# WEAK PION PRODUCTION
# AND RELATED PROCESSES


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ABSTRACT

The matrix element for the interaction W + N → $\pi$ + N is studied, where W is a virtual intermediate boson for the weak interactions (or just the weak current). Weak pion production — production of a pion by high energy neutrino collisions with nucleons, and intermediate boson production by pions are governed by this matrix element. The main case of interest is in the energy region where the pion-nucleon 3-3 resonance is dominant. Formulae are derived for solving the problem in this region.

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# I. THE WEAK INTERACTION CURRENTS

Introduction:

The decay of the muon is the only known example of a pure weak interaction. All the evidence supports the following interaction Lagrangian

$$\mathcal{L}_{int} = \frac{G_\mu}{\sqrt{2}} \left[\bar{\nu}\gamma_\alpha(1+\gamma_5)e\right]\left[\bar{\nu}'\gamma_\alpha(1+\gamma_5)\mu\right]^+ + H.C. \qquad (I.1)$$

( $\nu$ is written for the neutrino associated with the electron; $\nu'$ that associated with the muon.)

The decay rate of the muon from I.1 is

$$\Gamma_\mu = \frac{1}{192\pi^3} G_\mu^2 m_\mu^5 \ sec^{-1} \qquad (I.2)$$

Berman (1) has calculated that electromagnetic corrections to muon decay produce a factor of .9956 to the right hand side of I.2. Thus $G_\mu$ can be determined and

$$G_\mu M_p^2 = 1.011 \times 10^{-5} \qquad (I.3)$$

(with an error of around 1%)

Now let us consider weak interactions in which strongly interacting particles are involved, but limiting ourselves to nucleons and pions (i.e. considering only strangeness-preserving processes). Using the notation of Ref. 2 we now write the interaction Lagrangian in the form

$$\mathcal{L}_{int} = \frac{G}{\sqrt{2}} \left[V_\alpha + P_\alpha\right]\left[\bar{\nu}\gamma_\alpha(1+\gamma_5)e + \bar{\nu}'\gamma_\alpha(1+\gamma_5)\mu\right]^+ + H.C. \qquad (I.4)$$

where $V_\alpha$ and $P_\alpha$ are the vector and axial vector weak currents of the nucleons. Beta-decay experiments strongly suggest that

$G(V_\alpha + P_\alpha)$ is like its analogue in the leptonic case, namely $G_\mu \bar{p} \gamma_\alpha (1 + \gamma_5) n$ for neutron beta-decay. But we do not expect the two cases to be identical by any means; renormalisation effects due to the presence of strong interactions should produce new renormalised coupling constants $G_V$ and $(-G_A)$, i.e. in the limit of zero momentum transfer

$$
\begin{aligned}
G \langle p | V_\alpha | n \rangle &\rightarrow G_V \bar{u}_f \gamma_\alpha \tau_+ u_i & (a) \\
G \langle p | P_\alpha | n \rangle &\rightarrow -G_A \bar{u}_f \gamma_\alpha \gamma_5 \tau_+ u_i & (b)
\end{aligned}
\qquad (I.5)
$$

$G_V$ and $G_A$ are the usual Fermi and Gamow-Teller coupling constants of nuclear theory.

The Vector Current:

Surprisingly it turns out experimentally that $G_\mu = G_V$ to within about one per cent, and a 'universal' theory of weak interaction would require $G_\mu = G$. It would appear unlikely that the lack of renormalisation in the case of $G_V$ $(G_V = G)$ could be fortuitous. A theoretical reason for this lack of renormalisation was produced by Feynman and Gell-Mann (3) and some years earlier by Gershtein and Zeldovich (4). The argument follows by analogy with electromagnetism; because nucleons are coupled to the pion field one would expect a renormalisation of the coupling of the nucleon current to the photon field. But the charge of a proton, or of a charged pion, is numerically that of an electron. And the reason is simple; the total electric current of the strongly interacting particles is conserved, i.e.

$$
J_\mu^{el} = i \bar{\psi} \gamma_\mu \frac{1+\tau_z}{2} \psi + i \left[ \varphi^* T_z \partial_\mu \varphi - (\partial_\mu \varphi^*) T_z \varphi \right] \qquad (I.6)
$$

satisfies $\partial_\mu J_\mu^{el} = 0$.

Note that although the isotopic scalar nucleon term alone is conserved, the isotopic vector nucleon term is not, unless the pionic term is included. Pions, too, can carry electric charge.

So this suggests the explanation in the case of the weak vector current. If this current were conserved, one would expect no renormalisation of the coupling constant. Now under strong interactions the total isotopic spin current is conserved. In particular the (+) component is conserved

$$J_\mu^V = i\bar{\psi}\gamma_\mu\tau_+\psi + 2i\left[\varphi^* T_+ \partial_\mu \varphi - (\partial_\mu \varphi^*) T_+ \varphi\right] \qquad (I.7)$$

satisfies $\partial_\mu J_\mu^V = 0$

So, for low-energy transitions, just like electromagnetism, one finds a coupling constant which is unchanged by the presence of strong interactions. A further consequence of this choice is that the interaction at the second term of $J_\mu^V$ with the lepton current leads to the prediction of the process $\pi^- \rightarrow \pi^0 + e^- + \bar{\nu}$. This has not been observed. But as the theory implies no renormalisation of the direct interaction, the transition rate of this decay can be computed directly and the branching ratio to the normal mode of decay is of the order of $10^{-8}$, which is [redacted] small for existing experiments.

If we move from the low momentum transfer region, the strong interactions induce further terms in the matrix element $< p/V_\alpha/n >$. Again look at the electromagnetic current matrix element

We know that we can write

$$
\begin{aligned}
e\langle p|j_\mu^{el}|p\rangle &= ie\bar{\psi}_f\gamma_\mu\psi_i F_1^p(k^2) \\
&\quad - i\frac{e}{2M}\mu_p' k_\nu\bar{\psi}_f\sigma_{\mu\nu}\psi_i F_2^p(k^2) \\
e\langle n|j_\mu^{el}|n\rangle &= ie\bar{\psi}_f\gamma_\mu\psi_i F_1^n(k^2) \\
&\quad - i\frac{e}{2M}\mu_n k_\nu\bar{\psi}_f\sigma_{\mu\nu}\psi_i F_2^n(k^2) \\
F_1^n(0) &= 0
\end{aligned}
\tag{I.8}
$$

or in isotopic spin language

$$
\begin{aligned}
e\langle N|j_\mu^{el}|N\rangle &= \frac{ie}{2}\bar{\psi}_f\gamma_\mu\psi_i F_1^{S,V}(k^2) \\
&\quad - i\mu^{S,V}\frac{e}{4M}k_\nu\bar{\psi}_f\sigma_{\mu\nu}\psi_i F_2^{S,V}(k^2)
\end{aligned}
\tag{I.9}
$$

where S and V refer to the isotopic scalar and isotopic vector parts respectively. $F_{1,2}^{p}(k^2)$, $F_{1,2}^{n}(k^2)$ are the proton and neutron charge and magnetic moment form factors and have been measured by Hofstadter and his collaborators at Stanford. $\mu_p'$ refers to the anomalous proton magnetic moment.

$$
\begin{aligned}
F_1^S &= F_1^p + F_1^n \qquad & \mu^S F_2^S &= \mu_p' F_2^p + \mu_n F_2^n \\
F_1^V &= F_1^p - F_1^n \qquad & \mu^V F_2^V &= \mu_p' F_2^p - \mu_n F_2^n
\end{aligned}
\tag{I.10}
$$

$$
\mu^S = \mu_p' + \mu_n \qquad \mu^V = \mu_p' - \mu_n
$$

The measurements of the form factors have been carried out in the range $0 < k^2 < 40\, m_\pi^2$ at the present time.

When we come to the matrix element $<p/V_\alpha/n>$ we find from Lorentz invariance that

$$G<p|V_\mu|n> = i G_V \bar{\psi}_p \gamma_\mu \psi_n F_1(k^2) - \frac{iG_V}{2M} k_\nu \bar{\psi}_p \sigma_{\mu\nu} \psi_n F_2(k^2)$$

where $F_1$, $F_2$ are form factors and $F_1(0) = 1$.

We can go further and remember that the isotopic spin character of $< p/V_\alpha/n>$ is a (+) component of an isotropic vector. So as we only consider first order effects in the weak and electromagnetic coupling, and as the weak and electromagnetic currents are both relativistic currents, the matrix element $< p/V_\alpha/n>$ and the isotopic vector part of $<N/J_\mu^{el}/N>$ have exactly the same form. This could be seen before when $J_\mu^{el}$ and $J_\mu^{V}$ were explicitly written down.

So

$$G<p|V_\alpha|n> = i G_V \bar{\psi}_p \gamma_\alpha \tau_+ \psi_n F_1^V(k^2) - i \frac{G_V}{2M} \mu^V k_\nu \bar{\psi}_p \sigma_{\alpha\nu} \psi_n F_2^V(k^2) \qquad \text{(I.11)}$$

and $F_1^V$, $F_2^V$, $\mu^V$ are given by I.10.

The form factors $F_{1,2}^V(k^2)$ are taken to satisfy dispersion relations. The number of subtractions in those equations is not known but is often taken to be one for the charge form factor and none for the magnetic moment form factor, i.e.

$$F_1^V(k^2) = 1 - \frac{k^2}{\pi} \int_{4m_\pi^2}^{\infty} \frac{g_m\, F_1^V(k'^2)\, dk'^2}{k'^2(k'^2 + k^2 - i\epsilon)} \qquad \text{(I.12)}$$

$$F_2^V(k^2) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{g_m \, F_2^V(k'^2)\, dk'^2}{k'^2 + k^2 - i\epsilon} \tag{I.12}$$

These form factors are discussed in Appendix I.

The Axial Vector Current:

The first question which now must be asked for this current is whether it also is conserved. Experimentally $\frac{-G_A}{G} = 1.25$ so it appears that a strictly conserved axial vector current is impossible, although 1.25 is quite 'near' unity, and so perhaps the axial vector current is nearly conserved; that is to say, it is conserved in some limit. It should be recalled that even the vector current is only conserved neglecting electromagnetic mass differences and other electromagnetic effects. The concept of a partially conserved axial vector current is valid, but it can be shown (later) that $\partial_\alpha P_\alpha = 0$ implies that the charged pion does not decay in its normal mode. So any limit in which $\partial_\alpha P_\alpha$ were small would be one in which, for example, the energies involved were so large that the pion mass could be neglected.

Are there any simple alternatives other than one involving $m_\pi = 0$? It can be readily seen that $\partial_\alpha P_\alpha$ is a pseudoscalar and that the simplest pseudoscalar field is the pion field. So possibly

$$\partial_\alpha P_\alpha = \frac{ia}{\sqrt{2}} \pi^- \tag{I.13}$$

where a is a real constant and $\pi^-$ is the renormalised field operator which destroys a $\pi^-$. This possibility was exhaustively studied in Ref. 2 and some rather artificial models of the strong interactions in which it held were exhibited. But from I.13 immediately

$$\langle 0 | P_\alpha(x) | \pi^- \rangle = -\frac{a}{\sqrt{2}} \frac{k_\alpha}{m_\pi^2} \langle 0 | \pi^-(x) | \pi^- \rangle \qquad \text{(I.14)}$$

and from I.5b in the limit of small momentum transfer

$$\begin{aligned} G \langle p | \partial_\alpha P_\alpha | n \rangle &= -G_A(-ik_\alpha)\, \bar{u}_f \tau_+ \gamma_\alpha \gamma_5 u_i \\ &= 2M(-G_A)\, \bar{u}_f \tau_+ \gamma_5 u_i \end{aligned} \qquad \text{(I.15)}$$

But

$$\begin{aligned} \langle p | \partial_\alpha P_\alpha | n \rangle &= \frac{ia}{\sqrt{2}} \langle p | \pi^- | n \rangle \\ &\sim \frac{ia}{\sqrt{2}} \frac{i\sqrt{2}\, g_1}{k^2 + m_\pi^2}\, \bar{u}_f \tau_+ \gamma_5 u_i \end{aligned} \qquad \text{(I.16)}$$

where $g_1$ is the renormalised pion-nucleon coupling constant. I.16 should hold in the vicinity of $k^2 = -M_\pi^{\ 2}$.

So if $\langle p/\partial_\alpha P_\alpha/n\rangle$ is a slowly varying function of $k^2$ we have

$$a = -\frac{2M}{g_1} m_\pi^2 \left(-\frac{G_A}{G}\right) \qquad \text{(I.17)}$$

and from I.14 the decay rate of the charged pion is given by an easy calculation to be

$$\begin{aligned} \Gamma_{\pi\to\mu+\nu} &= \frac{G^2}{16\pi m_\pi} \frac{m_\mu^2}{m_\pi^2} \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 a^2 \\ &= \frac{G_A^2 M^4}{4\pi g_1^2} m_\pi \left(\frac{m_\mu}{M}\right)^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 \end{aligned} \qquad \text{(I.18)}$$

This formula was first derived by Goldberger and Treiman (5) using dispersion theoretic techniques and making some rather flagrant approximations.

But this theory is not satisfactory. Let us consider the most general matrix element of $P_\alpha$ between nucleon states

$$\langle p | P_\alpha | n \rangle = -\left(\frac{G_A}{G}\right) \bar{u}_f \tau_+ \gamma_\alpha \gamma_5 u_i \, \alpha(k^2)$$
$$+ i k_\alpha \bar{u}_f \tau_+ \gamma_5 u_i \, \beta(k^2)$$
$$+ i(p_\alpha^f + p_\alpha^i) \bar{u}_f \tau_+ \gamma_5 u_i \, \gamma(k^2)$$

$$k_\alpha = p_\alpha^f - p_\alpha^i \; ; \quad \alpha(0) = 1 \tag{I.19}$$

By time reversal invariance $\gamma(k^2) = 0$.

Define $K(k^2)$ by

$$\langle p | \lambda^- | n \rangle = i\sqrt{2} \, \bar{u}_f \tau_+ \gamma_5 u_i \, K(k^2) \tag{I.20}$$

where $\lambda^- = -\frac{i\sqrt{2}}{a} \partial_\alpha P_\alpha$

Then from I.19 and I.20,

$$2M\left(-\frac{G_A}{G}\right)\alpha(k^2) + k^2\beta(k^2) = a K(k^2) \tag{I.21}$$

So

$$-\frac{G_A}{G} = \frac{a}{2M} K(0) \tag{I.22}$$

$\alpha(k^2)$, $\beta(k^2)$, $K(k^2)$ should satisfy dispersion relations. $\beta(k^2)$ is the induced pseudoscalar term of Goldberger and Treiman (6) which is important for large momentum transfer and is significant in muon capture. Diagrammatically it is shown in Fig. I for this case. Thus $\beta(k^2)$ has a pion pole in its spectral representation and so therefore has $K(k^2)$.

a is chosen such that

$$K(k^2) = \frac{-g_1}{k^2 + m_\pi^2} + \frac{1}{\pi}\int_{9m^2}^{\infty} dM^2 \, \frac{\sigma_K(M^2)}{k^2 + M^2} \qquad \text{(I. 23(a))}$$

$$\alpha(k^2) = 1 - \frac{k^2}{\pi}\int_{9m_\pi^2}^{\infty} \frac{dM^2}{M^2} \, \frac{\sigma_\alpha(M^2)}{k^2 + M^2} \qquad \text{(b)}$$

$$\beta(k^2) = \frac{2g_1}{m_\pi^2} \, \frac{1}{k^2 + m_\pi^2} + \frac{1}{\pi}\int_{9m_\pi^2}^{\infty} dM^2 \, \frac{\sigma_\beta(M^2)}{k^2 + M^2} \qquad \text{(c)}$$

It has thus been assumed that $K(k^2)$ is sufficiently 'non-singular' to satisfy a dispersion relation with no subtractions. If, in addition, the second term in the equations is small in comparison with the first (due to the high mass terms in the denominators of the integrands) for low $k^2$, then a is given as before and the formal identity $\lambda^- = \pi^-$ is not required, but $\lambda^-$ in this region does behave like $\pi^-$. This more recent result was first worked out by Bernstein, Fubini, Gell-Mann, and Thirring (7). In any case it is somewhat arbitrary which pseudoscalar field with the quantum numbers of the pion is called the pion field.

## II. THE INTERMEDIATE BOSON HYPOTHESIS

Up to the present we have compared the weak currents to the electromagnetic current without commenting on a fundamental difference between the theories as they stand; namely that the interaction in the weak case is a four-farmion current-current interaction, whereas in electromagnetism the current is coupled to a boson field, which indicates the interaction. The other interaction that we know something about, the Yukawa interaction, also is by a boson. So it is tempting to ask whether in the case of weak interactions, the universality which seems to characterise all (or at least most) weak interactions comes from the presence of a boson field, coupled to the weak currents with the same (bare) coupling constant.

In such a case one would write

$$\mathcal{L}_{int} = g\sqrt{2}\, J_\alpha \phi_\alpha^{+} + H.C. \tag{II.1}$$

where $J_\alpha$ would represent the total weak interaction current, $\phi_\alpha$ would be the field operator appropriate to a charged spin one particle which we shall call $W^{\pm}$ and g would be a coupling constant.

If now we take the matrix element of $J_\alpha$ between nucleon states as before, we would write (for small momentum transfer)

$$g\langle p|J_\alpha|n\rangle = i g\sqrt{2}\, \bar{u}_f \gamma_\alpha \left(1 + \left(-\frac{G_A}{G}\right)\gamma_5\right)\tau_+ u_i \tag{II.2}$$

and using

$$g\langle e|J_\alpha|\nu\rangle = i g\sqrt{2}\, \bar{u}_f \gamma_\alpha (1+\gamma_5) u_i$$

for neutron β-decay we must have

$$\frac{2g^2}{4\pi} = \frac{G M_W^2}{4\pi\sqrt{2}} \qquad \text{(II.3)}$$

where $M_W$ is the mass of the boson.

$M_W$ must be larger than the K-meson mass, otherwise K would decay much faster than it is observed to do. W decays into leptons, $2\pi$, $3\pi$, etc. and the coupling strength is given by

$$\frac{2g^2}{4\pi} > 1.6 \times 10^{-7} \qquad \text{(II.4)}$$

Thus the decay rates into lepton are given (8) by

$$\Gamma_{W\to\mu+\nu} \sim \Gamma_{W\to e+\nu} = \frac{G M_W^3}{6\pi\sqrt{2}} > 8 \times 10^{16}\ sec^{-1} \qquad \text{(II.5)}$$

As the matrix element for muon decay is now of the form

$$\frac{G M_W^2}{\sqrt{2}} \; \frac{\bar{\nu}'\gamma_\alpha(1+\gamma_5)\mu\left(\delta_{\alpha\beta} + \frac{k_\alpha k_\beta}{M_W^2}\right)\bar{e}\,\gamma_\beta(1+\gamma_5)\,\nu}{k^2 + M_W^2} \qquad \text{(II.6)}$$

a nonlocality of size $M_W^{-1}$ would alter slightly the predictions of the four-fermion local theory; in general, effects would be most noted for large momentum transfer.

The Michel parameter in muon decay is now given by $\rho = .75 + \frac{m_\mu^2}{3M_W^2}$ approximately, which is consistent with present experimental results. In fact, the latest measurements (9) give $\rho = .780 \pm 0.025$ with $M_W = M_K$, $\rho = .765$, which is within the experimental error.

Also in muon capture, momentum transfer is appreciable and the effects of W existing might be noticed. Of course, in the limit of infinite $M_W$, the theory reduces to the theory of Ch. I.

Isotopic Character of W:

There are several possible theories of the intermediate bosons and really, for us, it does not matter which we choose, as the forthcoming calculations will refer only to a charged boson, or equivalently, a weak current carrying electric charge. But in order to indicate the possibilities of a theory of weak interactions with spin one bosons, one of the simplest theories, that of Lee and Yang (10), will be described briefly.

This theory assumes that all weak interactions (including the strangeness changing weak interactions, which will not concern us after this section) are transmitted through the boson field W. Then it is also required that the $\Delta I = \frac{1}{2}$ rule holds for strangeness non-conserving decays, and that $\Delta S = \pm 2$ interactions are absent. The equality of the coupling constant connecting $W^{\pm}$ with $(\bar{e}\nu)$, $(\bar{\mu}\nu')$, $(\bar{n}p)$ is known from experiment.

It is clear first of all that if the simplest nucleon and strangeness changing currents are written down

$$J_\alpha = f_1 (\bar{n} p)$$
$$S_\alpha = f_2 (\bar{\Lambda} p) \qquad \text{(II. 7)}$$

then under isotopic rotations $J_\alpha$ forms a component of a vector and $S_\alpha$ of an isotopic doublet. Then to satisfy a $|\Delta I| = \frac{1}{2}$ rule some decay such as $\Lambda \rightarrow p + \pi^-$ we will need a theory including a neutral current and hence a neutral W. Then it would seem possible that isotopic spin was conserved in the W-J coupling while $|\Delta \underline{I}| = \frac{1}{2}$ comes from the W-S coupling. So take the triplet ($W^+$, $W^o$, $W^-$) as an isotopic vector just like ($\pi^+$, $\pi^o$, $\pi^-$), i.e. the W-J coupling is

$$f_1 \left\{ (\bar{n} p) W^- + \frac{1}{\sqrt{2}} \left[ \bar{p} p - \bar{n} n \right] W^o + (\bar{p} n) W^+ \right\} \tag{II.8}$$

whereas the W-S coupling is

$$f_2 \left\{ (\bar{\Lambda} p) W^- - \frac{1}{\sqrt{2}} (\bar{\Lambda} n) W^o \right\} + H.C. \tag{II.9}$$

thus producing a $|\Delta \underline{I}| = \frac{1}{2}$ rule.

