Boundary Relative Entropy as Quasilocal Energy: Positive Energy Theorems and Tomography

Thesis by

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To Monica, Alexandra, my parents, and my grandparents.

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Abstract

We argue that for a spherical region R on the boundary, relative entropy between the vacuum and an arbitrary holographic excited state can be computed in the bulk as a quasilocal energy associated to the volume between R and the minimal surface $\tilde{B}$ ending on the boundary ∂R. Since relative entropy is monotonic and positive in any well-defined quantum theory, the associated quasilocal energy must also be positive and monotonic. This gives rise to an infinite number of constraints on the gravitational bulk, which must be satisfied in any theory of quantum gravity with a well-defined UV completion. For small regions R, these constraints translate into integrated positivity conditions of the bulk stress-energy tensor. When the bulk is Einstein gravity coupled to scalar fields, the boundary relative entropy can be related to an integral of the bulk action on the minimal surface $\tilde{B}$. Near the boundary, this expression can be inverted via the inverse Radon transform, to obtain the bulk stress energy tensor at a point in terms of the boundary relative entropy.

Published content and contributions

This thesis is based on paper [1] (DOI:10.1103/PhysRevLett.114.221601), which was an equal collaboration with Jennifer Lin, Matilde Marcolli and Hirosi Ooguri, and on paper [2], which was an equal collaboration with Jennifer Lin, Nima Lashkari, Hirosi Ooguri and Mark Van Raamsdonk, and it should not be cited without citing these two papers. Apart from a schematic (and qualitative) discussion in the conclusion, the results in this thesis come directly from these two papers, presented with the author's personal flavoring.

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Chapter 1

Brief introduction

This thesis covers a number of recent results in physics that live at the interface of gravity and information theory. These results fall under the broad umbrella of holography, which is a framework for describing a type of strong-weak duality between field theories and theories of gravity in one dimension higher. Over the last few years holography has seen rapid progress, as significant advances have been made toward better understanding its fundamental aspects. Although we are currently still far from a fundamental understanding of holography, it is the author's hope that rapid progress currently being made in the community will shed considerable more light in this direction within the next few years.

This thesis deals primarily with gravitational realizations of various types of entropy. Entropy is a macroscopic counting of degrees of freedom, and we will be concerned mostly with two types of entropy: entanglement entropy and relative entropy. The former can roughly be thought of as given by the degrees of freedom shared across a (fictitious) boundary, while the latter can be considered as a proxy for how different the degrees of freedom in a certain state are from those in another.

On the (dual) geometric side, we will be working with asymptotically anti-de Sitter spaces, which are manifolds obeying the Einstein equations that near the boundary asymptote to geometries of constant negative curvature. In this setting the entropies (and other field-theoretic quantities we will be concerned with) translate to geometric objects. Based on papers [1, 2], we will see that the relative entropy associated to a spherical boundary region can be viewed as a form of quasi-local energy, in the precise sense defined by Wald. The constraints obeyed by relative entropy map to an infinite family of constraints on the bulk, which hold for arbitrary spacetimes away from the vacuum. In various perturbative regimes, these constraints reduce to the linearized Einstein equations around vacuum, integrated positivity of the bulk stress-energy tensor and positivity of canonical energy.[1] Furthermore, our formula for relative entropy can be recast into a form which is amenable to bulk reconstruction from the boundary via techniques in integral geometry. While this cannot be done in general with the current technology, near the boundary inversion formulas exist and they reconstruct the bulk stress-energy tensor from the boundary, up to a certain order which will be made precise.

1.1 Outline of the thesis

The thesis proceeds as follows. In Chapter 2 we review some of the salient features of entanglement entropy, with emphasis on holography. Chapter 3 is the bulk of the thesis, and proceeds by introducing the necessary formalism (Wald's formalism), so that the computations we are interested in become natural. This will quickly get technical, but once it is established all our results will be readily obtained from one solid framework. We then present our results, by visiting quasilocal energy, bulk constraints, and bulk reconstruction, in this order. Finally, in Chapter 4 we present

[1] The linearized Einstein equations and canonical energy positivity stories have been derived independently, however our approach presents a unified method for obtaining them.

some future directions that arise from the work presented in this thesis, as well as some more speculative directions that could play important roles for holography in the coming years.

Chapter 2

Entanglement entropy and holography

The holographic principle has been around since the 70s, when 't Hooft noticed that in the large N limit (with, very importantly, g^2N held fixed[1]), a $U(N)$ gauge theory can be described in terms of a weakly coupled dual model, with dual coupling constant $1/N$ [3]. In its purest form it states that certain weakly coupled theories living on a spatial domain $\mathcal{M}$ have alternate descriptions as strongly coupled theories living on the boundary $\partial\mathcal{M}$.[2] The holographic principle was made much more precise by Maldacena [4], who suggested that Type IIB string theory on asymptotically $\mathrm{AdS}_5 \times S^5$ spacetime is the same theory as $\mathcal{N} = 4$ $SU(N)$ SYM in four dimensions. Paper [4] gave birth to the modern field of holography.

Since the field of holography has by now become a very vast subject, below we will only focus on the aspects we need. The aim is to give a self-contained presentation, which does not take too much space. Regarding the general workings of holography, all that we will need is that states in strongly-coupled CFTs are

[1]Here g is the gluon's charge.

[2]As a technical note, a more precise version would be that quantum gravity on $\mathrm{AdS}_{d+1} \times C$, with C a compact manifold, is dual to a field theory living on $\mathbb{R} \times S^{d-1}$, but such distinctions will not be important for the purposes of this thesis.

dual to weakly-coupled asymptotically AdS spacetimes, and bulk fields translate to boundary operators.

2.1 Entanglement entropy

For any quantum field theory (QFT), entanglement is a purely quantum feature that has no counterpart in classical physics. Given a partitioning of a Hilbert space $\mathcal{H}$ into subspaces $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$, the entanglement measures the failure of a state $|\psi\rangle \in \mathcal{H}$ to be written as a tensor product over the subspaces $\mathcal{H}_i$. There are at least two important features that make entanglement "quantum": (1) it is nonlocal, since the partitions of the Hilbert space can correspond to subsystems that are physically separated by large distances, and (2) there are no observables (or even quantities) that capture the structure of entanglement of a state in a faithful manner. Entanglement has several important roles in modern physics:[3] it is a crucial ingredient of quantum information theory and computing (for a review see e.g. [5]), and it is also a key player for topological order, where long-range entanglement is responsible for rich classes of physical phenomena [6,7]. However, the most important feature of entanglement is that it may still have things to teach us about fundamental physics [8].

One popular quantity which characterizes entanglement is entanglement entropy. For a partition $\mathcal{H} = \mathcal{H}_A \otimes H_B$ of the Hilbert space and a state $|\psi\rangle$ of density matrix ρ^ψ_{AB}, taking the partial trace $\rho^\psi_A = \mathrm{Tr}_B \rho^\psi_{AB}$, the entanglement entropy of A is defined as

$$S_{\mathrm{EE}}(A) = -\mathrm{Tr}\left(\rho^\psi_A \ln \rho^\psi_A\right) . \tag{2.1}$$

[3] There exist other equally important applications of entanglement omitted here due to spatial constraints.

This is just the Von Neumann entropy for density matrix ρ_A^ψ. Roughly speaking, the logic behind this definition is the following: When tracing out over subsystem B, what is the information in A we forget about? If it is zero then there is no correlation between A and B, otherwise there will be some.

Entanglement entropy has a number of features. It is non-negative, and for pure states ρ_{AB} it is reflexive, $S_{\text{EE}}(A) = S_{\text{EE}}(B)$. If the state is not pure then in general entanglement entropy is not reflexive, with the difference $S_{\text{EE}}(B) - S_{\text{EE}}(A)$ receiving a contribution from the classical (thermal) entropy. Finally, for QFTs living on spatial domains with A and B defined by partitioning the spatial domain into a region and its complement, entanglement entropy is UV divergent due to the degrees of freedom infinitesimally close to either side of the codimension-1 entangling surface.

2.2 Entanglement entropy in QFTs

A simple example in which entanglement entropy is easy to compute is that of two spin-1/2 degrees of freedom with the state given by a Bell pair,

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle \right), \tag{2.2}$$

in which case the entanglement entropy comes out to

$$S_{\text{EE}}(A) = S_{\text{EE}}(B) = \ln 2. \tag{2.3}$$

However, if the theory at hand is a QFT then entanglement entropy becomes notoriously difficult to compute, even for simple states and partitions. There are some techniques available, such as the replica trick [9], or (for spherical entangling surfaces) a clever mapping of the entanglement entropy to thermal entropy [10], but in general it remains a nontrivial endeavor.

One situation where the entanglement entropy can be computed via the replica trick is between a segment of length L and its complement in a $(1+1)$-dimensional CFT of central charge c, and it is equal to [11]

$$S = \frac{c}{3} \ln \frac{L}{\epsilon}, \tag{2.4}$$

with ϵ an UV cutoff.

It can also be shown that for general QFTs entanglement entropy obeys a number of inequalities. It is subadditive [12],

$$S(AB) \leq S(A) + S(B), \tag{2.5}$$

where A and B are two disjoint regions and AB is their union.[4] In fact, entanglement entropy obeys the considerably more powerful property of strong subadditivity (SSA) [13],

$$S(B) + S(ABC) \leq S(AB) + S(BC). \tag{2.6}$$

For arbitrary quantum field theories strong subadditivity is notoriously hard to prove [5, 14].

2.3 Holography: The Ryu-Takayanagi formula

Although entanglement entropy in QFTs is an unwieldy beast, the situation considerably improves if we restrict to theories which admit holographic duals. This

[4]Throughout this thesis writing $A_1 \dots A_n$ for disjoint regions stands for their union, $A_1 \sqcup \cdots \sqcup A_n$.

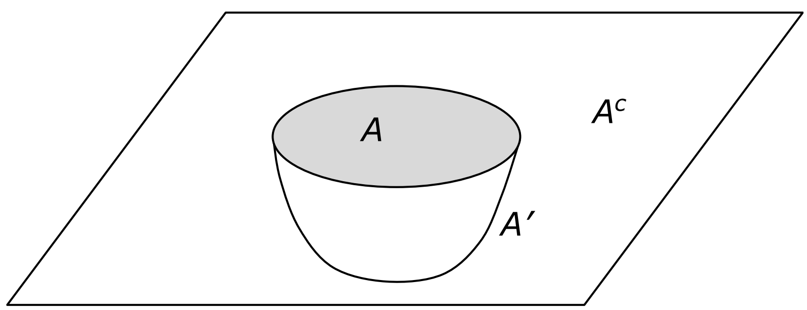
happens because holographic theories are a special class of theories living inside the space of all QFTs, which have more structure in the way the degrees of freedom are organized.

