Appendix A

SPECULAR BOUNDARIES

Section 3.3 derived the BTE solutions in thin films with diffuse boundary scattering. Here, we derive the governing equation for the problem of nonblack, specular boundaries with wall temperatures ΔT_1 and ΔT_2, respectively. The boundary conditions can be written as:

$$\widetilde{g}^{+}_{\omega}(x=0,\mu) = P_{\omega} = \epsilon_1 \frac{C_{\omega}}{4\pi}\Delta T_1 + (1-\epsilon_1)\widetilde{g}^{-}_{\omega}(x=0,-\mu) \tag{A.1}$$

$$\widetilde{g}^{-}_{\omega}(x=L,\mu) = B_{\omega} = \epsilon_2 \frac{C_{\omega}}{4\pi}\Delta T_2 + (1-\epsilon_2)\widetilde{g}^{+}_{\omega}(x=L,-\mu), \tag{A.2}$$

Applying the boundary conditions to Eqs. (5.6) & (5.7), we have

$$\begin{aligned}
\widetilde{g}^{+}_{\omega}(x) &= F_1\Delta T_1 \frac{C_{\omega}}{4\pi} e^{-\frac{\gamma_\omega}{\mu}x} + (1-\epsilon_1)F_2\Delta T_2 \frac{C_{\omega}}{4\pi} e^{-\frac{\gamma_\omega}{\mu}(L+x)} \\
&+ (1-\epsilon_1)F_2 \int_0^L \frac{C_\omega \Delta\widetilde{T}(x') + \widetilde{Q}_\omega(x')\tau_\omega}{4\pi\Lambda_\omega\mu} e^{-\frac{\gamma_\omega}{\mu}(x'+x)} dx' \\
&+ \int_0^x \frac{C_\omega \Delta\widetilde{T}(x') + \widetilde{Q}_\omega(x')\tau_\omega}{4\pi\Lambda_\omega\mu} e^{\frac{\gamma_\omega}{\mu}(x'-x)} dx' \ (\mu\in[0,1]),
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
\widetilde{g}^{-}_{\omega}(x) &= F_2\Delta T_2 \frac{C_{\omega}}{4\pi} e^{-\frac{\gamma_\omega}{\mu}(L-x)} + (1-\epsilon_2)F_1\Delta T_1 \frac{C_{\omega}}{4\pi} e^{-\frac{\gamma_\omega}{\mu}(2L-x)} \\
&+ (1-\epsilon_2)F_1 \int_0^L \frac{C_\omega \Delta\widetilde{T}(x') + \widetilde{Q}_\omega(x')\tau_\omega}{4\pi\Lambda_\omega\mu} e^{-\frac{\gamma_\omega}{\mu}(2L-x'-x)} dx' \\
&+ \int_x^L \frac{C_\omega \Delta\widetilde{T}(x') + \widetilde{Q}_\omega(x')\tau_\omega}{4\pi\Lambda_\omega\mu} e^{-\frac{\gamma_\omega}{\mu}(x'-x)} dx' \ (\mu\in[0,1]),
\end{aligned} \tag{A.4}$$

where $F_1 = \frac{\epsilon_1}{\epsilon_1+\epsilon_2-\epsilon_1\epsilon_2}$ and $F_2 = \frac{\epsilon_2}{\epsilon_1+\epsilon_2-\epsilon_1\epsilon_2}$.

To close the problem, we insert Eqs. (A.3) & (A.4) into Eq. (5.4) and nondimensionalize x by L. We then derive an integral equation for temperature for the specular boundary conditions, given by

$$\begin{aligned}
2\int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} d\omega \Delta\widetilde{T}(\widehat{x}) &= \int_0^{\omega_m} \frac{C_\omega}{\tau_\omega} H_\omega(\widehat{x}) d\omega + \int_0^1\int_0^{\omega_m} \widetilde{Q}_\omega(x') \frac{G_\omega(\widehat{x},\widehat{x}')}{\mathrm{Kn}_\omega} d\omega d\widehat{x}' \\
&+ \int_0^1 \Delta\widetilde{T}(\widehat{x}') \int_0^{\omega_m} \frac{C_\omega G_\omega(\widehat{x},\widehat{x}')}{\mathrm{Kn}_\omega \tau_\omega} d\omega d\widehat{x}',
\end{aligned} \tag{A.5}$$

where $\widehat{x} = x/L$, $\text{Kn}_\omega = \Lambda_\omega/L$ is the Knudsen number, $\widehat{\gamma}_\omega = \frac{1+i\eta\tau_\omega}{\text{Kn}_\omega}$ and

$$\begin{aligned} H_\omega(\widehat{x}) &= F_1\Delta T_1 E_2(\widehat{\gamma}_\omega\widehat{x}) + F_2\Delta T_2 E_2(\widehat{\gamma}_\omega(1-\widehat{x})) \\ &+ (1-\epsilon_1)F_2\Delta T_2 E_2(\widehat{\gamma}_\omega(1+\widehat{x})) + (1-\epsilon_2)F_1\Delta T_1 E_2(\widehat{\gamma}_\omega(2-\widehat{x})) \end{aligned} \tag{A.6}$$

and

$$G_\omega(\widehat{x},\widehat{x}') = (1-\epsilon_1)F_2E_1(\widehat{\gamma}_\omega(\widehat{x}+\widehat{x}')) + (1-\epsilon_2)F_1E_1(\widehat{\gamma}_\omega(2-\widehat{x}-\widehat{x}')) + E_1(\widehat{\gamma}_\omega|\widehat{x}-\widehat{x}'|). \tag{A.7}$$

In this case, the inhomogeneous function becomes

$$f(\widehat{x}) = \frac{1}{2\int_0^{\omega_m}\frac{C_\omega}{\tau_\omega}d\omega}\left[\int_0^{\omega_m}\frac{C_\omega}{\tau_\omega}H_\omega(\widehat{x})d\omega + \int_0^1\int_0^{\omega_m}\widetilde{Q}_\omega(x')\frac{G_\omega(\widehat{x},\widehat{x}')}{\text{Kn}_\omega}d\omega d\widehat{x}'\right] \tag{A.8}$$

and the kernel function becomes

$$K(\widehat{x},\widehat{x}') = \frac{1}{2\int_0^{\omega_m}\frac{C_\omega}{\tau_\omega}d\omega}\int_0^{\omega_m}\frac{C_\omega G_\omega(\widehat{x},\widehat{x}')}{\text{Kn}_\omega\tau_\omega}d\omega. \tag{A.9}$$

With these results, the problem can be solved by following the same procedures described in Sec.3.3 are followed to formulate a linear system of equations. The solution of this system then yields the temperature Fourier coefficients.