Unfortunately under this scheme

$$n \longleftrightarrow \Lambda + W^o \quad \text{and}$$

$$\Lambda \longleftrightarrow n + W^o$$

So $\quad n + n \longleftrightarrow \Lambda + W^o + n \longleftrightarrow \Lambda + \Lambda$

which contradicts $\Delta S \neq 2$ for first order weak interactions.

Hence we are led to the other simple alternative in which $\underline{I}$ is conserved in W-S couplings, and $|\Delta \underline{I}| = \frac{1}{2}$ arises out of the W-J couplings. So here $W^+$ and $W^o$ form an isotopic doublet (like the $K^+$, $K^o$ doublet) and $W^o$, $\overline{W}^o$ are distinct particles ($\overline{W}^o$, $W^-$ also form a doublet). So the W-S coupling is

$$f_2 \left\{ (\bar{\Lambda} p) W^- + (\bar{\Lambda} n) \overline{W}^o \right\} + f_2 \left\{ (\bar{n} \Lambda) W^o + (\bar{p} \Lambda) W^+ \right\} \tag{II.10}$$

The W-J coupling is now

$$f_1 \left\{ (\bar{n} p) W^- - \frac{1}{2} \left[ (\bar{p} p - \bar{n} n) \right] \overline{W}^o \right\} + H.C.$$

or

$$f_1\left( (\bar{n}p)W^- + \frac{1}{\sqrt{2}}\left[(\bar{p}p - \bar{n}n)\right]W_a^0 + (\bar{p}n)W^+ \right)$$

(II.11)

where $$W_a^0 = \frac{-W^0 - \overline{W}^0}{\sqrt{2}}$$

(II.12)

and define $$W_b^0 = \frac{i(W^0 - \overline{W}^0)}{\sqrt{2}}$$

$W_a{}^o$, $W_b{}^o$ are the same sort of entities as $K_1{}^o$ and $K_2{}^o$. We can even take $W^+$, $W_a{}^o$, $W^-$ as an isotopic vector which conserves isotopic spin in the W-J interaction, letting the $|\Delta \underline{I}| = \frac{1}{2}$ transition come from a transition among the W fields themselves.

i.e. $$(W^+, W_a^0, W^-) \longleftrightarrow \begin{matrix}(W^+, W^0) \\ (\overline{W}^0, W^-)\end{matrix}$$

The preceding difficulty of $n + n \leftrightarrow \Lambda + \Lambda$ does not now arise as

$$n + n \longleftrightarrow \Lambda + W_b^0 + n \longleftrightarrow \Lambda + \Lambda$$

exactly cancels

$$n + n \longleftrightarrow \Lambda + W_a^0 + n \longleftrightarrow \Lambda + \Lambda$$

The main point of this discussion for our purposes is to show that a theory of intermediate bosons can be developed in which some triplet of the W fields transform just like pions in isotopic spin space and form an isotopic vector. This is useful in calculations even though the charged W only will be required. $f_1$ of this chapter corresponds to $g\sqrt{2}$ of the last.

Further details of the properties of the neutral W particles and of the strangeness changing weak interactions implied by this scheme are to be found in Ref. 10.

## III. THE INTERACTION $W + N \rightarrow \pi + N$

Introduction:

Now that the matrix element of the weak current between nucleon states has been found, more involved processes can be considered. The simplest of these is pion production by the weak current off nucleons (from now, called weak pion production). In analogy with the case of pion electroproduction (production of pions by virtual photons--photons off the mass shell) we shall consider the process

$$W + N \longrightarrow \pi + N$$

where W represents an intermediate boson off its mass shell (in general). In the limit of the boson mass $M_W$ becoming infinite, the interaction becomes that expected from the four-fermion theory. If the W is virtual, the physical situation corresponds to a neutrino collision with a nucleon, forming a nucleon, a pion and a lepton (muon or electron) as shown in Fig. II. The scattering amplitude for this process is given by

$$T = \frac{-ig^2/2}{k^2 + M_W^2} \langle p_2 q | j_\mu | p_1 \rangle \left( \delta_{\mu\nu} + \frac{k_\mu k_\nu}{M_W^2} \right) \bar{u}_{t_2} \gamma_\nu (1 + \gamma_5) u_{t_1} \qquad \text{(III.1)}$$

where $p_1$ and $t_1$ are the four-momenta of the initial nucleon and neutrino, $p_2$, $t_2$ and q those of the final nucleon, lepton, and pion; $j_\mu$ is the total weak current, and $k = t_2 - t_1$ is four-momentum transfer to the pion-nucleon system, in other words the momentum of the virtual W.

Let us write

$$\mathcal{E}_\mu = i\bar{u}_{\ell_2} \gamma_\mu (1+\gamma_5) u_{\ell_1} \tag{III. 2}$$

and

$$H_\mu = \langle p_2 q | J_\mu | p_1 \rangle \tag{III. 3}$$

is the matrix element we want to evaluate. For convenience we will write

$$\mathcal{M} = H \cdot e = \mathcal{M}_V + \mathcal{M}_A \tag{III. 4}$$

where $e_\mu$ is an arbitrary four-vector, and $\mathcal{M}_V$, $\mathcal{M}_A$ are the parts of the matrix element coming from the vector and axial vector currents.

In the case of the final lepton being an electron, and neglecting the mass of the electron as an 'electrical mass difference' between the mass of it and that of the neutrino, similar to the mass difference between the proton and neutron (which we also neglect), we have from III. 2

$$k \cdot \mathcal{E} = 0 \tag{III. 5}$$

Thus, in this case

$$T = \frac{-g^2\sqrt{2}\; H \cdot \mathcal{E}}{k^2 + M_W^2} \tag{III. 6}$$

and

$$\mathcal{E}_0 = \frac{\underline{k} \cdot \underline{\mathcal{E}}}{k_0} \tag{III. 7}$$

a relation which is useful in calculations. The amplitude T is a function of the scalars

$$\nu = -\frac{P\cdot k}{M}, \quad \nu_B = \frac{q\cdot k}{2M}, \quad \lambda^2 = k^2$$
$$P = \frac{1}{2}(p_1 + p_2) \qquad \text{(III. 8)}$$

We learned in Ch. II that we could take W as a vector in isotopic spin space. So the isotopic dependence of $\mathcal{M}$ is just like pion-nucleon scattering; that is, let β be the isotopic state of the W and α that of the pion (α, β = 1, 2, 3).

Then

$$O_i^{\alpha\beta} = \delta_{\alpha\beta}\, O_i^{+} + \frac{1}{2}[\tau_\alpha, \tau_\beta]\, O_i^{-} \qquad \text{(III. 9)}$$

In terms of total isotopic spin

$$O_i^{+} = \frac{1}{3}\left( O_i^{(1/2)} + 2O_i^{(3/2)} \right)$$
$$O_i^{-} = \frac{1}{3}\left( O_i^{(1/2)} - O_i^{(3/2)} \right) \qquad \text{(III. 10)}$$

where we have put

$$\mathcal{M} = \sum_i M_i O_i \; ; \qquad M_i$$

are expressions involving gamma-matrices and $O_i$ are invariant scalar amplitudes.

Weak Current Theory

We can first see what general results can be obtained from the theories of the weak currents outlined in Chapter I. First consider

$$\mathcal{M}_V = \langle p_2 q | V_\mu | p_1 \rangle\, e_\mu = H^V \cdot e$$

together with $\partial_\mu V_\mu = 0$

So $$\langle p_2 q | \partial_\mu V_\mu | p_1 \rangle = 0 = -i k_\mu \langle p_2 q | V_\mu | p_1 \rangle$$

or $$k \cdot H^V = 0 \qquad \text{(III.11)}$$

This is analogous to gauge invariance in electromagnetism. Stated formally, gauge invariance asserts here that whenever the polarisation vector $e_\mu$ is replaced by the photon momentum $k_\mu$ in the amplitude for an electromagnetic process, the amplitude vanishes. We have

$$\mathcal{M}_V = H^V \cdot e \qquad H^V \cdot k = 0$$

so $\mathcal{M}_V$ is a gauge invariant in precisely the way the concept is used in electromagnetism.

Also, even if $k \cdot \epsilon \neq 0$, the vector part of T is given by

$$T_V = \frac{-g^2/2 \; H^V \cdot \epsilon}{k^2 + M_W^2} \qquad \text{(III.12)}$$

Now consider

$$\mathcal{M}_A = \langle p_2 q | P_\mu | p_1 \rangle e_\mu$$

together with $\partial_\alpha \underline{P}_\alpha \simeq i a \underline{\pi}$ (III.13)

(The vector denoted here is an isotopic vector. The same underlining will be used where necessary to denote an ordinary, spacelike, three-vector. It should be obvious from the context which type of vector is meant.)

$$a = -\frac{2M}{g_1} m_\pi^2 \left( -\frac{G_A}{G} \right)$$ from eqn. (I.17)

It was shown in Chapter I that III.13 holds for small $k^2$ in the sense of eqn. I.23(a) being correct.

$$\text{So} \quad \langle p_2 q | \partial_\mu P_\mu | p_1 \rangle = -i k_\mu \langle p_2 q | P_\mu | p_1 \rangle$$

$$= i a \langle p_2 q | \pi | p_1 \rangle = + \frac{a \langle p_2 q | p_1 k \rangle}{k^2 + m_\pi^2}$$

or

$$k \cdot H^A = \frac{i a \; \langle p_2 q | p_1 k \rangle}{k^2 + m_\pi^2} \qquad \text{(III.14)}$$

where $< p_2 q/p_1 k >$ is the scattering amplitude for pion-nucleon scattering; the incoming pion having momentum k where $k^2$ is not necessarily $(-m_\pi^2)$. This relation connects the matrix element for weak pion production with that for pion-nucleon scattering.

The Born Approximation:

As we are interested in calculating an amplitude which involves a strong interaction, a dispersion theoretic technique will be used (Chapter IV). But the Born approximation is invaluable in providing a guide to the necessary pole terms and subtractions in the one-dimensional dispersion relation which will be employed.

The appropriate Feynman diagrams involving one-particle intermediate states for the vector part of the amplitude are shown in Fig. 3. Any strong two pion or three pion interaction would imply additional terms to those here considered (for simplicity, treating the resonances as particles in their own right, as indeed they may be). We will omit these terms now, and look at them later.

The expression for the Born approximation is thus

$$g_1 \frac{\gamma_5 \cdot i\gamma\cdot q}{2q\cdot p_2 - 1} \gamma_\alpha \gamma_\beta \left( \gamma_\mu F_1^V(\lambda^2) - \frac{\mu^V}{2M} \sigma_{\mu\nu} k_\nu F_2^V(\lambda^2) \right) e_\mu$$

$$+ g_1 \left( \gamma_\mu F_1^V(\lambda^2) - \frac{\mu^V}{2M} \sigma_{\mu\nu} k_\nu F_2^V(\lambda^2) \right) \gamma_\beta \gamma_\alpha e_\mu \frac{i\gamma\cdot q\, \gamma_5}{2p_1\cdot q + 1}$$

$$- g_1 [\gamma_\alpha, \gamma_\beta] i\gamma_5 F_\pi(\lambda^2) \frac{(2q-k)\cdot e}{(2q-k)\cdot k}$$

(III.15)

$F_\pi(\lambda^2)$ is the electromagnetic form factor of the pion (see Ref. 11) which comes from (c) in Fig. 3, and is further discussed in Appendix I. We have taken $m_\pi$ as unity. Expression III.15 is to be taken between Dirac spinors for the initial and final nucleon states, and isotopic spinors for these states.

We test the gauge invariance of this expression by replacing $e_\mu$ by $k_\mu$. We obtain

$$g_1 [\gamma_\alpha, \gamma_\beta] i\gamma_5 \left( F_1^V(\lambda^2) - F_\pi(\lambda^2) \right) \qquad \text{(III.16)}$$

There is no theoretical reason why these three terms should be gauge invariant on their own; only the total amplitude must be gauge invariant. It would however be very convenient if this were the case, especially as the calculation in the next chapter is done essentially using the Born terms as a first approximation; if gauge invariance is not present initially, we would not be able to impose it on the complete amplitude.

In the case of an electron being produced in our reaction, $k \cdot \epsilon = 0$, and $T^V$ only involves $H^V \cdot \epsilon$. So if we replace $e_\mu$ by $\epsilon_\mu$ in III.15 and add

$$g_1 [\gamma_\alpha, \gamma_\beta] \, i\gamma_5 \, \frac{F_\pi(\lambda^2) - F_1^V(\lambda^2)}{\lambda^2} \, k \cdot \epsilon \qquad \text{(III.17)}$$

we have not changed the value of the expression but we now have gauge invariance in this case.

If the lepton produced is a muon we cannot do this. But note that for $\lambda^2 = 0$, $F_1^V(0) = F_\pi(0)$. The structure of $F_\pi(\lambda^2)$ and $F_1^V(\lambda^2)$ are fairly similar; it is believed that for small $\lambda^2$ the dominant cause of the shape of both form factors is the $J = 1$, $I = 1$ pion-pion resonance (Appendix I). It would be nice if $F_\pi(\lambda^2)$ and $F_1^V(\lambda^2)$ were roughly equal for small $\lambda^2$; then we would not have this problem. A pseudo-reason for believing this would be that in similar reactions — e.g. photoproduction, electroproduction — the Born approximation corresponding to the diagrams in Fig. 3 conspires to be gauge invariant in fact.

The expression for the Born approximation for $\mathcal{M}_A$ is

$$g_1 \frac{\gamma_5 \, i\gamma \cdot q}{2p_2 \cdot q - 1} \gamma_\alpha \gamma_\beta \left( \gamma_\mu \gamma_5 \left( -\frac{G_A}{G} \right) \alpha(\lambda^2) + i k_\mu \gamma_5 \beta(\lambda^2) \right) e_\mu$$

$$+ \; g_1 \left( \gamma_\mu \gamma_5 \left( -\frac{G_A}{G} \right) \alpha(\lambda^2) + i k_\mu \beta(\lambda^2) \right) \gamma_\beta \gamma_\alpha \frac{i\gamma \cdot q \, \gamma_5}{2p_1 \cdot q + 1} e_\mu$$

(III.18)

corresponding to the diagrams in Fig. 4. (again ignoring pion-pion interactions).

Two and Three Pion Exchange:

We can also consider two pion and three pion exchange. This can be done if we take the view (supported by experiments on the electromagnetic form factors of the nucleon — see Appendix I) that there are resonances in the I = 1, J = 1 two pion system at an energy of about 4.7 $m_\pi$ in the centre of mass of the system, and a resonance I = 0, J = 1 three pion system at a C.M. energy of around 3 $m_\pi$. Alternatively, following the ideas of Gell-Mann and Sakurai (12, 13, 14) among others, we can consider these resonances as occurring through the existence of vector mesons ρ, ω, coupled to conserved currents.

The diagrams to be computed are shown in Fig. 5. Fig. 5(a) for the vector current coupling to the nucleon via an ω has an amplitude

$$\frac{f_{V\omega\pi}\, F_{V\omega\pi}(\lambda^2)}{(q-k)^2+m_\omega^2}\, \varepsilon_{\sigma\rho\mu\nu}\, k_\sigma\, q_\rho\, e_\mu\, \gamma_{\omega NN}\, \bar{u}_f \Big[ \gamma_\nu F_1^\omega((q-k)^2) - \frac{\mu^S}{2M} \sigma_{\nu\lambda} (q-k)_\lambda F_2^\omega((q-k)^2)\Big] \delta_{\alpha\beta}\, u_i \qquad \text{(III.19)}$$

where $f_{V\omega\pi}$ and $\gamma_{\omega NN}$ are the renormalised coupling constants for the $V\omega\pi$ and ωNN vertices, $F_{V\omega\pi}(\lambda^2)$ is the form factor for the $V\omega\pi$ vertex, and $F_{1,2}^\omega$ are the 'charge' and 'magnetic moment' form factors for the ωNN vertex. $\mu^S$ is just the scalar anomalous magnetic moment of the nucleon.

The expression III.19 can alternatively be written

$$\frac{f_{V\omega\pi}\,\gamma_{\omega NN}\,F_{V\omega\pi}(\lambda^2)}{(q-k)^2+m_\omega^2}\;\bar{u}_f\Big\{2\gamma_5\Big[\{P,\gamma\}+\tfrac{1}{2}iM\{\gamma,\gamma\}\Big]\times$$

$$F_1^{\omega}((q-k)^2)\;+\;\frac{\mu^S}{2M}\,i\gamma_5\Big[\frac{\lambda^2-4M\nu_B}{2}\;\{\gamma,\gamma\}$$

$$+\;\frac{2M\nu_B-\lambda^2}{M\nu_B}\;\{P,q\}\;-\;\frac{\nu}{\nu_B}\,\{k,q\}\Big]\,F_2^{\omega}((q-k)^2)\Big\}\,\delta_{\alpha\beta}\,u_i$$

(III. 20)

where $\{a,b\} = a\cdot e\; b\cdot k - a\cdot k\; b\cdot e$

Approximately, we can put $F_{V\omega\pi}(\lambda^2) = 1$. Also,

$$\frac{f_{V\omega\pi}}{g} = \frac{2f_{\gamma\omega\pi}}{e}$$

and $f_{\gamma\omega\pi}$ can be determined from the rate of $\pi^0$ decay (14). [redacted] $\gamma_{\omega NN}$ and $F_{1,2}$ are discussed in Ref. 14 and in Appendix I.

In order to compute the matrix element for Fig. 5(b) we need to know the amplitude $< \pi, \rho / P_\mu / 0 >$. This would be a useful quantity if it were known, as it is connected to the amplitude for $\pi$-$\rho$ scattering, and also to the axial vector form factors $\alpha(\lambda^2)$, $\beta(\lambda^2)$. All we know is its pion pole term at present; this gives

$$\frac{-ia\,k\cdot e}{m_\pi^2\,(\lambda^2+m_\pi^2)}\;\frac{\gamma_{\pi\pi\rho}\,F_{\pi\pi\rho}[(q-k)^2]}{(q-k)^2+m_\rho^2}\;(q+k)_\mu\;\times$$

(III. 21)

$$\gamma_{\rho NN} \bar{u}_f \left[ \gamma_\mu F_1^{\rho}((q-k)^2) - \frac{\mu^V}{2M} \sigma_{\mu\lambda} (q-k)_\lambda \right] [\tau_\alpha, \tau_\beta] u_i \qquad \text{(III. 21)}$$

All the quantities appearing in III. 21 are reasonably well-known (Ref. 14 and Appendix I).

Now that these terms have been written down they will be omitted in the calculations in the next chapter. The effects due to the ρ and ω have not yet been experimentally verified in photoproduction, a subject which has been exhaustively studied in recent years. They do not make any significant contribution to the pion-nucleon 3-3 resonance, and so very careful experiments have to be done in low-energy photoproduction to see the effects and to measure the relevant coupling constants. This will no doubt be done in the near future, but for weak pion production, with its vastly smaller cross-section, we will concentrate on the terms most likely to be dominant.

Terms involving $V\rho\pi$ and $P\omega\pi$ vertices have been omitted as we have made the tacit assumption of the GP invariance of the weak currents.

## IV. DISPERSION RELATIONS AND THE 3-3 RESONANCE

Vector Part:

We can write

$$\mathcal{M} = \sum M_i O_i$$

where $M_i$ are relativistic invariant forms involving gamma matrices and scalars formed from k, q, e, P; each $M_i$ is linear in e (as we take weak interaction only to first order). There are eight independent $M_i$ allowed, allowing for the Dirac equation for the initial and final nucleon spinors and energy-momentum conservation. $O_i$ are functions of $\nu$, $\nu_B$, $\lambda^2$ only and are taken to obey dispersion relations. In the case of $\mathcal{M}_V$ we have the further requirement of gauge invariance which reduces the number of $M_i$ to six.

We will take as fundamental forms for $\mathcal{M}_V$ (following F.N.W., Ref. 15)

$$
\begin{aligned}
M_A &= \tfrac{1}{2} i \gamma_5 \{\gamma, \gamma\} && (+) \\
M_B &= 2 i \gamma_5 \{P, q\} && (+) \\
M_C &= \gamma_5 \{\gamma, q\} && (-) \\
M_D &= 2\gamma_5 \left[ \{\gamma, P\} - \tfrac{1}{2} i M \{\gamma, \gamma\} \right] && (+) \\
M_E &= i \gamma_5 \{k, q\} && (-) \\
M_F &= \gamma_5 \{k, \gamma\} && (-)
\end{aligned}
\qquad \text{(IV.1)}
$$

where $\{a,b\} \equiv a \cdot e\ b \cdot k - a \cdot k\ b \cdot e$ is automatically gauge invariant and

$$\mathcal{M}_V = A M_A + B M_B + \cdots + F M_F \tag{IV.2}$$

The signs in parentheses in IV.1 refer to the crossing symmetry of the invariants.

From the isotopic spin decomposition (III.9) we see that (+) amplitudes are even and (-) are odd under crossing.

So we write one-dimensional dispersion relations for the energy variable $\nu$, keeping the momentum transfer variable $\nu_B$ constant, in the form

$$\begin{aligned} A_i(\nu, \nu_B, \lambda^2) &= C_i(\nu, \nu_B, \lambda^2) + R_i(\lambda^2)\left(\frac{1}{\nu_B - \nu} \pm \frac{1}{\nu_B + \nu}\right) + \frac{1}{\pi}\int_{\nu_0}^{\infty} d\nu'\, \mathrm{Im}\, A_i(\nu', \nu_B, \lambda^2)\left(\frac{1}{\nu' - \nu}\right. \\ &\left. \pm \frac{1}{\nu' + \nu}\right) \qquad \nu_0 = \nu_B + 1 + \frac{1}{2M} \\ &\qquad\qquad\qquad\quad i = 1, \ldots, 6 \end{aligned} \tag{IV.3}$$

and the $\pm$ sign depends on the crossing symmetry. We are guided to the values of $C_i$ and $R_i$ by the Born approximation III.15 together with the additional term III.17.

