ELECTROMAGNETIC FIELDS IN NONUNIFORM LOSSLESS TRANSMISSION LINES

Thesis by

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ABSTRACT

Methods are developed for electromagnetic field calculations in nonuniform lossless transmission lines which support quasi-one dimensional propagation of a single baseband wave species. Approximate solutions are obtained in perturbation series for smoothly tapered lines by expanding Maxwell's equations and boundary conditions in the dimensionless parameter η, given by the ratio of typical cross-section dimension to the length of the tapered section. The method emphasizes construction of a single "warped" field description rather than the local modal expansions of Schelkunoff's generalized telegraphist's equations.

Expansions in Cartesian coordinates yield the traditional distributed circuit parameter equations in the lowest approximation. Correction terms appear in even powers of η, and their effects are shown most clearly by calculating waveform aberrations introduced by line transitions of nominally constant characteristic impedance. Improved field descriptions in nonuniform regions are obtained by reformulating the exact equations in special nonorthogonal coordinate systems more closely related to the essential structure of the field problem. The lowest term of the ordered expansion is now exact for a uniform finite angle taper. New circuit level nonuniform line equations are obtained which reduce to the well-known forms for gradual tapers.

These techniques are extended to treat tapered plate lines with curved center lines and then to give a description of coaxial lines in which the field pattern is locally dominated by the

boundaries, and the electrical center line is located in the propagation region. Odd-sequence field distortion terms now appear in third and higher orders. In all the systems investigated, distributed circuit equations give results, outside the nonuniform region, that are valid to within second order terms in the taper scale parameter.

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I. INTRODUCTION

Every student of electrical engineering today acquires as part of his technical armory, a knowledge of the properties of uniform lossless transmission lines and waveguides and of how the engineering description of fields and wave propagation in these idealized systems is founded on rigorous solutions of the Maxwell equations and appropriate boundary conditions. In real life, for reasons which may be intentional or unavoidable, or both, the idealized conditions are never found, but the simple theory is often still adequate or provides a basis for a perturbation approximation. As an example conductor losses are adequately handled for many purposes by introducing appropriate loss constants into the mode propagation factors whereas a rigorous treatment would wash out the whole mode classification scheme. Yet it is still clear that for small losses, only slight distortions of the lossless description are needed to give useful results.

We shall be concerned here with lossless nonuniform transmission lines, a class of intentional disruptions of the framework of the simple theory, in which the boundary cross-section and direction of the basic axis of propagation are allowed to vary with distance along the axis, but where we wish to retain the basic sense of one-dimensional propagation, forward and reflected, of a single baseband wave species, corresponding to the well-known TEM mode in uniform two-conductor lines. Such transmission lines have found extensive applications for purposes such as impedance matching, pulse shaping, transitions between uniform lines of differing sizes, shapes and directions, and in microwave components such as directional couplers, filters and resonators. The problems of long distance information transmission through over-moded waveguides for millimeter wave communications, where deleterious effects arise from nonuniformities such as unintentional manufacturing and installation tolerances and deliberate bends, have led to extensive development of nonuniform waveguide theory based on Maxwell's equations. In contrast, beyond this, the foundation of baseband nonuniform line theory in Maxwell's equations is in a hazy almost non-existent state.

In Chapter II we survey the existing theoretical developments relevant to baseband nonuniform lines. Almost all the literature directly concerned with this class of problem uses a distributed circuit theory approach which is only approximate and does not contain enough physics to show the way to better approximations. The ideas of voltage and current measurement, rigorously justifiable in uniform line theory, need close re-examination in nonuniform lines. The generalized telegraphist equations, derived from the Maxwell equations are rigorous, given sufficient care with nonuniform convergence and coefficient singularities, and contain the circuit theory equations as the simplest approximation, when applied to this case. The main strength of the theory is that it provides a complete formalism for multimode problems to this same order of approximation, but it does not readily yield systematic improved approximations for even a single propagating mode. Convergence in the generalized telegraphist equation approach has been improved by use of better local basis functions. We then propose a systematic perturbation expansion procedure, starting from Maxwell's equations and boundary conditions, using as expansion parameter the dimensionless ratio of typical cross-sectional dimension to length of the nonuniform region, in order to describe the propagation of a single warped dominant mode.

Chapter III contains the development in Cartesian coordinates of the procedure for two-dimensional tapered plate transmission lines with straight center lines. It is shown that this yields the distributed circuit equations in the lowest order of approximation and higher order approximate equations are derived. An exact solution is available for the uniform wedge and it is shown that the perturbation series reproduces at least the first few terms of the convergent series expansion of the exact solution, term by term. When the plates remain equidistant, the line is uniform and reflectionless in the distributed circuit theory, but the perturbation approach generates higher order correction equations. The first correction is evaluated to show transmission aberrations and reflections in terms of the line profile.

The uniform wedge solution shows how successive approximations bend the calculated wave-front shape back from straight transverse towards a circular arc. In Chapter IV we formulate the approximation procedure in a way which gives exact results for a uniform finite angle wedge in its lowest order, thus telescoping an infinite number of terms of the Cartesian expansion. This is achieved by using a non-orthogonal "warped cylindrical" coordinate system which provides a better fit to the inherent structure of the electromagnetic problem. The geometry of this system is studied and it is used to express the Maxwell equations and boundary conditions before applying the expansion procedure. A modified sequence of approximate equations describing one dimensional single warped mode propagation is obtained. Perhaps even more important is the inverse view point, that for roughly the same effort in solving the lowest order approximate propagation equation, very much improved calculations of electric and magnetic fields are available, expecially off-axis and near the boundaries. A curious by-product of this investigation is that it shows that the simple distributed circuit equations work as well as they do because the wrong quantity is evaluated in the wrong place in such a way that the errors cancel for a uniform finite angle wedge line.

So far the expansions have proceeded in even powers of the length scale parameter. Two dimensional transmission lines with curved center lines are treated in Chapter V and it is shown that odd power terms now appear. Planar section theory is developed for transmission lines with boundaries defined symmetrically on the normals of a curved center line. The problem of defining an appropriate center line when only the boundaries are given is considered and a warped cylindrical description is developed. It is in fact easier and more natural to define than the normal center line planar description when only the boundaries are given. For a curved but otherwise uniform line the distinctions vanish.

In Chapter VI these techniques are applied to the analysis of the nonuniform coaxial line. The planar section expansion generates the traditional distributed circuit equation as the basic approximation. Higher order corrections are calculated and used to solve the important practical problem of finding waveform aberrations introduced by a transition with nominally constant characteristic impedance between coaxial lines of different sizes. A warped spherical coordinate system is then generated to give improved field descriptions, and the lowest term of this expansion is exact for concentric conical transmission lines. In this expansion the field pattern is maintained through the second order only for a constant impedance line. Especially in low impedance lines the influence of the walls locally dominates the fields rather more than does the coaxial center line. The curved center line descriptions of an earlier chapter are extended to provide a propagation region centered description of coaxial line.

II. BACKGROUND AND REVIEW

2.1 The Distributed Circuit Parameter Theory

The traditional method for the investigation of nonuniform lossless transmission lines has been the distributed circuit parameter approach, in which the low frequency circuit parameters, inductance $L(z)$ and capacitance $C(z)$, are defined for the element of transmission line between successive normal planar cross-sections at z and $z + dz$, with the field lines assumed to remain in the transverse plane. This method was first used by Kelvin to treat uniform lines, and later extended by Heaviside to nonuniform transmission lines, where L and C may vary with axial position z. An extensive bibliography, from Heaviside onwards, is given by Kaufman (1955). The circuit equations are

$$\frac{\partial V}{\partial z} = - L(z) \frac{\partial I}{\partial t} \tag{2.1}$$

$$\frac{\partial I}{\partial z} = - C(z) \frac{\partial V}{\partial t} \tag{2.2}$$

These first order coupled equations yield a pair of second order equations which are often more convenient to work with.

$$\frac{\partial^2 V}{\partial z^2} - \frac{1}{L}\frac{dL}{dz}\frac{\partial V}{\partial z} - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = 0 \tag{2.3}$$

$$\frac{\partial^2 I}{\partial z^2} - \frac{1}{C}\frac{dC}{dz}\frac{\partial I}{\partial z} - \mu\varepsilon \frac{\partial^2 I}{\partial t^2} = 0 \qquad (2.4)$$

When the line is uniform, these equations may be given an exact interpretation in terms of the principal TEM mode on a two conductor line. When applied to nonuniform lines the equations can only be regarded as an heuristic approximation to the exact field solution, of unknown nature and accuracy. Despite these shaky foundations, much effort continues to be devoted to the study of these equations. We may instance the construction of wide classes of analytic solutions by Holt and Ahmed (1968), studies of synthesis techniques and mathematical properties by Youla (1964), Gopinath and Sondhi (1971), and techniques for numerical solution, Gruner (1970). In particular, Gruner takes care to point out the limitations of the theoretical basis of the circuit equations.

In essence, the circuit parameter calculations assume a purely transverse field distribution like that of a TEM mode in uniform lines, and this is obviously not so in a nonuniform line even if the field variation with time is slow enough to give quasistatic behaviour over the cross-section. Secondly the axial rate of change of transverse dimensions, and so of characteristic impedance, must be small in some ill-defined sense.

We can show by an example that some restrictions are essential. Consider a transition between two coaxial lines which maintains a constant ratio of inner and outer diameters at each cross-section as shown in Figure 2.1. We expect, from practical

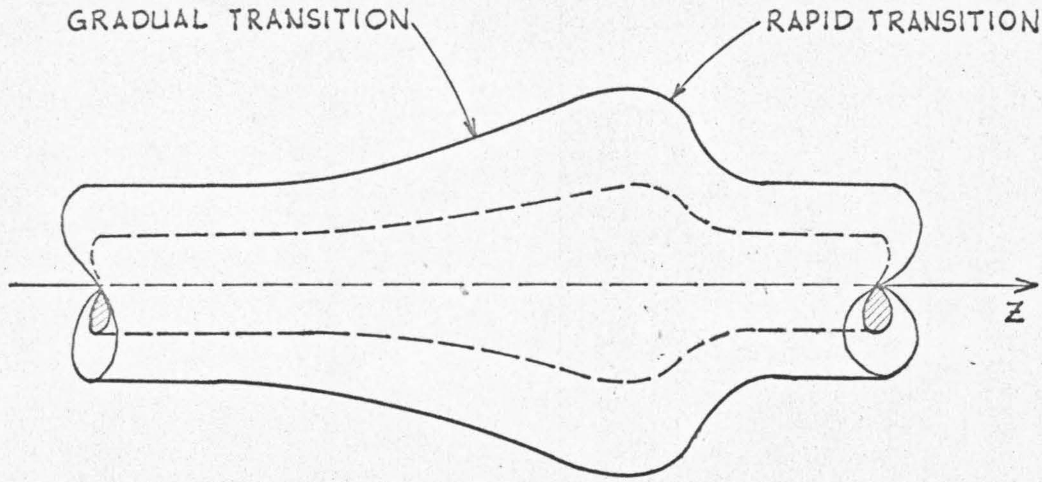

Figure 2.1 Gradual and Rapid Transitions in Nonuniform Coaxial line

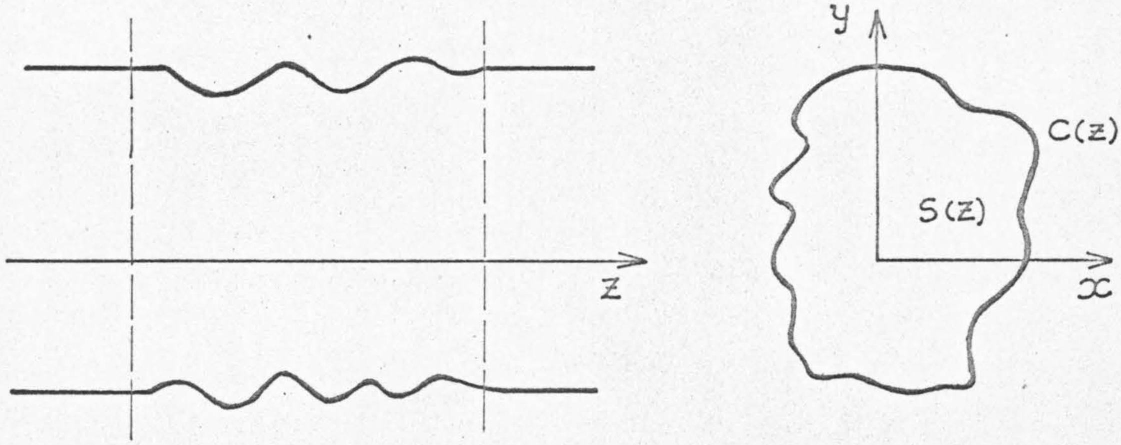

Figure 2.2 Nonuniform Waveguide Section

experience, little or no reflections from the gradual part of the transition, but significant reflections from the rapid transition, even though the distributed circuit theory predicts no reflections in either case. In the limit of a sharp transition there is a discontinuity capacitance which may be calculated by well-known methods. Thus a gradual taper is essential. It will be found in later chapters that the restriction to planar transverse fields can be lifted very profitably.

Gruner then suggests that the resolution of these problems is to be found in application of the generalized telegraphist's equations but does not attempt to carry this program through. In the next section we shall describe this formalism, and show that it does not provide a useful way to improve the distributed circuit analysis.

2.2 The Generalized Telegraphist's Equations

Widespread interest in mode conversion at the inevitable nonuniformities in highly overmoded circular waveguide was inspired by the lure of applications to wide-band long-haul communications. The best known version of the theories of mode conversion is due to Schelkunoff (1955) who popularized the terminology we are using. Similar developments took place in the Soviet Union by Katsenelenbaum (1955), where the term "method of cross-sections" is preferred, and also in Hungary by Reiter (1959). The method had been anticipated by Stevenson (1951) in his theory of electromagnetic horns. An extensive review of mode conversion effects and waveguide transitions is given by Tang (1969). The derivation of these equations will not be rehashed here, and the reader is referred to Reiter's exposition which is the easiest to follow.

At each cross-section the set of vector mode functions $\underline{e}_i$ with propagation coefficients β_i and wave impedances K_i that would exist in an infinite waveguide of the same cross-section, are calculated. Write V_i and I_i for the equivalent voltage and current for each suitably normalized mode. Then the generalized telegraphist's equations can be written in the form

$$\frac{dV_i}{dz} = + i \beta_i K_i I_i + \sum_j T_{ji} V_j \tag{2.5a}$$

$$\frac{dI_i}{dz} = + i \frac{\beta_i V_i}{K_i} - \sum_j T_{ij} I_j \tag{2.5b}$$

where

$$T_{ji} = \int_{(\text{cross-section})} \underline{e}_j \cdot \frac{\partial \underline{e}_i}{\partial z} \, dS \tag{2.6}$$

If this is applied to a two conductor line and everything neglected except the single dominant mode for each cross-section then we recover the ordinary circuit equations.

An interesting result is proved under reasonable conditions by Solymar (1959). He shows that if a single mode is incident on a gradual taper then the levels of any other modes generated in the taper can be made smaller than any given magnitude by increasing the length scale of the taper. Thus, if the taper profile is sufficiently gradual, a single nonuniform transmission line is an adequate representation. The result may be looked at from another point of view, that the standard nonuniform line equations yield the first term in an asymptotic expansion of the exact solution in terms of the length scale parameter. Further in a multimode guide, applying the coupled set of equations (2-5) to the modes of interest and neglecting terms for other modes, gives a similar lowest order asymptotic approximation for the multimode problem.

There are some difficulties associated with this treatment. The mode functions of an infinite cylindrical guide do not, in general, satisfy the boundary conditions, so convergence of the modal series is nonuniform at the boundary of each cross-section. The coupling coefficients are singular at cross-sections where the corresponding cylindrical modes are at cut off. A number of authors have presented modified theories to alleviate these problems. The basic scheme is to use local basis functions which individually satisfy the boundary conditions and so eliminate the nonuniform convergence. For tapered plate and circular waveguides explicit representations are available as the radially propagating modes of wedges and cones respectively. Pokrovskii, Ulinich, and Savvinykh (1959) use "natural coordinates" which locally are approximately cylindrical or spherical to obtain mode and amplitude solutions, while Amitay, Lavi, and Young (1961) and Bahar (1968) assume local modal expansions and match across the boundaries of successive elementary sections, thus generalizing Schelkunoff's treatment. Since wedge and cone modes inherently contain cut off phenomena, it might be thought that coefficient singularity difficulties are solved in principle, but as pointed out by Tang (1969), all these authors make approximations which bound the solution away from cut off transition regions. Amitay et al. in fact analyze a quasi-TEM problem that we also shall investigate, but their results are limited to a high frequency solution, which is adequate for some purposes.

Thus none of the extant treatments have really come to grips with the problems outlined in Section 2.1 and so a new outlook is in order.

2.3 Warped Mode Propagation

The standard derivation of transmission line ideas from cylindrical waveguide solutions leans heavily on the symmetry of the infinite guide theory, to the extent that it is both extremely rigid and misleadingly simple. Engineers and physicists know however, that their transmission line ideas hold quite well for flexible and non-uniform lines and cheerfully think in terms of field patterns that go around corners, and expand and contract while locally seeming like ideal modes. This concept has been dubbed "warped mode" or "quasi-normal" mode by Fox (1955) and Louisell (1955) in related work. In the generalized telegraphist's equations approach, the allowable form of the mode "warping" is specified ahead of time in terms of the running amplitudes of uniform waveguide modes redefined at each cross-section. It is this inflexibility that limits the description. Why should one expect the warping corrections to a lowest order single dominant mode solution to be well described by a sum of non-propagating modes? What is needed is a more free-form approach to building up the description of the warped mode species with the right properties, even if this does limit the class of problems solved.

The technique which will be employed is to use a perturbation expansion to reduce the two- or three-dimensional field description given by the full Maxwell equations to a basic one-dimensional propagation equation, with the warping corrections falling out as further one-dimensional propagating waves. The formulation involves scaling the space and time variables in the governing equations so that the ratio of transverse and axial scale lengths appears explicitly and can be used as the parameter in power series expansions. Cole (1968) has used a similar technique in a simpler problem of steady heat conduction, completed only to the lowest term. We assume, where necessary, sufficiently smooth transitions to uniform input and output lines. Section and slope discontinuities as shown in Fig. 2.3 would need to be incorporated by a local quasistatic analysis.

The perturbation method separates the problem into a quasi-static transverse variation and a longitudinal wave description, and so applies to the baseband mode in multiconductor lines or to waveguides just above cut off of their lowest mode. We do not expect the method to work for multimode waveguide problems. Thus there is no direct competition between the perturbation method and the generalized telegraphist's equation method, since they are each an optimum description of a physically distinct class of problems.

An entirely different method should also be mentioned. This is the use of conformal mappings to relate nonuniform boundary problems to nonuniform medium problems as reviewed by Borgnis and

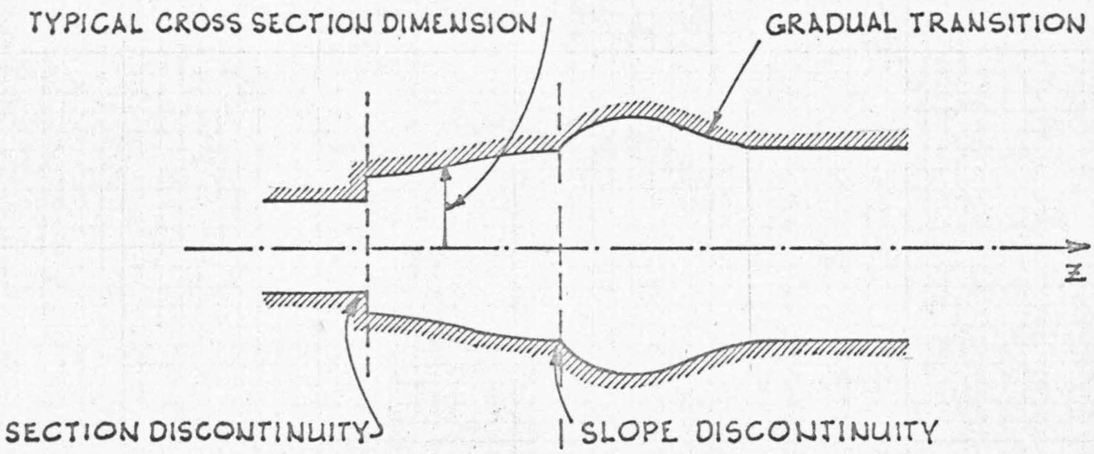

Figure 2.3 Gradual and Discontinuous Transitions

Papas (1958). This correspondence has been generalized by Baum (1968) with the use of three dimensional orthogonal coordinate system transformations and applied, among other things, to the synthesis of ideal line transitions. The aim of this work will instead be to obtain, with a minimum of calculation, good approximate answers for problems involving arbitrarily given nonuniform transmission line profiles.

III. PLANAR SECTION EXPANSIONS

3.1 The Symmetrical Tapered Plate Transmission Line

In this section the perturbation expansion is formulated for a two-dimensional tapered plate line as shown in Fig. 3.1. The line has perfectly conducting boundaries symmetrical about a center plane $x = 0$, and is uniform in the y-direction with the wave propagation in the z-direction. This is the simplest example which contains the essential features of nonuniform transmission line behaviour. With this boundary symmetry, the dominant solution will remain TM to z, reducing to the standard TEM dominant two conductor mode when the plate spacing is constant. The fields inside the line will have the form

$$\underline{E} = \hat{e}_x E_x(x,z,t) + \hat{e}_z E_z(x,z,t) \tag{3.1}$$

$$\underline{H} = \hat{e}_y H_y(x,z,t) \tag{3.2}$$

The Maxwell equations then reduce to

$$\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial z^2} - \mu\varepsilon \frac{\partial^2 H_y}{\partial t^2} = 0 \tag{3.3}$$

and the electric field components may be calculated from this single magnetic field component by

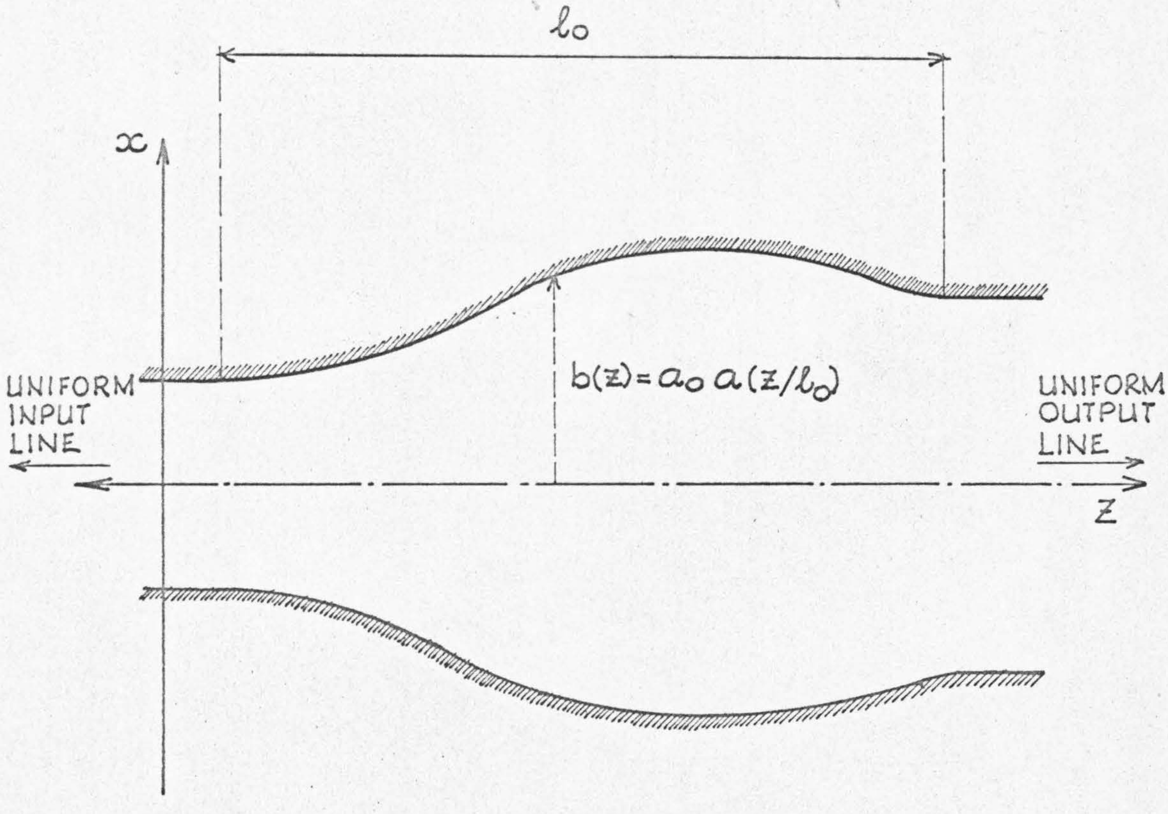

Figure 3.1 Symmetrical Tapered Plate Line Profile

$$- \varepsilon \frac{\partial E_x}{\partial t} = \frac{\partial H_y}{\partial z} \tag{3.4}$$

$$\varepsilon \frac{\partial E_z}{\partial t} = \frac{\partial H_y}{\partial x} \tag{3.5}$$

The boundary condition for perfectly conducting plates is that the tangential component electric field E_t vanishes on the boundary.

$$\underline{E}_t = 0 \quad \text{on} \quad x = \pm\, a_o\, a(\frac{z}{\ell_o}) \tag{3.6}$$

Then certainly also
$$\frac{\partial \underline{E}_t}{\partial t} = 0 \text{ on } x = \pm\, a_o\, a(\frac{z}{\ell_o}) \tag{3.7}$$

From symmetry, the lower half $x < 0$ of the structure is just the image of the upper half $x > 0$ in a perfectly conducting center plane, and we shall for convenience, consider only the region between the upper plane and the center plane where the electric field must also be normal.

The boundary conditions then reduce to

$$\frac{\partial H_y}{\partial x} = \frac{a_o}{\ell_o}\, a'(\frac{z}{\ell_o}) \frac{\partial H_y}{\partial z} \quad \text{on} \quad x = a_o\, a(\frac{z}{\ell_o}) \tag{3.8}$$

and

$$\frac{\partial H_y}{\partial x} = 0 \qquad \text{on} \quad x = 0 \tag{3.9}$$

where the prime, as usual, denotes differentiation with respect to the argument of the function.

Now introduce normalized variables

$$x^{\dagger} = \frac{x}{a_o} ; \quad z^{\dagger} = \frac{z}{\ell_o} ; \quad t^{\dagger} = t/\sqrt{\mu\varepsilon}\ \ell_o \tag{3.10}$$

The time scale is normalized to the wave transit time through the length of the nonuniform section. The dimensionless ratio $\eta = a_o/\ell_o$ measures the length scale of the taper, and is used as the expansion parameter in the perturbation series, in the limit $\eta \to 0$. Then in terms of the normalized variables equations (3.3), and (3.8,9) become

$$\frac{\partial^2 H_y}{\partial x^{\dagger 2}} + \eta^2 \frac{\partial^2 H_y}{\partial z^{\dagger 2}} - \eta^2 \frac{\partial^2 H_y}{\partial t^{\dagger 2}} = 0 \tag{3.11}$$

with

$$\frac{\partial H_y}{\partial x^{\dagger}} = \eta^2 a' \frac{\partial H_y}{\partial z^{\dagger}} \quad \text{on} \quad x^{\dagger} = a(z^{\dagger}) \tag{3.12}$$

and $$\frac{\partial H}{\partial x^{\dagger}} = 0 \quad \text{on} \quad x^{\dagger} = 0 \tag{3.13}$$

The limit $\eta \to 0$ of the taper parameter may be thought of as either $a_o \to 0$ or $\ell_o \to \infty$. In this chapter there will be no real reason to prefer one over the other. The version $a_o \to 0$ emphasizes directly the approach to transverse quasistatic behaviour for a fixed wavelength and nonuniform section length, while the limit $\ell_o \to \infty$ shows the behaviour for a given typical cross-section as the taper is made more gradual by stretching out its length uniformly.

The reason for our choice of time normalization is now apparent, as it couples axial and time derivatives in a wave operator, independently of η. In frequency domain language, the number of wavelengths in the nonuniform section is held constant as η is varied. This number, however, is not restricted except by the need to avoid waveguide mode propagation effects when the wavelength becomes comparable to the transverse dimension.

Assume that H_y may be expanded in a regular series in η^2 and that the extent and smoothness of the nonuniformity are such that the correction terms do not blow up:

$$H_y(x,z,t) = H_o(x^\dagger,z^\dagger,t^\dagger) + \eta^2 H_2(x^\dagger,z^\dagger,t^\dagger) + \eta^4 H_4(x^\dagger,z^\dagger,t^\dagger) + \ldots \tag{3.14}$$

Substitute this series into the equation (3.11) and its boundary conditions (3.12,13) to obtain, after classification by powers of η, an ordered sequence of problems

$$O(\eta^o) \qquad \frac{\partial^2 H_o}{\partial x^{\dagger 2}} = 0 \tag{3.15}$$

$$\text{with} \qquad \frac{\partial H_o}{\partial x^\dagger} = 0 \quad \text{on} \quad \begin{cases} x^\dagger = a(z^\dagger) & (3.16) \\ x^\dagger = 0 & (3.17) \end{cases}$$

$$O(\eta^2) \qquad \frac{\partial^2 H_2}{\partial x^{\dagger 2}} = -\left(\frac{\partial^2}{\partial z^{\dagger 2}} - \frac{\partial^2}{\partial t^{\dagger 2}}\right) H_o \tag{3.18}$$

with
$$\frac{\partial H_2}{\partial x^\dagger} = \begin{cases} a'(z^\dagger)\ \dfrac{\partial H_o}{\partial z^\dagger} & \text{on } x^\dagger = a(z^\dagger) \quad (3.19) \\[2ex] 0 & \text{on } x^\dagger = 0 \quad (3.20) \end{cases}$$

with subsequent entries for $O(\eta^4)$ and higher orders similar to that for $O(\eta^2)$. We now obtain a selfconsistent set of solutions of equations (3.15-20) such that the lowest term H_o is identical to the circuit solution and higher order solutions yield corrections to this.

$$\frac{\partial H_o}{\partial x^\dagger} = B_o(z^\dagger, t^\dagger) = 0 \tag{3.21}$$

Integrate once more to obtain

$$H_o(x^\dagger, z^\dagger, t^\dagger) = A_o(z^\dagger, t^\dagger) \tag{3.22}$$

where A_o is an as yet arbitrary function of $z^\dagger$ and $t^\dagger$, and independent of $x^\dagger$. It can already be seen that the lowest approximation is constant across each planar cross-section just as envisaged in the distributed circuit approximation for this structure. To determine A_o more precisely, we need to consider the next higher approximation. Only a partial solution is necessary for this purpose. Substitute in (3.18) and (3.19) to obtain

$$\frac{\partial^2 H_2}{\partial x^{\dagger 2}} = - \left\{ \frac{\partial^2}{\partial z^{\dagger 2}} - \frac{\partial^2}{\partial t^{\dagger 2}} \right\} A_o(z^\dagger, t^\dagger) \tag{3.23}$$

with $$\frac{\partial H_z}{\partial x} = \begin{cases} a' \dfrac{\partial A_o(z^\dagger, t^\dagger)}{\partial z^\dagger} & \text{on } x^\dagger = a(z^\dagger) \\ 0 & \text{on } x^\dagger = 0 \end{cases} \tag{3.24}$$

Equation (3.23) may be integrated immediately to give

$$\frac{\partial H_2}{\partial x^\dagger} = - x^\dagger \left\{ \frac{\partial^2}{\partial z^{\dagger 2}} - \frac{\partial^2}{\partial t^{\dagger 2}} \right\} A_o(z^\dagger, t^\dagger) + B_2(z^\dagger, t^\dagger) \tag{3.25}$$

From the lower boundary condition $B_2 = 0$, and the upper boundary condition yields

$$a'(z^\dagger) \frac{\partial A_o(z^\dagger, t^\dagger)}{\partial z^\dagger} = - a(z^\dagger) \left\{ \frac{\partial^2}{\partial z^{\dagger 2}} - \frac{\partial^2}{\partial t^{\dagger 2}} \right\} A_o(z^\dagger, t^\dagger). \tag{3.26}$$

A little rearrangement yields a familiar equation

$$\frac{\partial^2 A_o}{\partial z^{\dagger 2}} + \frac{a'(z^\dagger)}{a(z^\dagger)} \frac{\partial A_o}{\partial z^\dagger} - \frac{\partial^2 A_o}{\partial t^{\dagger 2}} = 0 \tag{3.27}$$

which becomes when the variables are denormalized

$$\frac{\partial^2 H_o}{\partial z^2} + \frac{a'(z/\ell_o)}{\ell_o\, a(z/\ell_o)} \frac{\partial H_o}{\partial z} - \mu\varepsilon \frac{\partial^2 H_o}{\partial t^2} = 0 \tag{3.28}$$

In the distributed circuit analysis the distributed capacity C is inversely related to the spacing and so

$$-\frac{1}{C}\frac{dC}{dz} = -\frac{1}{a}\left\{\frac{-a'}{\ell_o a^2}\right\} = \frac{a'}{\ell_o a} \tag{3.29}$$

and so the nonuniform transmission line equation for the current is

$$\frac{\partial^2 I}{\partial z^2} + \frac{a'(z/\ell_o)}{\ell_o\, a(z/\ell_o)} \frac{\partial I}{\partial z} - \mu\varepsilon \frac{\partial^2 I}{\partial t^2} = 0 \tag{3.30}$$

Since the longitudinal currents are proportional to the magnetic field given by (3.28) and are equal and opposite on the two plates at each planar section, then equation (3.28) for the lowest order approximation in the perturbation theory recreates exactly the distributed circuit parameter nonuniform line equation. Observe also from equation (3.5) that the electric field remains purely transverse in this approximation as assumed in the circuit theory, and the voltage can be defined by integrating the electric field across the section.