In the rest of the thesis, when working with holographic theories we will ignore both stringy α' corrections and quantum corrections.

Consider a QFT on the boundary of a gravitational bulk, and divide the boundary into a region A and its complement A^c. Ryu and Takayanagi [15,16] conjectured that the entanglement entropy of A (with A^c) is given by the area of the bulk minimal surface ending on the boundary ∂A that is homologous with A (see Figs. 2.1 and 2.2),

$$S_{\mathrm{EE}}(A) = \frac{\min A'}{4G_N}. \tag{2.7}$$

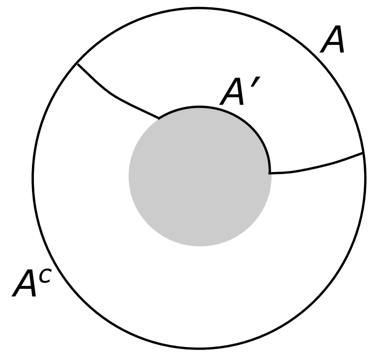
Figure 2.1: The bulk minimal area A' for a region A on a planar boundary.



There is by now ample evidence behind this conjecture, and it has been proven to various degrees [10,17–20], although a fundamental understanding of why it holds is still lacking. We only note that in the case of $\mathrm{AdS}_3/\mathrm{CFT}_2$ (with AdS radius ℓ) and planar boundary, the length of a geodesic ending on two points separated by distance L on the boundary is

$$S = 2\ell \ln \frac{L}{\epsilon}, \tag{2.8}$$

Figure 2.2: A black hole in the bulk (gray area) corresponds to a mixed (thermal) state in the CFT. The minimal area A' for domain A picks up a contribution from the horizon's thermal entropy and is no longer equal to the complement's minimal area.



so that Eqs. (2.4) and (2.8) agree provided that we identify the CFT central charge c with

$$c = \frac{3\ell}{2G_N}. \tag{2.9}$$

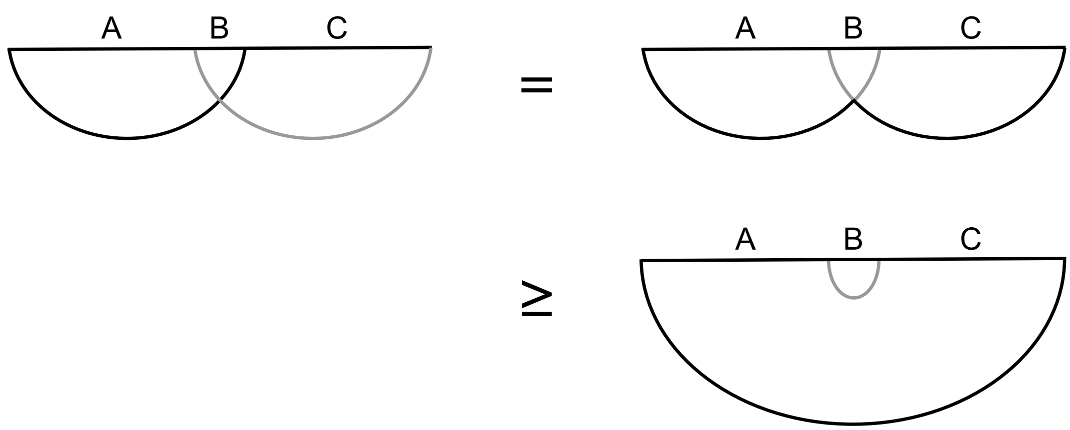
Brown and Henneaux [21] have argued from asymptotic symmetry analysis that this is indeed the correct identification to make.

2.4 Strong subadditivity revisited

The geometric interpretation of entanglement entropy makes manifest many of its properties. In particular, it is immediate to prove that SSA (2.6) holds, as we now review [22]. Suppose regions A, B and C are adjacent as in Fig. 2.3. Then it is possible to partition the minimal surfaces corresponding to $S(AB) + S(BC)$ into a surface subtending B and a surface subtending ABC; these repartitioned surfaces are not necessarily minimal, so they will be greater than or equal to $S(B) + S(ABC)$. This covers the case when the boundary regions are adjacent; the proof proceeds similarly when they are not.

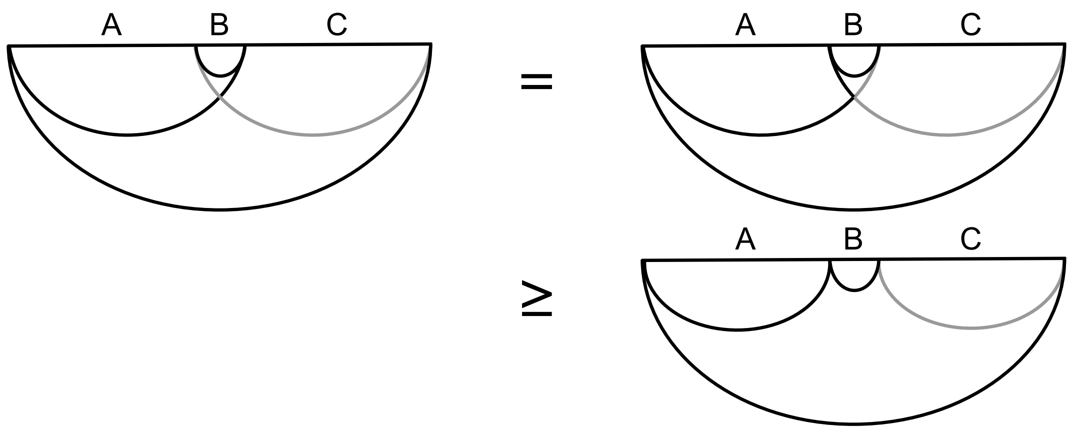
While this extra complication is essentially just bookkeeping, and for SSA can be dealt with by inspection (see [22]), there exists a general method for dealing with the cutting and pasting of minimal surfaces [23]: For the boundary intervals, draw all possible pieces of the minimal surfaces, labeling them by intersections of bulk regions and their complements. The cutting and pasting can be encoded by making use of a contraction mapping, which is a function with the property that the distances between any two points it maps get contracted. If such a function can be found, such that the contraction property holds, then the inequality is valid. This provides a combinatorial method for proving holographic inequalities; although a priori it is computationally expensive (doubly exponential complexity in the number of boundary regions), it turns out to work quite well when the number of regions is small. See [23] for more details.

Figure 2.3: Pictorial proof of strong subadditivity.



An interesting feature of this proof is that it only requires the existence of a ge-

Figure 2.4: Pictorial proof of mutual information monogamy.



ometric dual.[5] This dual does not have to be physically sound, and in particular it does not have to satisfy any energy conditions. This is because for static configurations the Ryu-Takayanagi formula only has access to the system at one snapshot in time, and it does not see any singularities or other problems that may develop when evolving forward in time. However, in time-dependent situations Wall [24] showed that SSA is implied by an integrated bulk null energy condition (NEC), and Lashkari *et al.* [25] showed that in some situations SSA implies an integrated bulk NEC.[6]

[5]What it means for the holographic dual to exist is made precise in [23], as the existence of a graph model on which cutting and pasting can be done.

[6]In time-dependent situations the entanglement entropy in the bulk is conjectured to be calculated via the Hubeny-Rangamani-Takayanagi prescription [26].

2.5 A holographic inequality: Mutual information monogamy

It turns out that in the large N limit (and ignoring stringy α' corrections) entanglement entropy in holographic theories obeys additional inequalities that do not hold for general CFTs. One such inequality is mutual information monogamy [27], and it states that

$$S(AB) + S(BC) + S(CA) \geq S(A) + S(B) + S(C) + S(ABC), \tag{2.10}$$

or in terms of the mutual information $I(A:B) \equiv S(A) + S(B) - S(AB)$,

$$I(A:BC) \geq I(A:B). \tag{2.11}$$

The proof of this inequality is similar to that of holographic strong subadditivity, and we sketch it in Fig. 2.4 for adjacent boundary regions. The general case is covered in the same manner as for SSA, by carefully labeling the bulk regions and cutting and pasting minimal regions (see [23, 27]).

The holographic proof of mutual information monogamy shares some features with the proof of SSA: the static case only relies on the existence of a bulk dual, and if time evolution is considered it can be derived from an integrated null energy condition [24]. However, it is not clear if there exist any situations where it implies any type of integrated bulk NEC.

Quantum entanglement between two parties has the property that it cannot be shared with a third party. Since mutual information in arbitrary QFTs need not be monogamous (due to the presence of classical correlations), the fact that holographic mutual information is monogamous can be thought of as indicating that the correlations in holographic theories are quantum.

Another feature of the monogamy inequality is that it provides a way of differentiating states (and theories) which have smooth geometric duals: If a QFT state admits a partitioning of the spatial domain such that the entanglement entropies do not obey Eq. (2.10), then it cannot be a holographic state (in the semiclassical limit).

Chapter 3

Holographic relative entropy and constraints on geometry

This chapter is based on paper [1], in collaboration with J. Lin, M. Marcolli, and H. Ooguri, and on paper [2], in collaboration with N. Lashkari, J. Lin, H. Ooguri, and M. Van Raamsdonk. It has significant overlap with these two papers (although it may emphasize things differently), and should not be cited without them. Most of the content comes directly from [1] and [2] with only minor edits.

CFT counterparts to the Einstein equations have also been considered in [28,29], and other papers which have investigated the connection between relative entropy, modular Hamiltonians and holography are [25,30–36].

The story in this chapter is built around a holographic computation of boundary relative entropy. This computation can be performed in two ways: either using Wald's formalism, or via a replica trick similar to that in [20]. In this thesis, we will only cover the Wald method; the replica trick approach is described in [2].

The setup consists of two CFT states, the vacuum and an arbitrary excited state (with a smooth holographic dual), and a region A in the spatial CFT domain (that is the same for both states). Using Wald's formalism, we will find that up to a universal offset, the relative entropy between the vacuum and the arbitrary state (for region A) is given by the integral of the diffeomorphism Noether current on the bulk region between A and the minimal surface ending on ∂A, plus a boundary term, and can be interpreted as a form of quasilocal energy associated to region A. Because relative entropy obeys certain universal properties, this apparently innocuous statement turns out to have some significant consequences in the bulk, in that it imposes constraints that any geometry dual to a well-defined CFT state must satisfy. For small regions these constraints turn into integrated positive energy conditions, and it is possible to turn them back into the CFT and obtain an additional constraint that relative entropy must satisfy (in theories with geometric duals). Furthermore, using the inverse Radon transform near the boundary it is possible to reconstruct the bulk stress energy tensor from relative entropy, up to a certain order which will be made precise. The formula we will derive also indicates a possible path to reconstructing the bulk action from relative entropy at points deep in the bulk, although more mathematical advances will be required in order to make this work.