Then we have

$$\begin{aligned} R[A^{\pm}] &= -f F_1^V(\lambda^2) \\ R[B^{\pm}] &= f F_1^V(\lambda^2) / 2M\nu_B \\ R[C^{\pm}] &= R[D^{\pm}] = \frac{f F_2^V(\lambda^2)}{2M} \mu^V \\ R[E^{\pm}] &= R[F^{\pm}] = 0 \end{aligned} \tag{IV.4}$$

and

$$C_A = C_B = C_C = C_D = C_F = 0$$

$$C_{E^+} = 0$$

$$C_{E^-} = -\frac{4Mf}{\lambda^2}\left(\frac{-2F_\pi(\lambda^2)}{2q\cdot k-\lambda^2} + \frac{F_1^V(\lambda^2)}{q\cdot k}\right) \qquad \text{(IV.5)}$$

[ $C_A^+$, $C_B^+$, $C_D^+$ and $C_E^+$ have contributions from the three pion intermediate state; these can be read off from equation III.20] .

From the well-known theorem of F.N.W. we know that the phases of the matrix elements for this process in the centre of mass system are just the pion-nucleon phase shifts for corresponding energies. We limit ourselves to low-energy final states so that the 3-3 resonance is dominant. Then below and in the resonance region, only the 3-3 component of $A_i$ has any appreciable imaginary part. So clearly, the first step in the evaluation of the dispersion relations is the substitution of the 3-3 contributions in place of the total amplitudes in the dispersion integrals.

We thus seem to be working in the same spirit as Chew and his collaborators, in the original papers using dispersion techniques for pion-nucleon scattering and photoproduction in the 3-3 resonance region (Ref. 16). This can be justified for this region, as the more modern studies using the double dispersion approach by the Berkeley and CERN groups (Ref. 17-21) show that the solutions to the partial-wave dispersion relations for the 3-3 state in the static limit and to first order in $\frac{1}{M}$ as in CGLN are essentially the same. Further, the CERN groups using the method of Cini and

Fubini (Ref. 19) show that the one-dimensional dispersion relation in CGLN (written down in the same way as our equations) are correct in general in the low-energy region provided the strong pion-pion effects are included — and these can simply be included by taking these effects as due to two spin one particles $\rho$, $\omega$ and just calculating the Born approximation as we did in Chapter 3. Bowcock, Cottingham and Lurie (Ref. 21) show that in pion-nucleon scattering, the term involving the exchange of a $\rho$-meson brought agreement with the observed s and p-wave non-resonant scattering phase-shifts, but contribute little to the 3-3 phase-shifts.

We should emphasize, however, that the form of the dispersion equations that we have assumed is very much an assumption. In order to proceed with any semblance of rigour, one should first assume some form of double dispersion relation (with no subtractions, say) and derive from these the one-dimensional relations with accompanying pole terms and subtractions. But for our case, in practice, this is not really necessary for a computation, as the actual structure of the dispersion relations is not as important as the existence of the final state resonance. When we put this into the equations from the start, we essentially are using a not-very-glorified Breit-Wigner technique. Presumably, in order to obtain theoretically the position of the resonance we would have to be rather more careful. But this has not yet been done, even in the pion-nucleon scattering problem.

In order not to make the approximations of CGLN and FNW (at least at first) we use the method of Blankenbecler and Gartenhaus

(22) discussed in Appendix II. This method assumes the dominance of the 3-3 final state, but treats crossing and recoil exactly, and does not expand in either partial waves or in powers of $\frac{1}{M}$. It is worse to make expansions in $\frac{1}{M}$ in weak production (and electroproduction) than in photoproduction because no longer are there only two terms of order $\frac{1}{M}$ to consider, namely $\frac{1}{M}$ and $\frac{\omega}{M}$ ($\omega = W-M$, W total C.M. energy) which are small. Terms like $\lambda^2/M\omega$ also appear in our case and for wide-angle scattering, which is of importance for the measurement of form-factors, this term is not small. It is possible for $\lambda \sim 1$ Bev and still produce a resonant pion-nucleon final state.

The method assumes the phases of the amplitudes in the dispersion relation known. Then the dispersion relations can be solved formally, and a first approximation which could be iterated if necessary to obtain a better approximation is given by

$$A_i(x, \nu_B, \lambda^2) = A_i^{B.A}(x, \nu_B, \lambda^2) + \frac{e^{i\delta_{33}(x)}}{\pi} \int_{1+\frac{1}{2M}}^{\infty} dy \sin\delta_{33}(y)\, a_i(y, \nu_B, \lambda^2)\, e^{\Delta(y, \nu_B, \lambda^2)} \times \left( \frac{1}{y-x-i\epsilon} \pm \frac{1}{y+x+2\nu_B} \right) \qquad \text{(IV.6)}$$

where

$$x = \nu - \nu_B = \frac{W^2 - M^2}{2M}$$

$\delta_{33}(x)$ is the 3-3 phase shift

$$\Delta(x', x, \nu_B) = \rho(x, \nu_B) - \rho(x', \nu_B)$$

$$\rho(x, \nu_B) = \frac{P}{\pi} \int_{1+\frac{1}{2M}}^{\infty} dy\, \delta_{33}(y) \left[ \frac{1}{y-x} + \frac{1}{y+x+2\nu_B} \right]$$

$a_i(x, \nu_B, \lambda^2)$ is the 3-3 projection of $A_i^{B.A.}(x, \nu_B, \lambda^2)$

(IV.7)

We see that

$$e^{\Delta(y,x,\nu_B)} = e^{\rho(x,\nu_B) - \rho(y,\nu_B)}$$

and for x in the resonance region, this function can be expanded in the form

$$e^{\Delta} = 1 + a(y-x) + \ldots\ldots$$

where under the integral the second term in the expansion is small compared with the first; as the integral is sharply peeked around $y = x_r$.

So now our enhancement term is approximately

$$\frac{e^{i\delta_{33}(x)}}{\pi} \int_{1+\frac{1}{2M}}^{\infty} dy \sin\delta_{33}(y)\, a_i(y) \left( \frac{1}{y-x-i\epsilon} \pm \frac{1}{y+x+2\nu_B} \right)$$

$$= i e^{i\delta_{33}(x)} a_i(x) \sin\delta_{33}(x)$$

$$+ \frac{e^{i\delta_{33}(x)}}{\pi} P \int_{1+\frac{1}{2M}}^{\infty} dy \sin\delta_{33}(y)\, a_i(y)$$

$$\times \left( \frac{1}{y-x} \pm \frac{1}{y+x+2\nu_B} \right)$$

(IV.8)

Now the first term gives the right phase as demanded by unitarity to the result; when put with the 3-3 part of $A_i^{B.A}$ one gets

$$a_i(x) + i e^{i\delta_{33}(x)} a_i(x) \sin\delta_{33}(x) = e^{i\delta_{33}(x)} \sin\delta_{33}(x) \cos\delta_{33}(x) \times a_i(x)$$

which vanishes at resonance. Also the crossed term under the integral is small in this region and so around resonance we have

$$A_i(x, \nu_B, \lambda^2) = A_i^{B.A.}(x, \nu_B, \lambda^2) + \frac{e^{i\delta_{33}(x)}}{\pi} P\int_{1+\frac{1}{2m}}^{\infty} dy \, \frac{a_i(y)\sin\delta_{33}(y)}{y-x} \qquad \text{(IV.9)}$$

The principal value integral evidently gives the enhancement to the 3-3 state, and in general will give the enhancement for a resonance with width $\Gamma$ say. But we know that a Breit-Wigner expression satisfies the dispersion equations in the resonance region approximately; hence we must expect that eq. IV.9 simulates

$$A_i(x, \nu_B, \lambda^2) = A_i^{B.A.}(x, \nu_B, \lambda^2) + \frac{a_i(x, \nu_B, \lambda^2)}{1 - \frac{x}{x_r} - i\Gamma} \qquad \text{(IV.10)}$$

or even in the resonance region

$$A_i(x, \nu_B, \lambda^2) = \frac{a_i(x, \nu_B, \lambda^2)}{1 - \frac{x}{x_r} - i\Gamma} \qquad \text{(IV.11)}$$

that is, just the Born approximation to go into the 3-3 state with an enhancement factor.

So all we have to do to use any of these expressions for the complete amplitude is to calculate the functions $a_i(x, \nu_B, \lambda^2)$.

First we project out the 3/2 spin states of $A_i(x, \nu_B, \lambda^2)$. So we write the matrix element $\mathcal{M}_V = \sum A M_A$ in terms of two-component spinors

$$\bar{u}_f \mathcal{M}_V u_i = \chi_f^* \mathcal{F}_V \chi_i$$

where $$\mathcal{F}_V = \sum_{i=1}^{6} \mathcal{F}_i \Sigma_i \tag{IV.12}$$

and $\Sigma_i$ are defined by Ref. 23.

$$\Sigma_1 = i\underline{\sigma}\cdot\underline{a} \qquad \Sigma_2 = \frac{\underline{\sigma}\cdot\underline{q}\;\underline{\sigma}\cdot(\underline{k}\times\underline{a})}{qk}$$

$$\Sigma_3 = \frac{i\underline{\sigma}\cdot\underline{k}\;\underline{q}\cdot\underline{a}}{qk} \qquad \Sigma_4 = \frac{i\underline{\sigma}\cdot\underline{q}\;\underline{q}\cdot\underline{a}}{q^2}$$

$$\Sigma_5 = \frac{i\underline{\sigma}\cdot\underline{k}\;\underline{k}\cdot\underline{a}}{k^2} \qquad \Sigma_6 = \frac{i\underline{\sigma}\cdot\underline{q}\;\underline{k}\cdot\underline{a}}{qk} \tag{IV.13}$$

where $\underline{a}$ is the gauge-invariant three vector given by

$$\underline{a} = \underline{e} - \frac{e_0}{k_0}\underline{k} \quad \left( = \underline{e} - \frac{\underline{k}\cdot\underline{e}}{k_0^2}\underline{k} \text{ when } \underline{k}\cdot\underline{e} = k_0 e_0 \right) \tag{IV.14}$$

and $q = |\underline{q}|,\; k = |\underline{k}|,$ etc.

The relations between the $\mathcal{F}_i$ and A, B, ...F are (compare Ref. 23)

$$F_1 = \frac{2M}{\omega O_1} \mathcal{F}_1 = A - \frac{2M\nu_B}{\omega}(C-D) + \omega D + \frac{\lambda^2}{\omega} F$$

$$F_2 = \frac{2M(E_1+M)}{qkO_2(W+M)} \mathcal{F}_2 = -A - \frac{2M\nu_B}{W+M}(C-D) + (W+M)D + \frac{\lambda^2}{W+M} F$$

$$F_3 = \frac{2M(E_1+M)}{qkO_1(W+M)} \mathcal{F}_3 = \omega B + C - D - \frac{\lambda^2}{W+M} E$$

$$F_4 = \frac{2M}{q^2 O_2 \omega} \mathcal{F}_4 = -(W+M)B + C - D + \frac{\lambda^2}{\omega} E$$

$$F_5 = \frac{2M(E_1+M)}{k^2 O_1} \mathcal{F}_5 = -A - 2M\nu_B B - \omega D + 2M\nu_B E - (W+M)F$$

$$F_6 = \frac{2M}{qkO_2} \mathcal{F}_6 = -\frac{k_0}{E_1+M} A + 2M\nu_B B - \frac{2M\nu_B}{E_1+M}(C-D) + k_0 \frac{W+M}{E_1+M} D - 2M\nu_B E - k_0 \frac{W+M}{E_1+M} F$$

(IV.15)

where $E_1$ and $E_2$ are the initial and final nucleon energies, $e_o$, $k_o$, $q_o$ are the time components of $e_\mu$, $k_\mu$, $q_\mu$ and

$$O_1 = \sqrt{(E_1+M)(E_2+M)} \qquad O_2 = \sqrt{\frac{E_1+M}{E_2+M}}$$

Now writing the isotopic 3/2 part of $A_i^{B.A.}(x, V_B, \lambda^2)$ in terms of $F_1, \ldots F_6$ we have

$$F_\mu^{B.A} = -\frac{2f}{qk}\mu_V^c(\lambda^2)\frac{1}{\cos\theta+a}\left(M, -M, 1, 1, \frac{\omega}{2}, \frac{(W+M)(E_1-M)}{2(E_1+M)}\right)$$

$$F_g^{B.A} = \frac{2f}{qk}F_1^V(\lambda^2)\frac{1}{\cos\theta+a}\left(0, 0, \frac{\omega}{W+M}, \frac{W+M}{\omega}, \frac{\omega}{2}, \frac{W+M}{2}\right)$$

$$F_\pi^{B.A.} = \frac{2f}{qk}F_\pi(\lambda^2)\frac{1}{\cos\theta-b}\left(0, 0, \frac{2M}{W+M}, -\frac{2M}{\omega}, -M, M\right)$$

(IV.16)

where we have split up the results into terms linear in

$$\mu_V^c(\lambda^2),\ F_1^V(\lambda^2),\ F_\pi^V(\lambda^2)$$

Further

$$\underline{q}\cdot\underline{k} = qk\cos\theta$$

$$\mu_V^c(\lambda^2) = \mu^V F_2^V(\lambda^2) + F_1^V(\lambda^2)$$

$$a = \frac{2k_0E_2+\lambda^2}{2qk} \qquad b = \frac{2q_0k_0+\lambda^2}{2qk} \qquad \text{(IV.17)}$$

Relations between $E_1$, $E_2$, $k_o$, $q_o$ and W and $\lambda^2$ are

$$E_1 = \frac{W^2 + M^2 + \lambda^2}{2W}$$

$$E_2 = \frac{W^2 + M^2 - 1}{2W}$$

$$q_0 = \frac{W^2 - M^2 + 1}{2W}$$

$$k_0 = \frac{W^2 - M^2 - \lambda^2}{2W}$$

(IV.18)

Now we can project the spin 3/2 state out of IV.16. We use

$$\mathcal{F}^{J=3/2}(\underline{q}, \underline{k}, W) = \frac{1}{4\pi} \int \frac{d\Omega_{q'}}{q\, q'} \left( 3\, \underline{q} \cdot \underline{q}' - \underline{\sigma} \cdot \underline{q}\ \underline{\sigma} \cdot \underline{q}' \right) \mathcal{F}(\underline{q}', \underline{k}, W)$$

(IV.19)

Useful formulae are

$$\frac{1}{4\pi} \int \frac{d\Omega_{q'}}{\hat{\underline{q}}' \cdot \hat{\underline{k}} + a} = \frac{1}{2} \log \frac{a+1}{a-1} \qquad \left( \hat{\underline{k}} = \frac{\underline{k}}{k} \text{ etc.} \right)$$

$$\frac{1}{4\pi} \int \frac{\hat{\underline{q}}'\, d\Omega_{q'}}{\hat{\underline{q}}' \cdot \hat{\underline{k}} + a} = \hat{\underline{k}}\ \bar{\alpha}(a)$$

$$\bar{\alpha}(a) = 1 - \frac{a}{2} \log \frac{a+1}{a-1}$$

$$\frac{1}{4\pi} \int \frac{\hat{q}'_\alpha \hat{q}'_\beta}{\hat{q}' \cdot \hat{\underline{k}} + a} d\Omega_{q'} = \frac{1}{2} \bar{\beta}(a) \delta_{\alpha\beta} - \frac{1}{2} \bar{\gamma}(a) \hat{k}_\alpha \hat{k}_\beta$$

$$\bar{\beta}(a) = a - \frac{a^2-1}{2} \log \frac{a+1}{a-1}$$

$$\bar{\gamma}(a) = 3a - \frac{3a^2-1}{2} \log \frac{a+1}{a-1}$$

$$\frac{1}{4\pi} \int \frac{\hat{q}'_\alpha \hat{q}'_\beta \hat{q}'_\gamma}{\hat{q}' \cdot \hat{\underline{k}} + a} d\Omega_{q'} = \frac{1}{3} \left( \delta_{\alpha\beta} \hat{k}_\gamma + \delta_{\alpha\gamma} \hat{k}_\beta + \delta_{\beta\gamma} \hat{k}_\alpha \right) \times$$

$$\left(1 - \frac{3}{2} a \bar{\beta}(a)\right) + \hat{k}_\alpha \hat{k}_\beta \hat{k}_\gamma f(a)$$

$$f(a) = \frac{1}{3} + a^2 \bar{\alpha}(a) - \left(1 - \frac{3}{2} a \bar{\beta}(a)\right)$$

(IV.20)

Doing the projections and collecting terms we get for the 'magnetic moment' terms (compare Ref. 23)

$$\frac{F_\mu^{1}}{A_\mu \cos\theta} = 3M\bar{\alpha}(a) - \frac{3}{2}\frac{Mqk(W+M)}{\omega(E_1+M)(E_2+M)}\bar{\tau}(a) + \frac{q^2}{E_2+M}\left(1-\frac{3}{2}a\bar{\beta}(a)\right)$$

$$\frac{F_\mu^{2}}{A_\mu} = \frac{M(E_1+M)(E_2+M)}{qk(W+M)}\omega\bar{\alpha}(a) - \frac{E_2+M}{2}\bar{\beta}(a) - \frac{M}{2}\bar{\tau}(a)$$

$$\frac{F_\mu^{3}}{A_\mu} = \frac{3}{2}\bar{\beta}(a) + \left(1-\frac{3}{2}a\bar{\beta}(a)\right)\frac{E_1+M}{E_2+M}\frac{\omega}{W+M}\frac{q}{k}$$

$$F_\mu^{4} = 0$$

$$\frac{F_\mu^{5}}{A_\mu \cos\theta} = \frac{3}{2}\omega\bar{\alpha}(a) - \frac{3}{4}\frac{q}{k}\frac{W+M}{E_2+M}(2E_2+E_1-M)\bar{\tau}(a) + \frac{q^2}{k^2}\frac{\omega(E_1+M)}{E_2+M}\left(\frac{5}{2}a\bar{\tau}(a)-2\bar{\alpha}(a)\right)$$

$$\frac{F_\mu^{6}}{A_\mu} = -\frac{\omega(E_1+M)(E_2+M)}{2qk}\bar{\alpha}(a) + \left(\frac{(W+M)(E_1-M)}{4(E_1+M)} - \frac{\omega a}{2}\frac{q}{k}\right)\bar{\tau}(a)$$

$$A_\mu = -\frac{2f}{qk}\mu_V^c(\lambda^2) \tag{IV.21}$$

For the 'charge' terms:

$$\frac{F_g^1}{B_g \cos\theta} = \frac{q^2}{E_2+M}\ \frac{W+M}{\omega}\left(1 - \frac{3}{2}a\bar{\beta}(a)\right)$$

$$\frac{F_g^2}{B_g} = -\frac{1}{2}\ \frac{\omega}{W+M}\ (E_2+M)\bar{\beta}(a)$$

$$\frac{F_g^3}{B_g} = \frac{3}{2}\ \frac{\omega}{W+M}\bar{\beta}(a) + \frac{W+M}{\omega}\ \frac{qk}{(E_2+M)(k_0+\frac{\lambda^2}{\omega})}\left(1-\frac{3}{2}a\bar{\beta}(a)\right)$$

$$F_g^4 = 0$$

$$\frac{F_g^5}{B_g\cos\theta} = \frac{3}{2}\omega\bar{\alpha}(a) - \frac{3}{4}\ \frac{q}{k}\left[\frac{(W+M)(E_1+M)}{E_2+M} + 2\omega\right]\bar{\tau}(a)$$
$$+ \frac{q^2}{k^2}\ \frac{(E_1+M)(W+M)}{E_2+M}\left(\frac{5}{2}a\bar{\tau}(a) - 2\bar{\alpha}(a)\right)$$

$$\frac{F_g^6}{B_g} = -\frac{k\omega}{2q}\ \frac{E_2+M}{E_1+M}\bar{\alpha}(a) - \frac{\omega}{2}\ \frac{E_2+M}{E_1+M}\bar{\beta}(a)$$
$$+\left[-\frac{\omega}{2}\ \frac{E_2+M}{E_1+M} + \frac{W+M}{4} - \frac{q}{2k}(W+M)a\right]\bar{\tau}(a)$$

$$B_g = \frac{2f}{qk}F_1^V(\lambda^2) \qquad \text{(IV.22)}$$

Finally, from the meson current we have

$$\frac{F_\pi^1}{B_\pi \cos\theta} = -\frac{2M}{\omega}\frac{q^2}{E_2+M}\left(1-\frac{3}{2}b\bar{\beta}(b)\right)$$

$$\frac{F_\pi^2}{B_\pi} = \frac{M(E_2+M)}{W+M}\bar{\beta}(b)$$

$$\frac{F_\pi^3}{B_\pi} = \frac{-3M}{W+M}\bar{\beta}(b) - \frac{2M}{\omega}\frac{qk}{E_2+M}\frac{1}{k_0+\lambda^2/\omega}\left(1-\frac{3}{2}b\bar{\beta}(b)\right)$$

$$F_\pi^4 = 0$$

$$\frac{F_\pi^5}{B_\pi \cos\theta} = -M\bar{\alpha}(b) + \frac{3}{2}M\frac{q}{k}\left[\frac{E_1+M}{E_2+M}+2\right]\bar{\gamma}(b)$$

$$-2M\frac{q^2}{k^2}\frac{E_1+M}{E_2+M}\left(\frac{5}{2}b\bar{\gamma}(b)-2\bar{\alpha}(b)\right)$$

$$\frac{F_\pi^6}{B_\pi} = M\frac{k}{q}\frac{E_2+M}{E_1+M}\bar{\alpha}(b) + M\frac{E_2+M}{E_1+M}\bar{\beta}(b)$$

$$+M\left[\frac{q}{k}b-\frac{1}{2}-\frac{E_2+M}{E_1+M}\right]\bar{\gamma}(b)$$

$$B_\pi = \frac{2f}{qk}F_\pi(\lambda^2) \tag{IV.23}$$

So the 3-3 projections $a_i\,(x, v_B, \lambda^2)$ are found by taking these equations, substituting them on the left hand side of IV.15 and then solving for a,b,...f. The solutions are given explicitly by

$$a = 2Md + \frac{1}{2W}\left[\omega F_1 - (W+M)F_2\right]$$

$$b = \frac{\lambda^2}{4MWk_0 v_B}\left[\frac{\omega}{2W}F_1 + \frac{k_0(W+M)}{2W(E_1+M)}F_2 + \frac{Mv_B}{W\lambda^2}(W^2-M^2)(F_3-F_4) + \frac{\omega}{2W}F_5 - \frac{W+M}{2W}F_6\right]$$

$$c = d + \frac{\omega}{2W}F_4 + \frac{W+M}{2W}F_3$$

$$d = \frac{1}{4W^2k_0}\left[(W^2-M^2)F_1 + \frac{k_0}{E_1+M}(W+M)^2F_2 + 2Mv_B\left((W+M)F_3 + \omega F_4\right) + \lambda^2[F_5+F_6]\right]$$

$$e = \frac{W^2-M^2}{\lambda^2}b + \frac{W^2-M^2}{2W\lambda^2}(F_4-F_3)$$

$$f = -d + \frac{W+M}{2W(E_1+M)}F_2 - \frac{F_5+F_6}{2W}$$

(IV.24)

Remember $a^{+} = \frac{2}{3} a^{(3/2)}$, $a^{-} = -\frac{1}{3} a^{(3/2)}$ etc. finally.

