) \right].$$

This in turn gives

$$A = \sum_{x_1=0}^{p^a-1} \cdots \sum_{x_n=0}^{p^a-1} \sum\nolimits^*_{\chi \bmod p^a} \chi(r')\chi(x_1)\cdots\chi(x_n) e\left(\frac{x_1 + \cdots + x_n}{p^a} \right).$$

Thus,

$$A = \sum_{x_1=0}^{p^a-1} \cdots \sum_{x_n=0}^{p^a-1} \left[\sum\nolimits^*_{\chi \bmod p^a} \chi(r' x_1 \cdots x_n) \right] e\left(\frac{x_1 + \cdots + x_n}{p^a} \right)$$

which by equation (2.24) gives

$$A = \sum_{x_1=0}^{p^a-1} \cdots \sum_{x_n=0}^{p^a-1} e\left(\frac{x_1 + \cdots + x_n}{p^a} \right) (p^{a-1}\delta_{S_a}(r' x_1 \cdots x_n) - p^{a-2}\delta_{S_{a-1}}(r' x_1 \cdots x_n)).$$

We get that

$$A = p^{a-1} \sum_{b \in S_a} T(br, p^a) - p^{a-2} \sum_{c \in S_{a-1}} \sum_{i=0}^{p-1} T(cr + ip^{a-1}, p^a), \tag{2.27}$$

where

$$T(u,p^a) = \sum_{\substack{x_1,\cdots,x_{n-1} (\text{mod } p^a)\\(x_i,p)=1}} e\left(\frac{x_1+\cdots+x_{n-1}+ux_1'\cdots x_{n-1}'}{p^a}\right).$$

From Lemma 2.1, for $(u,p)=1$ and a sufficiently large

$$|T(u,p^a)| \ll p^{1/2+a(n-1)/2}. \tag{2.28}$$

From (2.27) and (2.28) it follows that

$$|A| \ll p^{a-1}(p-1)p^{1/2+a(n-1)/2} + p^{a-2}(p-1)^2 p^{1/2+a(n-1)/2}.$$

Thus $|A| \ll p^a p^{1/2+a(n-1)/2}$. □

2.3.2 Non-vanishing of p-power twists on GL$(n,\mathbb{A}_\mathbb{Q})$

Here we will show the following result on isobaric sums of unitary cuspidal automorphic representations of GL$(n,\mathbb{A}_\mathbb{Q})$ for $n \geq 3$:

Theorem 2.1. *Let π be an isobaric sum of unitary cuspidal automorphic representations of GL$(n,\mathbb{A}_\mathbb{Q})$ with $n \geq 3$ and s,r be integers relatively prime to p. If $L(\pi\otimes\chi,s)$ and $L(\pi'\otimes\chi,s)$ are entire for all χ p-power order characters of conductor p^a for some a, then*

$$\lim_{a\to\infty} p^{-a} {\sum}^*_{\chi \bmod p^a} \overline{\chi}(s)\chi(r)L(\pi\otimes\chi,\beta) = \frac{1}{p}\left(1-\frac{1}{p}\right)\frac{a_\pi(s/r)}{(s/r)^\beta}, \tag{2.29}$$

where $\sum^*$ denotes the sum over primitive p-power order characters of conductor p^a and $1 \geq \beta > \frac{n-1}{n+1}$ if π is an isobaric sum of tempered unitary cuspidal automorphic representations and $1 \geq \beta > \frac{n-1}{n}$ in general. Here the elements $a_\pi(s/r)$ represent *the coefficients of the Dirichlet series that defines $L(\pi, s)$ in the right half-plane $Re(s) > 1$, with $a_\pi(1) = 1$ and $a_\pi(s/r) := 0$ if $r \not| s$.*

Let s, r be integers relatively prime to p. For π an isobaric sum of unitary cuspidal automorphic representations of $\mathrm{GL}(n, \mathbb{A}_\mathbb{Q})$ define

$$S_{s/r}(p^a, \pi, \beta) = p^{-a} \sum\nolimits^*_{\chi \bmod p^a} \overline{\chi}(s)\chi(r)L(\pi \otimes \chi, \beta), \tag{2.30}$$

where $\sum^*$ denotes the sum over primitive wild characters of conductor p^a.

Hence, we will show that:

$$\lim_{a\to\infty} S_{s/r}(p^a, \pi, \beta) = \frac{1}{p}\left(1 - \frac{1}{p}\right)\frac{a_\pi(s/r)}{(s/r)^\beta} \tag{2.31}$$

for $\beta > \frac{n-1}{n+1}$ if π is tempered, and for $\beta > \frac{n-1}{n}$ unconditionally.

Here by π tempered we will mean an isobaric sum of tempered (unitary) cuspidal automorphic representations. If $r \not| s$ above, then we define $a_\pi(s/r)$ to be zero.

Proof. The following approximate functional equation holds (see Section 2.2.2):

$$\begin{aligned} L(\pi \otimes \chi, \beta) &= \sum_{m=1}^{\infty} \frac{a_\pi(m)\chi(m)}{m^\beta} F_1\left(\frac{my}{f_\pi p^{an}}\right) \\ &+ \omega_\pi(p^a)\epsilon(0,\pi)\tau(\chi)^n (f_\pi p^{an})^{-\beta} \sum_{m=1}^{\infty} \frac{a_{\tilde\pi}(m)\overline{\chi}(mf'_\pi)}{m^{1-\beta}} F_2\left(\frac{m}{y}\right), \end{aligned}$$

where χ is a character of conductor p^a and f'_π is the multiplicative inverse of f_π modulo p^a.

Define x such that $xy = p^{an}$. Write

$$S_{s/r}(p^a, \beta) = S_{1,s/r}(p^a, \beta) + S_{2,s/r}(p^a, \beta), \tag{2.32}$$

where

$$S_{1,s/r}(p^a,\beta) = p^{-a}\sum\nolimits^*_{\chi \bmod p^a}\sum_{m=1}^{\infty}\frac{a_\pi(m)\chi(ms'r)}{m^\beta}F_1\left(\frac{m}{f_\pi x}\right) \tag{2.33}$$

and

$$\begin{aligned} S_{2,s/r}(p^a,\beta) &= p^{-a}\omega_\pi(p^a)\sum\nolimits^*_{\chi \bmod p^a}\epsilon(0,\pi)\tau(\chi)^n(f_\pi p^{an})^{-\beta}\times \\ &\times \sum_{m=1}^{\infty}\frac{a_{\tilde{\pi}}(m)\overline{\chi}(ms'rf'_\pi)}{m^{1-\beta}}F_2\left(\frac{m}{y}\right). \end{aligned} \tag{2.34}$$

Let

$$Z_{s/r}(p^a,\beta) = \sum_{b\in S_a}\sum_{\substack{rm\equiv bs(p^a)\\ m\geq 1}}\frac{a_\pi(m)}{m^\beta}F_1\left(\frac{m}{f_\pi x}\right). \tag{2.35}$$

Then applying equation (2.24) gives

$$S_{1,s/r}(p^a) = p^{-a}\sum_{m=1}^{\infty}\frac{a_\pi(m)}{m^\beta}F_1\left(\frac{m}{f_\pi x}\right)\left[p^{a-1}\delta_{S_a}(ms'r) - p^{a-2}\delta_{S_{a-1}}(ms'r)\right],$$

and hence

$$S_{1,s/r} = \frac{1}{p}\left[Z_{s/r}(p^a,\beta) - p^{-1}Z_{s/r}(p^{a-1},\beta)\right]. \tag{2.36}$$

If $r|s$, consider the term in (2.35) with $b=1$ and $m=s/r$. This is a solution to the equation $rm\equiv bs(\text{mod } p^a)$ for all a. We will want to set the necessary condition for this to be the only dominant contribution. If $r\not| s$ this term will not appear in the sum and the argument remains as below, requiring the condition that there is no dominant contribution and that the limit of $S_{s/r}(p^a,\pi,\beta)$ as $a\to\infty$ is zero.

Now if $m\neq s/r$, then $m = bs/r + kp^a$. If $k=0$ then $b\neq 1$ and since $b\in S_a$, it follows that $b \gg p^{a/(p-1)}$ which implies

$$m \gg p^{a/(p-1)}.$$

If $k \neq 0$, then $m \ll kp^a$.

Decompose

$$Z_{s/r}(p^a, \beta) = \Sigma_{1,a} + \Sigma_{2,a},$$

where

$$\Sigma_{1,a} = \frac{a_\pi(s/r)}{(s/r)^\beta} F_1\left(\frac{s}{rf_\pi x}\right) \tag{2.37}$$

and

$$\Sigma_{2,a} = \sum_{b \in S_a} \sum_{\substack{rm \equiv bs(p^a) \\ m \geq 1, m \neq s/r}} \frac{a_\pi(m)}{m^\beta} F_1\left(\frac{m}{f_\pi x}\right). \tag{2.38}$$

Since $F_1\left(\frac{m}{f_\pi x}\right) = 1 + O\left(\frac{m}{f_\pi x}\right)$,

$$\Sigma_{1,a} = \frac{a_\pi(s/r)}{(s/r)^\beta}\left(1 + O\left(\frac{1}{x}\right)\right). \tag{2.39}$$

Let

$$b_{m,a} := \begin{cases} 1 & \text{if } m = bs/r + kp^a \\ 0 & \text{otherwise.} \end{cases} \tag{2.40}$$

Then

$$\Sigma_{2,a} \ll \Big|\sum_{\substack{1 \leq m \ll x^{1+\epsilon} \\ m \neq s/r}} \frac{a_\pi(m)}{m^\beta} b_{m,a} F_1\left(\frac{m}{f_\pi x}\right)\Big| + \Big|\sum_{\substack{m \gg x^{1+\epsilon} \\ m \neq s/r}} \frac{a_\pi(m)}{m^\beta} b_{m,a} F_1\left(\frac{m}{f_\pi x}\right)\Big|. \tag{2.41}$$

Define

$$P_{2,a} = \Big| \sum_{\substack{1 \le m \ll x^{1+\epsilon} \\ m \ne s/r}} \frac{a_\pi(m)}{m^\beta} b_{m,a} F_1\left(\frac{m}{f_\pi x}\right)\Big| \quad \text{and} \quad Q_{2,a} = \Big| \sum_{\substack{m \gg x^{1+\epsilon} \\ m \ne s/r}} \frac{a_\pi(m)}{m^\beta} b_{m,a} F_1\left(\frac{m}{f_\pi x}\right)\Big|.$$

Since $F_1\left(\frac{m}{f_\pi x}\right) = 1 + O(x^\epsilon)$ for $m \ll x^{1+\epsilon}$ and $F_1\left(\frac{m}{f_\pi x}\right) \ll \frac{x^t}{m^t}$ for any integer t and $m \gg x^{1+\epsilon}$

$$P_{2,a} \ll x^\epsilon \Big| \sum_{\substack{1 \le m \ll x^{1+\epsilon} \\ m \ne s/r}} \frac{a_\pi(m)}{m^\beta} b_{m,a} \Big| \quad \text{and} \quad Q_{2,a} \ll x^t \Big| \sum_{\substack{m \gg x^{1+\epsilon} \\ m \ne s/r}} \frac{a_\pi(m)}{m^{\beta+t}} b_{m,a} \Big|. \tag{2.42}$$

If π is tempered then by (2.42)

$$P_{2,a} \ll x^\epsilon \sum_{\substack{1 \le m \ll x^{1+\epsilon} \\ m \ne s/r}} m^{\epsilon-\beta} b_{m,a} \ll p^{-a} x^{1-\beta+\epsilon} \quad \text{and} \quad Q_{2,a} \ll p^{-a} x^{1-\beta+\epsilon}, \tag{2.43}$$

and hence

$$\Sigma_{2,a} \ll p^{-a} x^{1-\beta+\epsilon}. \tag{2.44}$$

We want $\Sigma_{2,a} \to 0$ as $a \to \infty$. Substituting with $x = p^{an(1-\upsilon)}$ gives the condition

$$\upsilon > 1 - \frac{1}{n(1-\beta+\epsilon)}. \tag{2.45}$$

If π is not tempered, then applying the Cauchy-Schwarz inequality in (2.42) gives

$$P_{2,a} \ll \frac{x^{1/2+\epsilon}}{p^{a/2}} \left(\sum_{1 \le m \ll x^{1+\epsilon}} \frac{|a_\pi(m)|^2}{m^{2\beta}} \right)^{1/2}.$$

By inequality (2.6) and summation by parts we get

$$P_{2,a} \ll p^{-a/2} x^{1-\beta+\epsilon}. \tag{2.46}$$

Write $t = t_1 + t_2$ in (2.42), with t_1, t_2 large integers, and apply the Cauchy-Schwarz inequality:

$$\begin{aligned} Q_{2,a} &\ll x^{t_1+t_2} \left(\sum_{m \gg x^{1+\epsilon}} \frac{|a_\pi(m)|^2}{m^{2\beta+2t_1}} \right)^{1/2} \left(\sum_{m \gg x^{1+\epsilon}} \frac{b_{m,a}^2}{m^{2t_2}} \right)^{1/2} \\ &\ll x^{t_1+t_2} \left(\sum_{i \gg (1+\epsilon)\log(x)} \sum_{2^{i-1} < m \leq 2^i} \frac{|a_\pi(m)|^2}{m^{2\beta+2t_1}} \right)^{1/2} \left(\sum_{k \gg \frac{x^{1+\epsilon}}{p^a}} \frac{1}{(kp^a)^{2t_2}} \right)^{1/2} \end{aligned} \tag{2.47}$$

Using (2.6) gives

$$Q_{2,a} \ll p^{-at_2} x^{1-\beta+\epsilon} \tag{2.48}$$

and hence

$$\Sigma_{2,a} \ll p^{-a/2} x^{1-\beta+\epsilon}. \tag{2.49}$$

Since we want $\Sigma_{2,a} \to 0$, we get the condition

$$\upsilon > 1 - \frac{1}{2n(1-\beta+\epsilon)}. \tag{2.50}$$

For υ as above,

$$\lim_{a\to\infty} S_{1,s/r}(p^a, \beta) = \frac{p-1}{p^2} \cdot \frac{a_\pi(s/r)}{(s/r)^\beta}. \tag{2.51}$$

In (2.34) write

$$|S_{2,s/r}| \ll A_{2,s/r} + B_{2,s/r}, \tag{2.52}$$

where

$$A_{2,s/r} = p^{-a}p^{-an\beta} \sum_{m \ll y^{1+\epsilon}} \left[\frac{|a_{\tilde{\pi}}(m)|}{m^{1-\beta}} F_2\left(\frac{m}{y}\right) \left| \sum\nolimits^*_{\chi (\text{mod } p^a)} \overline{\chi}(ms'rf'_\pi)\tau^n(\chi) \right| \right] \quad (2.53)$$

and

$$B_{2,s/r} = p^{-a}p^{-an\beta} \sum_{m \gg y^{1+\epsilon}} \left[\frac{|a_{\tilde{\pi}}(m)|}{m^{1-\beta}} F_2\left(\frac{m}{y}\right) \left| \sum\nolimits^*_{\chi (\text{mod } p^a)} \overline{\chi}(ms'rf'_\pi)\tau^n(\chi) \right| \right]. \quad (2.54)$$

If π is tempered then $|a_{\tilde{\pi}}(m)| \ll m^\epsilon$. Also, $F_2\left(\frac{m}{y}\right) \ll 1+\left(\frac{m}{y}\right)^{1-\beta-\epsilon}$ for $m \ll y^{1+\epsilon}$, which gives $F_2\left(\frac{m}{y}\right) \ll y^{\epsilon(1-\beta)}$. Applying Lemma 2.2,

$$|A_{2,s/r}| \ll p^{-a}p^{-an\beta}p^{1/2+a(n+1)/2}y^{\epsilon(1-\beta)} \sum_{m=1}^{y^{1+\epsilon}} m^{\epsilon+\beta-1}$$

and hence for any $\epsilon > 0$

$$|A_{2,s/r}| \ll p^{-an\beta+a(n-1)/2}y^{\epsilon+\beta}. \quad (2.55)$$

Assume now that π is not tempered. By the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} |A_{2,s/r}| \ &\ll \ p^{-a}p^{-an\beta}y^\epsilon \left(\sum_{m \ll y^{1+\epsilon}} \frac{|a_{\tilde{\pi}}(m)|^2}{m^{2-2\beta}} \right)^{1/2} \times \\ &\times \ \left(\sum_{m=-\infty}^{\infty} H\left(\frac{m}{y}\right) \left| \sum\nolimits^*_{\chi \text{ mod } p^a} \overline{\chi}(ms'rf'_\pi)\tau^n(\chi) \right|^2 \right)^{1/2}, \end{aligned}$$

where

$$H(u) := \frac{1}{\pi(1+u^2)}.$$

A simple computation shows that

$$\sum_{m \ll y^{1+\epsilon}} \frac{|a_{\tilde{\pi}}(m)|^2}{m^{2-2\beta}} \ll y^{2\beta-1+\epsilon}. \tag{2.56}$$

Thus,

$$|A_{2,s/r}| \ll y^{\beta-1/2+\epsilon} p^{-a-an\beta} \left(\sum_{m=-\infty}^{\infty} H\left(\frac{m}{y}\right) \left| \sum\nolimits^*_{\chi \bmod p^a} \overline{\chi}(ms'rf'_\pi)\tau^n(\chi) \right|^2 \right)^{1/2}. \tag{2.57}$$

Define

$$D := \sum_{m=-\infty}^{\infty} H\left(\frac{m}{y}\right) \left| \sum\nolimits^*_{\chi \bmod p^a} \overline{\chi}(ms'rf'_\pi)\tau^n(\chi) \right|^2. \tag{2.58}$$

We have

$$D \ll \sum\nolimits^*_{\chi \bmod p^a} \sum\nolimits^*_{\psi \bmod p^a} \left| \tau^n(\chi)\tau^n(\overline{\psi}) \sum_{m=-\infty}^{\infty} \overline{\chi}\psi(ms'rf'_\pi) H\left(\frac{m}{y}\right) \right|.$$

Following the general approach of [Luo05, War], we consider the diagonal and off-diagonal contributions separately. The terms corresponding to $\chi = \psi$ give:

$$\sum\nolimits^*_{\chi \bmod p^a} \left| \tau^n(\chi)\tau^n(\overline{\chi}) \sum_{m=-\infty}^{\infty} H\left(\frac{m}{y}\right) \right| \ll p^{a+na} \sum_{m=-\infty}^{\infty} H\left(\frac{m}{y}\right)$$

since there are $\ll p^a$ primitive p-power characters and since $|\tau^n(\chi)| = p^{an/2}$ from the properties of the Gauss sum of a primitive character. Using the Fourier transform property $\mathcal{F}\{g(xA)\} = \frac{1}{A}\hat{g}\left(\frac{\nu}{A}\right)$ for $A > 0$ (see also [Luo05, War]) we get that

$$\sum_{m=-\infty}^{\infty} H\left(\frac{m}{y}\right) = y \sum_{\nu=-\infty}^{\infty} T(y\nu).$$

Function $T(\nu)$ is the Fourier transform of $H(m)$ and is given by $T(\nu) = e^{-2\pi|\nu|}$, and hence $\sum_{m\in\mathbb{Z}} H\left(\frac{m}{y}\right) \ll y$. Note that we have used the Poisson summation formula. Thus the contribution to D is $\ll p^{a+na}y$.

For the terms in D that have $\chi \neq \psi$, even if χ and ψ are primitive the product $\overline{\chi}\psi$ may be non-primitive because the conductors are not relatively prime. We have that for $g : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$:

$$\sum_{m=-\infty}^{\infty} g(m) f\left(\frac{m}{q}\right) = \sum_{b \bmod q} g(b) F\left(\frac{b}{q}\right) = \sum_{\nu=-\infty}^{\infty} \hat{g}(-\nu)\hat{f}(\nu),$$

where $F(x) = \sum_{\nu=-\infty}^{\infty} \hat{f}(\nu) e^{-2\pi i \nu x}$. Applying this in our case,

$$\sum_{m=-\infty}^{\infty} \overline{\chi}\psi(m) H\left(\frac{m}{y}\right) = \frac{y}{p^a} \sum_{\nu=-\infty}^{\infty} \left(\sum_{b \bmod p^a} \overline{\chi}\psi(b) e^{-2\pi i \nu b/p^a} \right) T\left(\frac{y\nu}{p^a}\right).$$

The interior sum is $\ll p^a$ since the number of characters is $\ll p^a$, and for $\nu = 0$ it is zero since $\overline{\chi}\psi$ is non-trivial. Thus,

$$\Big| \sum_{m=-\infty}^{\infty} \overline{\chi}\psi(m) H\left(\frac{m}{y}\right) \Big| \ll y \sum_{\nu\in\mathbb{Z},\nu\neq 0} T\left(\frac{y\nu}{p^a}\right).$$

Assuming $\upsilon > \frac{1}{n}$ (which will be part of our constraint) gives that $y/p^a \to \infty$. We have

$$\sum_{\nu\in\mathbb{Z},\nu\neq 0} T\left(\frac{y\nu}{p^a}\right) \ll \frac{2}{e^{2\pi y p^{-a}} - 1} \ll \frac{1}{y}.$$

Putting everything together, these terms of D contribute $\ll p^{2a+na}$. Thus, we conclude that the two contributions for $\chi = \psi$ and $\chi \neq \psi$ combined give

$$D \ll p^{a+na} y. \tag{2.59}$$

From (2.57) and (2.59), even if π is not tempered,

$$|A_{2,s/r}| \ll y^{\beta+\epsilon} p^{-an\beta+a(n-1)/2}. \tag{2.60}$$

For $m \gg y^{1+\epsilon}$, $F_2\left(\frac{m}{y}\right) \ll \frac{y^t}{m^t}$ for any integer $t \geq 1$, and applying Cauchy-Schwarz's inequality in (2.54) gives

$$|B_{2,s/r}| \ll p^{-a} p^{-an\beta} y^t \left(\sum_{m \gg y^{1+\epsilon}} \frac{|a_{\tilde{\pi}}|^2}{m^{2-2\beta+2t}} \right)^{1/2} D^{1/2}.$$

Using summation by parts and (2.6), as well as the bound in (2.59) gives

$$|B_{2,s/r}| \ll y^{\beta+\epsilon} p^{-an\beta+a(n-1)/2}. \tag{2.61}$$

From (2.52), (2.60), and (2.61) we conclude that

$$|S_{2,s/r}| \ll y^{\beta+\epsilon} p^{-an\beta+a(n-1)/2}. \tag{2.62}$$

We want $S_{2,s/r} \to 0$ as $a \to \infty$. Taking $y = p^{anv}$ in (2.62) gives the condition

$$v < \frac{1-n+2n\beta}{2n(\beta+\epsilon)}. \tag{2.63}$$

If π is tempered then we need to check that v satisfies conditions (2.45) and (2.63). Thus, for a general n, the desired condition is

$$\beta > \frac{n-1}{n+1}. \tag{2.64}$$

If π is not tempered, then conditions (2.50) and (2.63) need to be satisfied. This gives the condition

$$\beta > \frac{n-1}{n}.$$

□

As a consequence of Theorem 2.1, the following non-vanishing result holds:

Corollary 2.1. *Let π be an isobaric sum of unitary cuspidal automorphic representations of $GL(n, \mathbb{A}_{\mathbb{Q}})$ with $n \geq 3$. There are infinitely many primitive p-power order characters χ of conductor p^a for some a, such that $L(\pi \otimes \chi, \beta) \neq 0$ for any fixed $\beta \notin \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$. If π is an isobaric sum of tempered unitary cuspidal automorphic representations then the same holds for any fixed $\beta \notin \left[\frac{2}{n+1}, 1 - \frac{2}{n+1}\right]$.*

Proof. Take $s = r = 1$ in Theorem 2.1 and use the functional equation. Note that if $\beta > 1$, $L(\pi \otimes \chi, \beta)$ has an Euler product expansion and hence is nonvanishing. □

A similar nonvanishing result involving p-power twists of cuspidal automorphic representations of $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$ was proved in [War] for $\beta \notin \left[\frac{2}{n+1}, 1 - \frac{2}{2n+1}\right]$. In [BR94] a nonvanishing result for β in the same intervals as in Corollary 2.1 was proved for all twists of L-functions of $\mathrm{GL}(n)$, instead of just for p-power twists. In [Luo05], the result in [BR94] was further improved to the interval $\beta \notin \left[\frac{2}{n}, 1 - \frac{2}{n}\right]$. Note that the set of primitive characters of p-power order and conductor p^a for some a is more sparse than the set of characters considered in [BR94] and [Luo05].

It should be noted that for $n = 2$ Rohrlich [Roh84] proves that if f is a newform of weight 2, then for all but finitely many twists by Dirichlet characters the L-function is nonvanishing at the center.

2.3.3 Determination on GL(3) cusp forms

We show the following result on the determination of isobaric automorphic representations of GL(3) over $\mathbb{Q}$:

Theorem 2.2. *Suppose π and π' are two isobaric sums of unitary cuspidal automorphic representations of $GL(3, \mathbb{A}_{\mathbb{Q}})$ with the same central character ω. Let $X^w_{(p)}$ be the set of p-power order characters of conductor p^a for some a. Suppose $L(\pi \otimes \chi, s)$ and $L(\pi' \otimes \chi, s)$ are entire for all $\chi \in X^w_{(p)}$, and that there exist constants $B, C \in \mathbb{C}$ such that*

$$L(\pi \otimes \chi, \beta) = B^a C L(\pi' \otimes \chi, \beta) \tag{2.65}$$

for some fixed $1 \geq \beta > \frac{2}{3}$ and for all $\chi \in X^w_{(p),a}$ primitive p-power order characters of conductor p^a for all but a finite number of a. Then $\pi \cong \pi'$. Moreover, if π and π' are isobaric sums of tempered unitary cuspidal automorphic representations then the same result holds if (2.65) is satisfied for some fixed $1 \geq \beta > \frac{1}{2}$ (if the generalized Ramanujan conjecture is true this condition is automatically satisfied).

Note that in [MS] a result was proved concerning the determination of GL(3) forms by twists of characters of almost prime modulus of the central L-values. In our case, we twist over a more sparse set of characters.

