Appendix B

FOURIER COEFFICIENTS FOR NONBLACK DIFFUSE BOUNDARIES

Section 3.3 demonstrated an example calculation for steady-state heat conduction between two walls that are either black or non-black. In this section we derive the Fourier coefficients of the kernel function $K(\widehat{x}, \widehat{x}')$ and the inhomogeneous function $f(\widehat{x})$ for these two specific cases.

For steady-state heat conduction between two non-black walls as studied in Sec. 3.3, the inhomogeneous function becomes

$$f(\widehat{x}) = \frac{1}{2\int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} d\omega} \int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} \left[A_{1\omega} E_2 \left(\frac{\widehat{x}}{\mathrm{Kn}_\omega} \right) + A_{2\omega} E_2 \left(\frac{1-\widehat{x}}{\mathrm{Kn}_\omega} \right) \right] d\omega. \tag{B.1}$$

Its Fourier coefficients in Eq. (3.34) are then given by:

$$f_0 = \frac{1}{2\int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} d\omega} \int_0^{\omega_m} \frac{C_\omega \mathrm{Kn}_\omega}{\tau_\omega} (A_{1\omega} + A_{2\omega}) \left[1 - 2E_3 \left(\frac{1}{\mathrm{Kn}_\omega} \right) \right] d\omega, \tag{B.2}$$

and

$$f_n = \frac{1}{\int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} d\omega} \int_0^{\omega_m} \int_0^1 \frac{C_\omega}{\tau_\omega} \mathrm{Kn}_\omega \mu \frac{[A_{1\omega} + (-1)^n A_{2\omega}] - e^{-\frac{1}{\mathrm{Kn}_\omega \mu}} [(-1)^n A_{1\omega} + A_{2\omega}]}{1 + (\mathrm{Kn}_\omega \mu)^2 (n\pi)^2}, \tag{B.3}$$

providing the right-hand side of Eq. (3.39). Under the same assumption of diffuse, non-black walls, the kernel function becomes

$$K(\widehat{x}, \widehat{x}') = \frac{1}{2\int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} d\omega} \int_0^{\omega_m} \frac{C_\omega G_\omega(\widehat{x}, \widehat{x}')}{\mathrm{Kn}_\omega \tau_\omega} d\omega, \tag{B.4}$$

where

$$\begin{aligned} G_\omega(\widehat{x}, \widehat{x}') &= E_2 \left(\frac{\widehat{x}}{\mathrm{Kn}_\omega} \right) \left[D_\omega E_1 \left(\frac{1-\widehat{x}'}{\mathrm{Kn}_\omega} \right) + B_{1\omega} E_1 \left(\frac{\widehat{x}'}{\mathrm{Kn}_\omega} \right) \right] \\ &+ E_2 \left(\frac{1-\widehat{x}}{\mathrm{Kn}_\omega} \right) \left[D_\omega E_1 \left(\frac{\widehat{x}'}{\mathrm{Kn}_\omega} \right) + B_{1\omega} E_1 \left(\frac{1-\widehat{x}'}{\mathrm{Kn}_\omega} \right) \right] + E_1 \left(\frac{|\widehat{x} - \widehat{x}'|}{\mathrm{Kn}_\omega} \right). \end{aligned} \tag{B.5}$$

Its Fourier coefficients k_{mn} are given by Eq. (3.35), and can be evaluated as:

$$\begin{aligned} k_{00} &= \frac{2}{\int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} d\omega} \int_0^{\omega_m} \frac{C_\omega \mathrm{Kn}_\omega}{\tau_\omega} \left\{ \frac{2}{\mathrm{Kn}_\omega} - 1 + 2E_3\left(\frac{1}{\mathrm{Kn}_\omega}\right) \right. \\ &\left. +(2D_\omega + B_{1\omega} + B_{2\omega}) \left[\frac{1}{2} - E_3\left(\frac{1}{\mathrm{Kn}_\omega}\right) - \frac{1}{2}E_2\left(\frac{1}{\mathrm{Kn}_\omega}\right) + E_3\left(\frac{1}{\mathrm{Kn}_\omega}\right) E_2\left(\frac{1}{\mathrm{Kn}_\omega}\right) \right] \right\} d\omega, \end{aligned} \tag{B.6}$$

