Rota-Baxter algebras, renormalization on Kausz compactifications and replicating of binary operads

Thesis by

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Abstract

This thesis is divided into two parts:

In the first part, we consider Rota-Baxter algebras of meromorphic forms with poles along a (singular) hypersurface in a smooth projective variety and the associated Birkhoff factorization for algebra homomorphisms from a commutative Hopf algebra. In the case of a normal crossings divisor, the Rota-Baxter structure simplifies considerably and the factorization becomes a simple pole subtraction. We apply this formalism to the unrenormalized momentum space Feynman amplitudes, viewed as (divergent) integrals in the complement of the determinant hypersurface. We lift the integral to the Kausz compactification of the general linear group, whose boundary divisor is normal crossings. We show that the Kausz compactification is a Tate motive and the boundary divisor is a mixed Tate configuration. The regularization of the integrals that we obtain differs from the usual renormalization of physical Feynman amplitudes, and in particular it gives mixed Tate periods in cases that have non-mixed Tate contributions in the usual form. This part is based on joint work with Matilde Marcolli (see (80)).

In the second part, we consider the notions of the replicators, including the duplicator and triplicator, of a binary operad. We show that taking replicators is in Koszul dual to taking successors in (9) for binary quadratic operads and is equivalent to taking the white product with certain operads such as *Perm*. We also relate the replicators to the actions of average operators. After the completion of this work (in 2012; see (85)), we realized that the closely related notions di-Var-algebra and tri-Var-algebra have been introduced independently in (48) (in 2011; see also (63; 64)) by Kolesnikov and his coauthors. In fact their notions also apply to not necessarily binary operads (64). In this regard, the second part of this thesis provides an alternative and more detailed treatment of these notations for binary operads. This part is based on joint work with Chengming Bai, Li Guo, and Jun Pei (see (85)).

PUBLISHED CONTENT AND CONTRIBUTIONS

(a) M. Marcolli, X. Ni, *Rota-Baxter algebras, singular hypersurfaces, and renormalization on Kausz compactifications*, arXiv:1408.3754.

X.N. participated in proving the results and the writing of the paper.

(b) J. Pei, C. Bai, L. Guo and X. Ni, *Replicating of binary operads, Koszul duality, Manin products and average operators*, arXiv:1212.0177.

X.N. participated in proving the results and the writing of the paper.

Contents

Chapter 1

Introduction

1.1 Rota-Baxter algebras

A Rota–Baxter algebra of weight λ is a unital commutative algebra $\mathcal{R}$ together with a linear operator $T : \mathcal{R} \to \mathcal{R}$ satisfying the Rota–Baxter identity

$$T(x)T(y) = T(xT(y)) + T(T(x)y) + \lambda T(xy). \tag{1.1.1}$$

For example, Laurent polynomials $\mathcal{R} = \mathbb{C}[z, z^{-1}]$ with T the projection onto the polar part are a Rota–Baxter algebra of weight -1.

The Rota–Baxter operator T of a Rota–Baxter algebra of weight -1, satisfying

$$T(x)T(y) + T(xy) = T(xT(y)) + T(T(x)y), \tag{1.1.2}$$

determines a splitting of $\mathcal{R}$ into $\mathcal{R}_+ = (1 - T)\mathcal{R}$ and $\mathcal{R}_-$, the unitization of $T\mathcal{R}$, where both $\mathcal{R}_\pm$ are algebras. For an introduction to Rota–Baxter algebras we refer the reader to (49).

1.2 Singular hypersurfaces and renormalization on Kausz compactifications

In the first part of this thesis, we consider the problem of extracting periods of algebraic varieties from a class of divergent integrals arising in quantum field theory. The method we present here provides a regularization and extraction of finite values that differs from the usual (renormalized) physical Feynman amplitudes, but whose mathematical interest lies in the fact that it always gives a period of a mixed Tate motive, for all graphs satisfying a simple combinatorial relation that ensures the amplitude can be computed using (global) forms with logarithmic poles. For more general graphs, one also obtains a period, where the nature of the motive involved depends on how a certain hyperplane arrangement intersects the big cell in a compactification of the general linear group. More precisely, the motive considered here is provided by the Kausz compactification of the general linear group and by a hyperplane arrangement that contains the boundary of the chain of integration. The regularization procedure we propose is modeled on the algebraic renormalization method, based on Hopf algebras of graphs and Rota–Baxter algebras, as originally developed by Connes and Kreimer (26) and by Ebrahmi-Fard, Guo, and Kreimer (38). The main difference in our approach is that we apply the formalism to a Rota–Baxter algebra of (even) meromorphic differential forms instead of applying it to a regularization of the integral. The procedure becomes especially simple in cases where the deRham cohomology of the singular hypersurface complement is all realized by forms with logarithmic poles, in which case we replace the divergent integral with a family of convergent integrals obtained by a pole subtraction on the form and by (iterated) Poincaré residues. In (24) a similar approach was developed for integrals in configuration spaces.

In Section 2.1 we introduce Rota–Baxter algebras of even meromorphic forms, along the lines of (24), and we formulate a general setting for extraction of finite values (regularization and renormalization) of divergent integrals modeled on algebraic renormalization applied to these Rota–Baxter algebras of differential forms.

In Section 2.2 we discuss the Rota–Baxter algebras of even meromorphic forms in the case of a smooth hypersurface $Y \subset X$. We show that, when restricted to forms with logarithmic poles, the Rota–Baxter operator becomes simply a derivation, and the Birkhoff factorization collapses to a simple pole subtraction, as in the case of log divergent graphs. We show that this simple pole subtraction can lead to too much loss of information about the unrenormalized integrand and we propose considering the additional information of the Poincaré residue and an additional integral associated to the residue.

In Section 2.3 we consider the case of singular hypersurfaces $Y \subset X$ given by a simple normal crossings divisor. We show that, in this case, the Rota–Baxter operator satisfies a simplified form of the Rota–Baxter identity, which is not just a derivation, however. We show that this modified identity still suffices to have a simple pole subtraction $\phi_+(X) = (1 - T)\phi(X)$ in the Birkhoff factorization, even though the negative piece $\phi_-(X)$ becomes more complicated. Again, to avoid too much loss of information in passing from $\phi(X)$ to $\phi_+(X)$, we consider, in addition to the renormalized integral $\int_\sigma \phi_+(X)$, the collection of integrals of the form $\int_{\sigma\cap Y_I} \mathrm{Res}_{Y_I}(\phi(X))$, where Res_{Y_I} is the iterated Poincaré residue ((4), (3)), along the intersection $Y_I = \cap_{j\in I} Y_j$ of components of Y. These integrals are all periods of mixed Tate motives if $\{Y_I\}$ is a mixed Tate configuration, in the sense of (44). We discuss the question of further generalizations to more general types of singularities, beyond the normal crossings case, via Saito's theory of forms with logarithmic poles (90), by showing that one can also define a Rota–Baxter structure on the Saito complex of forms with logarithmic poles.

In Section 2.4 we present our main application, which is a regularization (different from the physical one) of the Feynman amplitudes in momentum space, computed on the complement of the determinant hypersurface as in (6). Since the determinant hypersurface has worse singularities than what we need, we pull back the integral computation to the Kausz compactification (60) of the general linear group, where the boundary divisor that replaces the determinant hypersurface is a simple normal crossings divisor. We show that the motive of the Kausz compactification is Tate, and that the components of the boundary divisor form a mixed Tate configuration. We discuss how one can replace the form η_Γ of the Feynman amplitude with a form with logarithmic poles. In general it is defined on the big cell of the Kausz compactification. We also discuss the nature of the periods. This part is based on joint work with Matilde Marcolli (see (80)).

1.3 Replicating of binary operads, Koszul duality, Manin products and average operators

Motivated by the study of the periodicity in algebraic K-theory, J.-L. Loday (69) introduced the concept of a Leibniz algebra twenty years ago as a non-skew-symmetric generalization of the Lie algebra. He then defined the diassociative algebra (70) as the enveloping algebra of the Leibniz algebra in analogue to the associative algebra as the enveloping algebra of the Lie algebra. The dendriform algebra was introduced as the Koszul dual of the diassociative algebra. These structures were studied systematically in the next few years in connection with operads (74), homology (41; 42), Hopf algebras (2; 57; 75; 88), arithmetic (71), combinatorics (40; 76), quantum field theory (40) and Rota-Baxter algebra (1).

The diassociative and dendriform algebras extend the associative algebra in two directions. While the diassociative algebra "doubles" the associative algebra in the sense that it has two associative operations with certain compatible conditions, the dendriform algebra "splits" the associative algebra in the sense that it has two binary operations with relations between them so that the sum of the two operations is associative.

Into this century, more algebraic structures with multiple binary operations emerged, beginning with the triassociative algebra that "triples" the associative algebra and the tridendriform algebra that gives a three way splitting of the associative algebra (75). Since then, quite a few dendriform related structures, such as the quadri-algebra (2), the ennea-algebra, the NS-algebra, the dendriform-Nijenhuis algebra, the octo-algebra (65–67), and eventually a whole class of algebras (36; 74) were introduced. All these dendriform type structures have a common property of "splitting" the associativity into multiple pieces. Furthermore, analogues of the dendriform algebra, quadri-algebra and octo-algebra for the Lie algebra, commutative algebra, Jordan algebra, alternative algebra, and Poisson algebra have been obtained (1; 10; 55; 68; 73; 84), such as the pre-Lie and Zinbiel algebras. More recently, these constructions can be put into the framework of operad products (Manin black square and black dot products) (35; 72; 93).

In (9), the notions of "successors" were introduced to give the precise meaning of two-way and three-way splitting of a binary operad and thus put the previous constructions in a uniform framework. This notion is also related to the Manin black products that had only been dealt with in special cases before, as indicated above. It is also shown to be related to the action of the Rota-Baxter operator, completing a long series of studies starting from the beginning of the century (1).

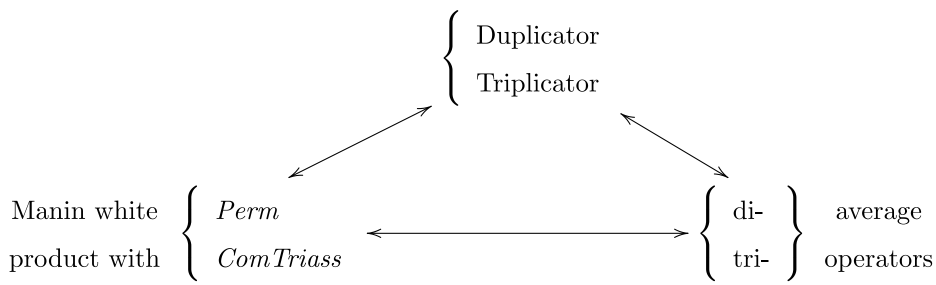
In this paper, we take a similar approach to the other class of structures starting from the diassociative (resp. triassociative) algebra. That is, we seek to understand the phenomena of "replicating" the operations in an operad. After the completion of this work (in 2012; see (85)), we realized that the closely related notions di-Var-algebra and tri-Var-algebra have been introduced independently in (48) (in 2011; see also (63; 64)) by Kolesnikov and his coauthors. In fact, their notions also apply to not necessarily binary operads (64). In this regard, the second part of this thesis provides an alternative and more detailed treatment of these notations for binary operads.

In Section 3.1 we set up a general framework to make precise the notion of "replicating" any binary algebraic operad. This provides a general framework to study the previously well-known di-type (resp. tri-type) algebras which are analogues of the diassociative (resp. triassociative) algebra associated to the associative algebra, including the Leibniz algebra for the Lie algebra and the permutative algebra for the commutative algebra, as well as the recently defined pre-Lie dialgebra (39). In general, it gives a "rule" to construct new di-type (resp. tri-type) algebraic structures associated to any other binary operads. This notion is simpler in formulation but turns out to be equivalent to the notion of di-Var-algebra in (48) for binary operads with nontrivial relations.

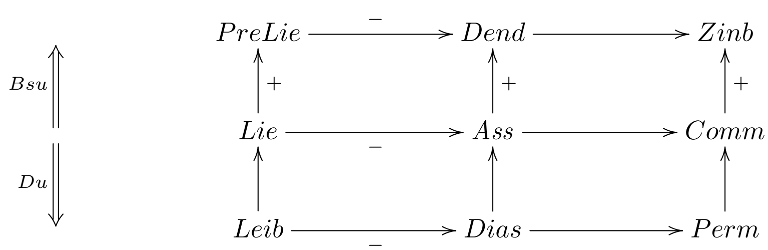
We show in Section 3.2 that taking the replicator of a binary quadratic operad is in Koszul dual with taking the successor of the dual operad. A direct application of this duality (Theorem 3.2.3 and 3.2.4) is to explicitly compute the Koszul dual of the operads of existing algebras, for example the Koszul dual of the commutative tridendriform algebra of Loday (73). We also relate replicating to the Manin white product in the case of binary quadratic operads. In fact taking the duplicator (resp. triplicator) of such an operad with nontrivial relations is isomorphic to taking the white product of the operad *Perm* (resp. *ComTriass*) with this operad, as in the case of taking di-Var-algebras and tri-Var-algebras (48), thus showing that the notations of duplicator and triplicator are equivalent to those of di-Var-algebras and tri-Var-algebras.

Finally, in Section 3.3, we relate the replicating process to the action of average operators on binary quadratic operads. Aguiar (1) showed that the action of the two-sided average operator on a commutative associative algebra (resp. associative algebra) gives a perm algebra (resp. associative dialgebra). In (92), Uchino extended the classical derived bracket construction to any algebra over a binary quadratic operad, showing that the derived bracket construction can be given by the Manin white product with the operad *Perm*.

Thus there is a relationship among the three operations applied to a binary operad $\mathcal{P}$: take its duplicator (resp. triplicator), and take its Manin white product with *Perm* (resp. *ComTriass*), when the operad is quadratic, and apply a di-average operator (resp. tri-average operator) to it, as summarized in the following diagram.



Combining the replicators with the successors introduced in (9) allows us to put the splitting and replicating processes together, as exemplified in the following commutative diagram of operads. The arrows should be reversed on the level of categories.



Here the vertical arrows in the upper half of the diagram are addition of the two operations given in (9, Proposition 2.31.(a)) while those in the lower half of the diagram are given in Proposition 3.1.19 (a). The horizontal arrows in the left half of the diagram are anti-symmetrization of the binary operations while those in the right half of the diagram are induced by the identity maps on the binary operations. In the diagram, the Koszul dual of an operad is the reflection across the center. A similar commutative diagram holds for the trisuccessors and triplicators.

This part is based on joint work with Chengming Bai, Li Guo, and Jun Pei (see (85)).

Chapter 2

Singular hypersurfaces and renormalization on Kausz compactifications

2.1 Rota–Baxter algebras of meromorphic forms

We generalize the algebraic renormalization formalism to a setting based on Rota–Baxter algebras of algebraic differential forms on a smooth projective variety with poles along a hypersurface.

2.1.1 Rota–Baxter algebras of even meromorphic forms

Let Y be a hypersurface in a projective variety X, with defining equation $Y = \{f = 0\}$. We denote by $\mathcal{M}^{\star}_X$ the sheaf of meromorphic differential forms on X, and by $\mathcal{M}^{\star}_{X,Y}$ the subsheaf of meromorphic forms on with poles (of arbitrary order) along Y. It is a graded-commulative algebra over the field of definition of the varieties X and Y. We can write forms $\omega \in \mathcal{M}^{\star}_{X,Y}$ as sums $\omega = \sum_{p\geq 0} \alpha_p/f^p$, where the α_p are holomorphic forms.

In particular, we consider forms of even degrees, so that $\mathcal{M}^{\mathrm{even}}_{X,Y}$ is a commutative algebra under the wedge product.

Lemma 2.1.1. *The commutative algebra $\mathcal{M}^{\mathrm{even}}_{X,Y}$, together with the linear operator $T : \mathcal{M}^{\mathrm{even}}_{X,Y} \to \mathcal{M}^{\mathrm{even}}_{X,Y}$ defined as the polar part*

$$T(\omega) = \sum_{p\geq 1} \alpha_p/f^p, \tag{2.1.1}$$

is a Rota–Baxter algebra of weight -1.

Proof. For $\omega_1 = \sum_{p\geq 0} \alpha_p/f^p$ and $\omega_2 = \sum_{q\geq 0} \beta_q/f^q$, we have

$$T(\omega_1 \wedge \omega_2) = \sum_{p\geq 0, q\geq 1} \frac{\alpha_p \wedge \beta_q}{f^{p+q}} + \sum_{p\geq 1, q\geq 0} \frac{\alpha_p \wedge \beta_q}{f^{p+q}} - \sum_{p\geq 1, q\geq 1} \frac{\alpha_p \wedge \beta_q}{f^{p+q}},$$

$$T(T(\omega_1) \wedge \omega_2) = \sum_{p\geq 1, q\geq 0} \frac{\alpha_p \wedge \beta_q}{f^{p+q}},$$

$$T(\omega_1 \wedge T(\omega_2)) = \sum_{p\geq 0, q\geq 1} \frac{\alpha_p \wedge \beta_q}{f^{p+q}},$$

$$T(\omega_1) \wedge T(\omega_2) = \sum_{p\geq 1, q\geq 1} \frac{\alpha_p \wedge \beta_q}{f^{p+q}},$$

so that (1.1.2) is satisfied. □

Equivalently, we have the following description of the Rota–Baxter operator.

Corollary 2.1.2. *The linear operator*

$$T(\omega) = \alpha \wedge \xi, \qquad \text{for } \omega = \alpha \wedge \xi + \eta, \tag{2.1.2}$$

acting on forms $\omega = \alpha \wedge \xi + \eta$, *with* α *a meromorphic form on* X *with poles on* Y *and* ξ *and* η *holomorphic forms on* X, *is a Rota–Baxter operator of weight* -1.