The circuit version stops at equation (3.30) and goes no further but the present theory leads onward. Integration of (3.25) yields

$$H_2(x^\dagger, z^\dagger, t^\dagger) = -\frac{x^{\dagger 2}}{2}\left\{\frac{\partial^2}{\partial x^{\dagger 2}} - \frac{\partial^2}{\partial t^{\dagger 2}}\right\} A_o(z^\dagger, t^\dagger) + A_2(z^\dagger, t^\dagger) \tag{3.31}$$

where $A_2(z^\dagger, t^\dagger)$ is a new arbitrary function that remains to be determined. It can already be seen in the second approximation (3.31), that the magnetic field is no longer constant over the planar cross-section, and so from (3.5) a longitudinal electric field component appears also. Alternatively, wavefronts in this approximation are no longer in the normal plane. In the half-problem, currents at the ends of a normal plane section are no longer equal and opposite, or in the full symmetrical problem where this is necessarily true, the ready identification with axial propagation distance has been lost.

Now continue with the analysis of the fourth order equations, writing L for the normalized wave operator for the sake of brevity. We find

$$\frac{\partial^2 H_4}{\partial x^{\dagger 2}} = -\frac{x^{\dagger 2}}{2} L \left\{\frac{a'}{a}\frac{\partial A_o}{\partial z^\dagger}\right\} - L\, A_2 \tag{3.32}$$

with

$$\frac{\partial H_4}{\partial x^{\dagger}} = \begin{cases} a' \left[\dfrac{a^2}{2} \cdot \dfrac{\partial}{\partial z^{\dagger}} \left\{ \dfrac{a'}{a} \dfrac{\partial A_o}{\partial z^{\dagger}} \right\} + \dfrac{\partial A_2}{\partial z^{\dagger}} \right] & \text{on } x^{\dagger} = a \\ 0 & \text{on } x^{\dagger} = 0 \end{cases} \quad (3.33)$$

This yields, after some manipulations, an equation for the second function $A_2(z^{\dagger}, t^{\dagger})$

$$\frac{\partial^2 A_2}{\partial z^{\dagger 2}} + \frac{a'(z^{\dagger})}{a(z^{\dagger})} \frac{\partial A_2}{\partial z^{\dagger}} - \frac{\partial^2 A_2}{\partial t^{\dagger 2}}$$

$$= - \frac{(a(z^{\dagger}))^2}{6} \left[\frac{\partial^2}{\partial z^{\dagger 2}} + 3 \frac{a'(z^{\dagger})}{a(z^{\dagger})} \frac{\partial}{\partial z^{\dagger}} - \frac{\partial^2}{\partial t^{\dagger 2}} \right] \left\{ \frac{a'}{a} \frac{\partial A_o}{\partial z} \right\} \quad (3.36)$$

This equation has the form

$$\left\{ \begin{matrix} \text{nonuniform line} \\ \text{operator} \end{matrix} \right\} = \left\{ \begin{matrix} \text{Source terms depending on lower} \\ \text{order solution and taper profile} \end{matrix} \right\}$$

The homogeneous solution of this equation will add nothing that is not already contained in the lower order solution and so may be set to zero without loss of generality. It does not seem to be a profitable exercise to attempt to identify the inhomogeneous terms explicitly as equivalent voltage or current sources in a nonuniform distributed circuit. Also we see that the leading term in E_z will be $O(\eta)$ as expected on physical grounds.

The analysis can be extended in this fashion to higher orders at the expense of rapidly escalating complexity. The results through the next order of approximation are given in Appendix A. For a uniform line, $a(z^{\dagger})$ = constant, the equation (3.27) for A_o reduces to the simple wave equation for the uniform line, and the inhomogeneous terms in (3.34) vanish, leaving the simple wave equation again. A more stringent and non-trivial test is furnished by the finite angle uniform wedge, for which an exact solution is available.

3.2 Exact and Approximate Solutions for the Uniform Wedge

A uniform finite angle wedge is shown in Fig. 3.3 defined in both Cartesian and cylindrical coordinates. For simplicity in this section we assume a harmonic time dependence $e^{-i\omega t}$. In terms of normalized time $t^{\dagger}$ this can be written $e^{-ik^{\dagger}t^{\dagger}}$ where

$$k^{\dagger} = \frac{\sqrt{\mu\varepsilon}}{\ell_o} = \omega \tag{3.35}$$

The well-known solution for an outgoing wave in cylindrical coordinates for the ϕ-independent TM_o wave with components $H_y(\rho)$ and $E_\rho(\rho)$ is

$$H_y(\rho) = H_o^{(1)}(k\rho) \tag{3.36}$$

where $H_o^{(1)}$ denotes a zero order Hankel function of the first kind and "outgoing" is interpreted as the direction in which the wedge

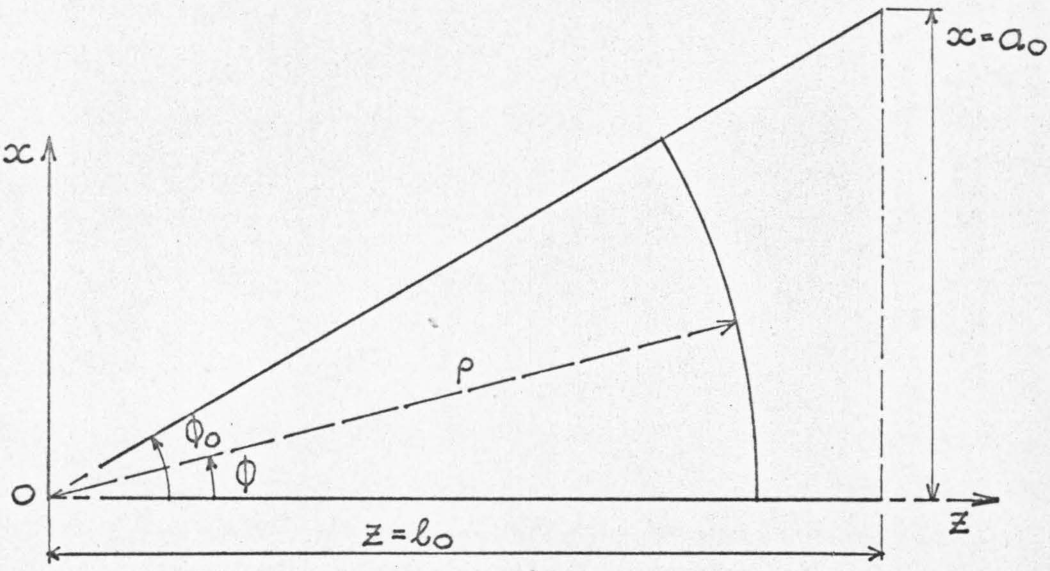

Figure 3.2 Uniform Wedge Coordinates

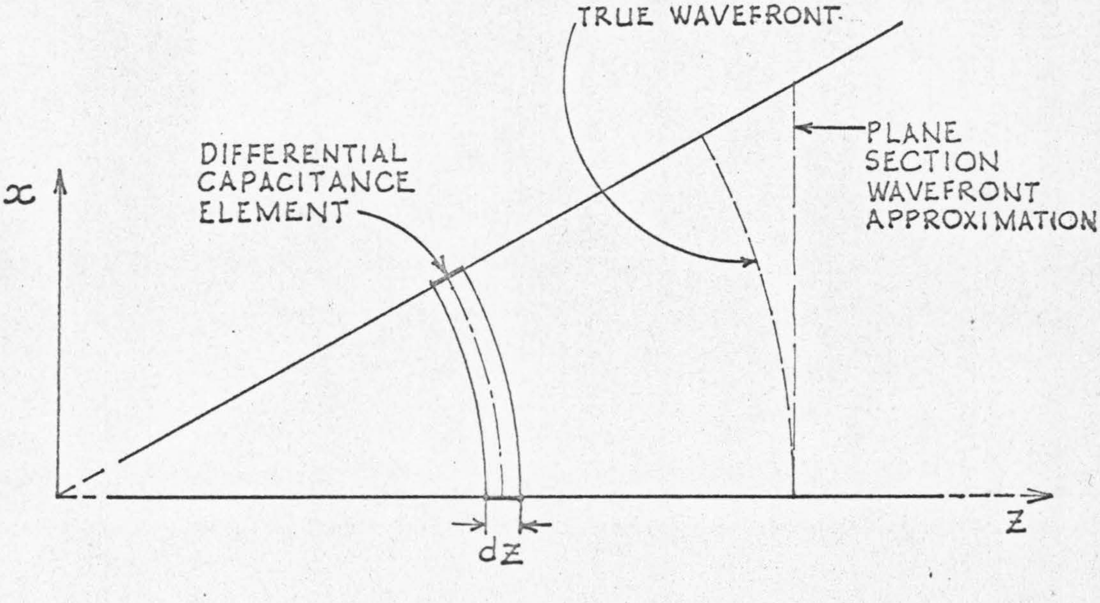

Figure 3.3 Uniform Wedge, Phase Fronts, and Field Lines

opens out, in this case the $z^\dagger$ axis direction. The phase fronts for this solution are circles with center at the wedge apex. Fig. 3.4. Expressed in terms of normalized Cartesian coordinates this becomes

$$H_y = H_o^{(1)}\left(k\sqrt{x^2 + z^2}\right)$$

$$= H_o^{(1)}\left(k^\dagger z^\dagger \sqrt{1 + \eta^2\left(\frac{x^\dagger}{z^\dagger}\right)^2}\right) \tag{3.37}$$

in the region $z > 0$. Since in normalized coordinates the upper boundary is $x^\dagger = z^\dagger$ then for any interior point $|x^\dagger| < z^\dagger$. Hence we may expand (3.37) by the multiplication theorem for Bessel functions for $\eta < 1$ (see Abramowitz and Stegun, section 9.1.74)

$$H_y = H_o^{(1)}(k^\dagger z^\dagger) - \frac{\eta^2}{2}\frac{x^{\dagger 2} k^\dagger}{z^\dagger} H_1^{(1)}(k^\dagger z^\dagger) + \frac{\eta^4}{8}\frac{x^{\dagger 4} k^{\dagger 2}}{z^{\dagger 2}} H_2^{(1)}(k^\dagger z^\dagger)$$

$$- \frac{\eta^6}{48}\frac{x^{\dagger 6} k^{\dagger 3}}{z^{\dagger 3}} H_3^{(1)}(k^\dagger z^\dagger) + \ldots. \tag{3.38}$$

The extreme $\eta = 1$, the steepest wedge for which the series (3.38) converges at all interior points, represents a wedge with interior angle of 45°.

We now proceed to the calculation of the first few terms of the perturbation series expansion for the uniform wedge. First note for reference the identity, with n integral

$$\left\{\frac{d^2}{dz^2} + k^2\right\} \left\{z^{-n}\, H_n^{(1)}(kz)\right\} = - \frac{(2n+1)k}{z^{n+1}}\, H_{n+1}^{(1)}(kz) \qquad (3.39)$$

The function A_o is given by the equation (for harmonic time variation)

$$\frac{d^2 A_o}{dz^{\dagger 2}} + \frac{1}{z^\dagger} \frac{dA_o}{dz^\dagger} + k^{\dagger 2} A_o = 0 \qquad (3.40)$$

The appropriate solution to this equation is just

$$A_o = H_o^{(1)}(k^\dagger z^\dagger) = H_o^{(1)}(kz) \qquad (3.41)$$

which on-axis $x^\dagger = 0$ is identical with the exact solution. The exact solution maintains this value on the circular arc while the approximate solution says the phase front is the normal plane.

The equation for A_2 becomes

$$\frac{d^2 A_2}{dz^{\dagger 2}} + \frac{1}{z^\dagger} + \frac{dA_2}{dz^\dagger} + k^{\dagger 2} A_2 = - \frac{a^2}{6} \left\{\frac{d^2}{dz^{\dagger 2}} + \frac{3}{z^\dagger} \frac{d}{dz^\dagger} + k^2\right\} \left\{\frac{1}{z^\dagger} \frac{d}{dz^\dagger} (H_o'(k^\dagger z^\dagger))\right\}$$

$$(3.42)$$

Since

$$\frac{d}{dz}\, H_o^{(1)}(kz) = - k\, H_1^{(1)}(kz)$$

application of (3.39) with $n = 1$ shows that the inhomogeneous term in (3.42) vanishes. Hence without losing anything essential we may take $A_2 = 0$.

The next higher order arbitrary function $A_4(z^\dagger)$ is defined by the equation, modified from (A1.2)

$$\frac{d^2 A_4}{dz^{\dagger 2}} + \frac{1}{z^\dagger}\frac{dA_4}{dz^\dagger} + k^{\dagger 2} A_4$$

$$= -\frac{a^2}{6}\left\{\frac{d^2}{dz^{\dagger 2}} + \frac{3}{z^\dagger}\frac{d}{dz^\dagger} + k^{\dagger 2}\right\}\left\{\frac{1}{z^\dagger}\frac{dA_2}{dz^\dagger}\right\}$$

$$+ \frac{a^4}{120}\left\{\frac{d^2}{dz^{\dagger 2}} + \frac{5}{z^\dagger}\frac{d}{dz^\dagger} + k^{\dagger 2}\right\}\left\{\frac{d^2}{dz^{\dagger 2}} + k^{\dagger 2}\right\}\left\{\frac{1}{z}\frac{dA_o}{dz^\dagger}\right\}$$

$$- \frac{a^2}{36}\left\{\frac{d^2}{dz^{\dagger 2}} + \frac{3}{z^\dagger}\frac{d}{dz^\dagger} + k^{\dagger 2}\right\}\left\{z^{\dagger 2}\left(\frac{d^2}{dz^{\dagger 2}} + \frac{3}{z^\dagger}\frac{d}{dz^\dagger} + k^{\dagger 2}\right)\right\}\left\{\frac{1}{z^\dagger}\frac{dA_o}{dz^\dagger}\right\} \tag{3.43}$$

Of the inhomogeneous terms, the first vanishes since $A_2 = 0$, and the third term vanishes also, the argument being exactly similar to that used for the A_2 equation. A double application of (3.39), first with $n = 1$ and then with $n = 2$ shows that the middle term vanishes as well. Hence A_4 also satisfies the homogeneous non-uniform line equation and may be taken as zero.

Further calculation indicates that the equation for A_6

will also turn out to be homogeneous, then we can complete the first four terms of the perturbation expansion, omitting all but the A_o terms from equation (A1.3)

$$H(x^\dagger,z^\dagger) = A_o(z^\dagger) + \eta^2 \frac{x^{\dagger 2}}{2}\left\{\frac{1}{z^\dagger}\frac{dA_o}{dz^\dagger}\right\} - \eta^4 \frac{x^{\dagger 4}}{24}\left\{\frac{d^2}{dz^{\dagger 2}} + k^{\dagger 2}\right\}\left\{\frac{1}{z^\dagger}\frac{dA_o}{dz^\dagger}\right\}$$

$$+ \eta^6 \frac{x^{\dagger 6}}{720}\left\{\frac{d^2}{dz^{\dagger 2}} + k^{\dagger 2}\right\}\left\{\frac{d^2}{dz^{\dagger 2}} + k^{\dagger 2}\right\}\left\{\frac{1}{z^\dagger}\frac{dA_o}{dz^\dagger}\right\}$$

$$- \eta^6 \frac{x^{\dagger 4}}{144}\left\{\frac{d^2}{dz^{\dagger 2}} + k^{\dagger 2}\right\}\left\{z^{\dagger 2}\left\{\frac{d^2}{dz^{\dagger 2}} + \frac{3}{z^\dagger}\frac{d}{dz^\dagger} + k^{\dagger 2}\right\}\right\}\left\{\frac{1}{z^\dagger}\frac{dA_o}{dz^\dagger}\right\} \tag{3.44}$$

The identity (3.39) may be employed to show that the last $O(\eta^6)$ term vanishes and that the others reduce to give

$$H(x^\dagger,z^\dagger) = H_o^{(1)}(k^\dagger z^\dagger) - \eta^2 \frac{x^{\dagger 2} k^\dagger}{2z^\dagger} H_1^{(1)}(k^\dagger z^\dagger) + \eta^4 \frac{x^{\dagger 4} k^{\dagger 2}}{8z^{\dagger 2}} H_2^{(1)}(k'z')$$

$$- \eta^6 \frac{x^{\dagger 6} k^{\dagger 3}}{48\, z^{\dagger 3}} H_3^{(1)}(k^\dagger z^\dagger) + \ldots \tag{3.45}$$

This matches term by term the convergent series expansion (3.38) of the exact solution. The higher order terms progressively correct the off-axis field distribution and can be regarded as successively bending back the predicted fronts towards the exact position. So far wave fronts have been clearly predicted in two cases, one the planar sections of the lowest approximation and second the cylindrical surfaces in the uniform wedge only. It is worthy of note that the lowest approximation, the circuit equation, gives exact results along the axis of a uniform wedge. We can see how this might happen by looking at the shape of the electric field lines (Fig. 3.3) in this exact solution. If the incremental capacitance is calculated with this circular field pattern then $C \propto 1/\rho$ and

$$-\frac{1}{C}\frac{dC}{d\rho} = \frac{1}{\rho} = \frac{1}{z} \quad \text{when} \quad \phi = 0 \qquad (3.46)$$

giving a circuit equation in the radial coordinate, which on axis is identical with the usual version. This process can be thought of as automatically incorporating the effects of neighboring capacitance elements, and for a uniform wedge it gives the exact solution, curved wavefront and all.

Amitay et al (1961) give a solution for the nonuniform wedge in which this idea is applied locally and the Hankel function solutions are matched from section to section as the wedge angle varies. They construct solutions in the process of their analysis but make the assumption $kr >> 1$ for asymptotic expansion of the $H_o^{(1),(2)}$, functions. For a finite angle wedge, θ, at a half spacing a, the free space wavelength must lie in the range

$$\frac{\pi}{\sin\theta} >> \frac{\lambda}{2a} >> 1 \tag{3.47}$$

There is thus only limited overlap between the theories. The lower bound on wavelength is to avoid higher mode effects and is a common limitation.

We shall show in a later chapter how this kind of solution may be imbedded in a new more powerful perturbation expansion. In the next section we shall consider a further example of a planar section expansion.

3.3 The Constant Impedance Tapered Plate Line

If the spacing between the plates of a tapered two dimensional transmission line is maintained constant in the x-direction (Fig. 3.5), then in the standard distributed circuit theory it is a constant impedance line which gives no reflections or transmission aberrations on an incident waveform. It is much easier to show the effects of higher order corrections in such a nominally perfectly transparent situation than it is when superimposed on already complex nonuniform line solutions. The notation and procedures will be very similar to those of 4.1 and will not be shown in detail. In normalized coordinates the new boundaries are

$$x^{\dagger} = 1 \pm a(z^{\dagger}) \tag{3.48}$$

In the lowest order $O(\eta^{o})$ we find again that

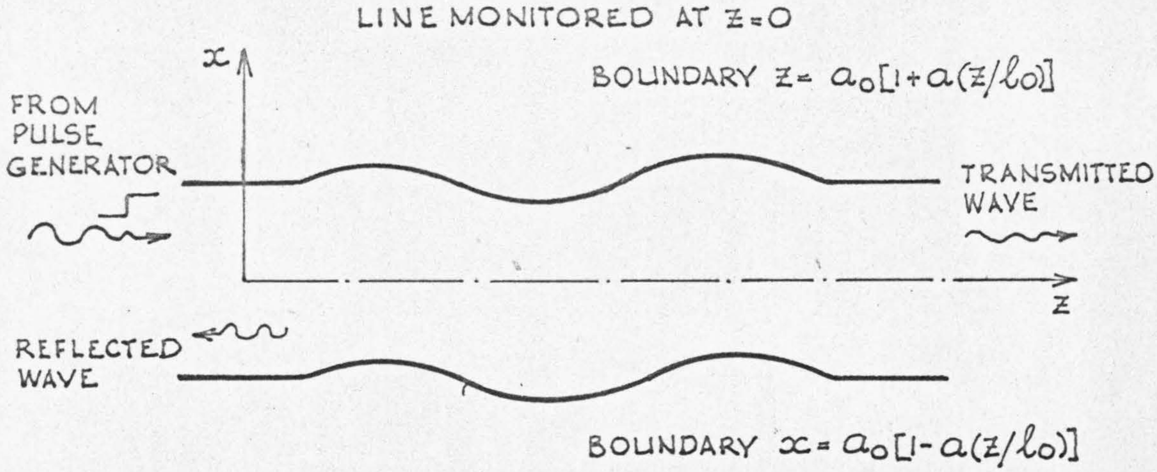

Figure 3.4 Constant Spacing Tapered Plate Line Profile

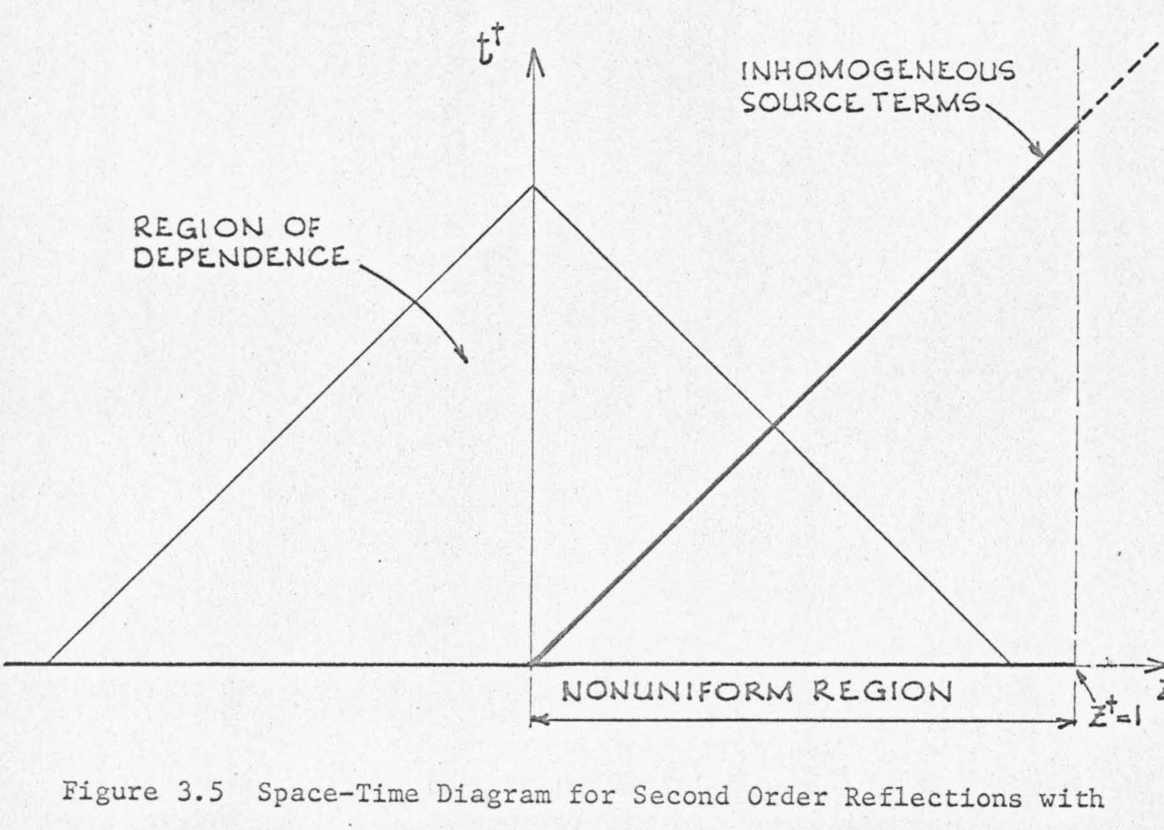

Figure 3.5 Space-Time Diagram for Second Order Reflections with Incident Step Function Excitation

$$H_o(x^\dagger, z^\dagger, t^\dagger) = A_o(z^\dagger, t^\dagger) \tag{3.49}$$

where A_o now satisfies the uniform line wave equation

$$\left\{\frac{\partial^2}{\partial z^{\dagger 2}} - \frac{\partial^2}{\partial t^{\dagger 2}}\right\} A_o(z^\dagger, t^\dagger) = 0 \tag{3.50}$$

In the next order $O(\eta^2)$ we have

$$H_2(x^\dagger, z^\dagger, t^\dagger) = x^\dagger a'(z^\dagger) \frac{\partial A_o}{\partial z^\dagger} + A_2(z^\dagger, t^\dagger) \tag{3.51}$$

where

$$\left\{\frac{\partial^2}{\partial z^{\dagger 2}} - \frac{\partial^2}{\partial t^{\dagger 2}}\right\} A_2(z^\dagger, t^\dagger) = -a(z^\dagger) \left\{\frac{\partial^2}{\partial z^{\dagger 2}} + \frac{a'(z^\dagger)}{a(z^\dagger)} \frac{\partial}{\partial z^\dagger} - \frac{\partial^2}{\partial t^{\dagger 2}}\right\} \left\{a'(z^\dagger) \frac{\partial A_o}{\partial z^\dagger}\right\} \tag{3.52}$$

Thus the first correction term H_2 is composed of a field distortion term and a new propagating term which satisfies an inhomogeneous uniform line equation. The complexity of the equations increase rapidly as we move on to higher order systems. We expect in any case that the theory of this section will apply well only to small deviations $a \ll 1$, with larger deviations being better described by the curved center line theory to be developed in a later chapter.

This problem has the great virtue that it is simple enough, even for an arbitrary profile, to allow an explicit, easily interpretable solution. A typical measurement set-up using a time domain reflectometer is shown in Fig. 3.5. The TDR applies a unit step to the line and monitors the sum of incident and reflected waves at the point $z = 0$ which is taken as the beginning of the nonuniform section. Adequate smoothness of transition is obtained if the profile $a(z^\dagger)$ and its derivatives a', a'' vanish at $z^\dagger = 0$. The solution for A_o, a uniform line problem may be taken as a propagating unit step.

$$A_o(z^\dagger, t^\dagger) = H(t^\dagger - z^\dagger) \tag{3.53}$$

The equation for the new propagating term A_2, in the next approximation is of the form

$$\frac{\partial^2 A_2}{\partial z^{\dagger 2}} - \frac{\partial^2 A_2}{\partial t^{\dagger 2}} = f(z^\dagger, t^\dagger) \tag{3.54}$$

an inhomogeneous one dimensional wave equation, which also has zero initial conditions. It is well-known that the solution to this equation may be written

$$A_2(z^\dagger, t^\dagger) = \frac{1}{2} \iint_D f(\zeta, \tau) \, d\zeta \, d\tau \tag{3.55}$$

where D is the triangular domain of dependence shown in Fig. 3.6. The inhomogeneous term from (3.52) may be written, on plugging in (3.53) as

$$f(z^\dagger,t^\dagger) = -a'^2\ \delta'(t^\dagger-z^\dagger) - 2aa''\ \delta'(t^\dagger-z^\dagger) + (aa'')'\ \delta(t^\dagger-z^\dagger) \tag{3.56}$$

where the a's are all functions of $z^\dagger$. This exists along the line $t^\dagger-z^\dagger=0$ Fig. 3.6 and vanishes elsewhere. For the choice of D shown the solution is evaluated at time $t^\dagger$ at position $z^\dagger = 0$ and it is easily seen that there are no contributions to the solution from $\tau > t^\dagger/2$. Thus the reflected wave seen on the reflectometer is given by

$$A_2(0^\dagger,t^\dagger) = -\frac{1}{2}\iint_D (a'(\zeta))^2\ \delta'(\tau-\zeta)\,d\tau\ d\zeta$$

$$+\frac{1}{2}\iint_D \left\{-2a'(\zeta)a''(\zeta)\delta'(\tau-\zeta) + \frac{d}{d\zeta}(a(\zeta)a''(\zeta))\delta(\tau-\zeta)\right\}\ d\tau\ d\zeta \tag{3.57}$$

the need to make the substitutions

$$u = \frac{1}{2}(\tau + \zeta) \quad \rightarrow \quad \tau = u + v \tag{3.58}$$

$$v = \frac{1}{2}(\tau - \zeta) \quad \rightarrow \quad \zeta = u - v$$

to bring the integrals to a form where integrals involving δ-functions may be given a standard interpretation.

The Jacobian of this transformation has the value $J = 2$ and the argument of the δ-functions is now $\tau - \zeta = 2v$. Use the results, with $a > 0$, for integrals involving δ-functions

$$\int \delta(ax)\ f(-x)\ dx = \frac{1}{a} f(0) \tag{3.59}$$

$$\int \delta'(ax)\ f(-x)\ dx = \frac{1}{a^2} f'(0)$$

The second term in (3.57) then vanishes after the v-integration leaving

$$A_2(0,t^\dagger) = \frac{1}{4}\int_0^{t^\dagger/2} \frac{d}{du}\left\{\left(\frac{da}{du}\right)^2\right\} du$$

$$= \left(\frac{da}{dx}\right)^2_{x = \frac{t^\dagger}{2}} \tag{3.60}$$

The reflected wave A_2 observed on the reflectometer at time $t^\dagger$ is given by equation (3.60) as proportional to the square of the line profile slope at a round trip propagation time of $t^\dagger$ down the line.

In later chapters other examples involving the simple wave operator for a nominally transparent line are solved by transform methods, which are better adapted to finding a general solution for any point in the line.

It should be remarked that A_o and A_2 are magnetic field components which are proportional to the conductor currents and could be directly observed with a current transformer. Present day reflectometer techniques for fast time domain observations generally use voltage sampling techniques. Since the transverse electric field is obtained by a taking a z-derivative, the relative sign of the voltage contribution of a reflected wave will be reversed compared to that for the forward wave, for currents in the same direction. Thus (3.60) predicts a dip in the reflectometer trace. This is certainly consistent with the well-known quasistatic representation of a sudden discontinuity as a extra shunt capacitance due to fringing fields.