and

$$\begin{aligned} k_{m0} &= \frac{2}{\int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} d\omega} \int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} \int_0^1 \frac{\mathrm{Kn}_\omega \mu [(-1)^m + 1] \left[e^{-\frac{1}{\mathrm{Kn}_\omega \mu}} - 1 \right]}{1 + (\mathrm{Kn}_\omega \mu)^2 (m\pi)^2} d\mu d\omega \\ &+ \frac{2}{\int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} d\omega} \int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} (D_\omega + B_{1\omega}) \left[1 - E_2\left(\frac{1}{\mathrm{Kn}_\omega}\right) \right] \int_0^1 \frac{\mathrm{Kn}_\omega \mu \left[1 - (-1)^m e^{-\frac{1}{\mathrm{Kn}_\omega \mu}} \right]}{1 + (\mathrm{Kn}_\omega \mu)^2 (m\pi)^2} d\mu d\omega \\ &+ \frac{2}{\int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} d\omega} \int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} (D_\omega + B_{2\omega}) \left[1 - E_2\left(\frac{1}{\mathrm{Kn}_\omega}\right) \right] \int_0^1 \frac{\mathrm{Kn}_\omega \mu \left[(-1)^m - e^{-\frac{1}{\mathrm{Kn}_\omega \mu}} \right]}{1 + (\mathrm{Kn}_\omega \mu)^2 (m\pi)^2} d\mu d\omega, \end{aligned} \tag{B.7}$$

and

$$\begin{aligned} k_{0n} &= \frac{2}{\int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} d\omega} \int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} \int_0^1 \frac{\mathrm{Kn}_\omega \mu [(-1)^n + 1] \left[e^{-\frac{1}{\mathrm{Kn}_\omega \mu}} - 1 \right]}{1 + (\mathrm{Kn}_\omega \mu)^2 (n\pi)^2} d\mu d\omega \\ &+ \frac{2}{\int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} d\omega} \int_0^{\omega_m} \frac{C_\omega \mathrm{Kn}_\omega}{\tau_\omega} (D_\omega + B_{1\omega}) \left[\frac{1}{2} - E_3\left(\frac{1}{\mathrm{Kn}_\omega}\right) \right] \int_0^1 \frac{\left[1 - (-1)^n e^{-\frac{1}{\mathrm{Kn}_\omega \mu}} \right]}{1 + (\mathrm{Kn}_\omega \mu)^2 (n\pi)^2} d\mu d\omega \\ &+ \frac{2}{\int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} d\omega} \int_0^{\omega_m} \frac{C_\omega \mathrm{Kn}_\omega}{\tau_\omega} (D_\omega + B_{2\omega}) \left[\frac{1}{2} - E_3\left(\frac{1}{\mathrm{Kn}_\omega}\right) \right] \int_0^1 \frac{\left[(-1)^n - e^{-\frac{1}{\mathrm{Kn}_\omega \mu}} \right]}{1 + (\mathrm{Kn}_\omega \mu)^2 (n\pi)^2} d\mu d\omega, \end{aligned} \tag{B.8}$$

and for $m \neq n$

$$\begin{aligned} k_{mn} &= \frac{2}{\int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} d\omega} \int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} \int_0^1 \frac{\mathrm{Kn}_\omega \mu \left\{ e^{-\frac{1}{\mathrm{Kn}_\omega \mu}} [(-1)^m + (-1)^n] - [1 + (-1)^{m+n}] \right\}}{[1 + (\mathrm{Kn}_\omega \mu)^2 (m\pi)^2][1 + (\mathrm{Kn}_\omega \mu)^2 (n\pi)^2]} d\mu d\omega \\ &+ \frac{2}{\int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} d\omega} \int_0^{\omega_m} \frac{C_\omega D_\omega}{\tau_\omega} \int_0^1 \frac{\mathrm{Kn}_\omega \mu \left[1 - (-1)^m e^{-\frac{1}{\mathrm{Kn}_\omega \mu}} \right]}{1 + (\mathrm{Kn}_\omega \mu)^2 (m\pi)^2} d\mu \int_0^1 \frac{\left[(-1)^n - e^{-\frac{1}{\mathrm{Kn}_\omega \mu}} \right]}{1 + (\mathrm{Kn}_\omega \mu)^2 (n\pi)^2} d\mu d\omega \\ &+ \frac{2}{\int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} d\omega} \int_0^{\omega_m} \frac{C_\omega D_\omega}{\tau_\omega} \int_0^1 \frac{\mathrm{Kn}_\omega \mu \left[(-1)^m - e^{-\frac{1}{\mathrm{Kn}_\omega \mu}} \right]}{1 + (\mathrm{Kn}_\omega \mu)^2 (m\pi)^2} d\mu \int_0^1 \frac{\left[1 - (-1)^n e^{-\frac{1}{\mathrm{Kn}_\omega \mu}} \right]}{1 + (\mathrm{Kn}_\omega \mu)^2 (n\pi)^2} d\mu d\omega \\ &+ \frac{2}{\int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} d\omega} \int_0^{\omega_m} \frac{C_\omega B_{1\omega}}{\tau_\omega} \int_0^1 \frac{\mathrm{Kn}_\omega \mu \left[1 - (-1)^m e^{-\frac{1}{\mathrm{Kn}_\omega \mu}} \right]}{1 + (\mathrm{Kn}_\omega \mu)^2 (m\pi)^2} d\mu \int_0^1 \frac{\left[1 - (-1)^n e^{-\frac{1}{\mathrm{Kn}_\omega \mu}} \right]}{1 + (\mathrm{Kn}_\omega \mu)^2 (n\pi)^2} d\mu d\omega \\ &+ \frac{2}{\int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} d\omega} \int_0^{\omega_m} \frac{C_\omega B_{2\omega}}{\tau_\omega} \int_0^1 \frac{\mathrm{Kn}_\omega \mu \left[(-1)^m - e^{-\frac{1}{\mathrm{Kn}_\omega \mu}} \right]}{1 + (\mathrm{Kn}_\omega \mu)^2 (m\pi)^2} d\mu \int_0^1 \frac{\left[(-1)^n - e^{-\frac{1}{\mathrm{Kn}_\omega \mu}} \right]}{1 + (\mathrm{Kn}_\omega \mu)^2 (n\pi)^2} d\mu d\omega, \end{aligned} \tag{B.9}$$