Proof. For $\omega_i = \alpha_i \wedge \xi_i + \eta_i$, with $i = 1, 2$, we have

$$T(\omega_1 \wedge \omega_2) = (-1)^{|\alpha_2|\,|\xi_1|} \alpha_1 \wedge \alpha_2 \wedge \xi_1 \wedge \xi_2 + \alpha_1 \wedge \xi_1 \wedge \eta_2 + (-1)^{|\eta_1|\,|\alpha_2|} \alpha_2 \wedge \eta_1 \wedge \xi_2$$

while

$$T(T(\omega_1) \wedge \omega_2) = (-1)^{|\alpha_2|\,|\xi_1|} \alpha_1 \wedge \alpha_2 \wedge \xi_1 \wedge \xi_2 + \alpha_1 \wedge \xi_1 \wedge \eta_2$$

$$T(\omega_1 \wedge T(\omega_2)) = (-1)^{|\alpha_2|\,|\xi_1|} \alpha_1 \wedge \alpha_2 \wedge \xi_1 \wedge \xi_2 + (-1)^{|\eta_1|\,|\alpha_2|} \alpha_2 \wedge \eta_1 \wedge \xi_2$$

and

$$T(\omega_1) \wedge T(\omega_2) = (-1)^{|\alpha_2|\,|\xi_1|} \alpha_1 \wedge \alpha_2 \wedge \xi_1 \wedge \xi_2,$$

where all signs are positive if the forms are of even degree. Thus, the operator T satisfies (1.1.2). $\square$

The following statement is proved exactly as in Theorem 6.4 of (24) and we omit the proof here.

Lemma 2.1.3. *Let (X_ℓ, Y_ℓ) for $\ell \geq 1$ be a collection of smooth projective varieties X_ℓ with hypersurfaces Y_ℓ, all defined over the same field of definition. Then the commutative algebra $\bigwedge_\ell \mathcal{M}^{\mathrm{even}}_{X_\ell, Y_\ell}$ is a Rota–Baxter algebra of weight -1 with the polar projection operator T determined by the T_ℓ on each $\mathcal{M}^{\mathrm{even}}_{X_\ell, Y_\ell}$.*

2.1.2 Renormalization via Rota–Baxter algebras

In (26), the BPHZ renormalization procedure of perturbative quantum field theory was reinterpreted as a Birkhoff factorization of loops in the pro-unipotent group of characters of a commutative Hopf algebra of Feynman graphs. This procedure of *algebraic renormalization* was reformulated in more general and abstract terms in (38), using Hopf algebras and Rota–Baxter algebras.

We summarize here quickly the basic setup of algebraic renormalization. We refer the reader to (26), (27), (38), and (79) for more details.

The Connes–Kreimer Hopf algebra of Feynman graphs $\mathcal{H}$ is the free commutative algebra with generators 1PI Feynman graphs Γ of the theory, with grading by loop number (or better by number of internal edges)

$$\deg(\Gamma_1 \cdots \Gamma_n) = \sum_i \deg(\Gamma_i), \quad \deg(1) = 0$$

and with coproduct

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \in \mathcal{V}(\Gamma)} \gamma \otimes \Gamma/\gamma, \tag{2.1.3}$$

where the class $\mathcal{V}(\Gamma)$ consists of all (possibly multiconnected) divergent subgraphs γ such that the quotient graph (identifying each component of γ to a vertex) is still a 1PI Feynman graph of the theory. The antipode is constructed inductively as

$$S(X) = -X - \sum S(X')X''$$

for $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$, with the terms X', X'' of lower degrees.

An *algebraic Feynman rule* $\phi : \mathcal{H} \to \mathcal{R}$ is a *homomorphism of commutative algebras* from the Hopf algebra $\mathcal{H}$ of Feynman graphs to a Rota–Baxter algebra $\mathcal{R}$ of weight -1,

$$\phi \in \mathrm{Hom}_{\mathrm{Alg}}(\mathcal{H}, \mathcal{R}).$$

The morphism ϕ by itself does not know about the coalgebra structure of $\mathcal{H}$ and the Rota–Baxter structure of $\mathcal{R}$. These enter in the factorization of ϕ into divergent and finite part.

The Birkhoff factorization of an algebraic Feynman rule consists of a pair of commutative algebra homomorphisms

$$\phi_\pm \in \mathrm{Hom}_{\mathrm{Alg}}(\mathcal{H}, \mathcal{R}_\pm),$$

where $\mathcal{R}_\pm$ is the splitting of $\mathcal{R}$ induced by the Rota–Baxter operator T, with $\mathcal{R}_+ = (1-T)\mathcal{R}$ and $\mathcal{R}_-$ the unitization of $T\mathcal{R}$, satisfying

$$\phi = (\phi_- \circ S) \star \phi_+,$$

where the product $\star$ is dual to the coproduct in the Hopf algebra, $\phi_1 \star \phi_2(X) = \langle \phi_1 \otimes \phi_2, \Delta(X) \rangle$.

As shown in (26), there is an inductive formula for the Birkhoff factorization of an algebraic Feynman rule, of the form

$$\phi_-(X) = -T(\phi(X) + \sum \phi_-(X')\phi(X'')) \quad \text{and} \quad \phi_+(X) = (1-T)(\phi(X) + \sum \phi_-(X')\phi(X'')), \tag{2.1.4}$$

where $\Delta(X) = 1 \otimes X + X \otimes 1 + \sum X' \otimes X''$.

In the original Connes–Kreimer formulation, this approach is applied to the unrenormalized Feynman amplitudes regularized by dimensional regularization, with the Rota–Baxter algebra consisting of germs of meromorphic functions at the origin, with the operator of projection onto the polar part of the Laurent series.

In the following, we consider the following variant on the Hopf algebra of Feynman graphs.

Definition 2.1.4. *As an algebra, $\mathcal{H}_{\mathrm{even}}$ is the commutative algebra generated by Feynman graphs of a given scalar quantum field theory that have an even number of internal edges, $\#E(\Gamma) \in 2\mathbb{N}$. The coproduct (2.1.3) on $\mathcal{H}_{\mathrm{even}}$ is similarly defined with the sum over divergent subgraphs γ with* *even $\#E(\gamma)$, with 1PI quotient.*

Notice that in dimension $D \in 4\mathbb{N}$ all log divergent subgraphs $\gamma \subset \Gamma$ have an even number of edges, since $Db_1(\gamma) = 2\#E(\gamma)$ in this case.

2.1.3 Rota–Baxter algebras and Atkinson factorization

In the following we will discuss some interesting properties of algebraic Birkhoff decomposition when the Rota-Baxter operator satisfies the identity $T(T(x)y) = T(x)y$.

Let $e : H \to A$ be the unit of $\mathrm{Hom}(H, A)$ (under the convolution product) defined by $e(1_H) = 1_A$ and $e(X) = 0$ on $\oplus_{n>0} H_n$.

The main observation can be summarized as follows:

(a) If the Rota-Baxter operator T on A also satisfy the identity $T(T(x)y) = T(x)y$, then on $\ker(e) = \oplus_{n>0} H_n$, the negative part of the Birkhoff factorization φ_- takes the following form:

$$\phi_- = -T(\phi(X)) - \sum T(\phi(X'))\phi(X''), \quad \text{for } \Delta(X) = 1 \otimes X + X \otimes 1 + \sum X' \otimes X''.$$

(b) If T also satisfies $T(xT(y)) = xT(y)$, $\forall x, y \in A$, then the positive part is given by $\phi_+ = (1 - T)(\phi(X))$, $\forall X \in \ker e = \oplus_{n>0} H_n$.

This follows from the properties of the Atkinson Factorization in Rota–Baxter algebras, which we recall below.

Proposition 2.1.5. (Atkinson Factorization, (50)) *Let (A, T) be a Rota-Baxter algebra of weight $\lambda \neq 0$. Let $\tilde{T} = -\lambda \mathrm{id} - T$ and let $a \in A$. Assume that b_l and b_r are a solution of the fixed point equations*

$$b_l = 1 + T(b_l a), \quad b_r = 1 + \tilde{T}(a b_r). \tag{2.1.5}$$

Then

$$b_l(1 + \lambda a)b_r = 1.$$

Thus

$$1 + \lambda a = b_l^{-1} b_r^{-1} \tag{2.1.6}$$

if b_l and b_r are invertible.

A Rota-Baxter algebra (A, T) is called complete if there are algebras $A_n \subseteq A, n \geq 0$, such that (A, A_n) is a complete algebra and $T(A_n) \subseteq A_n$.

Proposition 2.1.6. (Existence and uniqueness of the Atkinson Factorization, (50)) *Let (A, T, A_n) be a complete Rota-Baxter algebra of weight $\lambda \neq 0$. Let $\tilde{T} = -\lambda \mathrm{id} - T$ and let $a \in A_1$.*

(a) *Equations (2.1.5) have unique solutions b_l and b_r. Furthermore, b_l and b_r are invertible. Thus Atkinson Factorization (2.1.6) exists.*

(b) *If $\lambda \neq 0$ and $T^2 = -\lambda T$ (in particular if $T^2 = -\lambda T$ on A), then there are unique $c_l \in 1+T(A)$ and $c_r \in 1 + \tilde{T}(A)$ such that*

$$1 + \lambda a = c_l c_r.$$

Define

$$(Ta)^{[n+1]} := T((Ta)^{[n]}a) \quad \text{and} \quad (Ta)^{\{n+1\}} = T(a(Ta)^{\{n\}})$$

with the convention that $(Ta)^{[1]} = T(a) = (Ta)^{\{1\}}$ and $(Ta)^{[0]} = 1 = (Ta)^{\{0\}}$.

Proposition 2.1.7. *Let (A, A_n, T) be a complete filtered Rota-Baxter algebra of weight -1 such that $T^2 = T$. Let $a \in A_1$. If T also satisfies the following identity*

$$T(T(x)y) = T(x)y, \quad \forall x, y \in A, \tag{2.1.7}$$

then the equation

$$b_l = 1 + T(b_l a). \tag{2.1.8}$$

has a unique solution

$$1 + T(a)(1-a)^{-1}.$$

Proof. First, we have $(Ta)^{[n+1]} = T(a)a^n$ for $n \geq 0$. In fact, the case when $n = 0$ just follows from the definition. Suppose it is true up to n, then $(Ta)^{[n+2]} = T((Ta)^{[n+1]}a) = T((T(a)a^n)a) = T(T(a)a^{n+1}) = T(a)a^{n+1}$. Arguing as in ((37)), $b_l = \sum_{n=0}^{\infty}(Ta)^{[n]} = 1 + T(a) + T(T(a)a) + \cdots +$

$(Ta)^{[n]} + \cdots$ is the unique solution of (2.1.8). So

$$\begin{aligned} b_l &= 1 + T(a) + T(a)a + T(a)a^2 + \cdots \\ &= 1 + T(a)(1 + a + a^2 + \cdots) \\ &= 1 + T(a)(1-a)^{-1}. \end{aligned}$$

□

A bialgebra H is called a connected, filtered cograded bialgebra if there are subspaces H_n of H such that (*a*) $H_pH_q \subseteq \sum_{k\le p+q} H_k$; (*b*) $\Delta(H_n) \subseteq \oplus_{p+q=n} H_p \oplus H_q$; (*c*) $H_0 = \mathrm{im}u (= \mathbb{C})$, where $u : \mathbb{C} \to H$ is the unit of H.

Proposition 2.1.8. *Let H be a connected filtered cograded bialgebra (hence a Hopf algebra) and let (A, T) be a (not necessarily commutative) Rota-Baxter algebra of weight $\lambda = -1$ with $T^2 = T$. Suppose that T also satisfies (2.1.7). Let $\phi : H \to A$ be a character, i.e. an algebra homomorphism. Then there are unique maps $\phi_- : H \to T(A)$ and $\phi_+ : H \to \tilde{T}(A)$, where $\tilde{T} = 1 - T$, such that*

$$\phi = \phi_-^{*(-1)} * \phi_+,$$

where $\phi^{*(-1)} = \phi \circ S$, with S the antipode. ϕ_- takes the following form on $\ker e = \oplus_{n>0} H_n$*:*

$$\begin{aligned} \phi_-(X) &= -T(\phi(X)) - \sum_{n=1}^{\infty} (-1)^n \sum T(\phi(X^{(1)}))\phi(X^{(2)})\phi(X^{(3)}) \cdots \phi(X^{(n+1)}) \\ &= -T(\phi(X)) - \sum_{n=1}^{\infty} (-1)^n ((T\phi)\tilde{*}\phi^{\tilde{*}^n})(X). \end{aligned}$$

Here we use the notation $\tilde{\Delta}^{n-1}(X) = \sum X^{(1)} \otimes \cdots \otimes X^{(n)}$, and $\tilde{\Delta}(X) := \Delta(X) - X \otimes 1 - 1 \otimes X$ (which is coassociative), and $\tilde{*}$ is the convolution product defined by $\tilde{\Delta}$. Furthermore, if T satisfies

$$T(xT(y)) = xT(y), \quad \forall x, y \in A, \tag{2.1.9}$$

then ϕ_+ takes the form on $\ker e = \oplus_{n>0} H_n$*:*

$$\phi_+ = (1 - T)(\phi(X)).$$

Proof. Define $R := \mathrm{Hom}(H, A)$ and

$$P : R \to R, \quad P(f)(X) = T(f(X)),\ f \in \mathrm{Hom}(H, A), X \in H.$$

Then by (50), R is a complete algebra with filtration $R_n = \{f \in \mathrm{Hom}(H, A) | f(H^{(n-1)}) = 0\}, n \geq 0$, and P is a Rota-Baxter operator of weight -1 and $P^2 = P$. Moreover, since T satisfies (2.1.7), it is easy to check that $P(P(f)g) = P(f)g$ for any $f, g \in \mathrm{Hom}(H, A)$. Let $\phi : H \to A$ be a character. Then $(e - \phi)(1_H) = e(1_H) - \phi(1_H) = 1_A - 1_A = 0$. So $e - \phi \in A_1$. Set $a = e - \phi$, by Proposition 2.1.6, we know that there are unique $c_l \in T(A)$ and $c_r \in (1 - T)(A)$ such that $\phi = c_l c_r$. Moreover, by Proposition 2.1.7 we have $\phi_- = b_l = c_l^{-1} = e + T(a)(e - a)^{-1} = e + T(e - \phi) \sum_{n=0}^{\infty}(e - \phi)^n$. We also have $\sum_{n=0}^{\infty}(e - \phi)^n(1_H) = 1_A$ and for any $X \in \ker e = \oplus_{n>0} H_n$, we have $(e - \phi)^0(X) = e(X) = 0$; $(e - \phi)^1(X) = -\phi(X)$; $(e - \phi)^2(X) = \sum (e - \phi)(X')(e - \phi)(X'') = \sum \phi(X')\phi(X'')$. More generally, we have $(e - \phi)^n(X) = (-1)^n \sum \phi(X^{(1)})\phi(X^{(2)}) \cdots \phi(X^{(n)}) = (-1)^n \phi^{\tilde{*}^n}(X)$. So for $X \in \ker e = \oplus_{n>0} H_n$,

$$\begin{aligned}
\phi_-(X) &= (T(e - \phi) \sum_{n=0}^{\infty} (e - \phi)^n)(X) \\
&= T(e - \phi)(1_H) \sum_{n=0}^{\infty} (e - \phi)^n(X) + T(e - \phi)(X) \sum_{n=0}^{\infty} (e - \phi)^n(1_H) \\
&\quad + \sum T((e - \phi)(X')) \sum_{n=1}^{\infty} (e - \phi)^n(X'') \\
&= -T(\phi(X)) - \sum T(\phi(X')) \sum_{n=1}^{\infty} (-1)^n \sum \phi((X'')^{(1)})\phi((X'')^{(2)}) \cdots \phi((X'')^{(n)}) \\
&= -T(\phi(X)) - \sum_{n=1}^{\infty} (-1)^n \sum T(\phi(X^{(1)}))\phi(X^{(2)})\phi(X^{(3)}) \cdots \phi(X^{(n+1)}) \\
&= -T(\phi(X)) - \sum_{n=1}^{\infty} (-1)^n ((T\phi)\tilde{*}\phi^{\tilde{*}^n})(X).
\end{aligned}$$

Suppose that T also satisfies Equation (2.1.9), then for any $a, b \in A$, we have $(1 - T)(a)(1 - T)(b) = ab - T(a)b - aT(b) + T(a)T(b) = ab - T(T(a)b) - T(aT(b)) + T(a)T(b) = ab - T(ab) = (1 - T)(ab)$, as T is a Rota-Baxter operator of weight -1. As shown in ((26)) and ((37)), $\phi_+ = (1 - T)(\phi(X) + \sum \phi_-(X')\phi(X''))$. So $\phi_+ = (1 - T)(\phi(X)) + \sum (1 - T)(\phi_-(X'))(1 - T)(\phi(X''))$ by the previous computation. But ϕ_- is in the image of T and $T^2 = T$, so we must have $(1 - T)(\phi_-(X')) = 0$, which shows that $\phi_+ = (1-T)(\phi(X))$. $\square$

2.1.4 A variant of algebraic renormalization

We consider now a setting inspired by the formalism of the Connes–Kreimer renormalization recalled above. The setting generalizes the one considered in (24) for configuration space integrals and our main application will be to extend the approach of (24) to momentum space integrals.

The main difference with respect to the Connes–Kreimer renormalization is that, instead of renormalizing the Feynman amplitude (regularized so that it gives a meromorphic function), we propose to renormalize the differential form, before integration, and then integrate the renormalized form to obtain a period.

The result obtains by this method differs from the physical renormalization, as we will see in explicit examples in Section 2.4.11 below. Whenever non-trivial, the convergent integral obtained by the method described here will be a mixed Tate period even in cases where the physical renormalization is not.

The main steps required for our setup are the following.

- For each $\ell \geq 1$, we construct a pair (X_ℓ, Y_ℓ) of a smooth projective variety X_ℓ (defined over $\mathbb{Q}$) whose motive $\mathfrak{m}(X_\ell)$ is mixed Tate (over $\mathbb{Z}$), together with a (singular) hypersurface $Y_\ell \subset X_\ell$.
- We describe the Feynman integrand as a morphism of commutative algebras

$$\phi : \mathcal{H}_{\text{even}} \to \bigwedge_\ell \mathcal{M}^{\text{even}}_{X_\ell, Y_\ell}, \quad \phi(\Gamma) = \eta_\Gamma,$$

  with η_Γ an algebraic differential form on X_ℓ with polar locus Y_ℓ, for $\ell = b_1(\Gamma)$, and with the Rota–Baxter structure of Lemma 2.1.3 on the target algebra.
- We express the (unrenormalized) Feynman integrals as a (generally divergent) integral $\int_\sigma \eta_\Gamma$, over a chain σ in X_ℓ.
- We construct a divisor, $\Sigma_\ell \subset X_\ell$, that contains the boundary $\partial\sigma$, whose motive $\mathfrak{m}(\Sigma_\ell)$ is mixed Tate (over $\mathbb{Z}$) for all $\ell \geq 1$.
- We perform the Birkhoff decomposition $\phi_\pm$ obtained inductively using the coproduct on $\mathcal{H}$ and the Rota–Baxter operator T (polar part) on $\mathcal{M}^*_{X_\ell, Y_\ell}$.