Let $\pi \in \mathcal{A}_u(3)$ be an isobaric sum of unitary cuspidal automorphic representations of $\mathrm{GL}(3, \mathbb{A}_{\mathbb{Q}})$. The local components π_ℓ are determined by the set of nonzero complex numbers $\{\alpha_\ell, \beta_\ell, \gamma_\ell\}$, which we represent by the diagonal matrix $A_\ell(\pi)$.

The L-factor of π at a prime ℓ is given by

$$L(\pi_\ell, s) = \det(I - A_l(\pi)\ell^{-s})^{-1} = \prod_{j=1}^{n} (1 - \alpha_\ell \ell^{-s})^{-1}(1 - \beta_\ell \ell^{-s})^{-1}(1 - \gamma_\ell \ell^{-s})^{-1}. \tag{2.66}$$

Let $S_0 = \{\ell : \pi_\ell \text{ unramified and tempered}\}$, and let $S_1 = \{\ell : \pi_\ell \text{ is ramified}\}$. Note that S_1 is finite. Take the union

$$S = S_0 \cup S_1 \cup \{\infty\}.$$

Since π is unitary, π_ℓ is tempered iff $|\alpha_\ell| = |\beta_\ell| = |\gamma_\ell| = 1$.

Lemma 2.3. *If $\ell \notin S$ then*

$$A_\ell(\pi) = \{u\ell^t, u\ell^{-t}, w\}, \tag{2.67}$$

with $|u| = |w| = 1$ and $t \neq 0$ a real number. If $\ell \in S_0$ then

$$A_\ell(\pi) = \{\alpha, \beta, \gamma\}$$

with $|\alpha| = |\beta| = |\gamma| = 1$.

Proof. Suppose first that $\ell \notin S$. We may assume that $|\alpha_\ell| \neq 1$. Then $\alpha_\ell = u\ell^t$, for some $|u| = 1$ complex and $t \neq 0$ real. By unitarity,

$$\{\overline{\alpha}_\ell, \overline{\beta}_\ell, \overline{\gamma}_\ell\} = \{\alpha_\ell^{-1}, \beta_\ell^{-1}, \gamma_\ell^{-1}\}.$$

Clearly $\overline{\alpha}_\ell \neq \alpha_\ell^{-1}$. Without loss of generality, take $\beta_\ell^{-1} = \overline{\alpha}_\ell$. This gives $\beta_\ell = u \cdot \ell^{-t}$. So, we must have $\overline{\gamma}_\ell = \gamma_\ell^{-1}$, hence $\gamma_\ell = w$ with $|w| = 1$. Thus

$$A_\ell(\pi) = \{u\ell^t, u\ell^{-t}, w\}$$

with $|u| = |w| = 1$.

Now suppose that $\ell \in S_0$. Then $|\alpha_\ell| = |\beta_\ell| = |\gamma_\ell| = 1$. □

Proof of Theorem 2.2. Let $T = \{\ell | \pi_\ell \text{ or } \pi'_\ell \text{ is ramified}\}$. This is a finite set.

Consider $\ell \notin T$ an arbitrary finite place with $\ell \neq p$. Let $A_\ell(\pi) = \{\alpha_\ell, \beta_\ell, \gamma_\ell\}$ and $A_\ell(\pi') = \{\alpha'_\ell, \beta'_\ell, \gamma'_\ell\}$. Applying Theorem 2.1, $a_\pi(n) = B^a C a_{\pi'}(n)$ for all $(n, p) = 1$ and all but finitely many a. Since $a_\pi(1) = a_{\pi'}(1)$, we conclude that $B = C = 1$. Thus, $a_\pi(\ell) = a_{\pi'}(\ell)$.

We want to show that $A_l(\pi) = A_l(\pi')$. Indeed,

$$\alpha_\ell + \beta_\ell + \gamma_\ell = \alpha'_\ell + \beta'_\ell + \gamma'_\ell \tag{2.68}$$

and since π and π' have the same central character

$$\alpha_\ell \beta_\ell \gamma_\ell = \alpha'_\ell \beta'_\ell \gamma'_\ell. \tag{2.69}$$

To show that $\{\alpha_\ell\, \beta_\ell, \gamma_\ell\} = \{\alpha'_\ell, \beta'_\ell, \gamma'_\ell\}$, by Vieta's formulas (cf. [Vie]) and the above two relations, it is enough to check that

$$\alpha_\ell\beta_\ell + \alpha_\ell\gamma_\ell + \beta_\ell\gamma_\ell = \alpha'_\ell\beta'_\ell + \alpha'_\ell\gamma'_\ell + \beta'_\ell\gamma'_\ell.$$

Suppose $A_\ell(\pi) = \{u\ell^t, u\ell^{-t}, w\}$ with $|u| = |w| = 1$. Then

$$\alpha_\ell\beta_\ell + \alpha_\ell\gamma_\ell + \beta_\ell\gamma_\ell = u^2 + uw(\ell^t + \ell^{-t}) = \frac{1}{u^2} + \frac{1}{uw}(\ell^t + \ell^{-t}) = \frac{w + u(\ell^t + \ell^{-t})}{u^2 w},$$

and hence $\alpha_\ell\beta_\ell + \alpha_\ell\gamma_\ell + \beta_\ell\gamma_\ell = \frac{\alpha_\ell+\beta_\ell+\gamma_\ell}{\alpha_\ell\beta_\ell\gamma_\ell}$.

Now suppose that $A_\ell(\pi) = \{\alpha_\ell, \beta_\ell, \gamma_\ell\}$ with $|\alpha_\ell| = |\beta_\ell| = |\gamma_\ell| = 1$. Then

$$\alpha_\ell\beta_\ell + \alpha_\ell\gamma_\ell + \beta_\ell\gamma_\ell = \frac{1}{\alpha_\ell\beta_\ell} + \frac{1}{\alpha_\ell\gamma_\ell} + \frac{1}{\beta_\ell\gamma_\ell} = \frac{\alpha_\ell + \beta_\ell + \gamma_\ell}{\alpha_\ell\beta_\ell\gamma_\ell}.$$

Thus, whenever $\alpha_\ell + \beta_\ell + \gamma_\ell = \alpha'_\ell + \beta'_\ell + \gamma'_\ell$ and $\alpha_\ell\beta_\ell\gamma_\ell = \alpha'_\ell\beta'_\ell\gamma'_\ell$, we obtain that $\alpha_\ell\beta_\ell + \alpha_\ell\gamma_\ell + \beta_\ell\gamma_\ell = \alpha'_\ell\beta'_\ell + \alpha'_\ell\gamma'_\ell + \beta'_\ell\gamma'_\ell$.

We have thus shown that for $\ell \notin T \cup \{p\} \cup \{\infty\}$, $A_l(\pi) = A_\ell(\pi')$, and hence $\pi_\ell \cong \pi'_\ell$. Since $T \cup \{p\} \cup \{\infty\}$ is a finite set, this implies that $\pi \cong \pi'$ by the Generalized Strong Multiplicity One Theorem. □

Let π be a unitary cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_\mathbb{Q})$ with $A_\ell(\pi) = \{\alpha_\ell, \beta_\ell\}$. At an unramified place ℓ, it has $a_\ell = \alpha_\ell + \beta_\ell$ and central character $\omega(\varpi_\ell) = \alpha_\ell\beta_\ell$, with ϖ_ℓ the uniformizer at ℓ. There exists an isobaric automorphic representation $Ad(\pi)$ of $\mathrm{GL}(3, \mathbb{A}_\mathbb{Q})$ (cf. [GJ78]) such that at an unramified place ℓ,

$$a_\ell(Ad(\pi)) = \alpha_\ell/\beta_\ell + \beta_\ell/\alpha_\ell + 1.$$

Using Theorem 4.1.2 in [Ram00], the following is a consequence of Theorem 2.2:

Theorem 2.3. *Suppose π and π' are two unitary cuspidal automorphic representations of $GL(2, \mathbb{A}_\mathbb{Q})$ with the same central character ω. Suppose there exist constants $B, C \in \mathbb{C}$ such that*

$$L(Ad(\pi) \otimes \chi, \beta) = B^a C L(Ad(\pi') \otimes \chi, \beta) \tag{2.70}$$

for some $1 \geq \beta > \frac{2}{3}$ and for all $\chi \in X^w_{(p),a}$ primitive p-power order characters of conductor p^a for all but a finite number of a. Then there exists a quadratic character ν such that $\pi \cong \pi' \otimes \nu$. If π and π' are tempered then the same result holds if (2.70) *is true for some $1 \geq \beta > \frac{1}{2}$.*

Proof of Theorem 2.3. Theorem 2.2 implies that $Ad(\pi) \cong Ad(\pi')$. Then, by Theorem 4.1.2 in [Ram00], we deduce that since π and π' have the same central character, there exists a quadratic character ν such that $\pi \cong \pi' \otimes \nu$. □

2.4 Application: Adjoint p-adic L-function of an elliptic curve

2.4.1 Complex adjoint L-functions of an elliptic curve

Let $E/\mathbb{Q}$ be an elliptic curve with conductor N given by a global minimal Weierstrass equation over $\mathbb{Z}$:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6. \tag{2.71}$$

Define the complex L-function of E by the Euler product for $\mathrm{Re}(s) > \frac{3}{2}$:

$$L(E,s) = \prod_{r|N} \frac{1}{1 - a_r r^{-s}} \prod_{r \nmid N} \frac{1}{1 - a_r r^{-s} + r^{1-2s}},$$

where $a_r = r + 1 - \#E(\mathbb{F}_r)$ if $r \nmid N$. If $r|N$ then a_r depends on the reduction of E at r in the following way: $a_r = 1$ if E has split multiplicative reduction at r, $a_r = -1$ if E has non-split multiplicative reduction at r and $a_r = 0$ if E has additive reduction at r.

Let f be the holomorphic newform of weight 2 and level N associated to E. The Fourier coefficients c_r of f at $r \nmid N$ prime coincide with the coefficients a_r in the Euler product of E and the L-function of E is given by

$$L(E,s) = \sum_{n=1}^{\infty} c_n n^{-s}.$$

If $\Lambda(E,s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(E,s)$, then the following functional equation holds:

$$\Lambda(E,s) = \pm\Lambda(E, 2-s),$$

where the sign varies, depending on E. If we associate to f a unitary cuspidal automorphic form π of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$ with trivial central character and conductor N then we want to have $L(\pi, s)$ unitarily normalized by setting

$$L_u(\pi, s) = (2\pi)^{-s-1/2} L\left(E, s + \frac{1}{2}\right).$$

We define the complex L-function associated to the symmetric square of an elliptic curve in the following way (cf. [CS12]). Let l be an odd prime number. Take $E[l^n]$ to be the l^n-torsion and

$$T_l(E) = \varprojlim E[l^n]$$

to be the l-adic Tate module of E. Consider the $V_l(E) = T_l(E) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$, which is 2-dimensional over $\mathbb{Q}_l$. There is a continuous natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on V_l. Let $\Sigma_l(E) = Sym^2 H_l^1(E)$, where $H_l^1(E) = \mathrm{Hom}(V_l(E), \mathbb{Q}_l)$. Consider the representation

$$\rho_l : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(\Sigma_l(E)). \tag{2.72}$$

The L-function of $Sym^2(E)$ is given by the Euler product

$$L(Sym^2 E, s) = \prod_{r \text{ prime}} P_r(r^{-s})^{-1} \tag{2.73}$$

in the half-plane $\mathrm{Re}(s) > 2$. The polynomial $P_r(X)$ is

$$P_r(X) := \det(1 - \rho_l(\mathrm{Frob}_r^{-1})X | \Sigma_l(E)^{I_r}),\ l \neq r, \tag{2.74}$$

with I_r the inertia subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}_r}/\mathbb{Q}_r)$ and Frob_r an arithmetic Frobenius element at r. By the Néron-Ogg-Shafarevich criterion we have that

$$P_r(X) = (1 - \alpha_r^2 X)(1 - \beta_r^2 X)(1 - rX)$$

when E has good reduction at r (see [CS12]). The elements α_r and β_r are the roots of the polynomial

$$X^2 - a_r X + r$$

with a_r the trace of Frobenius at r.

Let $L(Sym^2 E, \chi, s)$ denote the L-function associated to the twist of the l-adic representations by a Dirichlet character χ. Note that $L(Sym^2 E, \chi, 1) = 0$ for χ odd (cf. [DD97]). The critical points for $Sym^2 E$ are $s = 1$ and $s = 2$.

Let χ be a primitive even Dirichlet character with conductor c_χ. Let C denote the conductor of the l-adic representation (2.72). If $\tau(\chi)$ is the Gauss sum of character χ, define

$$W(\chi) = \chi(C) c_\chi^{1/2} \frac{\tau(\chi)}{\tau(\overline{\chi})^2}.$$

Then, by Theorem 2.2 in [CS12], which is based on results in [GJ78], if the conductor N of E satisfies $(c_\chi, N) = 1$, then

$$\Lambda(Sym^2 E, \chi, s) = (C \cdot c_\chi^3)^{s/2} (2\pi)^{-s} \pi^{-\frac{s}{2}} \Gamma(s) \Gamma\left(\frac{s}{2}\right) L(Sym^2 E, \chi, s)$$

has a holomorphic continuation over $\mathbb{C}$ and satisfies the functional equation

$$\Lambda(Sym^2 E, \chi, s) = W(\chi) \Lambda(Sym^2 E, \overline{\chi}, 3 - s). \tag{2.75}$$

2.4.2 Adjoint p-adic L-functions of an elliptic curve

Fix p an odd prime and let E be an elliptic curve over $\mathbb{Q}$ with semistable reduction at p. We now describe a construction of a p-adic analogue to $L(Sym^2E, s)$ by the Mazur-Mellin transform of a p-adic measure μ_p on $\mathbb{Z}_p^\times$ as introduced in [DD97].

Consider the real and imaginary periods of a Néron differential of a minimal Weierstrass equation for E over $\mathbb{Z}$ which we denote by $\Omega^\pm(E)$. Let

$$\Omega^+(Sym^2E(1)) := (2\pi i)^{-1}\Omega^+(E)\Omega^-(E) \text{ and } \Omega^+(Sym^2E(2)) := 2\pi i\Omega^+(E)\Omega^-(E)$$

be the periods for Sym^2E at the critical twists. In [DD97] two p-adic distributions $\mu_p(\Omega^+(Sym^2E(1)))$ and $\mu_p(\Omega^+(Sym^2E(2)))$ are defined. In this thesis we will use the latter distribution.

Let X_p be the set of continuous characters of $\mathbb{Z}_p^\times$ into $\mathbb{C}_p^\times$. For $\chi \in X_p$, let p^{m_χ} be the conductor of χ. Since $\mathbb{Z}_p^\times \cong (1+p\mathbb{Z}_p) \times (\mathbb{Z}/p)^\times$, we can write $X := X_p$ as the product of $X((\mathbb{Z}/p)^\times)$ with $X_0 = X(1+p\mathbb{Z}_p)$. The elements of X_0 are called wild p-adic characters. By Section 2.1 in [Vis76] we can give X_0 a $\mathbb{C}_p$-structure through the isomorphism of X_0 to the disk

$$U := \{u \in \mathbb{C}_p^\times || u-1| < 1\} \tag{2.76}$$

constructed by mapping $\nu \in X_0$ to $\nu(1+p)$, with $1+p$ a topological generator of $1+p\mathbb{Z}_p$.

We follow the definition of the p-adic distribution $\mu_p(\Omega^+(Sym^2E(2)))$ on $\mathbb{Z}_p^\times$ in [DD97]. Suppose E has good reduction at p. Let $\chi \in X_0$ be a non-trivial wild p-adic character, with conductor p^{m_χ} which can be identified with a primitive Dirichlet character. Then given $\alpha_p(E)$ the root of $X^2 - a_pX + p$ with a_p the trace of the

Frobenius at p, we define

$$\int_{\mathbb{Z}_p^\times} \chi d\mu_p(\Omega^+(Sym^2E(2))) := \alpha_p(E)^{-2m_\chi} \cdot \tau(\overline{\chi})^2 p^{m_\chi} \cdot \frac{L(Sym^2E, \chi, 2)}{\Omega^+(Sym^2E(2))}. \tag{2.77}$$

If E has good ordinary reduction at p then the distributions $\mu_p(\Omega^+(Sym^2E(2)))$ are bounded measures on $\mathbb{Z}_p^\times$. If E has supersingular reduction at p then the distributions $\mu_p(\Omega^+(Sym^2E(2)))$ give h-admissible measures on $\mathbb{Z}_p^\times$, with $h = 2$. Note that the set of h-admissible measures with $h = 1$ is larger, but contains the bounded measures.

Now suppose that E has bad multiplicative reduction at p (either split or non-split). Let $\chi \in X_0$ denote a Dirichlet character of conductor p^{m_χ} as above. Then

$$\int_{\mathbb{Z}_p^\times} \chi d\mu_p(\Omega^+(Sym^2E(2))) := \tau(\overline{\chi})^2 p^{m_\chi} \cdot \frac{L(Sym^2E, \chi, 2)}{\Omega^+(Sym^2E(2))} \tag{2.78}$$

and the distributions $\mu_p(\Omega^+(Sym^2E(2)))$ are bounded measures on $\mathbb{Z}_p^\times$.

Consider μ an h-admissible measure as above. Then

$$\chi \to L_\mu(\chi) := \int_{\mathbb{Z}_p^\times} \chi d\mu \tag{2.79}$$

is an analytic function of type $o(\log^h)$ (cf. [Vis76]). Note that for an analytic function F to be of type $o(\log^h)$ it must satisfy

$$\sup_{|u-1|_p<r} \|F(u)\| = o\left(\sup_{|u-1|_p<r} |\log_p^h(u)|\right) \text{ for } r \to 1_-.$$

Consider the p-adic distribution $\mu = \mu_p(\Omega^+(Sym^2E(2)))$ as defined above. Denote by L_p the corresponding p-adic L-function. We have

$$L_p(Sym^2 E, \chi, s) := \int_{\mathbb{Z}_p^\times} \chi(x)\langle x\rangle^s d\mu,$$

where $\langle\cdot\rangle : \mathbb{Z}_p^\times \to 1 + p\mathbb{Z}_p$, $\langle x\rangle = \frac{x}{\omega(x)}$, with $\omega : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ the Teichmüller character.

2.4.3 Main result on the determination of elliptic curves

We now prove the main result of this chapter concerning the p-adic L-function of the symmetric square of an elliptic curve over $\mathbb{Q}$. Theorem 2.4 below gives a generalization of the result obtained in [LR97] concerning p-adic L-functions of elliptic curves over $\mathbb{Q}$:

Theorem 2.4. *Let p be an odd prime and E, E' be elliptic curves over $\mathbb{Q}$ with semistable reduction at p. Suppose*

$$L_p(Sym^2 E, n) = C L_p(Sym^2 E', n) \tag{2.80}$$

for all $(n, p) = 1$ elements of an arbitrary infinite set Y and some constant $C \in \overline{\mathbb{Q}}$. Then E' is isogenous to a quadratic twist E_D of E. If E and E' have square free conductors, then E and E' are isogenous over $\mathbb{Q}$.

We first prove the following lemma:

Lemma 2.4. *Let p be an odd prime. Let E, E' be elliptic curves over $\mathbb{Q}$ with semistable reduction at p such that $L_p(Sym^2 E, n) = C L_p(Sym^2 E', n)$, for an infinite number of integers n prime to p in some arbitrary set Y, and some constant*

$C \in \overline{\mathbb{Q}}$. Then for every finite order wild p-adic character χ,

$$L_p(Sym^2 E, \chi, s) = C L_p(Sym^2 E', \chi, s)$$

holds for all $s \in \mathbb{Z}_p$.

Proof. We follow the approach in [LR97]. Let

$$G(\nu) = L_p(Sym^2 E, \nu) - C L_p(Sym^2 E', \nu)$$

for every $\nu \in X_0$. G vanishes on $X_1 = \{\alpha_n = \langle x \rangle^n | n \in Y\}$ by hypothesis; we want to show that G vanishes on X_0. We use the fact that G is an analytic function on X_0 of type $o(\log^h)$ (as in (2.79)). G considered as an analytic function on U (see (2.76)) vanishes on the subset

$$U_1 = \{(1+p)^n | n \in Y\}.$$

There exists $r = 1/p$ such that the number of zeros z of G such that $|z - 1| = r$ is infinite. Indeed, for all $n \in Y$ elements in an infinite set with n relatively prime to p as above, $z_n := (1+p)^n \in U_1$ is a zero of G and

$$|z_n - 1| = |(1+p)^n - 1|_p = \Big|\sum_{j=1}^{n} \binom{n}{j} p^j\Big|_p = \frac{1}{p}.$$

By Section 2.5 in [Vis76], G is identically zero on U. □

Proof of Theorem 2.4 in the non-CM case. By Lemma 2.4, for every finite order wild p-power character χ, the identity

$$L_p(Sym^2 E, \chi, s) = C L_p(Sym^2 E', \chi, s) \tag{2.81}$$

holds for all $s \in \mathbb{Z}_p$. By equation (2.77), if E has good reduction at p then

$$\alpha_p(E)^{-2m_\chi} L(Sym^2 E, \chi, 2) = C' \alpha_p(E')^{-2m_\chi} L(Sym^2 E', \chi, 2) \tag{2.82}$$

for some $C' \in \overline{\mathbb{Q}}$. If E has bad multiplicative reduction at p, then by (2.78),

$$L(Sym^2 E, \chi, 2) = C' L(Sym^2 E', \chi, 2). \tag{2.83}$$

Let π and π' be the unitary cuspidal automorphic representations over $\mathrm{GL}(3, \mathbb{A}_{\mathbb{Q}})$ associated to $Sym^2 E$ and $Sym^2 E'$ respectively. Then the unitarized L-functions L_u corresponding to π and π' satisfy $L_u(\pi, s) = L(Sym^2 E, s+1)$. Thus, if E has semistable reduction at p, from (2.82) and (2.83) there exist constants $C_1, C_2 \in \mathbb{C}$ such that

$$L(\pi \otimes \chi, 1) = C_1 C_2^{m_\chi} L(\pi' \otimes \chi, 1)$$

for all wild p-power characters χ of conductor p^{m_χ} with m_χ sufficiently large. Then by Theorem 2.2, we conclude that $\pi \cong \pi'$ and thus $\mathrm{Ad}(\eta) \cong \mathrm{Ad}(\eta')$ where η and η' are the unitary cuspidal automorphic representations of $\mathrm{GL}(2, \mathbb{Q})$ associated to E. By Theorem 4.1.2 in [Ram00] we conclude that $\eta' = \eta \otimes \nu$ with ν a quadratic character since $\omega_\eta = \omega_{\eta'} = 1$. Write $\nu(\cdot) = \left(\frac{\cdot}{D}\right)$. It then follows by Faltings' isogeny theorem that E' is isogenous to E_D, where for the elliptic curve E given by the equation $y^2 = f(x)$ we have that E_D is given by the equation $Dy^2 = f(x)$. Clearly if the conductors of E and E' are square free, then E and E' are isogenous. $\square$

An elliptic curve E over $\mathbb{Q}$ is of CM-type if $\mathrm{End}(E) \otimes \mathbb{Q} = K$, with $K = \mathbb{Q}(\sqrt{-D})$ an imaginary quadratic number field. We have that $L(E, s) = L(\eta, s - 1/2)$ for some unitary Hecke character η of the idele class group C_K. Let $\pi = I_K^{\mathbb{Q}}(\eta)$ be the associated dihedral representation of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$. Denote by π' the cuspidal automorphic representation $I_K^{\mathbb{Q}}(\eta^2)$ of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$. By (2.22) we have

$$L(Sym^2\pi, s) = L(\pi', s)L(\eta_0, s),$$

where η_0 is the restriction of η to $C_{\mathbb{Q}}$. Twisting by some character χ gives

$$L(Sym^2\pi \otimes \chi, s) = L(\pi' \otimes \chi, s)L(\eta_0 \otimes \chi, s).$$

Note that $L(\pi' \otimes \chi, s)L(\eta_0 \otimes \chi, s)$ is entire unless $\eta_0 \otimes \chi$ is trivial, in which case

$$L(Sym^2\pi \otimes \eta_0^{-1}, s) = L(\pi' \otimes \eta_0^{-1}, s)\zeta(s)$$

has a pole at $s = 1$. Thus, we have that $L(Sym^2\pi \otimes \chi, s)$ is entire unless $\chi = \eta_0^{-1}$.

Proof of Theorem in CM case. Let π and π' be the isobaric sums of unitary cuspidal automorphic representations over $\mathrm{GL}(3, \mathbb{A}_{\mathbb{Q}})$ associated to Sym^2E and Sym^2E' respectively. Just as in the non-CM case, it follows that if E has semistable reduction at p we have that

$$L(\pi \otimes \chi, 1) = C_1 C_2^{m_\chi} L(\pi' \otimes \chi, 1)$$

for all wild p-power characters χ of conductor p^{m_χ} with m_χ sufficiently large and by the discussion above, the twisted L-functions are entire. Then by Theorem 2.2 we conclude that $\pi \cong \pi'$, and the proof proceeds as in the non-CM case. □

Chapter 3

Average result for the degree 4 L-function on GSp(4) using the relative trace formula

3.1 Introduction

The relative trace formula was first introduced by Jacquet to study period integrals of autormorphic forms, which can be in some cases related to values of L-functions. The trace formula, in the usual and relative versions, is most commonly used to prove functoriality. In such situations, one compares the geometric sides of the relative trace formulas for two different groups. The idea is to show that the relative traces for these two groups are equal with respect to suitably chosen test functions, without actually computing either geometric side. The equality of the relative trace formulas, together with some global work, has allowed the proof of several cases of functoriality of automorphic representations.

However, another way the relative trace formula can be used is to fix just one group and explicitly evaluate the geometric side and then deduce results for the spectral side. The difficulty of this method lies in computing the geometric side, which in general can be very hard to do. In my case, I identify the spectral side with an appropriate weighted average of L-values at the center and then work on explicitly evaluating the terms that appear on the geometric side. I want to show that there exist leading terms that are nonzero in a suitable limit.