The Axial Vector Part:

We report the procedure for $\mathcal{M}_A$. We no longer have gauge invariance, so this time there are eight invariant amplitudes

$$-i\mathcal{M}_A = A M_A + \ldots + H M_H \tag{IV.25}$$

where this time

$$\begin{aligned}
M_A &= \tfrac{1}{2}(\gamma\cdot q\,\gamma\cdot e - \gamma\cdot e\,\gamma\cdot q) && (-) \\
M_B &= 2P\cdot e && (-) \\
M_C &= q\cdot e && (+) \\
M_D &= iM\gamma\cdot e && (-) \\
M_E &= i\gamma\cdot k\, 2P\cdot e && (+) \\
M_F &= i\gamma\cdot k\; q\cdot e && (-) \\
M_G &= k\cdot e && (+) \\
M_H &= i\gamma\cdot k\; k\cdot e && (-)
\end{aligned} \tag{IV.26}$$

From the Born approximation III.18 we find the residues for the dispersion relations for the A, . . . H(IV.3) as follows:

$$R[A] = f\alpha(\lambda^2)\left(-\frac{G_A}{G}\right)$$

$$R[C] = f\alpha(\lambda^2)\left(-\frac{G_A}{G}\right)$$

$$R[H] = -f\beta(\lambda^2)$$

(IV.27)

All other R's are zero, and no subtraction terms $C_i$. [If the two pion intermediate state were included, we see from III.21 that a subtraction in at least $H^-$ would have been necessary.]

In the case of k·e being zero, amplitudes G and H do not contribute. We will project out the 3-3 states as before.

In terms of two-component spinors

$$\bar{u}_f \mathcal{M}_A u_i = i\chi_f^* \mathcal{F}_A \chi_i$$

where $\mathcal{F}_A = \sum_{i=1}^{\infty} \mathcal{F}_i \Sigma_i$ (IV.28)

and

$$\Sigma_1 = \frac{\underline{\sigma}\cdot\underline{q}\;\underline{\sigma}\cdot\underline{e}}{q} \qquad \Sigma_2 = \frac{\underline{k}\cdot\underline{e}\;\underline{\sigma}\cdot\underline{q}\;\underline{\sigma}\cdot\underline{k}}{qk^2}$$

$$\Sigma_3 = \frac{\underline{q}\cdot\underline{e}\;\underline{\sigma}\cdot\underline{q}\;\underline{\sigma}\cdot\underline{k}}{q^2 k} \qquad \Sigma_4 = \frac{\underline{\sigma}\cdot\underline{q}\;\underline{\sigma}\cdot\underline{k}}{qk}\,e_0$$

$$\Sigma_5 = \frac{\underline{\sigma}\cdot\underline{e}\;\underline{\sigma}\cdot\underline{k}}{k} \qquad \Sigma_6 = \frac{\underline{q}\cdot\underline{e}}{q}$$

$$\Sigma_7 = \frac{\underline{k}\cdot\underline{e}}{k} \qquad \Sigma_8 = e_0$$

(IV.29)

Then the relations for the $\mathcal{F}$'s in terms of A, ... H are

$$F_1 = \frac{2M}{q\,O_2}\,\mathcal{F}_1 = (W+M)A - MD$$

$$F_2 = \frac{2MO_1}{q\,k^2}\,\mathcal{F}_2 = B + (W+M)E - G - (W+M)H$$

$$F_3 = \frac{2MO_1}{q^2 k}\,\mathcal{F}_3 = A + B - C + (W+M)E - (W+M)F$$

$$F_4 = \frac{2MO_1}{qk}\,\mathcal{F}_4 = (E_2+M)A + (E_1+E_2)B + q_0 C - MD$$
$$+ k_0 G + (W+M)\left[(E_1+E_2)E + q_0 F + k_0 H\right]$$

$$F_5 = \frac{2MO_2}{k}\,\mathcal{F}_5 = -\omega A - MD$$

$$F_6 = \frac{2M}{q\,O_1}\,\mathcal{F}_6 = -A - B + C + \omega(E-F)$$

$$F_7 = \frac{2M}{kO_1}\,\mathcal{F}_7 = -B + \omega E + G - \omega H$$

$$F_8 = \frac{2M}{O_1}\,\mathcal{F}_8 = -(E_2-M)A - (E_1+E_2)B - q_0 C - MD$$
$$- k_0 G + \omega\left[(E_1+E_2)E + q_0 F + k_0 H\right]$$

(IV.30)

When $k \cdot e = 0$, $F_8$ becomes incorporated in $F_7$ and $F_4$ in $F_2$. We will project out the 3-3 part of only the terms linear in $\alpha(\lambda^2)$; if $k \cdot e = 0$, these are the only terms of interest, and if not, there is a simple way to relate the $\beta$ terms to pion-nucleon scattering which will be shown later.

So the isotopic 3/2 part of the amplitudes are given by

$$F_\alpha^{B.A.} = -\frac{2Mf\alpha(\lambda^2)\left(-\frac{G_A}{G}\right)}{qk(a+\cos\theta)}\Big[W+M,\ 0,\ 2,\ E_2+M-q_0,\ -\omega,\ -2,\ 0,\ -(E_2-q_0-M)\Big] \quad \text{(IV.31)}$$

Doing the spin 3/2 projections

$$\frac{F_\alpha^1}{A_\alpha} = -\omega\,\frac{k(E_2+M)}{q(E_1+M)}\,\bar{\alpha}(a) + (E_2+M)\bar{\beta}(a) + \frac{W+M}{2}\bar{\gamma}(a)$$

$$\frac{F_\alpha^2}{A_\alpha} = \frac{2\omega(E_2+M)}{qk}\,\bar{\alpha}(a) - \left(\frac{q}{k}a + \frac{E_2+M}{E_1-M}\right)\bar{\gamma}(a)$$

$$F_\alpha^3 = 0$$

$$\frac{F_\alpha^4}{A_\alpha} = \frac{(E_1+M)(E_2+M)}{qk}(E_2-q_0-M)\,\bar{\alpha}(a) + \frac{E_2+M-q_0}{2}\,\bar{\gamma}(a)$$

$$\frac{F_\alpha^5}{A_\alpha \cos\theta} = -3\omega\,\bar{\alpha}(a) + 2(E_2-M)\left(1-\tfrac{3}{2}a\bar{\beta}(a)\right) + \frac{3}{2}\,\frac{q}{k}\,\frac{E_1+M}{E_2+M}(W+M)\,\bar{\gamma}(a)$$

$$\frac{F_\alpha^6}{A_\alpha} = -3\bar{\beta}(a) + \frac{2qk}{(E_1+M)(E_2+M)}\left(1-\tfrac{3}{2}a\bar{\beta}(a)\right)$$

$$\frac{F_\alpha^7}{A_\alpha \cos\theta} = \frac{-3qq_0}{k(E_2+M)}\,\bar{\gamma}(a) + 6\,\frac{E_2-M}{E_1+M}\,f(a)$$

$$\frac{F_\alpha^8}{A_\alpha \cos\theta} = -3(E_2-M-q_0)\,\bar{\alpha}(a) - \frac{3}{2}\,\frac{qk}{(E_1+M)(E_2+M)}\,\bar{\gamma}(a)$$

$$A_\alpha = \frac{-2Mf}{qk}\left(\frac{-G_A}{G}\right)\alpha(\lambda^2)$$

(IV.32)

In the general formulation, equation IV.30 must now be solved for a, b, ...h in terms of $F'_\alpha$, ...$F_\alpha^8$ as before.

We now have to compute the terms linear in $\beta(\lambda^2)$. These are not present in a situation where $k \cdot \epsilon = 0$, as in weak pion production with electron. But if a muon is produced they are, and could be appreciable for high momentum transfer. We could compute them by the same methods that have been used up to now, but there is a simple way to relate any term involving $\beta(\lambda^2)$ to a similar term involving a pion which can be used more generally than the other methods (the dominance of a resonance is not required).

In our case, consider the diagram Fig. 6. For it

$$\langle \pi p_2 | P_\mu | p_1 \rangle = \langle \pi p_2 | \pi_k | p_1 \rangle \langle \pi_k | P_\mu | 0 \rangle$$

But
$$\langle \pi p_2 | \pi_k | p_1 \rangle = \frac{-i}{\lambda^2 + m_\pi^2} \langle q p_2 | k p_1 \rangle$$

where as in III.14 $\langle q p_2 | k p_1 \rangle$ describes pion-nucleon scattering with the initial pion off the mass shell.

Also, from the theory of the axial vector current, for $\lambda^2$ not too large

$$\langle \pi_k | P_\mu | 0 \rangle = \frac{-ik_\mu}{m_\pi^2} \langle \pi | \partial_\mu P_\mu | 0 \rangle = \frac{k_\mu}{m_\pi^2} a \langle \pi | \pi(x) | 0 \rangle$$

The last matrix element is just a phase factor which we can take equal to one.

So

$$H_\beta = \langle \pi p_2 | P_\mu | p_1 \rangle_\beta \, e_\mu$$

$$= \frac{ia}{m_\pi^2} k \cdot e \, \frac{\langle q p_2 | k p_1 \rangle}{\lambda^2 + m_\pi^2} \qquad \text{(IV.33)}$$

But

$$\beta(\lambda^2) = \frac{2g_1}{m_\pi^2}\,\frac{1}{\lambda^2+m_\pi^2} + \ldots \qquad \text{(I.23c)}$$

So
$$H_\beta = \frac{-i\beta(\lambda^2)}{g_1}\, k\cdot e \,\langle q\, p_2 | k\, p_1\rangle \qquad \text{(IV.34)}$$

Now
$$\langle q\, p_2 | k\, p_1\rangle = \bar{u}_f\left(-A + i\gamma\cdot k\, B\right) u_i$$

where
$$A = A(\nu, \nu_B, \lambda^2), \quad B = B(\nu, \nu_B, \lambda^2)$$

For $\lambda^2 = -m_\pi^2$, we know that there are no one-nucleon poles in A and that the residue for these poles in B is $g_1^2/2M$. So we expect no poles here for our amplitude G, and a pole of residue

$$\frac{-i\beta(\lambda^2)}{g_1}\,\frac{g_1^2}{2M} = -i\beta(\lambda^2)\, f \qquad \text{for H}$$

(eq. IV.27) thus showing our choice of phase factor is correct.

Let us put

$$i\,\langle p_2\, q | \pi_R | p_1\rangle = \bar{u}_f\left(-A_0 + i\gamma\cdot k\, B_0\right) u_i$$

$$= \frac{\bar{u}_f\left(-A + i\gamma\cdot k\, B\right) u_i}{\lambda^2 + m_\pi^2} \qquad \text{(IV.35)}$$

We expect $A_0$, $B_0$ to be analytic in $\nu$, $\nu_B$, and $\lambda^2$. Write new variables

$$s = -(p_1 + k)^2 \qquad t = -(p_1 - p_2)^2$$

instead of $\nu$, $\nu_B$. Then for $s < (M+1)^2, t < 9$; $A_0(s,t,\lambda^2)$, $B_0(s,t,\lambda^2)$ are real for $\lambda^2 < 9$ and we expect the following dispersion equations to hold:

$$A_0(s,t,\lambda^2) = \frac{A(s,t,-m_\pi^2)}{\lambda^2 + m_\pi^2} + \frac{1}{\pi}\int_{9m_\pi^2}^{\infty} \frac{\mathrm{Im}\, A_0(s,t,\lambda'^2)\, d\lambda'^2}{\lambda'^2 + \lambda^2}$$

$$B_0(s,t,\lambda^2) = \frac{B(s,t,-m_\pi^2)}{\lambda^2 + m_\pi^2} + \frac{1}{\pi}\int_{9m_\pi^2}^{\infty} \frac{\mathrm{Im}\, B_0(s,t,\lambda'^2)\, d\lambda'^2}{\lambda'^2 + \lambda^2}$$

(IV.36)

In the same way that we discussed the functions $\beta(\lambda^2), K(\lambda^2)$, we expect for reasonably small $\lambda^2$ to be able to neglect the integrals in these equations in comparison with the pion pole terms.

Now let us continue analytically in s to the resonance region; $s \sim (M+2)^2$. The functions A, B immediately become complex as s becomes greater than $(M+1)^2$, and presumably so do the continuations of $\mathrm{Im}\, A_0$, $\mathrm{Im}\, B_0$. We know that A and B contain a large imaginary part in the resonance region, but there is no reason to expec t either the real or imaginary parts of $\mathrm{Im}\, A_0$, $\mathrm{Im}\, B_0$ to become appreciably larger for these values of s than they were before. We have not continued very far. So the obvious approximation is to write

$$A_0(s,t,\lambda^2) = \frac{A(s,t,-m_\pi^2)}{\lambda^2 + m_\pi^2}$$

$$B_0(s,t,\lambda^2) = \frac{B(s,t,-m_\pi^2)}{\lambda^2 + m_\pi^2}$$

(IV.37)

So

$$H_\beta = \frac{-i\beta(\lambda^2)}{g_1} k\cdot e\ \bar{u}_f \Big[ -A(\nu,\nu_B,-m_\pi^2) + i\gamma\cdot k\, B(\nu,\nu_B,-m_\pi^2)\Big] u_i \qquad \text{(IV.38)}$$

Now from the study of the pion-nucleon problem by CGLN we have

$$A^{33} = \left(3\,\frac{W+M}{E_2+M}\cos\theta + \frac{\omega}{E_2-M}\right)\frac{4\pi e^{i\delta_{33}}\sin\delta_{33}}{q}$$

$$B^{33} = \left(\frac{3}{E_2+M}\cos\theta - \frac{1}{E_2-M}\right)\frac{4\pi e^{i\delta_{33}}\sin\delta_{33}}{q}$$

(IV. 39)

and $$g^{33} = \frac{i\beta(\lambda^2)}{g_1}A^{33}, \quad h^{33} = -\frac{i\beta(\lambda^2)}{g_1}B^{33} \qquad \text{(IV. 40)}$$

Static Limit

The simplest way to see what eq. IV, 21, 22, 23 and 32 are about is to go to the static limit, expand in $\frac{\omega}{M}$, and keep $\lambda^2$ small. We will need the following expansions

$$\bar{\alpha}(a) = \frac{-r^2s^2}{3M^2\omega^2}\left(1-\frac{\omega}{M}\right) \qquad 1-\frac{3}{2}a\bar{\beta}(a) = -\frac{r^2s^2}{5M^2\omega^2}\left(1-\frac{\omega}{M}\right)$$

$$\bar{\beta}(a) = \frac{2rs}{3M\omega}\left(1-\frac{\omega}{2M}\right) \qquad \frac{5}{2}a\bar{\tau}(a) - 2\bar{\alpha}(a) = -\frac{6r^4s^4}{35M^4\omega^4}\left(1-\frac{\omega}{2M}\right)$$

$$\bar{\tau}(a) = -\frac{4r^3s^3}{15M^3\omega^3}\left(1-\frac{3\omega}{2M}\right)$$

$$q = r\left(1-\frac{\omega}{2M}\right) \qquad E_1 = M + \frac{s^2}{2M}$$

$$k = s\left(1-\frac{\omega}{2M}\right) \qquad E_2 = M + \frac{r^2}{2M}$$

$$r = \sqrt{\omega^2-1} \qquad s = \sqrt{\omega^2+\lambda^2} \qquad \text{(IV. 41)}$$

We have three sets of terms $\mathcal{F}_\mu$, $\mathcal{F}_g$, $\mathcal{F}_\pi$ for the vector amplitude. The charge terms $\mathcal{F}_g$ are well known to be essentially recoil terms and can be neglected in the static limit. For the terms $\mathcal{F}_\mu$; $\mathcal{F}_\mu^4 = 0$, $\mathcal{F}_\mu^5$ and $\mathcal{F}_\mu^6$ are small if $\lambda^2$ is small. So we are left with $\mathcal{F}_\mu^1$, $\mathcal{F}_\mu^2$ and $\mathcal{F}_\mu^3$.

Explicitly the leading terms in powers of $\frac{\omega}{M}$ (and $\frac{1}{M}$) are given by

$$\frac{F_\mu^1}{A_\mu \cos\theta} = 3M\bar{\alpha}(a) \qquad \frac{\mathcal{F}_\mu^1}{A_\mu \cos\theta} = \frac{\omega O_1}{2M}\left[3M\bar{\alpha}(a)\right] = 3M\omega\bar{\alpha}(a)$$

$$\frac{F_\mu^2}{A_\mu} = -\frac{E_2+M}{2}\bar{\beta}(a) + \frac{M(E_1+M)(E_2+M)}{qk(W+M)}\omega\bar{\alpha}(a)$$

$$\frac{\mathcal{F}_\mu^2}{A_\mu} = \frac{qkO_2(W+M)}{2M(E_1+M)}\frac{F_\mu^2}{A_\mu} = -\frac{qk}{2}\bar{\beta}(a) + M\omega\bar{\alpha}(a)$$

$$\frac{F_\mu^3}{A_\mu} = \frac{3}{2}\bar{\beta}(a)$$

$$\frac{\mathcal{F}_\mu^3}{A_\mu} = \frac{qkO_1(W+M)}{2M(E_1+M)}\left[\frac{3}{2}\bar{\beta}(a)\right] = \frac{3}{2}qk\bar{\beta}(a)$$

(IV.42)

From IV.41, to first order

$$\bar{\alpha}(a) = -\frac{3}{4}\left[\bar{\beta}(a)\right]^2 \tag{IV.43}$$

So to zero order

$$\frac{\mathcal{J}_\mu^1}{A_\mu \cos\theta} = -\frac{q}{4} M\omega \left[\bar{\beta}(a)\right]^2 = -\frac{3}{2} q k \bar{\beta}(a)$$

$$\frac{\mathcal{J}_\mu^2}{A_\mu} = -\frac{qk}{2}\bar{\beta}(a) - \frac{qk}{2}\bar{\beta}(a) = -qk\bar{\beta}(a)$$

So

$$\frac{\mathcal{J}_\mu^{33}}{A_\mu} = \Big[-\frac{3}{2} i \underline{\sigma}\cdot\underline{a}\, \underline{q}\cdot\underline{k} - \underline{\sigma}\cdot\underline{q}\, \underline{\sigma}\cdot(\underline{k}\times\underline{a}) + \frac{3}{2} i \underline{\sigma}\cdot\underline{k}\, \underline{q}\cdot\underline{a}\Big]\bar{\beta}(a)$$

If we talk about weak pion production with electron, then

$$\underline{a} = \underline{\varepsilon} - \frac{\underline{k}\cdot\underline{\varepsilon}}{k_0^2}\underline{k} \tag{IV.14}$$

so

$$\frac{\mathcal{J}_\mu^{33}}{A_\mu} = \Big[-\frac{3}{2} i \underline{\sigma}\cdot\Big(\underline{\varepsilon} - \frac{\underline{k}\cdot\underline{\varepsilon}}{k_0^2}\underline{k}\Big)\underline{q}\cdot\underline{k} - \underline{q}\times\underline{k}\cdot\underline{\varepsilon}$$

$$- i\underline{\sigma}\cdot\left[\underline{q}\cdot\underline{\varepsilon}\,\underline{k} - \underline{q}\cdot\underline{k}\,\underline{\varepsilon}\right] + \frac{3}{2} i \underline{\sigma}\cdot\underline{k}\, \underline{q}\cdot\Big(\underline{\varepsilon} - \frac{\underline{k}\cdot\underline{\varepsilon}}{k_0^2}\underline{k}\Big)\Big]\bar{\beta}(a)$$

$$= -\Big(2\underline{q}\times\underline{k}\cdot\underline{\varepsilon} + i\underline{\sigma}\cdot\underline{\varepsilon}\,\underline{q}\cdot\underline{k} - i\underline{\sigma}\cdot\underline{k}\,\underline{q}\cdot\underline{\varepsilon}\Big)\frac{\bar{\beta}(a)}{2} \tag{IV.44}$$

Notice that the longitudinal terms have dropped out. It is clear that this term is the magnetic dipole term. Indeed $\overline{\beta}(a)$ is a multiple of the function termed $F_M$ by CGLN.