- This gives a holomorphic form $\phi_+(\Gamma)$ on X_ℓ. The divergent Feynman integral is then replaced by the integral
$$\int_{\Upsilon(\sigma)} \phi_+(\Gamma),$$
which is a period of the mixed Tate motive $\mathfrak{m}(X_\ell, \Sigma_\ell)$.

- In addition to the integral of $\phi_+(\Gamma)$ on X_ℓ we consider integrals on the strata of the complement $X_\ell \smallsetminus Y_\ell$ of the polar part $\phi_-(\Gamma)$, which under suitable conditions will be interpreted as Poincaré residues.

If convergent, the Feynman integral $\int_\sigma \eta_\Gamma$ would be a period of $\mathfrak{m}(X_\ell \smallsetminus Y_\ell, \Sigma_\ell \smallsetminus (\Sigma_\ell \cap Y_\ell))$. The renormalization procedure described above replaces it with a (convergent) integral that is a period of the simpler motive $\mathfrak{m}(X_\ell, \Sigma_\ell)$. By our assumptions on X_ℓ and Σ_ℓ, the motive $\mathfrak{m}(X_\ell, \Sigma_\ell)$ is mixed Tate for all ℓ.

Thus, this strategy eliminates the difficulty of analyzing the motive $\mathfrak{m}(X_\ell \smallsetminus Y_\ell, \Sigma_\ell \smallsetminus (\Sigma_\ell \cap Y_\ell))$ encountered for instance in (6). The form of renormalization proposed here always produces a mixed Tate period, but at the cost of incurring in a considerable loss of information with respect to the original Feynman integral.

Indeed, a difficulty in the procedure described above is ensuring that the resulting regularized form

$$\phi_+(\Gamma) = (1-T)(\phi(\Gamma) + \sum_{\gamma \subset \Gamma} \phi_-(\gamma) \wedge \phi(\Gamma/\gamma))$$

is nontrivial. This condition may be difficult to control in explicit cases, although we will discuss below an especially simple situation, when one can reduce the problem to forms with logarithmic poles, where using the pole subtraction together with Poicaré residues one can obtain nontrivial periods (although the result one obtains is not equivalent to the physical renormalization of the Feynman amplitude).

An additional difficulty that can cause loss of information with respect to the Feynman integral is coming from the combinatorial conditions on the graph given in (6) that we will use for the embedding into the complement of the determinant hypersurface; see Section 2.4.11.

2.2 Rota–Baxter algebras and forms with logarithmic poles

We now focus on the case of meromorphic forms with logarithmic poles, where the Rota–Baxter structure and the renormalization procedure described above drastically simplify.

Lemma 2.2.1. *Let X be a smooth projective variety and $Y \subset X$ a smooth hypersurface with defining equation $Y = \{f = 0\}$. Let $\Omega^\star_X(\log(Y))$ be the sheaf of algebraic differential forms on X with logarithmic poles along Y. The Rota–Baxter operator T of Lemma 2.1.1 preserves $\Omega^{\rm even}_X(\log(Y))$ and the pair $(\Omega^{\rm even}_X(\log(Y)), T)$ is a graded Rota–Baxter algebra of degree -1 with the property that, for all $\omega_1, \omega_2 \in \Omega^{\rm even}_X(\log(Y))$, the wedge product $T(\omega_1) \wedge T(\omega_2) = 0$.*

Proof. Forms $\omega \in \Omega^\star_X(\log(Y))$ can be written in canonical form

$$\omega = \frac{df}{f} \wedge \xi + \eta,$$

with ξ and η holomorphic, so that $T(\omega) = \frac{df}{f} \wedge \xi$. We then have (1.1.2) as in Corollary 2.1.2 above, with $T(\omega_1) \wedge T(\omega_2) = (-1)^{|\xi_1|+1} \alpha \wedge \alpha \wedge \xi_1 \wedge \xi_2$ where α is the 1-form $\alpha = df/f$ so that $\alpha \wedge \alpha = 0$. ☐

Lemma 2.2.1 shows that, when restricted to $\Omega^\star_X(\log(Y))$, the operator T satisfies the simpler identity

$$T(xy) = T(T(x)y) + T(xT(y)). \tag{2.2.1}$$

This property greatly simplifies the decomposition of the algebra induced by the Rota–Baxter operator. In particular, we get a simplified form of the general result of Proposition 2.1.8, when taking into account the vanishing $T(x)T(y) = 0$, as shown in Lemma 2.2.1.

Lemma 2.2.2. *Let $\mathcal{R}$ be a commutative algebra and $T : \mathcal{R} \to \mathcal{R}$ a linear operator that satisfies the identity (2.2.1) and such that, for all $x, y \in \mathcal{R}$ the product $T(x)T(y) = 0$. Let $\mathcal{R}_+ = {\rm Range}(1 - T)$. Then the following properties hold.*

(a) *$\mathcal{R}_+ \subset \mathcal{R}$ is a subalgebra.*

(b) *Both T and $1 - T$ are idempotent, $T^2 = T$ and $(1 - T)^2 = 1 - T$.*

Proof. (1) The product of elements in $\mathcal{R}_+$ can be written as $(1-T)(x) \cdot (1-T)(y) = xy - T(x)y - xT(y) = xy - T(x)y - xT(y) - (T(xy) - T(T(x)y) - T(xT(y))) = (1-T)(xy - T(x)y - xT(y))$.

(2) The identity (2.2.1) gives $T(1) = 0$, since taking $x = y = 1$ one obtains $T(1) = 2T^2(1)$ while taking $x = T(1)$ and $y = 1$ gives $T^2(1) = T^3(1)$. Then (2.2.1) with $y = 1$ gives $T(x) = T(xT(1)) + T(T(x)1) = T^2(x)$ for all $x \in \mathcal{R}$. For $1-T$ we then have $(1-T)^2(x) = x-2T(x)+T^2(x) = (1-T)(x)$, for all $x \in \mathcal{R}$. □

Lemma 2.2.3. *Let $\mathcal{R}$ be a commutative algebra and $T : \mathcal{R} \to \mathcal{R}$ a linear operator that satisfies the identity (2.2.1) and such that, for all $x, y \in \mathcal{R}$ the product $T(x)T(y) = 0$. If, for all $x, y \in \mathcal{R}$, the identity $T(x)y + xT(y) = T(T(x)y) + T(xT(y))$ holds, then the operator $(1 - T) : \mathcal{R} \to \mathcal{R}_+$ is an algebra homomorphism and the operator T is a derivation on $\mathcal{R}$.*

Proof. We have $(1 - T)(xy) = xy - T(T(x)y) - T(xT(y))$ while $(1 - T)(x) \cdot (1 - T)(y) = xy - T(x)y - xT(y)$. Assuming that, for all $x, y \in \mathcal{R}$, we have $T(T(x)y) + T(xT(y)) = T(x)y + xT(y)$ gives $(1 - T)(xy) = (1 - T)(x) \cdot (1 - T)(y)$. Moreover, the identity (2.2.1) can be rewritten as $T(xy) = T(x)y + xT(y)$, and hence T is just a derivation on $\mathcal{R}$. □

Consider then again the case of a smooth hypersurface Y in $\mathbb{P}^n$. We have the following properties.

Proposition 2.2.4. *Let $Y \subset X$ be a smooth hypersurface in a smooth projective variety. The Rota–Baxter operator $T : \mathcal{M}^{\text{even}}_{\mathbb{P}^n,Y} \to \mathcal{M}^{\text{even}}_{X,Y}$ of weight -1 on meromorphic forms on X with poles along Y restricts to a derivation on the graded algebra $\Omega^{\text{even}}_X(\log(Y))$ of forms with logarithmic poles. Moreover, the operator $1 - T$ is a morphism of commutative algebras from $\Omega^{\text{even}}_X(\log(Y))$ to the algebra of holomorphic forms Ω^{even}_X.*

Proof. It suffices to check that the polar part operator $T : \Omega^{\text{even}}_X(\log(Y)) \to \Omega^{\text{even}}_X(\log(Y))$ satisfies the hypotheses of Lemma 2.2.3. We have seen that, for all $\omega_1, \omega_2 \in \Omega^{\text{even}}_X(\log(Y))$, the product $T(\omega_1) \wedge T(\omega_2) = 0$. Moreover, for $\omega_i = d\log(f) \wedge \xi_i + \eta_i$, we have $T(\omega_1) \wedge \omega_2 = d\log(f) \wedge \xi_1 \wedge \eta_2$ and $\omega_1 \wedge T(\omega_2) = (-1)^{|\eta_1|} d\log(f) \wedge \eta_1 \wedge \xi_2$, where the ξ_i and η_i are holomorphic, so that we have $T(T(\omega_1) \wedge \omega_2) = T(\omega_1) \wedge \omega_2$ and $T(\omega_1 \wedge T(\omega_2)) = \omega_1 \wedge T(\omega_2)$. Thus, the hypotheses of Lemma 2.2.3 are satisfied. □

2.2.1 Birkhoff factorization and forms with logarithmic poles

In cases where the pair (X, Y) has the property that the deRham cohomology $H^*_{dR}(X \smallsetminus Y)$ can always be realized by algebraic differential forms with logarithmic poles, the construction above simplifies significantly. Indeed, the Birkhoff factorization becomes essentially trivial, because of Proposition 2.2.4. In other words, all graphs behave "as if they were log divergent". This can be stated more precisely as follows.

Proposition 2.2.5. *Let $Y \subset X$ be a smooth hypersurface inside a smooth projective variety and let $\Omega_X^{\mathrm{even}}(\log(Y))$ denote the commutative algebra of algebraic differential forms on X of even degree with logarithmic poles on Y. Let $\phi : \mathcal{H} \to \Omega_X^{\mathrm{even}}(\log(Y))$ be a morphism of commutative algebras from a commutative Hopf algebra $\mathcal{H}$ to $\Omega_X^{\mathrm{even}}(\log(Y))$ with the operator T of pole subtraction. Then for every $X \in \mathcal{H}$ one has*

$$\phi_+(X) = (1 - T)\phi(X),$$

while the negative part of the Birkhoff factorization takes the form

$$\phi_-(X) = -T(\phi(X)) - \sum \phi_-(X')\phi(X''),$$

where $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$. Moreover, ϕ_- takes the following nonrecursive form on $\ker e = \oplus_{n>0} H_n$:

$$\begin{aligned} \phi_-(X) &= -T(\phi(X)) - \sum_{n=1}^{\infty} (-1)^n \sum T(\phi(X^{(1)}))\phi(X^{(2)})\phi(X^{(3)}) \cdots \phi(X^{(n+1)}) \\ &= -T(\phi(X)) - \sum_{n=1}^{\infty} (-1)^n ((T\phi)\tilde{*}\phi^{\tilde{*}^n})(X). \end{aligned}$$

Proof. The operator T of pole subtraction is a derivation on $\Omega_X^{\mathrm{even}}(\log(Y))$. By (2.1.4) we have $\phi_+(X) = (1 - T)(\phi(X) + \sum \phi_-(X')\phi(X''))$. By Proposition 2.2.4 we know that, in the case of forms with logarithmic poles along a smooth hypersurface, $1 - T$ is an algebra homomorphism, hence $\phi_+(X) = (1 - T)(\phi(X)) + \sum (1 - T)(\phi_-(X'))(1 - T)(\phi(X'')))$, but $\phi_-(X')$ is in the range of T and, again by Proposition 2.2.4, we have $T^2 = T$, so that the terms in the sum all vanish, since $(1 - T)(\phi_-(X')) = 0$. By (2.1.4) we have $\phi_-(X) = -T(\phi(X) + \sum \phi_-(X')\phi(X'')) = -T\phi(X) - \sum T(\phi_-(X'))\phi(X'') - \sum \phi_-(X')T(\phi(X''))$, because by Proposition 2.2.4 T is a derivation. The last sum vanishes because $\phi_-(X')$ is in the range of T and we have $T(\eta) \wedge T(\xi) = 0$ for all $\eta, \xi \in \Omega^*_{X_\ell}(\log(Y_\ell))$. Thus, we are left with $\phi_-(X) = -T\phi(X) - \sum T(\phi_-(X'))\phi(X'') = -T\phi(X) - \sum \phi_-(X')\phi(X'')$. The last part follows from Proposition 2.1.8, since $T(T(\eta) \wedge \xi) = T(\eta) \wedge \xi$. □

Notice that this is compatible with the property that $\phi(X) = (\phi_- \circ S \star \phi_+)(X)$ (with the $\star$-product dual to the Hopf algebra coproduct). In fact, this identity is equivalent to $\phi_+ = \phi_- \star \phi$, which means that $\phi_+(X) = \langle \phi_- \otimes \phi, \Delta(X)\rangle = \phi_-(X) + \phi(X) + \sum \phi_-(X')\phi(X'') = (1-T)\tilde{\phi}(X)$ as above. Equivalently, all the nontrivial terms $\phi_-(X')\phi(X'')$ in $\tilde{\phi}(X)$ satisfy $T(\phi_-(X')\phi(X'')) = \phi_-(X')\phi(X'')$, because of the simplified form (2.3.3) of the Rota–Baxter identity.

Corollary 2.2.6. *If one has a construction of a character $\phi : \mathcal{H} \to \Omega_X^{\mathrm{even}}(\log(Y))$, of the Hopf algebra of Feynman graphs, where $X = X_\ell$ and $Y = Y_\ell$ independently of the number of loops $\ell \geq 1$, then the negative part of the Birkhoff factorization of Proposition 2.2.5 would take on the simple form*

$$\phi_-(\Gamma) = -\frac{dh}{h} \wedge \left(\xi_\Gamma + \sum_{N\geq 1} (-1)^N \sum_{\gamma_N \subset \cdots \subset \gamma_1 \subset \gamma_0 = \Gamma} \xi_{\gamma_N} \wedge \bigwedge_{j=1}^{N} \eta_{\gamma_{j-1}/\gamma_j} \right), \tag{2.2.2}$$

where $\phi(\Gamma) = \frac{dh}{h} \wedge \xi_\Gamma + \eta_\Gamma$, and $Y = \{h = 0\}$.

Proof. The result follows from the expression

$$\phi_-(\Gamma) = -T(\phi(\Gamma)) - \sum_{\gamma \subset \Gamma} \phi_-(\gamma)\phi(\Gamma/\gamma),$$

obtained in Proposition 2.2.5, where $\phi(\Gamma) = \omega_\Gamma = \frac{dh}{h} \wedge \xi_\Gamma + \eta_\Gamma$, so that $T(\phi(\Gamma)) = \frac{dh}{h} \wedge \xi_\Gamma$ and $\phi(\Gamma/\gamma) = \frac{dh}{h} \wedge \xi_{\Gamma/\gamma} + \eta_{\Gamma/\gamma}$. The wedge product of $\phi_-(\gamma) = -T(\phi(\gamma)) - \sum_{\gamma_2 \subset \gamma} \phi_-(\gamma_2)\phi(\gamma/\gamma_2)$ with $\phi(\Gamma/\gamma)$ will give a term $\frac{dh}{h} \wedge \xi_\gamma \wedge \eta_{\Gamma/\gamma}$ and additional terms $\phi_-(\gamma_2)\phi(\gamma/\gamma_2) \wedge \eta_{\Gamma/\gamma}$. Proceeding inductively on these terms, one obtains (2.2.2). ☐

In the more general case, where X_ℓ and Y_ℓ depend on the loop number $\ell \geq 1$, the form of the negative piece $\phi_-(\Gamma)$ is more complicated, as it will contain forms on the products $X_{\ell(\gamma)} \times X_{\ell(\Gamma/\gamma)}$ with logarithmic poles along $Y_{\ell(\gamma)} \times X_{\ell(\Gamma/\gamma)} \cup X_{\ell(\gamma)} \times Y_{\ell(\Gamma/\gamma)}$.

2.2.2 Polar subtraction and the residue

We have seen that, in the case of a smooth hypersurface $Y \subset X$, the Birkhoff factorization in the algebra of forms with logarithmic poles reduces to a simple pole subtraction, $\phi_+(X) = (1-T)\phi(X)$. If the unrenormalized $\phi(X)$ is a form written as $\alpha + \frac{df}{f} \wedge \beta$, with α and β holomorphic, then $\phi_+(X)$

vanishes identically whenever $\alpha = 0$. In that case, all information about $\phi(X)$ is lost in the process of pole substraction. Suppose that $\int_\sigma \phi(X)$ is the original unrenormalized integral. To maintain some additional information, it is preferable to consider, in addition to the integral $\int_\sigma \phi_+(X)$, also an integral of the form

$$\int_{\sigma\cap Y} \mathrm{Res}_Y(\eta),$$

where $\mathrm{Res}_Y(\eta) = \beta$ is the Poincaré residue of $\eta = \alpha + \frac{df}{f} \wedge \beta$ along Y. It is dual to the Leray coboundary, in the sense that

$$\int_{\sigma\cap Y} \mathrm{Res}_Y(\eta) = \frac{1}{2\pi i} \int_{\mathcal{L}(\sigma\cap Y)} \eta,$$

where the Leray coboundary $\mathcal{L}(\sigma \cap Y)$ is a circle bundle over $\sigma \cap Y$. In this way, even when $\alpha = 0$, one can still retain the nontrivial information coming from the Poincaré residue, which is also expressed as a period.

2.3 Singular hypersurfaces and meromorphic forms

In our main application, we will need to work with pairs (X, Y) where X is smooth projective, but the hypersurface Y is singular. Thus, we now discuss extensions of the results above to more general situations where $Y \subset X$ is a singular hypersurface in a smooth projective variety X.

Again we denote by $\mathcal{M}^*_{X,Y}$ the sheaf of meromorphic differential forms on X with poles along Y, of arbitrary order, and by $\Omega^*_X(\log(Y))$ the sub-sheaf of forms with logarithmic poles along Y. Let h be a local determination of Y, so that $Y = \{h = 0\}$. We can then locally represent forms $\omega \in \mathcal{M}^*_{X,Y}$ as finite sums $\omega = \sum_{p\geq 0} \omega_p/h^p$, with the ω_p holomorphic. The polar part operator $T : \mathcal{M}^{\mathrm{even}}_{X,Y} \to \mathcal{M}^{\mathrm{even}}_{X,Y}$ can then be defined as in (2.1.1).