A holomorphic cusp form π on $\mathrm{GSp}(4)/\mathbb{Q}$ gives rise to a holomorphic differential form ω on the Siegel threefold X, for instance, when π has scalar weight 3, ω is a $(3,0)$-form on X. The holomorphic cuspidal automorphic representations on $\mathrm{GSp}(4,\mathbb{A}_{\mathbb{Q}})$ are not generic, since at the infinite place the holomorphic discrete series do not admit a Whittaker model. Piatetski-Shapiro [PS97] gave a Rankin-Selberg type integral construction for the degree 4 L-function of automorphic representations of $\mathrm{GSp}(4)$ which works for representations that are not necessarily generic.

Using the integral representation I will define a suitable relative trace formula whose spectral side is an average of twisted central L-values of holomorphic Siegel eigenforms weighted by Fourier-Bessel coefficients.

Let $G = \mathrm{GSp}(4)/\mathbb{Q}$ and let Z denote the center of G. Fixing an imaginary quadratic field F, I consider a relative trace formula of G with respect to $H \times U$, where H is the group of matrices in $\mathrm{GL}(2)/F$ with rational determinant and U is the unipotent radical of the Siegel parabolic subgroup. More precisely, I consider the linear functional

$$I(f) := \int_{Z(\mathbb{A})H(\mathbb{Q})\backslash H(\mathbb{A})} \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} K_f(h,u)\mathcal{E}_s^{\phi}(h)\psi(u)\mathrm{d}u\mathrm{d}h,$$

where $\mathcal{E}_s^{\phi}(h)$ is an Einsenstein series on GL_2/F, ψ is a nontrivial character of $U(\mathbb{Q})\backslash U(\mathbb{A})$ and K_f is the kernel associated to a convenient test function $f \in C_c^{\infty}(Z(\mathbb{A})\backslash G(\mathbb{A}))$.

The linear functional can be written as

$$I(f) = \sum_{\delta} I(\delta, f),$$

where the sum is over the double coset representatives $\delta \in \tilde{H}(\mathbb{Q})\backslash\tilde{G}(\mathbb{Q})/U(\mathbb{Q})$, with $\tilde{H} = Z\backslash H$ and $\tilde{G} = Z\backslash G$.

In Section 3.4 we show that there are two types of cosets indexed by parameters $\lambda, \rho, \mu \in \mathbb{Q}$ with $\lambda \neq 0$ and $\mu \neq 0$ given by

$$\eta(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda \\ 0 & 0 & \lambda & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \text{ and } \xi(\rho, \mu) = \begin{pmatrix} 0 & 0 & \mu & \rho \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ \rho & -\mu & 0 & 0 \end{pmatrix}.$$

In Section 3.6, I argue that the leading contribution comes from the double cosets $\xi(0, 1)$ and the oscillating contribution comes from the double cosets $\xi(\rho, 1)$ with $\rho \in \mathbb{Z} \setminus \{0\}$. The remaining double cosets have a contribution of zero for an appropriate choice of data.

I am currently working towards showing that the oscillating terms have an overall smaller contribution than the leading terms. This can be done by taking a fixed weight at infinity, say $k = 3$ and verifying that under the variation of the prime parahoric level N, the leading terms is nonzero as $N \to \infty$.

The ultimate goal would be to prove the following non-vanishing result:

($\star$) Fix a Siegel weight $k \geq 3$. Then for a suitable fixed character χ_0, there exist infinitely many genus 2 holomorphic Siegel eigenforms π of trivial central character and weight k such that

$$L(\pi \otimes \chi_0, 1/2) \neq 0.$$

as we vary the level $N \to \infty$.

Alternatively, I expect the procedure to work when fixing a suitable level $N > 1$ and varying the weight k at infinity.

I now give an outline of the rest of the chapter. In Section 3.2 I give a detailed description of the relative trace formula, as well as other relevant notions necessary to set up the problem, such as the integral representation for the degree 4 L-function. In Section 3.3, I present a suitable test function, and compute its value at the archimedian place. In Section 3.4 I compute the double cosets, while in Section 3.5 I show that the relative trace formula gives a desirable weighted average of central L-values on the spectral side. Finally, in Section 3.6 I express each $I(\delta, f)$ as a (sum of) factorizable integrals and determine which terms contribute to the sum with respect to the double coset representatives. The results obtained thus far are summarized in Section 3.7.

3.2 Setup

3.2.1 GSp(4) and its subgroups

Let $G = \mathrm{GSp}(4) = \{g \in \mathrm{GL}(4) : \exists \lambda(g) \in \mathrm{GL}(1)\ g^t J g = \lambda(g) J\}$, where

$$J = \begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix}.$$

The map $\lambda : G \to \mathrm{GL}(1)$ is called the multiplier homomorphism. Its kernel is the symplectic group $Sp(4)$. If we let

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$$

be an arbitrary element, then the blocks A, B, C and D satisfy

$$C^t A = A^t C,\ D^t B = B^t D \text{ and } D^t A - B^t C = \lambda I_2. \tag{3.1}$$

The strong approximation theorem gives

$$\mathrm{GSp}(4, \mathbb{A}) = \mathrm{GSp}(4, \mathbb{Q})\mathrm{GSp}(4, \mathbb{R})^+ \prod_{p<\infty} K_p, \tag{3.2}$$

where $\mathrm{GSp}(4, \mathbb{R})^+$ is the subgroup of elements of $\mathrm{GSp}(4, \mathbb{R})$ with positive determinant and $K_p = \mathrm{GSp}(4, \mathbb{Z}_p)$.

The Weyl group of G is the dihedral group with eight elements. G has three standard parabolic subgroup: the Borel subgroup B, the Siegel subgroup P, and the Klingen subgroup Q. The Borel subgroup B has Levi decomposition

$$B = \left\{ \begin{pmatrix} a & & & \\ & b & & \\ & & \lambda a^{-1} & \\ & & & \lambda b^{-1} \end{pmatrix} \begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{pmatrix} \begin{pmatrix} 1 & & s & t \\ & 1 & t & u \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\},$$

while P has the Levi decomposition $P = MU$ with reductive part

$$M = \left\{ \begin{pmatrix} A & \\ & \lambda \cdot (A^{-1})^t \end{pmatrix} | A \in \mathrm{GL}(2,\mathbb{Q}), \lambda \in \mathbb{Q}^\times \right\}$$

and

$$U = \left\{ \begin{pmatrix} I_2 & X \\ & I_2 \end{pmatrix} | X \in Sym_2(\mathbb{Q}) \right\}.$$

the unipotent radical.

Proposition (Bruhat decomposition) We have

$$G = \overline{P}P \cup \overline{P}w_1P \cup \overline{P}w_2P, \tag{3.3}$$

where $\overline{P}$ is the transpose of the Siegel parabolic and

$$w_1 = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

The subgroup $\overline{P}$ of G is given by the Levi decomposition $\overline{P} = M\overline{U}$ where

$$\overline{U} = \left\{ \begin{pmatrix} I_2 & 0 \\ Y & I_2 \end{pmatrix} | Y \in Sym_2(\mathbb{Q}) \right\}.$$

Proof. See for example Section 4.1 in [FS03]. □

Let $F = \mathbb{Q}(\sqrt{d})$ be a quadratic imaginary field with d a square-free integer.

Consider the group

$$H = \{h \in \mathrm{GL}(2, F) | \det(h) \in \mathbb{Q}^\times\}. \tag{3.4}$$

For $x = (x_1, x_2), y = (y_1, y_2) \in F^2$ we consider a skew-symmetric form

$$\rho(x, y) = \mathrm{Tr}_{F/\mathbb{Q}}(x_1 y_2 - x_2 y_1).$$

We can then define

$$\mathrm{GSp}_\rho = \{g \in GL(4, \mathbb{Q}) | \rho(xg, yg) = \lambda(g)\rho(x, y)\}.$$

Since H preserves ρ up a factor in $\mathbb{Q}^\times$, more precisely $\rho(xh, yh) = \det(h)\rho(x, y)$ for $h \in H$, there exists a natural embedding

$$H \hookrightarrow \mathrm{GSp}_\rho.$$

By Proposition 2.1 in [PS97], there exists an isomorphism $\varphi : \mathrm{GSp}_\rho \to \mathrm{GSp}(4)$ satisfying certain properties. The map φ is defined on H by

$$\varphi\left(\begin{pmatrix} a_1 + b_1\sqrt{d} & a_2 + b_2\sqrt{d} \\ a_3 + b_3\sqrt{d} & a_4 + b_4\sqrt{d} \end{pmatrix}\right) = \begin{pmatrix} a_1 & b_1 d & \frac{a_2}{2} & \frac{b_2}{2} \\ b_1 & a_1 & \frac{b_2}{2} & \frac{a_2}{2d} \\ 2a_3 & 2b_3 d & a_4 & b_4 \\ 2b_3 d & 2a_3 d & b_4 d & a_4 \end{pmatrix}.$$

The above map can be viewed in terms of a change of basis. Indeed, consider the linear transformation $T : F^2 \to F^2$ defined on the standard basis elements $\{e_1, e_2\}$

of F^2 as a F-vector space:

$$T(e_1) = (a_1 + b_1\sqrt{d})e_1 + (a_3 + b_3\sqrt{d})e_2$$

$$T(e_2) = (a_2 + b_2\sqrt{d})e_1 + (a_4 + b_4\sqrt{d})e_2.$$

Then

$$A = \begin{pmatrix} a_1 + b_1\sqrt{d} & a_2 + b_2\sqrt{d} \\ a_3 + b_3\sqrt{d} & a_4 + b_4\sqrt{d} \end{pmatrix}$$

is the matrix associated to this linear transformation with respect to $\{e_1, e_2\}$. Furthermore, assume that $\det(A) \in \mathbb{Q}^{\times}$. This implies that $a_1b_4 + b_1a_4 - a_2b_3 - b_2a_3 = 0$.

Consider the basis of F^2 as a 4-dimensional $\mathbb{Q}$-vector space given by $\{f_1, f_2, f_3, f_4\}$ with $f_1 = e_1$, $f_2 = e_1\sqrt{d}$, $f_3 = \frac{e_2}{2}$, $f_4 = \frac{e_2}{2\sqrt{d}}$. The matrix of T in the basis $\{f_1, f_2, f_3, f_4\}$ is then

$$\begin{pmatrix} a_1 & b_1 d & \frac{a_2}{2} & \frac{b_2}{2} \\ b_1 & a_1 & \frac{b_2}{2} & \frac{a_2}{2d} \\ 2a_3 & 2b_3 d & a_4 & b_4 \\ 2b_3 d & 2a_3 d & b_4 d & a_4 \end{pmatrix} \in \mathrm{GSp}(4). \tag{3.5}$$

From now on we let H be the subgroup of GSp(4) consisting of matrices as in (3.5) with

$$a_1b_4 + b_1a_4 - a_2b_3 - b_2a_3 = 0.$$

3.2.2 Holomorphic Siegel eigenforms of degree 2

Just as before, we let $G = \mathrm{GSp}(4)$ and we consider the space $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ of measurable functions $\varphi : G(\mathbb{A}) \to \mathbb{C}$ that have the property that

$$\varphi(z\gamma g) = \varphi(g) \text{ for all } z \in Z(\mathbb{A}), \gamma \in G(\mathbb{Q}), g \in G(\mathbb{A})$$

and such that $\int_{G(\mathbb{Q})\backslash G(\mathbb{A})} |\varphi(g)|^2 \mathrm{d}g < \infty$. We say that a function φ in this space is cuspidal if for any parabolic subgroup P with Levi decomposition $P = MU$ we have

$$\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug)\mathrm{d}u = 0.$$

The subspace of cuspidal functions is denoted $L_0^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. The right regular representation of $G(\mathbb{A})$ on L_0^2 decomposes as a direct sum of cuspidal automorphic representations of $G(\mathbb{A})$.

A holomorphic cuspidal Siegel eigenform of degree 2, weight $k \geq 3$ and level N generates a cuspidal automorphic representation of $G(\mathbb{A})$ with the property that π_∞ is a holomorphic discrete series $\mathcal{D}_k$ with scalar minimal K-type $\tau_{k,k}$. This means that the Harish-Chandra parameter of π_∞ is $(k-1, k-2)$.

The maximal compact subgroup of $G(\mathbb{R})$ is

$$K_\infty = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A^tA + B^tB = I \text{ and } A^tB = B^tA \right\} \to \{A - iB\} \in U(2).$$

We have that $\tau_{k,k}$ is an irreducible representation of $K_\infty \cong U(2)$ with highest weight (k, k). Then we must have that $\tau_{k,k} \cong \det^k$. We let v_0 be a lowest weight (unit) vector which generates the minimal K-type $\tau_{k,k}$. This will be computed explicitly in

Section 3.3.

For a finite place v of $\mathbb{Q}$, we let $K_v = \mathrm{GSp}(4, \mathbb{Z}_v)$ with measure normalized such that $\mathrm{meas}(K_v) = 1$, and if N is a positive integer, we define the subgroup

$$K_0(N)_v = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in K_v | C \equiv 0 (\mathrm{mod}\ N\mathbb{Z}_v) \right\} \tag{3.6}$$

and then consider

$$K_0(N) = \prod_{v<\infty} K_0(N)_v. \tag{3.7}$$

Note that if v is a place prime to N then $K_0(N)_v = K_v$.

For a cuspidal representation π of $G(\mathbb{A})$, we can write $\pi = \pi_\infty \otimes \pi_{\mathrm{fin}}$, with π_{fin} a representation of $G(\mathbb{A}_{\mathrm{fin}})$. We let $S_k(N)$ denote the subspace of cuspidal representations of $G(\mathbb{A})$ given by

$$S_k(N) = \bigoplus_{\substack{\pi_\infty = \mathcal{D}_k \\ \pi_{\mathrm{fin}}^{K_0(N)} \neq 0}} \mathbb{C}v_0 \otimes \pi_{\mathrm{fin}}^{K_0(N)}, \tag{3.8}$$

where $\pi_{\mathrm{fin}}^{K_0(N)}$ is the space of $K_0(N)$-fixed vectors in π_{fin}.

We have that the forms in $S_k(N)$ are exactly the classical holomorphic Siegel eigenforms of weight k and level N (see for example [AS01]).

A fact that will be useful later on is that for an automorphic form φ on $G(\mathbb{A})$ we have a Fourier expansion (see [PS97]):

$$\varphi(g) = \sum_{\substack{\psi \text{ character of } U(\mathbb{A}\backslash U(\mathbb{A}) \\ \psi \text{ nontrivial}}} \varphi_\psi(g), \tag{3.9}$$

where the Fourier coefficients are given by

$$\varphi_\psi(g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug)\psi(u)\mathrm{d}u. \tag{3.10}$$

3.2.3 Principal series of $\mathbf{GL}(2)$ over a local field

Let F be a local field with ring of integers $\mathcal{O}$ and $\mathfrak{p}$ the maximal ideal of $\mathcal{O}$. Let χ_1, χ_2 be characters of F and let $V(\chi_1, \chi_2, s)$ for $s \in \mathbb{C}$ be the space of locally constant functions $\phi_s : \mathrm{GL}(2, F) \to \mathbb{C}$ such that

$$\phi_s\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g\right) = \chi_1(a)\chi_2(b)\left|\frac{a}{b}\right|^{s+1/2}\phi_s(g).$$

Let the action of $\mathrm{GL}(2, F)$ on this vector space be $g.\phi_s(x) = \phi_s(xg)$. This gives an admissible representation. Also note that

$$V(\chi_1, \chi_2, s)^\vee = V(\chi_1^{-1}|\cdot|^{-2s}, \chi_2^{-1}|\cdot|^{2s}, s).$$

We have that $V(\chi_1, \chi_2, s)$ is irreducible if and only if $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1-2s}$. If $\chi_1\chi_2^{-1} = |\cdot|^{1-2s}$ then $V(\chi_1, \chi_2, s)$ is a twisted Steinberg representation.

If $V(\chi_1, \chi_2, s)$ is irreducible then it is called a principal series. We have that two irreducible representations $V(\chi_1, \chi_1, s)$ and $V(\chi_1', \chi_2', s')$ are equal if and only if $\chi_1|\cdot|^s = \chi_1'|\cdot|^{s'}$ and $\chi_2|\cdot|^{-s} = \chi_2'|\cdot|^{-s'}$ or vice versa.

Let

$$\Gamma_2(n) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathcal{O}_F) : c \in \mathfrak{p}^n \text{ and } d \in 1 + \mathfrak{p}^n\right\}.$$

If $\pi = V(\chi_1, \chi_2, s)$ then the conductor $c(\pi)$ of π is the minimal n such that $V^{\Gamma_2(n)} \neq$

$\{0\}$. The space $V^{\Gamma_2(c(\pi))}$ is 1-dimensional. The conductor of π is the sum of the conductors of χ_1 and χ_2.

Lemma 3.1.
$$GL(2,\mathcal{O}) = \bigsqcup_{i=0}^{n} \tilde{B}(\mathcal{O})\gamma_i\Gamma_2(n),$$
where
$$\gamma_i = \begin{pmatrix} 1 & \\ \varpi^i & 1 \end{pmatrix}$$
for $0 \le i \le n-1$ *and* $\gamma_n = I_2$*. Here* $\tilde{B}$ *denotes the standard Borel subgroup of* $GL(2)$ *and* ϖ *a uniformizer for* $\mathcal{O}$*.*

Proof. If $v(c) = i > 0$ then we have that
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{ad\varpi_v^i}{c} - b\varpi^i & b \\ & d \end{pmatrix} \begin{pmatrix} 1 & \\ \varpi_v^i & 1 \end{pmatrix} \begin{pmatrix} \frac{c}{\varpi_v^i d} & \\ & 1 \end{pmatrix}.$$

If $v(c) = 0$ then
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{ad}{c} - b & a + \left(b - \frac{ad}{c}\right)(1+\varpi^n) \\ & c \end{pmatrix} \begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1+\varpi^n & (1+\varpi^n)\frac{d}{c} - 1 \\ -\varpi^n & 1 - \varpi^n\frac{d}{c} \end{pmatrix}.$$

To check that the union is disjoint, consider $i \ne j$, with $0 \le i < j \le n$ and suppose
$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \varpi^j & 1 \end{pmatrix} \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix},$$
with
$$\begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \in \Gamma_2(n).$$

This gives

$$a + m\varpi^i = k_1,\ m = k_2,\ b\varpi^i = k_1\varpi^j + k_3 \text{ and } b = k_2\varpi^j + k_4.$$

From the last two equations we get that $k_2\varpi^{j+i} + k_4\varpi^i = k_1\varpi^j + k_3$, which is a contradiction, because since $k_4 \in 1 + \mathfrak{p}^n$ and $k_3 \in \mathfrak{p}^n$, and hence the left hand side has valuation i, while the right hand side has valuation at least j.

□

Consider a principal series representation $V(\chi_1, \chi_2, s)$. Let n_1 and n_2 be the conductors of χ_1 and χ_2 respectively, and assume $n_2 \geq n_1 > 0$. Let $n = n_1 + n_2$. A nontrivial $\Gamma_2(n)$-invariant vector is given by (see for example [Sch02]):

$$\phi_s(g) = \begin{cases} \chi_1(a)\chi_2(b)\left|\frac{a}{b}\right|^{s+1/2}\chi_1(\varpi^{-n_2})q^{n_2 s} & \text{if } g \in \begin{pmatrix} a & \star \\ & b \end{pmatrix}\begin{pmatrix} 1 & \\ \varpi^{n_2} & 1 \end{pmatrix}\Gamma_2(n) \\ 0 & \text{if } g \notin \tilde{B}(\mathcal{O})\begin{pmatrix} 1 & \\ \varpi^{n_2} & 1 \end{pmatrix}\Gamma_2(n), \end{cases} \tag{3.11}$$

where $q = |\mathcal{O}/\mathfrak{p}|$ is the size of the residue field.

3.2.4 Eisenstein series on GL(2) over a quadratic field

Let $F = \mathbb{Q}(\sqrt{d})$ be a quadratic field, (χ_1, χ_2) a pair of characters of $F^\times\backslash\mathbb{A}_F^\times$ and $s \in \mathbb{C}$. Let $V(\chi_1, \chi_2, s)$ denote the representation of $\mathrm{GL}(2, \mathbb{A}_F)$ by right translation on the space of classes of functions ϕ_s satisfying

$$\phi_s\left[\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g\right] = \chi_1(a)\chi_2(b)\left|\frac{a}{b}\right|^{s+1/2}\phi_s(g)$$

and

$$\int_\Gamma \left|\phi_s(k)\right|^2 \mathrm{d}k < \infty,$$

with $\Gamma = \prod_v \Gamma_v$ the maximal compact subgroup of $\mathrm{GL}(2, \mathbb{A}_F)$.

We can view $V(\chi_1, \chi_2, s)$ as a fibre bundle over the space of pairs (χ_1, χ_2). By Iwasawa decomposition, $V(\chi_1, \chi_2, s)$ can be viewed as the subspace of functions in $L^2(\Gamma)$ such that

$$\phi_s\left[\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} k\right] = \chi_1(a)\chi_2(b)\left|\frac{a}{b}\right|^{s+1/2}\phi_s(k)$$

for all $k \in \Gamma$ and

$$\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in \Gamma.$$

For $\Phi \in S(\mathbb{A}_F^2)$ a Schwarz-Bruhat function, the function

$$\phi_s(g) = \chi_1(\det g)|\det g|^{s+1/2}\int_{I_F} \Phi[(0,t)g]|t|^{2s+1}\chi_1\chi_2^{-1}(t)\mathrm{d}^\times t \tag{3.12}$$

is in the space $V(\chi_1, \chi_2, s)$, and we can define the corresponding Eisenstein series which converges for $\mathrm{Re}(s) > 1$ as

$$\mathcal{E}_s^\phi(g) = \sum_{\gamma \in \tilde{B}(F)\backslash \mathrm{GL}(2,F)} \phi_s(\gamma g) \tag{3.13}$$

where $\tilde{B}$ is the standard Borel subgroup of $\mathrm{GL}(2)$.

3.2.5 The intertwining operator

We continue to use the notations from Section 3.2.4. We write $\phi_s = \otimes\phi_{s,v}$ and for each place v we define the local intertwining operator to be

$$A_v(s, w_0)\phi_{s,v}(g) = \int_{N_v} \phi_{s,v}(w_0 ng)\mathrm{d}n,$$

with

$$N = \begin{pmatrix} 1 & \star \\ & 1 \end{pmatrix}$$

the unipotent subgroup of GL(2) and

$$w_0 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

We let

$$M(s) = \otimes_v A_v(s, w_0).$$

As a consequence of the theory of Eisenstein series, $M(s)$ extends to a meromorphic function on $\mathbb{C}$ with a finite number of simple poles.