IV. CYLINDRICAL SECTION APPROXIMATIONS

4.1 The Tapered Plate Line Revisited

In the Cartesian expansions of Chapter 3, only the lowest term provides universal wavefront, voltage, and current definitions that may be readily disentangled from the details of particular solutions. Higher order terms cannot be readily described in this fashion, but only as field distributions and by their contributions to the waves in uniform line sections at input and output. The uniform wedge, either in an infinite sum of approximate terms, or exactly, or from highly heuristic circuit considerations, has been shown to admit voltage and current definitions on the supports of cylindrical wavefronts. We might hope then, for a nonuniform wedge, to find an approximation sequence, whose lowest term reproduces the cylindrical wavefronts of the uniform wedge, just as the lowest term of the Cartesian expansion becomes exact for a uniform line. The technique for generating this expansion will be based on the construction of a nonorthogonal coordinate system, which is close to a cylindrical geometry in any small section. Unger (1965) has attempted to use a closely related system of locally spherical coordinates in setting up generalized telegraphist's equations, but his treatment contains errors in both principle and detail. Appendix C gives some details for this and related systems.

4.2 Nature of Warped Cylindrical Coordinates

Once again we consider the upper half of a symmetrical

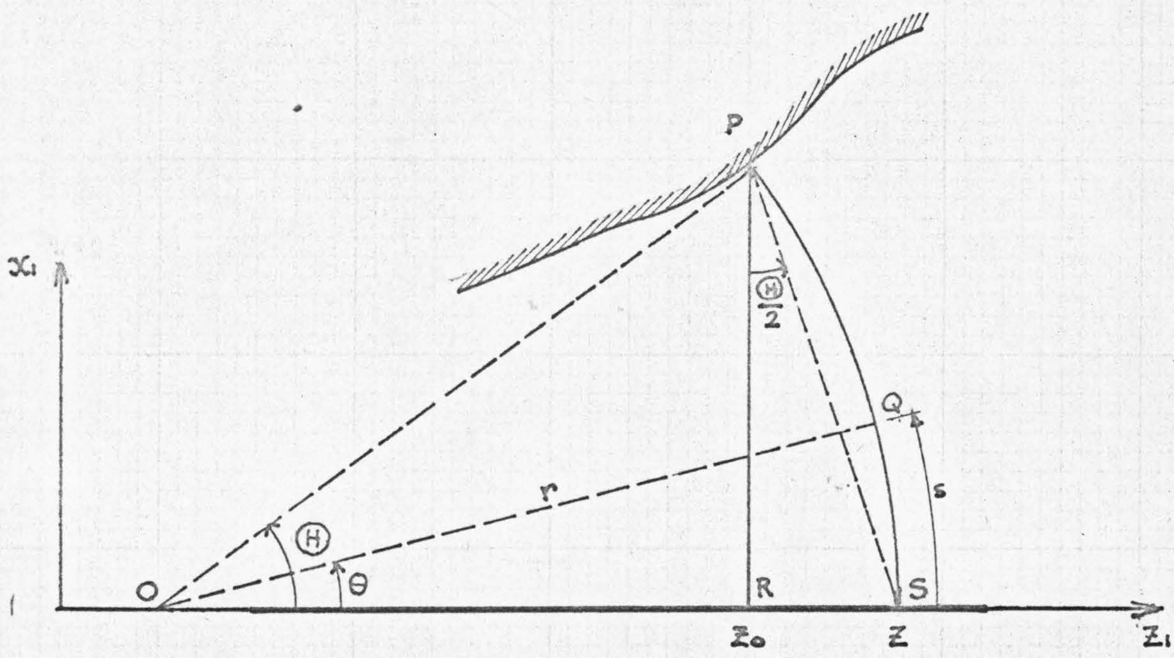

Figure 4.1 Coordinate Arc Definitions

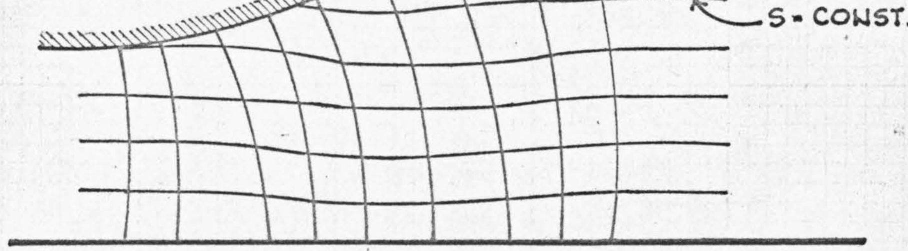

Figure 4.2 Warped Cylindrical Coordinate Net

taper profile as shown in Fig. 4.1. At any point P on the curve, construct the tangent to its intersection with the z-axis at O. Then draw an arc with center O and radius OP to meet the axis at S. This is done for each point on the taper profile, and the resulting family of arcs is used to define one set of coordinate surfaces, each of which is a longitudinal section of a cylindrical surface. These surfaces can be labelled in a variety of ways, the most useful being the z-coordinate of the foot S of the arc PS. The lateral coordinates now need to be defined. The principal choice for this work will be the arc length s from the axis measured along each arc with plus sign in the upper half. A closely related choice is the angle $\theta = s/r$. It is worthy of note, that with neither choice is the boundary a coordinate surface, except in very special cases. Previous authors seem always to have been constrained by the desire to use the boundary profile as a coordinate surface. This is found to be an unnecessary and even hobbling restriction in the present development.

The (z,θ) system for a uniform wedge is exactly equivalent to cylindrical coordinates when the z-origin is chosen at the apex of the wedge. The (z,s) systems becomes Cartesian for plane parallel boundaries and is closely related to cylindrical coordinates for a uniform wedge. The scaling procedure used in setting up the perturbation expansion requires that both coordinates have dimension length,so that the ratio of typical dimensions in each coordinate will form the dimensionless expansion parameter η. Thus we expect the (z,s) system to be a more natural description for our purposes than (z,θ) despite the increased algebraic complexity of the metric.

Also in the limit of parallel-plane boundaries the θ coordinate becomes indeterminate and this limit is almost always included in input and output lines. Obviously the possible coordinate choices have by no means been exhausted.

The introduction of exotic coordinate systems does bring some new apparent problems. In a concave inwards section of the profile there is the possibility that coordinate arcs may overlap as shown in Figure 4.3. For a smooth profile the condition that this does not occur is that at every point $dz/dz_o > 0$ which will be shown later to imply that the circle of curvature at each boundary point must intersect the center line. This leaves a useful range of profiles, and the statement of Bahar (1968) on this point is in error.

If this condition is violated in a large scale sense, then a plausible interpretation is that such regions should be treated with a new center line or as special end-zones. Compare this to a T-junction in rectangular coordinates. When we apply the coordinate analysis to a specific situation at finite η, then there may be local small scale irregularities that would be smoothed out in the limit $\eta \to 0$, but which break the coordinate uniqueness condition. Physical considerations would indicate that the field behaviour over most of the line cross-section is largely determined by the general trend of the boundary. We then define the arc coordinate system on some smooth average curve as shown in Fig. 4.4. and evaluate the boundary conditions on the actual profile in terms of the smoothed coordinate system. The coordinate arc system is extended a small

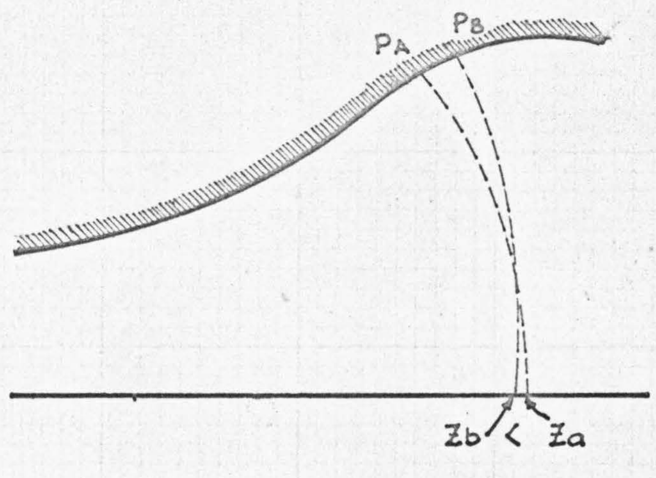

Figure 4.3 Coordinate Arc Overlap

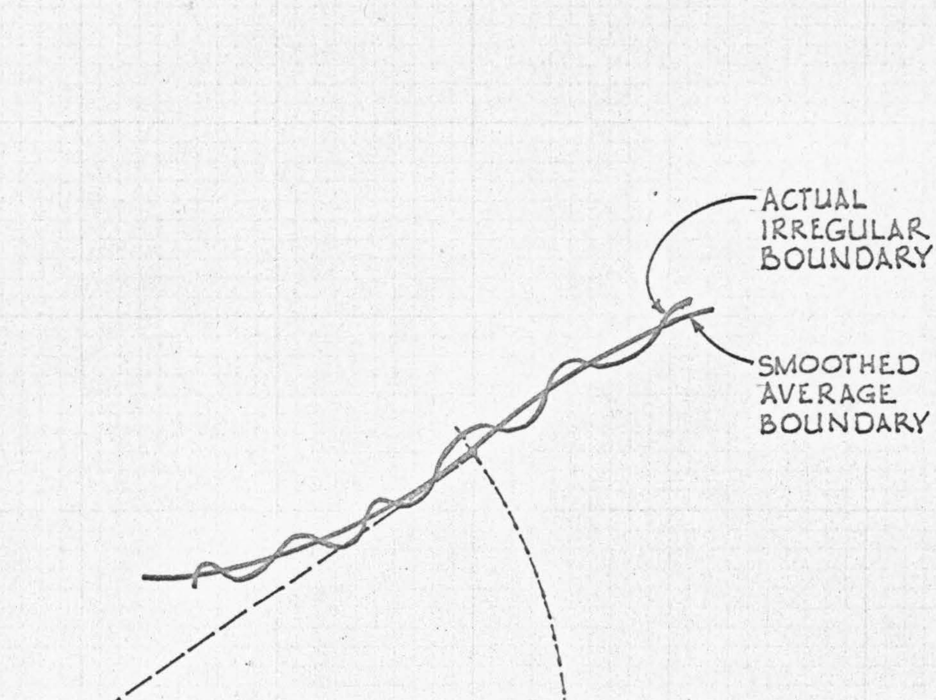

Figure 4.4 Warped Coordinates Defined on a Smoothed Boundary

distance as necessary outside its defining boundary to intersect the irregular boundary. This will set a lower bound on the radius of curvature of concave outwards sections if the irregular boundary is outside such a segment of the smoothed boundary.

4.3 Geometry of Warped Cylindrical Coordinates

Let z_o be the Cartesian z-coordinate of point P on the boundary profile and z be the z-coordinate of the foot S of the new coordinate arc through P as shown in Fig. 4.1. Also let the taper profile be given by

$$x = b(z_o) = a_o \, a\left(\frac{z_o}{\ell_o}\right) \tag{4.2}$$

and let r be the length of the tangent intercept OP. Then the slope of the tangent is given by

$$\tan \Theta = \frac{db}{dz_o} = \eta a' \tag{4.3}$$

and also, with primes denoting differentiation with respect to z_o

$$r = \frac{b}{\sin \Theta} = \frac{b}{b'} \sqrt{1 + b'^2} \tag{4.4}$$

and

$$z - z_o = b \tan \frac{\Theta}{2} = r - \frac{b}{b'}$$

$$z-z_o = \frac{b}{b'}\left\{\sqrt{1+b'^2}-1\right\} \tag{4.5}$$

Formulas for the derivative dr/dz will be useful for future application. From (4.5) we find

$$\frac{dz}{dz_o} = \frac{dr}{dz_o} + \frac{bb''}{b'^2} \tag{4.6}$$

and by an indirect calculation

$$\frac{dr}{dz} = \frac{\frac{dr}{dz_o}}{\frac{dz}{dz_o}} = \frac{\frac{dr}{dz_o}}{\frac{dr}{dz_o}+\frac{bb''}{b'^2}} \tag{4.7}$$

so then

$$\frac{dr}{dz} - 1 = \frac{-\frac{bb''}{b'^2}}{\frac{dr}{dz_o}+\frac{bb''}{b'^2}} \tag{4.8}$$

$$= -\frac{r^2}{b^2}\frac{bb''}{(1+b'^2)\frac{dz}{dz_o}} \tag{4.9}$$

Note that his expression vanishes for a uniform wedge.

The derivative dz/dz_o that determines the uniqueness of the coordinate description can be written as

$$\frac{dz}{dz_o} = \sqrt{1 + b'^2} \left\{ 1 + \frac{\text{sgn } b''}{r_c} \frac{b}{b'^2} \left(1 + b'^2 - \sqrt{1 + b'^2} \right) \right\}$$

$$= \sqrt{1 + b'^2} \left\{ 1 + \frac{\text{sgn } b''}{r_c} \frac{r^2}{b} (1 - \cos \Theta) \right\} \tag{4.10}$$

where r_c is the radius of curvature of the taper profile at P:

$$\frac{1}{r_c} = \left| \frac{b''}{(1 + b'^2)^{3/2}} \right| \tag{4.11}$$

From (4.10) it is easily seen that for $b'' > 0$, a concave outwards profile, $dz/dz_o > 0$ always. If the taper profile is concave inwards $b'' < 0$ and for coordinate uniqueness the radius of curvature is limited to the range

$$r_c > \frac{r^2}{b} (1 - \cos \Theta) \tag{4.12}$$

The radius of a circle tangent to OP at P that is also tangent to the z axis is, in terms of the half angle

$$r_c = r \tan \frac{\Theta}{2}$$

and so from (4.4)

$$r_o = \frac{r^2}{b} (1 - \cos \Theta) = r_{c\ \min}. \tag{4.13}$$

Hence a necessary and sufficient condition that the warped coordinate system be unique, when the profile is concave inwards, is that the circle of curvature at any point on the profile intersects the center line axis. Then we can write

$$\frac{dr}{dz} - 1 = - \frac{r^2}{b^2} \frac{b}{(r_c \operatorname{sgn} b'' + r_{c\,min})} \tag{4.14}$$

It can be seen that r and dr/dz blow up when the line is locally parallel. In future calculations these infinities will be cancelled by zeros of angular terms to give very well behaved results, but it means that fairly large groups of terms must be treated as a whole.

We have now collected enough definitions and results to proceed directly with the calculation of metric coefficients for the warped cylindrical system. It can be seen that the Cartesian coordinates (x_1, z_1) of a point Q are related to the warped cylindrical coordinates (z,s) by

$$\left.\begin{aligned} x_1 &= r \sin \frac{s}{r} \\ z_1 &= z_o - r + r \cos \frac{s}{r} \end{aligned}\right\} \tag{4.15}$$

in terms of the intermediary functions z_o and r. Note that Θ is not explicitly involved. First calculate the derivatives

$$\left.\begin{aligned} \frac{\partial z_1}{\partial z} &= 1 - \frac{dr}{dz} + \frac{dr}{dz}\cos\frac{s}{r} + \frac{s}{r}\frac{dr}{dz}\sin\frac{s}{r} \\ \frac{\partial x_1}{\partial z} &= \frac{dr}{dz}\sin\frac{s}{r} - \frac{s}{r}\frac{dr}{dz}\cos\frac{s}{r} \end{aligned}\right\} \qquad (4.16)$$

$$\left.\begin{aligned} \frac{\partial z_1}{\partial s} &= -\sin\frac{s}{r} \\ \frac{\partial x_1}{\partial s} &= \cos\frac{s}{r} \end{aligned}\right\} \qquad (4.17)$$

The metric coefficients are then found from

$$\left.\begin{aligned} g_{zz} &= \left(\frac{\partial z_1}{\partial z}\right)^2 + \left(\frac{\partial x_1}{\partial z}\right)^2 \\ g_{ss} &= \left(\frac{\partial z_1}{\partial s}\right)^2 + \left(\frac{\partial x_1}{\partial s}\right)^2 \\ g_{zs} &= \frac{\partial z_1}{\partial z}\cdot\frac{\partial z_1}{\partial s} + \frac{\partial x_1}{\partial z}\cdot\frac{\partial x_1}{\partial s} = g_{sz} \end{aligned}\right\} \qquad (4.18)$$

and the determinant of the metric g by

$$g = g_{zz}\, g_{ss} - g_{zs}^2 \qquad (4.19)$$

These can be written explicitly, after much tiresome algebra, in a form where the individual terms are well behaved, even though this does not correspond to a maximal algebraic reduction

$$g_{zz} = 1 + \frac{s^2}{r^2} + \left(\frac{dr}{dz} - 1\right)\frac{s^2}{r^2}$$

$$+ 2\left(\frac{dr}{dz} - 1\right)\left\{1 - \cos\frac{s}{r} - \frac{s}{r}\sin\frac{s}{r} + \frac{1}{2}\frac{s^2}{r^2}\right\}$$

$$+ 2\left(\frac{dr}{dz} - 1\right)^2\left\{1 - \cos\frac{s}{r} - \frac{s}{r}\sin\frac{s}{r} + \frac{1}{2}\frac{s^2}{r^2}\right\} \qquad (4.20)$$

$$g_{ss} = 1 \qquad (4.21)$$

$$g_{zs} = -\frac{s}{r} - \left(\frac{dr}{dz} - 1\right)\left\{1 - \cos\frac{s}{r}\right\} \qquad (4.22)$$

$$\sqrt{g} = 1 + \left(\frac{dr}{dz} - 1\right)\left\{1 - \cos\frac{s}{r}\right\} \qquad (4.23)$$

These expressions are easily shown to be well behaved for small slopes, since then, for example

$$\left(\frac{dr}{dz} - 1\right) = 0\left(\frac{1}{\Theta^2}\right)$$

and since $s/r \leq \Theta$

$$\left\{1 - \cos\frac{s}{r} - \frac{s}{r}\sin\frac{s}{r} + \frac{1}{2}\frac{s^2}{r^2}\right\} = \frac{1}{8}\frac{s^4}{r^4} + 0\left(\frac{s^6}{r^6}\right) = 0(\Theta^4) \qquad (4.24)$$

Thus even the final term of (4.20) remains finite under all conditions. For a uniform finite angle wedge these expressions reduce to a metric closely related to cylindrical coordinates.

$$\left.\begin{aligned} g_{zz} &= 1 + \frac{s^2}{r^2} \\ g_{sz} &= -\frac{s}{r} \end{aligned}\right\} \tag{4.25}$$

and $$g = g_{ss} = 1$$

As a further simplification, for a parallel plate line $r \to \infty$ and we are left with a simple Cartesian metric. From the expression for g_{zs} it is seen that only in this special case is $g_{zs} = 0$. Thus in general the (z,s) warped coordinate system is nonorthogonal. The (z,θ) version reduces to ordinary orthogonal cylindrical coordinates for a uniform finite angle wedge. The metric properties of this and related systems are given in Appendix B.

4.4 The Field Equations in Warped Coordinates

The use of nonorthogonal coordinate systems is relatively rare in the electrical engineering literature, so we shall derive from first principles expressions for the Maxwell equations and boundary conditions in the warped coordinate system. A review of tensor analysis adequate to handle field computation problems in stationary media is given in the classic text of Stratton (1941). For the purposes of this section we shall follow Stratton in writing the various sets of components of a vector $\underline{E}$ as

E_z, E_s, E_y physical components

e^z, e^s, e^y contravariant components

e_z, e_s, e_y covariant components

The covariant and contravariant components in the (z,s,y) system of a vector $\underline{F}$ are related by

$$\left.\begin{aligned} f_z &= g_{zz}\, f^z + g_{zs}\, f^s \\ f_z &= g_{zs}\, f^z + g_{ss}\, f^s \\ f_y &= f^y \end{aligned}\right\} \tag{4.26}$$

The contravariant components g^i of a vector $\underline{G} = \nabla \times \underline{F}$ are

$$g^z = \frac{1}{\sqrt{g}}\left\{\frac{\partial f_y}{\partial s} - \frac{\partial f_s}{\partial y}\right\} ; \quad g^s = \frac{1}{\sqrt{g}}\left\{\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right\}; \; g^y = \frac{1}{\sqrt{g}}\left\{\frac{\partial f_s}{\partial z} - \frac{\partial f_z}{\partial s}\right\} \tag{4.27}$$

In the tapered two-dimensional wedge line problem, all the fields are assumed y-independent and also $E_y = 0$, and $H_z = H_s = 0$. So then the first Maxwell equation

$$\nabla \times \underline{E} = -\mu \frac{\partial H}{\partial t} \tag{4.28}$$

transforms to

$$\frac{1}{\sqrt{g}}\left\{\frac{\partial e_s}{\partial z} - \frac{\partial e_z}{\partial s}\right\} = -\mu \frac{\partial h^y}{\partial t} = -\mu \frac{\partial h_y}{\partial t} = -\mu \frac{\partial H_y}{\partial t} \tag{4.29}$$

The second Maxwell equation

$$\nabla \times H = \varepsilon \frac{\partial E}{\partial t} \tag{4.30}$$

may be written as the two equations

$$\frac{1}{\sqrt{g}} \frac{\partial H_y}{\partial s} = \varepsilon \frac{\partial e^z}{\partial t} \tag{4.31}$$

$$-\frac{1}{\sqrt{g}} \frac{\partial H_y}{\partial z} = \varepsilon \frac{\partial e^s}{\partial t} \tag{4.32}$$

Now transform (4.31) and (4.32) to the covariant components of $\underline{E}$ and then substitute into the first Maxwell equation (4.29). The divergence equations have already been used up in the assumptions on the form of the solution and provide no further simplification. We obtain

$$\frac{\partial}{\partial z}\left\{\frac{g_{ss}}{\sqrt{g}}\frac{\partial H_y}{\partial z}\right\} + \frac{\partial}{\partial s}\left\{\frac{g_{zz}}{\sqrt{g}}\frac{\partial H_y}{\partial s}\right\} - \frac{\partial}{\partial z}\left\{\frac{g_{sz}}{\sqrt{g}}\frac{\partial H_y}{\partial s}\right\} - \frac{\partial}{\partial s}\left\{\frac{g_{sz}}{\sqrt{g}}\frac{\partial H_y}{\partial z}\right\}$$

$$- \mu\varepsilon \sqrt{g} \frac{\partial H_y}{\partial t^2} = 0 \tag{4.33}$$

Next consider the boundary conditions on the curved boundary. Figure 4.5 shows the covariant and contravariant basis vectors at a point on the boundary. The definiton of the (z,s) coordinate system is such that the tangent vector $\hat{t}$ to the boundary curve at P is parallel to the covariant basis vector $\underline{e}^{[z]}$.

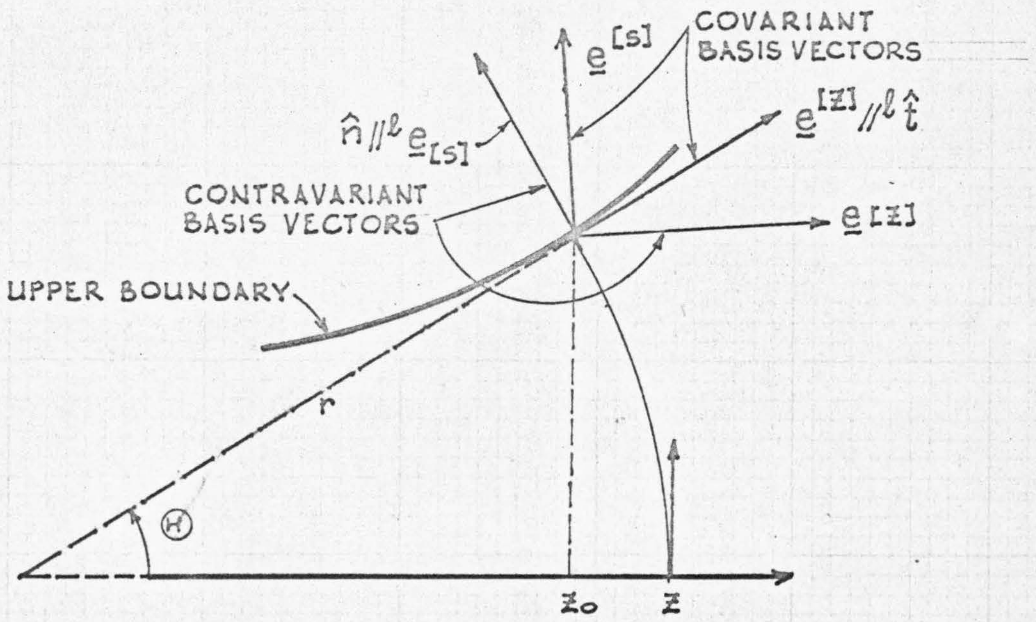

Figure 4.5 Basis Vector Sets on Boundary

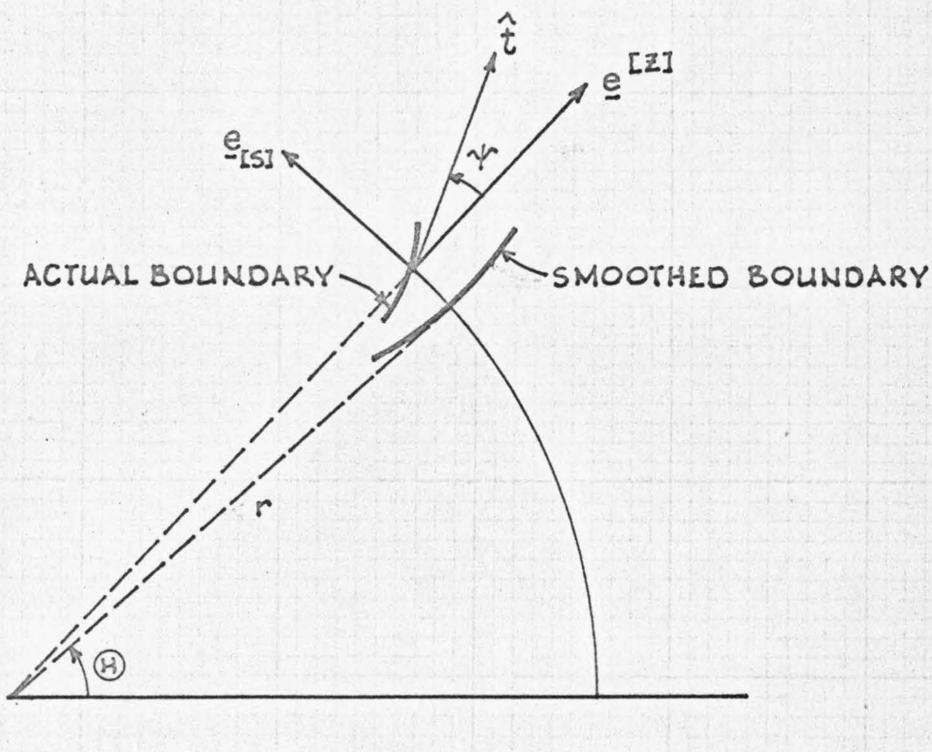

Figure 4.6 Boundary Conditions for an Irregular Boundary

For a perfectly conducting boundary the tangential component of the electric field on the boundary must vanish. Then

$$0 = \hat{t} \cdot \underline{E}$$

$$= \underline{e}^{[z]} \cdot \left\{ \underline{e}_{[z]}\, e^{z} + \underline{e}_{[s]}\, e^{s} \right\}$$

$$= e^{z} \tag{4.34}$$

Hence the boundary condition is equivalent to a requirement that the contravariant z-component of the electric field vanish on the boundary. From equation 4.31 it follows that the boundary condition may be written

$$\frac{\partial H_y}{\partial s} = 0 \quad \text{on} \quad s = \begin{cases} r\Theta & \text{upper boundary} \\ 0 & \text{centerline} \end{cases} \tag{4.35}$$

This is simpler than the condition (3.8,9) in Cartesian coordinates and is more akin to a cylindrical coordinate expression.

4.5 Expansions in the Taper Parameter

The transmission line problem is now rigorously described by the complicated equation (4.33) and the simple boundary condition (4.35). The need for simplifying approximations is apparent, preferably based on a systematic procedure. This scaling procedure will no longer be as simple as that used for the Cartesian expansions where we considered a family of taper profiles differing only on length scale. The transverse description is now no longer purely transverse, so the shape similarity will have to be sacrificed.

Introduce the normalized variables by $z_o = z_o^\dagger \ell_o$ and $z = z^\dagger \ell_o$. Then from (4.5)

$$z^\dagger - z_o^\dagger = \frac{a(z_o^\dagger)}{a'(z_o^\dagger)} \left\{ \sqrt{1 + \eta^2 a'^2(z_o^\dagger)} - 1 \right\} \tag{4.36}$$

Suppose we consider the familiar set of similar shapes obtained by scaling z_o. Then from (4.36) it is evident that z will not scale uniformly with η for a given basic shape function $a(z_o^\dagger)$. A first approach is to do just this and let z fall where it may, but then the z-derivatives in the differential equation (4.33) must undergo the full transformation (4.36) and even in power series expansion this is excessively complicated and restrictive.

An alternative view is to consider the particular problem under consideration as giving a specification for the shape function $a(z_o^\dagger)$ at some particular value of $\eta = \eta_o$. Then under scaling of z we can regard (4.36) as an equation for $z_o^\dagger$ to be solved in terms of $z^\dagger$ and η. In general this process will result in a change of the nonuniform profile by a topological distortion of the support of $a(z_o^\dagger)$. If $\eta a' << 1$ then approximately

$$z^\dagger - z_o^\dagger = \frac{1}{2} \eta^2 a a' \tag{4.37}$$

and so the relative shifts of $z^{\dagger}$ and $z_o^{\dagger}$ will be of second order $O(\eta^2)$. If we were to insist that both $z^{\dagger}$ and $z_o^{\dagger}$ keep their places in $z_1^{\dagger}\varepsilon(0,1)$ then (4.36) shows that a would have to scale in the same proportion, thus defeating the whole purpose.

With this viewpoint we now can simply scale the derivatives in (4.33). The scaling procedure can be regarded as basically a device for ordering the approximation sequence when a given problem is described in normalized variables, and once the approximation sequence is established the variables may be denormalized and the original profile inserted in the solutions.

In its Cartesian version the expansion proceeded strictly in powers of η, the taper scale parameter. In this section that purist approach will be abandoned and the terms ordered in the limit of small η by the dominant power that they contain, with the form of the terms being chosen to give the best description of the physics at finite η. The principle here is that perturbation procedures are generally improved by including all available structural and progress information as early as possible in the proceedings. It is well-known from elementary analysis that in general a convergent power series gives only a limited description of a function and it certainly seems preferable to modify the formulation from the beginning to improve the radius of convergence, rather than by arbitrary telescoping transformations, when, as is almost always the case in perturbation expansions, only the first few terms are available. The criterion to be adopted for selecting the lowest order approximation will be that it give the exact solution for a uniform finite angle wedge independently of η.