Finally then

$$\mathcal{F}_\mu^{33} = \frac{f\mu_v^c(\lambda^2)}{qk}\overline{\beta}(a)\left(2\underline{q}\times\underline{k}\cdot\underline{\varepsilon} + i\underline{\sigma}\cdot\underline{\varepsilon}\ \underline{q}\cdot\underline{k} - i\underline{\sigma}\cdot\underline{k}\ \underline{q}\cdot\underline{\varepsilon}\right)$$

$$= \frac{2f\mu_v^c(\lambda^2)}{3M\omega}\left(2\underline{q}\times\underline{k}\cdot\underline{\varepsilon} + i\underline{\sigma}\cdot\underline{\varepsilon}\ \underline{q}\cdot\underline{k} - i\underline{\sigma}\cdot\underline{k}\ \underline{q}\cdot\underline{\varepsilon}\right) \tag{IV.45}$$

Next there is $\mathcal{F}_\pi$. These terms are difficult to deal with convincingly. The treatment of photoproduction is fairly satisfactory in its agreement with experiment without including the pionic current terms at resonance. Yet the $F_\pi$ at first glance appear as big as $F_\mu$. Indeed $F_\mu{}'$ for example is of order $\frac{1}{M}$ whereas $F_\pi{}'$ is of order one. However $1-\frac{3}{2}b\overline{\beta}(b)$ is fairly small although it does not involve M, and it turns out that $F_\mu{}'$ is greater than $F_\pi{}'$ by about a factor of 3/2 for the region we are interested in. Also, because of the factor $1-\frac{3}{2}b\overline{\beta}(b)$, $F_\pi{}'$ and even more so $\mathcal{F}_\pi{}'$ is a decreasing function of $\omega$ whereas $\mathcal{F}_\mu{}'$ is roughly constant in $\omega$. The same considerations apply to $\mathcal{F}^2$ and $\mathcal{F}^3$. In view of this the enhancement integral

$$P\int_{1+\frac{1}{2n}}^{\infty}\frac{a_i(y)\sin\delta_{33}(y)}{y-x}\,dy\ , \quad y = \frac{\omega^2+2M\omega}{2M} \tag{IV.9}$$

should contribute rather less to the pionic terms than to the magnetic terms. Finally and most important

$$A_\mu = -\frac{2f}{qk}\mu_V^c(\lambda^2) \qquad B_\pi = \frac{2f}{qk}F_\pi(\lambda^2)$$

and so $A_\mu$ is about 4.7 times as large as $B_\pi$. In view of all this, it should be no worse to neglect the pionic terms than to neglect terms of order $\frac{1}{M}$.

So the simplest approximation to $\mathcal{F}_V$ is just the magnetic dipole term.

$$\mathcal{F}_V^{33} = \mathcal{F}_\mu^{33} = \frac{2f\mu_V^c(\lambda^2)}{3M\omega}\Big(2\underline{q}\times\underline{k}\cdot\underline{\varepsilon} + i\underline{\sigma}\cdot\underline{\varepsilon}\,\underline{q}\cdot\underline{k} - i\underline{\sigma}\cdot\underline{k}\,\underline{q}\cdot\underline{\varepsilon}\Big) \tag{IV.46}$$

Next is $\mathcal{F}_A$. We are considering the case where $k\cdot\epsilon = 0$ so the terms in $\beta(\lambda^2)$ vanish. By inspection of IV.30 and IV.32 we see that the largest terms are

$$\frac{F_\alpha^1}{A_\alpha} = (E_2+M)\bar{\beta}(a) \qquad \frac{\mathcal{F}_\alpha^1}{A_\alpha} = \frac{qQ_2}{2M}\frac{F_\alpha^1}{A_\alpha} = q\bar{\beta}(a)$$

and

$$\frac{F_\alpha^6}{A_\alpha} = -3\bar{\beta}(a) \qquad \frac{\mathcal{F}_\alpha^6}{A_\alpha} = \frac{qQ_1}{2M}\left(-3\bar{\beta}(a)\right) = -3q\bar{\beta}(a) \tag{IV.47}$$

to zero order in $\frac{1}{M}$.

So

$$\frac{\mathcal{F}_\alpha^{33}}{A_\alpha} = -\left(3\underline{q}\cdot\underline{\varepsilon} - \underline{\sigma}\cdot\underline{q}\,\underline{\sigma}\cdot\underline{\varepsilon}\right)\bar{\beta}(a) \tag{IV.48}$$

and

$$\mathcal{F}_A^{33} = \mathcal{F}_\alpha^{33} = \frac{4f\left(-\frac{G_A}{G}\right)\alpha(\lambda^2)}{3\omega}\left(3\, \underset{\sim}{q}\cdot\underset{\sim}{\varepsilon} - \underset{\sim}{\sigma}\cdot\underset{\sim}{q}\,\underset{\sim}{\sigma}\cdot\underset{\sim}{\varepsilon}\right) \qquad \text{(IV.49)}$$

Thus the principal term is of 'pseudomagnetic dipole' form. Also $\mathcal{F}_A^{33}$ looks appreciably larger than $\mathcal{F}_V^{33}$. This is not surprising, as the direct vector weak interaction — the charge term — does not contribute in the static limit to a spin 3/2 state of the final system; it is the weak magnetism which contributes. On the other hand, the axial vector term can go directly in the static limit.

The next task is to find the actual amplitudes to be used in a calculation. If we are interested only in the resonance region we can drop both the Born terms and the crossed term in the complete expressions for the amplitudes, and we can write the resonant term in the form

$$A_i(x, \nu_B, \lambda^2) = \frac{a_i(x, \nu_B, \lambda^2)}{1 - \frac{x}{x_r} - i\Gamma} \qquad \text{(IV.11)}$$

where $\Gamma$ refers to the width of the resonance. Now in the static limit $x \simeq \omega$, and we recall the Chew-Low formula for the enhancement factor for the pion-nucleon system in the static limit (Ref. 24):

$$e^{i\delta_{33}} \sin\delta_{33} = \frac{\frac{4}{3}\left(\frac{f^2}{4\pi}\right)\frac{q^3}{\omega}}{1 - \frac{\omega}{\omega_r} - i\left(\frac{4}{3}\right)\left(\frac{f^2}{4\pi}\right) q^3/\omega} \qquad \text{(IV.50)}$$

So we can write

$$A_i(\omega, \nu_B, \lambda^2) = \frac{a_i(\omega, \nu_B, \lambda^2)}{\frac{4}{3}\frac{f^2}{4\pi} q^3/\omega} e^{i\delta_{33}} \sin\delta_{33} \tag{IV.51}$$

Now the $A_i$ are linear in the $F_i$ and so our final expressions will be

$$\mathcal{F}_V = \frac{2f\mu_V^c(\lambda^2)}{3M\omega} \frac{1}{\frac{4}{3}\frac{f^2}{4\pi}\frac{q^3}{\omega}} \Big(2\underline{q}\times\underline{k}\cdot\underline{\varepsilon}$$

$$+ i\underline{\sigma}\cdot\underline{\varepsilon}\,\underline{q}\cdot\underline{k} - i\underline{\sigma}\cdot\underline{k}\,\underline{q}\cdot\underline{\varepsilon}\Big) e^{i\delta_{33}} \sin\delta_{33}$$

$$= \frac{2\pi\mu_V^c(\lambda^2)}{fMq^3} \Big(2\underline{q}\times\underline{k}\cdot\underline{\varepsilon} + i\underline{\sigma}\cdot\underline{\varepsilon}\,\underline{q}\cdot\underline{k}$$

$$- i\underline{\sigma}\cdot\underline{k}\,\underline{q}\cdot\underline{\varepsilon}\Big) e^{i\delta_{33}} \sin\delta_{33}$$

(IV.52)

This expression is given in F.N.W.

And

$$\mathcal{F}_A = \frac{4f}{3\omega}\left(-\frac{G_A}{G}\right)\alpha(\lambda^2) \frac{1}{\frac{4}{3}\frac{f^2}{4\pi}\frac{q^3}{\omega}}$$

$$\left(3\underline{q}\cdot\underline{\varepsilon} - \underline{\sigma}\cdot\underline{q}\,\underline{\sigma}\cdot\underline{\varepsilon}\right) e^{i\delta_{33}} \sin\delta_{33}$$

$$= \frac{4\pi\left(-\frac{G_A}{G}\right)\alpha(\lambda^2)}{fq^3}\left(3\underline{q}\cdot\underline{\varepsilon} - \underline{\sigma}\cdot\underline{q}\,\underline{\sigma}\cdot\underline{\varepsilon}\right) e^{i\delta_{33}} \sin\delta_{33} \tag{IV.52}$$

Let us write

$$M_+ = \frac{2\pi \mu_V^c(\lambda^2)}{f M q^3} \qquad M_- = \frac{4\pi\left(-\frac{G_A}{G}\right)\alpha(\lambda^2)}{f q^3} \tag{IV.53}$$

## Cross Section Calculations

In general, for weak pion production with electron, we can write

$$\mathcal{F}_V = \underline{\mathcal{F}}_V \cdot \underline{\mathcal{E}} \qquad \mathcal{F}_A = \underline{\mathcal{F}}_A \cdot \underline{\mathcal{E}}$$

where

$$\underline{\mathcal{F}}_V = \underline{P} + i\underline{\sigma}Q + i(\underline{\sigma}\cdot\underline{u}_1\,\underline{q} + \underline{\sigma}\cdot\underline{u}_2\,\underline{k})$$

$$\underline{\mathcal{F}}_A = \underline{X} + i(\underline{\sigma}\cdot\underline{v}_1\,\underline{q} + \underline{\sigma}\cdot\underline{v}_2\,\underline{k}) + \underline{\sigma}\cdot\underline{Y}\,\underline{\sigma}$$

(IV.54)

We can find $\underline{P}$, Q etc. from the definitions of $\mathcal{F}_V$ and $\mathcal{F}_A$ in IV.13 and IV.29.

Then providing all the quantities $\underline{P}$, Q,.... are real, we can write (averaging over initial and summing over final nucleon spin)

$$T_{\alpha\beta} = \frac{1}{2}\sum_{i,f} \chi_i^*(\mathcal{F}_{V\alpha}^* - i\,\mathcal{F}_{A\alpha}^*)\chi_f\,\chi_f^*(\mathcal{F}_{V\beta} + i\,\mathcal{F}_{A\beta})\chi_i$$

$$= \frac{1}{2}\,Sp\left[(\mathcal{F}_{V\alpha}^* - i\,\mathcal{F}_{A\alpha}^*)(\mathcal{F}_{V\beta} + i\,\mathcal{F}_{A\beta})\right]$$

$$
\begin{aligned}
= \; & P_\alpha P_\beta + X_\alpha X_\beta + q_\alpha q_\beta (\underline{u}_1^2 + \underline{v}_1^2) \\
& + k_\alpha k_\beta (\underline{u}_2^2 + \underline{v}_2^2) + (k_\alpha q_\beta + q_\alpha k_\beta)(\underline{u}_1 \cdot \underline{u}_2 + \underline{v}_1 \cdot \underline{v}_2) \\
& + (Q^2 + Y^2)\delta_{\alpha\beta} + X_\alpha Y_\beta + X_\beta Y_\alpha \\
& + Q(u_{1\alpha} q_\beta + u_{1\beta} q_\alpha + u_{2\alpha} k_\beta + u_{2\beta} k_\alpha) \\
& + Y_\delta \left[ \varepsilon_{\lambda\delta\beta} (v_{1\lambda} q_\alpha + v_{2\lambda} k_\alpha) - \varepsilon_{\alpha\delta\lambda} (v_{1\lambda} q_\beta + v_{2\lambda} k_\beta) \right] \\
& + i \Big\{ (P_\alpha X_\beta - P_\beta X_\alpha) + (P_\alpha Y_\beta - P_\beta Y_\alpha) \\
& \quad + (k_\alpha q_\beta - k_\beta q_\alpha)(\underline{u}_2 \cdot \underline{v}_1 - \underline{u}_1 \cdot \underline{v}_2) \\
& \quad + 2Q\, \varepsilon_{\alpha\delta\beta} Y_\delta \\
& \quad + Q \left[ (v_{1\alpha} q_\beta - v_{1\beta} q_\alpha) + (v_{2\alpha} k_\beta - v_{2\beta} k_\alpha) \right] \\
& \quad + Y_\delta \left[ \varepsilon_{\lambda\delta\beta} (u_{1\lambda} q_\alpha + u_{2\lambda} k_\alpha) + \varepsilon_{\alpha\delta\lambda} (u_{1\lambda} q_\beta + u_{2\lambda} k_\beta) \right] \Big\}
\end{aligned}
\tag{IV.54}
$$

All Greek indices from now on take on the values 1, 2, 3. Roman will take 1, 2, 3, 4. $\epsilon_{\alpha\beta\gamma}$ is the 3-index permutation symbol. Notice that $T_{\alpha\beta}$ has a symmetric term which does not mix vector and axial vector components, and a completely antisymmetric term which does.

$T_{\alpha\beta}$ must be contracted with the leptonic contribution

$$
\begin{aligned}
e_{\alpha\beta} &= Sp\left[ \not{t}_1 \gamma_\alpha (1+\gamma_5) \not{t}_2 \gamma_\beta (1+\gamma_5)\right] \\
&= 8 \Big[ t_{1\alpha} t_{2\beta} + t_{2\alpha} t_{1\beta} - t_1 \cdot t_2 \, \delta_{\alpha\beta} \\
&\qquad - \varepsilon_{\alpha p \beta q} t_{2p} t_{1q} \Big] \\
&= 8 \Big[ 2 t_{1\alpha} t_{2\beta} - t_{1\alpha} k_\beta - k_\alpha t_{1\beta} + \tfrac{1}{2}\lambda^2 \delta_{\alpha\beta} \\
&\qquad + \varepsilon_{\alpha p \beta q} k_p t_{1q} \Big]
\end{aligned}
\tag{IV.55}
$$

$t_1, t_2$ are the four-momenta of the neutrino and electron respectively, $\not{t}_1 = \gamma \cdot t_1$, $k = t_1 - t_2$, $\varepsilon_{\alpha p \beta q}$ is the 4-index completely antisymmetric permutation symbol.

Write

$$|m|^2 = T_{\alpha\beta} e_{\alpha\beta} \tag{IV.56}$$

Now the differential cross section can be written in the form

$$
\sigma v = \frac{2 g^4 M^2}{8 E_1 E_2 q_0 t_1 t_2} \frac{|m|^2}{(\lambda^2 + M_W^2)^2} (2\pi)^4 \delta^4 (p_2 + q_2 + t_2 - t_1 - p_1)
$$
$$
\frac{d^3 p_2}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \frac{d^3 t_2}{(2\pi)^3} \tag{IV.57}
$$

We will be concerned with an experiment in which the initial nucleon is at rest; that is $E_1 = M$ and $v = 1$. However $|m|^2$ must be evaluated in the centre of mass system of the final nucleon and pion. So let us use letters as before for C.M. quantities, and capital letters for the quantities evaluated in the laboratory frame; i.e. $T_1$, $T_2$, $K$, $K_o$, $P_2$, $Q$, $Q_o$ ($E_1^{Lab} = M$ and $E_2^{Lab}$ never appears).

Then the differential cross section for production of a pion in solid angle $d\Omega_q$ electron in solid angle $d\Omega_{t_2}$ is

$$\sigma = \frac{g^4 M}{128\pi^5 Q_o T_1} \frac{|m|^2}{(\lambda^2+M_w^2)^2} \frac{Q^2 T_2^2 \, d\Omega_q \, d\Omega_{t_2} \, dQ}{T_2[T_1+M-Q_o] - \underline{T}_2\cdot(\underline{T}_1-\underline{Q})} \tag{IV.57}$$

It is possible, following Dalitz and Yennie (25), to obtain the cross section for inelastic lepton scattering in a simple form. (Inelastic lepton scattering implies that only the final lepton is observed.)

First notice that

$$|m|^2 \frac{d^3p_2}{E_2} \frac{d^3q}{q_o} \delta^4(p_2+q-k-p_1) \tag{IV.58}$$

is a Lorentz scalar and hence can be evaluated in any frame; in particular in the centre of mass frame of the final pion and nucleon. So it is just

$$|m|^2 \frac{d^3p_2}{E_2} \frac{d^3q}{q_o} \delta^3(p_2+q)\,\delta(E_2+q_o-k_o-E_1) \tag{IV.59}$$

This can be integrated to

$$\frac{q}{W} |m|^2 d\Omega_q = \frac{4\pi q}{W} d\bar{\Omega}_q \, |m|^2 \tag{IV.60}$$

where $d\bar{\Omega}_q$ implies that we are going to average over the directions of the final pion.

Write

$$\langle m^2 \rangle = \int \frac{4\pi q}{W} |m|^2 \, d\bar{\Omega}_q \qquad \text{(IV.61)}$$

Then

$$\frac{d^2\sigma}{d\Omega \, dT_2} = \frac{g^4 M}{128\pi^5} \frac{T_2}{T_1} \frac{\langle m^2 \rangle}{(\lambda^2 + M_W^2)^2} \qquad \text{(IV.62)}$$

If we are not interested in looking for intermediate boson effects, remember

$$g^2 = \frac{G M_W^2}{2\sqrt{2}} \qquad \text{(II.3)}$$

so

$$\frac{g^4}{(\lambda^2 + M_W^2)^2} \sim \frac{G^2}{8} \qquad \text{(IV.63)}$$

Now we have to find the form of $|m|^2$ in the static limit. Here

$$\begin{aligned}
\underline{P} &= 2(\underline{q} \times \underline{k}) M_+ e^{i\delta_{33}} \sin \delta_{33} \\
\varphi &= \underline{q} \cdot \underline{k} \, M_+ e^{i\delta_{33}} \sin \delta_{33} \\
\underline{u}_1 &= -\underline{k} \, M_+ e^{i\delta_{33}} \sin \delta_{33} \qquad \underline{u}_2 = 0 \\
\underline{X} &= 3\underline{q} \, M_- e^{i\delta_{33}} \sin \delta_{33} \\
\underline{Y} &= -\underline{q} \, M_- e^{i\delta_{33}} \sin \delta_{33} \\
\underline{V}_1 &= \underline{V}_2 = 0
\end{aligned} \qquad \text{(IV.64)}$$

Then

$$\frac{|m|^2}{\sin^2\delta_{33}} = 8q^2k^2M_+^2\Big[8t_1^2\sin^2\alpha\sin^2\phi\sin^2\psi$$
$$+ 2t_1^2\cos^2\alpha + 2t_1^2\cos^2\theta - 4t_1^2\cos\psi\cos\alpha\cos\theta + \tfrac{1}{2}\lambda^2(5-3\cos^2\theta)\Big]$$
$$+ 8q^2M_-^2\big[2t_1^2 + 6t_1^2\cos^2\alpha - 2t_1k\cos\psi$$
$$- 6t_1k\cos\alpha\cos\theta + 3\lambda^2\big]$$
$$+ \frac{176}{3}q^2kt_1M_+M_-(k - k_0\cos\psi) \qquad \text{(IV.65)}$$

$\alpha$ is the angle between $\underline{t}_1$ and $\underline{q}$, $\phi$ is the angle between the $\underline{t}_1, \underline{k}$ and $\underline{t}_1, \underline{q}$ planes, $\psi$ is the angle between $\underline{t}_1$ and $\underline{k}$. Clearly,

$$\cos\theta = \cos\alpha\cos\psi + \sin\alpha\sin\psi\cos\phi \qquad \text{(IV.66)}$$

Now $\langle m^2\rangle$ can be found by averaging over $\alpha$ and $\phi$. It is

$$\Big\{16q^2k^2M_+^2(2t_1^2\sin^2\psi + \lambda^2)$$
$$+ 8q^2M_-^2(4t_1^2 + 3\lambda^2 - 4t_1k\cos\psi)$$
$$+ \frac{176}{3}q^2kt_1M_+M_-(k - k_0\cos\psi)\Big\}\frac{4\pi q}{W}\sin^2\delta_{33} \qquad \text{(IV.67)}$$

In order for the final pion-nucleon system to be resonant $k_o$ must have the right magnitude. However k and hence $\lambda^2$ are not limited. Hence it is possible, in theory at least, to gain information on the form factors as functions of $\lambda^2$, either by observing both the final electron and pion (this sort of experiment has not yet even been done for electroproduction) or by finding the energy distribution at fixed angle of the electron. This curve is peaked at the energy of the electron which leads to the resonant energy of the pion-nucleon system. The cross-section evidently increases as $t_1^{\,2}$, but in non-forward directions $k^2$ and $\lambda^2$ also increase and so the form factors limit the cross section. In the forward direction (of electron) the $M_-^{\,2}$ term remains and increases as $t_1^{\,2}$; $\lambda^2 = 0$, and so at high energies this direction becomes increasingly dominant (as one would expect). The pion distribution is isotropic eventually at high energy. Curves are drawn in Figs. 7 and 8 for inelastic scattering of the neutrino, observing the electron at $10^{\circ}$, for incident energies of about 1 and 5 Bev. At 5 Bev the resonant energy can just be reached for $10^{\circ}$ deflection of the lepton, and thus the cross-section is about maximum for this particular angle. At higher energy $K_o$ will have increased beyond 200 Mev. The calculations were done keeping $\alpha(\lambda^2)$ as unity. This is of course incorrect, but if three pions in an $I = 1$, $J = 1$ pseudovector state do not interact strongly, $\alpha(\lambda^2)$ should vary slowly. As forecast previously, the axial vector term is dominant at low energy, and less dominant at higher energies, especially if $\alpha(\lambda^2)$ varies more like $F_1^{\,V}(\lambda^2)$ than it is assumed to do here. Fig. 9 shows the pion angular distribution at resonance with a $10^{\circ}$ deflection of the lepton at incident energy around 5 Bev. The pion, electron and incident neutrino are taken to be in the same plane. In this case only the axial vector term varies significantly with the angle $\alpha$. Both the vector and the interference only vary with $\alpha$ through the phase space factor. So as the interference contribution is also about the same size as the vector term, it is not drawn in Fig. 9.