Using (3.12) we get

$$\begin{aligned} M(s)\phi_s(g) &= \int_{N_{\mathbb{A}}} \phi_s(w_0 ng)\mathrm{d}n \\ &= \chi_1(\det\, g)|\det\, g|^{s+1/2} \int_{\mathbb{A}_F}\int_{I_F} \Phi[(t, tn)g]|t|^{2s+1}\chi_1\chi_2^{-1}(t)\mathrm{d}^\times t\ \mathrm{d}n \\ &= \chi_1(\det\, g)|\det\, g|^{s+1/2} \int_{I_F} \left(\int_{\mathbb{A}_F} \Phi[(t, n)g]\mathrm{d}n\right) |t|^{2s}\chi_1\chi_2^{-1}(t)\mathrm{d}^\times t. \end{aligned}$$

We have

$$\phi_{s,v}(g_v) = \frac{\chi_{1,v}(\det\ g)|\det\ g|_v^{s+1/2}}{L(2s+1,\chi_{1,v}\chi_{2,v}^{-1})}\int_{F_v^\times}\Phi[(0,t)g]|t|_v^{2s+1}\chi_{1,v}\chi_{2,v}^{-1}(t)\mathrm{d}^\times t$$

and hence

$$\begin{aligned} A_v(s,w_0)\phi_{s,v}(g_v) &= \chi_{1,v}\chi_{2,v}^{-1}(-1)\frac{\chi_{1,v}(\det\ g)|\det\ g|_v^{s+1/2}}{L(2s+1,\chi_{1,v}\chi_{2,v}^{-1})}\times \\ &\times \int_{F_v}\int_{F_v^\times}\Phi[(t,tn)g]|t|_v^{2s+1}\chi_{1,v}\chi_{2,v}^{-1}(t)\mathrm{d}^\times t\ \mathrm{d}n, \end{aligned}$$

which, after the change of variables $n \to t^{-1}n$, gives

$$\begin{aligned} A_v(s,w_0)\phi_{s,v}(g_v) &= \chi_{1,v}\chi_{2,v}^{-1}(-1)\frac{\chi_{1,v}(\det\ g)|\det\ g|_v^{s+1/2}}{L(2s+1,\chi_{1,v}\chi_{2,v}^{-1})}\times \\ &\times \int_{F_v^\times}\left[\int_{F_v}\Phi[(t,n)g]\mathrm{d}n\right]|t|_v^{2s}\chi_{1,v}\chi_{2,v}^{-1}(t)\mathrm{d}^\times t. \end{aligned}$$

Let $\Phi'(t) := \int_{F_v}\Phi[(t,n)g]\mathrm{d}n$. Then we have:

$$A_v(s,w_0)\phi_{s,v}(g_v) = \chi_{1,v}\chi_{2,v}^{-1}(-1)\frac{\chi_{1,v}(\det\ g)|\det\ g|_v^{s+1/2}}{L(2s+1,\chi_{1,v}\chi_{2,v}^{-1})}Z(\Phi',\chi_{1,v}\chi_{2,v}^{-1},2s).$$

But by Tate's thesis

$$\frac{Z(\hat{\Phi}',\chi_{1,v}^{-1}\chi_{2,v},1-2s)}{L(1-2s,\chi_{1,v}^{-1}\chi_{2,v})} = \epsilon(2s,\chi_{1,v}\chi_{2,v}^{-1},\upsilon_v)\frac{Z(\Phi',\chi_{1,v}\chi_{2,v}^{-1},2s)}{L(2s,\chi_{1,v}\chi_{2,v}^{-1})}.$$

Note that for almost all v, $\epsilon(2s,\chi_{1,v}\chi_{2,v}^{-1},\upsilon_v) \equiv 1$ for a fixed υ_v. From the above we get

$$A_v(s,w_0)\phi_{s,v}(g_v) = \frac{L(2s,\chi_{1,v}\chi_{2,v}^{-1})}{L(2s+1,\chi_{1,v}\chi_{2,v}^{-1})\epsilon(2s,\chi_{1,v}\chi_{2,v}^{-1},\upsilon_v)}\widetilde{\phi_{s,v}}(g), \tag{3.14}$$

where

$$\begin{aligned}\widetilde{\phi_{s,v}(g)} &= \chi_{1,v}\chi_{2,v}^{-1}(-1)\frac{\chi_{1,v}(\det\ g)|\det\ g|_v^s}{L(1-2s,\chi_{1,v}^{-1}\chi_{2,v})}Z(\hat{\Phi}',\chi_{1,v}^{-1}\chi_{2,v},1-2s) \\ &= \chi_{1,v}\chi_{2,v}^{-1}(-1)\frac{\chi_{1,v}(\det\ g)|\det\ g|_v^{s+1/2}}{L(1-2s,\chi_{1,v}^{-1}\chi_{2,v})}\int_{F_v^\times}\hat{\Phi}'(t)\chi_{1,v}^{-1}\chi_{2,v}(t)|t|_v^{1-2s}\mathrm{d}^\times t.\end{aligned} \tag{3.15}$$

Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and

$$\hat{\Phi}(x) = \int_{F_v}\Phi(u)\upsilon_v(xu)\mathrm{d}u.$$

We get

$$\begin{aligned}\hat{\Phi}'(t) &= \int_{F_v}\Phi'(u)\upsilon_v(tu)\mathrm{d}u \\ &= \iint_{F_v^2}\Phi[(u,v)g]\upsilon_v(tu)\mathrm{d}u\mathrm{d}v \\ &= \iint_{F_v^2}\Phi(au+cv,bu+dv)\upsilon_v(tu)\mathrm{d}u\mathrm{d}v.\end{aligned}$$

Taking $u' = au + cv$ and $v' = bu + dv$ gives

$$u = \frac{u'd - v'c}{ad-bc} = \frac{u'd - v'c}{\det(g)}$$

and

$$J = \begin{pmatrix} \partial u'/\partial u & \partial u'/\partial v \\ \partial v'/\partial u & \partial v'/\partial v \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Thus,

$$\hat{\Phi}'(t) = \iint_{F_v^2} \Phi(u', v')\upsilon_v(t(u'd - v'c)(\det(g))^{-1})\frac{\mathrm{d}u'\mathrm{d}v'}{|\det(g)|}.$$

Now, if we let

$$\hat{\Phi}(x, y) := \int_{F_v^2} \Phi(u, v)\upsilon_v(yu - xv)\mathrm{d}u\mathrm{d}v,$$

we conclude that

$$\hat{\Phi}'(t) = |\det(g)|^{-1}\hat{\Phi}[(\det(g)^{-1})(0, t)g].$$

This gives

$$\begin{aligned}
\widetilde{\phi_{s,v}(g)} &= \chi_{1,v}\chi_{2,v}^{-1}(-1)\frac{\chi_{1,v}(\det g)|\det g|_v^{s+1/2}}{L(1-2s, \chi_{1,v}^{-1}\chi_{2,v})} \times \\
&\times \int_{F_v^\times} |\det(g)|^{-1}\hat{\Phi}[(\det(g)^{-1})(0, t)g]\chi_{1,v}^{-1}\chi_{2,v}(t)|t|^{1-2s}\mathrm{d}^\times t \\
&= \chi_{1,v}\chi_{2,v}^{-1}(-1)\frac{\chi_{1,v}(\det g)|\det g|_v^{s-1/2}}{L(1-2s, \chi_{1,v}^{-1}\chi_{2,v})} \times \\
&\times \int_{F_v^\times} \hat{\Phi}[(\det(g)^{-1})(0, t)g]\chi_{1,v}^{-1}\chi_{2,v}(t)|t|^{1-2s}\mathrm{d}^\times t.
\end{aligned}$$

Now we do the change of variables $t \to \det(g)t$ which gives

$$\begin{aligned}
\widetilde{\phi_{s,v}(g)} &= \frac{\chi_{1,v}\chi_{2,v}^{-1}(-1)\chi_{1,v}(\det\ g)|\det\ g|_v^{s-1/2}}{L(1-2s, \chi_{1,v}^{-1}\chi_{2,v})} \times \\
&\times \int_{F_v^\times} \hat{\Phi}[(0, t)g]\chi_{1,v}^{-1}\chi_{2,v}(t)|t|^{1-2s}\chi_{1,v}^{-1}\chi_{2,v}(\det\ g)|\det\ g|^{2-2s}\mathrm{d}^\times t.
\end{aligned}$$

Thus,

$$\widetilde{\phi_{s,v}(g)} = \chi_{1,v}\chi_{2,v}^{-1}(-1)\frac{\chi_{2,v}(\det\ g)|\det\ g|_v^{3/2-s}}{L(1-2s, \chi_{1,v}^{-1}\chi_{2,v})} \int_{F_v^\times} \hat{\Phi}[(0, t)g]\chi_{1,v}^{-1}\chi_{2,v}(t)|t|^{1-2s}\mathrm{d}^\times t.$$

Note that the map $\phi_{s,v} \to \widetilde{\phi_{s,v}}$ takes $(\Phi, \chi_{1,v}, \chi_{2,v}) \to (\hat{\Phi}, \chi_{2,v}|\cdot|^{1-2s}, \chi_{1,v}|\cdot|^{1+2s})$.

3.2.6 The degree 4 L-function

The Langlands dual group of G is ${}^LG = \mathrm{GSp}(4,\mathbb{C})$. Given ρ a finite dimensional representation of ${}^LG$, and π a cuspidal automorphic representation of $G(\mathbb{A})$, Langlands defined an L-function $L(\pi, s, \rho)$ as a certain Euler product convergent in some right half plane. For G, we can consider the irreducible representations of dimension 4 and 5, which give the spinor L-function and the standard L-function repsectively. If for each prime p, the local representation π_p of the automorphic representation π has Satake parameters $\alpha_p, \beta_p, \gamma_p$, the degree 4 (spinor) L-function has the Euler product

$$L(\pi, s, \rho_4) = \prod_p \left((1-\alpha_p p^{-s})(1-\alpha_p\beta_p p^{-s})(1-\alpha_p\gamma_p p^{-s})(1-\alpha_p\beta_p\gamma_p p^{-s})\right)^{-1},$$

while the degree 5 (standard) L-function has the Euler product

$$L(\pi, s, \mathrm{std}) = \prod_p \left((1-p^{-s})(1-\beta_p p^{-s})(1-\beta_p^{-1} p^{-s})(1-\gamma_p p^{-s})(1-\gamma_p^{-1} p^{-s})\right)^{-1}.$$

In this section, we review an integral representation of Rankin-Selberg type for the degree 4 L-function that is due to Piatetski-Shapiro [PS97] and that is applicable even in the case of non-generic representations. In the next section, we will further review how this integral representation unfolds in terms of a generalized global Whittaker model, and how a generalized local Whittaker can be defined, which leads to construction of the local L-factors. We will use some of the notations in [PS97] for simplicity.

We let ν be a character of I_F and μ and Grössencharacter on $\mathbb{Q}$ and consider

$$\phi_s(g) = \mu(\det\, g)|\det\, g|^{s+1/2} \int_F \Phi[(0,t)g]|t\bar{t}|^{s+1/2}\mu(t\bar{t})\nu(t)\mathrm{d}t,$$

which satisfies the property that $\phi_s \in \mathrm{Ind}_{\tilde{B}_{\mathbb{A}}}^{H_{\mathbb{A}}}\chi$, where χ is a character of $\tilde{B}_{\mathbb{A}}$ defined by

$$\chi\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \bar{t} & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\right) = \mu(x)|x|^{s+1/2}\nu^{-1}(t).$$

The Eisenstein series $\mathcal{E}_s^{\phi}$ is then defined as in Section 3.2.4. Note that in the notations of Section 3.2.4, we have that characters μ and ν correspond to the pair (χ_1, χ_2) as follows: $\mu(x) = \chi_1(x)$ and $\nu^{-1}(t) = \chi_1(\bar{t})\chi_2(t)$.

If $\mu(t\bar{t})\nu(t) \neq 1$ (or equivalently if $\chi_1 \neq \chi_2$), we have that $\mathcal{E}_s^{\phi}$ has no poles. Otherwise, it has a pole at $s = -\frac{1}{2}$ and $s = \frac{3}{2}$.

Let π be a holomorphic cuspidal automorphic representation on $\mathrm{GSp}(4, \mathbb{A})$, and consider $\varphi \in \pi$. We define the Rankin Selberg type integral

$$L(\varphi; \Phi, \mu, \nu, s) := \int_{Z(\mathbb{A})H(\mathbb{Q})\backslash H(\mathbb{A})} \varphi(g)\mathcal{E}_s^{\Phi}(g; \mu, \nu)\mathrm{d}g. \tag{3.16}$$

The meromorphic continuation of the Eisenstein series gives the meromorphic continuation of $L(\varphi; \Phi, \mu, \nu, s)$. The twisted L-function $L(\pi, \mu, s)$ of π can be defined to be so that $\frac{L(\varphi;\Phi,\mu,\nu,s)}{L(\pi,\mu,s)}$ is entire for all choices of ν and Φ. To do this we describe in the next section a generalized local Whittaker model introduced in [PS97], which will allow us to define local L-factors.

3.2.7 Generalized Whittaker models on GSp(4)

While in the GL(2) case we can define the local L-factors by using a Whittaker model, a Whittaker model does not exist for holomorphic cuspidal automorphic representations of G. However, a generalized (local) Whittaker model can be defined, not with respect to the maximal unipotent U, but rather with respect to another subgroup R of G.

Let k be a local field. Consider a non-degenerate linear form

$$l_\beta(u) = \mathrm{tr}(\beta X), \tag{3.17}$$

where

$$u = \begin{pmatrix} I & X \\ & I \end{pmatrix},$$

and $\beta \in \mathrm{GL}(2,k)$ with $\beta^t = \beta$. If we let T_β be the connected component of the stabilizer of l_β in M, where M is the reductive part of the Siegel parabolic $P = MU$ (see Section 3.2.1) then there exists a unique semisimple algebra K over k such that $(K : k) = 2$ and $T_\beta \cong K^\star$. In each orbit of M we can find a representative l_β corresponding to

$$\beta = \begin{pmatrix} 1 & \\ & -d \end{pmatrix},$$

with a d square free integer. Then $K = k \oplus k$ if $d = 1$ and $K = k(\sqrt{d})$ if $d \neq 1$. We also let

$$N_\beta = \{u \in U : l_\beta(u) = 0\} \text{ and } H_\beta = \left\{g \in GL(2,K) | \det(g) \in k^\times\right\}.$$

If $K = k(\sqrt{d})$ we have that

$$T_\beta = \left\{ \begin{pmatrix} \bar{t} & 0 \\ 0 & t \end{pmatrix} : t \in K^\star \right\} \cong \left\{ \begin{pmatrix} t_1 & -t_2 d & & \\ -t_2 & t_1 & & \\ & & t_1 & t_2 \\ & & t_2 d & t_1 \end{pmatrix} : t_1 + t_2\sqrt{d} \in K^\star \right\}$$

and

$$N_\beta = \left\{ \begin{pmatrix} 1 & n_1 + n_2\sqrt{d} \\ 0 & 1 \end{pmatrix} \right\} \cong \left\{ \begin{pmatrix} 1 & & \frac{n_1}{2} & \frac{n_2}{2} \\ & 1 & \frac{n_2}{2} & \frac{n_1}{2d} \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\}.$$

We let $R_\beta = T_\beta U$. We omit the index β from now on for simplicity. If we take ν to be a character of T and ψ_β a character of U, with

$$\psi_\beta(u) = \psi_0(l_\beta(u)), \tag{3.18}$$

we can define a character $\alpha_{\nu,\psi}$ of R as

$$\alpha_{\nu,\psi}(r) = \alpha_{\nu,\psi}(tu) = \nu(t)\psi(u), \tag{3.19}$$

where $r = tu$ with $t \in T$ and $u \in U$.

By Theorem 3.1 in [PS97], we have that if π is an irreducible smooth admissible representation of $\mathrm{GSp}(4, k)$ and $\alpha_{\nu,\psi}$ a character of R as in (3.19), then there exists at most one linear functional (up to scalar multiplication) $l : V_\pi \to \mathbb{C}$ such that

$$l(\pi(r)\upsilon) = \alpha_{\nu,\psi}(r)l(\upsilon), \tag{3.20}$$

for all $r \in R(k)$ and $\varphi \in \pi$. In addition, Howe showed that if $k \neq \mathbb{C}$ then a functional as in (3.20) always exists for π an infinite-dimensional representation, and that for $k = \mathbb{C}$ the only exceptions are the Weil representations.

For $\upsilon \in V_\pi$ define the generalized Whittaker function

$$W_\upsilon(g) = l(\pi(g)\upsilon),$$

and let $\mathcal{W}^{\nu,\psi}$ be the space of generalized Whittaker functions. We can define the representation by right translation on this space. We have $\pi \cong \mathcal{W}^{\nu,\psi}$ and for $r \in R$ and $\upsilon \in V_\pi$

$$W_\upsilon(rg) = \alpha_{\upsilon,\psi}(r) W_\upsilon(g).$$

For $W \in \mathcal{W}^{\nu,\psi}$, μ a character of $k^\star$ and Φ a Schwarz function on K^2 define

$$L(W, \Phi, \mu, s) = \int_{N\backslash H} W(h)\Phi[(0,1)g]\mu(\det\ g)|\det\ g|_k^{s+1/2}\mathrm{d}g, \tag{3.21}$$

and if to Φ we associate the function

$$\phi(g; \mu, \nu, s) = \mu(\det\ g)|\det\ g|^{s+1/2} \int_{K^\times} \Phi[(0,t)g]|t|^{2s+1}\mu(t\bar{t})\nu(t)\mathrm{d}^\times t, \tag{3.22}$$

we have

$$L(W, \Phi, \mu, s) = \int_{TN\backslash H} W(g)\phi(g; \mu, \nu, s)\mathrm{d}g. \tag{3.23}$$

This function converges in some right half plane of s and admits a meromorphic continuation to the entire plane and a functional equation.

We can make an analogous construction for global fields. More specifically, we define a generalized global Whittaker model as follows. For a global field k, a cuspidal automorphic representation π of $G(\mathbb{A})$ and character $\alpha_{\nu,\psi}$ of $R(\mathbb{A})$ there exists a cusp form $\varphi \in \pi$ such that

$$\int_{Z_{\mathbb{A}}R(k)\backslash R(\mathbb{A})} \varphi(r)\alpha_{\nu,\psi}^{-1}(r)\mathrm{d}r \neq 0, \tag{3.24}$$

and we can define

$$W_{\varphi}(g) = \int_{Z_{\mathbb{A}}R(k)\backslash R(\mathbb{A})} \varphi(rg)\alpha_{\nu,\psi}^{-1}(r)\mathrm{d}r \tag{3.25}$$

for $g \in \mathrm{GSp}(4,\mathbb{A})$. The function W_{φ} has the property that

$$W_{\varphi}(rg) = \alpha_{\nu,\psi}(r)W_{\varphi}(g).$$

Let $\mathcal{W}^{\nu,\psi}$ be the space of Whittaker functions W_{φ} and consider the representation of $\mathrm{GSp}(4,\mathbb{A})$ on this space by right translation. This representation is isomorphic to π.

If we write $\pi = \otimes' \pi_v$, then for each π_v there exists a unique generalized local Whittaker model corresponding to α_{ν_v,ψ_v}, with ν_v and ψ_v the local components at v of ν and ψ respectively. The global model is then the restricted tensor product of the corresponding local models.

The integral representation in equation (3.16) unfolds in terms of the generalized global Whittaker model to give

$$\int_{T(\mathbb{A})N(\mathbb{A})\backslash H(\mathbb{A})} W_{\varphi}(g)\phi(g;\mu,\nu,s), \tag{3.26}$$

which can be further expressed as

$$\int_{N(\mathbb{A})\backslash H(\mathbb{A})} W_{\varphi}(g)\Phi[(0,1)g]\mu(\det\, g)|\det\, g|^{1/2+s}\mathrm{d}g. \tag{3.27}$$

Note that if π is a cuspidal automorphic representation of $\mathrm{GSp}(4,\mathbb{A}_{\mathbb{Q}})$ and $\varphi \in \pi$

we have a Fourier expansion (see for example [Har04]):

$$\varphi(ug) = \sum_{\beta} \varphi_\beta(g)\psi_\beta(u)$$

for $u \in U(\mathbb{A})$ and ψ_β as in (3.18), with the sum over $\beta \in Sym_2(\mathbb{Q})$. The Fourier coefficients φ_β are smooth functions on $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$. If φ_β is nonzero then we say that β is in the support of π.

If π is holomorphic then β must be a positive definite matrix so that K is an imaginary quadratic field. If π is not holomorphic then K is either a real quadratic field or $\mathbb{Q} \oplus \mathbb{Q}$.

For this reason, the subgroup $H = H_\beta$ as it appears in (3.16) corresponds to an imaginary quadratic field F.

3.2.8 The relative trace formula

Jacquet's relative trace formula is a generalization of the Arthur-Selberg trace formula. The setup of the relative trace formula consists of integrating the kernel over non-diagonal subgroups.

For $G = \mathrm{GSp}(4)/\mathbb{Q}$ we define

$$C_c^\infty(G(\mathbb{A})) = C_c^\infty(G(\mathbb{R})) \otimes C_c^\infty(G(\mathbb{A}_{\mathrm{fin}})),$$

where $C_c^\infty(G(\mathbb{R}))$ is the space of smooth compactly supported functions on $G(\mathbb{R})$ with values in $\mathbb{C}$, and $C_c^\infty(G(\mathbb{A}_{\mathrm{fin}}))$ is the space of locally constant and compactly supported functions on $G(\mathbb{A}_{\mathrm{fin}})$ with values in $\mathbb{C}$.

Let Z denote the center G. For a test function $f \in C_c^\infty(Z(\mathbb{A})\backslash G(\mathbb{A}))$ we associate the kernel function

$$K(x,y) := \sum_{\gamma \in Z(\mathbb{Q})\backslash G(\mathbb{Q})} f(x^{-1}\gamma y). \tag{3.28}$$

Let ρ be the right regular representation

$$(\rho(y)\varphi)(x) = \varphi(xy)$$

of $Z(\mathbb{A})\backslash G(\mathbb{A})$ on the Hilbert space $L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))$. We define

$$\rho(f) : L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})) \to L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))$$

given by

$$(\rho(f)\varphi)(x) = \int_{Z(\mathbb{A})\backslash G(\mathbb{A})} f(y)\rho(y)\varphi(x)\mathrm{d}y = \int_{Z(\mathbb{A})\backslash G(\mathbb{A})} f(y)\varphi(xy)\mathrm{d}y. \tag{3.29}$$

Setting some conditions on f_∞, the component of f at the archimedian place (see Sections 3.3 and 3.5), the operator $\rho(f)$ will decompose into a direct sum of cuspidal representations:

$$\rho(f) = \oplus_\pi m_\pi \pi(f).$$

Note that $\pi(f)$ is defined as:

$$(\pi(f)\varphi)(x) = \int_{Z(\mathbb{A})\backslash G(\mathbb{A})} f(y)\pi(y)\varphi(x)\mathrm{d}y = \int_{Z(\mathbb{A})\backslash G(\mathbb{A})} f(y)\varphi(xy)\mathrm{d}y,$$

for $\varphi \in \pi$. Associated to this decomposition of $\rho(f)$ into a direct sum of $\pi(f)$, we have an alternative expression of the kernel

$$K(x,y) = \sum_{\pi} K_\pi(x,y).$$

Let $F = \mathbb{Q}(\sqrt{d})$ be a fixed auxiliary imaginary quadratic field with $d < 0$ square-free. Let H be the group of matrices in $\mathrm{GL}(2)/F$ with rational determinant, viewed as a subgroup of $\mathrm{GSp}(4)/\mathbb{Q}$ (see Section 3.2.1) and U be the unipotent radical of the Siegel parabolic P of G.

We consider the linear functional on $C_c^\infty(Z(\mathbb{A})\backslash G(\mathbb{A}))$

$$I(f) := \int_{Z(\mathbb{A})H(\mathbb{Q})\backslash H(\mathbb{A})} \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} K_f(h,u)\mathcal{E}_s^\phi(h)\psi(u)\mathrm{d}u \, \mathrm{d}h, \tag{3.30}$$

where $\mathcal{E}_s^\phi(x)$ is an Einsenstein series on $\mathrm{GL}(2)/F$ and ψ is a nontrivial character of $U(\mathbb{Q})\backslash U(\mathbb{A})$. The character ψ is given by

$$\psi(u) = \psi_0(\mathrm{tr} SX) \tag{3.31}$$

with

$$u = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix},$$

and X a symmetric 2×2 matrix over $\mathbb{Q}$. Here ψ_0 is the standard additive character of $\mathbb{Q}\backslash\mathbb{A}$ and S is a symmetric 2×2 matrix over $\mathbb{Q}$. The character ψ_0 is defined as follows. We define the local component $\psi_{0,p}$ at p prime to be

$$\psi_{0,p} = \left[\mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}/\mathbb{Z} \to S^1\right].$$

Now every $x \in \mathbb{Q}_p$ can be represented in the form

$$x = x_{-r}p^{-r} + x_{1-r}p^{1-r} + \cdots + x_{-1}p^{-1} + x_0 + x_1 p + \cdots = \frac{a}{p^r} + \sum_{i=0}^{\infty} x_i p^i$$

with $0 \leq x_n \leq p-1$. We say that $\frac{a}{p^r}$ is the fractional part $\{x\}_p$ of x. We set

$$\psi_{0,p}(x) = e^{2\pi i \{x\}_p}.$$

Note that $\psi_{0,p}(x) = 1$ iff $x \in \mathbb{Z}_p$. In addition, we let

$$\psi_{0,\infty} : \mathbb{R} \to S^1, \ x \to e^{-2\pi i x}$$

be the standard non-trivial character at infinity.

From now on, we will sometimes use when convenient the notations $\tilde{G} = Z\backslash G$ and $\tilde{H} = Z\backslash H$.

3.3 Test function

We take the test function f to be a factorizable, smooth function in $C_c^\infty(G(\mathbb{A}))$ with suitable properties. We write:

$$f = f_\infty \times f_{\text{fin}} = f_\infty \times f_N \times f_S \times f^{S_0}$$

with $S_0 = S \cup \{N\}$ with N some fixed prime.

The places in S correspond to a finite set of places p where $\chi_{1,p}$ and $\chi_{2,p}$ defining the Eisenstein series $\mathcal{E}_s^\phi$ (see eq. (3.12) and (3.13)) are ramified with conductor n_p.

Let $\mathcal{D}_k$ denote the holomorphic discrete series representation of $\mathrm{PGSp}(4,\mathbb{R})$ of lowest weight k. We choose the function f_∞ to be (up to a constant) the complex conjugate of the matrix coefficient

$$\langle \mathcal{D}_k(g)v_0, v_0\rangle,$$

where v_0 is a vector that generates the K-type $\tau_{k,k}$. We can take v_0 to be a unit vector.

We take f_N to be the characteristic function of $Z_N\backslash K_0(N)_N Z_N$ divided by the measure V_N of $Z_N\backslash K_0(N)_N Z_N$. Here

$$K_0(N)_v = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}(4,\mathbb{Z}_v) | C \equiv 0 (\mathrm{mod}\ N\mathbb{Z}_v) \right\}.$$

We take f^{S_0} to be the characteristic function of $\prod_{v\notin S_0} \mathrm{GSp}(4,\mathbb{Z}_v)Z_v$. Finally we take f_p with $p \in S$ to be 1 on the coset

$$K(2n_p)_p \begin{pmatrix} -\frac{1}{2} & & & \\ & -\frac{1}{2d} & & \\ -\varpi_p^n & & 2 & \\ & -\varpi_p^n & & 2d \end{pmatrix} K(2n_p)_p,$$

and zero otherwise. Here

$$K(2n_p)_p = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}(4,\mathbb{Z}_p) | \begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} (\mathrm{mod}\ \varpi_p^{2n_p}) \right\}.$$

Finally, let us compute $f_\infty(g)$ through a series of lemmas. The group $G^+(\mathbb{R})$ consisting of those element with $\lambda(g) > 0$ acts on the Siegel upper half plane $\mathcal{H}_2 =$

$\{Z \in \mathrm{Mat}_2(\mathbb{C}) : Z^t = Z, \mathrm{Im}(Z) > 0\}$ in the usual way:

$$g \cdot Z := (AZ + B)(CZ + D)^{-1}$$

for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $Z \in \mathcal{H}_2$. We write

$$Z = \begin{pmatrix} X_1 + iY_1 & X_2 + iY_2 \\ X_2 + iY_2 & X_3 + iY_3 \end{pmatrix} \in \mathcal{H}_2$$

with

$$\mathrm{Im}(Z) = \begin{pmatrix} Y_1 & Y_2 \\ Y_2 & Y_3 \end{pmatrix} > 0,$$

which is equivalent to $Y_1 + Y_3 > 0$ and $Y_1 Y_3 - Y_2^2 > 0$, so in particular we have $Y_1, Y_3 > 0$.

Consider the space of holomorphic $\mathbb{C}$-valued functions F on $\mathcal{H}_2$ and let

$$\mathcal{D}_k(g)F(Z) = \lambda(g)^k (\det(CZ + D))^{-k} F\left((AZ + B)(CZ + D)^{-1}\right).$$

The measure is

$$\mathrm{d}Z = \det(Y)^{-3} \prod_{i \leq j} \mathrm{d}X_{ij} \mathrm{d}Y_{ij}.$$

Take $v_0(Z) := \det((Z + iI)^{-k})$ for $Z \in \mathcal{H}_2$. A straightforward computation verifies that $v_0(Z)$ is well-defined on $\mathcal{H}_2$.