and for $m \neq 0$

$$k_{mm} = \frac{2}{\int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} d\omega} \left\{ \int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} \frac{\tan^{-1}(m\pi \mathrm{Kn}_\omega)}{m\pi \mathrm{Kn}_\omega} d\omega + 2\int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} \int_0^1 \frac{\mathrm{Kn}_\omega \mu \left[e^{-\frac{1}{(-1)^m \mathrm{Kn}_\omega \mu}} - 1 \right]}{[1 + (\mathrm{Kn}_\omega \mu)^2 (m\pi)^2]^2} d\mu d\omega \right\}$$
$$+ \frac{2}{\int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} d\omega} \int_0^{\omega_m} \frac{C_\omega D_\omega}{\tau_\omega} \int_0^1 \frac{\mathrm{Kn}_\omega \mu \left[1 - (-1)^m e^{-\frac{1}{\mathrm{Kn}_\omega \mu}} \right]}{1 + (\mathrm{Kn}_\omega \mu)^2 (m\pi)^2} d\mu \int_0^1 \frac{\left[(-1)^m - e^{-\frac{1}{\mathrm{Kn}_\omega \mu}} \right]}{1 + (\mathrm{Kn}_\omega \mu)^2 (m\pi)^2} d\mu d\omega$$
$$+ \frac{2}{\int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} d\omega} \int_0^{\omega_m} \frac{C_\omega D_\omega}{\tau_\omega} \int_0^1 \frac{\mathrm{Kn}_\omega \mu \left[(-1)^m - e^{-\frac{1}{\mathrm{Kn}_\omega \mu}} \right]}{1 + (\mathrm{Kn}_\omega \mu)^2 (m\pi)^2} d\mu \int_0^1 \frac{\left[1 - (-1)^m e^{-\frac{1}{\mathrm{Kn}_\omega \mu}} \right]}{1 + (\mathrm{Kn}_\omega \mu)^2 (m\pi)^2} d\mu d\omega$$
$$+ \frac{2}{\int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} d\omega} \int_0^{\omega_m} \frac{C_\omega B_{1\omega}}{\tau_\omega} \int_0^1 \frac{\mathrm{Kn}_\omega \mu \left[1 - (-1)^m e^{-\frac{1}{\mathrm{Kn}_\omega \mu}} \right]}{1 + (\mathrm{Kn}_\omega \mu)^2 (m\pi)^2} d\mu \int_0^1 \frac{\left[1 - (-1)^m e^{-\frac{1}{\mathrm{Kn}_\omega \mu}} \right]}{1 + (\mathrm{Kn}_\omega \mu)^2 (m\pi)^2} d\mu d\omega$$
$$+ \frac{2}{\int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} d\omega} \int_0^{\omega_m} \frac{C_\omega B_{2\omega}}{\tau_\omega} \int_0^1 \frac{\mathrm{Kn}_\omega \mu \left[(-1)^m - e^{-\frac{1}{\mathrm{Kn}_\omega \mu}} \right]}{1 + (\mathrm{Kn}_\omega \mu)^2 (m\pi)^2} d\mu \int_0^1 \frac{\left[(-1)^m - e^{-\frac{1}{\mathrm{Kn}_\omega \mu}} \right]}{1 + (\mathrm{Kn}_\omega \mu)^2 (m\pi)^2} d\mu d\omega. \quad \text{(B.10)}$$

These equations specify the matrix elements of $\bar{\bar{A}}$ in Eq. (3.39). With the linear system specified, the coefficients of the temperature profile x_m can be easily obtained by solving a linear system.