# V. INTERMEDIATE BOSON PRODUCTION IN PION-NUCLEON COLLISIONS

Another process which is covered by the same amplitudes that we have found is

$$\pi + N \longrightarrow W + N$$

It is just necessary to put $\lambda^2 = - M_W^{\;2}$. The value of $M_W$ is at least $M_K$ so there is no possibility of the initial pion-nucleon state being resonant. So let us just consider the Born approximation IV.5 and IV.27 for this process. We could just as well use the amplitude $\mathcal{F}$ as the amplitude $\mathcal{M}$, and then the differential cross-section in the centre of mass system is

$$\frac{d\sigma}{d\Omega} = \frac{g^2}{16\pi^2} \frac{k}{q} \frac{M^2}{W^2} \left| \langle 2 | \mathcal{F} | 1 \rangle \right|^2 \tag{V.1}$$

where the matrix element includes the sum over final and average over initial spin states.

Now

$$\left| \langle 2 | \mathcal{F} | 1 \rangle \right|^2 = \left| \langle 2 | \mathcal{F}_V + i \mathcal{F}_A | 1 \rangle \right|^2$$

By virtue of the gauge-invariance of the vector current, the polarisation vector of W can be taken to be spacelike only

i.e. $$e_\mu = (\underline{e}, 0), \qquad \underline{e}^2 = 1 \tag{V.2}$$

Also in the Born approximation we can write

$$\begin{aligned} \mathcal{F}_V &= P + i \underline{\sigma} \cdot \underline{Q} \\ \mathcal{F}_A &= X + i \underline{\sigma} \cdot \underline{Y} \end{aligned} \tag{V.3}$$

where P, X are real scalars, linear in $\underline{e}$ and $\underline{Q}$, $\underline{Y}$ are real vectors, linear in $\underline{e}$. Then $|\langle 2|\mathcal{J}|1\rangle|^2 =$

$$\frac{1}{2}\sum_{i,f} \chi_i^*(P - iX - i\underline{\sigma}\cdot(\underline{Q} - i\underline{Y}))\chi_f \chi_f^*(P + iX + i\underline{\sigma}\cdot(\underline{Q} + i\underline{Y}))\chi_i$$

$$= \frac{1}{2} Sp[(P - iX - i\underline{\sigma}(\underline{Q} - i\underline{Y}))(P + iX + i\underline{\sigma}\cdot(\underline{Q} + i\underline{Y}))]$$

$$= P^2 + X^2 + \underline{Q}^2 + \underline{Y}^2$$

or

$$|\langle 2|\mathcal{J}|1\rangle|^2 = |\langle 2|\mathcal{J}_V|1\rangle|^2 + |\langle 2|\mathcal{J}_A|1\rangle|^2 \tag{V.4}$$

Thus we can consider $\mathcal{J}_V$, $\mathcal{J}_A$ separately. For definiteness, let us just consider

$$\pi^+ + p \rightarrow W^+ + p \tag{V.5}$$

So we only need the isotopic 3/2 state

$$A^{(3/2)} = A^+ - A^- \quad etc.$$

For the vector amplitude $\mathcal{M}_V = \sum A\, M_A$ (IV.2) we have from IV.4

$$A^{3/2} = -\frac{2f\, F_1^V(\lambda^2)}{\nu_B + \nu}$$

$$B^{3/2} = \frac{f\, F_1^V(\lambda^2)}{M\nu_B(\nu_B + \nu)}$$

$$
\begin{aligned}
C^{(3/2)} &= \frac{-f\mu^V F_2^V(\lambda^2)}{M(\nu_B + \nu)} \\
D^{(3/2)} &= \frac{f\mu^V F_2^V(\lambda^2)}{M(\nu_B + \nu)} \\
E^{(3/2)} &= \frac{4Mf}{\lambda^2}\left( -\frac{2F_\pi(\lambda^2)}{4M\nu_B - \lambda^2} + \frac{F_V'(\lambda^2)}{2M\nu_B} \right) \\
F^{(3/2)} &= 0
\end{aligned}
\qquad (V.6)
$$

and $\lambda^2 = -M_W^2$

Using equations IV.15 we can now find the amplitudes $\mathcal{J}_1 \ldots \mathcal{J}_6$.

Then

$$
\begin{aligned}
|\langle 2|\mathcal{J}_V|1\rangle|^2 &= \mathcal{J}_1^2 + \mathcal{J}_2^2 \\
&+ (\mathcal{J}_1 + \mathcal{J}_4)^2 + (\mathcal{J}_1 + \mathcal{J}_5)^2 + (\mathcal{J}_2 + \mathcal{J}_3)^2 + \mathcal{J}_6^2 \\
&+ 2\cos\theta \left[ (\mathcal{J}_1 + \mathcal{J}_4 + \mathcal{J}_5)(\mathcal{J}_3 + \mathcal{J}_6) - 2\mathcal{J}_1\mathcal{J}_2 \right] \\
&+ 2\cos^2\theta \left[ \mathcal{J}_4\mathcal{J}_5 + \mathcal{J}_3\mathcal{J}_6 - \mathcal{J}_2\mathcal{J}_3 \right]
\end{aligned}
\qquad (V.7)
$$

The calculation with all form factors equal to unity is shown in Fig. 10 for total cross-section as a function of $M_W$ at fixed energy. The order of magnitude of the cross-section is what one would expect. However, we have not included the form factors, discussed in detail in Appendix I. As is shown there, in order to explain the observed electromagnetic form factors for $\lambda^2 > 0$, a strong I = 1, J = 1 $\pi$-$\pi$ resonance is needed (or a spin one boson) with energy of about 4.5 pion masses. Hence all the form factors occurring in the vector amplitude become extremely large in the vicinity of $\lambda^2 \sim -22M_\pi^2$. From the drawn form factors, Figs. 12, 13, we can construct the following table of the enhancement of the vector part of the total cross section for various values of $M_W$ due to the rapid variation of the form factors in the region around $\lambda^2 \sim -22m_\pi^2$.

Table 1. Multiplicative factor given by $\pi$-$\pi$ resonance to curve in Fig. 10 for various values of $M_W$

| | | | | | |
|---|---|---|---|---|---|
| $M_W$ | 3.5 | 4 | 4.5 | 5 | 6 |
| Factor | 7 | 13 | 39 | 36 | 5 |

Now we come to the axial vector part of the problem. It is to be expected that the contribution from this part putting $\alpha(-M_W^2)=1$ and $\beta(-M_W^2) = -\frac{2g_1}{m_\pi^2}\frac{1}{M_W^2-m_\pi^2}$ is the same order of magnitude as the uncorrected calculation for the vector part. Is there then any reason for expecting $\alpha(\lambda^2)$ also to vary quickly for low ($<1$ Bev$^2$)$\lambda^2$? The answer at present is that there is no such reason. The strong three-pion interaction predicted to explain the scalar electromagnetic form factors has no place in the isotopic vector $\alpha(\lambda^2)$. This of course does not mean that no enhancement or otherwise of $\alpha(\lambda^2)$ takes place as nothing at all is yet known of its structure. (A strong $\pi$-$\rho$ s-wave interaction would certainly show up in the structure of both $\alpha(\lambda^2)$ and $\beta(\lambda^2)$).

It is also possible to investigate the effect of the 3-3 resonance on the amplitude. The simplest way to do this (for the vector case, say) is to replace the resonant curve by a delta-function, i.e.

$$\frac{1}{\pi}\,\mathcal{I}m\ A(x) = A_{33}(x)\,\delta\left(-\frac{\omega}{\omega_r}\right) \quad \text{etc.} \tag{V.8}$$

where $\omega_r$ refers to the position of the resonance and $A_{33}(x)$ is the appropriate 3-3 projection of the Born term for A and is given by equations IV.21, 22, 23, 24. For reasons given in Chapter IV it is permissible to omit the Born terms generated by the pion current IV.23 in such an approximation, and the charge terms IV.22 are small. So we are left as before with the magnetic terms IV.21. There is one snag. For $\lambda^2 = -M_W^2$ we are well above the 3-3 resonance energy in the centre of mass system, and equations IV.21 must be analytically continued to this region. $k = \sqrt{k_o^2 - M_W^2}$ is pure imaginary and so we have to introduce the following changes:

$$\tilde{k} = \sqrt{|k_0^2 - M_W^2|}$$

$$\tilde{a} = \frac{2k_0E_2 - M_W^2}{2q\tilde{k}}$$

$$\tilde{\alpha}(\tilde{a}) = 1 - \frac{\tilde{a}}{2}(\pi - 2\tan^{-1}\tilde{a})$$

$$\tilde{\beta}(\tilde{a}) = -\tilde{a} + \frac{1+\tilde{a}^2}{2}(\pi - 2\tan^{-1}\tilde{a})$$

$$\tilde{\gamma}(\tilde{a}) = -3\tilde{a} + \frac{1+3\tilde{a}^2}{2}(\pi - 2\tan^{-1}\tilde{a})$$

$$\cos\tilde{\theta} = \frac{-2M\nu_B + q_0k_0}{q\tilde{k}}$$

$$\tilde{A}_\mu = -\frac{2f}{q\tilde{k}}\,\mu_v^c \tag{V.9}$$

and replace equations IV.21 by

$$\frac{F_\mu^1}{\tilde{A}_\mu \cos\tilde{\theta}} = 3M\tilde{\alpha} + \frac{3}{2}\,\frac{Mq\tilde{k}(W+M)\tilde{\gamma}}{\omega(E_1+M)(E_2+M)} + \frac{q^2}{E_2+M}\left(1-\frac{3}{2}\tilde{a}\tilde{\beta}\right)$$

$$\frac{F_\mu^2}{\tilde{A}_\mu} = -\frac{M(E_1+M)(E_2+M)}{q\tilde{k}(W+M)}\,\omega\tilde{\alpha} - \frac{E_2+M}{2}\tilde{\beta} - \frac{M}{2}\tilde{\gamma}$$

$$\frac{F_\mu^3}{\tilde{A}_\mu} = \frac{3}{2}\tilde{\beta} - \left(1-\frac{3}{2}\tilde{a}\tilde{\beta}\right)\frac{E_1+M}{E_2+M}\,\frac{\omega}{W+M}\,\frac{q}{\tilde{k}}$$

$$F_\mu^4 = 0$$

$$\frac{F_\mu^5}{\tilde{A}_\mu \cos\tilde{\theta}} = \frac{3}{2}\omega\tilde{\alpha} - \frac{3}{4}\,\frac{q}{\tilde{k}}\,\frac{W+M}{E_2+M}(2E_2+E_1-M)\tilde{\gamma} - \frac{q^2}{\tilde{k}^2}\,\frac{\omega(E_1+M)}{E_2+M}\left(\frac{5}{2}\tilde{a}\tilde{\gamma}-2\tilde{\alpha}\right)$$

$$\frac{F_\mu^6}{\tilde{A}_\mu} = \frac{\omega(E_1+M)(E_2+M)}{2q\tilde{k}}\tilde{\alpha} + \left\{\frac{(W+M)(E_1-M)}{4(E_1+M)} - \frac{\omega\tilde{a}}{2}\,\frac{q}{\tilde{k}}\right\}\tilde{\gamma}$$

(V.10)

A large increase in cross-section is obtained for the energies and values of $M_W$ so far considered. But this result should not be taken seriously. It is only an indication that the Born approximation is not necessarily correct, even with the predicted form factors included.

One reason why the result is rather absurd comes from looking at the value of $\cos\tilde{\theta}$ in this unphysical region. For example, when $N = 3.5\ m_\pi$ and for $W = 12\ m_\pi$, $-1 \leqslant \cos\theta \leqslant +1$ in the real world implies $.75 \leqslant \cos\tilde{\theta} \leqslant 7.1$ in the equations for the effect of the 3-3 resonance.

Thus, the dangers of analytic continuation, starting with an approximate formula, are shown.

It is thus concluded that if $M_W$ lies between 4 and 5 pion masses, the cross-section for its production off nucleons (by any means) is much higher than would at first be thought. This effect is due to the W resonating with the $\rho$-meson.

# APPENDIX I: The Form Factors

We expect all form factors to satisfy dispersion equations. For example, the nucleon electromagnetic form factors:

$$F_1^S(s) = \frac{1}{\pi}\int_9^\infty \frac{g_i^S(s')}{s'-s}\,ds' \qquad \text{(A.I.1)}$$

$$F_1^V(s) = \frac{1}{\pi}\int_4^\infty \frac{g_i^V(s')}{s'-s}\,ds' \qquad \text{(A.I.2)}$$

$i = 1$ or $2$, $s = -\lambda^2$, and we have put $m_\pi = 1$. Also the electromagnetic form factor of the pion:

$$F_\pi(s) = \frac{1}{\pi}\int_4^\infty \frac{g_\pi(s')}{s'-s}\,ds' \qquad \text{(A.I.3)}$$

Experimental information on the nucleon form factors for negative s yields (26)

$$F_1^V(s) = F_2^V(s) = -.20 + \frac{1.20}{1-\frac{s}{22.4}}$$

$$F_1^S(s) = .44 + \frac{0.56}{1-\frac{s}{10.5}}$$

$$F_2^S(s) = 4.0 + \frac{-3.0}{1-\frac{s}{10.5}} \qquad \text{(A.I.4)}$$

Fraser and Fulco (27) pointed out that in order to obtain theoretically equations like A.I.4 for $F_i^V$ it was necessary to postulate a resonance in the $I = 1$, $J = 1$ system of two pions. They showed that with such a resonance the weight functions $g_i^V(s)$ in A.I.2 are given approximately by

$$g_i^V(s) = |F_\pi(s)|^2 [g_i^V(s)]_0 \qquad \text{(A.I.5)}$$

where $[g_i^V(s)]_0$ is given in the papers on nucleon structure neglecting such $\pi$-$\pi$ effects (28).

Bowcock, Cottingham, and Lurie (29) then assumed a Breit-Wigner form for the I = 1, J = 1 scattering amplitude. They wrote

$$f_{\pi\pi}^{11} = \frac{\gamma}{s_r - s - i\gamma\nu^3}, \quad \nu^2 = \frac{1}{4}s - 1 \qquad \text{(A.I.6)}$$

Then

$$|F_\pi(s)|^2 = \frac{s_r^2}{(s_r - s)^2 + \gamma^2\nu^6} \qquad \text{(A.I.7)}$$

B.C.L. now determined $s_r$ and $\gamma$ by comparing their final expressions for $F_i^V(s)$ with the experimental curves. They also computed the contribution given by $f_{\pi\pi}^{11}$ to such quantities as the non-resonant phase shifts in $\pi$-N scattering. Their best values were

$$s_r = 22.4 \qquad \gamma = .376 \qquad \text{(A.I.8)}$$

$|F_\pi(s)|^2$ is plotted with these parameters in Fig. 11. For W production (Chapter 5) we need $F_i^V(s)$ in the region of positive s. At resonance, the real part of $F_i^V(s)$ will vanish, so comparing A.I.4 with A.I.6 we can expect in the resonance region

$$F_i^V = 1.20 \frac{22.4}{22.4 - s - i\gamma\nu^3} \qquad \text{(A.I.9)}$$

The real and imaginary parts of $F_i^V$ (omitting the factor of 1.20) are plotted in Figs. 12 and 13.

All this can be viewed in a different way according to the vector meson theory of Gell-Mann and Zacharosen (14). We start with an unstable I = 1, J = 1 meson of mass $m_\rho = \sqrt{22.4}$, and decay rate into two pions given by

$$\Gamma_\rho = \frac{1}{3} \frac{\gamma_{\rho\pi\pi}^2}{4\pi} \frac{(m_\rho^2 - 4)^{3/2}}{m_\rho^2} \tag{A.I.10}$$

Near $s = m_\rho^2$, the form factor for the $\rho\pi\pi$ vertex (real pions) is given by

$$F_{\rho\pi\pi}(s) = \frac{s - m_\rho^2}{s - m_\rho^2 + i m_\rho \Gamma_\rho} \tag{A.I.11}$$

Similarly, for the $\rho NN$ vertex we have the form factor

$$F_1^\rho(s) = \frac{s - m_\rho^2}{s - m_\rho^2 + i m_\rho \Gamma_\rho}, \quad s \sim m_\rho^2 \tag{A.I.12}$$

for the 'charge' term.

Now, in general, the electromagnetic form factors are related to the $\rho$ form factors by

$$F_{el}^V(s) = \frac{-m_\rho^2}{s - m_\rho^2} \frac{F_\rho(s)}{F_\rho(0)} \tag{A.I.13}$$

where $F_{el}^V$ is any isotopic vector electromagnetic form factor, and $F_\rho$ is the corresponding $\rho$ form factor. According to the theories of Sakurai (12) and Gell-Mann (13) the $\rho$-meson is coupled to the isotopic spin current which is conserved. So at zero momentum transfer, it should have a universal interaction with the isotopic spin $\underline{I}$.

This can be expressed by

$$\gamma_\rho = \gamma_{\rho\pi\pi} F_{\rho\pi\pi}(0) = \gamma_{\rho NN} F_{\rho NN}(0) \quad etc. \tag{A.I.14}$$

Now, we can say that

$$F_\pi(s) = \frac{\gamma_\rho}{\gamma_{\rho\pi\pi}} \frac{-m_\rho^2}{s - m_\rho^2 + i m_\rho \Gamma_\rho} \tag{A.I.15}$$

Comparing with A.I.7 we can say that we require

$$\frac{\gamma_\rho}{\gamma_{\rho\pi\pi}} \simeq 1 \tag{A.I.16}$$

and

$$m_\rho \Gamma_\rho = \gamma \left(v(m_\rho)\right)^3$$

or

$$\frac{1}{3} \frac{\gamma_{\rho\pi\pi}^2}{4\pi} \frac{(m_\rho^2 - 4)^{3/2}}{m_\rho} = \gamma \left(\tfrac{1}{4} m_\rho^2 - 1\right)^{3/2}$$

or

$$\frac{\gamma_{\rho\pi\pi}^2}{4\pi} = \frac{3}{8} \gamma m_\rho = .66 \tag{A.I.17}$$

Also

$$F_1^V(s) = \frac{\gamma_\rho}{\gamma_{\rho NN}} \frac{-m_\rho^2}{s - m_\rho^2 + i m_\rho \Gamma_\rho} \tag{A.I.18}$$

for s near $m_\rho^2$.

For $s < 0$, $F_1^V(s)$ is real and the small imaginary term in the denominator of A.I.18 can be neglected.

Then

$$F_1^V(s) = \frac{\gamma_\rho}{\gamma_{\rho NN}} \frac{-m_\rho^2}{s - m_\rho^2} \tag{A.I.19}$$

Comparing with A.I.4 we need

$$\frac{\gamma_\rho}{\gamma_{\rho NN}} \simeq 1.2 \tag{A.I.20}$$

and an additional additive term (-.2). Perhaps this number represents a slowly varying contribution from states other than the two-pion.

We note in finishing this discussion that $F_\pi(s)$ and $F_1^V(s)$ are very nearly equal for s near $m_\rho^2$, and that experimentally $F_1^V$ and $F_2^V$ seem to be equal.

Similar considerations can be applied to the scalar form factors, either considering an $I = 0$, $J = 1$, three pion resonance (30); or a vector meson $\omega$ with these quantum numbers (12), (13), (14). From A.I.4 we see that $m_\omega \sim 3.2\ m_\pi$.

We know very little about the axial vector form factors $\alpha(s)$, $\beta(s)$. A three pion $I = 1$, $J = 1$ interaction would determine much of their structure (or in vector meson language, a strong $\pi - \rho$ coupling in an s state, for $I = 1$).

# APPENDIX II

## EVALUATION OF DISPERSION RELATIONS WHEN PHASE SHIFT IS KNOWN

A simplified version of that given in Reference 20 will be presented.