We have for

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}(4, \mathbb{R}) \tag{3.32}$$

with $\lambda(g) > 0$,

$$\begin{aligned}\mathcal{D}_k(g)v_0(Z) &= \lambda(g)^k(\det(CZ+D))^{-k}\det((AZ+B)(CZ+D)^{-1}+iI)^{-k}\\ &= \lambda(g)^k\det(AZ+B+i(CZ+D))^{-k}.\end{aligned}$$

Now consider $k_\infty \in K_\infty$,

$$k_\infty = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

We have

$$\mathcal{D}_k(k_\infty)v_0(Z) = \lambda(g)^k\det(A-iB)^{-k}v_0(Z),$$

and hence the vector v_0 generates the K-type $\tau_{k,k}$.

Let g as in (3.32). We want to compute

$$\begin{aligned}s_k(g) &= \langle \mathcal{D}_k(g)v_0, v_0\rangle\\ &= \lambda(g)^k\int_{\mathcal{H}_2}\det(AZ+B+i(CZ+D))^{-k}\det(\overline{Z}-iI)^{-k}\det(Y)^{k-3}\times\\ &\times\ \mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3\mathrm{d}Y_1\mathrm{d}Y_2\mathrm{d}Y_3\\ &= \lambda(g)^k\det(A+iC)^{-k}\int_{\mathcal{H}_2}\det(Z+(A+iC)^{-1}(B+iD))^{-k}\det(\overline{Z}-iI)^{-k}\times\\ &\times\ \det(Y)^{k-3}\mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3\mathrm{d}Y_1\mathrm{d}Y_2\mathrm{d}Y_3. \qquad (3.33)\end{aligned}$$

We will compute $s_k(g)$ by using the Cartan decomposition of GSp(4). To do that we first show the following result:

Lemma 3.2. $\langle D_k(g)F, F\rangle = \langle F, D_k(g^{-1}\mathit{det}\ g)F\rangle$

Proof. If g is an in (3.32) then

$$\begin{aligned}\langle D_k(g)F, F\rangle &= \int_{\mathcal{H}_2} \lambda(g)^k (\det(CZ+D))^{-k} F\left((AZ+B)(CZ+D)^{-1}\right) \times \\ &\times \overline{F(Z)} \det(Y)^{k-3} \mathrm{d}X_1 \mathrm{d}X_2 \mathrm{d}X_3 \mathrm{d}Y_1 \mathrm{d}Y_2 \mathrm{d}Y_3.\end{aligned}$$

Now let $Z' = (AZ+B)(CZ+D)^{-1}$ be a change of variables. Then we have that $Z = (A - Z'C)^{-1}(Z'D - B)$. Let

$$g^{-1} = \begin{pmatrix} M & N \\ P & Q \end{pmatrix}.$$

It is easy to verify that

$$Z = (A - Z'C)^{-1}(Z'D - B) = (MZ' + N)(PZ' + Q)^{-1}. \tag{3.34}$$

Thus, we get

$$\begin{aligned}\langle D_k(g)F, F\rangle &= \int_{\mathcal{H}_2} \lambda(g)^k (\det(C(MZ'+N)(PZ'+Q)^{-1} + D))^{-k} F(Z') \times \\ &\times \overline{F((MZ'+N)(PZ+Q)^{-1})} \det(Y)^{k-3} \mathrm{d}X_1 \mathrm{d}X_2 \mathrm{d}X_3 \mathrm{d}Y_1 \mathrm{d}Y_2 \mathrm{d}Y_3.\end{aligned} \tag{3.35}$$

We want to show that this equals

$$\begin{aligned}\langle F, D_k(g^{-1}\det\, g)F\rangle &= \int_{\mathcal{H}_2} \lambda(g^{-1}\det(g))^k \overline{\det(PZ'+Q)^{-k}} \det(g)^{-2k} \det(Y')^{k-3} \times \\ &\times F(Z') \overline{F((MZ'+N)(PZ'+Q)^{-1})} \mathrm{d}X_1' \mathrm{d}X_2' \mathrm{d}X_3' \mathrm{d}Y_1' \mathrm{d}Y_2' \mathrm{d}Y_3'.\end{aligned} \tag{3.36}$$

Now note that $\lambda(g)^k = \lambda(g^{-1}\det(g))^k \cdot \det(g)^{-k}$, and hence it is enough to show that

$$(\det(C(MZ'+N)(PZ'+Q)^{-1}+D))^{-k}\det(Y)^{k-3}\mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3\mathrm{d}Y_1\mathrm{d}Y_2\mathrm{d}Y_3 = \overline{\det(PZ'+Q)^{-k}}\det(g)^{-2k}\det(Y')^{k-3}\mathrm{d}X_1'\mathrm{d}X_2'\mathrm{d}X_3'\mathrm{d}Y_1'\mathrm{d}Y_2'\mathrm{d}Y_3'. \tag{3.37}$$

Now

$$\det(C(MZ'+N)(PZ'+Q)^{-1}+D))^{-k} = \det(C(MZ'+N)+D(PZ'+Q))^{-k}\det(PZ'+Q)^k. \tag{3.38}$$

But since

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} M & N \\ P & Q \end{pmatrix} = I_4,$$

we have that $CM + DP = 0$ and $CN + DQ = I$, which implies that

$$\det(C(MZ'+N)+D(PZ'+Q))^{-k} = 1.$$

Thus, it is enough to show the following two relations:

$$\det(PZ'+Q)\det(P\overline{Z'}+Q)\det(Y) = \det(g^{-1})\det(Y') \tag{3.39}$$

and

$$\det(Y)^{-3}\mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3\mathrm{d}Y_1\mathrm{d}Y_2\mathrm{d}Y_3 = \det(Y')^{-3}\mathrm{d}X_1'\mathrm{d}X_2'\mathrm{d}X_3'\mathrm{d}Y_1'\mathrm{d}Y_2'\mathrm{d}Y_3'. \tag{3.40}$$

Since $g^{-1} \in \mathrm{GSp}(4)$, we get that $M^tP = P^tM$, $N^tQ = Q^tN$ and $Q^tM - N^tP =$

$\lambda(g^{-1})I$. Then, (3.39) follows from the following matrix identity

$$\begin{aligned}(Q^tM - N^tP)(Z' - \overline{Z'})(PZ' + Q)^{-1} &= (P\overline{Z'} + Q)^t((MZ' + N)(PZ' + Q)^{-1} \\ &- (M\overline{Z'} + N)(P\overline{Z'} + Q)^{-1}),\end{aligned} \tag{3.41}$$

by applying the determinant (note that $Y = (Z - \overline{Z})/2i$). To show (3.41), note that it is equivalent to

$$(Q^tM - N^tP)(Z' - \overline{Z'}) = (P\overline{Z'} + Q)^t(MZ' + N - (M\overline{Z'} + N)(P\overline{Z'} + Q)^{-1}(PZ' + Q)).$$

But $(M\overline{Z'} + N)(P\overline{Z'} + Q)^{-1} = ((P\overline{Z'} + Q)^{-1})^t(M\overline{Z'} + N)^t$ since $g^{-1} \in \mathrm{GSp}(4)$. Thus (3.41) is equivalent to

$$\begin{aligned}\lambda(g^{-1})(Z' - \overline{Z'}) &= (Q^tM - N^tP)(Z' - \overline{Z'}) \\ &= (\overline{Z'}P^t + Q^t)(MZ' + N) - (\overline{Z'}M^t + N^t)(PZ' + Q),\end{aligned}$$

which can be verified through a direct computation.

To show (3.40) we use Proposition 2.9, Chapter 1 in [AZ90], which states that the volume element on the Siegel upper half plane $\mathcal{H}_2$ is invariant under all symplectic transformations. □

Corollary 3.1. *Let $k_1, k_2 \in K_\infty$ such that*

$$k_1 = \begin{pmatrix} A_1 & B_1 \\ -B_1 & A_1 \end{pmatrix}$$

and

$$k_2^{-1} \cdot det(k_2) = \begin{pmatrix} M_2 & N_2 \\ -N_2 & M_2 \end{pmatrix}.$$

Then we have that

$$\begin{aligned} s_k(k_1 g k_2) &= \overline{\lambda(k_1)^k det(A_1 - iB_1)^{-k} s_k(g) \times} \\ &\times \ \lambda\left(\begin{pmatrix} M_2 & N_2 \\ -N_2 & M_2 \end{pmatrix}\right)^k det(M_2 - iN_2)^{-k}. \end{aligned}$$

Proof. By Lemma 3.2 we have

$$\begin{aligned} s_k(k_1 g) &= \langle \mathcal{D}_k(k_1 g) v_0, v_0 \rangle = \langle \mathcal{D}_k(g) \circ \mathcal{D}_k(k_1) v_0, v_0 \rangle, \\ &= \lambda(k_1)^k \det(A_1 - iB_1)^{-k} \langle \mathcal{D}_k(g) v_0, v_0 \rangle = \lambda(k_1)^k \det(A_1 - iB_1)^{-k} s_k(g), \end{aligned}$$

and

$$\begin{aligned} s_k(g k_2) &= \langle \mathcal{D}_k(g k_2) v_0, v_0 \rangle = \langle v_0, \mathcal{D}_k(k_2^{-1} \cdot \det\ k_2 \cdot g^{-1} \det\ g) v_0 \rangle \\ &= \overline{\langle v_0, \mathcal{D}_k(g^{-1} \det\ g) \circ \mathcal{D}_k(k_2^{-1} \cdot \det(k_2)) v_0 \rangle} \\ &= \overline{\lambda\left(\begin{pmatrix} M_2 & N_2 \\ -N_2 & M_2 \end{pmatrix}\right)^k \det(M_2 - iN_2)^{-k}} \langle v_0, \mathcal{D}_k(g^{-1} \det\ g) v_0 \rangle \\ &= \lambda\left(\begin{pmatrix} M_2 & N_2 \\ -N_2 & M_2 \end{pmatrix}\right)^k \det(M_2 - iN_2)^{-k} \langle \mathcal{D}_k(g) v_0, v_0 \rangle \\ &= \lambda\left(\begin{pmatrix} M_2 & N_2 \\ -N_2 & M_2 \end{pmatrix}\right)^k \overline{\det(M_2 - iN_2)^{-k}} s_k(g). \end{aligned}$$

Putting the two identities together we get the desired result. $\square$

If $g \in \mathrm{GSp}(4)$ as in (3.32), we have that $A + iC$ is invertible and we can define

$$M + iN := (A + iC)^{-1}(B + iD).$$

We have by (3.33)

$$\begin{aligned} s_k(g) &= \lambda^k(g) \det(A + iC)^{-k} \int \det(Z + M + iN)^{-k} \det\left(\bar{Z} - iI\right)^{-k} \det Y^{k-3} \times \\ &\times \ dX_1 dX_2 dX_3 dY_1 dY_2 dY_3. \end{aligned} \tag{3.42}$$

By using the Cartan decomposition of GSp(4) and Corollary 3.1, we can reduce the computation of $s_k(g)$ to the case when g is diagonal.

Theorem 3.1. *If*

$$g = \begin{pmatrix} A & \\ & D \end{pmatrix}, \quad A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix},$$

then we have

$$s_k(g) = \frac{\pi^3 4^{3-k}}{(k-1)(k-2)(2k-3)} \frac{\det g^{k/2}}{(a_1 + d_1)^k (a_4 + d_4)^k}. \tag{3.43}$$

Proof. In this case $M = 0$ and

$$N = \begin{pmatrix} a_1^{-1} d_1 & 0 \\ 0 & a_2^{-1} d_2 \end{pmatrix} \equiv \begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix},$$

and the condition $g^T J g = \lambda(g) J$ translates to

$$a_1 d_1 = a_4 d_4 = \lambda(g).$$

The integral we want to compute is

$$\begin{aligned} I &= \lambda^k(g) \det(A)^{-k} \int \det(Z+iN)^{-k} \det\left(\bar{Z}-iI\right)^{-k} \det Y^{k-3} \times \\ &\times \; dX_1 dX_2 dX_3 dY_1 dY_2 dY_3. \end{aligned}$$

We first perform the X_1 integral. The second determinant has a pole at

$$X_1 \to \frac{-X_2^2 + 2iX_2Y_2 - iX_3Y_1 - iX_3 - Y_1Y_3 - Y_1 + Y_2^2 - Y_3 - 1}{-X_3 + iY_3 + i},$$

which is always in the upper half-plane, while the first determinant has a pole at

$$X_1 \to \frac{n_1n_2 - in_1X_3 + n_1Y_3 + n_2Y_1 + X_2^2 + 2iX_2Y_2 - iX_3Y_1 + Y_1Y_3 - Y_2^2}{in_2 + X_3 + iY_3},$$

which is always in the lower half-plane. We can thus perform the contour integral using the residue theorem and obtain

$$\begin{aligned} I &= 2\pi(-1)^{k-1}\binom{2(k-1)}{k-1} \det A^{-k}\lambda^k(g) \int \frac{(-X_3+iY_3+i)^{k-1}(in_2+X_3+iY_3)^{k-1}}{(\text{denominator}_1)^{2k-1}} \times \\ &\times \; \det Y^{k-3} dX_2 dX_3 dY_1 dY_2 dY_3. \end{aligned} \tag{3.44}$$

The denominator above has two poles, at

$$X_2 \to \frac{Y_2(2X_3 + i(n_2-1)) \pm \sqrt{\text{sqrt}_1}}{n_2 + 2Y_3 + 1},$$

where sqrt_1 is given by

$$\begin{aligned} \text{sqrt}_1 &= (iX_3 + Y_3 + 1)(-n_2 + iX_3 - Y_3) \times \\ &\times \; \left(n_1n_2 + 2Y_3(n_1 + 2Y_1 + 1) + n_1 + 2n_2Y_1 + n_2 + 2Y_1 - 4Y_2^2 + 1\right). \end{aligned}$$

We need to argue that these poles are always in different half-planes. Since dealing with the imaginary part of the square root is difficult, we will use an alternative argument. Suppose there is a value of the parameters $\{X_3, Y_1, Y_2, Y_3, n_1, n_2\}$ such that the poles are in the same half-plane. Then by closing the contour around the other half-plane the integral

$$I' = \int_{-\infty}^{\infty} \frac{dx_2}{\text{denominator}_1}$$

must equal 0. However, the real part of denominator$_1$ is

$$n_1n_2 + n_1X_3^2 + n_2X_2^2 + 2n_2Y_1 - n_2Y_2^2 + n_2 + X_2^2 - 4X_2X_3Y_2 + 2X_3^2Y_1 + X_3^2 - Y_2^2 > 0,$$

so the real part of I' cannot be zero. Thus for any value of $\{X_3, Y_1, Y_2, Y_3, n_1, n_2\}$, the two poles of $1/\text{denominator}_1$ must be in different half-planes, so the two poles of $1/\left(\text{denominator}_1\right)^{2k-1}$ must also be in different half-places (since the locations coincide). Thus, to compute I up to a sign it suffices to close the contour around either pole. With $p = 2k - 1$ we obtain that I is equal to

$$\begin{aligned} I &= i\pi^2 \frac{(-1)^{k-1}}{2^{2p-3}} \binom{2(k-1)}{k-1} \binom{2(p-1)}{p-1} \det A^{-k} \lambda^k(g) \times \\ &\times \int \frac{(1+n_2+2y_3)^{2k-2}}{\text{denominator}_2} \det Y^{k-3} dX_3 dY_1 dY_2 dY_3, \end{aligned}$$

where

$$\begin{aligned} \text{denominator}_2 &= (-X_3 + iY_3 + i)^{k-\frac{1}{2}} (in_2 + X_3 + iY_3)^{k-\frac{1}{2}} (1 + n_1 + n_2 + n_1n_2 + 2Y_1 \\ &\quad + 2n_2Y_1 - 4Y_2^2 + 2Y_3 + 2n_1Y_3 + 4Y_1Y_3)^{2k-\frac{3}{2}}. \end{aligned}$$

We now perform the X_3 integral. For $n_2 > 0$, $Y_3 > 0$, we have

$$\int_{-\infty}^{\infty} \frac{dX_3}{[-n_2 + i(1-n_2)X_3 - X_3^2 - Y_3 - n_2 Y_3 - Y_3^2]^{k-\frac{1}{2}}} =$$
$$i\sqrt{\pi}(-1)^k 2^{2k-2} \frac{\Gamma(k-1)}{\Gamma\left(k-\frac{1}{2}\right)} (n_2 + 2Y_3 + 1)^{2-2k},$$

so that I becomes

$$\begin{aligned} I &= \pi^{5/2} 2^{3-2k} \binom{2(k-1)}{k-1} \binom{2(p-1)}{p-1} \frac{\Gamma(k-1)}{\Gamma\left(k-\frac{1}{2}\right)} \det A^{-k} \lambda^k(g) \times \\ &\times \int \frac{\left(Y_1 Y_3 - Y_2^2\right)^{k-3}}{\text{denominator}_3} dY_1 dY_2 dY_3, \end{aligned}$$

with

$$\begin{aligned} \text{denominator}_3 &= \left(1 + n_1 + n_2 + n_1 n_2 + 2Y_1 + 2n_2 Y_1 \right. \\ &\quad \left. -4Y_2^2 + 2Y_3 + 2n_1 Y_3 + 4Y_1 Y_3\right)^{2k-\frac{3}{2}}. \end{aligned}$$

Denote

$$I_y = \int \frac{\left(Y_1 Y_3 - Y_2^2\right)^{k-3}}{\text{denominator}_3} dY_1 dY_2 dY_3.$$

We use the fact that

$$\int_{Y_2^2/Y_3}^{\infty} \frac{\left(Y_1 Y_3 - Y_2^2\right)^{k-3}}{\text{denominator}_3} dY_1 = -\frac{8^{2-k}(2k-3)(2k-1)Y_3^{2k-\frac{5}{2}}}{\sqrt{\pi}(4k-5)} \Gamma\left(\frac{7}{2} - 2k\right) \times$$
$$\times \quad \Gamma(2k-4)(n_2 + 2Y_3 + 1)^{2-k} \left((n_1+1)Y_3(n_2 + 2Y_3 + 1) + 2(n_2+1)Y_2^2\right)^{-k-\frac{1}{2}}.$$

Performing the Y_2 integral we obtain

$$\begin{aligned} I_y &= -2^{\frac{11}{2}-3k}(2k-3)(2k-1)(n_1+1)^{-k}\Gamma\left(\frac{7}{2}-2k\right)\Gamma(k)\Gamma(2k-4)\times \\ &\times \int_0^\infty \frac{Y_3^{k-\frac{5}{2}}(n_2+2Y_3+1)^{2-2k}}{(4k-5)\sqrt{n_2+1}\Gamma\left(k+\frac{1}{2}\right)}dY_3, \end{aligned}$$

which can be integrated to give

$$I_y = \sqrt{\pi}2^{11-6k}(k-1)\Gamma\left(\frac{5}{2}-2k\right)\Gamma(2k-4)(n_1+1)^{-k}(n_2+1)^{-k}.$$

Thus

$$\begin{aligned} I &= \pi^3 2^{14-8k}(k-1)\binom{2(k-1)}{k-1}\binom{2(p-1)}{p-1}\frac{\Gamma(k-1)\Gamma\left(\frac{5}{2}-2k\right)\Gamma(2k-4)}{\Gamma\left(k-\frac{1}{2}\right)}\times \\ &\times \frac{\det A^{-k}\lambda^k(g)}{(n_1+1)^k(n_2+1)^k}, \end{aligned}$$

which can be written as the expression in equation (3.43). □

Corollary 3.2. *For* $k \geq 3$ *and* g *as in* (3.32)*. We have*

$$s_k(g) = \frac{\pi^3 4^{3-k}}{(k-1)(k-2)(2k-3)}\frac{\det g^{k/2}}{\det\left[A+D-i(B-C)\right]^k}. \tag{3.45}$$

Proof. Follows through straightforward computations by Theorem 3.1, Lemma 3.2 and the Cartan decomposition. □

As a consequence, we will have

$$f_\infty(g) = c_k\frac{\det\, g^{k/2}}{\det[A+D+i(B-C)]^k} \tag{3.46}$$

if $\lambda(g) > 0$, and $f_\infty(g) = 0$ if $\lambda(g) < 0$, where c_k is some fixed constant that depends on the weight k.

3.4 Computing the double cosets

In this section, we will determine the double coset representatives for $H\backslash G/U$, where U is the unipotent radical of the Siegel parabolic. More specifically, we will show the following result:

Lemma 3.3. *The elements*

$$\eta(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda \\ 0 & 0 & \lambda & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \xi(\rho,\mu) = \begin{pmatrix} 0 & 0 & \mu & \rho \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ \rho & -\mu & 0 & 0 \end{pmatrix},$$

for $\lambda \in \mathbb{Q}^\star$, $\rho \in \mathbb{Q}$, $\mu \in \mathbb{Q}^\star$ constitute a complete list of double coset representatives for the space $H\backslash G/U$.

Proof. We have

$$P = \left\{ \begin{pmatrix} A & \\ & \lambda(A^{-1})^t \end{pmatrix} \begin{pmatrix} I_2 & X \\ & I_2 \end{pmatrix} : A \in \mathrm{GL}(2), \lambda \in \mathrm{GL}(1), X \in Sym_2 \right\}$$

denote the Siegel parabolic of G and $\overline{P}$ be is transpose. It has a Levi decomposition of the form

$$\overline{P} = \left\{ \begin{pmatrix} A & \\ & \mu(A^{-1})^t \end{pmatrix} \begin{pmatrix} I_2 & \\ Y & I_2 \end{pmatrix} : A \in \mathrm{GL}(2), \mu \in \mathrm{GL}(1), Y \in Sym_2 \right\}.$$

Let $H_1 = H \cap P$ and $H_2 = H \cap \overline{P}$.

A computation shows that

$$P = \bigcup_{s_2, t_2 \neq 0, y_1} \begin{pmatrix} 1 & s_2 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2 & 0 \\ 0 & 0 & -s_2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & y_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot H_1.$$

Indeed, let

$$p = \begin{pmatrix} m_1 & n_1 & m_1x_1 + n_1x_2 & m_1x_2 + n_1x_3 \\ p_1 & q_1 & p_1x_1 + q_1x_2 & p_1x_2 + q_1x_3 \\ 0 & 0 & \lambda q_1 & -\lambda p_1 \\ 0 & 0 & -\lambda n_1 & \lambda m_1 \end{pmatrix}$$

be an arbitrary element in P. This can be written as a product

$$\begin{pmatrix} 1 & \frac{dm_1p_1 - n_1q_1}{n_1p_1 - m_1q_1} & 0 & 0 \\ 0 & \frac{dp_1^2 - q_1^2}{n_1p_1 - m_1q_1} & 0 & 0 \\ 0 & 0 & \frac{dp_1^2 - q_1^2}{n_1p_1 - m_1q_1} & 0 \\ 0 & 0 & -\frac{dm_1p_1 - n_1q_1}{n_1p_1 - m_1q_1} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \frac{x_1 - dx_3}{\lambda} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot h_1,$$

where

$$h_1 = \begin{pmatrix} \frac{n_1p_1q_1 - m_1q_1^2}{dp_1^2 - q_1^2} & \frac{n_1p_1^2 - m_1p_1q_1}{dp_1^2 - q_1^2}d & \frac{d(n_1p_1 - m_1q_1)(p_1x_2 + q_1x_3)}{dp_1^2 - q_1^2} & \frac{(n_1p_1 - m_1q_1)(p_1x_1 + q_1x_2)}{dp_1^2 - q_1^2} \\ \frac{n_1p_1^2 - m_1p_1q_1}{dp_1^2 - q_1^2} & \frac{n_1p_1q_1 - m_1q_1^2}{dp_1^2 - q_1^2} & \frac{(n_1p_1 - m_1q_1)(p_1x_1 + q_1x_2)}{dp_1^2 - q_1^2} & \frac{(n_1p_1 - m_1q_1)(p_1x_2 + q_1x_3)}{dp_1^2 - q_1^2} \\ 0 & 0 & \frac{n_1p_1q_1\lambda - m_1q_1^2\lambda}{dp_1^2 - q_1^2} & -\frac{p_1(n_1p_1 - m_1q_1)\lambda}{dp_1^2 - q_1^2} \\ 0 & 0 & -\frac{p_1(n_1p_1 - m_1q_1)\lambda}{dp_1^2 - q_1^2}d & \frac{n_1p_1q_1\lambda - m_1q_1^2\lambda}{dp_1^2 - q_1^2} \end{pmatrix}.$$

Similarily, we can show that

$$\overline{P} = \bigcup_{s_1, t_1 \neq 0, y_3} H_2 \begin{pmatrix} 1 & s_1 & 0 & 0 \\ 0 & t_1 & 0 & 0 \\ 0 & 0 & t_1 & 0 \\ 0 & 0 & -s_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & y_3 & 0 & 1 \end{pmatrix}.$$

Thus, using Bruhat's decomposition (3.3), we have the following (non-disjoint) union

$$\begin{aligned} G &= \bigcup_i \bigcup_{s_1, t_1 \neq 0, y_3} \bigcup_{s_2 v_2 - u_2 t_2 \neq 0, \lambda \neq 0} H_2 \begin{pmatrix} 1 & s_1 & 0 & 0 \\ 0 & t_1 & 0 & 0 \\ 0 & 0 & t_1 & 0 \\ 0 & 0 & -s_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & y_3 & 0 & 1 \end{pmatrix} \times \\ &\times \ w_i \begin{pmatrix} s_2 & t_2 & 0 & 0 \\ u_2 & v_2 & 0 & 0 \\ 0 & 0 & \lambda v_2 & -\lambda u_2 \\ 0 & 0 & -\lambda t_2 & \lambda s_2 \end{pmatrix} U, \end{aligned}$$

with $i = 0, 1, 2$ and $w_0 := I_4$. So the distinct double coset representatives for $H \backslash G / U$ can be chosen from among elements of the form $T_i(s_1, t_1, y_3, s_2, t_2, u_2, v_2, \lambda)$, which are defined to be

$$\begin{pmatrix} 1 & s_1 & 0 & 0 \\ 0 & t_1 & 0 & 0 \\ 0 & 0 & t_1 & 0 \\ 0 & 0 & -s_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & y_3 & 0 & 1 \end{pmatrix} \cdot w_i \begin{pmatrix} s_2 & t_2 & 0 & 0 \\ u_2 & v_2 & 0 & 0 \\ 0 & 0 & \lambda v_2 & -\lambda u_2 \\ 0 & 0 & -\lambda t_2 & \lambda s_2 \end{pmatrix}.$$

So it is enough to show that each of these matrices are in the same equivalence class as one of the two types of cosets $\eta(\lambda)$ or $\xi(\rho,\mu)$.