In the case we considered above, with $Y \subset X$ as a smooth hypersurface, forms with logarithmic poles can be represented in the form

$$\omega = \frac{dh}{h} \wedge \xi + \eta, \qquad (2.3.1)$$

with ξ and η holomorphic. The Leray residue is given by $\mathrm{Res}(\omega) = \xi$. It is well defined, as the restriction of ξ to Y is independent of the choice of a local equation for Y.

In the next subsection we discuss how this case generalizes to a normal crossings divisor $Y \subset X$

inside a smooth projective variety X. The complex of forms with logarithmic poles extend to the normal crossings divisor case as in (29). For more general singular hypersurfaces, an appropriate notion of forms with logarithmic poles was introduced by Saito in (90). The construction of the residue was also generalized to the case where Y is a normal crossings divisor in (29) and for more general singular hypersurfaces in (90).

2.3.1 Normal crossings divisors

The main case of singular hypersurfaces that we focus on for our applications will be simple normal crossings divisors. In fact, while our formulation of the Feynman amplitude in momentum space is based on the formulation of (6), where the unrenormalized Feynman integral lives on the complement of the determinant hypersurface, which has worse singularities, we will reformulate the integral on the Kausz compactification of GL_n where the boundary divisor of the compactification is normal crossings.

If $Y \subset X$ is a simple normal crossings divisor in a smooth projective variety, with Y_j the components of Y, with local equations $Y_j = \{f_j = 0\}$, the complex of forms with logarithmic poles $\Omega^*_X(\log(Y))$ spanned by the forms $\frac{df_j}{f_j}$ and by the holomorphic forms on X.

As in Theorem 6.3 of (24), we obtain that the Rota–Baxter operator of polar projection $T : \mathcal{M}^{\mathrm{even}}_{X,Y} \to \mathcal{M}^{\mathrm{even}}_{X,Y}$ restricts to a Rota–Baxter operator $T : \Omega^{\mathrm{even}}_X(\log(Y)) \to \Omega^{\mathrm{even}}_X(\log(Y))$ given by

$$T : \eta \mapsto T(\eta) = \sum_j \frac{df_j}{f_j} \wedge \mathrm{Res}_{Y_j}(\eta), \tag{2.3.2}$$

where the holomorphic form $\mathrm{Res}_{Y_j}(\eta)$ is the Poincaré residue of η restricted to Y_j.

Unlike the case of a single smooth hypersurface, for a simple normal crossings divisor the Rota–Baxter operator operator T does not satisfy $T(x)T(y) \equiv 0$, since we now have terms like $\frac{df_j}{f_j} \wedge \frac{df_k}{f_k} \neq 0$, for $j \neq k$, so the Rota–Baxter identity for T does not reduce to a derivation, but some of the properties that simplify the Birkhoff factorization in the case of a smooth hypersurface still hold in this case.

Proposition 2.3.1. *The Rota–Baxter operator T of (2.3.2) satisfies $T^2 = T$ and the Rota–Baxter* *identity simplifies to the form*

$$T(\eta \wedge \xi) = T(\eta) \wedge \xi + \eta \wedge T(\xi) - T(\eta) \wedge T(\xi). \tag{2.3.3}$$

The operator $(1 - T) : \mathcal{R} \to \mathcal{R}_+$ *is an algebra homomorphism, with* $\mathcal{R} = \Omega_X^{\rm even}(\log(Y))$ *and* $\mathcal{R}_+ = (1 - T)\mathcal{R}$*. The Birkhoff factorization of a commutative algebra homomorphism* $\phi : \mathcal{H} \to \mathcal{R}$*, with* $\mathcal{H}$ *a commutative Hopf algebra is given by*

$$\begin{aligned} \phi_+(X) &= (1 - T)\phi(X) \\ \phi_-(X) &= -T(\phi(X) + \sum \phi_-(X')\phi(X'')). \end{aligned} \tag{2.3.4}$$

Moreover, ϕ_- *takes the following form on* $\ker e = \oplus_{n>0} H_n$*:*

$$\begin{aligned} \phi_-(X) &= -T(\phi(X)) - \sum_{n=1}^{\infty} (-1)^n \sum T(\phi(X^{(1)}))\phi(X^{(2)})\phi(X^{(3)}) \cdots \phi(X^{(n+1)}) \\ &= -T(\phi(X)) - \sum_{n=1}^{\infty} (-1)^n ((T\phi)\tilde{*}\phi^{\tilde{*}^n})(X). \end{aligned}$$

Proof. The argument is the same as in the proof of Theorem 6.3 in (24). It is clear by construction that T is idempotent and the simplified form (2.3.3) of the Rota–Baxter identity follows by observing that $T(T(\eta)\wedge\xi) = T(\eta)\wedge\xi$ and $T(\eta\wedge T(\xi)) = \eta\wedge T(\xi)$ as in Theorem 6.3 in (24). Then one sees that $(1-T)(\eta)\wedge(1-T)(\xi) = \eta\wedge\xi - T(\eta)\wedge\xi - \eta\wedge T(\xi) + T(\eta)\wedge T(\xi) = \eta\wedge\xi - T(\eta\wedge\xi)$ by (2.3.3). Consider then the Birkhoff factorization. We write $\tilde{\phi}(X) := \phi(X) + \sum \phi_-(X')\phi(X'')$. The fact that $(1-T)$ is an algebra homomorphism then gives $\phi_+(X) = (1-T)(\tilde{\phi}(X)) = (1-T)(\phi(X) + \sum \phi_-(X')\phi(X'')) = (1-T)(\phi(X)) + \sum (1-T)(\phi_-(X'))(1-T)(\phi(X'')))$, with $(1-T)(\phi_-(X')) = -(1-T)T(\tilde{\phi}_-(X')) = 0$, because T is idempotent. The last statement again follows from Proposition 2.1.8, since we have $T(T(\eta) \wedge \xi) = T(\eta) \wedge \xi$. □

2.3.2 Multidimensional residues

In the case of a simple normal crossings divisor $Y \subset X$, we can proceed as discussed in Section 2.2.2 for the case of a smooth hypersurface. Indeed, as we have seen in Proposition 2.3.1, we also have in this case a simple pole subtraction $\phi_+(X) = (1-T)\phi(X)$, even though the negative term $\phi_-(X)$ of the Birkhoff factorization can now be more complicated than in the case of a smooth hypersurface. The unrenormalized $\phi(X)$ is a form $\eta = \alpha + \sum_j \frac{df_j}{f_j} \wedge \beta_j$, with α and β_j holomorphic and $Y_j = \{f_j = 0\}$ the components of Y. Again, if $\alpha = 0$ we lose all information about $\phi(X)$ in our renormalization of the logarithmic form. To avoid this problem, we can again consider, instead of the single renormalized integral $\int_\sigma \phi_+(X)$, an additional family of integrals

$$\int_{\sigma \cap Y_I} \mathrm{Res}_{Y_I}(\eta),$$

where $Y_I = \cap_{j \in I} Y_j$ is an intersection of components of the divisor Y and $\mathrm{Res}_{Y_I}(\eta)$ is the iterated (or multidimensional, or higher) Poincaré residue of η, in the sense of (4), (3). These are dual to the iterated Leray coboundaries,

$$\int_{\sigma \cap Y_I} \mathrm{Res}_{Y_I}(\eta) = \frac{1}{(2\pi i)^n} \int_{\mathcal{L}_I(\sigma \cap Y_I)} \eta,$$

where $\mathcal{L}_I = \mathcal{L}_{j_i} \circ \cdots \circ \mathcal{L}_{j_n}$ for $Y_I = Y_{j_1} \cap \cdots \cap Y_{j_n}$.

If arbitrary intersections Y_I of components of Y are all mixed Tate motives, then all these integrals are also periods of mixed Tate motives.

2.3.3 Saito's logarithmic forms

Given a singular reduced hypersurface $Y \subset X$, a differential form ω with logarithmic poles along Y, in the sense of Saito (90), can always be written in the form ((90), (1.1))

$$f\,\omega = \frac{dh}{h} \wedge \xi + \eta, \tag{2.3.5}$$

where $f \in \mathcal{O}_X$ defines a hypersurface $V = \{f = 0\}$ with $\dim(Y \cap V) \leq \dim(X) - 2$, and with ξ and η holomorphic forms.

In the following, we use the notation ${}^S\Omega^\star_X(\log(Y))$ to denote the forms with logarithmic poles along Y in the sense of Saito, to distinguish it from the more restrictive notion of forms with logarithmic poles $\Omega^\star_X(\log(Y))$ considered above for the normal crossings case.

Following (3), we say that a (reduced) hypersurface $Y \subset X$ has *Saito singularities* if the modules of logarithmic differential forms and vector fields along Y are free. The condition that $Y \subset X$ has

Saito singularities is equivalent to the condition that ${}^S\Omega^n_X(\log(Y)) = \bigwedge^n \, {}^S\Omega^1_X(\log(Y))$, (90).

Let $\mathcal{M}_Y$ denote the sheaf of germs of meromorphic functions on Y. Then setting

$$\mathrm{Res}(\omega) = \frac{1}{f}\,\xi\,|_Y \tag{2.3.6}$$

defines the residue as a morphism of $\mathcal{O}_X$-modules, for all $q \geq 1$,

$$\mathrm{Res} : {}^S\Omega^q_X(\log(Y)) \to \mathcal{M}_Y \otimes_{\mathcal{O}_Y} \Omega^{q-1}_Y. \tag{2.3.7}$$

A refinement of (2.3.7) is given by the following result, (3). For $Y \subset X$ a reduced hypersurface, and for all $q \geq 1$, there is an exact sequence of $\mathcal{O}_X$-modules

$$0 \to \Omega^{q+1}_X \to {}^S\Omega^{q+1}_X(\log(Y)) \xrightarrow{\mathrm{Res}} \omega^q_Y \to 0. \tag{2.3.8}$$

Unlike the case of normal crossings divisors, the Saito residue of forms with logarithmic poles is not a holomorphic form, but a *meromorphic* form on Y.

It is natural to ask whether the extraction of polar part from forms with logarithmic poles that we considered here for the case of smooth hypersurfaces and normal crossings divisors extends to more general singular hypersurfaces using Saito's formulation.

Question 2.3.2. *For more general singular hypersurfaces $Y \subset X$ with Saito singularities, is the Rota–Baxter operator T on even meromorphic forms expressible in terms of Saito residues in the case of forms with logarithmic poles?*

We describe here a possible approach to this question. We introduce an analog of the Rota–Baxter operator considered above, given by the extraction of the polar part. The "polar part" operator, in this more general case, does not maps $\Omega^{\mathrm{even}}_X(\log(Y))$ to itself, but we show below that it gives a well defined Rota-Baxter operator of weight -1 on the space of Saito forms ${}^S\Omega^{\mathrm{even}}_X(\log(Y))$, and that this operator is a derivation.

Lemma 2.3.3. *The set $S_Y := \{f : \dim(\{f = 0\} \cap Y) \leq \dim(X) - 2\}$ is a multiplicative set. Localization of the Saito forms with logarithmic poles gives $S_Y^{-1}\, {}^S\Omega_X(\log(Y)) = {}^S\Omega_X(\log(Y))$.*

Proof. We have $V_{12} = \{f_1 f_2 = 0\} = \{f_1 = 0\} \cup \{f_2 = 0\}$ and $\dim(Y \cap V_{12}) = \dim((Y \cap \{f_1 = 0\}) \cup (Y \cap \{f_2 = 0\})) \leq \dim(X) - 2$, since $\dim(Y \cap \{f_i = 0\}) \leq \dim(X) - 2$ for $i = 1, 2$. Thus, for any $f_1, f_2 \in S_Y$, we have $f_1 f_2 \in S_Y$. Moreover, we have $1 \in S_Y$, and hence S_Y is a multiplicative set. The localization of ${}^S\Omega^\star_X(\log(Y))$ at S_Y is just ${}^S\Omega^\star_X(\log(Y))$ itself: in fact, for $\tilde{f}^{-1}\omega \in S_Y^{-1}\, {}^S\Omega^\star_X(\log(Y))$, with $\tilde{f} \in S_Y$ and $\omega \in {}^S\Omega^\star_X(\log(Y))$, expressed as in (2.3.5), we have

$$f\tilde{f}(\tilde{f}^{-1}\omega) = f\omega = \frac{dh}{h} \wedge \xi + \eta,$$

where $f\tilde{f} \in S_Y$, hence $\tilde{f}^{-1}\omega \in {}^S\Omega_X(\log(Y))$. □

Given a form $\omega \in {}^S\Omega^\star_X(\log(Y))$, which we can write as in (2.3.5), the residue (2.3.6) is the image under the restriction map $S_Y^{-1}\Omega^\star_X \to S_Y^{-1}\Omega^\star_Y$ of the form $f^{-1}\xi \in S_Y^{-1}\Omega^\star_X$. Moreover, we have an inclusion $\Omega^\star_X \hookrightarrow {}^S\Omega^\star_X(\log(Y))$, which induces a corresponding map of the localizations $S_Y^{-1}\Omega^\star_X \hookrightarrow S_Y^{-1}\, {}^S\Omega^\star_X(\log(Y)) = {}^S\Omega^\star_X(\log(Y))$. We can then define a linear operator

$$T : {}^S\Omega^\star_X(\log(Y)) \to {}^S\Omega^\star_X(\log(Y)) \wedge S_Y^{-1}\,\Omega^\star_X \hookrightarrow {}^S\Omega^\star_X(\log(Y)) \wedge S_Y^{-1}\, {}^S\Omega^\star_X(\log(Y)) = {}^S\Omega^\star_X(\log(Y))$$

given by

$$T(\omega) = \frac{dh}{h} \wedge \frac{\xi}{f}, \quad \text{for} \quad f\,\omega = \frac{dh}{h} \wedge \xi + \eta. \tag{2.3.9}$$

Lemma 2.3.4. *The operator T of (2.3.9) is a Rota–Baxter operator of weight -1 on ${}^S\Omega^{\rm even}_X(\log(Y))$, which is just given by a derivation, satisfying the Leibniz rule $T(\omega_1 \wedge \omega_2) = T(\omega_1) \wedge \omega_2 + \omega_1 \wedge T(\omega_2)$.*

Proof. Let

$$f_1\,\omega_1 = \frac{dh}{h} \wedge \xi_1 + \eta_1 \quad f_2\,\omega_2 = \frac{dh}{h} \wedge \xi_2 + \eta_2.$$

Then

$$f_1\, f_2\, \omega_1 \wedge \omega_2 = (\frac{dh}{h} \wedge \xi_1 + \eta_1) \wedge (\frac{dh}{h} \wedge \xi_2 + \eta_2) = \frac{dh}{h} \wedge (\xi_1 \wedge \eta_2 + (-1)^p \eta_1 \wedge \xi_2) + \eta_1 \wedge \eta_2,$$

where $\eta_1 \in \Omega^p(X)$. By Lemma 2.3.3, we know that $f_1 f_2 \in S_Y$. We have

$$T(\omega_1 \wedge \omega_2) = \frac{dh}{h} \wedge (\frac{\xi_1}{f_1} \wedge \frac{\eta_2}{f_2} + (-1)^p \frac{\eta_1}{f_1} \wedge \frac{\xi_2}{f_2}).$$

Since

$$T(\omega_1) = \frac{dh}{h} \wedge \frac{\xi_1}{f_1}, \quad \text{and} \quad T(\omega_2) = \frac{dh}{h} \wedge \frac{\xi_2}{f_2},$$

we obtain

$$T(\omega_1) \wedge T(\omega_2) = \frac{dh}{h} \wedge \frac{\xi_1}{f_1} \wedge \frac{dh}{h} \wedge \frac{\xi_2}{f_2} = 0.$$

Moreover, we have

$$T(\omega_1) \wedge \omega_2 = (\frac{dh}{h} \wedge \frac{\xi_1}{f_1}) \wedge \frac{dh}{h} \wedge \frac{\xi_2}{f_2} + \frac{dh}{h} \wedge \frac{\xi_1}{f_1} \wedge \frac{\eta_2}{f_2} = \frac{dh}{h} \wedge \frac{\xi_1}{f_1} \wedge \frac{\eta_2}{f_2},$$

with

$$f_1 f_2 (T(\omega_1) \wedge \omega_2) = \frac{dh}{h} \wedge \xi_1 \wedge \eta_2,$$

and similarly,

$$\omega_1 \wedge T(\omega_2) = (-1)^p \frac{dh}{h} \wedge \frac{\eta_1}{f_1} \wedge \frac{\xi_2}{f_2},$$

and hence T satisfies the Leibnitz rule. The operator T also satisfies $T(T(\omega_1) \wedge \omega_2) = T(\omega_1) \wedge \omega_2$, and $T(\omega_1 \wedge T(\omega_2)) = \omega_1 \wedge T(\omega_2)$, hence the condition that T is a derivation is equivalent to the condition that it is a Rota-Baxter operator of weight -1. □

Correspondingly, we have

$$(1 - T)\omega = \omega - \frac{dh}{h} \wedge \frac{\xi}{f} = \frac{\eta}{f} \in S_Y^{-1}\, \Omega_X^{\text{even}}.$$

Under the restriction map $S_Y^{-1}\, \Omega_X^{\text{even}} \to S_Y^{-1}\, \Omega_Y^{\text{even}}$ we obtain a form $(1 - T)(\omega)|_Y$. It follows that we can define a "subtraction of divergences" operation on $\phi : \mathcal{H} \to {}^S\Omega_X^{\text{even}}(\log(Y))$ by taking $\phi_+ : \mathcal{H} \to \mathcal{R}_X^{\text{even}}(\log(Y))$ given by $\phi_+(a) = (1 - T)\phi(a)|_Y$, for $a \in \mathcal{H}$, which maps $\phi(a) = \omega$ to $(1-T)\omega|_Y = f^{-1}\eta|_Y$, where $f\,\omega = \frac{dh}{h} \wedge \xi + \eta$. While this has subtracted the logarithmic pole along Y, it has also created a new pole along $V = \{f = 0\}$. Thus, it results again in a meromorphic form. If we consider the restriction to Y of $\phi_+(a) = f^{-1}\,\eta|_Y$, we obtain a meromorphic form with first order poles along a subvariety $V \cap Y$, which is by hypothesis of codimension at least one in Y. Thus, we can conceive of a more complicated renormalization method that progressively subtracts poles on subvarieties of increasing codimension, inside the polar locus of the previous pole subtraction, by iterating this procedure.