We start with a dispersion relation in the form

$$A(x, \nu_B) = B(x, \nu_B) + \frac{1}{\pi}\int_{x_0}^{\infty} dy \, \mathrm{Im}\, A(y, \nu_B) \left( \frac{1}{y - x - i\epsilon} \pm \frac{1}{y + x + 2\nu_B} \right) \tag{A. II. 1}$$

Suppose that A has a known phase $\delta(x)$. Then following Omnes (24) define a function F(z) of the complex variable z by

$$e^{\Delta(z)} F(z) = \frac{1}{2\pi i}\int_{x_0}^{\infty} dy \, \mathrm{Im}\, A(y, \nu_B) \left( \frac{1}{y - z} \pm \frac{1}{y + z + 2\nu_B} \right) \tag{A. II. 2}$$

where

$$\Delta(z) = \frac{1}{\pi}\int_{x_0}^{\infty} dy \, \delta(y) \left[ \frac{1}{y - z} + \frac{1}{y + z + 2\nu_B} \right] \tag{A. II. 3}$$

From this last equation we can see that as z approaches the real axis from above or below we may write

$$\Delta_{\pm}(x, \nu_B) = \rho(x, \nu_B) \pm i\delta(x) \tag{A. II. 4}$$

where

$$\rho(x, \nu_B) = \frac{P}{\pi}\int_{x_0}^{\infty} dy \, \delta(y) \left( \frac{1}{y - x} + \frac{1}{y + x + 2\nu_B} \right) \tag{A. II. 5}$$

From the original dispersion relation A. II. 1 we have

$$A(x, \nu_B) - B(x, \nu_B) = 2i \underset{z \to x + i\epsilon}{Lt} e^{\Delta(z)} F(z)$$

$$= 2i\, F_+(x, \nu_B)\, e^{\rho + i\delta}$$

or $$|A(x\nu_B)|\, e^{i\delta} = B(x, \nu_B) + 2i\, F_+(x, \nu_B)\, e^{\rho + i\delta} \qquad \text{(A. II. 6)}$$

Also

$$\underset{z \to x+i\epsilon}{Lt} e^{\Delta(z)} F(z) - \underset{z \to x-i\epsilon}{Lt} e^{\Delta(z)} F(z)$$

$$= \mathcal{I}m\, A(x, \nu_B) = |A(x, \nu_B)| \sin\delta$$

or $$e^{\rho}\left[F_+(x, \nu_B)\, e^{i\delta} - F_-(x, \nu_B)\, e^{-i\delta}\right] = |A| \sin\delta \qquad \text{(A. II. 7)}$$

Now we can eliminate $|A(x, \nu_B)|$ which is unknown and obtain

$$\left[B(x, \nu_B) + 2i\, F_+(x, \nu_B)\, e^{\rho + i\delta}\right] e^{-i\delta} \sin\delta$$

$$= e^{\rho}\left[F_+(x, \nu_B)\, e^{i\delta} - F_-(x, \nu_B)\, e^{-i\delta}\right]$$

or $$F_+(x, \nu_B) - F_-(x, \nu_B) = e^{-\rho} \sin\delta\, B(x, \nu_B) \qquad \text{(A. II. 8)}$$

So, up to the addition of a function which is continuous across the cuts along the real axis, F(z) is given by

$$F(z) = \frac{1}{2\pi i} \int_{x_0}^{\infty} dy\, B(y, \nu_B) \sin\delta(y)\, e^{-\rho(y, \nu_B)} \times \left(\frac{1}{y - z} + \frac{g}{y + z + 2\nu_B}\right) \qquad \text{(A. II. 9)}$$

The function g is arbitrary except it must have the continuity property just mentioned. However we appeal to crossing symmetry and say that only when $g = \pm 1$ can we have a physical solution.

Then the solution of our dispersion equations is

$$A(x, \nu_B) = B(x, \nu_B) + \frac{e^{\rho + i\delta}}{\pi} \int_{x_0}^{\infty} e^{-\rho(y)} \sin\delta(y) B(y, \nu_B) \left( \frac{1}{y - x - i\epsilon} \pm \frac{1}{y + x + 2\nu_B} \right) dy \qquad \text{(A.II.10)}$$

In practice, we will want to use this equation when a given state deominates, for example the 3-3 pion-nucleon resonant state. Then clearly the easiest thing to do is to use $\delta_{33}$ for $\delta$ and $b_{33}(y, \nu_B)$ for $B(y, \nu_B)$ under the integral.

(The solution A.II.10 is not unique as it stands. In general, a quantity $fe^{\rho + i\delta}$ can be added, where f is a polynomial in x. The choice of $f = 0$ gives the simplest theory.)

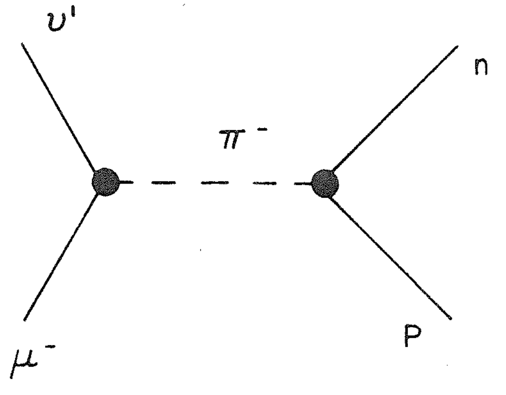


FIG. 1

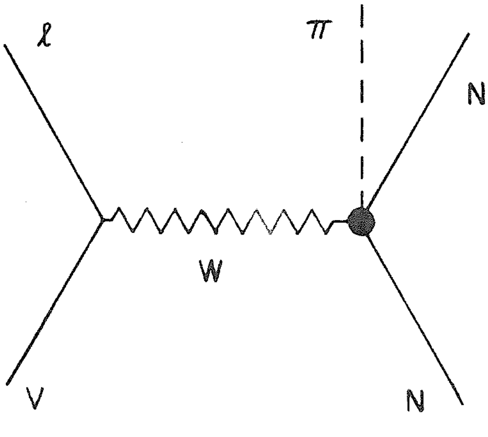


FIG. 2.

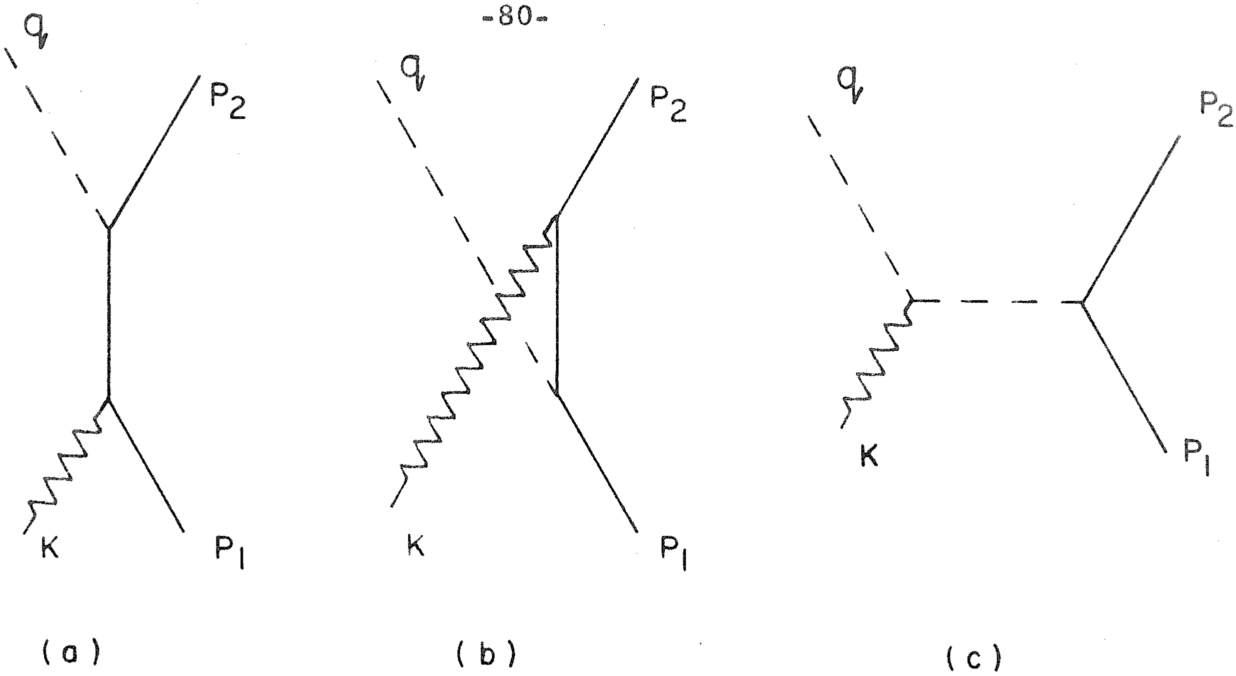


FIG. 3.

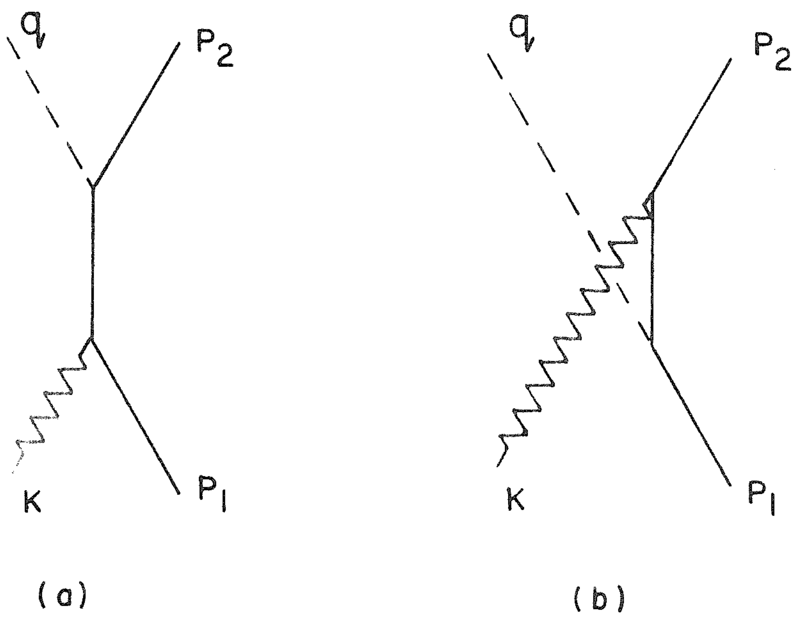


FIG. 4.

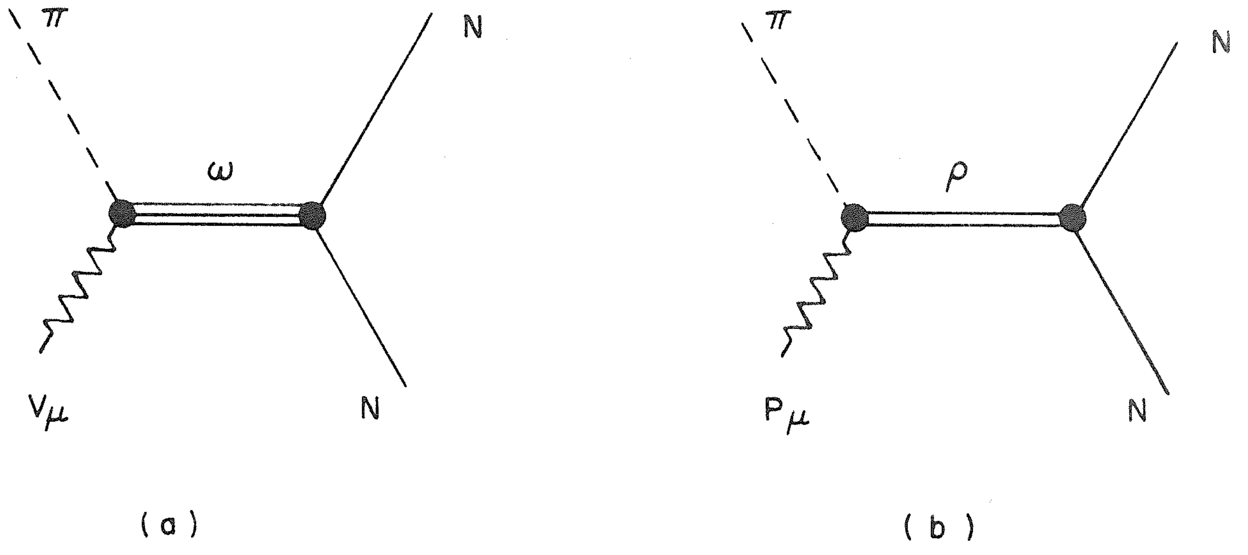


FIG. 5.

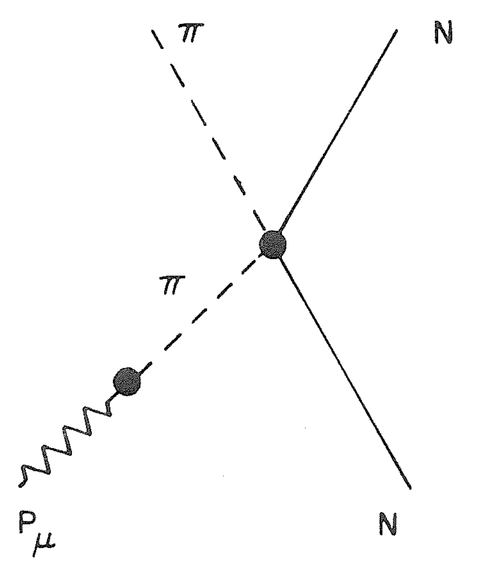


FIG. 6.

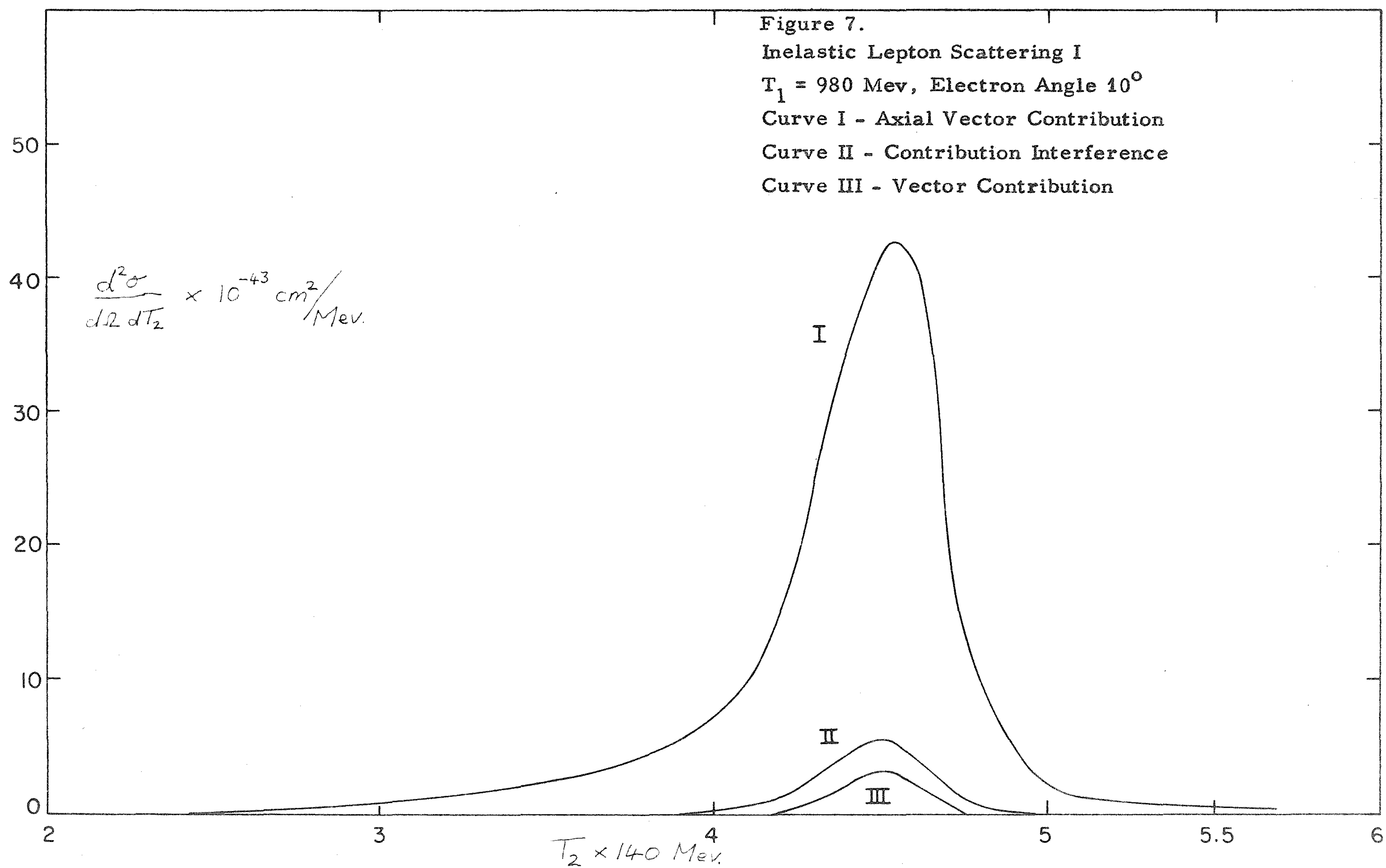


Figure 7.
Inelastic Lepton Scattering I
$T_1$ = 980 Mev, Electron Angle $10^{o}$
Curve I - Axial Vector Contribution
Curve II - Contribution Interference
Curve III - Vector Contribution

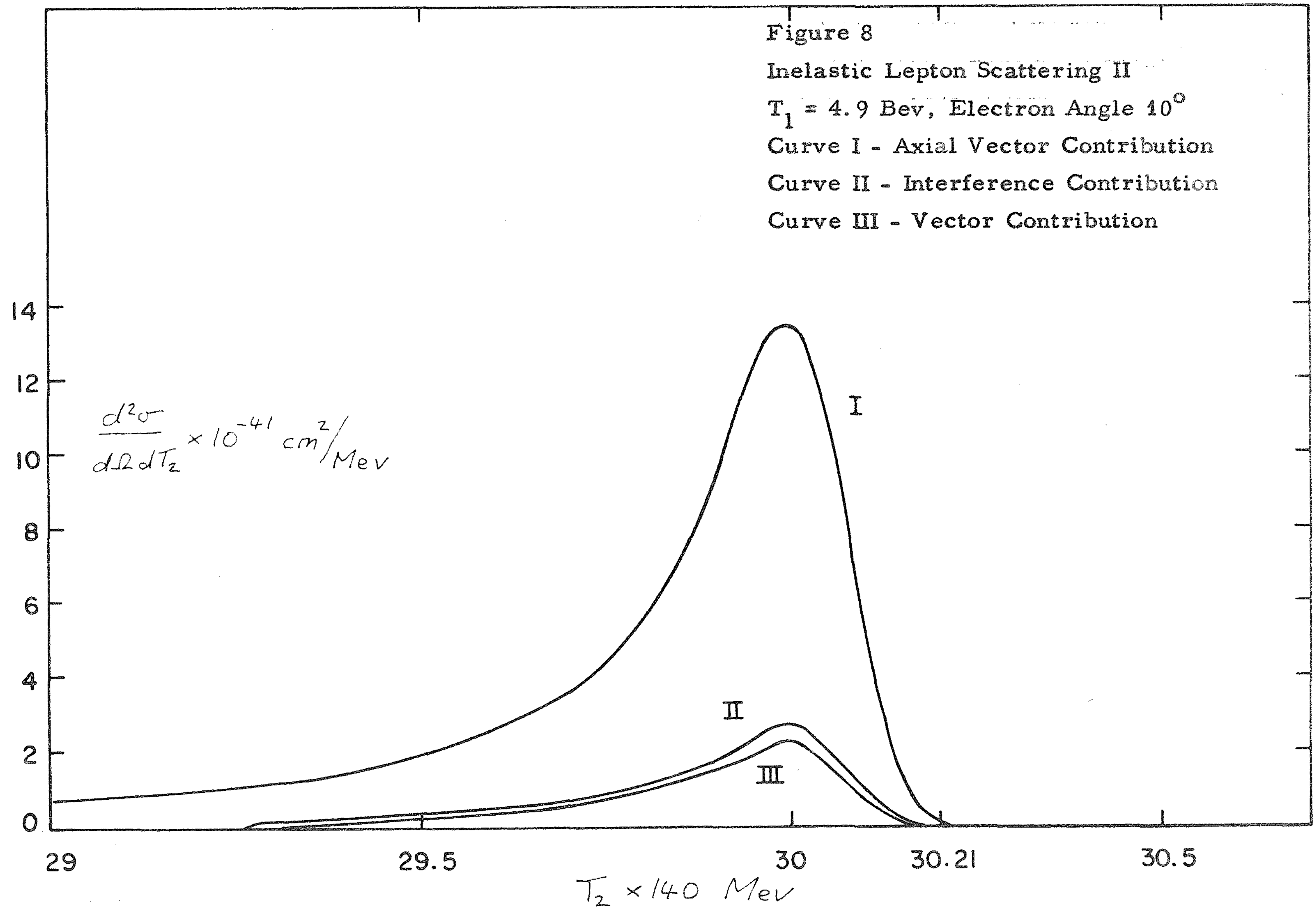


Figure 8

Inelastic Lepton Scattering II

$T_1$ = 4.9 Bev, Electron Angle $10^{\circ}$

Curve I - Axial Vector Contribution

Curve II - Interference Contribution

Curve III - Vector Contribution

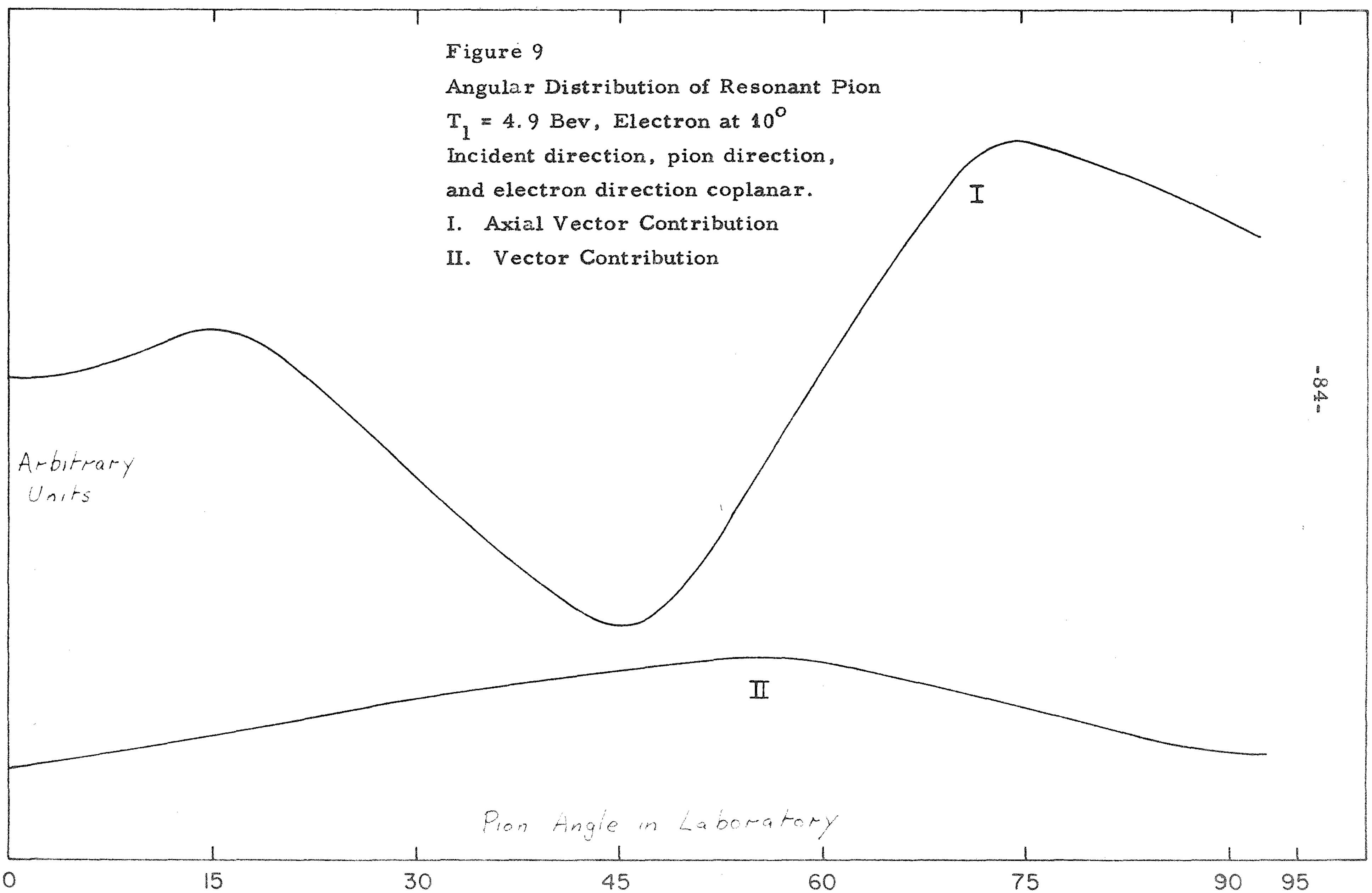


Figure 9
Angular Distribution of Resonant Pion
$T_1$ = 4.9 Bev, Electron at $10^o$
Incident direction, pion direction, and electron direction coplanar.
I. Axial Vector Contribution
II. Vector Contribution