3.1 Relative entropy

Relative entropy is an information-theoretic concept which, intuitively speaking, can be thought of as a proxy for how distinguishable two states are.[1] For two density matrices ρ^ψ and ρ^ϕ, it is defined as

$$S(\rho^\phi|\rho^\psi) = \mathrm{Tr}\left(\rho^\phi \ln \rho^\phi\right) - \mathrm{Tr}\left(\rho^\phi \ln \rho^\psi\right). \tag{3.1}$$

Relative entropy obeys a number of properties:

[1]However, it is not reflexive and thus not a metric; the metric on the state space is given by the Fisher information [25].

- It is non-negative, $S(\rho^{\phi}|\rho^{\psi}) \geq 0$, and $S(\rho^{\phi}|\rho^{\psi}) = 0$ iff $\rho^{\phi} = \rho^{\psi}$. In words, relative entropy vanishes only if the states coincide, otherwise it is positive because the states have to be at least a little bit distinguishable. The non-negativity is essentially a convexity statement and can be proven from Jensen's inequality.

- It is monotonic under increase of the spatial domain,

$$S\left(\rho_A^{\phi}|\rho_A^{\psi}\right) \leq S\left(\rho_B^{\phi}|\rho_B^{\psi}\right), \tag{3.2}$$

  if $A \subseteq B$. In particular, under an infinitesimal increase of the size R of the spatial domain A,

$$\partial_R S(\rho_A^{\phi}|\rho_A^{\psi}) \geq 0. \tag{3.3}$$

  This is also clear intuitively: If we look at more of the spatial domain, distinguishability between the two states cannot decrease.

- *The first law of entanglement entropy.* For density matrices ρ and $\rho + \epsilon\lambda$ that are infinitesimally close, relative entropy vanishes to first order in the small parameter ϵ,

$$S(\rho|\rho + \epsilon\lambda) = \mathcal{O}(\epsilon^{1+\delta}), \qquad \delta > 0. \tag{3.4}$$

  This happens because $S(\rho|\rho) = 0$, and moving away from ρ in any direction increases relative entropy, so that $\epsilon = 0$ is a local minimum and the gradient must vanish there.

It will be useful to add and subtract the term $\mathrm{Tr}\left(\rho^{\psi} \ln \rho^{\psi}\right)$ to the definition of relative entropy,

$$S(\rho^\phi|\rho^\psi) = \left[\mathrm{Tr}\left(\rho^\phi \ln \rho^\phi\right) - \mathrm{Tr}\left(\rho^\psi \ln \rho^\psi\right)\right] - \left[\mathrm{Tr}\left(\rho^\phi \ln \rho^\psi\right) - \mathrm{Tr}\left(\rho^\psi \ln \rho^\psi\right)\right] \quad (3.5)$$

$$= -\Delta S_{\mathrm{EE}} + \Delta\langle H_{\mathrm{mod}}\rangle. \quad (3.6)$$

The first bracket is (minus) the difference in entanglement entropy between the two states, and the second is the difference in expectation value of the modular Hamiltonian (for state ρ^ψ), which is an operator defined by relation (3.8) below. We will discuss the modular Hamiltonian in Sec. 3.2. Here, we only note that from Eq. (3.6) it is immediate to prove that the first law of entanglement entropy holds, since

$$\delta S_{\mathrm{EE}}(A) = -\delta\,\mathrm{Tr}\left(\rho_A \ln \rho_A\right) = -\mathrm{Tr}\left(\delta\rho_A \ln \rho_A\right) = \mathrm{Tr}\left(\delta\rho_A H_{\mathrm{mod}}(A)\right) = \delta\langle H_{\mathrm{mod}}(A)\rangle, \quad (3.7)$$

where we have used that the trace of any density matrix is normalized to unity, so $\mathrm{Tr}\delta\rho_A = 0$.

3.2 The modular Hamiltonian

Since any density matrix ρ is positive-semidefinite, it can be formally written as the exponential of some operator (normalized by its trace),

$$\rho = \frac{e^{-H_{\mathrm{mod}}}}{\mathrm{Tr}\, e^{-H_{\mathrm{mod}}}}. \quad (3.8)$$

This defines implicitly the modular Hamiltonian H_{mod}. In general the modular Hamiltonian is a nonlocal operator that is hard to compute, but there exist a few examples where it is tractable.

One case for which the modular Hamiltonian can be computed explicitly is that of a thermal state of temperature T, for which we have

$$H_{\text{mod}} = \frac{H}{T}. \tag{3.9}$$

Consider now the vacuum state of a QFT restricted to a Rindler wedge. Since the reduced density matrix is thermal with respect to the boost generator K [37], the modular Hamiltonian is [38, 39]

$$H_{\text{mod}} = 2\pi K = 2\pi \int\limits_{x>0} d^{d-1}x x T_{00}(\vec{x}), \tag{3.10}$$

with $T_{\mu\nu}$ the boundary stress-energy tensor. Since the Rindler wedge can be conformally mapped to the causal development of a ball $|\vec{x}| < R$, expression (3.10) can be mapped to the modular Hamiltonian corresponding to a reduced density matrix for the ball $|\vec{x}| < R$ in the vacuum state of a CFT [40, 41],

$$H_{\text{mod}} = 2\pi \int\limits_{|x|<R} d^{d-1}x \frac{R^2 - |\vec{x}|^2}{2R} T_{00}(\vec{x}). \tag{3.11}$$

Eq. (3.11) is the expression we will be working with in the rest of the thesis.

3.3 Wald's formalism

Wald's formalism loosely refers to a collection of quantities, tools and methods originally developed by Wald and collaborators as a way of computing black hole entropy in dynamical settings, for $f(R)$ theories of gravity, and ensuring that it obeys the first law of thermodynamics. Over time Wald's formalism has outgrown its original purpose, and it now has a broad range of applicability. In this thesis we will only be interested in how it can be applied to compute modular Hamiltonians and entanglement entropies, for which the relevant papers are [42–47].

It should not be surprising that Wald's formalism can be used to compute entanglement entropies and modular Hamiltonians, since minimal surfaces in aAdS are reminiscent of black hole horizons, and since for the case we are interested in (Eq. (3.11)), the modular Hamiltonian is a close cousin of the usual one.

3.3.1 Basics

We now review the aspects of Wald's formalism we are interested in. Our conventions are as follows. We work in $(d+1)$-dimensional asymptotically anti-de Sitter, and we write differential forms in bold. Latin indices $a, b, \ldots$ run over the bulk directions, Greek indices run over the boundary ones, and Latin indices $i, j, \ldots$ run over the spatial boundary indices. We denote the bulk fields (metric + matter) collectively by g. The $(d+1)$-dimensional volume form is denoted by $\boldsymbol{\varepsilon}$,

$$\boldsymbol{\varepsilon} = \frac{1}{(d+1)!}\sqrt{|g|}\epsilon_{a_1\ldots a_{d+1}} dx^{a_1} \wedge \cdots \wedge dx^{a_{d+1}}, \tag{3.12}$$

with $\epsilon_{a_1\ldots a_{d+1}}$ the antisymmetric symbol, and we can also define lower forms,

$$\boldsymbol{\varepsilon}_a = \frac{1}{d!}\sqrt{|g|}\epsilon_{a a_1\ldots a_d} dx^{a_1} \wedge \cdots \wedge dx^{a_d}, \tag{3.13}$$

$$\boldsymbol{\varepsilon}_{ab} = \frac{1}{(d-1)!}\sqrt{|g|}\epsilon_{ab a_1\ldots a_{d-1}} dx^{a_1} \wedge \cdots \wedge dx^{a_{d-1}}, \tag{3.14}$$

and so on.

Consider a theory of Einstein gravity plus matter specified by Lagrangian $\mathbf{L}$. Under an arbitrary variation δg of the bulk fields, the variation of the Lagrangian vanishes on-shell up to a boundary term $\mathbf{\Theta}$,

$$\delta \mathbf{L} = \mathbf{E}^g \delta g + d\mathbf{\Theta}(\delta g), \tag{3.15}$$

where $\mathbf{E}^g$ collectively denotes the bulk equations of motion (EOMs).

Using the identity

$$\mathcal{L}_X \mathbf{\Lambda} = i_X d\mathbf{\Lambda} + d\left(i_X \mathbf{\Lambda}\right), \tag{3.16}$$

which holds for arbitrary vector fields X and differential forms $\mathbf{\Lambda}$, the variation of the Lagrangian under a diffeomorphism generated by an arbitrary vector field X is

$$\delta_X \mathbf{L} = \mathcal{L}_X \mathbf{L} = d\left(i_X \mathbf{L}\right). \tag{3.17}$$

Here $\mathcal{L}_X$ is the Lie derivative and $i_X \mathbf{\Lambda}$ denotes contraction of X^a on the first index of an arbitrary form $\mathbf{\Lambda}$. Defining the Noether current $\mathbf{J}_X$ for the diffeomorphism generated by X as

$$\mathbf{J}_X = \mathbf{\Theta}(\mathcal{L}_X g) - i_X \mathbf{L}, \tag{3.18}$$

it will be conserved on-shell,

$$d\mathbf{J}_X = -\mathbf{E}^g \delta_X g. \tag{3.19}$$

Thus off-shell the Noether current can be written as

$$\mathbf{J}_X = d\mathbf{Q}_X + \mathbf{C}_X, \tag{3.20}$$

where $\mathbf{Q}_X$ is the diffeomorphism Noether charge and $\mathbf{C}_X$ are a combination of the equations of motion called the constraints, which vanish on-shell.

3.3.2 Symplectic structure

The d-form $\mathbf{\Theta}$ is a symplectic potential, and can be used to construct the theory's conserved charges (when they exist). Starting from $\mathbf{\Theta}$, one can define a symplectic current as

$$\boldsymbol{\omega}\left(\delta_1 g, \delta_2 g\right) = \delta_1 \mathbf{\Theta}\left(\delta_2 g\right) - \delta_2 \mathbf{\Theta}\left(\delta_1 g\right), \tag{3.21}$$

for arbitrary variations δ_1, δ_2.