2.4 Compactifications of GL_n and momentum space Feynman integrals

In this section, we restrict our attention to the case of compactifications of PGL_ℓ and of GL_ℓ and we use a formulation of the parametric Feynman integrals of perturbative quantum field theory in terms of (possibly divergent) integrals on a cycle in the complement of the determinant hypersurface (6), to obtain a new method of regularization and renormalization, which always gives rise to a renormalized integral that is a period of a mixed Tate motive, even though a certain loss of information can occur with respect to the physical Feynman integral.

2.4.1 The determinant hypersurface

In the following we use the notation $\hat{\mathcal{D}}_\ell$ and $\mathcal{D}_\ell$, respectively, for the affine and the projective determinant hypersurfaces. Namely, we consider in the affine space $\mathbb{A}^{\ell^2}$, identified with the space of all $\ell \times \ell$-matrices, with coordinates $(x_{ij})_{i,j=1,\dots,\ell}$, the hypersurface

$$\hat{\mathcal{D}}_\ell = \{\det(X) = 0 \,|\, X = (x_{ij})\} \subset \mathbb{A}^{\ell^2}.$$

Since $\det(X) = 0$ is a homogeneous polynomial in the variables (x_{ij}), we can also consider the projective hypersurface $\mathcal{D}_\ell \subset \mathbb{P}^{\ell^2-1}$.

The complement $\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell$ is identified with the space of invertible $\ell \times \ell$-matrices, namely with GL_ℓ.

2.4.2 The Kausz compactification of GL_n

We recall here some basic facts about the Kausz compactification $K\mathrm{GL}_n$ of GL_n, following (60) and the exposition in §11 of (81).

We first recall the Vainsencher compactification (94) of PGL_ℓ. Let $X_0 = \mathbb{P}^{\ell^2-1}$ be the projectivization of the space $\mathbb{A}^{\ell^2}$ of square $\ell \times \ell$-matrices. Let Y_i be the locus of matrices of rank i and consider the iterated blowups $X_i = \mathrm{Bl}_{\bar{Y}_i}(X_{i-1})$, with $\bar{Y}_i$ the closure of Y_i in X_{i-1}. It is shown in Theorem 1 and (2.4) of (94) that the X_i are smooth, and that $X_{\ell-1}$ is a wonderful compactification of PGL_ℓ, in the sense of (28). Moreover, the Y_i are PGL_i-bundles over a product of Grassmannians. One denotes by $\overline{\mathrm{PGL}_\ell}$ the wonderful compactification of PGL_ℓ obtained in this way. We also refer the reader to §12 of (81) for a quick overview of the main properties of the Vainsencher compactification.

The Kausz compactification (60) of GL_ℓ is similar. One regards $\mathbb{A}^{\ell^2}$ as the big cell in $\mathcal{X}_0 = \mathbb{P}^{\ell^2}$. The iterated sequence of blowups is given in this case by setting $\mathcal{X}_i = \mathrm{Bl}_{\bar{\mathcal{Y}}_{i-1}\cup\bar{\mathcal{H}}_i}(\mathcal{X}_{i-1})$, where $\mathcal{Y}_i \subset \mathbb{A}^{\ell^2}$ are the matrices of rank i and $\mathcal{H}_i$ are the matrices at infinity (that is, in $\mathbb{P}^{\ell^2-1} = \mathbb{P}^{\ell^2} \smallsetminus \mathbb{A}^{\ell^2}$) of rank i. It is shown in Theorem 9.1 of (60) that the $\mathcal{X}_i$ are smooth and that the blowup loci are disjoint unions of loci that are, respectively, a $\overline{\mathrm{PGL}_i}$-bundle and a $K\mathrm{GL}_i$-bundle over a product of Grassmannians. An overview of these properties and of the relation between the Vainsencher and the Kausz compactifications is given in §12 of (81).

As observed in (81), the Kausz compactification is then the closure of GL_ℓ inside the wonderful compactification of $\mathrm{PGL}_{\ell+1}$, see also (54), Chapter 3, §1.3. The compactification $K\mathrm{GL}_\ell$ is smooth and projective over $\mathrm{Spec}\mathbb{Z}$ (Corollary 4.2 (60)).

The other property of the Kausz compactification that we will be using in the following is the fact that the complement of the dense open set GL_ℓ inside the compactification $K\mathrm{GL}_\ell$ is a normal crossing divisor (Corollary 4.2 (60)).

2.4.3 The virtual motive of the Kausz compactification

We can use the description recalled above of the Kausz compactification, together with the blowup formula, to check that the virtual motive (class in the Grothendieck ring) of the Kausz compactification is Tate.

Proposition 2.4.1. *Let $K_0(\mathcal{V})$ be the Grothendieck ring of varieties (defined over $\mathbb{Q}$ or over $\mathbb{Z}$) and let $\mathbb{Z}[\mathbb{L}] \subset K_0(\mathcal{V})$ be the Tate subring generated by the Lefschetz motive $\mathbb{L} = [\mathbb{A}^1]$. For all $\ell \geq 1$ the class $[K\mathrm{GL}_\ell]$ is in $\mathbb{Z}[\mathbb{L}]$. Moreover, let $\mathcal{Z}_\ell$ be the normal crossings divisor $\mathcal{Z}_\ell = K\mathrm{GL}_\ell \smallsetminus \mathrm{GL}_\ell$. Then all the unions and intersections of components of $\mathcal{Z}_\ell$ have Grothendieck classes in $\mathbb{Z}[\mathbb{L}]$.*

Proof. We use the blowup formula for classes in the Grothendieck ring: if $\tilde{\mathcal{X}} = \mathrm{Bl}_{\mathcal{Y}}(\mathcal{X})$, where $\mathcal{Y}$ is of codimension $m+1$ in $\mathcal{X}$, then the classes satisfy

$$[\tilde{\mathcal{X}}] = [\mathcal{X}] + \sum_{k=1}^{m} [\mathcal{Y}]\mathbb{L}^k. \tag{2.4.1}$$

The Kausz compactification is obtained as an iterated blowup, starting with a projective space whose class is in $\mathbb{Z}[\mathbb{L}]$ and blowing up at each step a smooth locus that is a bundle over a product of Grassmannians with fiber either a $K\mathrm{GL}_i$ or a $\overline{\mathrm{PGL}}_i$ for some $i < \ell$. The Grothendieck class of a bundle is the product of the class of the base and the class of the fiber. Classes of Grassmannians (and products of Grassmannians) are in $\mathbb{Z}[\mathbb{L}]$. The classes of the wonderful compactifications $\overline{\mathrm{PGL}}_i$ of PGL_i are also in $\mathbb{Z}[\mathbb{L}]$, since it is known that the motive of these wonderful compactifications is mixed Tate (see for instance (52)). Thus, it suffices to assume, inductively, that the classes $[K\mathrm{GL}_i] \in \mathbb{Z}[\mathbb{L}]$ for all $i < \ell$, and conclude via the blowup formula that $[K\mathrm{GL}_\ell] \in \mathbb{Z}[\mathbb{L}]$.

Consider then the boundary divisor $\mathcal{Z}_\ell = K\mathrm{GL}_\ell \smallsetminus \mathrm{GL}_\ell$. The geometry of the normal crossings divisor $\mathcal{Z}_\ell$ is described explicitly in Theorems 9.1 and 9.3 of (60). It has components Y_i and Z_i, for $0 \leq i \leq \ell$, that correspond to the blowup loci described above. The multiple intersections $\cap_{i\in I} Y_i \cap \cap_{j\in J} Z_j$ of these components of $\mathcal{Z}_\ell$ are described in turn in terms of bundles over products of flag varieties with fibers that are lower dimensional compactifications $K\mathrm{GL}_i$ and $\overline{\mathrm{PGL}}_i$ and products. Again, flag varieties have cell decompositions, and hence their Grothendieck classes are in $\mathbb{Z}[\mathbb{L}]$ and the rest of the argument proceeds as in the previous case. If arbitrary intersections of the components of $\mathcal{Z}_\ell$ have classes in $\mathbb{Z}[\mathbb{L}]$ then arbitrary unions and unions of intersections also do by inclusion-exclusion in $K_0(\mathcal{V})$. □

2.4.4 The numerical motive of the Kausz compactification

Knowing that the Grothendieck class $[K\mathrm{GL}_\ell]$ is in the Tate subring $\mathbb{Z}[\mathbb{L}] \subset K_0(\mathcal{V})$ determines the motive in the category of pure motives with the numerical equivalence. More precisely, we have the following.

Proposition 2.4.2. *Let $h_{\mathrm{num}}(K\mathrm{GL}_\ell)$ denote the motive of the Kausz compactification $K\mathrm{GL}_\ell$ in the category of pure motives over $\mathbb{Q}$, with the numerical equivalence relation. Then $h_{\mathrm{num}}(K\mathrm{GL}_\ell)$ is in the subcategory generated by the Tate object. The same is true for arbitrary unions and intersections of the components of the boundary divisor $\mathcal{Z}_\ell$ of the compactification.*

Proof. The same argument used in Proposition 2.4.1 can be upgraded at the level of numerical motives. We replace the blowup formula (2.4.1) for Grothendieck classes with the corresponding formula for motives, which follows (already at the level of Chow motives) from Manin's identity principle, (78):

$$h(\tilde{X}) = h(X) \oplus \bigoplus_{r=1}^{m} h(Y) \otimes \mathbb{L}^{\otimes r}, \tag{2.4.2}$$

with $\tilde{X} = \mathrm{Bl}_Y(X)$ the blowup of a smooth subvariety $Y \subset X$ of codimension $m+1$ in a smooth projective variety X, and with $\mathbb{L} = h^2(\mathbb{P}^1)$ is the Lefschetz motive. Moreover, we use the fact that, for numerical motives, the motive of a locally trivial fibration $X \to S$ with fiber Y is given by the product

$$h_{\mathrm{num}}(X) = h_{\mathrm{num}}(Y) \otimes h_{\mathrm{num}}(S). \tag{2.4.3}$$

See Exercise 13.2.2.2 of (7). The decomposition (2.4.3) allows us to describe the numerical motives of the blowup loci of the iterated blowup construction of $K\mathrm{GL}_\ell$ as products of numerical motives of Grassmannians and of lower dimensional compactifications $K\mathrm{GL}_i$ and $\overline{\mathrm{PGL}}_i$. The motive of a Grassmannian can be computed explicitly as in (62), already at the level of Chow motives. If $G(d,n)$ denotes the Grassmannian of d-planes in k^n, the Chow motive $h(G(d,n))$ is given by

$$h(G(d,n)) = \bigoplus_{\lambda \in W^d} \mathbb{L}^{\otimes |\lambda|}, \tag{2.4.4}$$

where

$$W^d = \{\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{N}^d \,|\, n-d \geq \lambda_1 \geq \cdots \geq \lambda_d \geq 0\}$$

and $|\lambda| = \sum_i \lambda_i$; see Theorem 2.1 and Lemma 3.1 of (62). The same decomposition into powers of the Lefschetz motive holds at the numerical level. Moreover, we know (also already for Chow motives) that the motives $h(\overline{\mathrm{PGL}}_i)$ of the wonderful compactifications are Tate (see (52)), and we conclude the argument as in Proposition 2.4.1 by assuming inductively that the motives $h_{\mathrm{num}}(K\mathrm{GL}_i)$ are Tate, for $i < \ell$. The argument for the loci $\cap_{i\in I} Y_i \cap \cap_{j \in J} Z_j$ in $\mathcal{Z}_\ell$ is analogous. $\square$

Remark 2.4.3. Proposition 2.4.2 also follows from Proposition 2.4.1 using the general fact that two numerical motives that have the same class in $K_0(\mathrm{Num}(k)_{\mathbb{Q}})$ are isomorphic as objects in $\mathrm{Num}(k)_{\mathbb{Q}}$, because of the semi-simplicity of the category of numerical motives, together with the existence, for $char(k) = 0$, of a unique ring homorphism (the motivic Euler characteristic) $\chi_{\mathrm{mot}} : K_0(\mathcal{V}_k) \to K_0(\mathrm{Num}(k)_{\mathbb{Q}})$. This is such that, for a smooth projective variety X, $\chi_{\mathrm{mot}}([X]) = [h_{\mathrm{num}}(X)]$, where $h_{\mathrm{num}}(X)$ is the motive of X in $\mathrm{Num}(k)_{\mathbb{Q}}$; see Corollary 13.2.2.1 of (7).

2.4.5 The Chow motive of the Kausz compactification

Manin's blowup formula (2.4.2) and the computation of the motive of Grassmannians and of the wonderful compactifications $\overline{\mathrm{PGL}}_i$ already hold at the level of Chow motives. However, if we want to extend the argument of Proposition 2.4.2 to Chow motives, we run into the additional difficulty that one no longer necessarily has the decomposition (2.4.3) for the motive of a locally trivial fibration. Under some hypotheses on the existence of a cellular structure, one can still obtain a decomposition for motives of bundles, and more generally locally trivial fibrations, the fibers of which have cell decompositions with suitable properties, see (59), and also (51), (52), (58), and (86). We obtain an unconditional result on the Chow motive of the Kausz compactification, by analyzing its cellular structure.

Recall that, for G a connected reductive algebraic group and B a Borel subgroup, a *spherical variety* is is a normal algebraic variety on which G acts with a dense orbit of B, (16). Spherical varieties can be regarded as a generalization of toric varieties: when G is a torus, one recovers the usual notion of toric variety.

Proposition 2.4.4. *The Chow motive $h(K\mathrm{GL}_\ell)$ of the Kausz compactification is a Tate motive.*

Proof. The result follows by showing that $K\mathrm{GL}_\ell$ has a cellular structure for all $\ell \geq 1$, which allows us to extend the decomposition of the motive used in Proposition 2.4.2 from the numerical to the Chow case.

As shown in §3.1 of (16), it follows from the work of Bialynicki–Birula (13) that any complete, smooth, and spherical variety X has a cellular decomposition. This is determined by the decomposition of the spherical variety into B-orbits and is obtained by considering a one-parameter subgroup $\lambda : \mathbb{G}_m \hookrightarrow X$ in general position, with X^λ the finite set of fixed points, with cells given by

$$X(\lambda, x) = \{z \in X \mid \lim_{t \to 0} \lambda(t) z = x\}, \quad \text{for} \quad x \in X^\lambda. \tag{2.4.5}$$

The Kausz compactification $K\mathrm{GL}_\ell$ is a smooth toroidal equivariant compactification of GL_ℓ; see Proposition 1.15 of §3 of (54) and also Proposition 9.1 and Proposition 12.1 of (81). In particular, it is a spherical variety (see Proposition 9.1 of (81)), and hence it has a cellular structure as above. A relative cellular variety, in the sense of (59), is a smooth and proper variety with a decomposition into affine fibrations over proper varieties. The blowup loci of the Kausz compactification $K\mathrm{GL}_\ell$

are relative cellular varieties in this sense, since they are bundles over products of Grassmannians, with fiber a lower dimensional compactification $K\mathrm{GL}_i$, with $i < \ell$. Using the cell decomposition of the fibers $K\mathrm{GL}_i$, we obtain a decomposition of these blowup loci as relative cellular varieties, with pieces of the decomposition being fibrations over a product of Grassmannians, with fibers the cells of the cellular structure of $K\mathrm{GL}_i$.

There is an embedding of the category of pure Chow motives in the category of mixed motives (with some subtleties involved in passing from the cohomological formulation of pure motives to the homological formulation of mixed motives); see (7). By viewing the Chow motives of these blowup loci as elements in the Voevodsky category of mixed motives, Corollary 6.11 of (59) shows that they are direct sums of motives of products of Grassmannians (which are Tate motives), with twists and shifts which depend on the dimensions of the cells of $K\mathrm{GL}_i$. We conclude from this that all the blowup loci are Tate motives. We can then repeatedly applying the blowup formula for Chow motives to conclude (unconditionally) that the Chow motive of $K\mathrm{GL}_\ell$ is itself a Tate motive. Note that the blowup formula also holds in the Voevodsky category, Proposition 3.5.3 of (95), in the form

$$\mathfrak{m}(\mathrm{Bl}_Y(X)) = \mathfrak{m}(X) \oplus \bigoplus_{r=1}^{\mathrm{codim}_X(Y)-1} \mathfrak{m}(Y)(r)[2r],$$

which corresponds to the usual formula of (78) in the case of pure motives, after viewing them as objects in the category of mixed motives. The result can also be obtained, in a similar way, using Theorem 2.10 of (52) instead of Corollary 6.11 of (59). □

Remark 2.4.5. Given the existence of a cellular decomposition of $K\mathrm{GL}_\ell$, as above, it is possible to give a quicker proof that the Chow motive is Tate, by using distinguished triangles in the Voevodsky category associated to the inclusions of unions of cells, showing that $\mathfrak{m}(K\mathrm{GL}_\ell)$ is mixed Tate, then using the inclusion of pure motives in the mixed motives to conclude $h(K\mathrm{GL}_\ell)$ is Tate. In Proposition 2.4.4 above we chose to maintain the structure of the argument more similar to the cases of the virtual and the numerical motive, for better direct comparison.

Remark 2.4.6. Notice that a *conditional* result about the Chow motive would follow directly from Proposition 2.4.2 or Remark 2.4.3, if one assumes the Kimura–O'Sullivan conjecture (or Voevodsky's nilpotence conjecture, which implies it). For the precise statement and implications of the Kimura–O'Sullivan conjecture, and its relation to Voevodsky's nilpotence, we refer the reader to the survey

(8). By arguing as in Lemma 13.2.1.1 of (7), that would extend the result of Proposition 2.4.2 to the Chow motive. At the level of Grothendieck classes, the conjecture in fact implies that the K_0 of Chow motives and the K_0 of numerical motives coincide, and hence one can argue as in Remark 2.4.3 and conclude that, in order to know that the Chow motive is mixed Tate, it suffices to know that the Grothendieck class is mixed Tate.