Now examine the order of magnitude for small η, of the various intermediary functions that occur in the metric calculations. Introduce the normalized variables $r^\dagger = r/\ell_o$ and $r_c^\dagger = r_c/\ell_o$, then

$$\ell_o r^\dagger = \frac{a_o a}{\eta a'} \sqrt{1 + \eta^2 a'^2} \quad \to \ O(\eta^{-1}) \text{ as } \eta \to 0. \tag{4.38}$$

$$\frac{\ell_o}{a_o} \frac{r_c^\dagger}{a} = \frac{r_c}{b} = \frac{(1 + \eta^2 a'^2)^{3/2}}{\eta^2 aa''} \to O(\eta^{-2}) \text{ as } \eta \to 0 \tag{4.39}$$

$$\Theta = \tan^{-1} \eta a' \ \to \ O(\eta) \quad \text{as} \quad \eta \to 0 \tag{4.40}$$

$$\frac{dr}{dz} - 1 = -\frac{r^2}{b^2} \frac{1}{\frac{r_c}{b} \operatorname{sgn} b'' + \frac{r_{min}}{b}} \to O(\eta^{-2}) \text{ as } \eta \to 0 \tag{4.41}$$

Note also that $s/r \leq \Theta$ on any arc z. The next part of the program is to expand the metric quantities. The function $g^{-\frac{1}{2}}$ of the metric determinant occurs frequently so we expand in a geometric series

$$\frac{1}{\sqrt{g}} = \frac{1}{1 + \left(\frac{dr}{dz} - 1\right)\left(1 - \cos \frac{s}{r}\right)}$$

$$= 1 - \left(\frac{dr}{dz} - 1\right)\left(1 - \cos \frac{s}{r}\right) + \left(\frac{dr}{dz} - 1\right)^2\left(1 - \cos \frac{s}{r}\right)^2 \ \ldots \tag{4.42}$$

This expansion is convergent provided

$$\left|\left(\frac{dr}{dz} - 1\right)\left(1 - \cos\frac{s}{r}\right)\right| < 1 \qquad (4.43)$$

This condition may be written using (4.13) as

$$\left|r_c \operatorname{sgn} b'' + r_{c\ min}\right| > r_{c\ min} - \frac{r^2}{b}\left(\cos\frac{s}{r} - \cos\Theta\right) \qquad (4.44)$$

For a concave outwards line, with $\operatorname{sgn} b'' = +1$ this is always true. For a concave inwards line, $\operatorname{sgn} b'' = -1$ and the inequality becomes

$$r_c > 2\, r_{c\ min} - \frac{r^2}{b}\left(\cos\frac{s}{r} - \cos\Theta\right) \qquad (4.45)$$

When $s = o$ the second term is equal to $r_{c\ min}$. Hence a sufficient condition for the series (4.42) to be convergent is that the line profile be concave outwards, or, if it is concave inwards that the radius of curvature of the boundary at the same z-coordinate be $>$ twice the minimum allowable radius of curvature, $r_{c\ min}$, for no coordinate overlap, where $r_{c\ min}$ is the radius of a circle that touches the boundary at z and also the center line.

Several assortments of trigonometric functions occur in the metric expressions (4.20) to (4.22). These have convergent power series expansion for all s/r

$$\left.\begin{aligned}
&1 - \cos\frac{s}{r} = \frac{s^2}{2r^2} - \frac{s^4}{24r^4} + \ldots \\
&\frac{s}{r} - \sin\frac{s}{r} = \frac{s^3}{6r^3} - \frac{s^5}{120\ r^5} + \ldots \\
&1 - \cos\frac{s}{r} - \frac{s}{r}\sin\frac{s}{r} + \frac{1}{2}\frac{s^2}{r^2} = \frac{s^4}{8r^4} - \frac{s^6}{144\ r^6} + \ldots
\end{aligned}\right\} \quad (4.46)$$

The series (4.42) and (4.46) are then used in the metric expressions to expand them into enough terms to allow a solution of Maxwell's equation through $O(\eta^4)$. These expansions are, for reference

$$\sqrt{g} = 1 - \frac{s^2}{2b^2}\left\{\frac{b}{r_c \operatorname{sgn} b'' + r_{c\ min}}\right\} + \ldots \qquad (4.47)$$

$$\frac{1}{\sqrt{g}} = 1 + \frac{s^2}{2b^2}\left\{\frac{b}{r_c \operatorname{sgn} b'' + r_{c\ min}}\right\} + \ldots \qquad (4.48)$$

$$\frac{g_{zs}}{\sqrt{g}} = -\frac{s}{r} - \frac{1}{3}\frac{s^3}{rb^2}\left\{\frac{b}{r_c \operatorname{sgn} b'' + r_{c\ min}}\right\} - \ldots \qquad (4.49)$$

$$\frac{g_{zz}}{\sqrt{g}} = 1 + \frac{s^2}{r^2} - \frac{s^2}{2b^2}\left\{\frac{b}{r_c \operatorname{sgn} b'' + r_{min}}\right\} \qquad (4.50)$$

$$+ \frac{5}{24}\frac{s^4}{r^2b^2}\left\{\frac{b}{r_c \operatorname{sgn} b'' + r_{min}}\right\} + \frac{s^4}{2b^4}\left\{\frac{b}{r_c \operatorname{sgn} b'' + r_{min}}\right\}^2 + \ldots$$

The propagation equation for H_y can now be written down with the terms grouped by dominant order though still written in unnormalized coordinates. The only extra condition needed for this ordering to be valid is that r_c is finite, that is no sharp kinks in the taper profile. For the sake of brevity we write

$$F(z) = \frac{b}{r_c \operatorname{sgn} b'' + r_{c\,min}} \tag{4.51}$$

and from (4.39) the behaivour under scale change is given by $F = \eta^2 F^{\dagger}$ for the dominant behaviour.

$$0 = \left\{\frac{\partial^2 H_y}{\partial s^2}\right\}$$

$$+ \left\{\begin{array}{l} \dfrac{\partial^2 H_y}{\partial z^2} + \dfrac{1}{r}\dfrac{\partial H_y}{\partial z} - \mu\varepsilon\,\dfrac{\partial^2 H_y}{\partial t^2} + \dfrac{\partial}{\partial z}\;\dfrac{s}{r}\dfrac{\partial H_y}{\partial s} \\[2ex] + \dfrac{\partial}{\partial s}\;\dfrac{s^2}{r^2} - \dfrac{s^2}{2b^2}\,F(z)\;\dfrac{\partial H_y}{\partial s} + \dfrac{s}{r}\dfrac{\partial}{\partial z}\;\dfrac{\partial H_y}{\partial s} \end{array}\right\}$$

$$\frac{\partial}{\partial z}\left(\frac{s^2}{2b^2} F \frac{\partial H_y}{\partial z}\right) + \frac{s^2}{rb^2} F \frac{\partial H_y}{\partial z} + \mu\varepsilon \frac{s^2}{2b^2} F \frac{\partial^2 H_y}{\partial t^2}$$

$$+ \left\{ + \frac{5}{24} \frac{\partial}{\partial s}\left(\frac{s^4}{r^2 b^2} F \frac{\partial H_y}{\partial s}\right) + \frac{\partial}{\partial s}\left(\frac{s^4}{2b^4} F^2 \frac{\partial H_y}{\partial s}\right) \right\}$$

$$+ \frac{1}{3} \frac{\partial}{\partial z}\left(\frac{s^3}{rb^2} F \frac{\partial H_y}{\partial s}\right) + \frac{1}{3} \frac{s^3}{rb^2} F \frac{\partial}{\partial z} \frac{\partial H_y}{\partial s}$$

+ higher order terms. (4.52)

Since the basic equation is ordered in even powers we again expand H_y in a series of even powers of η

$$H_y(z,s,t) = h_o(z^\dagger, s^\dagger, t^\dagger) + \eta^2 h_2(z^\dagger, s^\dagger, t^\dagger) + \ldots$$

$$= H_o + H_2 + \ldots \qquad (4.53)$$

The boundary condition becomes, with $s_o = r\Theta = a_o s_o^\dagger$, and n integral

$$\frac{\partial h_{2n}}{\partial s^\dagger} = 0 \quad \text{on} \quad s^\dagger = \begin{cases} 0 \\ s_o^\dagger \end{cases} \qquad (4.54)$$

Rather than write out the complete set of equations, separated by order, we shall work through them term by term, taking advantage of simplifications as they arise. In the lowest order $O(\eta^o)$ we have

$$\frac{\partial^2 h_o}{\partial s^{\dagger 2}} = 0 \tag{4.55}$$

with $$\frac{\partial h_o}{\partial s^\dagger} = 0 \quad \text{on } s^\dagger = \begin{cases} 0 \\ s_o^\dagger \end{cases}$$

This implies $\partial H_o/\partial s = 0$ everywhere and $H_o = A_o(z^\dagger, t^\dagger)$ where A_o is an as yet undetermined function of $z^\dagger$ and $t^\dagger$. Substitute these results in the $Q(\eta^2)$ equation to obtain

$$\frac{\partial^2 h_2}{\partial s^{\dagger 2}} = -\left\{\frac{\partial^2 A_o}{\partial z^{\dagger 2}} + \frac{1}{r^\dagger}\frac{\partial A_o}{\partial z^\dagger} - \frac{\partial^2 A_o}{\partial t^{\dagger 2}}\right\} \tag{4.56}$$

with $$\frac{\partial h_2}{\partial s^\dagger} = 0 \quad \text{on } s^\dagger = \begin{cases} 0 \\ s_o^\dagger \end{cases}$$

The inhomogeneous term is independent of s. Integrate once and apply the boundary conditions. This and another integration yield

$$\frac{\partial^2 A_o}{\partial z^{\dagger 2}} + \frac{1}{r}\frac{\partial A_o}{\partial z^\dagger} - \frac{\partial^2 A_o}{\partial t^{\dagger 2}} = 0 \tag{4.57}$$

and $$h_2(z^\dagger, s^\dagger, t^\dagger) = A_2(z^\dagger, t^\dagger) \tag{4.58}$$

In the original variables, (4.57) becomes

$$\frac{\partial^2 H_o}{\partial z^2} + \frac{1}{r}\frac{\partial H_o}{\partial z} - \mu\varepsilon \frac{\partial^2 H_o}{\partial t^2} = 0 \tag{4.59}$$

This is the new nonuniform transmission line equation, rather similar in form to the distributed circuit nonuniform line equation also obtained in the Cartesian expansion, and identical when applied to a uniform wedge although the content is different. The first term solution given by (4.59) yields the exact fields everywhere in a uniform finite angle wedge, while the Cartesian expansion has to generate an infinite series of terms to get this result everywhere in the wedge, and this series converges off-axis only in a wedge of 45° half angle. On axis the series has only one term, identical to the solution of (4.59) evaluated on axis.

The first correction term in the earlier formulation for a general profile contains field warping terms as well as a new propagation correction which has the original plane wave fronts. Equation (4.58) shows the new nonuniform line equation in warped coordinates leaves a first correction term with the same transverse field distribution as the basic solution, so that the description of wave fronts on coordinate arcs is good through $O(\eta^2)$. Thus suitably modified distributed circuit current and voltage interpretations can be extended to this order.

A curious feature worth noting is that the new nonuniform line equation shows that the part of the taper profile at $z_1 = z_o$ determines the local propagation behaviour at $z_1 = z$ on axis. If the coefficient r^{-1} had been expanded strictly in powers of η, then the exact solution for the uniform wedge would not have been reproduced by the lowest term, though the phase fronts would have remained correct to the same order. In this light we must regard the on-axis success for the uniform wedge obtained from the standard nonuniform line equation or Cartesian expansion as somewhat fortuitous, the result of the wrong quantity being evaluated at the wrong place, so that the errors cancel exactly for a uniform wedge only.

We now return to the perturbation series. In physical variables the fourth order equation is

$$\frac{\partial^2 H_4}{\partial s^2} = - s^2 \left[\left\{\frac{\partial}{\partial z} \frac{F}{2b^2} + \frac{F}{2rb^2}\right\} \frac{\partial A_o}{\partial z} + \frac{F}{b^2} \cdot \mu\varepsilon \frac{\partial^2 A_o}{\partial t^2}\right]$$

$$- \left\{\frac{\partial^2 H_2}{\partial z^2} + \frac{1}{r}\frac{\partial H_2}{\partial z} - \mu\varepsilon \frac{\partial^2 H_2}{\partial t^2}\right\} \tag{4.60}$$

Integrate once and apply the boundary condition to find

$$\frac{\partial^2 H_2}{\partial z^2} + \frac{1}{r}\frac{\partial H_2}{\partial z} - \mu\varepsilon \frac{\partial^2 H_2}{\partial t^2} = - \frac{r^2\Theta^2}{3} \left\{\frac{\partial}{\partial z}\left(\frac{F}{2b^2}\right) + \frac{F}{2rb^2}\right\} \frac{\partial A_o}{\partial z} + \frac{F}{b^2} \mu\varepsilon \frac{\partial^2 A_o}{\partial t^2} \tag{4.61}$$

One more integration yields

$$H_4 = -\frac{s}{3}\left(r^2\Theta^2 - \frac{s^2}{3}\right)\left[\left\{\frac{\partial}{\partial z}\frac{F}{2b^2} + \frac{F}{2rb^2}\right\}\frac{\partial A_o}{\partial z} + \frac{F}{b^2}\mu\varepsilon\frac{\partial^2 A_o}{\partial t^2}\right] + A_4(z,t) \tag{4.62}$$

We see that the second order term satisfied the same improved non-uniform line equation, now with source terms depending on the line profile and lowest order solution. The expression for H_4 itself contains field distortion terms and a new propagating component $A_4(z,t)$.

The new nonuniform line equation (4.59) has much the same form as the well-known less accurate circuit or planar section version (3.28). This means that analysis and synthesis techniques developed in recent years will still be applicable in so far as they do not depend critically on treating the coefficient of the first order derivative as a logarithmic derivative. We shall regard this type of calculation as largely outside the scope of the present investigation.

When the circuit equation is used for computations it can be regarded as describing a nonuniform line other than the one intended. We have already seen that the coefficients differ in second order of the taper scale parameter. The exponential line with $a \propto \exp(z/m\ell_o)$ gives a constant coefficient in the circuit level equation and is thus a popular example. We now find the family of line profiles which has this same equation in the circular section approximation.

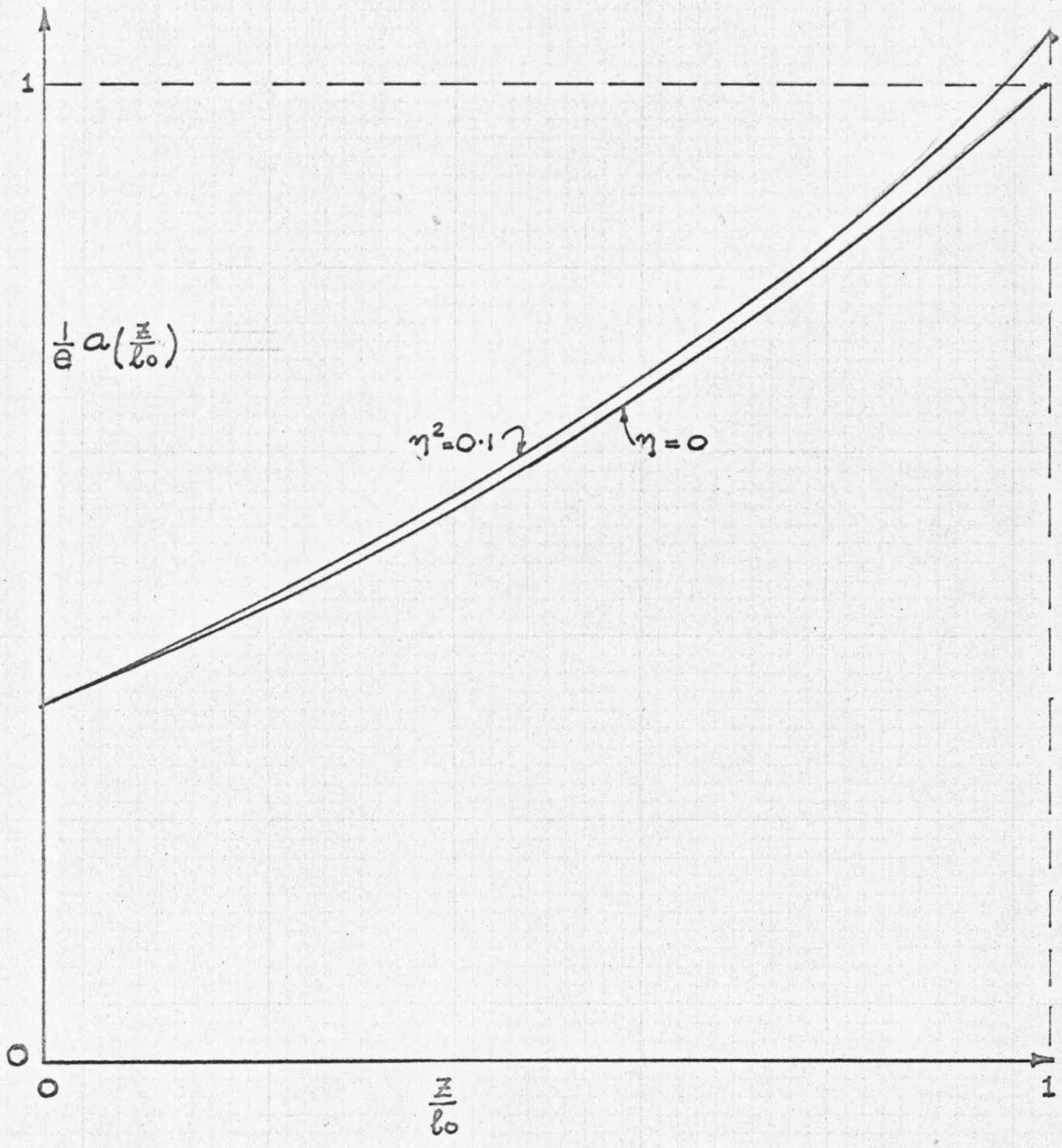

Figure 4.7 Line Profiles for $r = \ell_o$ = constant with η as parameter

$$r = m\ell_o = \frac{\ell_o a}{a'} \sqrt{1 + \eta^2 a'^2} \tag{4.63}$$

The solution of this differential equation is given by (4.64) in which the constant of integration has been chosen so that the profiles for any $\eta > 0$ all pass through $z = 0$, $a = 1$

$$\frac{z}{\ell_o} = m \left\{ \sqrt{1 - \frac{\eta^2 a^2}{m^2}} - 1 + \ell n \frac{2m}{\eta} - \cosh^{-1} \frac{m}{\eta a} \right\} \tag{4.64}$$

This equation may be expanded for small η to show the second order deviations clearly

$$\frac{z}{\ell_o} \approx m \left\{ \ell n\ a + \eta^2 \frac{a^2}{2m^2} \right\} \tag{4.65}$$

Figure 4.7 shows a normalized plot of the line profiles in the range $0 \leq z/\ell_o \leq 1$ for several values of η with $a_o = 1/e$ for convenience and with $m = 1$.

Now consider an irregular outline of the kind illustrated in Figure 4.4. The dominant field pattern over most of the interior is described by the warped cylindrical coordinates based on the smoothed boundary curve $s_c = r\Theta$. Let the irregular real boundary be $s = s_1(z)$. Figure 4.6 shows that the tangent vector to the real boundary is parallel to the vector $\underline{T}$

$$\underline{T} = \sqrt{g}\ \underline{e}^{[z]} + \tan\psi\ \underline{e}[s] \tag{4.66}$$

where ψ is the angle measured from the covariant basis vector $\underline{e}^{[z]}$, normal to the arc, to $\underline{T}$. The result $g_{ss} = 1$ is used in finding the lengths of the basis vectors. Then the boundary condition may be written

$$\underline{T} \cdot \underline{E} = 0 \tag{4.67}$$

From this, with $\underline{E}$ expressed in contravariant components, we find

$$\frac{\partial H_y}{\partial s} = \frac{\tan\psi}{\sqrt{g} + g_{sz}\tan\psi}\frac{\partial H_y}{\partial z} \tag{4.68}$$

Under scaling ψ will be at most $O(\eta)$ so that the boundary condition (4.68) is analogous to the inhomogeneous condition (3.12) in the Cartesian analysis. We find on expanding the denominator in (4.68) to terms in $O(\eta^2)$ that

$$\frac{\partial H_y}{\partial s} = \frac{\tan\psi}{1 - \frac{s^2 F}{2b^2} - \frac{s}{r}\tan\psi}\frac{\partial H_y}{\partial z} \tag{4.69}$$

or more simply in $O(1)$

$$\frac{\partial H_y}{\partial s} = \tan\psi\,\frac{\partial H_y}{\partial z} \tag{4.70}$$

The perturbation analysis will not be repeated here in detail. The new two-dimensional nonuniform transmission line equation for the basic solution becomes

$$\frac{\partial^2 H_o}{\partial z^2} + \left\{\frac{1}{r} - \frac{\tan \psi}{s_1}\right\} \frac{\partial A_o}{\partial z} - \mu\varepsilon \frac{\partial^2 A_o}{\partial t^2} = 0 \qquad (4.71)$$

V. TRANSMISSION LINES WITH CURVED CENTERLINES

5.1 Line Profiles Defined on a Normal Centerline

The examples studied so far of nonuniform transmission lines have been such that the essential feature of one dimensional wave propagation could be interpreted as occurring along a straight centerline axis. In this chapter we shall be concerned with the effects of gradual bends of the centerline, suitably defined, on wave propagation in our two dimensional tapered plate transmission line. Thus the theory will be extended to treat that important class of practical problems where the direction of the transmission line axis is changed deliberately.

The procedure that will be adopted in this section is first to specify a centerline and then to define the profile of the transmission line symmetrically on the normals of this centerline. This gives the curved centerline version of the Cartesian definitions of Chapter III.

Figure 5.1 shows a smooth open curve C_n, a transition curve between straight line segments at either end. C_n is defined by its parametric equations $x = x_o(\zeta)$; $z = z_o(\zeta)$ where ζ is the arc length measured along C_n. At each point of C_n the normal is constructed and the signed distance ξ measured along each normal can then be used to define a point in a (ζ,ξ) coordinate system. The smooth curves C_n and C_2 defined by $\xi = \pm a_o a(\zeta/\ell_o)$ respectively, are used to define

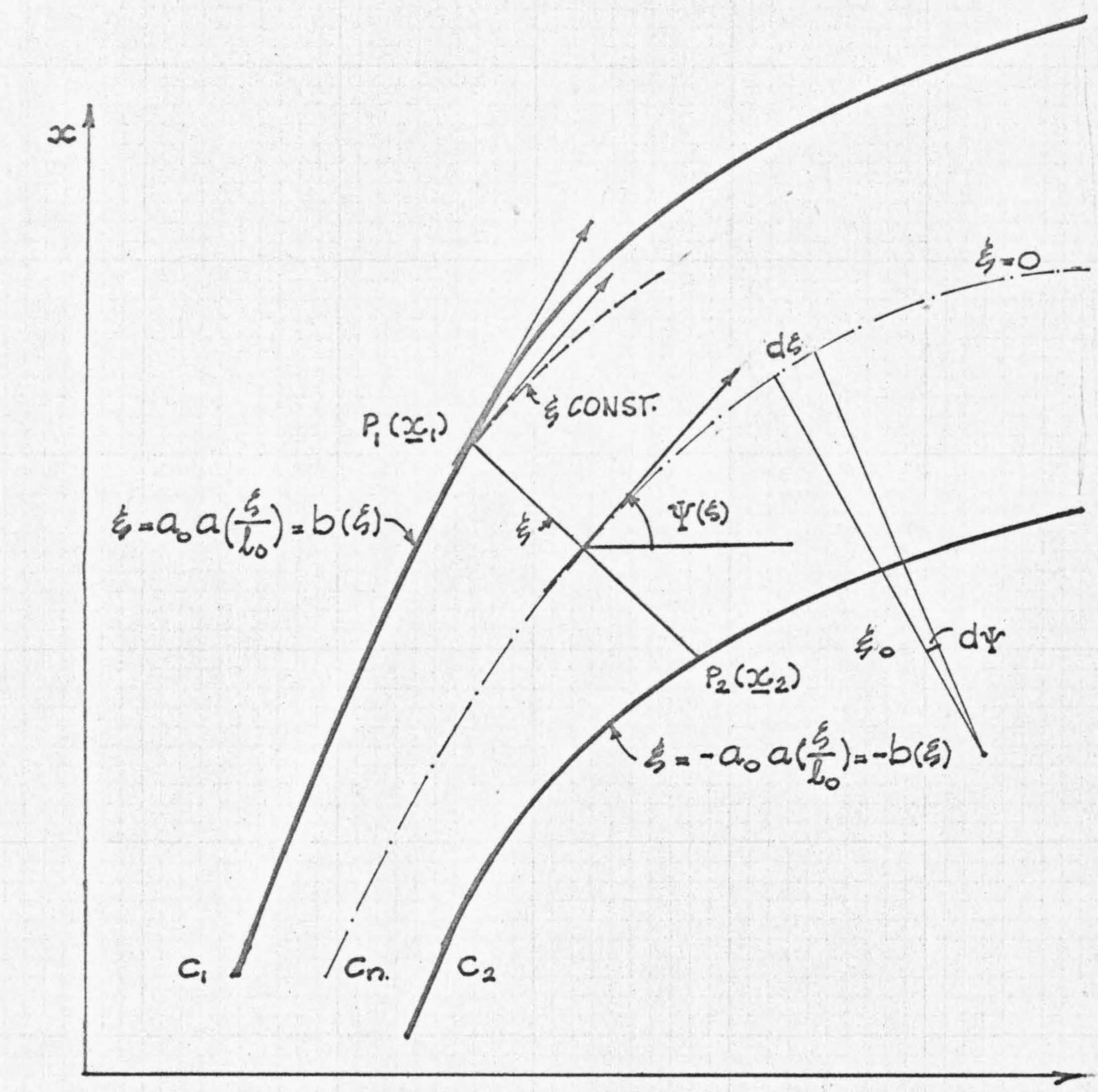

Figure 5.1 Transmission Line Profile with Normal Center Line

the upper and lower boundaries of the curved nonuniform transmission line profile. It is assumed that the successive normals to the normal centerline C_n do not intersect within the area containing C_n, enclosed by C_1 and C_2. Then a (ζ,ξ) description of the transmission line profile and interior is locally unique.

The slope tan ψ of the tangent to the centerline at ζ is given by

$$\left.\begin{aligned} \sin \Psi(\zeta) &= \frac{dx_o}{d\zeta} \\ \cos \Psi(\zeta) &= \frac{dz_o}{d\zeta} \end{aligned}\right\} \tag{5.1}$$

and the Cartesian coordinates (x,z) of a point (ζ,ξ) are given by

$$\left.\begin{aligned} x &= x_o(\zeta) + \xi \cos \Psi(\zeta) \\ z &= z_o(\zeta) - \xi \sin \Psi(\zeta) \end{aligned}\right\} \tag{5.2}$$

We can follow, with appropriate simplifications, the formalism of Chapter IV. In fact it is easily shown that the metric coefficient $g_{\zeta\xi} = 0$ so the (ζ,ξ) coordinate system is orthogonal everywhere. We also find

$$g_{\xi\xi} = 1 \tag{5.3}$$

$$g_{\zeta\zeta} = (1 - \xi\Psi'(\zeta))^2 \tag{5.4}$$

$$g^{\frac{1}{2}} = (1 - \xi\Psi'(\zeta)) \tag{5.5}$$

The derivative $\Psi'(\zeta)$, the arc rate of turning of the tangent, is just the centerline curvature. We can then write the Maxwell equation for the transverse magnetic field as

$$\frac{1}{(1-\xi\Psi')}\frac{\partial}{\partial\xi}\left\{(1-\xi\Psi')\frac{\partial H}{\partial\xi}\right\} + \frac{1}{(1-\xi\Psi')}\frac{\partial}{\partial\zeta}\left\{\frac{1}{(1-\xi\Psi')}\frac{\partial H}{\partial\zeta}\right\} - \mu\varepsilon\frac{\partial^2 H}{\partial t^2} = 0 \qquad (5.6)$$

From the insert picture in Figure 5.1 we see that when

$$\xi_o \frac{d\Psi}{d\zeta} = 1 \qquad (5.7)$$

successive normals intersect. We have already assumed that $\xi_{max} < \xi_o$ everywhere on and inside the line profile, so that the denominator terms in equation (5.6) can always be expanded in a convergent geometric series.

In general the line boundary C_1 or C_2 will intersect the lines of constant ξ at some angle θ given by

$$\tan\theta = \frac{b'(\zeta)}{(1 - \xi\Psi'(\zeta))} \qquad (5.8)$$

where the factor in the denominator takes into account the relation between the coordinate differential $d\zeta$ and the corresponding arc length element. From equations 4.31 and 4.32 we find

$$\frac{\partial H}{\partial\xi} = \frac{\partial E_\zeta}{\partial t} \qquad (5.9)$$

$$\frac{\partial H}{\partial\zeta} = -(1 - \xi\Psi'(\zeta))\frac{\partial E_\xi}{\partial t} \qquad (5.10)$$

The desired boundary condition, as ever, for perfectly conducting boundaries is that the tangential component of electric field vanish there. This becomes

$$\begin{cases} \dfrac{\partial H}{\partial \xi} = \dfrac{\pm b'(\zeta)}{(1 - \xi\Psi'(\zeta))^2} \dfrac{\partial H}{\partial \zeta} \\ \\ \text{on} \quad \xi = \pm b(\zeta) \end{cases} \tag{5.11}$$

Now apply scaling procedures to equations (5.6) and (5.11) to obtain solutions for quasi-one dimensional propagation along the curved center line. As before, we introduce normalized variables by $\xi = \xi^{\dagger} a_o$ and $\xi = \xi^{\dagger} \ell_o$ with the angle of the tangent written as $\psi(\zeta/\ell_o)$. Then we have, with $\eta = a_o/\ell_o$

$$\frac{\partial^2 H}{\partial \xi^{\dagger 2}} - \frac{\eta\psi'}{(1-\eta\xi^{\dagger}\psi')} \frac{\partial H}{\partial \xi^{\dagger}} + \frac{\eta^2}{(1-\eta\xi^{\dagger}\psi')^2} \frac{\partial^2 H}{\partial \zeta^{\dagger 2}} + \frac{\eta^3 \xi\psi''}{(1-\eta\xi^{\dagger}\psi')^3} \frac{\partial H}{\partial \zeta^{\dagger}} - \frac{\partial^2 H}{\partial t^{\dagger 2}} = 0 \tag{5.12}$$

with

$$\begin{cases} \dfrac{\partial H}{\partial \xi^{\dagger}} = \dfrac{\pm \eta^2 a'(\zeta^{\dagger})}{(1 - \eta\xi^{\dagger}\psi')^2} \dfrac{\partial H}{\partial \zeta^{\dagger}} \\ \\ \text{on} \quad \xi^{\dagger} = \pm a(\zeta^{\dagger}) \end{cases} \tag{5.13}$$

The scaling procedure can be thought of as either a shrinking of the transverse dimensions a_o on a given centerline or a stretching of the centerline scale ℓ_o. It can be seen from these equations (5.12) and (5.13) that odd powers of η will occur in the expansions of both differential equation and boundary conditions, so the appropriate perturbation expansion for H will contain a new odd-power sequence of terms that did not appear in the straight centerline analysis. We shall not reproduce all the details but simply give the equations for each order as it arises.

$$H = H_o + \eta H_1 + \eta^2 H_2 + \eta^3 H_3 + \ldots \qquad (5.14)$$

In the lowest order $O(\eta^o)$

$$\frac{\partial^2 H_o}{\partial \xi^{\dagger 2}} = 0 \qquad (5.15)$$

with

$$\frac{\partial H_o}{\partial \xi^{\dagger}} = 0 \quad \text{on } \xi^{\dagger} = \pm a \qquad (5.16)$$

This has the solution $H_o = A_o(\xi^{\dagger}, t^{\dagger})$ where the function A_o has yet to be determined. The problem for the $O(\eta)$ term H_1 also has the form

$$\frac{\partial^2 H_1}{\partial \xi^{\dagger 2}} = 0 \qquad (5.17)$$

$$\text{with} \quad \frac{\partial H_1}{\partial \xi^{\dagger}} = 0 \quad \text{on } \xi^{\dagger} = \pm a \qquad (5.18)$$

This also has a solution in the form $H_1 = A_1(\zeta^{\dagger}, t^{\dagger})$ another unknown function which has to be carried in the analysis until its governing equation is determined.