We now derive an expression for the symplectic current. The variation of $\mathbf{J}_X$ is

$$\delta \mathbf{J}_X = \delta \mathbf{\Theta}\left(\mathcal{L}_X g\right) - i_X \delta \mathbf{L}. \tag{3.22}$$

Using definition (3.18), off-shell this is

$$\delta \mathbf{J}_X = \delta \mathbf{\Theta}\left(\mathcal{L}_X g\right) - i_X\left(\mathbf{E}^g \cdot \delta g\right) - i_X d\mathbf{\Theta}\left(\delta g\right). \tag{3.23}$$

With identity (3.16) for the third term in Eq. (3.23), we have

$$\delta \mathbf{J}_X = \delta \mathbf{\Theta}\left(\mathcal{L}_X g\right) - i_X\left(\mathbf{E}^g \cdot \delta g\right) - \mathcal{L}_X \mathbf{\Theta}\left(\delta g\right) + d\left[i_X \mathbf{\Theta}\left(\delta g\right)\right]. \tag{3.24}$$

Employing definition (3.21), with $\delta_1 = \delta$, $\delta_2 = \mathcal{L}_X$, Eq. (3.24) can be written as

$$\delta \mathbf{J}_X = \boldsymbol{\omega}\left(\delta g, \mathcal{L}_X g\right) - i_X\left(\mathbf{E}^g \cdot \delta g\right) + d\left[i_X \mathbf{\Theta}\left(\delta g\right)\right]. \tag{3.25}$$

Finally, making use of relation (3.20), off-shell the symplectic current becomes

$$\boldsymbol{\omega}\left(\delta g, \mathcal{L}_X g\right) = \delta \mathbf{C}_X + i_X\left(\mathbf{E}^g \cdot \delta g\right) + d\left[\delta \mathbf{Q}_X - i_X \mathbf{\Theta}\left(\delta g\right)\right]. \tag{3.26}$$

3.3.3 Conserved charges

For a d-dimensional region $\mathcal{M}$, the symplectic current can be integrated into a symplectic form,

$$W_{\mathcal{M}}\left(\delta_1 g, \delta_2 g\right) = \int\limits_{\mathcal{M}} \boldsymbol{\omega}\left(\delta_1 g, \delta_2 g\right). \tag{3.27}$$

Provided that certain boundary conditions are met (which will be explained in detail in Ch. 3), for a diffeomorphism generated by a vector X which is a symmetry the symplectic form, $W_{\mathcal{M}}$ provides the associated conserved charge, denoted H_X,

$$\delta H_X = \int\limits_{\mathcal{M}} \boldsymbol{\omega}\left(\delta g, \mathcal{L}_X g\right), \tag{3.28}$$

which using Eqs. (3.15), (3.17) and (3.20) can be rewritten as a boundary integral,

$$\delta H_X = \int\limits_{\partial\mathcal{M}} \left(\delta \mathbf{Q}_X - i_X \boldsymbol{\Theta}(\delta g)\right). \tag{3.29}$$

If there exists a d-form $\mathbf{B}$ such that on the boundary $\partial\mathcal{M}$ we have[2]

$$\delta \int\limits_{\partial\mathcal{M}} i_X \mathbf{B} = \int\limits_{\partial\mathcal{M}} i_X \boldsymbol{\Theta}\left(\delta g\right), \tag{3.30}$$

then the conserved charge can be integrated to

$$H_X = \int\limits_{\partial\mathcal{M}} \left(\mathbf{Q}_X - i_X \mathbf{B}\right). \tag{3.31}$$

[2]For asymptotically AdS spaces, the intuitive explanation behind this condition is that due to the asymptotics of the fields, near the boundary it should be possible to pull the δ out of the $\boldsymbol{\Theta}$ and into a total derivative.

This equation has multiple uses. For example, if $\mathcal{M}$ is a constant-time slice of the spacetime manifold[3] and $X = t$ is an asymptotic time translation, then

$$\mathcal{E} = \int_{\partial\mathcal{M}} (\mathbf{Q}_t - i_t\mathbf{B}) \tag{3.32}$$

is the energy associated to the time translation. Similarly, Eq. (3.31) can also be used to define angular momentum [43].

Intuitively, Hamiltonian H_X can be defined when the integral of Eq. (3.29) in configuration space is independent of the path in configuration space along which we integrate. Form $\mathbf{B}$ such that Eq. (3.30) is satisfied then exists, and the result of the integral along any path is Eq. (3.31). To show integrability, it suffices to construct explicitly a d-form $\mathbf{B}$ such that Eq. (3.30) holds for arbitrary variations. Alternatively, another sufficient (but not necessary condition) for integrability is the Wald-Zoupas condition

$$\int_{\partial\mathcal{M}} i_X\,\boldsymbol{\omega}(\delta_1 g, \delta_2 g) = 0, \tag{3.33}$$

for any two variations $\delta_1 g$, $\delta_2 g$ on the boundary $\partial\mathcal{M}$. For more details see Refs. [45, 46].

3.3.4 An example: Einstein gravity

For pure gravity of Lagrangian

$$\mathbf{L} = \frac{1}{2\kappa^2} R\boldsymbol{\varepsilon}, \tag{3.34}$$

[3] Since we assume a notion of energy, we also assume that there exists a constant time slice on which it makes sense to compute it.

the Noether charge is

$$\mathbf{Q}_X = -\frac{1}{2\kappa^2}\left(D^a X^b\right)\epsilon_{ab}, \tag{3.35}$$

the boundary term is

$$\boldsymbol{\Theta}\left(\delta g\right) = \frac{1}{2\kappa^2}\left(g^{am}D^b\delta g_{mb} - g^{mn}D^a\delta g_{mn}\right)\boldsymbol{\varepsilon}_a, \tag{3.36}$$

and the constraints are

$$\mathbf{C}_X = 2X^a\left(E^g\right)_a^{\ b}\ \boldsymbol{\varepsilon}_b, \tag{3.37}$$

with $\kappa^2 \equiv 8\pi G_N$.

3.4 Holographic boundary relative entropy

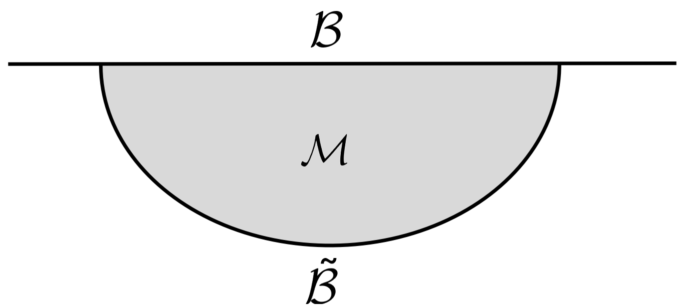
In this section we explain how to compute the boundary relative entropy from the bulk, for an arbitrary holographic state, using Wald's formalism. The same result can be obtained via the replica trick, which will not be explained here, see [2].

Consider a $(d+1)$-dimensional asymptotically AdS spacetime, and a CFT living on its boundary. Consider a surface of time-reflection symmetry through the spacetime, so that the Ryu-Takayanagi formula applies, and a spherical region $\mathcal{B}$ of radius R on the boundary.[4] On the constant-time slice, denote the d-dimensional bulk region between $\mathcal{B}$ and the minimal surface $\tilde{\mathcal{B}}$ ending on $\partial\mathcal{B}$ by $\mathcal{M}$ (see Fig. 3.1). We are interested in two states (and two corresponding geometries): the CFT vacuum, dual to empty AdS, and an arbitrary excited state, which has as holographic dual an asymptotically AdS space with bulk fields turned on. We denote the reduced density

[4]While the analysis of Sections 3.4 – 3.6 extends in a straightforward manner to maximin surfaces [24] in time-dependent situations that require the Hubeny-Rangamani-Takayanagi prescription [26], in this thesis we will only spell out the discussion for minimal surfaces embedded in surfaces of time-reflection symmetry, so that the Ryu-Takayanagi prescription applies.

matrix associated to region $\mathcal{B}$ by ρ^{vac} for the vacuum, and by ρ for the arbitrary state.

Figure 3.1: Computing the boundary relative entropy geometrically.



Suppose we can define a vector field X and a d-form $\mathbf{B}$ such that for both the vacuum and the arbitrary state we have

$$\int_{\mathcal{B}} \left(\mathbf{Q}_X - i_X \mathbf{B}\right) = \langle H_{\text{mod}} \rangle, \tag{3.38}$$

and

$$\int_{\tilde{\mathcal{B}}} \left(\mathbf{Q}_X - i_X \mathbf{B}\right) = S_{\text{EE}}. \tag{3.39}$$

Then the relative entropy between the vacuum and the arbitrary state (for the reduced density matrices) is

$$\begin{aligned} S\left(\rho|\rho^{\text{vac}}\right) &= \Delta\left(\langle H_{\text{mod}}\rangle - S_{\text{EE}}\right) & (3.40)\\ &= \Delta \int_{\partial\mathcal{M}} \left(\mathbf{Q}_X - i_X \mathbf{B}\right) & (3.41)\\ &= \Delta \int_{\mathcal{M}} \left[\mathbf{J}_X - d\left(i_X \mathbf{B}\right)\right]. & (3.42)\end{aligned}$$

Here Δ instructs us to take the difference between the excited geometry and empty AdS, and we have used that on-shell $\mathbf{C}_X = 0$.

Eq. (3.42) is one of our main results, expressing the boundary relative entropy in terms of the diffeomorphism Noether current and a boundary term. Its validity rests on the existence of X and $\mathbf{B}$ with the desired properties, which we will explain in Sec. 3.4.1 and 3.4.2. Since, apart from satisfying conditions (3.38) and (3.39), vector field X is unconstrained, we will choose it such that it asymptotes to the conformal AdS Killing vector near the boundary, and it vanishes on the minimal surface $\tilde{\mathcal{B}}$. Since the d-form $\mathbf{B}$ is determined only on the conformal boundary, Eq. (3.42) holds for all bulk values of $\mathbf{B}$ for which condition (3.38) holds.

3.4.1 Vector X on the minimal surface

We now detail the conditions X must satisfy on the minimal surface. Since X vanishes on $\tilde{\mathcal{B}}$, the condition we must impose is

$$\int_{\tilde{\mathcal{B}}} \mathbf{Q}_X = \frac{2\pi}{\kappa^2} A(\tilde{\mathcal{B}}) \tag{3.43}$$

for both the vacuum and the asymptotically AdS geometry, with $A(\tilde{\mathcal{B}})$ the area of $\tilde{\mathcal{B}}$. A sufficient (but not necessary) way of achieving this is by constraining the antisymmetrized derivative of X, so that our minimal surface conditions read[5]

$$D_{[a}X_{b]}\Big|_{\tilde{\mathcal{B}}} = 2\pi n_{ab}, \tag{3.44}$$

$$X\Big|_{\tilde{\mathcal{B}}} = 0. \tag{3.45}$$

[5]As explained by Iyer and Wald [43], using the freedom of adding total derivatives to $\boldsymbol{\Theta}$ and to the Lagrangian $\mathbf{L}$, the Noether charge can always be written as $-(1/2\kappa^2)D^a X^b \boldsymbol{\varepsilon}_{ab}$, even if the bulk contains matter.

Here n_{ab} is the binormal to the minimal surface, defined as

$$n_{ab} = n_a u_b - n_b u_a, \tag{3.46}$$

with n and u the spacelike and timelike normals to the surface respectively. Condition (3.44) ensures that

$$-\frac{1}{2\kappa^2}\left(D^a X^b\right)\boldsymbol{\varepsilon}_{ab} = \frac{2\pi}{\kappa^2} d\mathbf{S} \tag{3.47}$$

on the minimal surface, with $d\mathbf{S}$ the area $(d-1)$-form, so that Eq. (3.43) is satisfied.