We have

$$T_1(s_1,t_1,y_3,s_2,t_2,u_2,v_2,\lambda) := \begin{pmatrix} s_2+s_1u_2 & t_2+s_1v_2 & 0 & 0 \\ t_1u_2 & t_1v_2 & 0 & 0 \\ 0 & 0 & t_1v_2\lambda & -t_1u_2\lambda \\ u_2y_3 & v_2y_3 & -t_2\lambda - s_1v_2\lambda & s_2\lambda + s_1u_2\lambda \end{pmatrix}$$

and

$$T_2(s_1,t_1,y_3,s_2,t_2,u_2,v_2,\lambda) := \begin{pmatrix} 0 & 0 & -s_1t_2\lambda + v_2\lambda & s_1s_2\lambda - u_2\lambda \\ 0 & 0 & -t_1t_2\lambda & s_2t_1\lambda \\ -s_2t_1 & -t_1t_2 & 0 & 0 \\ s_1s_2-u_2 & s_1t_2-v_2 & -t_2y_3\lambda & s_2y_3\lambda \end{pmatrix}$$

and

$$T_3(s_1,t_1,y_3,s_2,t_2,u_2,v_2,\lambda) := \begin{pmatrix} s_2 & t_2 & -s_1t_2\lambda & s_1s_2\lambda \\ 0 & 0 & -t_1t_2\lambda & s_2t_1\lambda \\ 0 & 0 & t_1v_2\lambda & -t_1u_2\lambda \\ -u_2 & -v_2 & -s_1v_2\lambda - t_2y_3\lambda & s_1u_2\lambda + s_2y_3\lambda \end{pmatrix}.$$

For T_1 we have two cases. If $y_3 = 0$ then we have that $T_1(s_1,t_1,0,s_2,t_2,u_2,v_2,\lambda)$ for $\lambda \neq 0$ is always in the same coset as $T_1(s_1,t_1,0,s_2,t_2,u_2,v_2,1)$. If $y_3 \neq 0$ then we have that $T_1(s_1,t_1,y_3,s_2,t_2,u_2,v_2,\lambda)$ is always in the same double coset as $T_1(0,t_1,y_3,s_2,t_2,u_2,v_2,\lambda)$.

For T_2 we have that $T_2(s_1,t_1,y_3,s_2,t_2,u_2,v_2,\lambda)$ is in the same double coset as

$T_2(s_1,t_1,0,s_2,t_2,u_2,v_2,1)$. Finally, we have that $T_3(s_1,t_1,y_3,s_2,t_2,u_2,v_2,\lambda)$ is in the same coset as $T_3(0,t_1,0,s_2,t_2,u_2,v_2,\lambda)$. This is useful to simplify the computations.

Now, we have that T_3 is always in the same coset as some $\eta(\lambda')$ for an appropriate λ':

$$\begin{pmatrix} \frac{v_2}{s_2v_2-t_2u_2} & 0 & 0 & \frac{t_2}{s_2v_2-t_2u_2} \\ 0 & \frac{v_2}{s_2v_2-t_2u_2} & \frac{t_2}{s_2v_2-t_2u_2} & 0 \\ 0 & \frac{u_2}{s_2v_2-t_2u_2} & \frac{s_2}{s_2v_2-t_2u_2} & 0 \\ \frac{u_2}{s_2v_2-t_2u_2} & 0 & 0 & \frac{s_2}{s_2v_2-t_2u_2} \end{pmatrix} T_3(0,t_1,0,s_2,t_2,u_2,v_2,\lambda) = \eta(t_1\lambda).$$

In addition, that T_2 is always in the same coset as some $\xi(\rho,\mu)$ for an appropriate ρ and μ. Indeed,

$$\begin{aligned} & \begin{pmatrix} \frac{s_1t_2-v_2}{t_1t_2u_2-s_2t_1v_2} & \frac{dt_2}{s_2v_2-t_2u_2} & 0 & 0 \\ \frac{t_2}{s_2v_2-t_2u_2} & \frac{s_1t_2-v_2}{t_1t_2u_2-s_2t_1v_2} & 0 & 0 \\ \frac{ms_2+nt_2}{s_2v_2-t_2u_2} & \frac{ms_1s_2+ns_1t_2-mu_2-nv_2}{t_1t_2u_2-s_2t_1v_2} & \frac{s_1t_2-v_2}{t_1t_2u_2-s_2t_1v_2} & \frac{t_2}{t_2u_2-s_2v_2} \\ \frac{ms_1s_2+ns_1t_2-mu_2-nv_2}{t_1t_2u_2-s_2t_1v_2} & \frac{d(ms_2+nt_2)}{s_2v_2-t_2u_2} & \frac{dt_2}{t_2u_2-s_2v_2} & \frac{s_1t_2-v_2}{t_1t_2u_2-s_2t_1v_2} \end{pmatrix} \times \\ \times\quad & T_2(s_1,t_1,0,s_2,t_2,u_2,v_2,1) \times \\ \times\quad & \begin{pmatrix} 1 & 0 & m & n \\ 0 & 1 & n & \frac{2n\left(-s_2t_2s_1^2+t_2u_2s_1+s_2v_2s_1+ds_2t_1^2t_2-u_2v_2\right)-m\left(\left(s_1^2-dt_1^2\right)s_2^2-2s_1u_2s_2+u_2^2\right)}{\left(s_1^2-dt_1^2\right)t_2^2-2s_1v_2t_2+v_2^2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ =\quad & \xi\left(\frac{-ds_2t_1^2t_2+s_1^2s_2t_2-s_1s_2v_2-s_1t_2u_2+u_2v_2}{t_1(t_2u_2-s_2v_2)}, \frac{-dt_1^2t_2^2+s_1^2t_2^2-2s_1t_2v_2+v_2^2}{s_2t_1v_2-t_1t_2u_2}\right). \end{aligned}$$

If $y_3=0$ we have that T_1 is in the same coset as $\xi(\rho,\mu)$ for an appropriate ρ and μ.

Indeed, we have

$$\begin{pmatrix} 0 & 0 & \frac{dv_2}{s_2v_2-t_2u_2} & -\frac{t_2+s_1v_2}{t_1t_2u_2-s_2t_1v_2} \\ 0 & 0 & -\frac{t_2+s_1v_2}{t_1t_2u_2-s_2t_1v_2} & \frac{v_2}{s_2v_2-t_2u_2} \\ \frac{v_2}{t_2u_2-s_2v_2} & -\frac{t_2+s_1v_2}{t_1t_2u_2-s_2t_1v_2} & -\frac{ms_2+nt_2+ms_1u_2+ns_1v_2}{t_1t_2u_2-s_2t_1v_2} & \frac{mu_2+nv_2}{s_2v_2-t_2u_2} \\ -\frac{t_2+s_1v_2}{t_1t_2u_2-s_2t_1v_2} & \frac{dv_2}{t_2u_2-s_2v_2} & \frac{d(mu_2+nv_2)}{s_2v_2-t_2u_2} & -\frac{ms_2+nt_2+ms_1u_2+ns_1v_2}{t_1t_2u_2-s_2t_1v_2} \end{pmatrix} \times$$

$$\times \quad T_1(s_1,t_1,0,s_2,t_2,u_2,v_2,1)\times$$

$$\times \quad \begin{pmatrix} 1 & 0 & m & n \\ 0 & 1 & n & -\frac{m\left(s_2^2+2s_1u_2s_2+\left(s_1^2-dt_1^2\right)u_2^2\right)+2n\left(s_2(t_2+s_1v_2)+u_2\left(v_2s_1^2+t_2s_1-dt_1^2v_2\right)\right)}{t_2^2+2s_1v_2t_2+\left(s_1^2-dt_1^2\right)v_2^2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \quad \xi\left(\frac{-dt_1^2u_2v_2+s_1^2u_2v_2+s_1s_2v_2+s_1t_2u_2+s_2t_2}{s_2t_1v_2-t_1t_2u_2}, \frac{-dt_1^2v_2^2+s_1^2v_2^2+2s_1t_2v_2+t_2^2}{t_1(t_2u_2-s_2v_2)}\right).$$

Now if $y_3 \neq 0$ we can show that T_1 is in the same coset as $\eta(\lambda')$ for some λ. Indeed, we have

$$\begin{pmatrix} \frac{v_2}{s_2v_2-t_2u_2} & 0 & \frac{dt_1v_2}{(t_2u_2-s_2v_2)y_3} & \frac{t_2}{(t_2u_2-s_2v_2)y_3} \\ 0 & \frac{v_2}{s_2v_2-t_2u_2} & \frac{t_2}{(t_2u_2-s_2v_2)y_3} & \frac{t_1v_2}{(t_2u_2-s_2v_2)y_3} \\ 0 & -\frac{u_2}{t_2u_2-s_2v_2} & \frac{s_2}{(t_2u_2-s_2v_2)y_3} & \frac{t_1u_2}{(t_2u_2-s_2v_2)y_3} \\ -\frac{u_2}{t_2u_2-s_2v_2} & 0 & \frac{dt_1u_2}{(t_2u_2-s_2v_2)y_3} & \frac{s_2}{(t_2u_2-s_2v_2)y_3} \end{pmatrix} T_1(0,t_1,y_3,s_2,t_2,u_2,v_2,1)\times$$

$$\times \quad \begin{pmatrix} 1 & 0 & \frac{t_2^2-dt_1^2v_2^2}{(t_2u_2-s_2v_2)y_3} & -\frac{s_2t_2-dt_1^2u_2v_2}{(t_2u_2-s_2v_2)y_3} \\ 0 & 1 & -\frac{s_2t_2-dt_1^2u_2v_2}{(t_2u_2-s_2v_2)y_3} & -\frac{dt_1^2u_2^2-s_2^2}{(t_2u_2-s_2v_2)y_3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \eta\left(-\frac{t_1}{y_3}\right).$$

Finally, a straightforward computation shows that $\eta(\lambda)$ and $\eta(\lambda')$ for $\lambda \neq \lambda'$ are never in the same coset. Similarly, for $(\rho, \mu) \neq (\rho', \mu')$ we have that $\xi(\rho, \mu)$ and $\xi(\rho', \mu')$ are never in the same coset. □

3.5 Spectral side

Let f be the test function chosen in Section 3.3. In this section we will study the properties of $\rho(f)$ and give the spectral decomposition of the kernel $K_f(x, y)$ and of the linear functional $I(f)$.

Define the compact open subgroup $K_{N,S}$ of $\mathrm{GSp}(4, \mathbb{A}_{\mathrm{fin}})$ (using the notations from Section 3.3)

$$K_{N,S} = \prod_{p \notin S_0} K_p \times K_0(N) \cdot \times \prod_{p \in S} K(2n_p)_p. \tag{3.47}$$

We let $\mathcal{A}_k^S(N)$ denote the subspace of cuspidal representations of $G(\mathbb{A})$ given by

$$\mathcal{A}_k^S(N) = \bigoplus_{\substack{\pi_\infty = \mathcal{D}_k \\ \pi_{\mathrm{fin}}^{K_{N,S}} \neq 0}} \mathbb{C}v_0 \otimes \pi_{\mathrm{fin}}^{K_{N,S}}, \tag{3.48}$$

where $\pi_{\mathrm{fin}}^{K_{N,S}}$ is the space of $K_{N,S}$-fixed vectors in π_{fin} and v_0 is the lowest weight vector which generates the minimal K-type $\tau_{k,k}$ of the holomorphic discrete series $\mathcal{D}_k$.

For the test function f chosen in Section 3.3, we will find that $\rho(f)$ annihilates $(\mathcal{A}_k^S(N))^\perp$ and maps $\mathcal{A}_k^S(N)$ to itself. We will generalize some of the computations in Chapter 13 of [KL06] from GL(2) to our case. Recall that we can write $f = f_{\mathrm{fin}} \times f_\infty$.

Lemma 3.4. *For any $\varphi \in L^2(G(\mathbb{Q} \backslash G(\mathbb{A}))$, $\rho(f)\varphi$ is cuspidal.*

Proof. Since bounded functions φ in L^2 are dense in the space, it is enough to show that for bounded functions $\varphi \in L^2$ we have $\rho(f)\varphi \in L_0^2$.

Consider φ bounded. This assumption, along with the fact that f is L^1 integrable will provide a gurantee that

$$\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \rho(f)\varphi(ug)\mathrm{d}u$$

is absolutely convergent. We will show that the integral is in fact zero. By definition, we can write

$$\begin{aligned}
\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \rho(f)\varphi(ug)\mathrm{d}u &= \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \int_{Z(\mathbb{A})\backslash G(\mathbb{A})} f(x)\varphi(ugx)\mathrm{d}x\mathrm{d}u \\
&= \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \int_{Z(\mathbb{A})\backslash G(\mathbb{A})} f(g^{-1}u^{-1}x)\varphi(x)\mathrm{d}x\mathrm{d}u \\
&= \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \int_{U(\mathbb{Q})Z(\mathbb{A})\backslash G(\mathbb{A})} \sum_{\gamma \in U(\mathbb{Q})} f(g^{-1}u^{-1}\gamma x)\varphi(x)\mathrm{d}x\mathrm{d}u.
\end{aligned}$$

Switching the order of integration we get

$$\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \rho(f)\varphi(ug)\mathrm{d}u = \int_{U(\mathbb{Q})Z(\mathbb{A})\backslash G(\mathbb{A})} \left[\int_{U(\mathbb{A})} f(g^{-1}ux)\mathrm{d}u\right] \varphi(x)\mathrm{d}x. \tag{3.49}$$

But now note that

$$\int_{U(\mathbb{A})} f(g^{-1}ux)\mathrm{d}u = \int_{U(\mathbb{R})} f_\infty(g^{-1}ux)\mathrm{d}u \cdot \prod_{v<\infty} \int_{U(\mathbb{Q}_v)} f_v(g^{-1}u_v x)\mathrm{d}u_v.$$

If U_0 is any subgroup of U, we have

$$\int_{U(\mathbb{R})} f_\infty(g^{-1}ux)\mathrm{d}u = \int_{U_0(\mathbb{R})\backslash U(\mathbb{R})} \int_{U_0(\mathbb{R})} f_\infty(g^{-1}\gamma ux)\mathrm{d}\gamma\mathrm{d}u.$$

We choose $U_0(\mathbb{R})$ to be the set of matrices of the form

$$\gamma(t) := \begin{pmatrix} 1 & & t & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

with $t \in \mathbb{R}$. Then we have that the entries of $g^{-1}\gamma(t)ux$ are linear functions of t and in addition we have that $f_\infty(g^{-1}\gamma(t)ux)$ is a constant multiple of

$$\frac{1}{(\alpha t + \beta)^k}$$

with $\alpha, \beta \in \mathbb{C}$. Then we have

$$\int_{U_0(\mathbb{R})} f_\infty(g^{-1}\gamma ux)\mathrm{d}\gamma \mathrm{d}u = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{(\alpha t + \beta)^k} = 0,$$

and hence we conclude that $\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \rho(f)\varphi(ug)\mathrm{d}u = 0$.

□

Corollary 3.3. $\rho(f)$ *annihilates* $(L_0^2)^\perp$.

Proof. A straightforward computation shows that the adjoint $f^\star$ also satisfies the property $\rho(f)^\star = \rho(f^\star) : L^2 \to L_0^2$. This means that $\rho(f)$ annihilates $(L_0^2)^\perp$. □

Theorem 3.2. $\rho(f) \subset \mathcal{A}_k^S(N)$.

Proof. We may assume $\varphi \in L_0^2$ (since $\rho(f)$ annihilates $(L_0^2)^\perp$). Writing L_0^2 as a direct sum of irreducible cuspidal representations (π, V_π) and using the fact that the space $\mathcal{A}_k^S(N)$ is closed, we may assume that $\varphi \in \pi$ for some irreducible cuspidal representation π.

Writing $\varphi = \varphi_\infty \otimes \varphi_{\text{fin}}$, we have

$$\rho(f)\varphi = \pi_\infty(f_\infty)\varphi_\infty \otimes \pi_{\text{fin}}(f_{\text{fin}})\varphi_{\text{fin}}.$$

In order to have $\pi_\infty(f_\infty)\varphi_\infty \neq 0$ we must have $\pi_\infty \cong \mathcal{D}_k$, in which case $\mathcal{D}_k(f_\infty)\varphi_\infty \in \mathbb{C}v_0$ with v_0 the lowest weight vector of $\mathcal{D}_k$. Now, because f_{fin} is $K_{N,S}$-invariant we have that $\pi_{\text{fin}}(f_{\text{fin}})\varphi_{\text{fin}}$ is $K_{N,S}$-invariant thus we get that

$$\rho(f)\varphi \in \mathcal{A}_k^S(N).$$

$\square$

In fact, $\rho(f)$ annihilates $(\mathcal{A}_k^S(N))^\perp$. This is because $\rho(f)^\star$ also satisfies the property that $\rho(f)^\star \subset \mathcal{A}_k^S(N)$.

Now we give the spectral decomposition of the linear functional $I(f)$.

Theorem 3.3.

$$I(f) = \sum_\pi m_\pi \sum_{\varphi_i \in \pi} \frac{1}{\langle \varphi_i, \varphi_i \rangle} L(\varphi_i, \Phi, \mu, \nu, s)\overline{\varphi}_{i,\psi} \prod_{p\in S} a_i, p,$$

where $\varphi_{i,\psi}$ is the Fourier coefficient of φ_i with respect to character ψ (see eq. (3.10)) and $a_{i,p}$ is the eigenvalue such that $\rho(f_i)\varphi_{i,p} = a_{i,p}\varphi_{i,p}$ for $p \in S$. The outer sum is over π in $\mathcal{A}_k^S(N)$ and the inner sum is over an orthogonal basis of π.

Proof. We can write the spectral decomposition of the kernel as follows:

$$K(x, y) = \sum_\pi m_\pi \sum_{\varphi_i \in \pi} \frac{(\rho(f)\varphi_i)(x)\overline{\varphi_i(y)}}{\langle \varphi_i, \varphi_i \rangle_{\mathbb{A}}},$$

with the outer sum over π in $\mathcal{A}_k^S(N)$ and the inner sum over an orthogonal basis of π.

This gives a spectral decomposition

$$I(f) = \sum_{\pi} m_\pi \sum_{\{\varphi_i\}} \frac{1}{\langle \varphi_i, \varphi_i \rangle} \int_{\tilde{H}(\mathbb{Q})\backslash \tilde{H}(\mathbb{A})} \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} (\rho(f)\varphi_i)(x)\overline{\varphi_i(y)}\mathcal{E}_s^\Phi(x)\psi(y)\mathrm{d}x\ \mathrm{d}y.$$

After a separation of variables, we get

$$I(f) = \sum_{\pi} m_\pi \sum_{\{\varphi_i\}} \frac{1}{\langle \varphi_i, \varphi_i \rangle} \left(\int_{(\tilde{H}(\mathbb{Q})\backslash \tilde{H}(\mathbb{A}))} (\rho(f)\varphi_i)(x)\mathcal{E}_s^\Phi(x)\mathrm{d}x \right) \left(\int_{(N(\mathbb{Q})\backslash N(\mathbb{A})} \overline{\varphi_i(y)}\psi(y)\mathrm{d}y \right).$$

We can now compute $\rho(f)$ on an element $\varphi \in \mathcal{A}_k^S(N)$ belonging to a cuspidal representation π. We can write

$$f = f_\infty \times f^S \times \prod_{p\in S} f_p$$

and

$$\varphi = \varphi_\infty \otimes \varphi^S \otimes \bigotimes_{p\in S} \varphi_p.$$

Then we must have that

$$\rho(f)\varphi = \mathcal{D}_k(f_\infty)\varphi_\infty \otimes \pi^S(f^S)\varphi^S \otimes \bigotimes \pi_p(f_p)\varphi_p.$$

By the properties of the matrix coefficient we get $\mathcal{D}_k(f_\infty)\varphi_\infty = \varphi_\infty$. Now since $f^S = f_N \times f^{S_0}$ is such that f^{S_0} is the characteristic function of $\prod_{p\notin S_0} ZK_p$ and f_N is the characteristic function of $Z_N\backslash Z_N K_0(N)_N$ divided by the measure of

$Z_N \backslash Z_N K_0(N)_N$

$$\pi^S(f^S)\varphi^S = \varphi^S.$$

Finally, if we have that $p \in S$, we have that f_p is bi-$K(2n_p)_p$-invariant, and hence φ_p is an eigenvector and let's call the eigenvalue a_p. This eigenvalue is given by

$$a_p = f_p^\vee(t_p),$$

where $f_p^\vee$ is the Satake transform of f_p in the Hecke algebra of locally constant compactly supported bi-$K(2n_p)_p$-invariant functions on $\mathrm{GSp}(4, \mathbb{Q}_p)$, and t_p is the Satake parameter of π_p.

Thus, we can conclude

$$\rho(f)\varphi = \left(\prod_{p \in S} a_p\right)\varphi$$

and the conclusion follows. □

Corollary 3.4. *If for $s = 1/2$ and weight $k \to \infty$ we have that $I(f) \neq 0$ then it would imply that that $L(\pi \otimes \mu, 1/2) \neq 0$ for infinitely many Siegel eigenforms π.*

Proof. If π is a holomorphic cuspidal automorphic representation of G and $\varphi \in \pi$ then $L(\varphi, \Phi, \mu, \nu, s)$ has an Euler product with local factors that are at almost all places given by $L(\pi_v \otimes \mu, s)$. More precisely (see [Har04]),

$$\begin{aligned} L(\varphi, \Phi, \mu, \nu, s) &= \alpha(\pi, d, \nu) \prod_{v \in T} L_v(\varphi, \Phi, \mu, \nu, s) \prod_{v \notin T} L(\pi_v \otimes \mu, s) \\ &= \alpha(\pi, d, \nu) \prod_{v \in T} L_v(\varphi, \Phi, \mu, \nu, s) \frac{L(\pi \otimes \mu, s)}{\prod_{v \in T} L_v(\pi \otimes \mu, s)}. \end{aligned}$$

The conclusion then follows by Theorem 3.3. □

3.6 Geometric side

For $\delta \in \tilde{G}$ define the subgroup of $\tilde{H} \times U$:

$$C_\delta = \{(h,u) \in \tilde{H} \times U : h^{-1}\delta u = \delta\}.$$

We split the sum over γ in the kernel into sums over double cosets in the following way:

$$\sum_{\gamma \in \tilde{G}(\mathbb{Q})} f(h^{-1}\gamma u) = \sum_{\delta \in \tilde{H}(\mathbb{Q})\backslash \tilde{G}(\mathbb{Q})/U(\mathbb{Q})} \sum_{(h_0,u_0) \in C_\delta(\mathbb{Q})\backslash(\tilde{H}(\mathbb{Q})\times U(\mathbb{Q}))} f(h^{-1}h_0^{-1}\delta u_0 u).$$

We then get that

$$\begin{aligned} I(f) &= \sum_{\{\delta\}} \int_{Z(\mathbb{A})H(\mathbb{Q})\backslash H(\mathbb{A})} \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \sum_{(x_0,y_0) \in C_\delta(\mathbb{Q})\backslash(\tilde{H}(\mathbb{Q})\times U(\mathbb{Q}))} f((h_0h)^{-1}\delta(u_0u)) \times \\ &\times \mathcal{E}_s^\phi(h_0h)\psi(u_0u)\mathrm{d}u\ \mathrm{d}h, \end{aligned}$$

where the first sum is over the double coset representatives $\delta \in \tilde{H}(\mathbb{Q})\backslash\tilde{G}(\mathbb{Q})/U(\mathbb{Q})$. Then we get that

$$I(f) = \sum_{\{\delta\}} I(\delta, f),$$

where

$$I(\delta, f) = \int_{C_\delta(\mathbb{Q})\backslash \tilde{H}(\mathbb{A})\times U(\mathbb{A})} f(h^{-1}\delta u)\mathcal{E}_s^\phi(h)\psi(u)\mathrm{d}h\ \mathrm{d}u.$$

We write

$$\begin{aligned} I(\delta, f) &= \int_{C_\delta(\mathbb{A})\backslash(\tilde{H}(\mathbb{A})\times U(\mathbb{A}))} \int_{(C_\delta(\mathbb{Q})\backslash C_\delta(\mathbb{A}))_1} f((zh)^{-1}\delta(\delta^{-1}z\delta)u) \times \\ &\times \mathcal{E}_s^\phi(zh)\psi(\delta^{-1}z\delta y)\mathrm{d}h\ \mathrm{d}u\ \mathrm{d}z, \end{aligned}$$

where $(C_\delta(\mathbb{Q})\backslash C_\delta(\mathbb{A}))_1$ is the projection on the first component, and hence

$$I(\delta, f) = \int_{C_\delta(\mathbb{A})\backslash(\tilde{H}(\mathbb{A})\times U(\mathbb{A}))} f(h^{-1}\delta u)\cdot\left(\int_{(C_\delta(\mathbb{Q})\backslash C_\delta(\mathbb{A}))_1} \mathcal{E}_s^\phi(zh)\psi(\delta^{-1}z\delta u)\mathrm{d}z\right)\mathrm{d}u\ \mathrm{d}h.$$

In principle, there are two types of cosets: regular and singular. There are infinitely many regular cosets and finitely many singular ones, but the dominant terms typically come from the singular cosets.

Lemma 3.5. *$I(\eta(\lambda), f) = 0$ for all $\lambda \in \mathbb{Q}^\star$.*

Proof. Since $(C_{\eta(\lambda)}(\mathbb{Q})\backslash C_{\eta(\lambda)}(\mathbb{A}))_1$ is the identity matrix

$$I(\eta(\lambda), f) = \int_{\tilde{H}(\mathbb{A})\times U(\mathbb{A})} f(h^{-1}\eta(\lambda)u)\psi(u)\cdot\mathcal{E}_s^\phi(h)\mathrm{d}h\ \mathrm{d}u. \tag{3.50}$$

We have

$$\mathcal{E}_s^\phi(g) = \sum_{\gamma\in\tilde{B}(F)\backslash\mathrm{GL}(2,F)} \phi_s(\gamma g),$$

where $\tilde{B}$ is the Borel subgroup of GL(2).