In the second order of η, $O(\eta^2)$ we find that curvature terms have vanishing coefficients

$$\frac{\partial^2 H_2}{\partial \xi^{\dagger 2}} = - \left\{ \frac{\partial^2 H_o}{\partial \xi^{\dagger 2}} - \frac{\partial^2 A_o}{\partial t^{\dagger 2}} \right\} = - LA_o \qquad (5.19)$$

with

$$\frac{\partial H_2}{\partial \xi^\dagger} = \pm a' \frac{\partial A_o}{\partial \zeta^\dagger} \quad \text{on } \xi^\dagger = \pm a \tag{5.20}$$

These last equations are still identical to those in the straight centerline analysis. We find then the standard nonuniform line equation.

$$\frac{\partial^2 A_o}{\partial \xi^{\dagger 2}} + \frac{a'}{a} \frac{\partial A_o}{\partial \xi^\dagger} - \frac{\partial^2 A_o}{\partial t^{\dagger 2}} = 0 \tag{5.21}$$

and

$$H_2 = \frac{\xi^{\dagger 2}}{2} \frac{a'}{a} \frac{\partial H_o}{\partial \xi^\dagger} + A_2(\zeta^\dagger, t^\dagger) \tag{5.22}$$

The centerline curvature makes its first explicit appearance in the $O(\eta^3)$ system, that is not eliminated by lower order results.

$$\frac{\partial^2 H_3}{\partial \xi^{\dagger 2}} = -\left\{\frac{\partial^2 A_1}{\partial \zeta^{\dagger 2}} - \frac{\partial^2 A_1}{\partial t^{\dagger 2}}\right\} - 2\xi^\dagger \psi' \left\{\frac{\partial^2 A_o}{\partial \zeta^{\dagger 2}} - \frac{1}{2}\left(\frac{a'}{a} - \frac{\psi''}{\psi'}\right) \frac{\partial A_o}{\partial \zeta^\dagger}\right\}$$

$$= -LA_1 - 2\xi^\dagger \psi' MA_o \tag{5.23}$$

with

$$\frac{\partial H_3}{\partial \xi^\dagger} = \pm a' \frac{\partial H_1}{\partial \xi^\dagger} + 2aa' \psi' \frac{\partial H_o}{\partial \xi^\dagger} \qquad \text{on } \xi^\dagger = \pm a \tag{5.24}$$

The operators L and M are written solely for brevity and are easily identified. The boundary condition and differential equation now for the first time introduce even sequence source and boundary terms into the new odd sequence terms. Integrate (5.23) once and apply the boundary condition (5.24) to obtain

$$-aLA_1 = a' \frac{\partial A_1}{\partial \xi^\dagger} \tag{5.25}$$

Thus H_1 satisfies exactly the same homogeneous equation as H_o, and adds no new information, so we may take $H_1 = 0$. Then after another integration we obtain

$$H_3 = 2\xi^\dagger aa'\psi' \frac{\partial A_o}{\partial \xi^\dagger} + (a^2\xi^\dagger - \frac{\xi^{\dagger 3}}{3})\psi' MA_o + A_3(\zeta^\dagger, t^\dagger) \tag{5.26}$$

The third order term consists of field distortion terms proportional to the centerline curvature and depending on the basic solution A_o, and a new undetermined propagating term $A_3(\zeta^\dagger, t^\dagger)$. We shall briefly look at the fourth and fifth order problems in enough detail to fix the undetermined propagators in second and third order. The fourth order problem is

$$\frac{\partial^2 H_4}{\partial \xi^{\dagger 2}} = \psi' \frac{\partial H_3}{\partial \xi^\dagger} - \left\{LH_2 - \xi^\dagger \psi'^2 \frac{\partial H_2}{\partial \xi^\dagger}\right\} - 3\xi^{\dagger 2} \psi'^2 \left\{\frac{\psi''}{\psi'} \frac{\partial H_o}{\partial \zeta^\dagger} + \frac{\partial^2 H_o}{\partial \zeta^{\dagger 2}}\right\}$$

with (5.27)

$$\frac{\partial H_4}{\partial \xi^\dagger} = a' \frac{H_2}{\partial \xi^\dagger} + 3a^2 a' \frac{\partial H_o}{\partial \zeta^\dagger} \qquad \text{on } \xi^\dagger = \pm a \tag{5.28}$$

By the familiar process we find

$$\frac{\partial^2 A_2}{\partial \zeta^{\dagger 2}} + \frac{a'}{a}\frac{\partial A_2}{\partial \zeta^{\dagger}} - \frac{\partial^2 A_2}{\partial t^{\dagger 2}} = -\frac{a^2}{6}\left\{\frac{\partial^2}{\partial \zeta^{\dagger 2}} + \frac{3a'}{a}\frac{\partial}{\partial \zeta^{\dagger}} - \frac{\partial^2}{\partial t^{\dagger 2}}\right\}\left\{\frac{a'}{a}\frac{\partial A_o}{\partial \zeta^{\dagger}}\right\}$$

$$- \frac{1}{3} a^2 \psi'^2 \left\{\frac{\partial^2 A_o}{\partial \zeta^{\dagger 2}} + \left(\frac{3a'}{a} + \frac{2\psi''}{\psi'}\right)\frac{\partial A_o}{\partial \zeta^{\dagger}}\right\}$$

(5.29)

which is again an inhomogeneous nonuniform line equation, which now contains source terms depending on the centerline curvature as well as the familiar nonuniformity sources. Note that the curvature source terms in (5.29) are independent of the sign of the curvature. Physically, a bend in one direction should produce the same reflections as the same bend in the opposite sense.

From the ξ-symmetry of the fifth order equations it is quickly seen that A_3 also satisfies a homogeneous nonuniform line equation, and may be taken as zero, leaving just the first two terms of equation (5.26). The symmetry argument can be extended easily to show that the new propagating terms vanish in all odd sequence terms, leaving only local field distortion terms.

We can give a qualitative summary of the effects of centerline curvature as follows, remembering that (ζ,ξ) are no longer just simple Cartesian coordinates, but a more general orthogonal system that follows the center line curve. The basic zero order solution still satisfies the formal homogeneous nonuniform line equation, with no explicit appearance of center line curvature. The second order term looks the same formally as the straight line version; but now the inhomogeneous line equation governing the new propagating term contains source terms with explicit dependence on centerline curvature. Higher order even terms will contain field distortion and propagation terms that both explicitly depend on curvature. In addition a new odd sequence of terms appears, containing only local field distortions depending on curvature and which do not introduce any new propagating terms which could be distinguished from outside the curved section. Furthermore the first odd term vanishes altogether.

A special case of some interest is the curved uniform line. Then we have the simplified results, with the uniform line operator.

$$\frac{\partial^2 H_2}{\partial \zeta^{\dagger 2}} - \frac{\partial^2 H_2}{\partial t^{\dagger 2}} = - \frac{a^2 \psi'^2}{3} \left\{ \frac{\partial^2 H_o}{\partial \zeta^{\dagger 2}} + \frac{2\psi''}{\psi'} \frac{\partial H_o}{\partial \zeta^{\dagger}} \right\} \tag{5.30}$$

and

$$H_3 = \left\{ a^2 \xi^{\dagger} - \frac{\xi^{\dagger 3}}{3} \right\} \left\{ \frac{\partial^2 H_o}{\partial \zeta^{\dagger 2}} + \frac{\psi''}{2} \frac{\partial H_o}{\partial \zeta^{\dagger}} \right\} \tag{5.31}$$

where H_o is the solution of the ordinary wave equation and as before $H_1 = 0$. Hence reflections seen outside a curved uniform section are of second order in the length scale parameter η. Even for a uniform circular centerline the reflected second order wave does not vanish.

Consider for example, a curved transition section which blends in gradually with uniform straight lines at either end such that ψ'' is continuous. An example of such a centerline curvature function would be $\psi'(\zeta^{\dagger}) = \frac{\sqrt{\pi}}{2} e^{-\zeta^{\dagger 2}}$ which turns the centerline through a right angle. A circular section butted to the straight

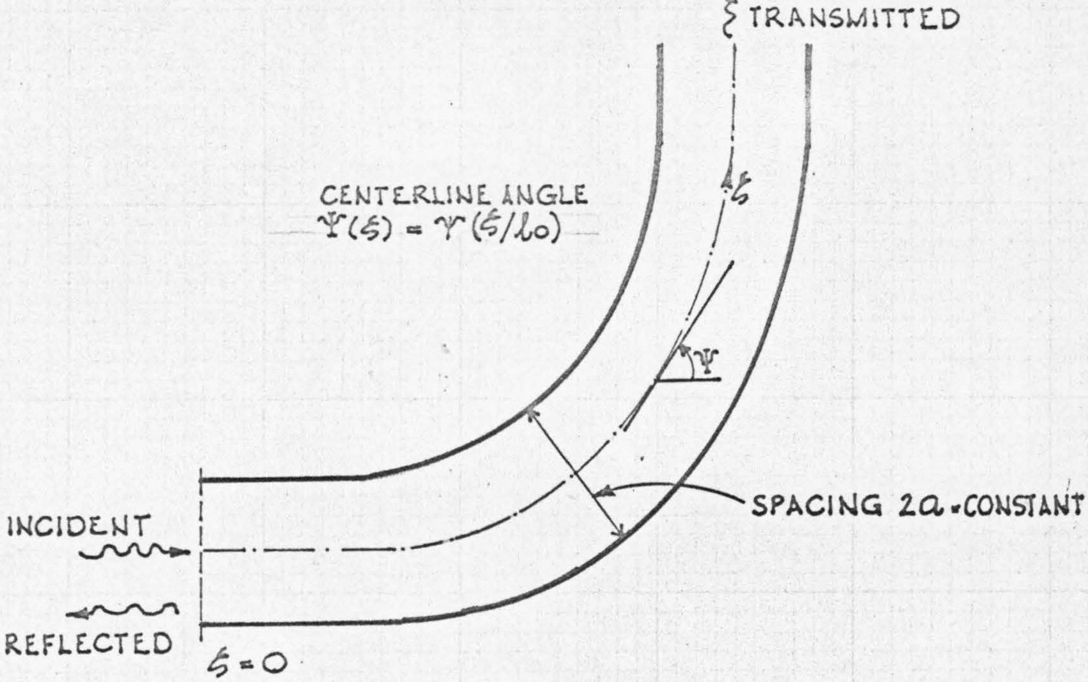

Figure 5.2 Curved Uniform line Section

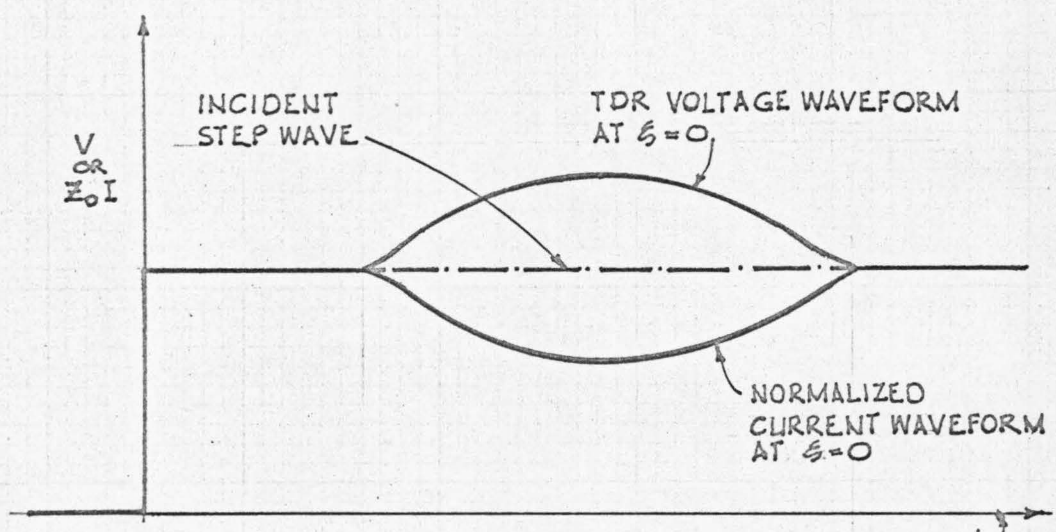

Figure 5.3 TDR Response at $\zeta = 0$ in Curved Uniform Line

sections would not be acceptable since the discontinuity in radius of curvature at the junction would give a δ-function in the source term coefficients. This is outside the gradual taper scheme of things and would have to be treated by a local expansion. Let the zero order solution be a wave propagating in the $+\zeta$ direction with arbitrary waveform $A_o(t^\dagger - \zeta^\dagger)$. Introduce the Laplace transform

$$\tilde{A}_2(\zeta^\dagger, s) = \int_{0-}^{\infty} A_2(\zeta^\dagger, t^\dagger)\, e^{-st^\dagger}\, dt^\dagger \tag{5.32}$$

Then the transform of equation (5.30) can be written

$$\frac{\partial^2 \tilde{A}_2}{\partial \zeta^{\dagger 2}} - s^2\, \tilde{A}_2 = \frac{1}{3}\, a^2 s\, \tilde{A}_o\, \frac{\partial}{\partial \zeta^\dagger} \left\{ (\psi'(\zeta^\dagger))^2\, e^{-s\zeta^\dagger} \right\} \tag{5.33}$$

Assume that $\psi' = 0$ as $\zeta = 0$ and that ψ''exists. Then the solution of equation (5.33) at $\zeta = 0$ may be written as

$$\begin{aligned}
\tilde{A}_2(\zeta^\dagger, s) &= -\frac{1}{6}\, a^2\, \tilde{A}_o \int_{0-}^{\infty} \frac{\partial}{\partial x} \left\{ (\psi'(x))^2\, e^{-sx} \right\} dx \\
&= -\frac{1}{6}\, a^2 s \tilde{A}_o \int_{0}^{\infty} (\psi'(x))^2\, e^{-2sx}\, dx \\
&= -\frac{1}{3}\, a^2 s \tilde{A}_o \int_{0}^{\infty} \left\{ \frac{d}{dx} \cdot \psi(\tfrac{x}{2}) \right\}^2 e^{-sx}\, dx
\end{aligned} \tag{5.34}$$

The integral in (5.34) has been massaged into the form of a Laplace transform, so that the inverse transform of A_2 may be immediately written as a convolution integral. We can take this calculation as a model of a time domain reflectometer (TDR) connected to the line at negative ζ with monitoring point at $\zeta = 0$. Then we can idealize A_o as a perfectly sharp unit step in the calculation leading to (5.34) without any distress and find

$$A_2(0^\dagger, t^\dagger) = -\frac{1}{3} a^2 \left\{\frac{d}{dx} \cdot \psi\left(\frac{x}{2}\right)\right\}^2_{x = t^\dagger} \qquad (5.35)$$

The effect of curvature of a uniform line defined in this way is always to produce a dip in the reflectometer trace whose amplitude is proportional to the square of the curvature at a round trip travel time $t^\dagger$ down the line.

5.2 Alternative Centerline Definitions

The line profiles considered in Section 5.1 are defined in terms of a centerline which is specified from the start. In the analysis of arbitrary curved line profiles, be it a physical line or a table of values in a computer memory, only the boundaries are known and the centerline must be calculated from these before the analysis already developed can be applied. The normal centerline shown in figure 5.4 has the property that the line profiles are symmetrically disposed along its normals. Hence the elementary increases in length at each end between neighboring normal sections, must be equal. This condition may be expressed as

$$d\zeta_1\ \hat{t}_1 \cdot (\underline{x}_1 - \underline{x}_2) = d\zeta_2\ \hat{t}_2 \cdot (\underline{x}_1 - \underline{x}_2) \qquad (5.36)$$

where $d\zeta_1$ and $d\zeta_2$ are increments of the profile arc lengths ζ_1 and ζ_2 and $\hat{t}_1$ and $\hat{t}_2$ are the respective tangent vectors. This equation

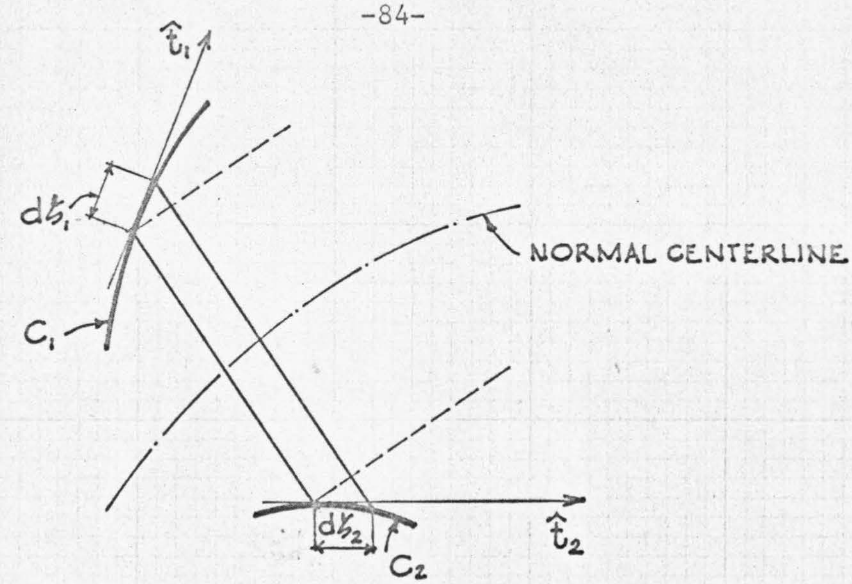

Figure 5.4 Normal Center Line Definition

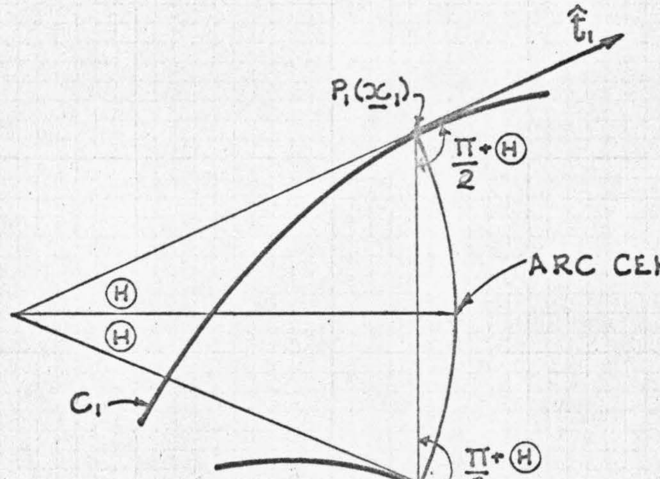

Figure 5.5 Equiangular Arc Center Line Definition

(5.36) can be regarded as a differential equation for $\zeta_1(\zeta_2)$ or $\zeta_2(\zeta_1)$. If there is somewhere where the normal section is known, such as in a straight section, then this provides an initial condition for the integration of the differential equation (5.36). From equation (5.8) it is evident that, in general, the boundary curves do not make equal angles with a normal section.

We might also adopt another point of view, more akin to that of Chapter III and look for cross sections that locally look like short segments of symmetrical wedge. For this we need cross section lines that make equal angles at each end with the boundary curves. Then an arc can be drawn normal to the boundary at each end, with its center on the normal bisector of the equiangular section. We then define the ends of this section by

$$(\underline{x}_1 - \underline{x}_2) \cdot (\hat{t}_1 + \hat{t}_2) = 0 \tag{5.37}$$

This relation can be used by taking a point $x_1(\zeta_1)$ and solving the algebraic equation (5.37) for ζ_2.

Then we define the locus of the center points of this family of arcs as the equiangular arc centerline. The centerline thus defined is in general not coincident with the normal centerline already used. For straight centerlines or uniform curved lines, where the arcs are straight lines, the normal and equiangular centerlines are identical. The arc length ζ is measured along this line. When the centerline is straight this reduces to the warped cylindrical coordinates introduced in Chapter IV.

The geometry of the equiangular arc centerline and the coordinate system based on it, are studied in Appendix B. The local wedge axis is not tangent to the centerline and it is shown there that the deviation angle is $O(\eta^2)$ in the taper scale parameter.

5.3 Field Solutions in the Equiangular Centerline Description

The general discussion of the two dimensional electromagnetic field problem in Section 4.4 is entirely adequate to handle the curved transmission line problem, when it is expressed, as shown in detail in Appendix B, in terms of warped cylindrical coordinates based on the curved equiangular centerline. In particular the differential equation (4.33) and boundary condition (4.35) are still valid, of course with (ζ,σ) written instead of the space variables (z,s). As is shown in Appendix B, equations (B-31) to (B-34), curvature dependent odd-sequence terms again make an appearance, with an effect analogous to that found in Section 5.1.

As in Section 4.5, we are more interested, because of the complex behavior of the coefficients under scaling, in obtaining solutions in physical variables for the particular profile being analyzed, using the scaling ideas only to establish the approximation sequence. So we expand

$$H(\zeta,\sigma,t) = h_o(\zeta^\dagger,\sigma^\dagger,t^\dagger) + \eta\, h_1(\zeta^\dagger,\sigma^\dagger,t^\dagger) + \ldots$$

$$= H_o(\zeta,\sigma,t) + H_1(\zeta,\sigma,t) + \ldots \qquad (5.38)$$

where the term H_n is of order $O(\eta^n)$. The boundary condition

$$\frac{\partial h_n}{\partial \sigma^\dagger} = \frac{\partial H_n}{\partial \sigma} = 0 \qquad (n = 0,1,\ldots) \tag{5.39}$$

is true for all orders and will not be repeated each time. The basic equation is

$$\frac{\partial^2 H_o}{\partial \sigma^2} = 0 \tag{5.40}$$

which again implies

$$H_o = A_o(\zeta,t) \tag{5.41}$$

The first order equation is

$$\frac{\partial^2 H_1}{\partial \sigma^2} = \frac{\partial}{\partial \sigma}(\sigma\psi' \frac{\partial H_o}{\partial \sigma}) = 0 \tag{5.42}$$

so that by the arguments of section 5.1, H_1 can be set to zero. The nonvanishing terms in the second order equation are

$$\frac{\partial^2 H_2}{\partial \sigma^2} = -\left\{\frac{\partial^2 A_o}{\partial \zeta^2} + \frac{1}{r}\frac{\partial A_o}{\partial \zeta} - \mu\varepsilon \frac{\partial^2 A_o}{\partial t^2}\right\} \tag{5.43}$$

The solution of this equation yields

$$\frac{\partial^2 H_o}{\partial \zeta^2} + \frac{1}{r}\frac{\partial H_o}{\partial \zeta} - \mu\varepsilon \frac{\partial^2 H_o}{\partial t^2} = 0 \tag{5.44}$$

and

$$H_2 = A_2\ (\zeta,t) \tag{5.45}$$

This last pair of results shows that the basic lowest approximation satisfies an equation formally similar to the improved nonuniform line equation, and that the first correction term is of second order in the nonuniformity parameter and still has the same wave-fronts as the basic solution. These important properties carry over unchanged from straight center line case. In the third order

$$\frac{\partial^2 H_3}{\partial \sigma^2} = -\sigma \, MA_o \tag{5.46}$$

where M is the operator defined by

$$MA_o = \left\{ \frac{\partial}{\partial \zeta} \left(\psi' \frac{\partial A_o}{\partial \zeta} \right) + \frac{\psi'}{r} \frac{\partial A_o}{\partial \zeta} + \psi' \sqrt{\mu\varepsilon} \frac{\partial^2 A_o}{\partial t^2} \right\} \tag{5.47}$$

The first integration yields with $\sigma_o = r\, H$

$$\frac{\partial H_3}{\partial \sigma} = \frac{1}{2} (\sigma_o{}^2 - \sigma^2) \, MA_o \tag{5.48}$$

The symmetry arguments again will apply to show that no new propagating term occurs. From the fourth order equation, the propagation equation that determines the second order correction can be found as

$$\frac{\partial^2 A_2}{\partial \zeta^2} + \frac{1}{r} \frac{\partial A_2}{\partial \zeta} - \mu\varepsilon \frac{\partial^2 A_2}{\partial t^2}$$

$$- \frac{1}{3} r^2 \; \Theta^2 \left\{ \left[\frac{\partial}{\partial z} \left(\frac{F}{2b^2} \right) + \frac{F}{2rb^2} \right] \frac{\partial A_o}{\partial z} + \frac{F}{b^2} \frac{\partial^2 A_o}{\partial t^2} \right\}$$

$$- \frac{1}{3} \sigma_o{}^2 \; \psi'^{\,2} \left\{ \frac{\partial^2 A_o}{\partial \zeta^2} + \frac{2\psi''}{\psi'} \frac{\partial A_o}{\partial \zeta} \right\} + \left\{ \frac{1}{2} \frac{\sigma_o{}^2 \psi'^{\,2}}{r} - \delta\psi' \right\} \frac{\partial A_o}{\partial \zeta} \tag{5.49}$$

In the limit of a curved uniform line $r \to \infty$, δ, $a' \to 0$ this equation becomes identical with the limit of equation (5.29) in orthogonal coordinates. The sources contributing new waves have been split up into straight line terms plus center line curvature terms. In the next section we shall apply (5.50) to analysis of the fields in a uniformly curved wedge, a relatively simple but definitely non-trivial example.

5.4 The Uniformly Curved Wedge

We define a uniformly curved wedge in terms of the equiangular arc center line analysis. If we have a line profile such that, when the analysis from equation (5.37) is made, $\Theta = \Theta_o$ = constant and the local centerline axis rotates uniformly in space with change of arc centerline coordinate ζ, ψ' = constant = $1/R_o$, then we describe this profile as a uniformly curved wedge. Since the local centerline axis is not tangent to the arc centerline, it follows that the arc centerline is not necessarily a circle, though it will be close for small η. We take the origin of ζ to be the point where $r = 0$ and confine attention to some region starting at $r = r_o > 0$. to avoid Hankel function singularities. Since $d\Theta = 0$ we can write (B.14) as

$$\tan \delta = \frac{d\zeta_1 - d\zeta_2}{d\zeta_1 + d\zeta_2} \tan \frac{\Theta_o}{2} \tag{5.50}$$

which gives
$$\delta = \frac{r \, \Theta_o^{\,2}}{2R_o} \{1 + O(\delta^2)\} \tag{5.51}$$

which follows from the geometry of the figure. Hence the final group of terms in (5.50) vanishes to a higher order than any of the individual terms, and so may be omitted entirely. The first group of terms vanishes for a uniform wedge so we are left with

$$\frac{\partial^2 A_2}{\partial \zeta^2} + \frac{1}{\zeta}\frac{\partial A_2}{\partial \zeta} + k^2 A_2 = -\frac{{\Theta_o}^2}{3{R_o}^2} \cdot \zeta^2 \cdot \frac{\partial^2 H_o^{(1)}(k\zeta)}{\partial \zeta^2} \qquad (5.52)$$

where the solution for the lowest order term A_o can be taken as a single outward propagating wave of form $H_o^{(1)}(k\zeta)$ when we assume harmonic time dependence.

The homogeneous solution of (5.52) adds nothing new. It may be verified that the particular integral of (5.52) is given by

$$A_2(\zeta,k) = -\frac{{\Theta_o}^2}{3\sigma}\left(\frac{\zeta}{R_o}\right)^2 \left\{5H_2^{(1)}(k\zeta) - 2k\zeta\ H_1^{(1)}(k\zeta)\right\} \qquad (5.53)$$

Since (5.52) is a linear second order equation, the particular solution may be obtained in a systematic fashion by the method of variation of parameters, or alternatively by the traditional guess and check method. The solution (5.53) was, in fact, obtained in part by trial and the more stubborn remainder by variation of parameters. Integrals of the form

$$I = \int^z x^3\ C_o^2\ (x)\ dx \qquad (5.54)$$

where C_o is a zero order cylinder function, were evaluated by Schafheitlin's reduction formula as quoted by Watson (1958).

Thus the second order correction term to the basic solution, showing the effects of axis curvature, is given by (5.53). As expected it depends only on the magnitude and not the sign of the curvature. A more subtle and interesting feature is that the Hankel functions in (5.54) are all of the first kind, representing outward propagation only, there being no reflected waves in the second order correction. Since the third order correction introduces field distortion terms only, the reflected wave in this profile geometry can be at most of fourth order magnitude $O(\eta^4)$ in the taper and curvature scale parameter η.

VI. THE NONUNIFORM COAXIAL LINE

6.1 Introduction

In previous chapters we have analyzed two dimensional nonuniform transmission lines. The two dimensional treatment has had the advantage of allowing the essential features of the coordinate systems and perturbation expansions to be exhibited with a minimum of distracting complications. Of course a price has been paid in the sense that strictly two dimensional problems rarely if ever occur in this three dimensional world, although it has been found from experience that a statement of two-dimensional results per unit width gives a good description of finite width lines, provided the spacing is much less than the width.

In this chapter we shall extend our methods to three-dimensional problems. A possible class of problems would be the practical completion of the two-dimensional theory to describe finite width strip lines, by including an estimate of edge effects. Even for uniform strip lines however, calculation of fringing fields requires considerable theoretical elaboration. Instead we shall make a natural extension of the techniques developed in previous chapters, to treat the coaxial line, an example of great technical importance. This treatment will be more realistic than the two-dimensional analysis in the sense that convenient laboratory realizations of coaxial lines require very little idealization to obtain a soluble theoretical model. The coaxial line also has the property, absent in two dimensions, that a grossly nonuniform line on a straight

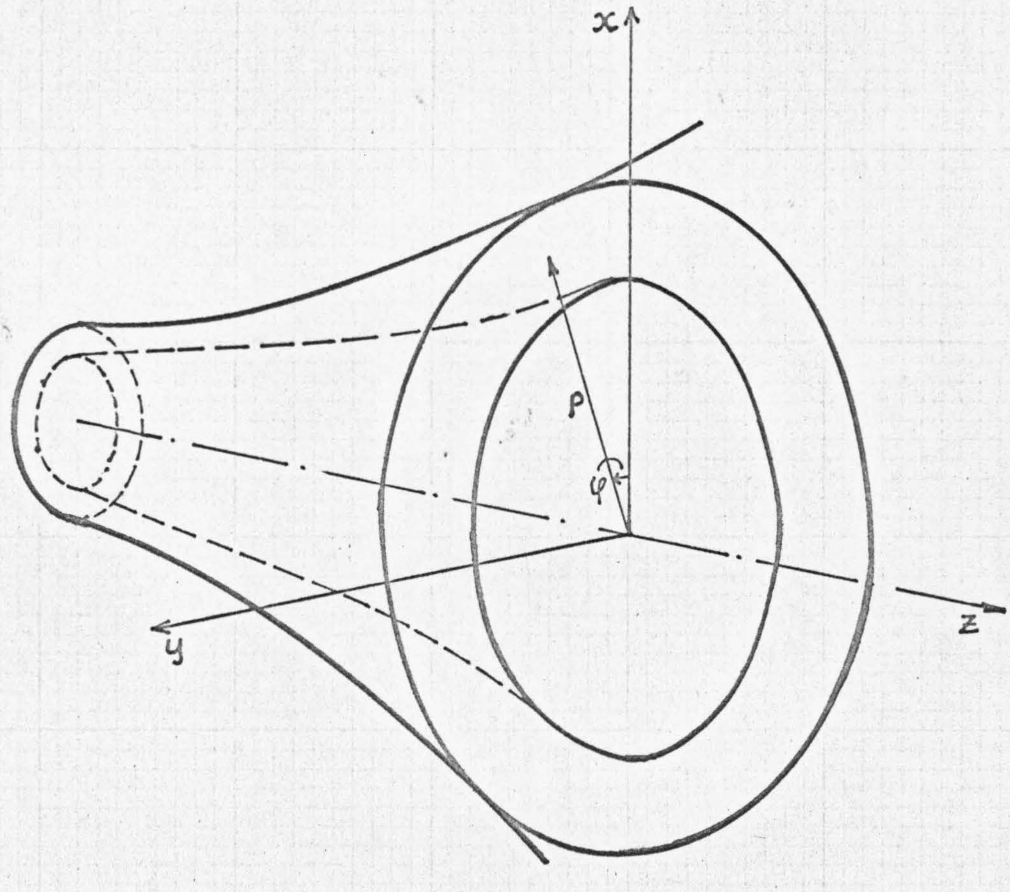

Figure 6.1 Nonuniform Coaxial Line with Planar Cross-sections

centerline can have constant impedance in the distributed circuit analysis.