3.4.2 Vector X on the boundary

When no fields are turned on in the bulk, there exists a bulk Killing vector on region $\mathcal{M}$,

$$X_{\text{asy}} = -\frac{2\pi}{R}(t-t_0)\left[z\partial_z + \left(x^i - x_0^i\right)\partial_i\right] + \frac{\pi}{R}\left[R^2 - z^2 - (t-t_0)^2 - \left(x^i - x_0^i\right)^2\right]\partial_t. \tag{3.48}$$

Here x_0^i is the center of the boundary ball $\mathcal{B}$. This Killing vector can be obtained from the time translation Killing vector in AdS-Rindler coordinates via a coordinate transformation [32]. On the boundary X_{asy} turns into a conformal Killing vector, which can be mapped to the Rindler wedge time translations [10].

When bulk fields are turned on, vector (3.48) will no longer be Killing, but it will remain an asymptotic symmetry as we approach the boundary. Specifically, we demand that in a neighborhood of the boundary we have

$$X = X_{\text{asy}} + \mathcal{O}\left(z^\delta\right) \tag{3.49}$$

with $\delta > 0$.[6]

With asymptotic condition (3.49) we can impose the precise version of Eq. (3.38), which reads

$$\Delta \int_{\mathcal{B}} (\mathbf{Q}_X - i_X \mathbf{B}) = \Delta \langle H_{\text{mod}} \rangle. \tag{3.50}$$

In words, we require the vacuum-subtracted charge associated to X to give the vacuum-subtracted modular Hamiltonian expectation value. This is a normalization issue, as without vacuum-subtraction the left-hand side and right-hand side of Eq. (3.50) would be off by a universal (divergent) term.

It is possible to argue the validity of Eq. (3.50) on general grounds, at least for certain matter content. This is because for pure gravity d-form $\mathbf{B}$ can be explicitly constructed as extrinsic curvature plus a cosmological constant (see Eq. (3.85) below, with $\mathbf{F} = 0$). In order for conformality to be preserved (and for expression (3.11) of the modular Hamiltonian to remain valid), any matter content must be normalizable; this implies that the matter fields decay rapidly towards the boundary, and so they will not contribute to the modular Hamiltonian expectation value in most, if not all, situations.[7]

In this section however, we will show Eq. (3.50) from geometry. The idea behind this method (first proposed in [32]) is that near the boundary the fields asymptote to their vacuum values, so for small balls the first law of entanglement entropy holds. This constrains the fields out of which the modular Hamiltonian is built in terms of the minimal area ending on small balls. For the sake of simplicity, for the rest of this section only we will work in Fefferman-Graham coordinates.

[6] This condition is again sufficient but not necessary. It is possible to weaken it somewhat, but there is no need to do so for our purposes.

[7] In cases in which the matter fields do contribute (should it be possible to arrange for such a scenario while respecting conformal symmetry), counterterms would need to be included in $\mathbf{B}$ through $\mathbf{F}$ in Eq. (3.11).

Consider an infinitesimal perturbation δ. From the first law of entanglement entropy, to leading order in the perturbation, in Fefferman-Graham coordinates we have[8]

$$\int_V \left(\delta \mathbf{Q}_X - i_X \mathbf{\Theta}(\delta g)\right) = \frac{d\ell^{d-3}}{2\kappa^2} \int_V X^\mu_{\text{cft}}\, \delta\Gamma_{\mu\nu}\, \epsilon^\nu, \tag{3.51}$$

where the left-hand side is the change in modular Hamiltonian expectation value, the right-hand side is the change in entanglement entropy, and the integral runs over the volume V between a small boundary ball and the minimal area ending on it. Here X_{cft} is the conformal Killing vector on the boundary,

$$X_{\text{cft}} = -\frac{2\pi}{R}(t-t_0)\left(x^i - x^i_0\right)\partial_i + \frac{\pi}{R}\left[R^2 - (t-t_0)^2 - \left(x^i - x^i_0\right)^2\right]\partial_t, \tag{3.52}$$

and $\delta\Gamma_{\mu\nu}$ is the change in induced metric on the minimal surface.

Eq. (3.50) can be recovered from Eq. (3.51) by integrating along any path in configuration space, from the vacuum to the state we're interested in. Since the right-hand side and the first term on the left-hand side are integrable, it follows that the $\mathbf{\Theta}$ term is also integrable, i.e. independent of the path in configuration space we take, and we denote the result of the integral by $\mathbf{B}$. This defines d-form $\mathbf{B}$ on the boundary.

3.5 Relative entropy from quasi-local energy

From Eq. (3.42), we have obtained that for region $\mathcal{B}$ the boundary relative entropy between the vacuum and an arbitrary excited state (with a holographic dual given

[8]This formula can be shown by explicit computation of both sides; for the details, see [32].

be a spacetime S) can be expressed in the bulk as

$$S(\rho|\rho^{\text{vac}}) = \text{Energy}_S(\mathcal{M}) - \text{Energy}_{\text{AdS}}(\mathcal{M}), \tag{3.53}$$

where

$$\text{Energy}_S(\mathcal{M}) = H_X(\mathcal{M}) = \int_{\mathcal{M}} [\mathbf{J}_X - d(i_X \mathbf{B})] \tag{3.54}$$

is the quasi-local energy given by expectation value of Hamiltonian H_X. Here X is any vector satisfying the previously mentioned boundary conditions; due to Stokes' theorem, all such vectors will correspond to the same Hamiltonian. Hamiltonian H_X exists because the Wald-Zoupas integrability condition (3.33) holds.

The existence of Hamiltonian H_X can be intuitively motivated by considering small perturbations of the geometry around vacuum AdS. In this case, $\mathcal{M}$ becomes (diffeomorphic to) a Rindler patch of AdS, and $H_X(\mathcal{M})$ becomes the Rindler energy.

Eq. (3.53) leads to an infinite set of constraints that aAdS spacetimes must satisfy. These come from the fact that in any well-defined CFT, relative entropy obeys the positivity and monotonicity constraints (3.2), (3.3). Thus, for any well-defined spacetime, the vacuum-subtracted quasi-local energy must be positive for any ball of any size,

$$\text{Energy}_S(\mathcal{M}) - \text{Energy}_{\text{AdS}}(\mathcal{M}) \geq 0, \tag{3.55}$$

and for any two balls contained in each other, $A \subseteq B$, the (vacuum-subtracted) quasi-local energy of the contained ball must be smaller than that of the containing ball,

$$\text{Energy}_S(\mathcal{M}_A) - \text{Energy}_{\text{AdS}}(\mathcal{M}_A) \leq \text{Energy}_S(\mathcal{M}_B) - \text{Energy}_{\text{AdS}}(\mathcal{M}_B). \tag{3.56}$$

Any low-energy effective theories which have as solutions spacetimes not obeying constraints (3.55), (3.56) cannot admit well-defined UV completions, and must instead lie in the swampland.

Conditions (3.55), (3.56) are very different from the constraints usually imposed on gravitational spacetimes. Indeed, to demand consistency of a spacetime it is common to demand a (pointwise or integrated) energy condition, which often (but not always, see [24, 35]) is postulated, rather than derived from some fundamental theory.[9] In contrast, our constraints are obtained by demanding the consistency of the CFT dual, and a priori have nothing to do with bulk energy conditions. However, in Sec. 3.6 we will see that in a certain limit, constraints (3.55), (3.56) do in fact reduce to integrated energy conditions. It remains an open question whether in general settings our relative entropy constraints can be implied by bulk energy conditions, and also whether they imply energy conditions when the spacetime is not close to vacuum.

An interesting connection to our results may be provided by Chen, Wang and Yau's work on quasilocal energy and mass [49–51]. Their work introduces a notion of quasilocal mass for arbitrary regions of spacetime in general relativity that obeys certain nice properties: (1) it is positive assuming the dominant energy condition, (2) it asymptotes to the ADM mass at spatial infinity in asymptotically flat spacetimes, (3) it is monotonic (but not additive) under increase of the spatial domain, and (4) it reduces to the Bel-Robinson tensor in vacuum, and to the matter density in non-vacuum. These properties are strongly reminiscent of the properties satisfied by our notion of quasi-local energy. However, the Wang-Yau mass is defined in a very different way, by minimizing over isometric embeddings in flat spacetime, so a priori

[9] In recent literature the null energy condition has been a popular choice, but other variants are the weak, strong or dominant energy conditions [48].

the connection to our quantity, if any, is unclear. It would be interesting to pursue this direction further, as it would help clarify, purely on the geometric side, which types of spacetimes satisfy our constraints. Quasi-local energy in asymptotically AdS spacetimes has recently been studied in [52, 53].

3.6 Perturbative bulk constraints

In this section we study perturbative consequences of Eqs. (3.55) and (3.56). These results have appeared in the literature as a series of papers, going to successively higher orders in the perturbation of the fields around vacuum. We will show how this hierarchy of results naturally follows from the nonperturbative equations (3.55) and (3.56).

3.6.1 The entanglement first law is equivalent to the linearized Einstein equations

When state ρ is arbitrarily close to the vacuum, the finite difference Δ turns into an infinitesimal difference, δ. The matter fields contribute at quadratic order in δ, thus to leading order in δ the bulk is pure gravity.

From Eq. (3.42), off-shell we have

$$S\left(\rho|\rho^{\mathrm{vac}}\right)=\delta\int\limits_{\mathcal{M}}\left[d\mathbf{Q}_X-d\left(i_X\mathbf{B}\right)\right]. \tag{3.57}$$

To first order in δ, the change in the minimal surface shape can be ignored, so the variation passes through the integral and using Eq. (3.26) we obtain

$$\delta S\left(\rho|\rho^{\mathrm{vac}}\right)=\int\limits_{\mathcal{M}}\left\{\boldsymbol{\omega}\left(\delta g,\mathcal{L}_X g\right)-\delta\mathbf{C}_X-i_X\left(\mathbf{E}^g\cdot\delta g\right)+d i_X\left[\boldsymbol{\Theta}\left(\delta g\right)-\delta\mathbf{B}\right]\right\}. \tag{3.58}$$

Since we work around empty AdS, X is given by expression (3.48), which is a Killing vector for the vacuum. Then $\mathcal{L}_X g=0$ everywhere and $\boldsymbol{\omega}$ vanishes since it is a bilinear form. The vacuum Einstein equations are $\mathbf{E}^g=0$ and by definition[10] $\delta\mathbf{B}=\boldsymbol{\Theta}\left(\delta g\right)$, so that Eq. (3.58) drastically simplifies to

$$\delta\langle H_{\mathrm{mod}}\rangle-\delta S_{\mathrm{EE}}=\int\limits_{\mathcal{M}}\delta\mathbf{C}_X, \tag{3.59}$$

where we have used Eq. (3.6). This shows that the linearized Einstein equations $\delta\mathbf{C}_X=0$ holding pointwise in the bulk implies the first law of entanglement entropy for spherical domains in the boundary, $\delta\langle H_{\mathrm{mod}}\rangle=\delta S_{\mathrm{EE}}$. Since the first law can be applied to balls of arbitrary size at arbitrary positions in the CFT (which correspond to minimal surfaces covering the entirety of the bulk), from Eq. (3.59) it also follows that the first law implies the linearized Einstein equations around the vacuum pointwise. Thus, the first law of entanglement entropy and the linearized Einstein equations are equivalent.