2.4.6 Feynman integrals in momentum space and non-mixed-Tate examples

It was shown in (14) that the parametric form of Feynman integrals in perturbative quantum field theory can be formulated as a (possibly divergent) period integral on the complement of a hypersurface defined by the vanishing of a combinatorial polynomial associated to the Feynman graphs. Namely, one writes the (unrenormalized) Feynman amplitudes for a *massless* scalar quantum field theory as integrals

$$U(\Gamma) = \frac{\Gamma(n - D\ell/2)}{(4\pi)^{\ell D/2}} \int_{\sigma_n} \frac{P_\Gamma(t,p)^{-n+D\ell/2}\omega_n}{\Psi_\Gamma(t)^{-n+D(\ell+1)/2}}, \tag{2.4.6}$$

where $n = \#E_\Gamma$ is the number of internal edges, $\ell = b_1(\Gamma)$ is the number of loops, and D is the spacetime dimension. Here we consider the "unregularized" Feynman integral, where D is just the integer valued dimension, without performing the procedure of dimensional regularization that analytically continues D to a complex number. The domain of integration is a simplex $\sigma_n = \{t \in \mathbb{R}^n_+ | \sum_i t_i = 1\}$. In the integration form, ω_n is the volume form, and P_Γ and Ψ_Γ are polynomials defined as follows. The graph polynomial is defined as

$$\Psi_\Gamma(t) = \sum_T \prod_{e \notin T} t_e,$$

where the summation is over spanning trees (assuming the graph Γ is connected). The polynomial P_Γ is given by

$$P_\Gamma(p,t) = \sum_{C \subset \Gamma} s_C \prod_{e \in C} t_e$$

with the sum over cut-sets C (complements of a spanning tree plus one edge) and with variables s_C depending on the external momenta of the graph, $s_C = (\sum_{v \in V(\Gamma_1)} P_v)^2$, where Γ_1 is one of the connected components after the cut (it does not matter which). The variables P_v are given by combinations of external momenta, $P_v = \sum_{e\in E_{ext}(\Gamma), t(e)=v} p_e$, where $\sum_{e\in E_{ext}(\Gamma)} p_e = 0$.

In the range $-n + D\ell/2 \geq 0$, which includes the log divergent case $n = D\ell/2$, the Feynman amplitude is therefore the integral of an algebraic differential form defined on the complement of the graph hypersurface $\hat{X}_\Gamma = \{t \in \mathbb{A}^n \mid \Psi_\Gamma(t) = 0\}$. Divergences occur due to the intersections of the domain of integration σ_n with the hypersurface. Some regularization and renormalization procedure is required to separate the chain of integration from the divergence locus. We refer the reader to (14) (or to (79) for an introductory exposition).

It was originally conjectured by Kontsevich that the graph hypersurfaces $\hat{X}_\Gamma$ would always be mixed Tate motives, which would have explained the pervasive occurrence of multiple zeta values in Feynman integral computations observed in (18). A general result of (12) disproved the conjecture, while more recent results of (20), (21), and (33) showed explicit examples of Feynman graphs that give rise to non-mixed-Tate periods.

2.4.7 Determinant hypersurface and parametric Feynman integrals

In (6) the computation of parametric Feynman integrals was reformulated by replacing the graph hypersurface complement by the complement of the determinant hypersurface.

More precisely, the (affine) graph hypersurface $\hat{X}_\Gamma$ is defined by the vanishing of the graph polynomial Ψ_Γ, which can be written as a determinant

$$\Psi_\Gamma(t) = \det M_\Gamma(t) = \sum_T \prod_{e\notin T} t_e$$

with

$$(M_\Gamma)_{kr}(t) = \sum_{i=1}^{n} t_i \eta_{ik} \eta_{ir}, \tag{2.4.7}$$

where the matrix η is given by

$$\eta_{ik} = \begin{cases} \pm 1 & \text{edge } \pm e_i \in \text{ loop } \ell_k \\ 0 & \text{otherwise.} \end{cases}$$

This definition of the matrix η involves the choice of a basis $\{\ell_k\}$ of the first homology $H_1(\Gamma;\mathbb{Z})$ and the choice of an orientation of the edges of the graph, with $\pm e$ denoting the matching/reverse orientation on the edge e. The resulting determinant $\Psi_\Gamma(t)$ is independent of both choices.

One considers then the map

$$\Upsilon : \mathbb{A}^n \to \mathbb{A}^{\ell^2}, \quad \Upsilon(t)_{kr} = \sum_i t_i \eta_{ik} \eta_{ir}$$

that realizes the graph hypersurface as the preimage

$$\hat{X}_\Gamma = \Upsilon^{-1}(\hat{\mathcal{D}}_\ell)$$

of the determinant hypersurface $\hat{\mathcal{D}}_\ell = \{\det(x_{ij}) = 0\}$.

It is shown in (6) that the map

$$\Upsilon : \mathbb{A}^n \smallsetminus \hat{X}_\Gamma \hookrightarrow \mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell \tag{2.4.8}$$

is an embedding whenever the graph Γ is 3-edge-connected with a closed 2-cell embedding of face width ≥ 3.

As discussed in §3 of (6), the 3-edge-connected condition on graphs can be viewed as a strengthening of the usual 1PI (one-particle-irreducible) condition assumed in physics, since the 1PI condition corresponds to 2-edge-connectivity. In perturbative quantum field theory, one considers 1PI graphs when computing the asymptotic expansion of the effective action. Similarly, one can consider the 2PI effective action (which is related to non-equilibrium phenomena in quantum field theory, see §10.5.1 of (87)) and restrict to 3-edge-connected graphs. The condition of having a closed 2-cell embedding of face width ≥ 3, on the other hand, is a strengthening of the analogous property with face width ≥ 2, which conjecturally is satisfied for all 2-vertex-connected graphs (strong orientable embedding conjecture; see Conjecture 5.5.16 of (82)). 2-vertex-connectivity is again a natural strengthening of the 1PI condition.

A detailed discussion of equivalent formulations and implications of these combinatorial conditions, as well as specific examples of graphs that fail to satisfy them, are given in §3 of (6).

When the map Υ is an embedding, one can, without loss of information, rewrite the parametric Feynman integral as

$$U(\Gamma) = \int_{\Upsilon(\sigma_n)} \frac{\mathcal{P}_\Gamma(x,p)^{-n+D\ell/2} \omega_\Gamma(x)}{\det(x)^{-n+(\ell+1)D/2}}. \tag{2.4.9}$$

Here $\omega_\Gamma(x)$ is a form on $\mathbb{A}^{\ell^2}$ satisfying

$$\omega_\Gamma(x) \wedge \langle \xi_\Gamma, dx \rangle = \omega_{\ell^2},$$

where the right hand side is the standard volume form on $\mathbb{A}^{\ell^2}$ and ξ_Γ is the $(\ell^2 - n)$-frame associated to the linear space $\Upsilon(\mathbb{A}^n)$, see Lemma 2.3 of (6).

The question on the nature of periods is then reformulated in (6) by considering a normal crossings divisor $\hat{\Sigma}_\Gamma$ in $\mathbb{A}^{\ell^2}$ with $\Upsilon(\partial\sigma_n) \subset \hat{\Sigma}_\Gamma$ and considering the motive

$$\mathfrak{m}(\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell, \hat{\Sigma}_\Gamma \smallsetminus (\hat{\Sigma}_\Gamma \cap \hat{\mathcal{D}}_\ell)). \tag{2.4.10}$$

The motive $\mathfrak{m}(\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell)$ of the determinant hypersurface complement belongs to the category of mixed Tate motives (see Theorem 4.1 of (6)), with Grothendieck class

$$[\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell] = \mathbb{L}^{\binom{\ell}{2}} \prod_{i=1}^{\ell} (\mathbb{L}^i - 1).$$

However, as shown in (6), the nature of the motive (2.4.10) is much more difficult to discern, because of the nature of the intersection between the divisor $\hat{\Sigma}_\Gamma$ and the determinant hypersurface. It is shown in Proposition 5.1 of (6), assuming the previous conditions on the graph, that one can consider a divisor $\hat{\Sigma}_{\ell,g}$ that only depends on $\ell = b_1(\Gamma)$ and on the minimal genus g of the surface S_g realizing the closed 2-cell embedding of Γ,

$$\hat{\Sigma}_{\ell,g} = L_1 \cup \cdots \cup L_{\binom{f}{2}}, \tag{2.4.11}$$

where $f = \ell - 2g + 1$ and the irreducible components $L_1, \dots, L_{\binom{f}{2}}$ are linear subspaces defined by the equations

$$\begin{cases} x_{ij} & = 0 \quad 1 \le i < j \le f-1 \\ x_{i1} + \cdots + x_{i,f-1} & = 0 \quad 1 \le i \le f-1. \end{cases}$$

It is also shown in (6) that the motives (2.4.10) are mixed Tate if the *varieties of frames*

$$\mathbb{F}(V_1, \dots, V_\ell) := \{(v_1, \dots, v_\ell) \in \mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell \mid v_k \in V_k\}$$

are mixed Tate. This question is closely related to the geometry of intersections of unions of Schubert cells in flag varieties and Kazhdan–Lusztig theory.

In this paper we will follow a different approach, which uses the same reformulation of parametric Feynman integrals in momentum space in terms of determinant hypersurfaces, as in (6), but instead of computing the integral in the determinant hypersurface complement, pulls it back to the Kausz compactification of GL_ℓ, following the model of computations of Feynman integrals in configuration space described in (24).

2.4.8 Cohomology and forms with logarithmic poles

Let $\mathcal{X}$ be a smooth projective variety and $\mathcal{Z} \subset \mathcal{X}$ a divisor. Let $\mathcal{M}^\star_{\mathcal{Z},\mathcal{X}}$ denote, as before, the complex of meromorphic differential forms on $\mathcal{X}$ with poles (of arbitrary order) along $\mathcal{Z}$, and let $\Omega^\star_{\mathcal{X}}(\log(\mathcal{Z}))$ be the complex of forms with logarithmic poles along $\mathcal{Z}$. Let $\mathcal{U} = \mathcal{X} \smallsetminus \mathcal{Z}$ and $j : \mathcal{U} \hookrightarrow \mathcal{X}$ be the inclusion.

Grothendieck's Comparison Theorem, (47), shows that the natural morphism (de Rham morphism)

$$\mathcal{M}^\star_{\mathcal{Z},\mathcal{X}} \to Rj_* \mathbb{C}_{\mathcal{U}}$$

is a quasi-isomorphism, and hence de Rham cohomology $H^\star_{dR}(\mathcal{U})$ is computed by the hypercohomology of the meromorphic de Rham complex. In particular, for $\mathcal{U}$ affine, the hypercohomology is not necessary and all classes are represented by closed global differential forms, with hypercohomology replaced by the cohomology of the complex of global sections.

The *Logarithmic Comparison Theorem* consists of the statement that, for certain classes of divisors $\mathcal{Z}$, the natural morphism

$$\Omega^\star_{\mathcal{X}}(\log(\mathcal{Z})) \to \mathcal{M}^\star_{\mathcal{Z},\mathcal{X}}$$

is also a quasi-isomorphism. This is known to hold for simple normal crossings divisors by (29), and for strongly quasihomogeneous free divisors by (23), and for a larger class of locally quasihomogeneous divisors in (53). For our purposes, we will focus only on the case of simple normal crossings divisors.

In combination with Grothendieck's Comparison Theorem, the Logarithmic Comparison Theorem of (29) for a simple normal crossings divisor implies that the de Rham cohomology of the divisor complement is computed by the hypercohomology of the logarithmic de Rham complex,

$$H^\star_{dR}(\mathcal{U}) \simeq \mathbb{H}^\star(\mathcal{X}, \Omega^\star_{\mathcal{X}}(\log \mathcal{Z})). \tag{2.4.12}$$

Remark 2.4.7. Even under the assumption that the complement $\mathcal{U}$ is affine, the hypercohomology on the right hand side of (2.4.12) cannot always be replaced by global sections and cohomology. For example, if $\mathcal{X}$ is a smooth projective curve of genus g, and $\mathcal{U}$ is the complement of n points in $\mathcal{X}$, then $H^1_{dR}(\mathcal{U})$ has dimension $2g+n-1$, but the dimension of the space of global sections of the sheaf of logarithmic differentials is only $g+n-1$ by Riemann-Roch.

Some direct comparisons between de Rham cohomology $H^\star_{dR}(\mathcal{U})$ and the cohomology of the logarithmic de Rham complex are known. We discuss in the coming subsections how these apply to our specific case. Our purpose is to replace the meromorphic form that arises in the Feynman integral computation with a cohomologous form with logarithmic poles along the divisor of the Kausz compactification. In doing so, we need to maintain explicit control of the motive of the variety over which cohomology is taken, and also maintain the algebraic nature of all the differential forms involved.

2.4.9 Pullback to the Kausz compactification, forms with logarithmic poles, and renormalization

For fixed $D, \ell \in \mathbb{N}$ (respectively the integer spacetime dimension and the loop number) and for assigned external momenta $p \in \mathbb{Q}^D$, we now consider the algebraic differential form

$$\eta_{\Gamma,D,\ell,p}(x) := \frac{\mathcal{P}_\Gamma(x,p)^{-n+D\ell/2}\omega_\Gamma(x)}{\det(x)^{-n+(\ell+1)D/2}}. \tag{2.4.13}$$

For simplicity, we write the above as $\eta_\Gamma(x)$. This is defined on the complement of the determinant hypersurface, $\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell = \mathrm{GL}_\ell$. Thus, by pulling back to the Kausz compactification, we can regard it as an algebraic differential form on

$$K\mathrm{GL}_\ell \smallsetminus \mathcal{Z}_\ell = \mathrm{GL}_\ell,$$

where $\mathcal{Z}_\ell$ is the normal crossings divisor at the boundary of the Kausz compactification.

2.4.9.1 Cellular decomposition approach

We consider a special case of a simple normal crossings divisor $\mathcal{Z}$ in a smooth projective variety $\mathcal{X}$, under the additional assumption that $\mathcal{X}$ has a cell decomposition. We denote by $\{X_{\alpha,i}\}$ the finite collection of cells of dimension i, and in particular we simply write $X_\alpha = X_{\alpha,\dim\mathcal{X}}$ for the top dimensional cells.

Proposition 2.4.8. *Let $\mathcal{Z} \subset \mathcal{X}$ be a pair as above, with $\{X_\alpha\}$ the top dimensional cells of the cellular decomposition. Given a meromorphic form $\eta \in \mathcal{M}^m_{\mathcal{X},\mathcal{Z}}$, there exist forms $\beta^{(\alpha)}$ on X_α with logarithmic poles along the normal crossings divisor $\mathcal{Z}$, such that*

$$[\beta^{(\alpha)}] = [\eta|_{X_\alpha}] \in H^*_{dR}(X_\alpha \smallsetminus \mathcal{Z}). \tag{2.4.14}$$

Proof. Lemma 2.5 of (23) shows that the Logarithmic Comparison Theorem is equivalent to the statement that, for all Stein open sets $\mathcal{V} \subset \mathcal{X}$, there are isomorphisms $H^\star(\Gamma(\mathcal{V}, \Omega^\star_{\mathcal{X}}(\log\mathcal{Z}))) \simeq H^\star_{dR}(\mathcal{V} \smallsetminus \mathcal{Z})$. Namely, the hypercohomology in the Logarithmic Comparison Theorem can be replaced by cohomology of the complex of sections, when restricted to Stein open sets. □

Remark 2.4.9. The forms $\beta^{(\alpha)}$ do not match consistently on the closures of the cells X_α, because of nontrivial Čech cocycles, and hence they are not restrictions of a unique form with logarithmic poles β defined on all of $\mathcal{X}$. In particular, the forms $\beta^{(\alpha)}$ obtained in this way depend on the cellular decomposition used.

Lemma 2.4.10. *Let $\mathcal{Z} \subset \mathcal{X}$ and $\{X_\alpha\}$ be as above, and suppose given a meromorphic form $\eta \in \mathcal{M}^N_{\mathcal{X},\mathcal{Z}}$, with $N = \dim\mathcal{X}$, and an N-chain $\sigma \subset \mathcal{X}$ with $\partial\sigma \subset \Sigma$, for a divisor Σ in $\mathcal{X}$. After performing a pole subtraction on the logaritmic forms on each cell X_α one can replace the integral $\int_\Sigma \eta$ with a renormalized version*

$$\int_\sigma \beta^+ := \sum_\alpha \int_{X_\alpha\cap\sigma} \beta^{(\alpha),+}, \tag{2.4.15}$$

where $\beta^{(\alpha),+}$ is a simple pole subtraction on $\beta^{(\alpha)}$. The integral (2.4.15) is a sum of periods of motives $\mathfrak{m}(X_\alpha, X_\alpha \cap \Sigma)$. The information contained in the subtracted polar part is recovered by the

Poincaré residues

$$\int_{\sigma\cap\mathcal{Z}_I} \mathrm{Res}_{\mathcal{Z}_I}(\beta) := \sum_\alpha \int_{\sigma\cap\mathcal{Z}_I\cap X_\alpha} \mathrm{Res}_{\mathcal{Z}_I}(\beta^{(\alpha)}) \tag{2.4.16}$$

along the intersections of components $\mathcal{Z}_I = Z_{i_1}\cap\cdots\cap Z_{i_k}$, $I=\{i_1,\ldots,i_k\}$ *of the divisor* $\mathcal{Z}$. *These are sums of periods of the motives* $\mathfrak{m}(\mathcal{Z}_I\cap X_\alpha)$.

Proof. Given the cell decomposition as above, we can write the integral as

$$\int_\sigma \eta = \sum_\alpha \int_{X_\alpha\cap\sigma} \eta|_{X_\alpha} = \sum_\alpha \int_{X_\alpha\cap\sigma} \beta^{(\alpha)}. \tag{2.4.17}$$

where each $\eta|_{X_\alpha}$ is replaced by the cohomologous $\beta^{(\alpha)}$ with logarithmic poles. After performing a pole subtraction on each $\beta^{(\alpha)}$ we obtain holomorphic forms $\beta^{(\alpha),+}$, and hence the resulting integral is a period of $\mathfrak{m}(X_\alpha, X_\alpha\cap\Sigma)$. For the relation between polar subtraction and the Poincaré residues, see the discussion in §2.2.2 and §2.3.2 above. ☐

In both (2.4.15) and (2.4.16), we use the notation on the left-hand-side, with a global integral and a global form β, purely as a formal shorthand notation for the sum of the integrals on the cells of the $\beta^{(\alpha)}$, since the latter are not restrictions of a global form β.

Remark 2.4.11. Notice that the resulting integral (2.4.15) obtained in this way can be identified with a period of $\mathfrak{m}(\mathcal{X},\Sigma)$ only in the case where the forms $\beta^{(\alpha),+}$ are restrictions $\beta^{(\alpha),+} = \beta^+|_{X_\alpha}$ of a single holomorphic form β^+ on $\mathcal{X}$. More generally, the resulting (2.4.15) is only a sum of periods of the motives $\mathfrak{m}(X_\alpha, X_\alpha\cap\Sigma)$.