6.2 Planar Section Approximations

The coaxial nonuniform line to be analyzed is shown in Figure 6.1. Define cylindrical coordinates (ρ,ϕ,z) with the z-axis along the common axis of the inner and outer conductors. We assume that all excitations and fields are ϕ-independent, and so the only nonzero field components are $E_\rho(\rho,z)$, $E_z(\rho,z)$ and $H_\phi(\rho,z)$. Maxwell's equations may be written

$$-\hat{e}_\rho \frac{\partial H_\phi}{\partial z} + \hat{e}_z \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H_\phi) = \varepsilon \frac{\partial}{\partial t}\left(\hat{e}_\rho E_\rho + \hat{e}_z E_z\right) \tag{6.1}$$

$$\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} = -\mu \frac{\partial H_\phi}{\partial t} \tag{6.2}$$

Let the inner and outer boundaries of the line be given by respectively

$$\rho = \begin{cases} a_1(z) = a_o\ a(z/\ell_o) \\ b_1(z) = a_o\ b(z/\ell_o) \end{cases} \tag{6.3}$$

where a_o is a typical cross-section radius, and ℓ_o the length scale of the nonuniform section, giving the familiar dimensionless taper parameter $\eta = a_o/\ell_o$. In terms of normalized variables $\rho^\dagger = \rho/a_o$, $z^\dagger = z/\ell_o$ and $t^\dagger = t/\sqrt{\mu\varepsilon}\ \ell_o$, Maxwell's equations and the boundary condition of zero tangential electric field can be written as

$$\frac{\partial}{\partial\rho^\dagger}\left(\frac{\partial H_\phi}{\partial\rho^\dagger} + \frac{H_\phi}{\rho^\dagger}\right) + \eta^2\left(\frac{\partial^2 H_\phi}{\partial z^{\dagger 2}} - \frac{\partial^2 H_\phi}{\partial t^{\dagger 2}}\right) = 0 \qquad (6.4)$$

with

$$\frac{\partial H_\phi}{\partial\rho^\dagger} + \frac{H_\phi}{\rho^\dagger} = \eta^2 \begin{Bmatrix} a'(z^\dagger) \\ \text{or} \\ b'(z^\dagger) \end{Bmatrix} \frac{\partial H_\phi}{\partial z^\dagger} \quad \text{on} \quad \rho^\dagger = \begin{Bmatrix} a(z^\dagger) \\ \text{or} \\ b(z^\dagger) \end{Bmatrix} \qquad (6.5)$$

The electric field components can both be calculated from the single transverse magnetic field component H_ϕ by means of equation (6.1). The differential equation (6.4) and its boundary condition (6.5) contain even powers only of η, so that a perturbation expansion of H_ϕ in even powers of η is appropriate.

$$\begin{aligned} H_\phi(\rho,z,t) &= H_o(\rho,z,t) + H_2(\rho,z,t) + \ldots \qquad (6.6) \\ &= h_o(\rho^\dagger,z^\dagger,t^\dagger) + \eta^2\, h_2(\rho^\dagger,z^\dagger,t^\dagger) + \ldots \end{aligned}$$

We shall not write out the collection of equations that define the perturbation series, but shall just quote and solve them as necessary. Then in order unity,

$$\frac{\partial}{\partial\rho^\dagger}\left(\frac{\partial h_o}{\partial\rho^\dagger} + \frac{h_o}{\rho^\dagger}\right) = 0 \qquad (6.7)$$

with
$$\frac{\partial h_o}{\partial \rho^\dagger} + \frac{h_o}{\rho^\dagger} = 0 \quad \text{on} \quad \rho^\dagger = a \text{ or } b \tag{6.8}$$

One integration yields

$$\frac{\partial h_o}{\partial \rho^\dagger} + \frac{h_o}{\rho^\dagger} = 0 \tag{6.9}$$

and a second integration

$$h_o = \frac{1}{\rho^\dagger} A_o(z^\dagger, t^\dagger) \tag{6.10}$$

where $A_o(z^\dagger, t^\dagger)$ is an as yet arbitrary function of $z^\dagger$ and $t^\dagger$. The equation for h_2 becomes

$$\frac{\partial}{\partial \rho^\dagger}\left(\frac{\partial h_2}{\partial \rho^\dagger} + \frac{h_2}{\rho^\dagger}\right) = -\frac{1}{\rho}\left\{\frac{\partial^2 A_o}{\partial z^{\dagger 2}} - \frac{\partial^2 A_o}{\partial t^{\dagger 2}}\right\} \tag{6.11}$$

with

$$\frac{\partial h_2}{\partial \rho^\dagger} + \frac{h_2}{\rho^\dagger} = \begin{Bmatrix} a' \\ b' \end{Bmatrix} \frac{\partial h_o}{\partial z^\dagger} \quad \text{on} \quad \rho^\dagger = \begin{Bmatrix} a \\ b \end{Bmatrix} \tag{6.12}$$

We find after one integration of (6.11) and application of the boundary conditions (6.12) that

$$\frac{\partial}{\partial\rho^\dagger}(\rho^\dagger h_2) = \rho^\dagger \left\{ \frac{\frac{b'}{b}\ \ell n\ \frac{\rho^\dagger}{a} - \frac{a'}{a}\ell\ \frac{\rho^\dagger}{b}}{\ell n\ b - \ell n\ a} \right\} \frac{\partial A_o}{\partial z^\dagger} \tag{6.13}$$

and

$$\frac{\partial^2 A_o}{\partial z^{\dagger 2}} + \frac{\frac{b'}{b} - \frac{a'}{a}}{\ell n\ b - \ell n\ a}\ \frac{\partial A_o}{\partial z^\dagger} - \frac{\partial^2 A_o}{\partial t^{\dagger 2}} = 0 \tag{6.14}$$

The equation (6.14) that determines the lowest order approximation to the transverse magnetic field may be compared to the distributed circuit equation for the current. The capacitance per unit length of an elementary plane transverse section of the coaxial line is given by

$$C(z) = \frac{2\pi\varepsilon}{\ell n\ \frac{b_1(z)}{a_1(z)}} \tag{6.15}$$

so that the coefficient of $\partial I/\partial z$ in the circuit equation is given by

$$-\frac{1}{Y}\frac{dY}{dz} = \frac{\frac{b_1'}{b_1} - \frac{a_1'}{a_1}}{\ell n\ \frac{b_1(z)}{a_1(z)}} \tag{6.16}$$

and is identical to the coefficient of $\partial A_o/\partial z$ in (6.14) after physical variables have been restored.

If the ratio of inner and outer conductor radii at any given cross-section remains constant, then the coefficient (6.16) vanishes and the line equation reduces to a simple wave equation. This is the well known constant impedance coaxial line transition, which is reflectionless in the distributed circuit theory. In general, the perturbation expansion will correct this with higher order terms that will show field distortions and new propagating waves, forward and reflected.

Equation (6.13) may be integrated again to give

$$h_2 = \frac{\rho^{\dagger}}{2}\left(\ell n\ \rho - \frac{1}{2}\right)\frac{\frac{b'}{b} - \frac{a'}{a}}{\ell n\ \frac{b}{a}}\frac{\partial A_o}{\partial z^{\dagger}} + \frac{\rho}{2}\,\frac{\frac{a'}{a}\ \ell n\ b - \frac{b'}{b}\ \ell n\ a}{\ell n\ \frac{b}{a}}\,\frac{\partial A_o}{\partial z} + \frac{1}{\rho}\,A_2(z^{\dagger},t^{\dagger}) \tag{6.17}$$

The complexity of the equations in higher order approximations escalates even more rapidly than it does in the tapered plate line calculations. The now familiar process may be used to find the equation satisfied by the first new propagating wave correction $A_2(z^{\dagger},t^{\dagger})$. This equation is given in its full glory in Appendix A(A.4). We shall concentrate in the main text on the nominal constant impedance line since the results are easier to interpret, and because it provides a good bench mark for evaluating the effects of higher order corrections. It also represents an important practical application in its own right. The equation for h_2 in the constant impedance line simplifies to

$$h_2 = \frac{\rho^\dagger}{2} \frac{a'}{a} \frac{\partial A_o}{\partial z^\dagger} + \frac{1}{\rho^\dagger} A_2(z^\dagger, t^\dagger) \tag{6.18}$$

where

$$\frac{\partial^2 A_2}{\partial z^{\dagger 2}} - \frac{\partial^2 A_2}{\partial t^{\dagger 2}} = - \frac{(d^2 - 1)}{4 \ln d} a^2(z^\dagger) \left\{ \frac{\partial^2}{\partial z^{\dagger 2}} + \frac{2a'}{a} \frac{\partial}{\partial z^\dagger} - \frac{\partial^2}{\partial t^{\dagger 2}} \right\} \left\{ \frac{a'}{a} \frac{\partial A_o}{\partial z^\dagger} \right\} \tag{6.19}$$

where $b/a = d > 1$ is the constant ratio of outer to inner radius. Since A_o is a solution of the homogeneous wave equation, (6.19) can be reduced to

$$\frac{\partial^2 A_2}{\partial z^{\dagger 2}} - \frac{\partial^2 A_2}{\partial t^{\dagger 2}} = - \frac{d^2 - 1}{4 \ln d} \left\{ aa'' \frac{\partial^2 A_o}{\partial z^{\dagger 2}} + a^2 \frac{\partial}{\partial z^\dagger} \left(\frac{a''}{a} \frac{\partial A_o}{\partial z^\dagger} \right) \right\} \tag{6.20}$$

An even simpler case occurs when the line boundaries are concentric cones. Then $a'' = b'' = 0$ and equation (6.20) is homogeneous, so its solution may be taken as zero. Then

$$H = \frac{1}{\rho^\dagger} A_o(t^\dagger - z^\dagger) + \frac{\eta^2 \rho^\dagger}{2z^\dagger} \frac{\partial}{\partial z^\dagger} A_o(t^\dagger - z^\dagger) + \ldots \tag{6.21}$$

This problem in fact has an exact solution for the principal mode in normalized coordinates

$$H = \frac{A_o(t^\dagger - r^\dagger)}{\rho^\dagger} = \frac{A_o(t^\dagger - r^\dagger)}{r \sin \theta} \tag{6.22}$$

where r and z are measured from the same origin. If we expand $r^\dagger$ in terms of $z^\dagger$ and $\rho^\dagger$ and then expand the function A_o in the Taylor series, we recover the first two terms of the perturbation expansion. The behaviour of the perturbation series here is similar to that in the uniform wedge analysis, comprising a propagating term in lowest order, with field distortions based on this alone in higher orders. The lowest order solution predicts fields on the normal planes z = constant as wave fronts. The exact solution for a uniform conical line shows that the wavefronts are spherical annuli, which coincide nowhere in the propagation region with the approximate predictions. This situation is analogous to that found for the uniform wedge, except that there the lowest order approximate solution was exact along the center plane. In later sections we shall remedy this situation by an extension of the nonorthogonal coordinate expansions and curved center line descriptions developed in earlier chapters.

6.3 Waveform Aberrations from Constant Impedance Transitions

An important application of nonuniform coaxial line sections is to provide transitions between uniform coaxial lines of the same characteristic impedance but different dimensions, while introducing a minimum of undesired reflections and transmission aberrations. The perturbation theory developed in the previous section for smooth transitions shows that if b/a is constant, the first visible effects of the nonuniformity appear in the second order equations as inhomogeneous terms. We now proceed to calculate by Laplace transform methods, from equation (6.19), the waveform aberrations that would

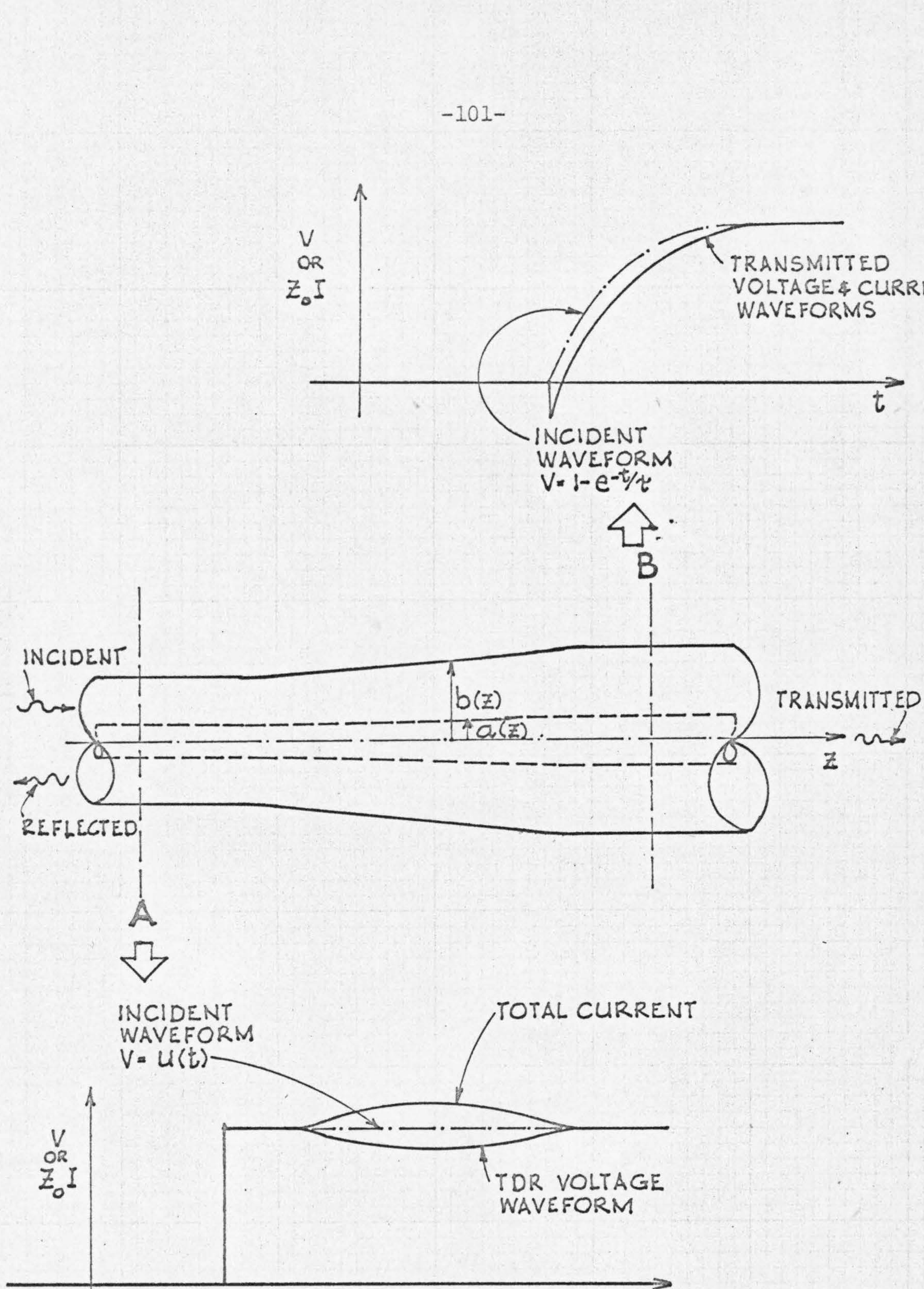

Figure 6.2 Coaxial Transition of Nominally Constant Impedance and Waveforms

be observed with TDR equipment. Rewrite equation (6.19) in physical variables

$$\left\{\frac{\partial^2}{\partial z^2} - \mu\varepsilon \frac{\partial^2}{\partial t^2}\right\} \left(\eta^2 A_2(z,t)\right) = - \frac{d^2-1}{4\ \ell n\ d} \left\{2a_1 a_1'' \frac{\partial^2 A_o}{\partial z^2} + \left((a_1 a_1'')' - (a_1'^2)\right)\frac{\partial A_o}{\partial z}\right\} \tag{6.23}$$

where a_1 is the inner boundary (as a function of z). Let the lowest order solution be a waveform $A_o(t - \sqrt{\mu\varepsilon}\ z)$ traveling in the positive z direction, for instance the fast step output from the pulse generator in a TDR system. Define the Laplace transform as

$$\tilde{A}_2(z,s) = \int_{0-}^{\infty} A_2(z,t)\ e^{-st}\ dt \tag{6.24}$$

Then we can write

$$A_o(z,s) = A_o\ e^{-\sqrt{\mu\varepsilon}\ zs} = \int_0^{\infty} A_o\ e^{-st}\ dt \tag{6.25}$$

where $\tilde{A}_o$ is the transform of the incident signal as viewed at the origin of z, which is the start of the transition section as shown in Figure 6.2. It will be necessary to assume that the first and second derivatives of $a_1(z)$ vanish at the end points $z = 0$, $z = \ell_o$. This will give a smooth enough joint to the nonuniform section for present purposes. The transform version of equation (6.28) is

$$\left\{\frac{d^2}{dz^2} - \mu\varepsilon\ s^2\right\}\left(\eta^2\ A_2(z,s)\right) = -\frac{d^2-1}{4\ \ell n\ d}\sqrt{\mu\varepsilon}\ s\ e^{-\sqrt{\mu\varepsilon}\ zs}\ \tilde{A}_o$$

$$\times\left\{2\sqrt{\mu\varepsilon}\ s\ a_1 a_1'' - (a_1 a_1'' - a_1' a_1'')\right\} \tag{6.26}$$

The solution of this ordinary differential equation may be found by means of the Green's function $G(z,z')$ which satisfies the equation

$$\left(\frac{d^2}{dz^2} - \mu\varepsilon\ s^2\right)G(z,z') = \delta(z - z') \tag{6.27}$$

and is given by

$$G(z,z') = -\frac{1}{2\sqrt{\mu\varepsilon}s}\ e^{-\sqrt{\mu\varepsilon}\ s\ |z - z'|} \tag{6.28}$$

We shall evaluate the solution of (6.31) at the generator end of the transition $z = 0$ to find the reflected wave, and at the far end, $z = \ell_o$, to find the aberrations on the transmitted wave. So we find at the far end

$$\eta^2 A_2 = +\frac{d^2-1}{8\ \ell n\ d}\ e^{-\sqrt{\mu\varepsilon}\ \ell_o s}\ \tilde{A}_o \int_{-\infty}^{\ell_o}\left\{2\sqrt{\mu\varepsilon}\ s\ ((a_1 a_1')' - a_1'^2)\right.$$

$$\left. - (a_1 a_1'')' + (a_1'^2)'\right\} dz' \tag{6.29}$$

where the integrand is a function of z'. All terms but one drop out after integration because of the assumed smoothness at $z = \ell_o$, leaving

$$\eta^2 A_2 = -\frac{d^2 - 1}{4 \ln d} e^{-\sqrt{\mu\varepsilon}\, \ell_o s} \sqrt{\mu\varepsilon}\, s\, \tilde{A}_o \int_0^{\ell_o} a_1'^2 \, dz \qquad (6.30)$$

Since the definite integral in (6.35) is just a constant once the taper profile is specified, the new transmitted waveform is proportional to the derivative of the incident waveform. In order then to have the correction remain small in comparison with the lowest order term, the rise time of the incident waveform must be restricted to a slow enough value. An ideal step transition of the incident waveform would lead to a δ-function in the transmitted correction term. So we choose the incident waveform as a single time constant approximation to an ideal step function.

$$A_o(0,t) = 1 - e^{-t/\tau} \qquad (6.31)$$

with time constant τ and rise time 2.2τ. Then the transmitted waveform aberration is given by

$$\eta^2 A_2(\ell_o,t) = -\frac{d^2 - 1}{4 \ln d} \cdot \frac{\sqrt{\mu\varepsilon}\, a_o}{\tau} \cdot \frac{a_o}{\ell_o} \cdot \int_0^1 (a_1'(z))^2 \, dz \cdot e^{-(t-\ell_o\sqrt{\mu\varepsilon})/\tau} \qquad (6.32)$$

where $a(z)$ is the normalized taper profile of the inner conductor. The quantity $\tau/\sqrt{\mu\varepsilon}$ is the distance λ that a plane wave would travel in the medium filling the coaxial line in one time constant of the incident pulse front. In wide band pulse techniques using coaxial lines, $a_o/\lambda << 1$ is maintained to avoid higher mode effects. The sign of the correction term (6.37) it such that is will always introduce a preshoot on the transmitted waveform as depicted in Figure 6.3.

In calculating the reflected wave, a different grouping of terms is appropriate. At $z = 0$ we find

$$\eta^2 A_2 = \frac{d^2 - 1}{8 \ \ell n \ d} \tilde{A}_o \int_0^\infty \left\{ 2\sqrt{\mu\varepsilon} \ s \ a_1 a_1'' - a_1 a_1''' - a_1' a_1'' + 2a_1' a_1'' \right\} e^{-2\sqrt{\mu\varepsilon} \ zs} \ dz \tag{6.33}$$

The first three terms of the integrand may be grouped as $d/dz\{aa'' \ e^{-2\sqrt{\mu\varepsilon} \ zs}\}$ and with the assumed boundary terms, this vanishes upon integration

$$\eta^2 A_2 = + \frac{d^2 - 1}{4 \ \ell n \ d} \ \tilde{A}_o \int_0^\infty a_1' a_1'' \ e^{-2\sqrt{\mu\varepsilon} \ zs} \ dz \tag{6.34}$$

Integrate by parts, and make variable changes to obtain

$$\eta^2 A_2 = \frac{d^2 - 1}{4 \ \ell n \ d} \frac{a_o^2}{\ell_o^2} s \ \tilde{A}_o \int_0 \left\{ \frac{d}{dx} \ a\left(\frac{x}{2}\right) \right\}^2 \ e^{-st} dt \tag{6.35}$$

where $\quad x = \dfrac{t}{2 \ \ell_o \sqrt{\mu\varepsilon}}$

The inverse transform is easily read off as a convolution. It is not necessary here to restrict the rise time of the incident wave and if A_o is a unit step function the reflected wave can immediately be written down as

$$\eta^2 A_2(0,t) = \frac{d^2 - 1}{4 \ln d} \cdot \frac{a_o^2}{\ell_o^2} \cdot \left\{\frac{d}{dx}\left(a\left(\frac{x}{2}\right)\right)\right\}^2 \qquad (6.36)$$

where $\quad x = \dfrac{t}{2\sqrt{\mu\varepsilon}\,\ell_o}$

and $a(z/\ell_o)$ is the normalized inner conductor profile. If we were to calculate the reflected wave when the incident pulse has finite rise time, the convolution process would smear out the details of (6.41) over a time interval $\approx \tau$, but no problems arise in the limit $\tau \to 0$ as they do in the transmitted wave. The reflected waveform shows a point by point dependence on the taper profile, while the transmitted waveform distortion depends only on the mean square of the slope of the taper profile.

An interesting diversion is furnished by the observation that, in the perturbation theory, second order variations of the taper profile will first appear in the second order results. Again we consider a nominally constant impedance coaxial transition b/a = constant = d, where the taper profile of the inner conductor is altered to $a + \eta^2\alpha$. The formulation of the problem in the perturbation description remains exactly the same through the fourth order equations with the exception that the fourth order boundary condition on the inner conductor becomes

$$\frac{\partial h_4}{\partial \rho^\dagger} + \frac{h_4}{\rho^\dagger} = a' \frac{\partial h_z}{\partial z^\dagger} + \alpha' \frac{\partial h_o}{\partial z^\dagger} \quad \text{on } \rho^\dagger = a \tag{6.37}$$

and the equation defining the second order wave correction becomes

$$\frac{\partial^2 A_2}{\partial z^{\dagger 2}} - \frac{\partial^2 A_2}{\partial t^{\dagger 2}} = - \frac{d^2-1}{4\ell nd} \left\{ 2aa'' \frac{\partial^2 A_o}{\partial z^{\dagger 2}} + (aa'''-a'a'') \frac{\partial A_o}{\partial z^\dagger} \right\} + \frac{\alpha'}{\ell nd} \frac{\partial A_o}{\partial z^\dagger} \tag{6.38}$$

If we choose

$$\alpha' = - \frac{1}{2}(d^2 - 1) a'a'' \tag{6.39}$$

then the term responsible for the reflected wave is cancelled. This may be written after integration

$$\alpha = - \frac{d^2 - 1}{4} \{a'(z^\dagger)\}^2 \tag{6.40}$$

The boundary shift $\alpha(z^\dagger)$ for no reflections vanishes when the line is uniform as would be expected. It is also seen that the aberrations on the transmitted waveform cannot be cancelled in this manner, and are not even affected in this order. Thus the absence of measured reflected waves does not guarantee perfect fidelity of the transmitted waveform to the same order.

6.4 Analysis in Axially Centered Warped Spherical Coordinates

The problem of poor wavefront prediction in the tapered plate line analysis was largely solved by introduction of the warped cylindrical coordinate system, which enabled finding of much improved field solutions for substantially the same effort applied to solving propagation equations. In Section 6.2 we have seen that the problem of poor field descriptions is even more acute in nonuniform coaxial line analysis, so we now develop a warped spherical coordinate description which merges into cylindrical coordinates for uniform coax in a fashion analogous to the passage from warped cylindrical to Cartesian coordinates.

The warped spherical coordinates will be based on the idea of rotating the two dimensional system about the center line axis as shown in Figure 6.4. The arc coordinate system (z,s) in a plane containing the axis is defined in terms of the outer conductor profile $b(z_1)$ after the fashion of Chapter IV, and then rotated about the center line to generate the azimuthal coordinate, ϕ. We can then write the Cartesian coordinates of an interior point in terms of its warped spherical coordinates

$$\left.\begin{aligned} z_1 &= z - r(1 - \cos \frac{s}{r}) \\ x_1 &= r \sin \frac{s}{r} \cos \phi \\ y_1 &= r \sin \frac{s}{r} \sin \phi \end{aligned}\right\} \qquad (6.41)$$

The metric now has the form

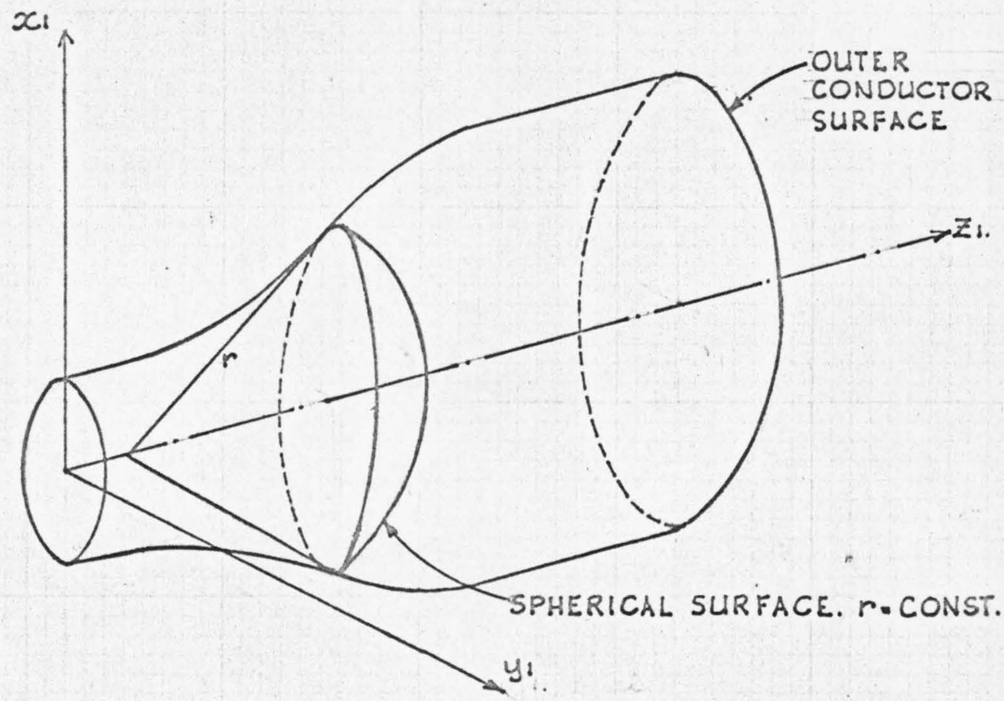

Figure 6.3 Warped Spherical Coordinate Definitions

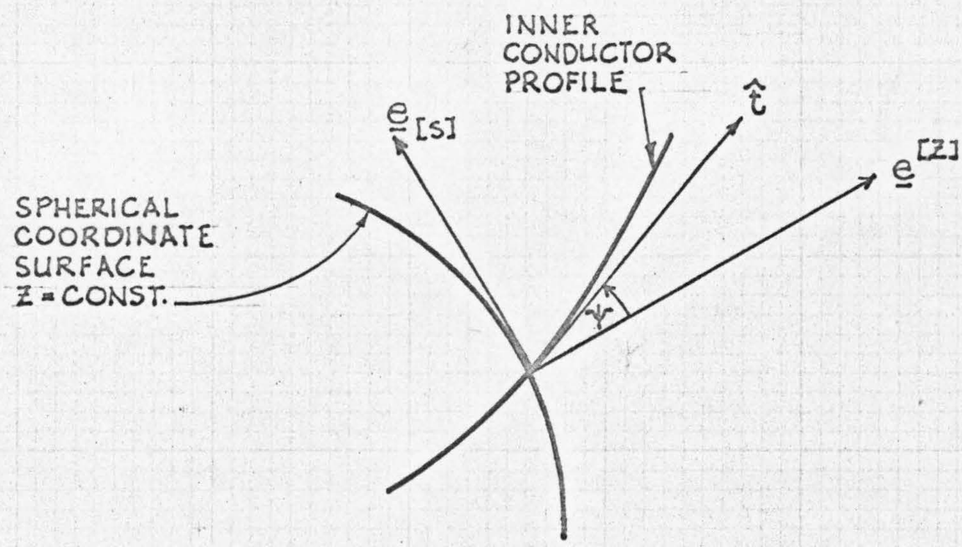

Figure 6.4 Boundary Conditions on Inner Conductor

$$g_{ij} = \begin{pmatrix} g_{zz} & g_{zs} & 0 \\ g_{zs} & g_{ss} & 0 \\ 0 & 0 & g_{\phi\phi} \end{pmatrix} \tag{6.42}$$

where
$$g_{\phi\phi} = r^2 \sin^2 \frac{s}{r} \tag{6.43}$$

and the other components are identical to their two dimensional counterparts given by (4.20) to (4.22). The metric determinant g_3 is given in terms of the two dimensional version g_2 by

$$g_3 = r^2\left(\sin^2 \frac{s}{r}\right)g_2 \tag{6.44}$$

The transverse magnetic field H_y is replaced by H_ϕ and the symmetries are otherwise unchanged. Since the off diagonal ϕ terms in the metric (6.42) are zero, the contravariant and covariant components of H_ϕ are parallel and are related by

$$h_\phi = r^2\left(\sin^2 \frac{s}{r}\right)h^\phi = r\left(\sin \frac{s}{r}\right)H_\phi = \rho H_\phi \tag{6.45}$$

With the assumed rotational symmetry, the first Maxwell equation becomes

$$\frac{1}{\sqrt{g_3}} \left\{\frac{\partial e_z}{\partial s} - \frac{\partial e_s}{\partial z}\right\} = - \mu \frac{\partial h^\phi}{\partial t} = - \frac{\mu}{g_{\phi\phi}} \frac{\partial h_\phi}{\partial t} \tag{6.46}$$

and the derivation thereafter is similar to that in Section 4.4. We obtain, in terms of familiar quantities, the propagation equation for the covariant component of transverse magnetic field

$$\frac{\partial}{\partial s}\left\{\frac{1}{r \sin \frac{s}{r} \sqrt{g_2}} \frac{\partial h_\phi}{\partial z}\right\} + \frac{\partial}{\partial s}\left\{\frac{g_{zz}}{r \sin \frac{s}{r} \sqrt{g_2}} \frac{\partial h_\phi}{\partial s}\right\} - \frac{\partial}{\partial z}\left\{\frac{g_{zs}}{r \sin \frac{s}{r} \sqrt{g_2}} \frac{\partial h_\phi}{\partial s}\right\}$$

$$- \frac{\partial}{\partial s}\left\{\frac{g_{zs}}{r \sin \frac{s}{r} \sqrt{g_2}} \frac{\partial h_\phi}{\partial z}\right\} - \mu\varepsilon \frac{\sqrt{g_2}}{r \sin \frac{s}{r}} \frac{\partial^2 h_\phi}{\partial t^2} = 0 \qquad (6.47)$$

The boundary condition on the outer conductor, which is also the defining boundary for the arc system, is

$$\frac{\partial h_\phi}{\partial s} = 0 \quad \text{on} \quad s = s_b = r\,\Theta_b \qquad (6.48)$$

The inner boundary has not yet been specified. The choice of inner conductor profile that most closely corresponds to the constant impedance line in planar transverse sections, is that which defines the same set of arcs as the outer boundary in their common interior region. Elementary sections of line on each spherical cross-section then belong to concentric cones. The inner boundary condition is then also

$$\frac{\partial h_\phi}{\partial s} = 0 \quad \text{on} \quad s = s_a = r\,\Theta_a \qquad (6.49)$$

In any longitudinal axial section, the set of such inner boundaries comprises the orthogonal trajectories of the arc system and, in principle, could be used to define an orthogonal coordinate system. In reality, exact calculations have proved prohibitively complex, and the nonorthogonal coordinate systems are far more practical.