We can define the β-th local Whittaker integral

$$W_\beta(\phi_v, g_v) = \int_{F_v} \phi_v\left(w_0\begin{pmatrix}1 & x_v\\ 0 & 1\end{pmatrix}g_v\right)\vartheta(-\beta x_v)\mathrm{d}x_v$$

and the intertwining operator

$$(M_{w_0}\phi_v)(g_v) = \int_{F_v} \phi_v\left(w_0\begin{pmatrix}1 & x_v\\ 0 & 1\end{pmatrix}g_v\right)\mathrm{d}x_v,$$

where

$$w_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then we have the Fourier expansion

$$\mathcal{E}_s^{\phi}(g) = \phi_s(g) + M_{w_0}\phi(g) + \sum_{\beta \in F} W_\beta(g), \tag{3.51}$$

where

$$M_{w_0}\phi(g) = \frac{1}{\sqrt{D_F}} \prod_v M_{w_0}\phi_v(g_v),$$

and

$$W_\beta(g) = \frac{1}{\sqrt{D_F}} \prod_v W_\beta(\phi_v, g_v),$$

with D_F is the discriminant of F.

Using the Fourier expansion (3.51), we have that (3.50) can be expressed as a sum of factorizable integrals

$$I(\eta(\lambda), f) = I_1 + I_2 + \sum_{\beta \in F} I_\beta,$$

where

$$I_1 = \int_{\tilde{H}(\mathbb{A}) \times U(\mathbb{A})} f(h^{-1}\eta(\lambda)u)\psi(u) \cdot \phi_s(h) \mathrm{d}h \ \mathrm{d}u = \prod_v I_{1,v},$$

$$I_2 = \int_{\tilde{H}(\mathbb{A}) \times U(\mathbb{A})} f(h^{-1}\eta(\lambda)u)\psi(u) \cdot M_{w_0}\phi(h) \mathrm{d}h \ \mathrm{d}u = \prod_v I_{2,v},$$

$$I_\beta = \int_{\tilde{H}(\mathbb{A}) \times U(\mathbb{A})} f(h^{-1}\eta(\lambda)u)\psi(u) \cdot W_\beta(h) \mathrm{d}h \ \mathrm{d}u = \prod_v I_{\beta,v}.$$

Consider now place $v = N$. We have that f_N is the characteristic function of

$K_0(N)Z_N$. We have by assumption that $(N,d)=1$ and $(N,2)=1$. We will show that $I_{1,N}=I_{2,N}=I_{\beta,N}=0$, which will imply that $I(\eta(\lambda,f)=0$.

We will show that $h^{-1}\eta(\lambda)u\notin K_0(N)Z_N$, which implies that $f_N(h^{-1}\eta(\lambda)u)=0$ for all $h\in\tilde{H}(\mathbb{Q}_N)$ and $u\in U(\mathbb{Q}_N)$. Writing, $h=h'\cdot z$ with $z\in Z_N$, we can reduce this to showing that $h^{-1}\eta(\lambda)u\notin K_0(N)\subset \mathrm{GSp}(4,\mathbb{Z}_N)$.

Let

$$h^{-1}=\begin{pmatrix} a_1 & b_1 d & \frac{a_2}{2} & \frac{b_2}{2} \\ b_1 & a_1 & \frac{b_2}{2} & \frac{a_2}{2d} \\ 2a_3 & 2b_3 d & a_4 & b_4 \\ 2b_3 d & 2a_3 d & b_4 d & a_4 \end{pmatrix}\in\tilde{H},$$

and

$$u=\begin{pmatrix} 1 & 0 & r & s \\ 0 & 1 & s & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\in U.$$

Then

$$h^{-1}\eta(\lambda)u=\begin{pmatrix} a_1 & -\frac{b_2}{2} & a_1 r-\frac{b_2 s}{2}+\frac{a_2\lambda}{2} & a_1 s-\frac{b_2 t}{2}+b_1 d\lambda \\ b_1 & -\frac{a_2}{2d} & b_1 r-\frac{a_2 s}{2d}+\frac{b_2\lambda}{2} & b_1 s-\frac{a_2 t}{2d}+a_1\lambda \\ 2a_3 & -b_4 & 2a_3 r-b_4 s+a_4\lambda & 2a_3 s-b_4 t+2b_3 d\lambda \\ 2b_3 d & -a_4 & 2b_3 dr-a_4 s+b_4 d\lambda & 2b_3 ds-a_4 t+2a_3 d\lambda \end{pmatrix}.$$

In order to have that $h^{-1}\eta(\lambda)u\in\mathrm{GSp}(4,\mathbb{Z}_v)$, with $v=N$, we need to have

$$v(a_1),v(b_1),v(a_2),v(b_2),v(a_3),v(b_3),v(a_4),v(b_4)\geq 0,$$

i.e. $h^{-1}\in M_4(\mathbb{Z}_v)\cap\mathrm{GSp}(4,\mathbb{Q}_v)$. The determinant of $h^{-1}\eta(\lambda)u$ has to be a unit, and hence $\det(h^{-1})\lambda^2$ is a unit. Then $2v(\lambda) = v(\det(h))$.

Now note that if $h^{-1} \in M_4(\mathbb{Z}_v) \cap \mathrm{GSp}(4, \mathbb{Q}_v)$ then $h \cdot \det(h^{-1}) \in M_4(\mathbb{Z}_v) \cap \mathrm{GSp}(4, \mathbb{Q}_v)$. We get

$$\eta(\lambda)u \cdot \det(h^{-1}) \in M_4(\mathbb{Z}_v) \cap \mathrm{GSp}(4, \mathbb{Q}_v).$$

So, we must have that

$$\eta(\lambda)u = \begin{pmatrix} 1 & 0 & r & s \\ 0 & 0 & 0 & \lambda \\ 0 & 0 & \lambda & 0 \\ 0 & -1 & -s & -t \end{pmatrix} \in \det(h)M_4(\mathbb{Z}_v) \cap \mathrm{GSp}(4, \mathbb{Q}_v).$$

But then $0, v(r), v(s), v(t), v(\lambda) \geq 2v(\lambda)$. Thus, we get the constraints that

$$v(\lambda) \leq 0 \text{ and } v(r), v(s), v(t) \geq 2v(\lambda),$$

the entries of matrix h have valuation $\geq 2v(\lambda)$ and the entries of h^{-1} have valuation ≥ 0 and $\det(h) = 2v(\lambda)$.

However, it is actually the case that λ must be a unit, i.e. $v(\lambda) = 0$. To see this we will use the Iwasawa decomposition. We have that $\mathrm{GSp}(4, \mathbb{Q}_v) = \mathrm{GSp}(4, \mathbb{Z}_v) \cdot B(\mathbb{Q}_v)$, where B is the Borel subgroup. An arbitrary element of B in the Iwasawa decomposition can be chosen to be

$$b = \begin{pmatrix} x_1 & 0 & x_3 & \frac{x_1y_2}{y_1} \\ 0 & y_1 & y_2 & y_3 \\ 0 & 0 & c_1 & 0 \\ 0 & 0 & 0 & u_4 \end{pmatrix}.$$

Since we are looking at elements in $\tilde{G}$, we can assume without loss of generality that $c_1 = 1$.

We can write $h^{-1} = kb$ with $k \in \mathrm{GSp}(4, \mathbb{Z}_v)$ and b an element of the Borel subgroup as above. So then the condition that $h^{-1}\eta(\lambda)u \in \mathrm{GSp}(4, \mathbb{Z}_v)$ is equivalent to $b\eta(\lambda)u \in \mathrm{GSp}(4, \mathbb{Z}_v)$. But this implies that

$$b\eta(\lambda)u = \begin{pmatrix} x_1 & -\frac{x_1y_2}{y_1} & rx_1 - \frac{sy_2x_1}{y_1} + \lambda x_3 & sx_1 - \frac{tx_1y_2}{y_1} \\ 0 & -y_3 & \lambda y_2 - sy_3 & \lambda y_1 - ty_3 \\ 0 & 0 & \lambda & 0 \\ 0 & -u_4 & -su_4 & -tu_4 \end{pmatrix} \in \mathrm{GSp}(4, \mathbb{Z}_v).$$

In particular, we conclude that $\lambda \in \mathbb{Z}_v$. But previously we had the condition that $v(\lambda) \leq 0$. Thus $\lambda \in \mathbb{Z}_v^\times$. Thus, the above conditions become

$$v(\lambda) = 0, h \in \mathrm{GSp}(4, \mathbb{Z}_v), u \in U(\mathbb{Z}_v).$$

To see when $h^{-1}\eta(\lambda)u \in K_0(N) \subset \mathrm{GSp}(4, \mathbb{Z}_v)$, we use the already known conditions that $v(\lambda) = 0$, $v(r), v(s), v(t) \geq 0$, and $h \in \mathrm{GSp}(4, \mathbb{Z}_v)$.

If

$$h^{-1}\eta(\lambda)u = \begin{pmatrix} a_1 & -\frac{b_2}{2} & a_1 r - \frac{b_2 s}{2} + \frac{a_2\lambda}{2} & a_1 s - \frac{b_2 t}{2} + b_1 d\lambda \\ b_1 & -\frac{a_2}{2d} & b_1 r - \frac{a_2 s}{2d} + \frac{b_2\lambda}{2} & b_1 s - \frac{a_2 t}{2d} + a_1\lambda \\ 2a_3 & -b_4 & 2a_3 r - b_4 s + a_4\lambda & 2a_3 s - b_4 t + 2b_3 d\lambda \\ 2b_3 d & -a_4 & 2b_3 dr - a_4 s + b_4 d\lambda & 2b_3 ds - a_4 t + 2a_3 d\lambda \end{pmatrix} \in K_0(N),$$

we get that $v(a_3), v(b_3), v(a_4), v(b_4) \geq 1$, so in particular $h \in K_0(N)$. This implies $\eta(\lambda)u \in K_0(N)$.

But then

$$\eta(\lambda)u = \begin{pmatrix} 1 & 0 & r & s \\ 0 & 0 & 0 & \lambda \\ 0 & 0 & \lambda & 0 \\ 0 & -1 & -s & -t \end{pmatrix} \in K_0(N),$$

which is a contradiction since $v(-1) = 0$.

□

Let's now consider the contribution from cosets $\xi(\rho, \mu)$ with $\rho, \mu \in \mathbb{Q}$ and $\mu \neq 0$. We have

$$\begin{aligned} I(\xi(\rho,\mu), f) &= \int_{C_{\xi(\rho,\mu)}(\mathbb{A})\backslash(\tilde{H}(\mathbb{A})\times U(\mathbb{A}))} f(h^{-1}\xi(\rho,\mu)u)\psi(u) \times \\ &\times \left(\int_{\overline{N}(\mathbb{Q})\backslash\overline{N}(\mathbb{A})} \mathcal{E}_s^{\phi}(zh)\psi(\xi(\rho,\mu)^{-1}z\xi(\rho,\mu))\mathrm{d}z\right) \mathrm{d}h\ \mathrm{d}u, \end{aligned}$$

where

$$\overline{N} = \begin{pmatrix} 1 & 0 \\ \star & 1 \end{pmatrix} /F.$$

Note that every element $z \in \overline{N}$ can be expressed as $w_0^{-1} n w_0$, where

$$n \in N = \begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix} / F.$$

We can write

$$\begin{aligned} I(\xi(\rho,\mu), f) &= \int_{C_{\xi(\rho,\mu)}(\mathbb{A})\backslash(\tilde{H}(\mathbb{A})\times U(\mathbb{A}))} f(h^{-1}\xi(\rho,\mu)u)\psi(u) \times \\ &\times \left(\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \mathcal{E}_s^{\phi}(n(w_0 h))\psi(\xi(\rho,\mu)^{-1} w_0^{-1} n w_0 \xi(\rho,\mu))\mathrm{d}n \right) \mathrm{d}h \ \mathrm{d}u. \end{aligned}$$

From the definition of the Eisenstein series we get

$$I(\xi(\rho,\mu), f) = \int_{C_{\xi(\rho,\mu)}(\mathbb{A})\backslash(\tilde{H}(\mathbb{A})\times U(\mathbb{A}))} f(h^{-1}\xi(\rho,\mu)u)\psi(u) \times$$
$$\times \left(\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \sum_{\gamma \in \tilde{B}(\mathbb{Q})\backslash H(\mathbb{Q})} \phi_s(\gamma n(w_0 h))\psi(\xi(\rho,\mu)^{-1} w_0^{-1} n w_0 \xi(\rho,\mu))\mathrm{d}n \right) \mathrm{d}h \ \mathrm{d}u.$$

We have $\phi_s(n w_0 h) = \phi_s(w_0 h)$ for all $n \in N(\mathbb{A})$. In addition, $w_0\xi(\rho,\mu)$ is an element of $M(\mathbb{Q})$, where $P = MU$ is the Levi decomposition of the Siegel parabolic, which we call m. We then get that $\psi(m^{-1}\gamma n m) = \psi(m^{-1}\gamma m m^{-1} n m) = \psi(m^{-1} n m)$ for all $\gamma \in N(\mathbb{Q})$ since $m^{-1}\gamma m \in N(\mathbb{Q})$. Applying the Bruhat decomposition we obtain

$$I(\xi(\rho,\mu)) = I_a(\rho,\mu) + I_b(\rho,\mu),$$

where

$$I_a(\rho,\mu) = \int_{C_{\xi(\rho,\mu)}(\mathbb{A})\backslash(\tilde{H}(\mathbb{A})\times U(\mathbb{A}))} f(h^{-1}\xi(\rho,\mu)u)\phi_s(w_0h)\psi(u)\times$$
$$\times \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \psi(\xi(\rho,\mu)^{-1}w_0^{-1}nw_0\xi(\rho,\mu))\mathrm{d}n\mathrm{d}h\ \mathrm{d}u, \quad (3.52)$$
$$I_b(\rho,\mu) = \int_{C_{\xi(\rho,\mu)}(\mathbb{A})\backslash(\tilde{H}(\mathbb{A})\times U(\mathbb{A}))} f(h^{-1}\xi(\rho,\mu)u)\psi(u)\times$$
$$\times \left(\int_{N(\mathbb{A})} \phi_s(w_0nw_0h)\psi(\xi(\rho,\mu)^{-1}w_0^{-1}nw_0\xi(\rho,\mu))\mathrm{d}n\right)\mathrm{d}h\ \mathrm{d}u. \quad (3.53)$$

The integrals $I_a(\rho,\mu)$ and $I_b(\rho,\mu)$ factorize at places, and we can write

$$I_a(\rho,\mu) = \prod_v I_{a,v}(\rho,\mu),\ I_b = \prod_v I_{b,v}(\rho,\mu).$$

Lemma 3.6. *Let v be a finite place and $h \in \tilde{H}(\mathbb{Q}_v)$ and $u \in U(\mathbb{Q}_v)$. Then $h^{-1}\xi(\rho,\mu)u \in GSp(4,\mathbb{Z}_v)$ implies $h \in \tilde{H}(\mathbb{Z}_v)$, $u \in U(\mathbb{Z}_v)$, $\rho \in \mathbb{Z}_v$, and $\mu \in \mathbb{Z}_v^\star$.*

Proof. We use the notations for h and u, and for the Iwasawa decomposition $h^{-1} = kb$ as in Lemma (3.5). We want to determine when $h^{-1}\xi(\rho,\mu)u \in \mathrm{GSp}(4,\mathbb{Z}_v)$. By the Iwasawa decomposition $h^{-1} = kb$, we get that

$$b\xi(\rho,\mu)u = \begin{pmatrix} x_1 & 0 & x_3 & \frac{x_1y_2}{y_1} \\ 0 & y_1 & y_2 & y_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & u_4 \end{pmatrix}\xi(\rho,\mu)u$$

is equal to

$$\begin{pmatrix} \frac{\rho x_1 y_2}{y_1} - x_3 & -\frac{\mu x_1 y_2}{y_1} & \mu x_1 - \frac{s\mu y_2 x_1}{y_1} + r\left(\frac{\rho x_1 y_2}{y_1} - x_3\right) & \rho x_1 - \frac{t\mu y_2 x_1}{y_1} + s\left(\frac{\rho x_1 y_2}{y_1} - x_3\right) \\ \rho y_3 - y_2 & -\mu y_3 & r\left(\rho y_3 - y_2\right) - s\mu y_3 & y_1 - t\mu y_3 + s\left(\rho y_3 - y_2\right) \\ -1 & 0 & -r & -s \\ \rho u_4 & -\mu u_4 & r\rho u_4 - s\mu u_4 & s\rho u_4 - t\mu u_4 \end{pmatrix}$$

and is an element in $\mathrm{GSp}(4, \mathbb{Z}_v)$. This implies $r, s \in \mathbb{Z}_v$.

Now if $h^{-1}\xi(\rho, \mu)u$, which is equal to

$$\begin{pmatrix} \frac{b_2\rho}{2} - \frac{a_2}{2} & -\frac{b_2\mu}{2} & a_1\mu - \frac{b_2 s\mu}{2} + r\left(\frac{b_2\rho}{2} - \frac{a_2}{2}\right) & b_1 d - \frac{b_2 t\mu}{2} + a_1\rho + s\left(\frac{b_2\rho}{2} - \frac{a_2}{2}\right) \\ \frac{a_2\rho}{2d} - \frac{b_2}{2} & -\frac{a_2\mu}{2d} & b_1\mu - \frac{a_2 s\mu}{2d} + r\left(\frac{a_2\rho}{2d} - \frac{b_2}{2}\right) & a_1 - \frac{a_2 t\mu}{2d} + b_1\rho + s\left(\frac{a_2\rho}{2d} - \frac{b_2}{2}\right) \\ b_4\rho - a_4 & -b_4\mu & 2a_3\mu - b_4 s\mu + r(b_4\rho - a_4) & 2b_3 d - b_4 t\mu + 2a_3\rho + s(b_4\rho - a_4) \\ a_4\rho - b_4 d & -a_4\mu & 2b_3 d\mu - a_4 s\mu + r(a_4\rho - b_4 d) & 2a_3 d + 2b_3\rho d - a_4 t\mu + s(a_4\rho - b_4 d) \end{pmatrix} \tag{3.54}$$

is an element in $\mathrm{GSp}(4, \mathbb{Z}_v)$, it implies that

$$h^{-1} \cdot \mu \in \mathrm{GSp}(4, \mathbb{Q}_v) \cap M_4(\mathbb{Z}_v). \tag{3.55}$$

This is because from the second column of (3.54) we get that $\frac{b_2\mu}{2}$, $\frac{a_2\mu}{2d}$, $b_4\mu$, and $a_4\mu$ are all elements in $\mathbb{Z}_v$. Now all the entries in the first column are also in $\mathbb{Z}_v$ and in addition $r, s \in \mathbb{Z}_v$ as we saw above. Thus, from the third column we also get that $a_1\mu, b_1\mu, 2a_3\mu$, and $2b_3 d\mu$ are all in $\mathbb{Z}_v$.

We have that $\det(h^{-1}) \cdot \mu^2 \in \mathbb{Z}_v^\times$ from (3.54) and $\det(h^{-1}) \cdot \mu^4 \in \mathbb{Z}_v$ from (3.55). This implies that $\mu^2 \in \mathbb{Z}_v$, and hence $\mu \in \mathbb{Z}_v$.

We apply Iwasawa decomposition in the following way. Write

$$h^{-1} = k'b'$$

with $k' \in \mathrm{GSp}(4, \mathbb{Z}_v)$ and

$$b' = \begin{pmatrix} 1 & 0 & x_3 & \frac{y_2}{y_1} \\ 0 & y_1 & y_2 & y_3 \\ 0 & 0 & z_1 & 0 \\ 0 & 0 & 0 & \frac{z_1}{y_1} \end{pmatrix}.$$

The condition that $h^{-1}\xi(\rho, \mu)u \in \mathrm{GSp}(4, \mathbb{Z}_v)$ is equivalent to the condition that

$$b'\xi(\rho,\mu)u = \begin{pmatrix} \frac{\rho y_2}{y_1} - x_3 & -\frac{\mu y_2}{y_1} & -\frac{s y_2 \mu}{y_1} + \mu + r\left(\frac{\rho y_2}{y_1} - x_3\right) & \rho - \frac{t\mu y_2}{y_1} + s\left(\frac{\rho y_2}{y_1} - x_3\right) \\ \rho y_3 - y_2 & -\mu y_3 & r\left(\rho y_3 - y_2\right) - s\mu y_3 & y_1 - t\mu y_3 + s\left(\rho y_3 - y_2\right) \\ -z_1 & 0 & -r z_1 & -s z_1 \\ \frac{\rho z_1}{y_1} & -\frac{\mu z_1}{y_1} & \frac{r\rho z_1}{y_1} - \frac{s\mu z_1}{y_1} & \frac{s\rho z_1}{y_1} - \frac{t\mu z_1}{y_1} \end{pmatrix}$$

is an element in $\mathrm{GSp}(4, \mathbb{Z}_v)$, which in particular implies that $z_1 \in \mathbb{Z}_v$. On the other hand, we must have that $\det(b')\mu^2 = z_1^2\mu^2 \in \mathbb{Z}_v^\times$ and we saw before that $\mu \in \mathbb{Z}_v$. Thus, $\mu \in \mathbb{Z}_v^\times$, and as a consequence, we get from (3.55) that

$$h^{-1} \in \mathrm{GSp}(4, \mathbb{Z}_v).$$

But then from $h^{-1}\xi(\rho, \mu)u \in \mathrm{GSp}(4, \mathbb{Z}_v)$ we conclude

$$\xi(\rho,\mu)u = \begin{pmatrix} 0 & 0 & \mu & \rho \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -r & -s \\ \rho & -\mu & r\rho - s\mu & s\rho - t\mu \end{pmatrix} \in \mathrm{GSp}(4,\mathbb{Z}_v),$$

and hence $\rho \in \mathbb{Z}_v$. In addition $t\mu \in \mathbb{Z}_v$ and since μ is a unit we get that $t \in \mathbb{Z}_v$.

Thus, the conditions that $h^{-1}\xi(\rho,\mu)u \in \mathrm{GSp}(4,\mathbb{Z}_v)$ implies that $h \in \tilde{H}(\mathbb{Z}_v)$, $\rho \in \mathbb{Z}_v$, $\mu \in \mathbb{Z}_v^\star$ and $u \in U(\mathbb{Z}_v)$.

□

Lemma 3.7. *We have that* $I_a(\rho,\mu) = I_b(\rho,\mu) = 0$ *if* $\rho \notin \mathbb{Z}$ *or* $\mu \neq \pm 1$.

Proof. At all finite places v we have that $f_v(h^{-1}\xi(\rho,\mu)u) = 0$ if $h^{-1}\xi(\rho,\mu)u \notin \mathrm{GSp}(4,\mathbb{Z}_v)$. Thus, if there exists h and u such that $f_v(h^{-1}\xi(\rho,\mu)u) \neq 0$ by Lemma 3.6 we get that $\rho \in \mathbb{Z}_v$ and $\mu \in \mathbb{Z}_v^\star$. In particular, if we must have $I_{a,v}(\rho,\mu) \neq 0$ for all v, it must be the case that $\rho \in \mathbb{Z}$ and $\mu = \pm 1$. The same is true for I_b. □

Thus, we only need to consider $I_a(\rho,\pm 1)$ and $I_b(\rho,\pm 1)$ with $\rho \in \mathbb{Z}$. We will further set restrictions by choosing the symmetric matrix S in (3.31) to be

$$S = \begin{pmatrix} -md & 0 \\ 0 & m \end{pmatrix}$$

with $m \in \mathbb{Z}^\star$. If

$$n = \begin{pmatrix} 1 & a_2 + b_2\sqrt{d} \\ 0 & 1 \end{pmatrix},$$

we have

$$\begin{aligned}\psi(\xi(\rho,\mu)^{-1}w_0^{-1}nw_0\xi(\rho,\mu)) &= \psi_0(\mathrm{tr}(S\xi(\rho,\mu)^{-1}w_0^{-1}nw_0\xi(\rho,\mu))) \\ &= \psi_0\left(\frac{2a_2(-\mu^2md+m(d+\rho^2))}{\mu}\right)\psi_0\left(\frac{4b_2dm\rho}{\mu}\right).\end{aligned}$$

Plugging this into the expression for I_a in (3.52) and using character orthogonality, we get that since

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})}\psi(\xi(\rho,\mu)^{-1}w_0^{-1}nw_0\xi(\rho,\mu))\mathrm{d}n \neq 0$$

implies $\rho = 0$ and $\mu = \pm 1$, it must be the case that if

$$I_a(\rho,\mu) \neq 0$$

then $I_a(0,\pm 1)$ are the only possibilities.