Remark 2.4.12. If the cellular decomposition of $\mathcal{X}$ has a single top dimensional cell X, then a unique form with logarithmic poles $\beta\in\Omega^\star_X(\log\mathcal{Z})$, satisfying $[\eta|_X]=[\beta]\in H^\star_{dR}(X\smallsetminus\mathcal{Z})$, suffices to regularize the integral $\int_\sigma\eta$, with regularized value $\int_{\sigma\cap X}\beta^+$.

As we discussed in Proposition 2.4.4, the Kausz compactification is a spherical variety (Proposition 1.15 of §3 of (54) and also Proposition 9.1 and Proposition 12.1 of (81)), and hence it has a cellular decomposition (§3.1 of (16)) into cells $X(\lambda,x)$ as in (2.4.5). Thus, we can apply the procedure described above, to regularize the integral $\int_{\Upsilon(\sigma)}\eta_\Gamma$. While this regularization procedure depends on the choice of the cell decomposition, the construction of (16) for spherical varieties provides a cellular structure that is intrinsically defined by the orbit structure of $K\mathrm{GL}_\ell$ and is quite naturally reflecting its geometry. We can then perform a renormalization procedure based on the pole subtraction procedure for forms with logarithmic poles described above.

Corollary 2.4.13. *The cell decomposition* $\{X(\lambda, x)\}$ *of* $K\mathrm{GL}_\ell$ *has a single big cell* X*. Given* $\eta_\Gamma = \eta_{\Gamma,D,\ell,p}$ *as in* (2.4.13)*, there is a form* $\beta_\Gamma = \beta_{\Gamma,D,\ell,p}$ *on the big cell* X*, with logarithmic poles along* $\mathcal{Z}_\ell$*, such that* $[\eta_\Gamma|_X] = [\beta_\Gamma] \in H^\star_{dR}(X \smallsetminus \mathcal{Z})$*. Applying the Birkhoff factorization for forms with logarithmic poles to* β_Γ*, we obtain a renormalized integral of the form*

$$R(\Gamma) = \int_{\tilde{\Upsilon}(\sigma_n)\cap X} \beta^+_{\Gamma,D,\ell,p}, \tag{2.4.18}$$

where β^+_Γ *is a simple pole subtraction on* β_Γ*.*

Proof. As mentioned in Proposition 2.4.4, the spherical variety $K\mathrm{GL}_\ell$ is a smooth toroidal equivariant compactification of GL_ℓ (Proposition 1.15 of §3 of (54) and Propositions 9.1 and 12.1 of (81)). By §2.3 of (17) and Proposition 9.1 of (81), it then follows that there is just one big cell X. We can then write the integral as

$$\int_{\tilde{\Upsilon}(\sigma_n)} \eta_\Gamma = \int_{X\cap\tilde{\Upsilon}(\sigma_n)} \eta_\Gamma|_X, \tag{2.4.19}$$

where $\tilde{\Upsilon}(\sigma_n)$ is the pullback to $K\mathrm{GL}_\ell$ of the domain of integration $\Upsilon(\sigma_n)$.

Let $\mathcal{H}$ be the Hopf algebra of Feynman graphs. The morphism $\phi : \mathcal{H} \to \mathcal{M}^*_{X,\mathcal{Z}_\ell\cap X}$ assigns to a Feynman graph Γ a meromorphic differential form $\beta_\Gamma = \beta_{\Gamma,D,\ell,p}$ with logarithmic poles along $\mathcal{Z}_\ell$ satisfying $[\eta_\Gamma|_X] = [\beta_\Gamma] \in H^\star_{dR}(X \smallsetminus \mathcal{Z})$.

We then perform the Birkhoff factorization, and we denote by β^+_Γ the regular differential form on $X \subset K\mathrm{GL}_\ell$ given by $\phi^+(\Gamma) = \beta^+_\Gamma$. Since we only have logarithmic poles, by Proposition 2.3.1 the operation becomes a simple pole subtraction and we have $\beta^+_\Gamma = (1-T)\beta_\Gamma$. □

If we assume that the external momenta p in the polynomial $\mathcal{P}_\Gamma(x,p)$ are rational, then the form $\eta_\Gamma = \eta_{\Gamma,D,\ell,p}(x)$ is an algebraic differential form defined over $\mathbb{Q}$, and hence we can also assume that the form with logarithmic poles β_Γ is also defined over $\mathbb{Q}$.

In addition to the integral (2.4.18), one also has the collection of the iterated Poincaré residues along the intersections of components of the divisor $\mathcal{Z}_\ell$. Namely, for any $\mathcal{Z}_{I,\ell} = \cap_{j\in I} Z_{j,\ell}$, with $Z_{j,\ell}$

the components of $\mathcal{Z}_\ell$, we have the additional integrals

$$\mathcal{R}(\Gamma)_I = \int_{\tilde{\Upsilon}(\sigma_n)\cap\mathcal{Z}_{I,\ell}\cap X} \mathrm{Res}_{\mathcal{Z}_I}(\beta_\Gamma). \tag{2.4.20}$$

2.4.9.2 Griffiths-Schmid approach

A global replacement of η_Γ by a single form $\beta_{\Gamma,D,\ell,p}$ on $K\mathrm{GL}_\ell$ with logarithmic poles along $\mathcal{Z}_\ell$ can be obtained if we use the $\mathcal{C}^\infty$-logarithmic de Rham complex instead of the algebraic or analytic one.

Proposition 2.4.14. *There is a $\mathcal{C}^\infty$-form β_Γ^∞ on $K\mathrm{GL}_\ell$ with logarithmic poles along $\mathcal{Z}_\ell$ such that*

$$[\beta_\Gamma^\infty] = [\eta_\Gamma] \in H^*_{dR}(K\mathrm{GL}_\ell \smallsetminus \mathcal{Z}_\ell; \mathbb{C}) = H^*_{dR}(\mathrm{GL}_\ell; \mathbb{C}). \tag{2.4.21}$$

Applying the Birkhoff factorization yields a renormalized integral

$$R^\infty(\Gamma) = \int_{\tilde{\Upsilon}(\sigma_n)} \beta^{\infty,+}_{\Gamma,D,\ell,p}, \tag{2.4.22}$$

where $\beta_\Gamma^{\infty,+}$ is a simple pole subtraction on β_Γ^∞, and iterated residues

$$\mathcal{R}^\infty(\Gamma)_I = \int_{\tilde{\Upsilon}(\sigma_n)\cap\mathcal{Z}_{I,\ell}} \mathrm{Res}_{\mathcal{Z}_I}(\beta_\Gamma^\infty). \tag{2.4.23}$$

Proof. For $\mathcal{X}$ is a complex smooth projective variety and $\mathcal{Z}$ a simple normal crossings divisor, let $\Omega_{\mathcal{C}^\infty(\mathcal{X})}(\log \mathcal{Z})$ be the $\mathcal{C}^\infty$-logarithmic de Rham complex. The Griffiths-Schmid theorem (Proposition 5.14 of (46)) shows that there is an isomorphism $H^*_{dR}(\mathcal{U}) = H^*(\Omega_{\mathcal{C}^\infty(\mathcal{X})}(\log \mathcal{Z}))$. □

Remark 2.4.15. With the Griffiths-Schmid theorem one loses the algebraicity of differential forms. Namely, the forms β_Γ^∞ and $\beta_\Gamma^{\infty,+}$ are only smooth and not algebraic or analytic differential forms. Even if the resulting form $\beta_\Gamma^{\infty,+}$, after pole subtraction, can then be replaced by an algebraic de Rham form in the same cohomology class in $H^*_{dR}(K\mathrm{GL}_\ell)$, it will remain, in general, only a form with $\mathbb{C}$-coefficients and not one defined over $\mathbb{Q}$. Thus, following this approach one obtains a consistent renormalization procedure, but one can lose control on the description of the resulting integrals as periods of motives defined over a number field.

2.4.9.3 The Hodge filtration approach

There is another case in which a form can be replaced globally by a cohomologous one with logarithmic poles on the complement of a normal crossings divisor, while only using algebraic or analytic forms. Indeed, there is a particular piece of the de Rham cohomology that is always realized by global sections of the (algebraic) logarithmic de Rham complex. This is the piece $F^n H^n_{dR}(\mathcal{U})$ of the Hodge filtration of Deligne's mixed Hodge structure, with $n = \dim \mathcal{X}$. This Hodge filtration on $\mathcal{U}$ is given by

$$F^p H^k_{dR}(\mathcal{U}) = \mathrm{Im}(\mathbb{H}^k(\mathcal{X}, \Omega^{\geq p}_{\mathcal{X}}(\log \mathcal{Z})) \to \mathbb{H}^k(\mathcal{X}, \Omega^{\star}_{\mathcal{X}}(\log \mathcal{Z}))).$$

Proposition 2.4.16. *Let $\mathcal{X}$ be a smooth projective variety with $N = \dim \mathcal{X}$, and let $\mathcal{Z}$ be a simple normal crossings divisor with affine complement $\mathcal{U} = \mathcal{X} \setminus \mathcal{Z}$. Then, for $n \leq N$, the Hodge filtration satisfies*

$$F^n H^n_{dR}(\mathcal{U}) = H^0(\mathcal{X}, \Omega^n_{\mathcal{X}}(\log \mathcal{Z})). \tag{2.4.24}$$

Proof. The Hodge filtration $F^p H^k_{dR}(\mathcal{U})$ is induced by the naive filtration on $\Omega^{\star}_{\mathcal{X}}(\log \mathcal{Z})$. Recall that (see Theorem 8.21 and Proposition 8.25 of (96)) the spectral sequence of a filtration F on a complex $K^\star$ that comes from a double complex $K^{p,q}$, with

$$F^p K^n = \oplus_{r \geq p, r+s=n} K^{r,s}$$

has terms

$$E_0^{p,q} = \mathrm{Gr}^F_p K^{p+q} = F^p K^{p+q}/F^{p+1}K^{p+q} = K^{p,q}$$

$$E_1^{p,q} = H^{p+q}(\mathrm{Gr}^F_p K^\star) = H^q(K^{p,\star})$$

$$E_\infty^{p,q} = \mathrm{Gr}^F_p H^{p+q}(K^\star).$$

Thus, the Frölicher spectral sequence associated to the Hodge filtration $F^p H^k_{dR}(\mathcal{U})$ has

$$E_1^{p,q} = H^q(\mathcal{X}, \Omega^p_{\mathcal{X}}(\log \mathcal{Z}))$$

$$E_\infty^{p,q} = F^p H^{p+q}_{dR}(\mathcal{U})/F^{p+1}H^{p+q}_{dR}(\mathcal{U}).$$

In particular, $E_1^{n,0} = H^0(\mathcal{X}, \Omega^n_{\mathcal{X}}(\log \mathcal{Z}))$ and $E_\infty^{n,0} = F^n H^n_{dR}(\mathcal{U})$.

Deligne proved in (29) (see also the formulation of the result given in Theorem 8.35 of (96)) that, in the case where $\mathcal{Z}$ is a normal crossings divisor, the spectral sequence of the Hodge filtration degenerates at the E_1 term. Thus, in particular, we obtain (2.4.24). □

Corollary 2.4.17. *Given a meromorphic form η with $[\eta] \in F^n H^n_{dR}(\mathrm{GL}_\ell)$, with $n \leq \ell^2 = \dim K\mathrm{GL}_\ell$, there is a form β on $K\mathrm{GL}_\ell$ with logarithmic poles along the normal crossings divisor $\mathcal{Z}_\ell$, such that*

$$[\beta] = [\eta] \in H^n_{dR}(K\mathrm{GL}_\ell \smallsetminus \mathcal{Z}_\ell) = H^n_{dR}(\mathrm{GL}_\ell). \tag{2.4.25}$$

Then after pole subtraction one obtains

$$\int_{\tilde{\Upsilon}(\sigma_n)} \beta^+, \tag{2.4.26}$$

which is a period of $\mathfrak{m}(K\mathrm{GL}_\ell, \Sigma_{\ell,g})$.

In this case also, in addition to the integral (2.4.26), we also have the iterated residues (which in this case exist globally),

$$\int_{\tilde{\Upsilon}(\sigma_n)\cap \mathcal{Z}_{I,\ell}} \mathrm{Res}_{\mathcal{Z}_I}(\beta). \tag{2.4.27}$$

In general, it is difficult to estimate where the form η_Γ lies in the Hodge filtration. One can give an estimate, based on the relation between the filtration by order of pole and the Hodge filtration, but it need not be accurate because exact forms can cancel higher order poles. The same issue was discussed, in the original formulation in the graph hypersurface complement, in §9.2 and Proposition 9.8 of (15).

Let $\mathcal{X}$ be a smooth projective variety and $\mathcal{Z} \subset \mathcal{X}$ a simple normal crossings divisor. As before, let $\mathcal{M}^\star_{\mathcal{Z},\mathcal{X}}$ denote the complex of meromorphic differential forms on $\mathcal{X}$ with poles (of arbitrary order) along $\mathcal{Z}$. This complex has a filtration $P^\star \mathcal{M}^\star_{\mathcal{Z},\mathcal{X}}$ by order of poles (polar filtration), where $P^k \mathcal{M}^m_{\mathcal{Z},\mathcal{X}}$ consists of the m-forms with pole of order at most $m - k + 1$, if $m - k \geq 0$ and zero otherwise. Deligne showed in §II.3, Proposition 3.13 of (30) and Proposition 3.1.11 of (29), that the filtration induced on the subcomplex $\Omega^\star_{\mathcal{X}}(\log \mathcal{Z})$ by the polar filtration on $\mathcal{M}^\star_{\mathcal{Z},\mathcal{X}}$ is the naive filtration (that is, the Hodge filtration), and that the natural morphism

$$(\Omega^\star_{\mathcal{X}}(\log \mathcal{Z}), F^\star) \to (\mathcal{M}^\star_{\mathcal{Z},\mathcal{X}}, P^\star)$$

is a filtered quasi-isomorphism. In particular (Theorem 2 of (31)) the image of $\mathbb{H}^\star(\mathcal{X}, P^k\mathcal{M}^\star_{\mathcal{X},\mathcal{Z}})$ inside $H^\star_{dR}(\mathcal{U})$ contains $F^kH^\star_{dR}(\mathcal{U})$. This means that we can use the order of pole of obtain at least an estimate of the position of $[\eta_\Gamma]$ in the Hodge filtration. We need to compute the order of pole of the pullback of the form η_Γ along the blowups in the construction of the compactification $K\mathrm{GL}_\ell$.

Proposition 2.4.18. *For a graph Γ with $n = \#E_\Gamma$ and $\ell = b_1(\Gamma)$, such that $n \geq \ell - 2$, and with spacetime dimension $D \in \mathbb{N}$, the position of $[\eta_\Gamma]$ in the Hodge filtration $F^kH^n_{dR}(\mathrm{GL}_\ell)$ is estimated by $k \geq n - (\ell-1)(-n + (\ell+1)D/2) + (\ell-1)^2 - 1$.*

Proof. At the first step in the construction of the compactification $K\mathrm{GL}_\ell$ we blow up the locus of matrices of rank one. We need to compare the order of vanishing of $\det(x)^{-n+(\ell+1)D/2}$ along this locus, with the order of zero acquired by the form ω_Γ along the exceptional divisor of this blowup. The determinant vanishes at order $\ell - 1$ on that stratum. The form ω_Γ, on the other hand, acquires a zero of order $c - 1$ where c is the codimension of the blowup locus. This can be seen in a local model: when blowing up a locus $L = \{z_1 = \cdots = z_c = 0\}$ in $\mathbb{C}^N$, the local coordinates w_i in the blowup can be taken as $w_iw_c = z_i$ for $i < c$ and $w_i = z_i$ for $i \geq c$, with $E = \{w_c = 0\}$ the exceptional divisor. Then for $n \geq c$, and a form $dz_1 \wedge \cdots \wedge dz_n$, the pullback satisfies

$$\pi^*(dz_1 \wedge \cdots \wedge dz_n) = d(w_cw_1) \wedge \cdots \wedge d(w_cw_{c-1}) \wedge d(w_c) \wedge \cdots \wedge d(w_n) = w_c^{c-1}dw_1 \wedge \cdots \wedge dw_n.$$

The codimension of the locus of rank one matrices is $c = (\ell-1)^2$. Thus, when performing the first blowup in the construction of $K\mathrm{GL}_\ell$, the pullback of the form η_Γ acquires a pole of order $(\ell-1)(-n + (\ell+1)D/2) - (\ell-1)^2 + 1$ along the exceptional divisor. Further blowups do not alter this pole order, and hence we can estimate that the pullback of the n-form η_Γ to the Kausz compactification is in the term P^k of the polar filtration, with $n - k + 1 = (\ell-1)(-n + (\ell+1)D/2) - (\ell-1)^2 + 1$. Taking into account the possibility of reductions of the order of pole, due to cancellations coming from exact forms, we obtain an estimate for the position in the polar and in the Hodge filtration, with $k \geq n - (\ell-1)(-n + (\ell+1)D/2) + (\ell-1)^2 - 1$. ☐

2.4.10 Nature of the period

We then discuss the nature of the period obtained by the evaluation of (2.4.26). We need a preliminary result.

Definition 2.4.19. *Let X be a smooth projective variety over a number field and $Y \subset X$ with irreducible components $\{Y_i\}_{i=1}^N$. Let $\mathcal{C}_Y = \{Y_I = Y_{i_1} \cap \cdots \cap Y_{i_k} \mid I = (i_1, \ldots, i_k),\, k \leq N\}$. Then Y is a mixed Tate configuration if all unions $Y_{I_1} \cup \cdots \cup Y_{I_r}$ of elements of the set $\mathcal{C}_Y$ have motives $\mathfrak{m}(Y_{I_1} \cup \cdots \cup Y_{I_r})$ contained in the Voevodsky derived category of mixed Tate motives.*

Let $\Sigma_{\ell,g}$ be the proper transform of the divisor given by the projective version of $\hat{\Sigma}_{\ell,g}$ described in (2.4.11), defined by the same equations.