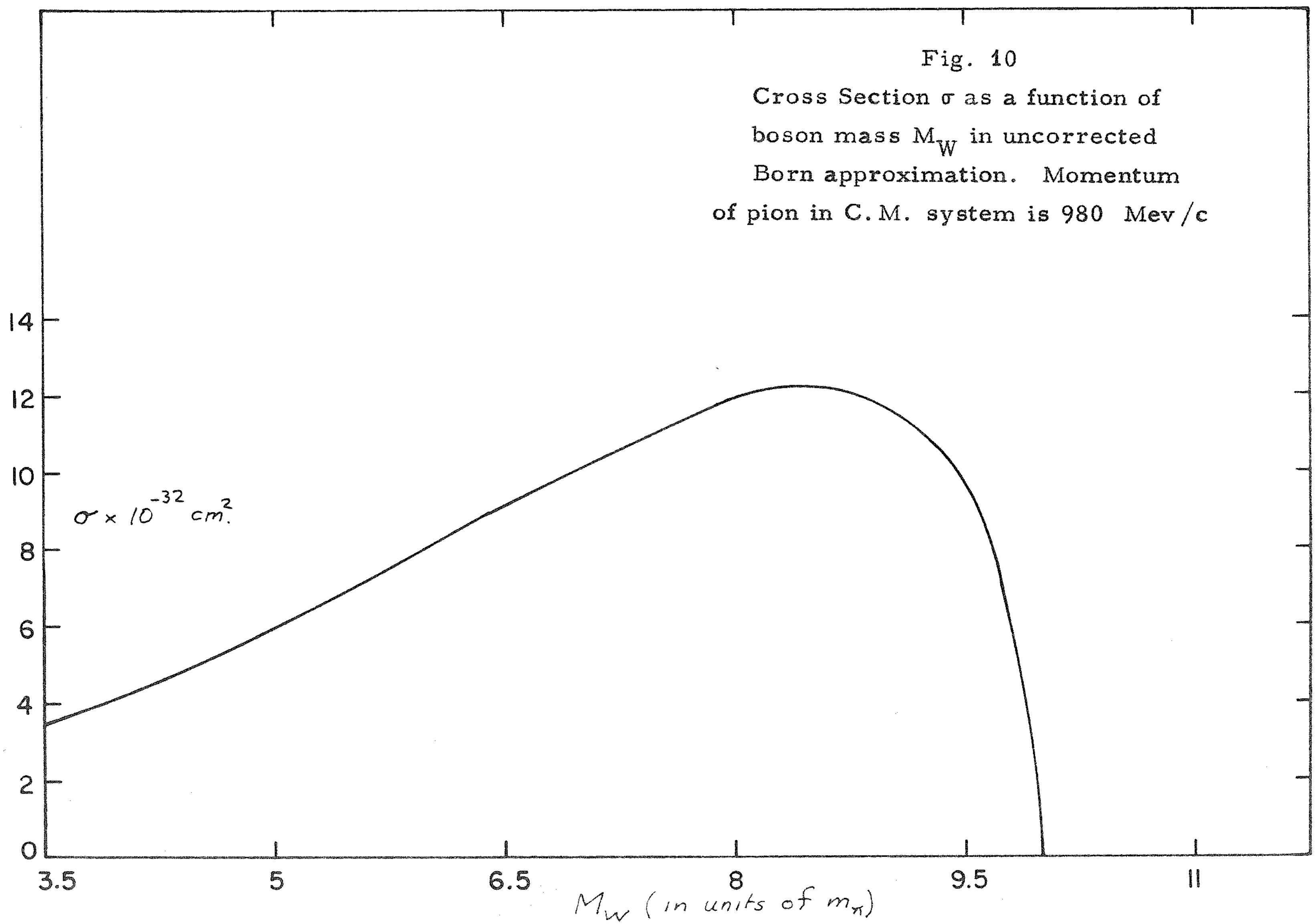


Fig. 10

Cross Section σ as a function of boson mass $M_W$ in uncorrected Born approximation. Momentum of pion in C. M. system is 980 Mev/c

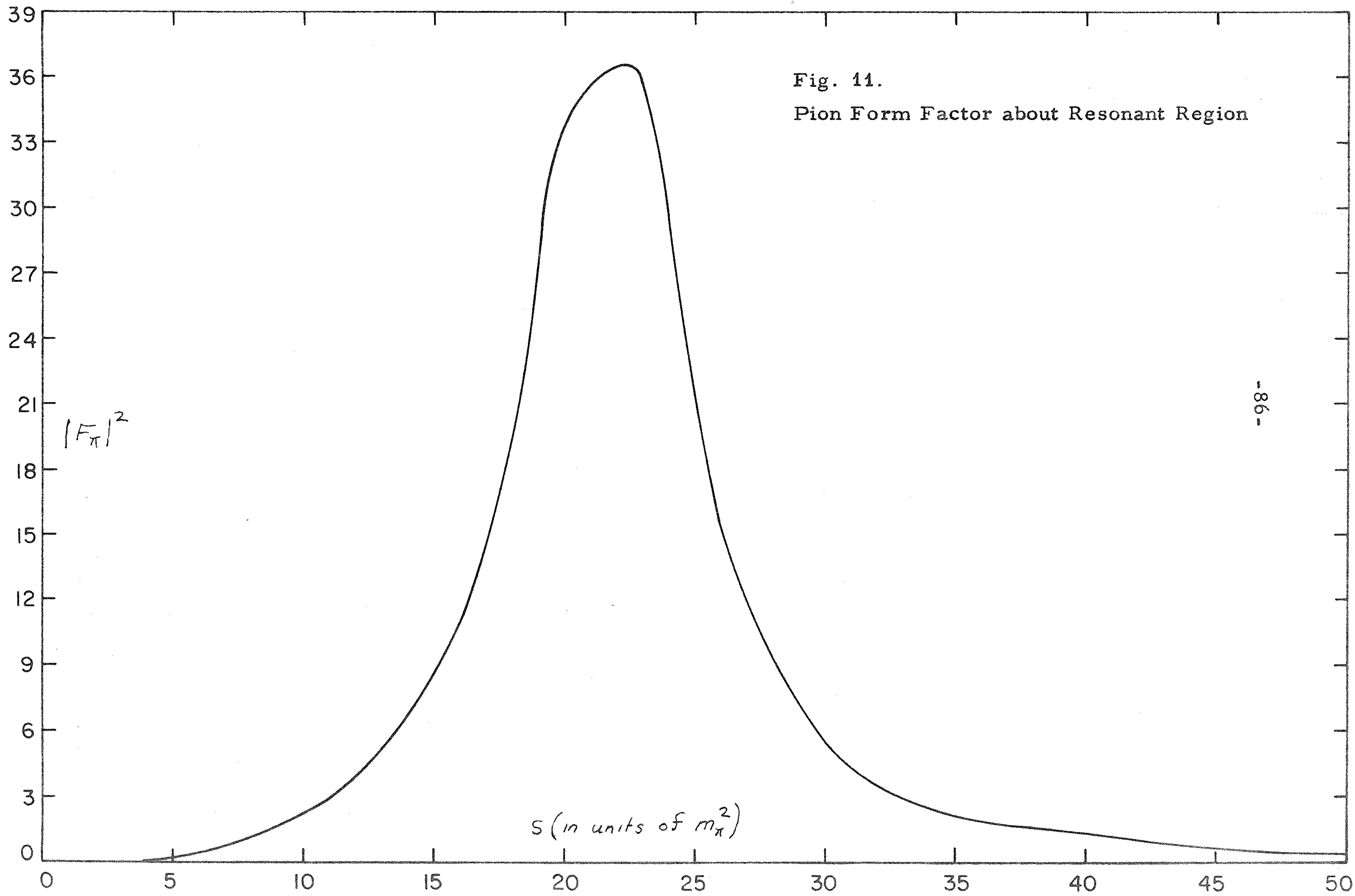


Fig. 11.
Pion Form Factor about Resonant Region

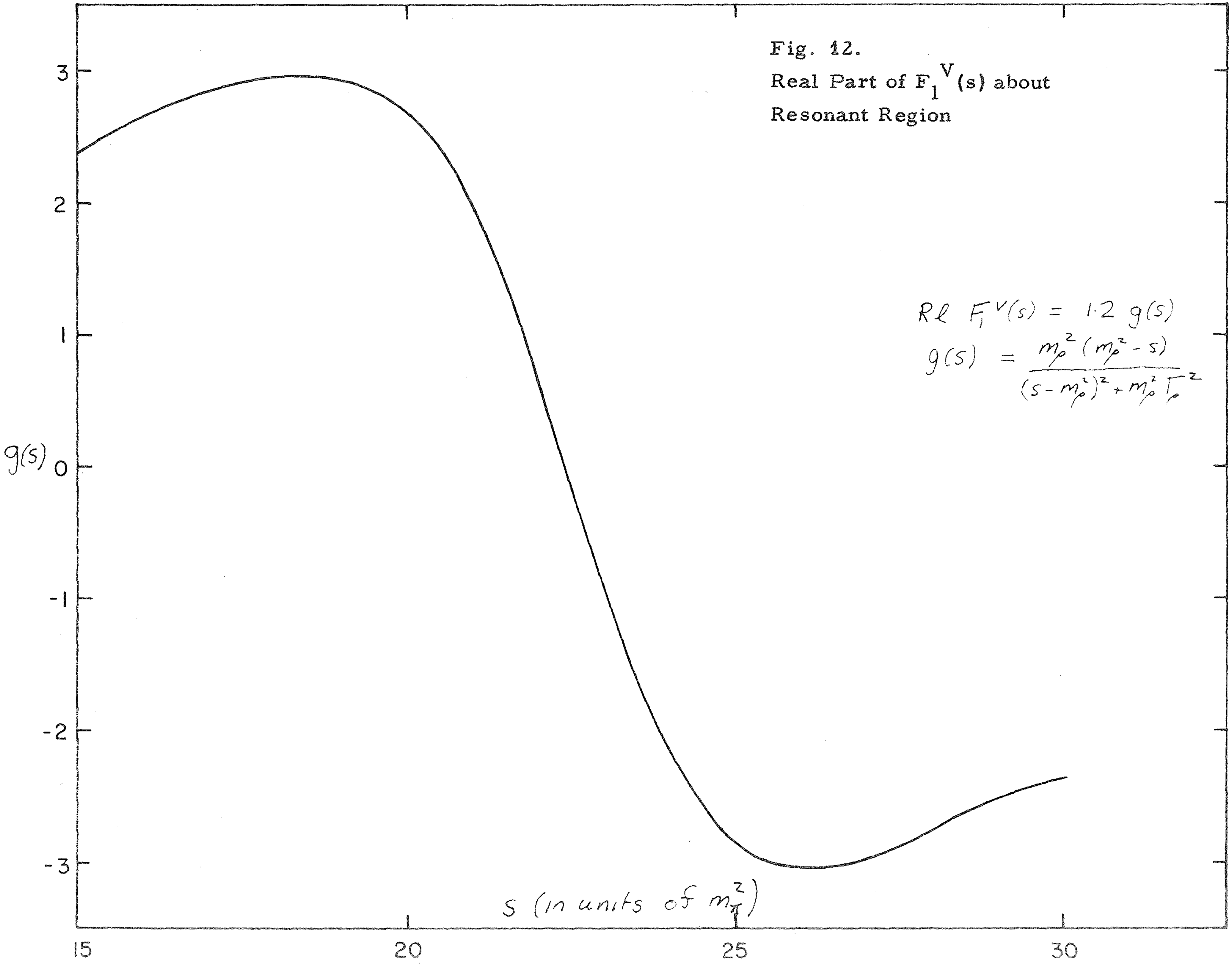
Fig. 12.
Real Part of $F_1^V(s)$ about
Resonant Region
$Re\ F_1^V(s) = 1.2\ g(s)$
$g(s) = \frac{m_\rho^2 (m_\rho^2 - s)}{(s-m_\rho^2)^2 + m_\rho^2 \Gamma_\rho^2}$
g(s)
3
2
1
0
-1
-2
-3
15
20
25
30
s (in units of $m_\pi^2$)

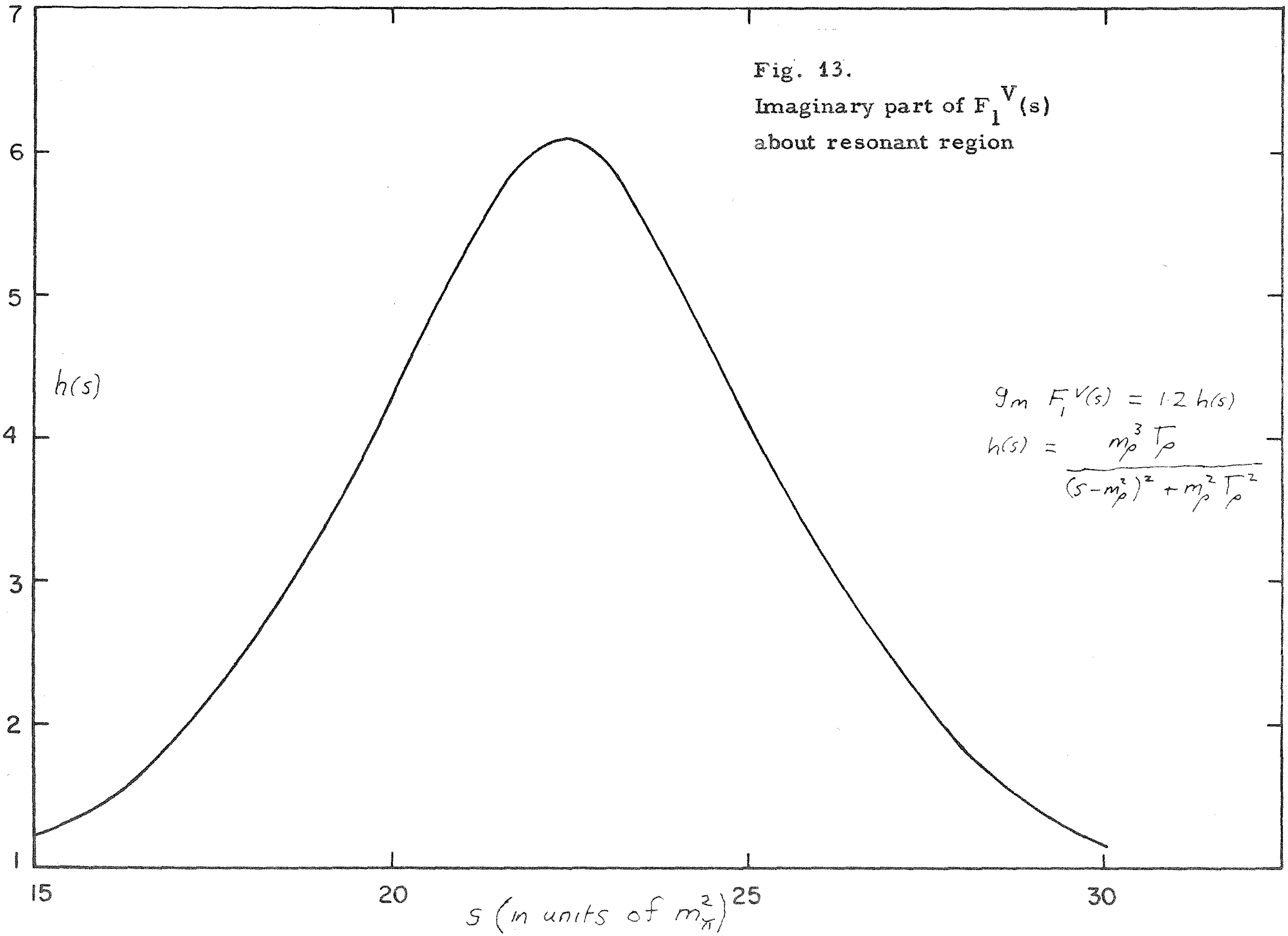


Fig. 13.
Imaginary part of $F_1^V(s)$ about resonant region

## REFERENCES


1. S. M. Berman, Phys. Rev., 112, p. 267, 1958.
2. M. Gell-Mann and M. Levy, Nuovo Cimento, 16, p. 705, 1960.
3. R. P. Feynman and M. Gell-Mann, Phys. Rev., 109, p. 193, 1958.
4. S. S. Gershtein and J. B. Zeldovich, Z.E.T.P., 29, p. 698, 1955.
5. M. L. Goldberger and S. B. Treiman, Phys. Rev., 110, p. 1178, 1958.
6. M. L. Goldberger and S. B. Treiman, Phys. Rev., 111, p. 354, 1958.
7. J. Bernstein, S. Fubini, M. Gell-Mann and W. Thirring, Nuovo Cimento, 17, p. 757, 1960.
8. T. D. Lee and C. N. Yang, Phys. Rev. Lett., 4, p. 307, 1960.
9. R. J. Plano, Phys. Rev., 119, p. 1400, 1960.
10. T. D. Lee and C. N. Yang, Phys. Rev., 119, p. 1410, 1960.
11. W. R. Frazer, Phys. Rev., 115, p. 1763, 1959.
12. J. J. Sakurai, Annals of Physics, 11, p. 1, 1960.
13. M. Gell-Mann, C.I.T. Synchrotron Report CTSL-20, 1961.
14. M. Gell-Mann and F. Zachariasen, C.I.T. Synchrotron Report CTSL-26, 1961.
15. S. Fubini, Y. Nambu, and V. Wataghin, Phys. Rev. 111, p. 329, 1958.
16. G. F. Chew, M. L. Goldberger, F. E. Low and Y. Nambu, Phys. Rev., 106, pp. 1337 and 1345, 1957.
17. S. C. Frautschi and J. D. Walecka, Phys. Rev., 120, p. 1486, 1960.

18. J. S. Ball, U.C.R.L. 9172.

19. M. Cini and S. Fubini, Theory of Low Energy Scattering in Field Theory, CERN report.

20. M. Gourdin, D. Lurie, A. Martin, CERN preprint.

21. J. Bowcock, N. Cottingham and D. Lurie, Nuovo Cimento (to be published).

22. R. Blankenbecler and S. Gartenhaus, Phys. Rev., 116, 1297, 1959.

23. R. Blankenbecler, S. Gartenhaus, R. Huff, and Y. Nambu, Nuovo Cimento, 17, p. 775, 1960.

24. G. F. Chew and F. E. Low, Phys. Rev., 101, p. 1570, 1956.

25. R. H. Dalitz and D. R. Yennie, Phys. Rev., 105, p. 1598, 1957.

26. R. Hofstadter and R. Herman, Phys. Rev. Lett., 6, p. 293, 1961.

27. W. R. Frazer and J. R. Fulco, Phys. Rev., 117, p. 1609, 1960.

28. G. F. Chew, R. Karplus, S. Gasiorowicz and F. Zachariasen, Phys. Rev. 110, p. 265, 1958.

29. J. Bowcock, W. N. Cottingham, and D. Lurie, Phys. Rev. Lett., 5, p. 386, 1960.

30. Y. Nambu, Phys. Rev., 106, p. 1366, 1957.

G. F. Chew, Phys. Rev. Lett., 4, p. 142, 1960.