We will evaluate $I_a(0,\pm 1)$ and $I_b(0,\pm 1)$ corresponding to the two cosets $\xi(0,\pm 1)$. We have

$$\begin{aligned}I_a(0,\pm 1) &= \mathrm{vol}(N(\mathbb{Q})\backslash N(\mathbb{A}))\int_{C_{\xi(0,\pm 1)}(\mathbb{A})\backslash(\tilde{H}(\mathbb{A})\times U(\mathbb{A}))} f(h^{-1}\xi(0,\pm 1)u)\times \\ &\times\ \phi_s(w_0h)\psi(u)\mathrm{d}h\ \mathrm{d}u \\ &= \mathrm{vol}(N(\mathbb{Q})\backslash N(\mathbb{A}))\prod_v\int_{C_{\xi(0,\pm 1)}(\mathbb{Q}_v)\backslash(\tilde{H}(\mathbb{Q}_v)\times U(\mathbb{Q}_v))} f_v(h^{-1}\xi(0,\pm 1)u)\times \\ &\times\ \phi_{s,v}(w_0h)\psi_v(u)\mathrm{d}h\ \mathrm{d}u \qquad (3.56) \\ &= \mathrm{vol}(N(\mathbb{Q})\backslash N(\mathbb{A}))\prod_v I_{a,v}\end{aligned}$$

and

$$\begin{aligned} I_b(0,\pm 1) &= \int_{C_{\xi(0,\pm1)}(\mathbb{A})\backslash(\tilde{H}(\mathbb{A})\times U(\mathbb{A}))} f(h^{-1}\xi(0,\pm1)u)\psi(u)\times \\ &\times \left(\int_{N(\mathbb{A})}\phi_s(w_0nw_0h)\mathrm{d}n\right)\mathrm{d}h\ \mathrm{d}u \\ &= \prod_v \int_{C_{\xi(0,\pm1)}(\mathbb{Q}_v)\backslash(\tilde{H}(\mathbb{Q}_v)\times U(\mathbb{Q}_v))} f(h^{-1}\xi(0,\pm1)u)\psi(u)\times \\ &\times \left(\int_{N(\mathbb{Q}_v)}\phi_s(w_0nw_0h)\mathrm{d}n\right)\mathrm{d}h\ \mathrm{d}u. \end{aligned} \tag{3.57}$$

Since $H = \mathrm{GL}(2)/F$ and $\tilde{H} = Z\backslash\tilde{G}$, we have the Iwasawa decomposition of $\tilde{H}(F_v)$ at each place v of F given by

$$\tilde{H}(F_v) = N(F_v)\tilde{A}(F_v)\Gamma_v,$$

where

$$A = \begin{pmatrix} \star & 0 \\ 0 & \star \end{pmatrix},$$

and $\tilde{A} = Z\backslash A$ and Γ_v is a maximal compact subgroup in $\tilde{H}(F_v)$. In particular, when v is a non-archimedian place, we have that $\Gamma_v = \tilde{H}(F_v)$. The Haar measure d$h$ in Iwasawa coordinates is given by

$$\mathrm{d}h = |a|^{-1}\mathrm{d}n\mathrm{d}a\mathrm{d}\gamma$$

such that if we have a measurable function f on $\tilde{H}(F_v)$, we have

$$\int_{\tilde{H}(F_v)} f(h)\mathrm{d}h = \int_{F_v}\int_{F_v^\times}\int_{\Gamma_v} f\left(\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}\begin{pmatrix} a & \\ & 1 \end{pmatrix}\gamma\right)|a|^{-1}\mathrm{d}\gamma\mathrm{d}^\times a\mathrm{d}n.$$

As a result we have the following lemma:

Lemma 3.8. *$C_{\xi(\rho,\mu)}\backslash\tilde{H}\times U$ over a local field has coset representatives given by the elements of*

$$\overline{N}\backslash\tilde{H}\times U$$

with the right $\tilde{H}$-invariant measure on $\overline{N}\backslash\tilde{H}$ given by $|a|^{-1}d^{\times}a\,d\gamma$, where $d^{\times}a$ is the Haar measure on $\tilde{A}$ and $d\gamma$ is the Haar measure on $\tilde{\Gamma}$.

Proof. A direct computation shows that

$$C_{\xi(\rho,\mu)}=\left\{(\overline{n},\xi(\rho,\mu)^{-1}\overline{n}^{-1}\xi(\rho,\mu))|\overline{n}\in\overline{N}\right\}.$$

Let's suppose that $\overline{N}\backslash\tilde{H}$ is given by a disjoint union of cosets $\cup_{i\in I}\overline{N}h_i$. Then we will check that (h_i,u) represent disjoint coset representatives for $C_{\xi(\rho,\mu)}\backslash\tilde{H}\times U$. Indeed, if we let $(h,u)\in H\times U$ be an arbitrary element, we can write it as

$$(h,u)=(\overline{n},\xi(\rho,\mu)^{-1}\overline{n}^{-1}\xi(\rho,\mu))(h_i,u')$$

with u' the unique solution to $u=\xi(\rho,\mu)^{-1}bh^{-1}\xi(\rho,\mu)u'$. It is now easy to check that (h_i,u') and (h_j,u'') represent different cosets for $i\neq j$. Indeed, suppose

$$(\overline{n},\xi(\rho,\mu)^{-1}\overline{n}^{-1}\xi(\rho,\mu))(h_i,u')=(h_j,u'').$$

Then we get $h_i=h_j$ which implies $\overline{n}=1$ and $u'=u''$.

Note that $N=w_0\overline{N}w_0^{-1}$, so that the Iwasawa decomposition of $\tilde{H}$ can be rewritten as $\tilde{H}=w_0\overline{N}w_0^{-1}\tilde{A}\Gamma$, or alternatively, $\tilde{H}=\overline{N}w_0^{-1}\tilde{A}\Gamma$. The measure on $\overline{N}\backslash\tilde{H}$ in Iwasawa coordinates can be deduced from this.

□

3.6.1 Non-archimedian computation of $I_{a,v}(0,\pm 1)$

In the following theorem we will evaluate all $I_{a,v}(0,\pm 1)$ in the relevant cases.

Theorem 3.4. *1. If v is an inert place where $\chi_{1,v}$ and $\chi_{2,v}$ are unramified and f_v is the characteristic function of $GSp(4,\mathbb{Z}_v)Z_v$ then*

$$I_{a,v}(0,\pm 1) = meas(\tilde{H}(\mathbb{Z}_v) \times U(\mathbb{Z}_v)).$$

2. If $v = N$ is an inert place where $\chi_{1,N}$ and $\chi_{2,N}$ are unramified and f_N is the characteristic function of $K_0(N)Z_N$ then

$$I_{a,N}(0,\pm 1) = meas(\Gamma_0(N)) \cdot (meas(K_0(N)))^{-1}.$$

3. If $(v,2) = 1$ is another inert place then

$$I_{a,v}(0,1) > 0$$

and

$$I_{a,v}(0,-1) = 0.$$

4. If $v = v_1 v_2$ is a split place with $(v,2) = 1$ such that $\chi_1^{(1)}, \chi_1^{(2)}, \chi_2^{(1)}, \chi_2^{(2)}$ are all unramified then

$$I_{a,v}(0,\pm 1) = meas(\tilde{H}(\mathbb{Z}_v) \times U(\mathbb{Z}_v)).$$

5. If $(v,2) = 1$ is a ramified place then assuming χ_1 and χ_2 are both unramified we get

$$I_{a,v}(0,\pm 1) = \chi_{1,s}(d^{-1})\chi_{2,s}(d^{-1}) \cdot meas(H_0 \times U(\mathbb{Z}_v)),$$

where H_0 is a subgroup of $\tilde{H}$ whose entries satisfy the inequalities in (3.60).

6. *If $v = 2$ then*

$$I_{a,v}(0,\pm 1) = \chi_{1,v}^{-1}(2d)\chi_{2,v}^{-1}(2d) \cdot meas(H_1 \times U(\mathbb{Z}_v)),$$

where H_1 is a subgroup of $\tilde{H}(\mathbb{Q}_v)$.

Proof. 1. We have by Lemma 3.8

$$I_{a,v}(0,\pm 1) = \int_{C_{\xi(0,\pm 1)}(\mathbb{Q}_v)\backslash(\tilde{H}(\mathbb{Q}_v)\times U(\mathbb{Q}_v))} f_v(h^{-1}\xi(0,\pm 1)u)\phi_{s,v}(w_0 h)\psi_v(u)\mathrm{d}h\ \mathrm{d}u.$$

As in Lemma 3.6, $f_v(h^{-1}\xi(0,\pm 1)u) \neq 0$ implies in particular $h \in \tilde{H}(\mathbb{Z}_v)$ and $u \in U(\mathbb{Z}_v)$. It is easy to see that for f_v the characteristic function of $\mathrm{GSp}(4,\mathbb{Z}_v)Z_v$ this is also a sufficient condition. Thus, taking into consideration Lemma 3.8, we have

$$\begin{aligned} I_{a,v}(0,\pm 1) &= \int_{\tilde{H}(\mathbb{Z}_v)\times U(\mathbb{Z}_v)} \phi_{s,v}(w_0 h)\psi_v(u)\mathrm{d}h\ \mathrm{d}u \\ &= \mathrm{meas}(\tilde{H}(\mathbb{Z}_v)\times U(\mathbb{Z}_v)) \end{aligned}$$

since when $\chi_{1,v}$ and $\chi_{2,v}$ are unramified we have that $\phi_{s,v}$ is right $\tilde{H}(\mathbb{Z}_v)$-invariant, and we also have $\psi_v(u) = 1$ for all $u \in U(\mathbb{Z}_v)$.

2. Since $\chi_{1,N}, \chi_{2,N}$ are unramified, $\phi_{s,N}$ is again right $\tilde{H}(\mathbb{Z}_N)$-invariant. We have that since f_N is the characteristic function of $Z_N\backslash K_0(N)Z_N$ divided by the volume, the necessary and sufficient conditions that we get from

$$h_N^{-1}\xi(0,\pm 1)u_N \in K_0(N)$$

are that $h_N \in \tilde{H}(\mathbb{Z}_N)$ and $u_N \in U(\mathbb{Z}_N)$, and in addition if

$$h_N = \begin{pmatrix} a_1 & b_1 d & \frac{a_2}{2} & \frac{b_2}{2} \\ b_1 & a_1 & \frac{b_2}{2} & \frac{a_2}{2d} \\ 2a_3 & 2b_3 d & a_4 & b_4 \\ 2b_3 d & 2a_3 d & b_4 d & a_4 \end{pmatrix}$$

then we must also have $v_N(a_1), v_N(b_1) > 0$. Consider the set of elements

$$\Gamma_0^\star(N) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \tilde{H}(\mathcal{O}_{F_N}), v_N(\alpha) > 0 \right\}.$$

We have

$$I_{a,N}(0, \pm 1) = \text{meas}(\Gamma_0^\star(N) \times U(\mathbb{Z}_N)).$$

Note however that

$$\Gamma_0^\star(N) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Gamma_0(N),$$

where

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathcal{O}) | \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} \star & \star \\ 0 & \star \end{pmatrix} \pmod{\varpi} \right\}.$$

Thus, we get that $\text{meas}(\Gamma_0^\star(N)) = \text{meas}(\Gamma_0(N))$.

3. In this case, $\chi_{1,v}$ and $\chi_{2,v}$ are not both unramified. Assume that $\text{cond}(\chi_{1,v}) = n$ and $\text{cond}(\chi_{2,v}) = n$ and that $\chi_{1,v}$ and $\chi_{2,v}$ are even characters. Just as before,

since f_v is zero outside $\mathrm{GSp}(4,\mathbb{Z}_v)$, we get

$$I_{a,v}(0,\pm 1) = \int_{\tilde{H}(\mathbb{Z}_v)\times U(\mathbb{Z}_v)} f_v(h^{-1}\xi(0,\pm 1)u)\cdot \phi_{s,v}(w_0 h)\mathrm{d}h\ \mathrm{d}u.$$

We have that f_v is 1 on the coset

$$K(2n)\begin{pmatrix} -\frac{1}{2} & & & \\ & -\frac{1}{2d} & & \\ -\varpi_v^n & & 2 & \\ & -\varpi_v^n & & 2d \end{pmatrix} K(2n),$$

and zero otherwise. Here

$$K(2n) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}(4,\mathbb{Z}_v) | \begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \pmod{\varpi_v^{2n}} \right\}$$

and the double coset consists of elements

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv \begin{pmatrix} -\frac{1}{2} & & & \\ & -\frac{1}{2d} & & \\ -\varpi_v^n & & 2 & \\ & -\varpi_v^n & & 2d \end{pmatrix} \pmod{\varpi_v^{2n}}.$$

If

$$h^{-1} = \begin{pmatrix} a_1 & b_1 d & \frac{a_2}{2} & \frac{b_2}{2} \\ b_1 & a_1 & \frac{b_2}{2} & \frac{a_2}{2d} \\ 2a_3 & 2b_3 d & a_4 & b_4 \\ 2b_3 d & 2a_3 d & b_4 d & a_4 \end{pmatrix},$$

then

$$h^{-1}\xi(0,\pm1)u = \begin{pmatrix} -\frac{a_2}{2} & \mp\frac{b_2}{2} & \pm a_1 \mp \frac{b_2 s}{2} - r\frac{a_2}{2} & b_1 d \mp \frac{b_2 t}{2} - s\frac{a_2}{2} \\ -\frac{b_2}{2} & \mp\frac{a_2}{2d} & \pm b_1 \mp \frac{a_2 s}{2d} - r\frac{b_2}{2} & a_1 \mp \frac{a_2 t}{2d} - s\frac{b_2}{2} \\ -a_4 & \mp b_4 & \pm 2a_3 \mp b_4 s - a_4 r & 2b_3 d \mp b_4 t - a_4 s \\ -b_4 d & \mp a_4 & \pm 2b_3 d \mp a_4 s - b_4 dr & 2a_3 d \mp a_4 t - b_4 ds \end{pmatrix}. \tag{3.58}$$

Note that

$$h^{-1}\xi(0,-1)u \in K(2n) \begin{pmatrix} -\frac{1}{2} & & & \\ & -\frac{1}{2d} & & \\ -\varpi_v^n & & 2 & \\ & -\varpi_v^n & & 2d \end{pmatrix} K(2n)$$

will give us a contradiction, hence $I_{a,v}(0,-1) = 0$.

If $h^{-1}\xi(0,1)u$ as in (3.58) is an element in

$$K(2n) \begin{pmatrix} -\frac{1}{2} & & & \\ & -\frac{1}{2d} & & \\ -\varpi_v^n & & 2 & \\ & -\varpi_v^n & & 2d \end{pmatrix} K(2n),$$

then $a_2 \equiv 1(\text{mod } \varpi_v^{2n})$, $b_2, b_4 \equiv 0(\text{mod } \varpi_v^{2n})$, $a_4 \equiv \varpi_v^n(\text{mod } \varpi_v^{2n})$, $a_3 \equiv 1(\text{mod } \varpi_v^n)$, and $b_3 \equiv 0(\text{mod } \varpi_v^n)$ are necessary conditions. The matrix h^{-1} corresponds to the matrix

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a_1 + b_1\sqrt{d} & a_2 + b_2\sqrt{d} \\ a_3 + b_3\sqrt{d} & a_4 + b_4\sqrt{d} \end{pmatrix} \in \mathrm{GL}(2, \mathcal{O})$$

so that $b', c' \equiv 1 (\mathrm{mod}\ \varpi_v^n)$, and $d' \equiv \varpi_v^n (\mathrm{mod}\ \varpi_v^{2n})$. Then

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{d'}{a'd'-b'c'} & -\frac{b'}{a'd'-b'c'} \\ -\frac{c'}{a'd'-b'c'} & \frac{a'}{a'd'-b'c'} \end{pmatrix}$$

with

$$a \equiv -\varpi_v^n (\mathrm{mod}\ \varpi_v^{2n}) \text{ and } b, c \equiv 1 (\mathrm{mod}\ \varpi_v^n). \tag{3.59}$$

Since $v(a) = n > 0$, we can write

$$w_0 h = \begin{pmatrix} c & d \\ -a & -b \end{pmatrix} = \begin{pmatrix} \frac{cb\varpi^n}{a} - d\varpi^n & d \\ & -b \end{pmatrix} \begin{pmatrix} 1 & \\ \varpi^n & 1 \end{pmatrix} \begin{pmatrix} \frac{a}{\varpi^n b} & \\ & 1 \end{pmatrix},$$

and for $\phi_{s,v}$ supported on

$$B \begin{pmatrix} 1 & \\ \varpi^n & 1 \end{pmatrix} \Gamma_2(2n)$$

a $\Gamma_2(2n)$-invariant map as in (3.11)

$$\phi_{s,v}(w_0 h) = \chi_{1,v}((cb - ad)\varpi^n / a)\chi_{2,v}(-b).$$

Since $\chi_{1,v}$ and $\chi_{2,v}$ are even characters and trivial on $1 + \mathfrak{p}^n$, we get that for h with entries satisfying (3.59), we get $\phi_{s,v}(w_0 h) = 1$.

Then the integral is just the volume of the elements $(h, u) \in \tilde{H}(\mathbb{Z}_v) \times U(\mathbb{Z}_v)$

with the property that

$$h^{-1}\xi(0,1)u \in K_1(2n)\begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}K_1(2n).$$

4. Here we consider the case when v splits into places v_1 and v_2 of F, and $(v,2) = 1$. We have that d is a square of F. We can write

$$\phi_s = \phi_s^{(1)} \cdot \phi_s^{(2)},$$

where $\phi_s^{(i)}$ represents the component at place v_i, and

$$\phi_s^{(i)}\left(\begin{pmatrix} a & \star \\ 0 & b \end{pmatrix} g\right) = \chi_{1,v}^{(i)}(a)\chi_{2,v}^{(i)}(b)\phi_s^{(i)}(g).$$

If f_v is the characteristic function of $Z_v\mathrm{GSp}(4,\mathbb{Z}_v)$ and $\chi_{1,v}^{(i)}$ and $\chi_{2,v}^{(i)}$ are all unramified, then similar to case 1 above we get that $I_{a,v}(0,\pm 1) = 1$.

5. If v is a ramified place such that $(v,2) = 1$ then if ϖ_F is a fixed uniformizer of F we have that $\varpi_F^2 = \varpi$. We may also assume that $v(\sqrt{d}) = 1/2$.

   Just like before, $h^{-1}\xi(0,\pm 1)u \in \mathrm{GSp}(4,\mathbb{Z}_v)$ implies $h \in \mathrm{GSp}(4,\mathbb{Z}_v)$ and $u \in U(\mathbb{Z}_v)$. If

$$h = \begin{pmatrix} a_1 & b_1 d & \frac{a_2}{2} & \frac{b_2}{2} \\ b_1 & a_1 & \frac{b_2}{2} & \frac{a_2}{2} \\ 2a_3 & 2b_3 d & a_4 & b_4 \\ 2b_3 d & 2a_3 d & b_4 d & a_4 \end{pmatrix} \in \mathrm{GSp}(4,\mathbb{Z}_v),$$

then

$$v(a_1), v(b_1), v(a_4), v(b_4), v(a_2), v(b_2), v(a_3) \geq 0 \text{ and } v(b_3) \geq -1. \tag{3.60}$$

We can write

$$\phi_s(w_0h) = \phi_s\left(\begin{pmatrix} d^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix}\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} w_0h\right),$$

and since

$$\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} w_0h \in \mathrm{GL}(2, \mathcal{O}),$$

we get that

$$\phi_s(w_0h) = \chi_{1,s}(d^{-1})\chi_{2,s}(d^{-1}).$$

Thus, we conclude that

$$I_{a,v}(0, \pm 1) = \chi_{1,s}(d^{-1})\chi_{2,s}(d^{-1}) \cdot \mathrm{meas}(H_0 \times U(\mathbb{Z}_v)),$$

where H_0 is the subgroup of $\tilde{H}(\mathbb{Q}_v)$ with entries satisfying the conditions in (3.60).

6. If $v = 2$ and $d \equiv 2 (\mathrm{mod}\ 4)$ then 2 ramifies, and using the usual notation we have $v(a_1), v(b_1), v(a_4), v(b_4) \geq 0$ and $v(b_2) \geq 1$, $v(a_3) \geq -1$ $v(a_2) \geq 2$, $v(b_3) \geq -2$.

   If $v = 2$ and $(d, 2) = 1$ and 2 is inert or ramified, which corresponds to $d \equiv 3, 5 (\mathrm{mod}\ 8)$, we have $v(a_1), v(b_1), v(a_4), v(b_4) \geq 0$ and $v(b_2) \geq 1$, $v(a_3) \geq -1$ $v(a_2) \geq 1$, $v(b_3) \geq -1$.

If $v=2$ and $(d,2)=1$ and 2 is split, i.e. $d\equiv 1(\mathrm{mod}\ 8)$, then we have that $\sqrt{d}\in\mathbb{Z}_2$ and $v(a_1),v(b_1),v(a_4),v(b_4)\geq 0$ and $v(b_2)\geq 1$, $v(a_3)\geq -1$ $v(a_2)\geq 1$, $v(b_3)\geq -1$.

In all cases, $\phi_{s,v}(w_0x)=\chi_{1,v}^{-1}(2d)\chi_{2,v}^{-1}(2d)$ from which the conclusion follows just like before.

□

3.6.2 Non-archimedean computation of $I_{b,v}(0,\pm 1)$

Recall

$$\begin{aligned} I_b(0,\pm 1) &= \int_{C_{\xi(0,\pm 1)}(\mathbb{A})\backslash(\tilde{H}(\mathbb{A})\times U(\mathbb{A}))} f(h^{-1}\xi(0,\pm 1)u)\psi(u)\times \\ &\times \left(\int_{N(\mathbb{A})}\phi_s(w_0nw_0h)\mathrm{d}n\right)\mathrm{d}h\ \mathrm{d}u \\ &= \prod_v \int_{C_{\xi(0,\pm 1)}(\mathbb{Q}_v)\backslash(\tilde{H}(\mathbb{Q}_v)\times U(\mathbb{Q}_v))} f(h^{-1}\xi(0,\pm 1)u)\psi(u)\times \\ &\times \left(\int_{N(\mathbb{Q}_v)}\phi_s(w_0nw_0h)\mathrm{d}n\right)\mathrm{d}h\ \mathrm{d}u. \end{aligned}$$

The inner integral is given by (see Section 3.2.5)

$$\int_{N(\mathbb{A})}\phi_s(w_0nw_0h)\mathrm{d}u = M(s)\phi_s(w_0h),$$

and hence

$$I_b(0,\pm 1)=\int_{C_{\xi(0,\pm 1)}(\mathbb{A})\backslash(\tilde{H}(\mathbb{A})\times U(\mathbb{A}))} f(h^{-1}\xi(0,\pm 1)u)\psi(u)\cdot M(s)\phi_s(w_0h)\mathrm{d}h\mathrm{d}u.$$

At place v, we have

$$\int_{C_{\xi(0,\pm1)}(\mathbb{Q}_v)\backslash(\tilde{H}(\mathbb{Q}_v)\times U(\mathbb{Q}_v))} f(h_v^{-1}\xi(0,\pm1)u)\psi_v(u_v)\cdot A_v(s)\phi_{s,v}(w_0h_v)\mathrm{d}h_v\mathrm{d}u_v.$$

We saw in Section 3.2.5 (see eq. (3.14)) that

$$A_v(s,w_0)\phi_{s,v}(g) = \frac{L(2s,\chi_{1,v}\chi_{2,v}^{-1})}{L(2s+1,\chi_{1,v}\chi_{2,v}^{-1})\epsilon(2s,\chi_{1,v}\chi_{2,v}^{-1},\upsilon_v)}\widetilde{\phi_{s,v}}(g).$$

Thus, we get $I_{b,v}(0,\pm1)$ equals

$$\begin{aligned} I_{b,v}(0,\pm1) &= \frac{L(2s,\chi_{1,v}\chi_{2,v}^{-1})}{L(2s+1,\chi_{1,v}\chi_{2,v}^{-1})\epsilon(2s,\chi_{1,v}\chi_{2,v}^{-1},\upsilon_v)} \times \\ &\times \int_{C_{\xi(0,\pm1)}(\mathbb{Q}_v)\backslash(\tilde{H}(\mathbb{Q}_v)\times U(\mathbb{Q}_v))} f(h_v^{-1}\xi(0,\pm1)u)\psi_v(u_v)\cdot\widetilde{\phi_{s,v}}(w_0h_v)\mathrm{d}h_v\mathrm{d}u_v, \end{aligned}$$

and we can replicate the computations in Theorem 3.4 for this integral.

3.7 Conclusion

We can summarize the results obtained thus far as follows:

Theorem. *For an appropriate choice of test function f and character ψ of $U(\mathbb{Q})\backslash U(\mathbb{A})$, the spectral side of the relative trace formula considered gives*

$$I(f) = \sum_{\pi} m_\pi \sum_{\varphi_i\in\pi} \frac{1}{\langle\varphi_i,\varphi_i\rangle} L(\varphi_i,\Phi,\mu,\nu,s)\overline{\varphi}_{i,\psi}\prod_{p\in S} a_i,p,$$

where $\varphi_{i,\psi}$ is the Fourier coefficient of φ_i with respect to character ψ (see eq. (3.10)) and $a_{i,p}$ is such that $\rho(f)\varphi_{i,p} = a_{i,p}\varphi_{i,p}$ for $p\in S$, with S a finite set of places. The outer sum is over π cuspidal representations in the space $\mathcal{A}_k^S(N)$, and the inner sum *is over an orthogonal basis of π.*

The corresponding geometric side is given by a sum

$$I(f) = I_a(0,1) + \sum_{\rho \in \mathbb{Z}} I_b(\rho, 1),$$

with $I_{a,v}(0,1) \neq 0$ and $I_{b,v}(0,1) \neq 0$ at all non-archimedean places v, as the prime level $N \to \infty$. The elements $I_a(\rho,\mu)$ and $I_b(\rho,\mu)$ represent the contribution from the coset representative $\xi(\rho,\mu)$.

The terms $I_b(\rho, 1)$ corresponding to double coset representatives $\xi(\rho, 1)$ with $\rho \in \mathbb{Z} \setminus \{0\}$ are equal to

$$\int_{C_{\xi(\rho,\mu)}(\mathbb{A})\backslash(\tilde{H}(\mathbb{A})\times U(\mathbb{A}))} f(h^{-1}\xi(\rho,\mu)u)\psi(u) \times$$
$$\times \left(\int_{N(\mathbb{A})} \phi_s(w_0 n w_0 h)\psi(\xi(\rho,\mu)^{-1}w_0^{-1} n w_0 \xi(\rho,\mu))\mathrm{d}n\right) \mathrm{d}h \ \mathrm{d}u.$$

In this case, the interior integral exhibits an oscillating behavior due to the factor $\psi(\xi(\rho,\mu)^{-1}w_0^{-1}nw_0\xi(\rho,\mu))$ which is not trivial unlike in the case when $\rho = 0, \mu = \pm 1$. It is our hope that we can use this fact to get a bound on these terms and show nonvanishing of $I(f)$.

On the spectral side it is enough that characters χ_1, χ_2 have just one prime place p (which we may assume is inert in F) where $\chi_{1,p}$ and $\chi_{2,p}$ are ramified. A computation similar to that in part (3) of Theorem 3.4 also shows that we can make some of the terms $I_b(\rho, 1)$ to be zero. Indeed, if we have a finite place p where $\chi_{1,p}$ and $\chi_{2,p}$ are ramified, then $I(\rho, 1)$ with ρ a multiple of p is zero.

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