The local geometry of the inner boundary when it is not orthogonal to an arc is shown in Figure 6.4. The contravariant basis vector $\underline{e}_{[s]}$ is a unit vector tangent to the arc. Let the inner boundary make an angle ψ with the normal to the arc which is parallel to the covariant basis vector $\underline{e}^{[z]}$ which has length $\sqrt{g_{ss}\, g_{\phi\phi}/g_3} = 1/\sqrt{g_2}$ which is the same as in the two-dimensional version of Chapter IV. The boundary condition that tangential electric field vanish on the inner conductor can then be expressed as

$$\left\{\tan\psi\, \underline{e}_{[s]} + \sqrt{g_2}\, \underline{e}^{[z]}\right\} \cdot \left\{e^z\, \underline{e}_{[z]} + e^s\, \underline{e}_{[s]}\right\} = 0 \qquad (6.50)$$

or in terms of h_ϕ

$$\frac{\partial h_\phi}{\partial s} = \frac{\tan\psi}{\sqrt{g_2} + g_{zs}\tan\psi} \cdot \frac{\partial h_\phi}{\partial z} \quad \text{on } s = s_a \qquad (6.51)$$

When $\tan\psi = 0$, the boundary is locally coincident with an orthogonal trajectory of the arc system, and the boundary condition reduces to the homogeneous form (6.50).

The propagation equation (6.48) will now be solved by the approximate methods of Section 4.5 with similar interpretation of the scaling procedure. Some useful coefficient expansions are given by (6.52) to (6.55) where $b = b(z_1)$ is the outer conductor profile.

$$\frac{\sqrt{g_2}}{r \sin \frac{s}{r}} = \frac{1}{s} \left\{ 1 + \frac{s^2}{6r^2} - \frac{s^2 F}{2b^2} + \dots \right\} \tag{6.52}$$

$$\frac{1}{\sqrt{g_2}\, r \sin \frac{s}{r}} = \frac{1}{s} \left\{ 1 + \frac{s^2}{6r^2} + \frac{s^2 F}{2b^2} + \dots \right\} \tag{6.53}$$

$$\frac{g_{zz}}{\sqrt{g_2}\, r \sin \frac{s}{r}} = \frac{1}{s} \left\{ 1 + \frac{7}{6} \cdot \frac{s^2}{r^2} - \frac{s^2 F}{2b^2} + \dots \right\} \tag{6.54}$$

$$\frac{g_{sz}}{\sqrt{g_2}\, r \sin \frac{s}{r}} = - \left\{ \frac{1}{r} + \frac{s^2}{6r^3} + \frac{s^2 F}{3rb^2} + \dots \right\} \tag{6.55}$$

The expansions have the convergence properties discussed in Section 4.5. The propagation equation, arranged by dominant orders of η, becomes

$$\left[s\frac{\partial}{\partial s}\left\{\frac{1}{s}\frac{\partial h_\phi}{\partial s}\right\}\right] + \left[\frac{\partial^2 h_\phi}{\partial z^2} - \mu\varepsilon\frac{\partial^2 h_\phi}{\partial t^2} + s\frac{\partial}{\partial s}\left\{\left(\frac{7}{6}\cdot\frac{s}{r^2} - \frac{sF}{2b^2}\right)\frac{\partial h_\phi}{\partial s}\right\} + \frac{25}{r}\frac{\partial}{\partial z}\left\{\frac{\partial h_\phi}{\partial s}\right\}\right]$$

$$+\left[\frac{\partial}{\partial z}\left\{\left(\frac{s^2}{6r^2} - \frac{s^2F}{2b^2}\right)\frac{\partial h_\phi}{\partial z}\right\} + s\frac{\partial}{\partial s}\left\{O(\eta^4)\frac{\partial h_\phi}{\partial s}\right\} + s\frac{\partial}{\partial s}\left\{\left(\frac{s^2}{6r^3} + \frac{s^2F}{3rb^2}\right)\frac{\partial h_\phi}{\partial z}\right\}\right.$$

$$\left. + s\frac{\partial}{\partial z}\left\{\left(\frac{s^2}{6r^3} + \frac{s^2F}{3rb^2}\right)\frac{\partial h_\phi}{\partial s}\right\} - \mu\varepsilon\left\{\frac{s^2}{6r^2} - \frac{s^2F}{2b^2}\right\}\frac{\partial^2 h_\phi}{\partial t^2}\right] + O(\eta^6) = 0 \qquad (6.56)$$

The angle ψ is of order at most $O(\eta)$. This corresponds to a coaxial line with gross impedance variations along its length or else small scale variations about a smooth profile. The boundary condition (6.52) will then be homogeneous only in the lowest order. If ψ is of order $O(\eta^3)$, the boundary condition is also homogeneous in second order $O(\eta^2)$, which corresponds to a constant impedance nonuniform line. The fourth order boundary condition then contains the $O(1)$ solution, A_o, which appears as an extra inhomogeneous term in the defining equation for A_2. Such an inner conductor profile can depart from a particular orthogonal trajectory of the arc system over the length of the taper, only by a spacing of order $O(\eta^2)$. This is very reminiscent of the effects of second order profile deviations noted in the previous section.

The lowest order problem in the perturbation sequence in order $O(1)$ is

$$s \frac{\partial}{\partial s}\left\{\frac{1}{s}\frac{\partial h_{\phi o}}{\partial s}\right\} = 0 \tag{6.57}$$

with
$$\frac{\partial h_{\phi o}}{\partial s} = 0 \quad \text{on} \quad s = s_a,\ s_b \tag{6.58}$$

This has solution $h_{\phi o} = A_o(z,t)$ where A_o is an undetermined function independent of s, which is to be found in solving the $O(\eta^2)$ problem

$$s \frac{\partial}{\partial s}\frac{1}{s}\frac{\partial h_{\phi 2}}{\partial s} = -\left\{\frac{\partial^2 A_o}{\partial z^2} - \mu\varepsilon \frac{\partial^2 A_o}{\partial t^2}\right\} \tag{6.59}$$

with

$$\frac{\partial h_{\phi 2}}{\partial s} = \begin{cases} 0 & \text{on} \quad s = s_b \\ \dfrac{\tan\psi}{\left\{1 - \dfrac{s}{r}\tan\psi - \dfrac{s^2 F}{2b^2}\right\}}\dfrac{\partial h_{\phi o}}{\partial z} & \text{on} \quad s = s_a \end{cases} \tag{6.60}$$

where the denominator of the inner boundary condition has been expanded to order $O(\eta^2)$. For consistency of ordering, it is sufficient to ignore even the $O(\eta)$ denominator term, but the more accurate version (6.61) still contains only quantities that will be calculated anyway. Integrate (6.60) once and apply the boundary conditions to find

$$\frac{\partial h_{\phi 2}}{\partial s} = s(\ell n\, s_b - \ell n\, s)\left\{\frac{\partial^2 A_o}{\partial z^2} - \mu\varepsilon \frac{\partial^2 A_o}{\partial t^2}\right\} \tag{6.61}$$

and

$$\frac{\partial^2 A_o}{\partial z^2} - \frac{\tan\psi}{s_a \, \ell n \frac{s_b}{s_a} \left(1 - \frac{s_a}{r}\tan\psi - \frac{s_a^2 F}{2b_1^2}\right)} \frac{\partial A_o}{\partial z} - \mu\varepsilon \, \frac{\partial^2 A_o}{\partial t^2} = 0 \tag{6.62}$$

Equation (6.63) is the new improved model nonuniform transmission line equation for circular coaxial transmission line. It can be seen from the geometry of the line that, as $\eta \to 0$

$$\lim_{\eta\to 0} \frac{-\tan\psi}{s_a\left(1 - \frac{s_a}{r}\tan\psi - \frac{s_a^2 F}{2b^2}\right) \ell n \frac{s_b}{s_a}} = \frac{\frac{b'}{b} - \frac{a'}{a}}{\ell n \frac{b}{a}} \tag{6.63}$$

which implies that both the defining equation for A_o and the coordinate system reduce to the Cartesian versions. The condition that (6.62) reduces to a homogeneous simple wave equation is that $\psi = 0$, independently of η. As already discussed $\psi = O(\eta^3)$ is a sufficient condition in the perturbation expansion for this to be true. In the special case of a uniform concentric conical line, $\psi = 0$ everywhere and the solutions for the physical component of magnetic field are

$$H_{\phi o} = \frac{A_o\,(t \pm \sqrt{\mu\varepsilon}\ z)}{r \sin\ \theta} \quad \text{on} \quad z = r \tag{6.64}$$

which are identical with the exact solutions. The warped spherical coordinate analysis telescopes into one term an infinite number of terms of the planar section expansion, yielding the exact solution in this special case.

Our foray into the next order of approximation, in the name of simplicity, will be limited to constant impedance lines with $\psi = 0$. From equation (6.62) it is seen that in this special case, that $h_{\phi 2} = A_2(z,t)$ only. This important result shows that for a constant impedance coaxial line, the second order correction has **exactly** the same wavefronts, spherical annuli r=constant, and transverse behaviour as the basic solution. Thus we can assign a transmission line interpretation with equal and opposite currents where the spherical sections intersect the conductors that is valid through second order corrections in the taper scale parameter. In the tapered plate analysis, this was true for all smooth profiles, but in the three-dimensional example of the coaxial line it is true only for the class of constant impedance lines $\psi \sim O(\eta^3)$. This restrictive result may be attributed to the loss of symmetry in an axial section even for a smooth gradual taper. Compare this with the irregular boundary analysis in two dimensions. In both this and the varying impedance coax line the local boundary does not fit exactly into the dominant behaviour.

The fourth order equation now reduces to

$$s \frac{\partial}{\partial s}\left\{\frac{1}{s}\frac{\partial h_{\phi 4}}{\partial s}\right\} = -\left\{\frac{\partial^2 A_2}{\partial z^2} - \mu\varepsilon \frac{\partial^2 A_2}{\partial t^2}\right\} - \frac{s^2}{2}\left\{\left(\frac{\partial}{\partial z}\left(\frac{F}{b_1^2}\right) + \frac{2F}{rb^2}\right)\frac{\partial A_o}{\partial z} + \frac{F}{b_1^2}\frac{\partial^2 A_o}{\partial z^2}\right\} \tag{6.65}$$

with

$$\frac{\partial h_{\phi 4}}{\partial s} = 0 \quad \text{on} \quad \begin{cases} s = s_a = r\,\Theta_a \\ s = s_b = r\,\Theta_b \end{cases} \tag{6.66}$$

Integrate and apply the boundary conditions to obtain an equation for A_z

$$\frac{\partial^2 A_2}{\partial z^2} - \mu\varepsilon \frac{\partial^2 A_2}{\partial t^2} = -\frac{s_b^2 - s_a^2}{4\,\ell n \frac{s_b}{s_a}}\left\{\frac{2F}{b_1^2}\frac{\partial^2 A_o}{\partial z^2} + \left[\frac{\partial}{\partial z}\left(\frac{F}{b_1^2}\right) + \frac{2F}{rb_1^2}\right]\frac{\partial A_o}{\partial z}\right\} \tag{6.67}$$

This equation has the same form as equation (6.23) for the planar section approximation and the solution methods developed there can be applied here also. Note that the $\partial^2 A_o/\partial z^2$ term is still responsible for transmitted waveform distortions and in the limit $\eta \to 0$ the coefficients of this term in (6.23) and (6.68) become identical.

The coordinate system used in this section has been based on the outer conductor profile, since the system, which has essentially interior definition, then has validity independent of the inner conductor profile. In later sections we treat low impedance lines where inner and outer conductors are of almost equal significance. In the extreme of a high impedance line with a uniform cylinder for outer conductor, the transverse sections are planar which expresses the field behaviour over most of the region and gives results identical to the original Cartesian expansions since then $b' = 0$, $\tan\psi = a'$ and $r = \infty$.

The principal result of this section can be summarized in the new nonuniform line equation

$$\frac{\partial^2 A_o}{\partial z^2} - \frac{\tan\psi}{s_a \ \ell n \frac{s_b}{s_a}} \frac{\partial A_o}{\partial z} - \mu\varepsilon \frac{\partial^2 A_o}{\partial t^2} = 0 \qquad (6.68)$$

6.5 Low Impedance Coaxial Line in a Warped Conical Description

Coaxial lines of low impedance have inner and outer conductor radii almost equal so that locally, from an azimuthal point of view, they look like strips of parallel plate line with spacing $b - a$, which may be paralleled to a total width equal to the mean circumference. A low impedance coaxial line of variable mean radius and spacing is illustrated in Figure 6.5. It would seem from an intuitive look at the propagation behaviour in such a line, that it would be best described by a curved center line axis in the propagation region, in an axial longitudinal section, rather than by the relatively distant coaxial center line axis, and with the field pattern dominated by the neighboring walls rather than by some construction based on a distant axis.

We have in fact met an analogous situation before. In Chapter II the constant impedance tapered plate line was treated with the restriction that the parallel undulations be only a small fraction of the plate spacing. This restriction was lifted in the

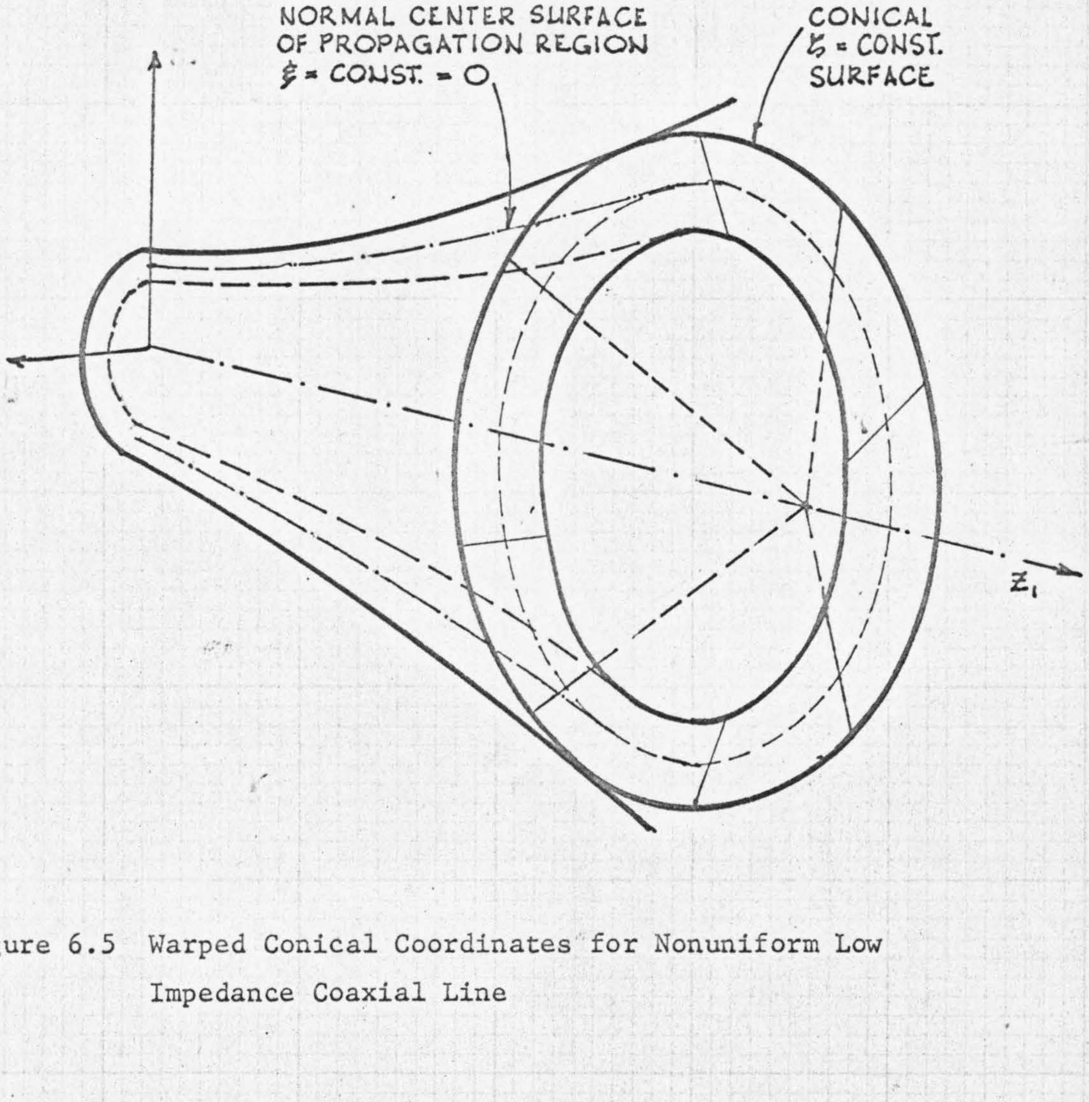

Figure 6.5 Warped Conical Coordinates for Nonuniform Low Impedance Coaxial Line

later chapter treating profiles with arbitrary curved center lines. The symmetrical tapered plate had a natural center line, a fact reflected in the correctness of the on-axis approximate solution, but in the extension to coaxial lines, it was an off axis section that was rotated to generate the coaxial line profile. We might expect then, that in low impedance coax lines, the off-axis asymmetrical behaviour of the generating section will be predominant.

The geometry for describing coaxial lines will now be generated by rotation of the planar curved center line figure about some axis which does not intersect the line profile, say the z-axis. In this section we shall use planar normal center line description, so that each transverse coordinate surface is a normal slice of a cone with axis along the coaxial center line. These could be called warped conical coordinates by extension of our earlier terminology.

The axial section coordinates (ζ,ξ) are as defined in Section 5.1 and the azimuthal coordinate angle ϕ is generated by rotation about the z-axis as in Figure 6.6. Then we have now writing ρ_o for x_o

$$\left.\begin{aligned} z &= z_o(\zeta) - \xi \sin \Psi(\zeta) \\ x &= (\rho_o(\zeta) + \xi \cos \Psi(\zeta)) \cos \phi \\ y &= (\rho_o(\zeta) + \xi \cos \Psi(\zeta)) \sin \phi \end{aligned}\right\} \qquad (6.69)$$

The diagonal metric of this orthogonal coordinate system is now given by

$$g_{\zeta\zeta} = (1 - \xi\,\Psi'(\zeta))^2 \tag{6.70}$$

$$g_{\xi\xi} = 1 \tag{6.71}$$

$$g_{\phi\phi} = (\rho_o + \xi \cos \Psi)^2 = \rho^2 \tag{6.72}$$

$$g_3^{\frac{1}{2}} = (\rho_o + \xi \cos \Psi)(1 - \xi\,\Psi'(\zeta)) = \rho g_2^{\frac{1}{2}} \tag{6.73}$$

The quantity $\rho = (\rho_o + \xi \cos \Psi)$ is simply the cylindrical radius of a point (ζ,ξ,ϕ) from the coax line axis. The conditions of Section 5.1 for good behaviour of the coordinates applies unchanged, and there is the additional condition that the z-axis not intersect the profile. We can then use the general formalism leading to equation (6.48) to write the equations for the covariant component of magnetic field h_ϕ, where the physical component H_ϕ is given by $h_\phi = \rho H_\phi$.

First consider the choice of normalization of the mean radius ρ_o, to avoid repetition in writing out of long equations. We shall adopt $\rho_o(\xi) = \ell_o\, \rho_o^\dagger(\zeta/\ell_o)$ so that for small η the line is of the kind discussed at the start of the section. The mean radius could be allowed to scale at some rate intermediate between a_o and ℓ_o and still yield the low impedance limit but the version stated is the most convenient. We expect from this that the results will be good for a wide range of $\rho_o^\dagger$. With this choice of normalization the governing equations can be written

$$\frac{\partial^2 h_\phi}{\partial \xi^{\dagger 2}} - \eta \frac{\cos \Psi}{x_o^\dagger + \eta\xi^\dagger \cos \psi} \frac{\partial h_\phi}{\partial \xi^\dagger} - \eta \frac{\psi'}{1 - \eta\xi^\dagger\psi'} \frac{\partial h_\phi}{\partial \xi^\dagger}$$

$$+ \eta^2 \frac{1}{(1 - \eta\xi^\dagger\psi')^2} \frac{\partial^2 h_\phi}{\partial \zeta^{\dagger 2}} - \eta^2 \frac{\sin \Psi}{(1 - \eta\xi^\dagger\psi')(x_o^\dagger + \eta\xi^\dagger \cos \Psi)} \frac{\partial h_\phi}{\partial \zeta^\dagger}$$

$$+ \eta^3 \frac{\xi^\dagger \psi''}{(1 - \eta\xi^\dagger\psi')^3} \frac{\partial h_\phi}{\partial \zeta^\dagger} - \eta^2 \frac{\partial h_\phi}{\partial t^{\dagger 2}} = 0 \tag{6.74}$$

with $$\frac{\partial h_\phi}{\partial \xi^\dagger} = \frac{\pm a'}{(1 - \eta\xi^\dagger\psi')^2} \frac{\partial h_\phi}{\partial \zeta^\dagger} \text{ on } \xi^\dagger = \pm a(\zeta^\dagger) \tag{6.75}$$

The expansion of h_ϕ needs to contain both even and odd powers of η

$$h_\phi = h_o + \eta h_1 + \eta^2 h_2 + \ldots \tag{6.76}$$

It is soon established by reasoning similar to that of Section 4.1 that h_o and h_1 are given by as yet undetermined functions $A_o(\zeta^\dagger, t^\dagger)$ and $A_1(\zeta^\dagger, t^\dagger)$ respectively. The second order problem is

$$\frac{\partial^2 h_2}{\partial \xi^{\dagger 2}} = -\left\{\frac{\partial^2 h_o}{\partial \zeta^{\dagger 2}} - \frac{\sin \Psi}{\rho_o^\dagger} \frac{\partial h_o}{\partial \zeta^\dagger} - \frac{\partial^2 h_o}{\partial t^{\dagger 2}}\right\} \tag{6.77}$$

with

$$\frac{\partial h_2}{\partial \xi^\dagger} = \pm a' \frac{\partial h_o}{\partial \zeta^\dagger} \quad \text{on} \quad \xi^\dagger \quad \pm a(\zeta^\dagger) \tag{6.78}$$

From these we find

$$h_2 = \frac{\xi^{\dagger 2}}{2} \frac{a'}{a} \frac{\partial A_o}{\partial \zeta^\dagger} + A_2(\zeta^\dagger, t^\dagger) \tag{6.79}$$

and

$$\frac{\partial^2 A_o}{\partial \zeta^{\dagger 2}} + \left\{ \frac{a'}{a} - \frac{\sin \Psi}{\rho_o^\dagger} \right\} \frac{\partial A_o}{\partial \zeta^\dagger} - \frac{\partial^2 A_o}{\partial t^{\dagger 2}} = 0 \tag{6.80}$$

It can further be shown that A_1 satisfies this same homogeneous equation and so adds nothing new and may be taken as zero, and that higher order odd sequence terms contribute only local field distortions on lower order even sequence solutions and so remain invisible from outside the nonuniform section. It can also be shown that A_2 satisfies an inhomogeneous version of the nonuniform line equation (6.82). Detailed statements of these results will be held over to Appendix A.

A physical interpretation of the new nonuniform line equation (6.80) is not difficult to find provided we recall that $\sin \psi = d\rho_o/d\zeta$. Since the mean circumference is proportional to ρ_o, this term measures the logarithmic rate of change of circumference. In view of our picture of a low impedance line as a tapered plate line section curled up transversely to a total width

$2\pi\rho_o$ we can interpret the condition for a perfectly transparent nonuniform line

$$\frac{a'}{a} = \frac{\sin \Psi}{\rho_o^{\dagger}} \tag{6.81}$$

as signifying that the increase in impedance due to increased spacing is offset by extra effective width due to increase in mean radius. Heuristic reasoning like this is frequently applied to variable width strip lines neglecting edge effects, but in this version edge effects take care of themselves.

Another special case where comparisons are easily made is when a is constant and $\Psi = \pi/2$, a radial transmission line between parallel plates. The exact equation may be written for this symmetry

$$\frac{\partial}{\partial\rho^2}(\rho H) - \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H) - \mu\varepsilon\,\frac{\partial^2(\rho H)}{\partial t^2} = 0 \tag{6.82}$$

In this case $\zeta \to \rho$ and $\sin \Psi \to 1$ leading to an identical equation, when it is recalled that A_o is a covariant component, ρH_ϕ. The new nonuniform line thus gives an exact description of the radial transmission line between parallel plates. It also gives a distributed circuit level description even when the local propagation axis bends back over.

From equation (A.5) for the second order correction term we see that the equation has the same general form as equation (6.23) and the solution method of Section 6.3 is directly applicable. No qualitatively new results are obtained so we shall not give all the details here. We remark that the coefficient of the derivative in transmitted correction is proportional to

$$\int_0^{\ell_o} (a^2\psi'^2 + a'^2)\, dz \tag{6.83}$$

This expression vanishes only for a uniform line, in agreement with our previous conclusion.

6.6 Low Impedance Coaxial Line in a Warped Toroidal Description

The normal center line description of a curved axis problem is known from earlier chapters to have the difficulty that if only the boundaries are given, that in general a differential equation must be solved to find the center line, though for gradual changes of spacing, the position differs only in second order from the equiangular arc center line of Section 5.2. It is also known that the predicted plane wavefronts are much inferior to the warped cylindrical version as a prediction of the true fields even in the lowest order approximation.

In the previous section the normal center line description has been extended to coaxial geometries giving conical transverse coordinate surfaces. The cylindrical description can be extended in the same fashion to give transverse coordinate surfaces that are sections around the major circumference of some toroid. This is a warped toroidal coordinate description in terminology consistent with our earlier usage. The system will in general be nonorthogonal.

We refer to the two dimensional calculations given in

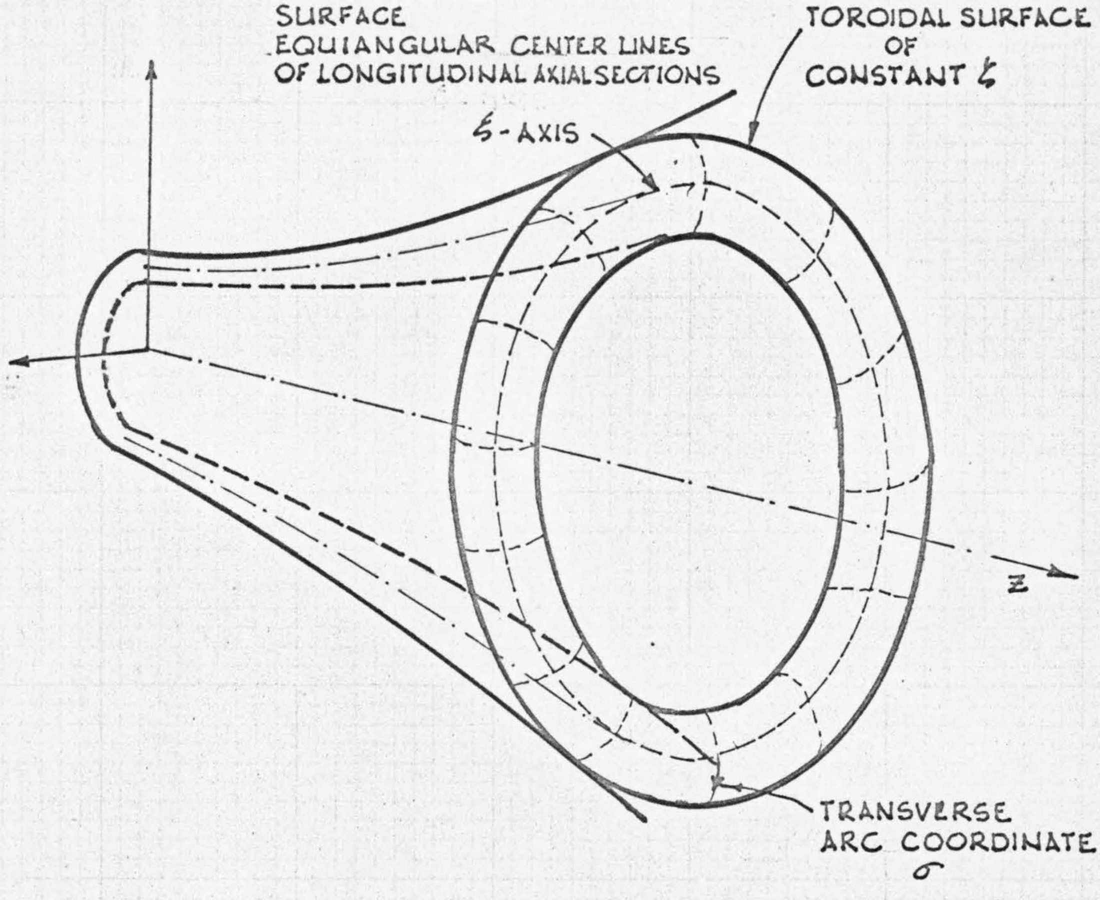

Figure 6.6 Warped Toroidal Coordinates for Low Impedance Coaxial Line

Appendix B. The rotation process, under the same conditions as in the last chapter, introduces a new diagonal term $g_{\phi\phi}$ into the metric with no cross terms (6.42). Then

$$g_{\phi\phi} = \left\{\rho_o + 2r \sin\frac{s}{2r} \cos\left(\frac{s}{2r} + \psi\right)\right\}^2 \tag{6.84}$$

$$\sqrt{g_3} = \left\{\rho_o + 2r \sin\frac{s}{2r} \cos\left(\frac{s}{2r} + \psi\right)\right\}\sqrt{g_2} \tag{6.85}$$

Write

$$A = 1 + \frac{2r}{\rho_o} \sin\frac{s}{2r} \cos\left(\frac{s}{2r} + \psi\right) \tag{6.86}$$

$$= 1 + O(\eta)$$

with ρ_o measured on the ℓ_o scale as discussed in the previous discussion. The full equations for propagation can then be written

$$\rho_o \frac{\partial}{\partial\zeta}\left\{\frac{1}{\rho_o A\sqrt{g_2}} \frac{\partial H_\phi}{\partial\zeta}\right\} + \frac{\partial}{\partial\sigma}\left\{\frac{g_{\xi\xi}}{A\sqrt{g_2}} \frac{\partial h_\phi}{\partial\sigma}\right\} - \frac{\partial}{\partial\sigma}\left\{\frac{g_{\zeta\sigma}}{A\sqrt{g_2}} \frac{\partial h_\phi}{\partial\zeta}\right\}$$

$$- \rho_o \frac{\partial}{\partial\zeta}\left\{\frac{g_{\zeta\sigma}}{\rho_o A\sqrt{g_2}} \frac{\partial h_\phi}{\partial\sigma}\right\} - \mu\varepsilon \frac{\sqrt{g_2}}{A} \frac{\partial^2 h_\phi}{\partial t^2} = 0 \tag{6.87}$$

with

$$\frac{\partial h_\phi}{\partial\sigma} = 0 \quad \text{on } \sigma = \pm\, r\, \Theta_o(\zeta) \tag{6.88}$$

We shall only briefly summarize the results of applying the perturbation procedure to these equations. Expand in a full sequence missing only an $O(\eta)$ term

$$h_\phi = h_o + h_2 + h_3 + \ldots \tag{6.89}$$

Then we find

$$h_o = A_o(\zeta,t) \tag{6.90}$$

$$h_2 = A_2(\zeta,t) \tag{6.91}$$

where A_o is defined by the homogeneous nonuniform line equation

$$\frac{\partial^2 A_o}{\partial \zeta^2} + \left\{\frac{1}{r} - \frac{\rho_o'}{\rho_o}\right\} \frac{\partial A_o}{\partial \zeta} - \mu\varepsilon \frac{\partial^2 A_o}{\partial t^2} = 0 \tag{6.92}$$

and the second order correction h_2 has the same wave fronts ζ = constant and is defined by an inhomogeneous version of equation (6.93). The simple wavefront prediction is now good through second order whether the line is constant impedance or not. We recall that in the spherical version based on the coaxial center line, this is valid in second order only for a constant impedance line.