The result in this section first appeared in [31, 32]; our analysis reproduces it in a natural manner.

[10] For an explicit construction of $\mathbf{B}$ see Sec. 3.7.2.

3.6.2 Positive energy conditions near the boundary

We now add normalizable matter to the discussion in the previous section. This section is based on paper [1]. We consider the bulk to consist of Einstein gravity plus matter fields dual to boundary operators of scaling dimension $\Delta > d/2$. We allow for the backreaction from the matter fields to be parametrically large, however, we will restrict our analysis to the bulk region close to the boundary. More precisely, we take the radius R of the boundary ball to be small compared to the largest energy scale $\mathcal{E}$ of the CFT,

$$\mathcal{E}^d R^d \ll 1, \tag{3.60}$$

so that in this region the bulk metric deviations from vacuum AdS are small.

A priori, under a change in the bulk metric, the change in minimal surface arises in two ways: (1) from the change of the induced metric, and (2) from the change in the position of the minimal surface. Because the surface is minimal, to first order in δg the change in area coming from the change in position vanishes (so we did not have to worry about it in the previous section), but it does enter at quadratic order, corresponding to terms of order $\mathcal{O}\left(\mathcal{E}^{2d} R^{2d}\right)$ and higher. Bulk matter dual to a boundary operator of dimension Δ contributes to δg at order $\mathcal{O}\left(\mathcal{E}^2 R^{2\Delta}\right)$ (because the fields are of order $\mathcal{O}\left(\mathcal{E} R^{\Delta}\right)$, and they backreact at order squared on the metric). Consequently, without accounting for the change in minimal surface position, we can keep control over operators of dimension $\Delta < d$, that is over relevant operators.

Our analysis in this section thus applies to operators with dimension $d/2 < \Delta < d$. For matter dual to such operators we can use the result of the previous section,

$$S\left(\rho|\rho^{\text{vac}}\right) = \int\limits_{\mathcal{M}} 2X^a_{\text{asy}} \left(E^g\right)_a^{\ b} \ \boldsymbol{\varepsilon}_b, \tag{3.61}$$

where E^g is now only the geometric part of the Einstein equation (including the cosmological constant). From the Einstein equations $(E^g)_a{}^b = T_a^b$, with T_a^b the bulk stress-energy tensor, so we have

$$S(\rho|\rho^{\text{vac}}) = \int_{\mathcal{M}} 2X^a_{\text{asy}}\, T_a{}^b\, \boldsymbol{\varepsilon}_b \tag{3.62}$$

$$= 8\pi G_N \int_{\mathcal{M}} \sqrt{|g|} X^t_{\text{asy}}\, \mathfrak{E} \geq 0, \tag{3.63}$$

where $\mathfrak{E}$ is the energy density in the bulk and $\sqrt{|g|}$ is the determinant of the full metric.

Eq. (3.63) implies that the positivity of relative entropy is equivalent to the integrated positivity of the bulk energy over the region between boundary and minimal surface. Taking an R derivative brings us to the monotonicity,[11]

$$\partial_R S(\rho|\rho^{\text{vac}}) = 8\pi^2 G_N \int_{\mathcal{M}} \sqrt{|g|} \left(1 + \frac{x^2+z^2}{R^2}\right) \mathfrak{E} \geq 0, \tag{3.64}$$

where we have used the explicit form (3.48) of X_{asy} (and that it vanishes on the minimal surface).

Eqns. (3.63) and (3.64) are dictionary entries between the positivity and monotonicity of relative entropy and an integrated positive energy condition in the bulk. Because we are working with inequalities, expressions (3.63) and (3.64) cannot be inverted to obtain the pointwise positivity of bulk energy density (and indeed the bulk energy density doesn't have to be pointwise positive in asymptotically AdS spaces).

We can deduce one more inequality by taking one more R derivative and assuming

[11] R is the radius of the boundary ball, as before.

that the bulk energy density is positive,

$$\left(\partial_R^2 + R^{-1}\partial_R - R^{-2}\right) S\left(\rho|\rho^{\text{vac}}\right) = 16\pi^2 G_N \int\limits_{\partial\mathcal{M}_2} \sqrt{|g|}\,\mathfrak{E} \geq 0. \tag{3.65}$$

Here $\partial\mathcal{M}_2$ denotes the minimal surface. Eq. (3.65) is a boundary prediction obtained from the bulk, for the operators and region sizes to which our analysis applies, and for holographic states.

Beyond the perturbative order discussed in this subsection, it is possible to push the analysis to second order in δ. This was done in [25], and the result is that Fisher information on the boundary is dual to canonical energy on the gravitational side,

$$\delta^2 S\left(\rho|\rho^{\text{vac}}\right) = W_{\mathcal{M}}\left(\delta g, \delta\left(\mathcal{L}_{X_{\text{asy}}} g\right)\right). \tag{3.66}$$

Fisher information is positive, so the canonical energy around the vacuum is positive. This implies that the bulk (in our case the AdS-Rindler wedge) is stable to linearized axisymmetric perturbations. For more details see [25, 47].

3.7 Towards holographic reconstruction

In this section and the next we discuss to what extent Eq. (3.42) can be inverted to recover bulk data from the boundary. There are two steps to such a reconstruction: (1) massaging Eq. (3.42) into a form suitable for inverting, and (2) performing the inversion. As we will see, step (1) can generally be accomplished. However, for the second step, only partial results currently exist in the mathematics literature on integral geometry, so we will only be able to invert our formula near the boundary (Sec. 3.8). Reconstruction deeper in the bulk will have to await further developments on the mathematical side.[12]

For the rest of section 3, we will restrict our discussion to spacetimes that are time-reflection symmetric around a slice $\mathcal{S}$ of constant time.[13]

3.7.1 The general argument

We parametrize the minimal surface ending on $\partial\mathcal{B}$, for a spherical domain $\mathcal{B}$ of radius R, by introducing a function $f(x^a)$ such that on the minimal surface

$$f(x^a) = R. \tag{3.67}$$

Since we assume time-reflection symmetry, f is an even function of t. Furthermore, time-reflection symmetry implies that in a neighborhood of $\mathcal{S}$ the metric obeys

$$\partial_t g_{tt} = \partial_t g_{\alpha\beta} = \mathcal{O}(t), \qquad g_{t\alpha} = g_{\alpha t} = \mathcal{O}(t), \tag{3.68}$$

with the Greek indices running over the spatial directions.

The introduction of f allows us to construct an explicit expression for the vector field X (that we naturally define through a 1-form),

$$X(R) = -2\pi t\sqrt{-g_{tt}}\,\frac{f\,df}{R||df||} - \left(R - \frac{f(x^a)^2}{R}\right)\frac{\pi\sqrt{-g_{tt}}\,dt}{||df||}. \tag{3.69}$$

[12] Or bold conjectures from physics.

[13] The analysis of Sec. 3.7 and 3.8 relies on the parity of fields under $t \to -t$ reflections (in situations of reflection symmetry), and as such it is not as straightforward to generalize to arbitrary time-dependent situations as it is for the discussion in Sections 3.4 – 3.6. However, it is likely that an appropriate generalization to time-dependent situations exists.

Here df is the gradient of (scalar) function field f,

$$df = \partial_a f dx^a, \tag{3.70}$$

and $||df||$ is its norm, $||df|| = \sqrt{g^{ab}\partial_a f \partial_b f}$. The gradient vector field ${}^*df = g^{ab}(\partial_a f)\partial_b$ is orthogonal to the minimal surface ending at radius R.

We now check that vector field X (Eq. (3.69)) satisfies the desired properties on the boundary and minimal surface (vanishing and giving the area element on the minimal surface, and asymptoting to the AdS Killing vector X_{asy} (3.48) near the boundary). First, we note that X is defined such that it vanishes on the minimal surface at $f = R$, $t = 0$. Near the boundary we have

$$f \to \sqrt{t^2 + z^2 + x^2}, \qquad \sqrt{-g_{tt}}||df|| \to 1, \tag{3.71}$$

so that $X^a \to X^a_{\text{asy}}$. Finally, with $d\mathbf{S}$ the area element $(d-1)$-form a direct computation shows that

$$-\frac{1}{4\kappa^2}\left(D^a X^b - D^b X^a\right)\boldsymbol{\varepsilon}_{ab} \overset{\text{RT}}{=} -\frac{\pi}{\kappa^2}\left(n^a u^b - n^b u^a\right)\boldsymbol{\varepsilon}_{ab} = \frac{2\pi}{\kappa^2}d\mathbf{S}, \tag{3.72}$$

where in the last equality we used identity

$$A^{a\ (n)}\boldsymbol{\varepsilon}_a = N_a A^{a\ (n-1)}\boldsymbol{\varepsilon} \tag{3.73}$$

for arbitrary vector A and unit normal N (see e.g. Appendix B of [54]). Thus, explicit construction (3.69) satisfies all the required conditions on the boundary and minimal surface.

Vector X is defined such that it behaves nicely under the action of various combinations of R and its derivatives. In particular, via direct computation we obtain

$$\left(\partial_R + R^{-1}\right) X = 2\pi T, \tag{3.74}$$

where T is a time-like vector defined (through a 1-form) as

$$T = -\frac{\sqrt{-g_{tt}}}{||df||} dt. \tag{3.75}$$

Vector T can be thought of as a kind of red-shifted time flow. Eq. (3.74) immediately implies

$$\left(\partial_R + R^{-1}\right) \mathbf{J}_X = 2\pi \mathbf{J}_T, \tag{3.76}$$

so that (since R-derivatives also act on the minimal surface) taking R-derivative of the integral on $\mathcal{M}$ we have

$$\left(\partial_R + R^{-1}\right) \int\limits_{\mathcal{M}} \mathbf{J}_X = 2\pi \int\limits_{\mathcal{M}} \mathbf{J}_T + \int\limits_{\partial\mathcal{M}} v \cdot \mathbf{J}_X. \tag{3.77}$$

Here v is a vector normal to the minimal surface with normalization given by $g^{ab} v_a (df)_b = 1$.