A special case of interest occurs when $\psi = \pi/2$ and the boundary angle is constant. Then $r = \rho_o = \zeta$ and

$$\frac{\partial^2 A_o}{\partial \zeta^2} - \mu\varepsilon \frac{\partial^2 A_o}{\partial t^2} = 0 \tag{6.93}$$

which has solutions

$$A_o = \rho H_o = f(\zeta \pm \mu\varepsilon t) \tag{6.94}$$

This is identical to the exact solution for a biconical transmission line.

VII. CONCLUSIONS

The classical distributed circuit analysis of nonuniform transmission lines is an ad hoc approximation. We have developed a perturbation expansion of the exact propagation equations and boundary conditions in a small dimensionless parameter η, which measures the length scale of a gradual taper. This yields equations essentially equivalent to the circuit equations in the lowest approximation and can be continued to provide systematic corrections. The physical nature of the approximation sequence is that it emphasizes quasistatic behaviour in transverse sections while preserving a one dimensional propagation description along the axis. This has been done in detail for the special geometries, which have great practical importance, of tapered plate lines and nonuniform coaxial line. The higher order terms are used to calculate aberrations on an incident waveform, beyond those calculated from the circuit equations. These are especially significant when the nonuniform line section is perfectly transparent in the distributed circuit theory.

It is shown that the generalized telegraphist's equation analysis of Schelkunoff and others, which has been suggested as the way to improve the circuit equations, is qualitatively unsuited for finding better approximations or for aiding physical interpretation in this class of problem, though it is the natural method for solving multimode waveguide problems. The different methods solve different problems with only a small region of overlap.

We then develop nonorthogonal coordinate techniques which lead to very much improved field descriptions in the nonuniform region, so that just the first term is exact for uniform finite angle profiles. The first propagating correction is also found to be the same basic wave species as the lowest order solution in this description. The analysis also leads to improved versions of the basic one-dimensional circuit type nonuniform line equations, which reduce to the well-known equations for very gradual transitions. The classical equation for the tapered wedge, for no apparent deep reason, gives the correct coefficient for a uniform wedge, but in the wrong coordinate system. With the improved understanding of the field solutions we can better attack problems such as making a clean, short circuit of well defined position in a nonuniform line.

The expansion techniques are also extended to tapered wedge lines with curved center lines and these results are later applied to generate conical and toroidal transverse field descriptions for low impedance coaxial lines where field behaviour is locally dominated by the closely spaced conductors. It is also seen that this analysis can be applied to stripline of variable height and width, provided edge effects can be neglected.

There are a number of ways open for extension of this work. Adequate smoothness has been assumed for line profiles and if this is not so, quasistatic boundary layer type calculations are required with matching to the wave solutions. We have also treated only cases where the basic field symmetry is maintained and so have not, for instance, treated transitions between coaxial lines of various cross-sectional shapes, where even more mode warping occurs. We can also anticipate the general nature of results for three dimensional curved lines such as flexible coax. In a more subtle extension, a class of nonuniform dielectric problems becomes amenable to approximate solution, when the dielectric constant varies only in directions normal to the transverse coordinate surfaces. The view from outside the nonuniform section will now be different depending on whether the dielectric distribution has been fitted to planar or to curved sections.

Further application of the nonorthogonal coordinate techniques for quasi-one-dimensional problems is not restricted to electromagnetic fields but looks promising for such diverse problems as heat conduction or wide angle deflection of electron beams.

APPENDIX A

Some Higher Order Perturbation Equations

The details of some higher order perturbation results, referred to in the main text, are given here in more detail, with the notation being that for the appropriate section.

In the planar section analysis of the two dimensional tapered plate line, we have, in normalized variables

$$H_4(x^\dagger, z^\dagger) = -\frac{x^{\dagger 4}}{24}\left(\frac{\partial^2}{\partial z^{\dagger 2}} - \frac{\partial^2}{\partial t^{\dagger 2}}\right)\left(\frac{a'}{a}\frac{\partial A_o}{\partial z^\dagger}\right) - \frac{x^{\dagger 2}}{2}\left(\frac{\partial^2}{\partial z^{\dagger 2}} - \frac{\partial^2}{\partial t^{\dagger 2}}\right)A_2 + A_4(z,t) \tag{A.1}$$

where the sixth order analysis shows that A_4 is given by

$$\frac{\partial^2 A_4}{\partial z^{\dagger 2}} + \frac{a'(z^\dagger)}{a(z^\dagger)}\frac{\partial A_4}{\partial z^\dagger} - \frac{\partial^2 A_4}{\partial t^{\dagger 2}}$$

$$= -\frac{a^2}{b}\left\{\frac{\partial^2}{\partial z^{\dagger 2}} + \frac{3a'}{a}\frac{\partial}{\partial z^\dagger} - \frac{\partial^2}{\partial t^{\dagger 2}}\right\}\left\{\frac{a'}{a}\frac{\partial A_o}{\partial z^\dagger}\right\}$$

$$+ \frac{a^4}{120}\left\{\frac{\partial^2}{\partial z^{\dagger 2}} + \frac{5a'}{a}\frac{\partial}{\partial z^\dagger} - \frac{\partial^2}{\partial t^{\dagger 2}}\right\}\cdot\left\{\frac{\partial^2}{\partial z^{\dagger 2}} - \frac{\partial^2}{\partial t^{\dagger 2}}\right\}\cdot\left\{\frac{a'}{a}\frac{\partial A_o}{\partial z^\dagger}\right\}$$

$$- \frac{a^2}{36}\left\{\frac{\partial^2}{\partial z^{\dagger 2}} + \frac{3a'}{a}\frac{\partial}{\partial z^\dagger} - \frac{\partial}{\partial t^{\dagger 2}}\right\}\left\{a^2\cdot\left\{\frac{\partial^2}{\partial z^{\dagger 2}} + \frac{3a'}{a}\frac{\partial}{\partial z^\dagger} - \frac{\partial^2}{\partial t^{\dagger 2}}\right\}\right\}\left\{\frac{a'}{a}\frac{\partial A_o}{\partial z^\dagger}\right\} \tag{A.2}$$

Equation (A.2) is an inhomogeneous nonuniform line equation. Also

$$H_6(x^\dagger, z^\dagger) = \frac{x^6}{720} L^2\left\{\frac{a'}{a}\frac{\partial A_o}{\partial z^\dagger}\right\} - \frac{x^6}{24} L\left\{\frac{a'}{a}\frac{\partial A_1}{\partial z^\dagger}\right\} - \frac{x^2}{2} L A_2$$

$$- \frac{x^4}{144} L\left\{a^2\left(L + \frac{3a'}{a}\frac{\partial}{\partial z^\dagger}\right)\left(\frac{a'}{a}\frac{A_o}{\partial z^\dagger}\right)\right\} + A_3(z^\dagger, t^\dagger) \qquad \text{(A.3)}$$

where

$$L \equiv \frac{\partial^2}{\partial z^{\dagger 2}} - \frac{\partial^2}{\partial t^{\dagger 2}}$$

In the planar section analysis of the general coaxial line, the first propagating correction A_2 is given by

$$\left\{\frac{\partial^2}{\partial z^{\dagger 2}} + \frac{\frac{b'}{b} - \frac{a'}{a}}{\ln\frac{b}{a}}\frac{\partial}{\partial z^\dagger} - \frac{\partial^2}{\partial t^{\dagger 2}}\right\} A_2(z^\dagger, t^\dagger) \qquad \text{(A.4)}$$

$$= -\frac{b^2 - a^2}{4\,\ell n\frac{b}{a}}\left\{\frac{b^2\ell nb - a^2\ell na}{b^2 - a^2} - 1\right\}\left\{\frac{\partial^2}{\partial z^{\dagger 2}} - \frac{\partial^2}{\partial t^{\dagger 2}}\right\}$$

$$+2\left\{\frac{b'b\ell nb - a'a\ell na}{b^2 - a^2} - 1\right\}\frac{\partial}{\partial z^\dagger}\Bigg\}\left\{\frac{\frac{b'}{b} - \frac{a'}{a}}{\ell n\frac{b}{a}}\frac{\partial A_o}{\partial z^\dagger}\right\}$$

$$+\left\{\frac{\partial^2}{\partial z^{\dagger 2}} - \frac{\partial^2}{\partial t^{\dagger 2}} + \frac{2(b'b - a'a)}{b^2 - a^2}\frac{\partial}{\partial z^\dagger}\right\}\left\{\frac{\frac{a'}{a}\ell nb - \frac{b'}{b}\ell na}{\ell n\frac{b}{a}}\frac{\partial A_o}{\partial z^\dagger}\right\}$$

The simplified version for a constant impedance line is given by (6.19).

In the special case of nominally uniform impedance coaxial line in the warped conical description of Section 6.5 the second order propagating correction A_2 is given by

$$\frac{\partial^2 A_2}{\partial \zeta^{\dagger 2}} - \frac{\partial^2 A_2}{\partial t^{\dagger t}} = -a^2 \left\{ \frac{a'}{a} \psi'^2 - \frac{5}{6} \psi'\psi'' - \frac{a'}{a} \psi' \frac{\cos\psi}{x_o} - \frac{11}{6} a\psi'' \frac{\cos\psi}{x_o} \right\} \frac{\partial A_o}{\partial \zeta^\dagger}$$

$$-a^2 \left\{ \frac{1}{3} \psi'^2 - \frac{2}{3} \psi' \frac{\cos\psi}{x_o} \right\} \frac{\partial^2 A_o}{\partial \zeta^{\dagger 2}}$$

$$-\frac{a^2}{6} \left\{ \frac{\partial^2}{\partial \zeta^{\dagger 2}} + 2 \frac{a'}{a} \frac{\partial}{\partial \zeta^\dagger} - \frac{\partial^2}{\partial t^{\dagger 2}} \right\} \left\{ \frac{a'}{a} \frac{\partial A_o}{\partial \zeta} \right\} \qquad (A.5)$$

Using the relation (6.83) we find the coefficient of $\frac{\partial^2 A_o}{\partial \zeta^{\dagger 2}}$ to be $-\frac{1}{3}\{a^2\psi'^2 + a'^2 - (aa')'\}$.

APPENDIX B

Geometry Associated with the Equiangular Arc Centerline

In this section we shall establish some geometrical properties of the arc coordinate system defined in Chapter V, where the properties have been quoted and used without proof. Refer to figure B.1 for details of point labelling. The typical transverse arc P_1RP_2 makes equal angles with the boundary curves C_1 and C_2. The governing equation for selection of points $P_1(\underline{x}_1)$ and $P_2(\underline{x}_2)$ is

$$(\underline{x}_1 - \underline{x}_2) \cdot \hat{t} = 0 \qquad \text{(B-1)}$$

where

$$\hat{t} = \frac{\hat{t}_1 + \hat{t}_2}{2 \cos \Theta} \qquad \text{(B-2)}$$

is a unit vector along the local axis OQRS and $\hat{t}_1$ and $\hat{t}_2$ arc tangent vectors at P_1 and P_2 respectively. We measure arc length ζ along the locus C of the arc centerpoint R and arc lengths ζ_1 and ζ_2 along the boundary curves C_1 and C_2. Then the vector description of the arc center R is given by

$$\vec{OR} = \frac{1}{2}(\underline{x}_1 + \underline{x}_2) + \hat{t}\, a \tan \frac{\Theta}{2} \qquad \text{(B-3)}$$

A unit vector $\hat{n}$ normal to $\hat{t}$ is given by

$$\hat{n} = \frac{1}{2a} (\underline{x}_1 - \underline{x}_2) \qquad \text{(B-4)}$$

In general OR will not be tangent to the locus of R, the equiangular centerline C. Let δ be the defect angle between OR and

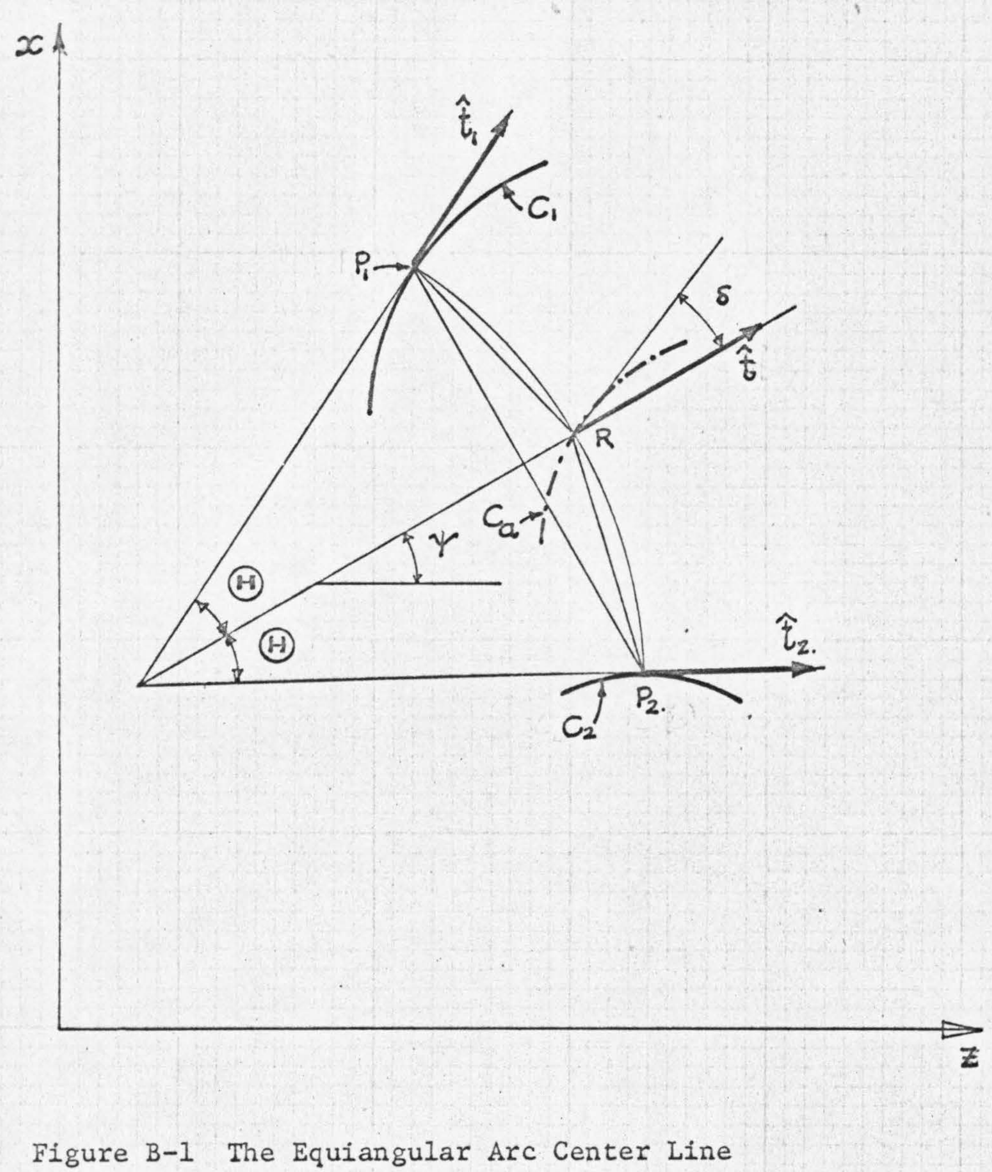

Figure B-1 The Equiangular Arc Center Line the tangent to C at R.

$$\delta = \phi - \psi \tag{B-5}$$

If we calculate the shift ΔR in the position of R on a nearby arc then we have

$$\tan \delta = \frac{\vec{\Delta R} \cdot \hat{n}}{\vec{\Delta R} \cdot \hat{t}} \tag{B-6}$$

$\vec{\Delta R}$ is obtained by differentiating (B-3). Since $\hat{t}$ is a unit vector the change in $\hat{t}$ will be normal to $\hat{t}$ and so parallel to $\hat{n}$. Then

$$\vec{\Delta R} = \frac{1}{2}(\hat{t}_1\ d\zeta_1 + \hat{t}_2\ d\zeta_2)$$

$$+ \Delta\hat{t}\ a \tan\frac{\Theta}{2} + \hat{t}\ \Delta(a \tan\frac{\Theta}{2}) \tag{B-7}$$

We find by differentiation of (B-1), that, after a little algebra

$$\hat{n} \cdot \Delta\hat{t} = -\frac{\cos\Theta}{2a}(d\zeta_1 - d\zeta_2) \tag{B-8}$$

so that, with the third term dropping out

$$\vec{\Delta R} \cdot \hat{n} = \frac{1}{2}\sin\Theta\ (d\zeta_1 - d\zeta_2) - \frac{1}{2}\cos\Theta \tan\frac{\Theta}{2}(d\zeta_1 - d\zeta_2)$$

$$= \frac{1}{2}(d\zeta_1 - d\zeta_2)(\sin\Theta - \cos\Theta\ \tan\frac{\Theta}{2}) \tag{B-9}$$

$$= \frac{1}{2}(d\zeta_1 - d\zeta_2)\tan\frac{\Theta}{2} \tag{B-10}$$

This is, in general, non-zero unless $\Theta = 0$, an equal spacing line, or $d\zeta_1 = d\zeta_2$ which implies a straight center line, so that $\delta \neq 0$ except in these special cases. A little reflection on the derivation will reveal that the point S where the normals intersect from the boundary curves at P_1 and P_2 on OR has its locus tangent to OR. This elegant property does not appear to have any direct relevance to electromagnetic field calculations.

It can also be readily shown that

$$\Delta a = \frac{1}{2} \sin \Theta \, (d\zeta_1 + d\zeta_2) \tag{B-11}$$

and

$$\Delta\Theta = \frac{1}{2} (\kappa_1 \, d\zeta_1 - \kappa_2 \, d\zeta_2) \tag{B-12}$$

where κ is the curvature of the boundary. Then

$$\Delta\vec{R}.\hat{t} = \frac{1}{2} \cos \Theta \, (d\zeta_1 + d\zeta_2) + \Delta(a \tan \frac{\Theta}{2})$$

$$= \frac{1}{2}(d\zeta_1 + d\zeta_2) + \frac{a}{4} \sec^2 \frac{\Theta}{2} \, (\kappa_1 d\zeta_1 - \kappa_2 d\zeta_2) \tag{B-13}$$

Substitute back into (B-6) to obtain

$$\tan \delta = \frac{(d\zeta_1 - d\zeta_2) \tan \frac{\Theta}{2}}{d\zeta_1 + d\zeta_2 + \frac{a}{2} (\kappa_1 d\zeta_1 - \kappa_2 d\zeta_2)(1 + \tan^2 \frac{\Theta}{2})} \tag{B-14}$$

Application of ordering estimate for $\eta \to 0$ shows that $\delta \sim O(\eta^2)$

The coordinates for the extended warped cylindrical coordinate system on a curved center line are defined as follows. Let the label for choosing the particular transverse arc be ζ, the arc length measured along the equiangular center line C_a, and let σ be the angle measured along each arc ζ from the center line. Then we can write the Cartesian coordinates (x,z) of a point $Q(\zeta,\sigma)$ by using equation (4.15) followed by rotation and translation.

$$\begin{aligned} x &= x_o + r \sin\frac{\sigma}{r} \cos\psi - r(1 - \cos\frac{\sigma}{r}) \sin\psi \\ z &= z_o - r \sin\frac{\sigma}{r} \sin\psi - r(1 - \cos\frac{\sigma}{r}) \cos\psi \end{aligned} \qquad \text{(B-15)}$$

The slope of the tangent to the equiangular center line is given by

$$\frac{\partial x_o}{\partial \zeta} = \sin\phi \; ; \qquad \frac{\partial z_o}{\partial \zeta} = \cos\phi \qquad \text{(B-16)}$$

It has been found by grinding experience that calculations go more smoothly when we work almost to the end in terms of half angle functions. If we write for brevity $s = \sin \sigma/2r$, $S = \sin(\sigma/2r+\psi)$ etc, then

$$\begin{aligned} x &= x_o + 2r \sin\frac{\sigma}{2r} \cos\left(\frac{\sigma}{2r} + \psi\right) \\ x &= x_o - 2r \sin\frac{\sigma}{2r} \sin\left(\frac{\sigma}{2r} + \psi\right) \end{aligned} \qquad \text{(B-17)}$$

and the partial derivatives become, with a prime denoting an ordinary ζ derivative

$$\frac{\partial x}{\partial \sigma} = \sin\phi + 2r'sC + (\theta r' - 2r\psi')sS - \theta r'cC$$

$$\frac{\partial z}{\partial \zeta} = \cos\phi - 2r'sS + (\theta r' - 2r\psi')sC + \theta r'cS \tag{B-18}$$

$$\frac{\partial x}{\partial \sigma} = cC - sS = \cos(\theta + \psi)$$

$$\frac{\partial}{\partial \sigma} = -sC - cS = -\sin(\theta + \psi) \tag{B-19}$$

After much algebraic manipulation we surface with the metric quantities

$$g_{\zeta\zeta} = 1 + 2r'^2(1 - \cos\frac{\sigma}{r} - \frac{\sigma}{r}\sin\frac{\sigma}{r} + \frac{1}{2}\frac{\sigma^2}{r^2}) - 2r'(\cos\delta - \cos\theta - \frac{\sigma}{r}\sin(\frac{\sigma}{r} - \delta)$$

$$+ 2(r\psi')^2(1 - \cos\theta) - 2(r'\theta)(r\psi')(1 - \cos\theta)$$

$$- 2r\psi'(\sin\delta + \sin(\frac{\sigma}{r} - \delta)) \tag{B-20}$$

$$g_{\zeta\zeta} = -\sin(\frac{\sigma}{r} - \delta) - r'(\frac{\sigma}{r} - \sin\frac{\sigma}{r}) + r\psi'(1 - \cos\frac{\sigma}{r}) \tag{B-21}$$

$$g_{\sigma\sigma} = 1 \tag{B-22}$$

$$g^{\frac{1}{2}} = \cos(\frac{\sigma}{r} - \delta) + r'(1 - \cos\frac{\sigma}{r}) - r\psi'\sin\frac{\sigma}{r} \tag{B-23}$$

No approximations have been made in deriving these expressions. Note that if we set $\psi' = \delta = 0$, then we recover the expressions for a nonuniform line with straight center line. The limit $\delta \to 0$, $r \to \infty$ represents an equally spaced line on a curved center line and the orthogonal coordinate expressions of Chapter V are recovered. For example

$$g_{\zeta\zeta} \to 1 + 2r^2\psi'^2 \frac{\sigma^2}{2r^2} - 2r\psi' \frac{\sigma}{r}$$

$$= (1 - \sigma\psi')^2 \qquad \text{(B-24)}$$

The expressions just given for the metric quantities have been separated into straight center line terms, curved axis terms, and correction terms in the defect angle δ. We now examine the series expansions required. The geometric series

$$g^{-\frac{1}{2}} = 1 - (r' - 1)(1 - \cos \frac{\theta}{r}) + r\psi' \sin \frac{\sigma}{r} + O(\eta^3) + \ldots \qquad \text{(B-25)}$$

will be convergent provided

$$\left|(r' - 1)(1 - \cos \frac{\sigma}{r}) - \frac{r}{R} \sin \frac{\sigma}{r}\right| < 1 \qquad \text{(B-26)}$$

neglecting the $O(\eta^3)$ terms in δ. This is true provided

$$\left|(r' - 1)(1 - \cos \frac{\sigma}{r})\right| < 1 - \left|\frac{b}{R}\right| \qquad \text{(B-27)}$$

where R is the center line radius of curvature.

Expansions of the metric quantities in increasing order of η may be written

$$\sqrt{g} = 1 - \sigma\psi' - \frac{\sigma^2 F}{2b^2} - \frac{\sigma^3\psi'}{6r^2} + \frac{\sigma\delta}{r} + \ldots \tag{B-28}$$

$$g^{-\frac{1}{2}} = 1 + \sigma\psi' + \frac{\sigma^2 F}{2b^2} + \sigma^2\psi'^2 + \ldots \tag{B-29}$$

$$g^{-\frac{1}{2}} g_{\zeta\zeta} = 1 - \sigma\psi' + \frac{\sigma^2}{r^2} - \frac{\sigma^2 F}{2b^2} + \ldots \tag{B-30}$$

$$g^{-\frac{1}{2}} g_{\zeta\sigma} = - \frac{\sigma}{r} + \delta - \frac{\sigma^2\psi'}{2r} - \frac{\sigma^3 F}{3rb^2} + \frac{\sigma^3\psi'^2}{2r} - \sigma\psi'\delta + \ldots \tag{B-31}$$

New terms, depending on center line curvature and angular offset, have appeared and these will give rise to odd order terms when substituted into the differential equation.

APPENDIX C

The Natural Coordinate System

Unger (1965) uses a coordinate system, related to our warped cylindrical coordinates, which he calls natural coordinates. The coordinate arcs are defined as in Chapter IV with ζ as parameter. Instead of arc length s, Unger uses the normalized angle $u \,\varepsilon(-1,1)$ where $u = \pm 1$ are the line boundaries

$$u = \frac{\theta}{\Theta} \tag{C-1}$$

Unger treats these coordinates as though they were orthogonal. The coordinate transformation is

$$\left.\begin{aligned} x_o &= r \sin \Theta u \\ z_o &= z - r + r \cos \Theta u \end{aligned}\right\} \tag{C-2}$$

Then we find, correcting Unger's version, with primes denoting z-derivatives in a two dimensional version

$$g_{zz} = 1 + r^2\, \Theta'^2 + 2(r'-1)\{r'(1-\cos \Theta u) + ur\, \Theta' \sin \Theta u\} \tag{C-3}$$

$$g_{uu} = r^2\, \Theta^2 \tag{C-4}$$

$$g_{zu} = ur^2\, \Theta\Theta' + r(r'-1)\Theta \sin \Theta u \tag{C-5}$$

This latter expression may be written

$$g_{zu} = \Theta \frac{dz_o}{dz} \frac{b^2 b''}{b'^2} \left\{ u - \frac{\sin \Theta u}{\sin \Theta} \right\} \tag{C-6}$$

Equation (C-6) makes it apparent that in general when $b'' \neq 0$ that g_{zu} vanishes only on axis or on the boundaries and so Unger's natural coordinate system is really nonorthogonal almost everywhere.

BIBLIOGRAPHY

Abramowitz, M., and I. A. Stegun, Handbook of Mathematical Functions, P. 362, National Bureau of Standards AMS-55, U.S. Department of Commerce, Washington 1964.

Amitay, N., A. Lavi, and F. Young, "Electromagnetic Field Theory Solution of the Infinite Tapered Plane Transmission Line", Z. angew. Math. u. Physik, 12, 1961 pp. 89-99.

Bahar, E., "Wave Scattering in Nonuniform Waveguides with Large Flare Angles and Near Cutoff", I.E.E.E. Trans. on Microwave Theory and Techniques, MTT-16, Aug. 1968, pp. 503-510.

Baum, C. E., "A Scaling Technique for the Design of Idealized Electromagnetic Lenses", Ph.D. Thesis, Caltech, 1969.

Borgnis, F. E., and C. H. Papas, "Electromagnetic Waveguides and Resonators", Sect. 22, pp. 358-364, in Handbuch der Physik Vd. XVI ed. S. Flügge, Springer Verlag, Berlin, 1958.

Cole, J. D., Perturbation Methods in Applied Mathematics, Ch. 4.5.3., Blaisdell, Waltham, 1968.

Fox, A. G., "Wave Coupling By Warped Normal Modes", Bell Sys. Tech. J. 34, July 1955, pp. 823-852.

Gopinath, B., and M. M. Sondhi, "Inversion of the Telegraph Equation and the Synthesis of Nonuniform Lines", Proc. I.E.E.E. 59, March 1971, pp. 383-392.

Gruner, L., "Analysis of Nonuniform Lossless Transmission Lines", Proc. I.R.E.E., Australia 31, Feb. 1970, pp. 36-41.

Holt, A.G.J., and K. U. Ahmed, "Exact Solutions of Generalized Nonuniform Lines and Their Classifications", Radio and Electronic Engr., Dec. 1968, pp. 373-379.

Katzenelenbaum, B. Z., "Nonuniform Waveguides with Slowly Changing Parameters", Dokl. Akad. Nauk SSSR, 12, 1955, pp. 711-714.

Kaufman, H., "Bibliography of Nonuniform Transmission Lines", IRE Trans. on Antennas and Propagation, AP-3, Oct. 1955, pp. 218-220.

Louisell, W. H., "Analysis of the Single Tapered Mode Coupler", Bell. Sys. Tech. J. 34, Jul. 1955, pp. 853-870.

Pokrovskii, V. L., F. R. Ulinich, and S. K. Savvinykh, "Theory of Tapered Waveguides:, Radio Eng. and Electronics, 4, Feb. 1959, pp. 161-171.

Reiter, G., "Generalized Telegraphist's Equation for Waveguides of Varying Cross-section", Proc. IEE (London) 106B, Suppl. 13, 1959, pp. 54-57.

Schelkunoff, S. A., "Conversion of Maxwell's Equations into Generalized Telegraphist's Equations", Bell Sys. Tech. J. 34, Sept. 1955, pp. 995-1044.

Solymar, L., "Spurious Mode Generation in Nonuniform Waveguide", IRE Trans. on Microwave Theory and Techniques, MTT-7, July 1959, pp. 379-383.

Stevenson, A. F., "General Theory of Electromagnetic Horns", J. Appl. Phys., 22, Dec. 1951, pp. 1447-1454.

Stratton, J. A., Electromagnetic Theory, pp. 38-47 , McGraw-Hill, New York, 1941.

Tang, C.C.H., "Mode Conversion in Nonuniform Multimode Waveguides and Transitions", in Advances in Microwaves, Vol. 4, 1969, ed. Leo Young, Academic Press.

Unger, H-G., "Wave Propagation in Horns and Through Horn Junctions", Arch. Elek. Übertragung, 19, 1965, pp. 459-468.

Watson, G. N., Theory of Bessel Functions, p. 137, 2nd ed. Cambridge University Press, Cambridge, 1958.

Youla, D. C., "Analysis and Synthesis of Arbitrarily Terminated Lossless Nonuniform Lines", I.E.E.E. Trans. on Circuit Theory, CT-11, Sept. 1964, pp. 363-377.