Expression (3.77) can be related to the relative entropy. Using Eq. (3.42), we have

$$\left(\partial_R + R^{-1}\right) S(\rho|\rho^{\text{vac}}) = 2\pi \Delta \mathcal{H}_T + \Delta \int\limits_{\partial\mathcal{M}} i_v \left(\mathbf{J}_X - d(i_X \mathbf{B})\right), \tag{3.78}$$

with quantity $\mathcal{H}_T$ defined as

$$\mathcal{H}_T = \int\limits_{\mathcal{M}} \mathbf{J}_T - \int\limits_{\partial\mathcal{M}} i_T \mathbf{B}. \tag{3.79}$$

Although object $\mathcal{H}_T$ has the form of Eq. (3.31), we write it with a curly $\mathcal{H}$ to emphasize it does not readily have the interpretation of a Hamiltonian associated to vector field T. This is because T does not vanish on the minimal surface, so the integrability condition (3.33) doesn't have to hold. Intuitively, this means that integrating δH_T in configuration space, from the vacuum to the finite state, could give different answers along different paths, which need not agree with Eq. (3.79). For the purposes of this thesis it will remain an open problem in which situations, if any, object $\mathcal{H}_T$ can be interpreted as a Hamiltonian. What matters for us is that $\mathcal{H}_T$ will allow writing a certain combination of R derivatives acting on the relative entropy as a surface integral, in a form which can be readily inverted using an inverse Radon transform (when one exists).

Vector T has the property that (with raised indices) it asymptotes to the usual time translation Killing vector near the boundary, $T \to \partial_t$, and it does not depend on R, although it does depend on the minimal surface. However, the precise physical meaning of T (other that it is a kind of red-shifted measure of time) is not completely clear.

We now massage the second term in Eq. (3.78) to show that it equals zero. We have

$$d(i_X \mathbf{B}) = \mathcal{L}_X \mathbf{B} = \mathbf{\Theta}(\mathcal{L}_X g), \tag{3.80}$$

where in the first equality we used that $X = 0$ on the minimal surface, and in the second inequality we used that since the Wald-Zoupas integrability condition (3.33) holds, the Lie derivative can be pulled out of $\mathbf{\Theta}$ (cf. Eq. (3.30)).[14] Thus, using definition (3.18) for $\mathbf{J}_X$ and the fact that X vanishes on the minimal surface, we

[14]The second equality in Eq. (3.80) can be thought of as the definition of $\mathbf{B}$. $\mathbf{B}$ so defined equals the $\mathbf{B}$ in Eq. (3.77), since under an infinitesimal variation $\delta H_X = \delta \mathbf{Q}_X - i_X \mathbf{\Theta}\,(\delta g)$.

have

$$\left(\partial_R + R^{-1}\right) S(\rho|\rho^{\text{vac}}) = 2\pi\Delta\mathcal{H}_T. \tag{3.81}$$

Applying one more R derivative to this equation gives[15]

$$\partial_R \left(\partial_R + R^{-1}\right) S(\rho|\rho^{\text{vac}}) = 2\pi\Delta \int_{\tilde{B}} i_v \left(\mathbf{J}_T - d(i_T \mathbf{B})\right). \tag{3.82}$$

The right-hand side of this formula is an integral over the minimal surface, which is the type of object that can be inverted via the inverse Radon transform, when such a construction is known to exist. We will cover such a case in Sec. 3.8.

3.7.2 Scalar fields in Einstein gravity

For a bulk theory of Einstein gravity plus scalar fields,

$$L = \frac{1}{2\kappa^2} R - \frac{1}{2}\left(\partial\phi^I\right)^2 - V(\phi^I), \tag{3.83}$$

the right-hand side of Eq. (3.82) can be simplified further. We employ the geometric identity (valid for arbitrary variations δ, and up to a total derivative term we discard)

$$\delta\left(K\boldsymbol{\varepsilon}^{(d)}\right) = \frac{1}{2}\boldsymbol{\varepsilon}^{(d)}\left(K_{ab} - \gamma_{ab}K\right)\delta\gamma^{ab} + \frac{1}{2}\boldsymbol{\varepsilon}^{(d)} n^a \left(-D^b \delta g_{ab} + g^{cd} D_a \delta g_{cd}\right). \tag{3.84}$$

Here γ_{ab}, K_{ab}, and $\epsilon^{(d)}$ are the induced metric, extrinsic curvature, and volume form on $\partial\mathcal{M}$ embedded in the slice of constant time. Vector n^a is the spacelike unit normal

[15] The physical meaning of derivative combination $\partial_R\left(\partial_R + R^{-1}\right)$ is not currently clear. Near the boundary, this combination annihilates the AdS Killing vector X_{asy} pointwise, however there is no obvious reason why this operator remains the correct object to apply to the relative entropy when the minimal surface goes deep in the bulk.

to $\partial\mathcal{M}$.[16]

For this choice of Lagrangian, the boundary term can be chosen to be the usual Gibbons-Hawking term plus counterterms,

$$\mathbf{B} = -\frac{1}{2\kappa^2}\left(K + \frac{d-1}{\ell}\right)\boldsymbol{\varepsilon}^{(d)} + \mathbf{F}(\phi^I), \tag{3.85}$$

with $\mathbf{F}(\phi^I)$ the scalar counterterms, if any, necessary to obtain $\Delta\langle H_{\text{mod}}\rangle$ when integrating $\mathbf{Q}_X - i_X\mathbf{B}$ on B.[17] With choice (3.85) for the boundary term, we have

$$\delta\mathbf{B} = \boldsymbol{\Theta}\left(\delta g, \delta\phi^I\right) + \frac{1}{2\kappa^2}\boldsymbol{\varepsilon}^{(d)}\left(K^{ab} - \gamma^{ab}K - \gamma^{ab}\frac{d-1}{\ell}\right)\delta\gamma_{ab} + \left(D^a\phi^I\right)\delta\phi^I\,\boldsymbol{\varepsilon}_a - \delta\mathbf{F}. \tag{3.86}$$

In general, the last three terms in this equation need not be zero. However, for the Lie derivative in the T direction, $\delta = \mathcal{L}_T$, parity conditions (3.68) (together with the fact that the ϕ^I's are scalars, rather than pseudoscalars), ensure that these three terms do vanish on the time-slice containing $\mathcal{M}$. Thus, only on the Ryu-Takayanagi surface we have

$$\mathcal{L}_T\mathbf{B} = \boldsymbol{\Theta}\left(\mathcal{L}_T g, \mathcal{L}_T\phi^I\right), \tag{3.87}$$

such that Eq. (3.82) simplifies to

$$\partial_R\left(\partial_R + R^{-1}\right)S(\rho|\rho^{\text{vac}}) = -2\pi\Delta\int_{\tilde{B}} i_v i_T\left(\mathbf{L} - d\mathbf{B}\right). \tag{3.88}$$

This shows that for Einstein gravity plus scalar fields, being able to perform the inverse Radon transform would recover the action at a bulk point (including the

[16]Identity (3.84) is useful when considering the variation of the Einstein-Hilbert action with Dirichlet boundary conditions; see e.g. Eq. (34) of [44].

[17]Since we are only interested in normalizable scalar fields, due to the rapid decay of the ϕ^I's towards the boundary, these counterterms should be ignorable in most situations, if not in all.

Gibbons-Hawking-like term) in terms of the boundary relative entropy.[18]

3.8 Inverse Radon transform near the boundary

Although Eq. (3.88) cannot generally be inverted with the current mathematical technology, if we restrict to points close to the boundary it turns out that it is possible to reconstruct bulk information from the boundary. In this case, the object recovered is the bulk stress-energy tensor. As in Sec. 3.6.2, we restrict our analysis to small radii $\mathcal{E}^d R^d \ll 1$, and to bulk fields dual to operators with scaling dimension $d/2 < \Delta < d$.[19]

We start from Eq. (3.65),

$$\left(\partial_R^2 + R^{-1}\partial_R - R^{-2}\right) S\left(\rho|\rho^{\text{vac}}\right) = 16\pi^2 G_N \int\limits_{\partial\mathcal{M}_2} \sqrt{|g|}\,\mathfrak{E}. \tag{3.89}$$

When the background is empty AdS, the minimal surface $\partial\mathcal{M}_2$ ending on a sphere is hyperbolic, and so it is totally geodesic (meaning that all geodesics on the minimal surface are also geodesics on the constant time slice in which the minimal surface is embedded). When $\partial\mathcal{M}_2$ is totally geodesic, the right-hand side of Eq. (3.89) is the Radon transform, and its inverse for hyperbolic spaces exists in the mathematics literature [55, 56], as we now explain.

[18]A related subtlety, as explained by Iyer and Wald [44], is that adding to $\mathbf{B}$ any function only dependent on the intrinsic geometry on $\partial\mathcal{M}$ does not change Eq. (3.87). On B, this gauge freedom is fixed by the requirement to recover the modular Hamiltonian expectation value, however there is no similar requirement on $\tilde{B}$. Any inverse Radon transform reconstruction, if it exists, would have to somehow perform this gauge fixing on $\tilde{B}$.

[19]It it not hard to see that in this limit we should recover the boundary stress-energy tensor, and not the action: Eq. (3.88) holds when the right-hand side is purely geometric (and off-shell), in which case taking the variation Δ turns the integrand into the geometric part of the equations of motion, which equals the bulk stress-energy tensor. This is another way of recovering Eq. (3.65).

For a D-dimensional space and a function f, the Radon transform $\mathcal{R}f$ integrates f on a surface that is a totally geodesic submanifold of dimension $n < D$, and associates the result to the surface in the space of totally geodesic submanifolds. The inverse Radon transform $\mathcal{R}^*\mathcal{R}f$ works backwards: It integrates over totally geodesic submanifolds and associates the result to a point in the usual space; the result is just the value of the function at that point.

For odd d and totally geodesic submanifolds of dimension $n = d-1$, according to [55], the (inverse and direct) Radon transforms obey the identity

$$f = \frac{1}{(-4)^{(d-1)/2}\pi^{d/2-1}\Gamma(d/2)} Q\left(\boldsymbol{\Delta}\right)\mathcal{R}^*\mathcal{R}f, \tag{3.90}$$

with f a test function (defined on the usual space) and $Q\left(\boldsymbol{\Delta}\right)$ a polynomial built out of the Laplace-Beltrami operator $\boldsymbol{\Delta}$,

$$Q\left(\boldsymbol{\Delta}\right) = \left[\boldsymbol{\Delta} + 1\cdot(d-2)\right]\left[\boldsymbol{\Delta} + 2\cdot(d-3)\right]\cdot\ldots\cdot\left[\boldsymbol{\Delta} + (d-2)\cdot 1\right].$$

Applying Eq. (3.90) to (3.89), we thus obtain the energy density at a point in the bulk in terms of the boundary relative entropy as

$$\mathfrak{E} = \frac{1}{(-4)^{(d+3)/2}\pi^{d/2+1}\Gamma(d/2)G_N} \times Q\left(\boldsymbol{\Delta}\right)\mathcal{R}^*\left(\partial_R^2 + R^{-1}\partial_R - R^{-2}\right)S(\rho|\rho^{\mathrm{vac}}). \tag{3.91}$$

Eq. (3.91) is a toy (and yet quite complicated) example of bulk data reconstruction in terms of boundary information. There exists a similar formula for even d, see [56].

At this point we should remember that Eq. (3.91) is approximate, and the approximations creep in several places: (1) the formula we inverted, Eq. (3.89), is approximate and only valid near the boundary, (2) the inversion formula (3.90) is valid for hyperbolic spaces (with no backreaction), and (3) there are totally geodesic surfaces that pass through the reconstruction point at z and go deep into the bulk (but their contributions are negligible when $\mathcal{E}z \ll 1$, with $\mathcal{E}$ the typical energy scale of the CFT). Thus, it would be interesting to obtain an exact inversion formula of Eq. (3.88).

Chapter 4

Conclusion

4.1 Future directions

We now discuss some of the natural follow-up directions that arise from the work in this thesis. Some of these ideas may not be too hard to put in practice, while others may be considerably more difficult.

- A straightforward (if not too exciting) direction would be to give examples of some theories that satisfy the inequality constraints derived in this thesis, and examples that don't. Apart from answering a very natural question, this would potentially shed light on what goes wrong when our constraints are not satisfied. It is tempting to speculate that the pathology has to do with too much negative energy density in the bulk, and consequently with a Hamiltonian becoming unbounded from below, but it would be desirable to make this precise.

- Another open thread is generalizing our discussion to time-dependent cases. Although most of the thesis can be extended to discussions on maximin surfaces (even if we did not spell this out), Sec. 3.7 on holographic reconstruction will need considerably more work. However, the computations needed are similar in spirit to the computations appearing in the HRT proposal, so there is some hope this could be made to work.

If the discussion is expanded to the time-dependent case, it will probably increase the discerning power of the constraints, since now they will also explicitly know about the dynamics of the theory.

- Obtaining the boundary dual of the Einstein equations in the nonlinear regime. Although this seems like a most natural question to ask, it may turn out to not be too closely related to the discussion in this thesis, and the linearized result may turn out to have been some sort of coincidence. This is because answering it will almost certainly require new ingredients on the CFT side, which reduce to the first law of entanglement entropy in the linearized regime, and it is not clear what these ingredients should be. A (possibly related) complication is the existence of entanglement shadows: bulk regions not probed by any minimal surface, which exist even in nice geometries (such as AdS stars) [57]. Since the Einstein equations hold everywhere, it is tempting to conjecture they should not be associated to minimal surfaces.

- Connecting our results to the Wang-Yau quasilocal energy and mass. This is a very exciting direction, but it may be hard to put in practice. Our results are (superficially at least) similar to the features of the Wang-Yau quasilocal mass, and it is tempting to conjecture they may be related. However, the definition of the Wang-Yau quasilocal mass is very different from our our definition of quasilocal energy, and it is not clear the two can be reconciled. It would be very interesting to do so, or to prove that no reconciliation exists. A related question is whether our constraints are implied by bulk energy conditions, or whether they imply some energy condition.

- Radon transform inversion for asymptotically AdS spaces. This is almost certainly very hard, but it may not be impossible, at least in certain cases. Since the mathematics literature on the subject is currently lacking, some radical approaches would be needed. Two ideas come to mind: (1) Based on expression (3.90), make some informed guesses as to what the inversion formula should be in general, and try to check these guesses against some examples, deferring proof for later, and (2) Try to use advanced number-theoretic machinery, using as inspiration what was done in [58]. The first approach has the disadvantage that it is accidental, so even if it works, it will not give immediate understanding, and the second approach has the disadvantage that it will probably be of considerable difficulty to set up the necessary machinery, if it even is possible. However, given sufficient time, if this second approach works it will provide a beautiful connection between general relativity, integral geometry and number theory.

- Is there a relation between our work and the "Complexity Equals Action" story [59,60]? The reason to suspect this is that the bulk action plus boundary term $\mathbf{L} - d\mathbf{B}$ plays a central role in both stories. However, the ways $\mathbf{L} - d\mathbf{B}$ enters are very different, and a priori there is no motivation to connect the two, since [59, 60] use the action in conjunction with non-minimal extremal surfaces, for which our results have no predictive power. But it may not be too far-fetched to speculate that with one more conceptual leap, a natural connection could be made.

4.2 The distant landscape

Before concluding, we should take some time to speculate what lies ahead for the field of holography. This section is highly speculative, and highly heretical. Unlike in the rest of the thesis, the tone in this section is lighthearted, and the section itself should not be taken too seriously.

Recent advances [61, 62] have pointed out that there exists a surprising connection between holography and quantum error-correcting codes (QECs). The reason for this is natural: bulk reconstruction from the boundary has a redundancy (and nonlocality) structure which is best captured by a QEC. However, QECs, like any error-correcting code, are most naturally associated to finite fields $\mathbb{F}_q$. Thus primes come into play, and an immediate question is whether there exists a natural bulk structure associated to the discreteness of a finite field, and with number-theoretic constructs in general. The answer turns out to be positive, and the structure is known as a Bruhat-Tits tree, which (for no field extensions) can be thought of as a discrete version of EAdS_3. Superficially an infinite tree of uniform valence $p+1$, and fundamentally a very deep object, a Bruhat-Tits tree can be thought of as the bulk dual of $\mathbb{Q}_p$. It turns out it is possible to do holography on a Bruhat-Tits tree, resulting in a p-adic analogue of $\mathrm{AdS}_3/\mathrm{CFT}_2$ [63–65]. Perhaps more surprisingly, it also turns out that for a QEC of type $[[p, 1, (p+1)/2]]_p$, the HaPPY tensor networks of [62] can be naturally associated to the Bruhat-Tits tree dual to $\mathbb{Q}_p$ [65]. Thus, HaPPY tensor networks naturally encode information about bulk reconstruction at finite places, and not about the bulk reconstruction of the usual anti-de Sitter space (the Archimedean place). In some sense, one can think that recovering the Archimedean place then corresponds to taking the limit $p \to \infty$, but this limit is not continuous in the usual sense of the word.

Up to now, all we have done is introduced a (very strange) kind of holography. However, there are three reasons to think this p-adic holography is worth studying:

1. Paper [65] argues that p-adic holography is as natural as HaPPY tensor networks. If we believe tensor networks are a useful tool for studying holography, then the p-adic version automatically follows.

2. The finite places, together with the Archimedean one, can naturally be grouped into an adelic structure. It may turn out that by studying finite place holography, results (such as entropic inequalities) can be derived at the Archimedean place via adelic formulas. This type of approach is used in certain branches of mathematics.

3. It may turn out that without considering the finite places, general relativity at the Archimedean place is not self-consistent at the quantum level. This may sound most heretical (and it is!), but it is motivated by the firewall paradox [66], and by the lore that in certain examples of $\mathrm{AdS}_3/\mathrm{CFT}_2$, there should be degrees of freedom not visible in any semi-classical treatment. Adelic holography may help here because by equipping each Archimedean place with finite counterparts, we are introducing additional degrees of freedom, as well as additional entanglement. If one is willing to entertain such an idea, then a natural suggestion would be that to obtain quantum gravity at the Archimedean place, one should first quantize the finite places, and then try to make sense of the large p limit. We caution the reader that this will still be quite an endeavor.[1]

Regardless of the veracity of the three points above, if we are willing to accept that p-adic holography is worth studying, a natural course of action can be outlined:

[1] As is customary in this type of wishful thinking, many facts presented here have been simplified, tamed or made to appear friendlier than they actually are.

1. Reference [65] does not discuss bulk dynamics beyond the probe limit. The tree should be dynamical, and its equations of motion should be meaningful in some way. This is a natural starting point, see references [67, 68].

2. Reference [65] is also stuck in Euclidean space. It is tempting to conjecture that going to Lorentzian signature should require imaginary quadratic extensions, and that whatever Lorentzian p-adic holography is, it should map in the Archimedean limit to the recent developments in kinematic space and tensor networks of [69–71]. Indeed, the two copies of kinematic space present in [71] look like they may naturally arise from a quadratic extension. This direction would marry the two leading tensor network models currently present in the literature, HaPPY tensor networks and kinematic space tensor networks. Once Lorentzian p-adic holography has been established, Euclidean holography for real extensions should be revisited to see if the extension introduces any new features. Lorentzian p-adic holography would also provide a p-adic window inside the black hole horizon. This should be interesting in conjunction with the Firewall, ER = EPR, and "Complexity Equals Action" stories.

3. To go to higher dimensions, one may want to consider Bruhat-Tits buildings, which have flats, apartments and chambers.

4. Once the interplay between Lorentzian and Euclidean p-adic holography is understood, one should also visit the dS/CFT correspondence [72], as there should be a p-adic realization of de Sitter space. The adelic norm comes with a cutoff, and it would be quite remarkable if a natural (high p) cutoff could be related to a small cutoff at the Archimedean place via adelic formulas.[2]

[2] This is yet more heresy, and up to evidence to the contrary should be treated as wishful thinking.

5. After p-adic holography is well established, it would be interesting to see how much of the holographic dictionary can be recast in terms of various types of trace formulas, both at the finite and Archimedean places. This is motivated by the fact that certain questions in holography naturally connect to questions in integral geometry, and both may be amenable to this type of technique (for an application of trace formulas to integral geometry see [58]).

6. If p-adic holography is to be taken seriously, there should be a string-theoretic realization of it. The adelic bosonic string has been studied in some detail (and is as pathological as its Archimedean counterpart), but literature on adelic strings accounting for fermions is nonexistent. It may be possible to write down such a theory (at least naively), in which case obvious immediate questions would be the critical dimension[3] and anomaly cancelation. Zabrodin [73] gives a guess for the fermionic string action; this may be a natural starting point.

7. On-shell scattering amplitudes. If the entanglement entropy cone [23] is reminiscent of the Amplituhedron polytope [74], then Bruhat-Tits trees should be the analogue of on-shell diagrams [75]. There are, of course, some differences: on-shell diagrams are finite, whereas trees are infinite (this may be because a diagram describes a local event, while the tree is the analogue of an infinite spacetime). Other differences include the valence of the vertices, and the connectivity, as well as other elements. But given the deep parallels between various versions of Yang-Mills theory and of gravity, it should be interesting to see if this naive analogy can be made precise.

[3] In the bosonic case an argument can be made that the critical dimensions at all places are equal, $D = 26$. It is not clear if this should still be the case for an adelic string admitting fermions.

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