Lemma 2.4.20. *The divisor $\Sigma_{\ell,g}$ is a mixed Tate configuration.*

Proof. By (2.4.11), $\Sigma_{\ell,g}$ and any arbitrary union of components are hyperplane arrangements. It is known from (11) that motives of hyperplane arrangements are mixed Tate, see also §1.7.1–1.7.2 and §3.1.1 of (34), where the computation of the motive in the Voevodsky category can be obtained in terms of Orlik–Solomon models. Using a characterization of the mixed Tate condition in terms of eigenvalues of Frobenius, the mixed Tate nature of hyperplane arrangements was also proved in Proposition 3.1.1 of (61). The mixed Tate property can be seen very explicitly at the level of the virtual motive. In fact, the Grothendieck class of an arrangement A in $\mathbb{P}^n$ is explicitly given (Theorem 1.1. of (5)) by

$$[A] = [\mathbb{P}^n] - \frac{\chi_{\hat{A}}(\mathbb{L})}{\mathbb{L} - 1},$$

where $\chi_{\hat{A}}(t)$ is the characteristic polynomial of the associated central arrangement $\hat{A}$ in $\mathbb{A}^{n+1}$. It then follows by inclusion-exclusion in the Grothendieck ring that all unions and intersections of components of A are mixed Tate. □

We then have the following conclusion:

Proposition 2.4.21. *When the form $\beta_{\Gamma,D,\ell,p}$ on the big cell extends to a logarithmic form in $\Omega^\star_{K\mathrm{GL}_\ell}(\log \mathcal{Z}_\ell)$, the integral $R(\Gamma) = \int_{\tilde{\Upsilon}(\sigma_n)} \beta^+_{\Gamma,D,\ell,p}$ is a period of a mixed Tate motive.*

Proof. In the globally defined case, this is an integral of an algebraic differential form defined on the compactification $K\mathrm{GL}_\ell$, and hence a genuine period, in the sense of algebraic geometry, of $K\mathrm{GL}_\ell$. By Proposition 2.4.4, we know that the Chow motive $h(K\mathrm{GL}_\ell)$ is Tate. We also know from Lemma 2.4.20 that the motive $\mathfrak{m}(\Sigma_{\ell,g})$ is mixed Tate. Under the embedding of pure motives into mixed motives we obtain objects $\mathfrak{m}(K\mathrm{GL}_\ell)$ and $\mathfrak{m}(\Sigma_{\ell,g})$ in the subcategory of mixed Tate motives $\mathcal{MTM}(\mathbb{Q})$ inside the Voevodsky triangulated category of mixed motives $\mathcal{DM}(\mathbb{Q})$. It then follows that the relative motive $\mathfrak{m}(K\mathrm{GL}_\ell, \Sigma_{\ell,g})$ is also mixed Tate, as it sits in a distinguished triangles in the Voevodsky triangulated category, where the other two terms are mixed Tate. □

Proposition 2.4.22. *In the case where one only has the form with logarithmic poles $\beta_{\Gamma,D,\ell,p}$ on the top cells X of the cellular decomposition of $K\mathrm{GL}_\ell$, if the motive $\mathfrak{m}(\Sigma_{\ell,g} \cap X)$ is mixed Tate, then the integral $R(\Gamma) = \int_{\tilde{\Upsilon}(\sigma_n)} \beta^+_{\Gamma,D,\ell,p}$ is a period of a mixed Tate motive.*

Proof. Using distinguished triangles in the Voevodsky category, we see that, if the motive $\mathfrak{m}(\Sigma_{\ell,g} \cap X)$ is mixed Tate, then the motive $\mathfrak{m}(X, \Sigma_{\ell,g} \cap X)$ also is, since the big cell has $\mathfrak{m}(X) = \mathbb{L}^{\ell^2}$. The result then follows, since the integral is by construction a period of the motive $\mathfrak{m}(X, \Sigma_{\ell,g} \cap X)$. □

Remark 2.4.23. The central difficulty in the approach of (6), which was to analyze the nature of the motive of $\mathfrak{m}(\Sigma_{\ell,g} \cap \mathcal{D}_\ell)$, is here replaced by the problem of identifying the nature of the motive $\mathfrak{m}(\Sigma_{\ell,g} \cap X)$, where X is the big cell of $K\mathrm{GL}_\ell$.

Remark 2.4.24. It may seem at first that we have simply substituted the problem of understanding for which range of (ℓ, g) the intersection of the divisor $\Sigma_{\ell,g}$ with GL_ℓ remains mixed Tate, with the very similar problem of when the intersection of $\Sigma_{\ell,g}$ with the big cell X of $K\mathrm{GL}_\ell$ remains mixed Tate. However, this reformulation makes it possible to use the explicit description of the cells $X(\lambda, x)$ of spherical varieties in terms of limits as in (2.4.5), to analyze this question.

One defines the category $\mathcal{MTM}(\mathbb{Z})$ of mixed Tate motives over $\mathbb{Z}$ as mixed Tate motives in $\mathcal{MTM}(\mathbb{Q})$ that are unramified over $\mathbb{Z}$. An object of $\mathcal{MTM}(\mathbb{Q})$ is unramified over $\mathbb{Z}$ if and only if, for any prime ℓ, its ℓ-adic realization is unramified outside of ℓ, see Proposition 1.8 of (32).

Proposition 2.4.25. *The motives $\mathfrak{m}(K\mathrm{GL}_\ell)$ are unramified over $\mathbb{Z}$.*

Proof. This question can be approached in a way analogous to our previous discussion of the Chow motive, namely using the description of $K\mathrm{GL}_\ell$ as an iterated blowup and the properties of the divisor of the compactification. The argument is similar to the one used in Theorem 4.1 and Proposition 4.3 of (45) to prove the analogous statement for the moduli spaces $\overline{\mathcal{M}}_{0,n}$ of rational curved with marked points. There, it is shown that $\overline{\mathcal{M}}_{0,n}$ is unramified over $\mathbb{Z}$ by showing that the combinatorics of the normal crossings divisor of the compactification is not altered by reductions mod p, see Definition 4.2 of (45). In our case, note that the description of the Kausz compactification and of the strata of its boundary divisor given in Theorems 9.1 and 9.3 of (60) hold over an arbitrary field, hence the same argument of Proposition 4.3 of (45) applies. □

Remark 2.4.26. Given that the unramified condition holds, one can conclude from Brown's theorem (19) and the previous Proposition 2.4.21 (and Proposition 2.4.22, when $\mathfrak{m}(\Sigma_{\ell,g} \cap X)$ is mixed Tate) that the integral (2.4.26) is a $\mathbb{Q}[2\pi i]$-linear combination of multiple zeta values.

2.4.11 Comparison with Feynman integrals

The result obtained in this way clearly differs from the usual computation of Feynman integrals, where non-mixed-Tate periods are known to occur, (20), (21). There are several reasons behind this difference, which we now discuss briefly.

There is loss of information in mapping the computation of the Feynman integral from the complement of the graph hypersurface (as in (14), (20), (21)) to the complement of the determinant hypersurface (as in (6)), when the combinatorial conditions on the graph recalled in §2.4.7 are not satisfied. Explicit examples of graphs that violate those conditions are given in §3 of (6). In such cases the map (2.4.8) need not be an embedding, hence part of the information contained in the Feynman integral calculation (2.4.6) will be lost in passing to (2.4.9).

However, this type of loss of information does not affect some of the cases where non-mixed Tate motives are known to appear in the momentum space Feynman amplitude.

Example 2.4.27. Let Γ be the graph with 14 edges that gives a counterexample to the Kontsevich polynomial countability conjecture, in Section 1 of (33). The map $\Upsilon : \mathbb{A}^n \to \mathbb{A}^{\ell^2}$ of (2.4.8) has $n = \#E(\Gamma) = 14$ and $\ell = b_1(\Gamma) = 7$. Let Υ_i denote the composition of the map Υ with the projection onto the i-th row of the matrix M_Γ of (2.4.7). In order to check if the embedding condition for Υ is satisfied, we know from Lemma 3.1 of (6) that it suffices to check that Υ_i is injective for i ranging over a set of loops such that every edge of Γ is part of a loop in that set. This can then be checked by computer verification for the matrix M_Γ of this particular graph.

Remark 2.4.28. The example above is a log divergent graph in dimension four. It is known to give a non-mixed Tate contribution with the usual method of computation of the Feynman integral; see (33) and (20).

The same verification method can be applied to the other currently known explicit counterexamples in (33), (20), (91), and (21).

Even for integrals where the map (2.4.8) is an embedding, it is clear that the regularization and renormalization procedure described here, using the Kausz compactification and subtraction of residues for forms with logarithmic poles, is not equivalent to the usual renormalization procedures of the regularized integrals. For instance, our regularized form (and hence our regularized integral) can be trivial in cases where the usual regularization and renormalization would give a non-trivial result. This may occur if the form β with logarithmic poles happens to have a nontrivial residue, but a trivial holomorphic part β^+.

In such cases, part of the information loss coming from pole subtraction on the differential form is compensated by keeping track of the residues. However, in our setting these also deliver only mixed Tate periods, so that even when this information is included, one still loses the richer structure of the periods arising from other methods of regularization and renormalization, adopted in the physics literature.

Chapter 3

Replicating of binary operads, Koszul duality, Manin products and average operators

3.1 The replicators of a binary operad

In this section, we first introduce the concepts of the replicators, namely the duplicator and triplicator, of a labeled planar binary tree, following the framework in (9) and in close resemblance with the concepts of the di-Var-algebra tri-Var-algebra in (48). These concepts are then applied to define similar concepts for a nonsymmetric operad and a (symmetric) operad. A list of examples is provided, followed by a study of the relationship among an operad, its duplicator and its triplicator.

3.1.1 The replicators of a planar binary tree

We first recall notions on operads represented by trees. For more details see (9; 77).

3.1.1.1 Labeled trees

Definition 3.1.1. (a) Let $\mathfrak{T}$ denote the set of planar binary reduced rooted trees together with the trivial tree $|$. If $t \in \mathfrak{T}$ has n leaves, we call t an n**-tree**. The trivial tree $|$ has one leaf.

(b) Let Ω be a set. By a **decorated tree** we mean a tree t of $\mathfrak{T}$ together with a decoration on the vertices of t by elements of Ω and a decoration on the leaves of t by distinct positive integers.

Let $t(\Omega)$ denote the set of decorated trees for t and denote

$$\mathfrak{T}(\Omega) := \coprod_{t\in\mathfrak{T}} t(\Omega).$$

If $\tau \in t(\Omega)$ for an n-tree t, we call τ a **labeled n-tree**.

(c) For $\tau \in \mathfrak{T}(\Omega)$, we let $\mathrm{V}in(\tau)$ (resp. $\mathrm{L}in(\tau)$) denote the set (resp. ordered set) of labels of the vertices (resp. leaves) of τ.

(d) Let $\tau \in \mathfrak{T}(\Omega)$ with $|\mathrm{L}in(\tau)| > 1$ be a labeled tree from $t \in \mathfrak{T}$. Then t can be written uniquely as the grafting $t_\ell \vee t_r$ of t_ℓ and t_r. Correspondingly, let $\tau = \tau_\ell \vee_\omega \tau_r$ denote the unique decomposition of τ as a grafting of τ_ℓ and τ_r in $\mathfrak{T}(\Omega)$ along $\omega \in \Omega$.

Let V be a vector space, regarded as an arity graded vector space concentrated in arity 2: $V = V_2$. Recall (77, Section 5.8.5) that the free nonsymmetric operad $\mathcal{T}_{ns}(V)$ on V is given by the vector space

$$\mathcal{T}_{ns}(V) := \bigoplus_{t\in\mathfrak{T}} t[V] \ ,$$

where $t[V]$ is the treewise tensor module associated to t, explicitly given by

$$t[V] := \bigotimes_{v\in \mathrm{V}in(t)} V_{|\mathrm{In}(v)|} \ .$$

Here $|\mathrm{In}(v)|$ denotes the number of incoming edges of v. A basis $\mathcal{V}$ of V induces a basis $t(\mathcal{V})$ of $t[V]$ and a basis $\mathfrak{T}(\mathcal{V})$ of $\mathcal{T}_{ns}(V)$. Consequently any element of $t[V]$ can be represented as a linear combination of elements in $t(\mathcal{V})$.

3.1.1.2 Duplicators

Definition 3.1.2. Let V be a vector space with a basis $\mathcal{V}$.

(a) Define a vector space

$$\mathrm{Du}(V) = V \otimes (\mathbf{k} \dashv \oplus\, \mathbf{k} \vdash) \ , \tag{3.1.1}$$

where we denote $(\omega \otimes \dashv)$ (resp. $(\omega \otimes \vdash)$) by $\binom{\omega}{\dashv}$ $\left(\text{resp. } \binom{\omega}{\vdash}\right)$ for $\omega \in \mathcal{V}$. Then $\bigcup_{\omega\in\mathcal{V}} \left\{ \binom{\omega}{\dashv}, \binom{\omega}{\vdash} \right\}$ is a basis of $\mathrm{Du}(V)$.

(b) For a labeled n-tree τ in $\mathcal{T}(\mathcal{V})$, define a subset $\mathrm{Du}(\tau)$ of $\mathcal{T}_{ns}(\mathrm{Du}(\mathcal{V}))$ by

- $\mathrm{Du}(|) = \{ \, | \, \}$,
- when $n \geq 2$, $\mathrm{Du}(\tau)$ is obtained by replacing each decoration $\omega \in \mathrm{V}in(\tau)$ by

$$\binom{\omega}{\dagger} := \left\{ \binom{\omega}{\dashv}, \binom{\omega}{\vdash} \right\} \; .$$

Thus $\mathrm{Du}(\tau)$ is a set of labeled trees.

Definition 3.1.3. Let V be a vector space with a basis $\mathcal{V}$. Let τ be a labeled n-tree in $\mathcal{T}(\mathcal{V})$. The **duplicator** $\mathrm{Du}_x(\tau)$ of τ with respect to a leaf $x \in \mathrm{L}in(\tau)$ is the subset of $\mathcal{T}_{ns}(\mathrm{Du}(\mathcal{V}))$ defined by induction on $|\mathrm{L}in(\tau)|$ as follows:

- $\mathrm{Du}_x(\, | \,) = \{|\} \quad ;$
- assume that $\mathrm{Du}_x(\tau)$ have been defined for τ with $|\mathrm{L}in(\tau)| \leq k$ for a $k \geq 1$. Then, for a labeled $(k+1)$-tree $\tau \in \mathcal{T}(\mathcal{V})$ with decomposition $\tau = \tau_\ell \vee_\omega \tau_r$, we define

$$\mathrm{Du}_x(\tau) = \mathrm{Du}_x(\tau_\ell \vee_\omega \tau_r) = \begin{cases} \mathrm{Du}_x(\tau_\ell) \vee_{\binom{\omega}{\dashv}} \mathrm{Du}(\tau_r), & x \in \mathrm{L}in(\tau_\ell), \\ \mathrm{Du}(\tau_\ell) \vee_{\binom{\omega}{\vdash}} \mathrm{Du}_x(\tau_r), & x \in \mathrm{L}in(\tau_r). \end{cases}$$

For labeled n-trees $\tau_i, 1 \leq i \leq r$, with the same set of leaf decorations and $c_i \in \mathbf{k}, 1 \leq i \leq r$, define

$$\mathrm{Du}_x\left(\sum_{i=1}^{r} c_i \tau_i\right) := \sum_{i=1}^{r} c_i \mathrm{Du}_x(\tau_i). \tag{3.1.2}$$

Here and in the rest of the paper we use the notation

$$\sum_{i=1}^{r} c_i W_i := \left\{ \sum_{i=1}^{r} c_i w_i \,\middle|\, w_i \in W_i, 1 \leq i \leq r \right\}, \tag{3.1.3}$$

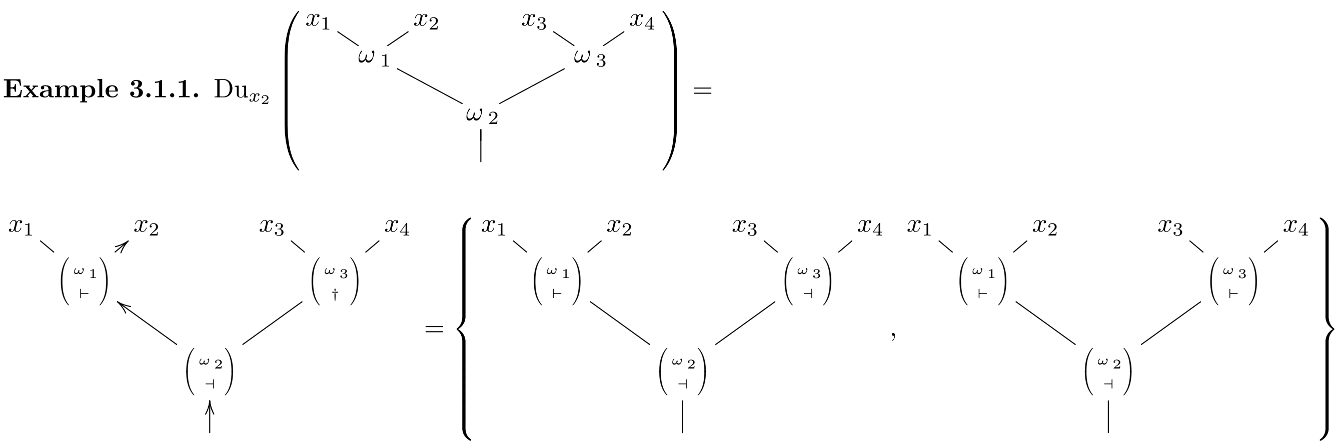
for nonempty subsets $W_i, 1 \leq i \leq r$, of a $\mathbf{k}$-module.

The next explicit description of the duplicator follows from an induction on $|\mathrm{L}in(\tau)|$.

Proposition 3.1.4. *Let V be a vector space with a basis $\mathcal{V}$, τ be in $\mathfrak{T}(\mathcal{V})$ and x be in* $\mathrm{L}in(\tau)$. *The duplicator* $\mathrm{Du}_x(\tau)$ *is obtained by relabeling a vertex ω of* $\mathrm{V}in(\tau)$ *by*

$$\begin{cases} \binom{\omega}{\dashv}, & \text{the path from the root of } \tau \text{ to } x \text{ turns left at } \omega\,; \\ \binom{\omega}{\vdash}, & \text{the path from the root of } \tau \text{ to } x \text{ turns right at } \omega\,; \\ \binom{\omega}{\dagger} := \left\{\binom{\omega}{\dashv}, \binom{\omega}{\vdash}\right\}, & \text{the path from the root of } \tau \text{ to } x \text{ does not pass } \omega\,. \end{cases}$$

Example 3.1.1. Du_{x_2}



3.1.1.3 Triplicators

Definition 3.1.5. Let V be a vector space with a basis $\mathcal{V}$.

(a) Define a vector space

$$\mathrm{Tri}(V) = V \otimes (\mathbf{k} \dashv \oplus\, \mathbf{k} \vdash \oplus\, \mathbf{k} \perp)\,, \tag{3.1.4}$$

where we denote $(\omega \otimes \dashv)$ (resp. $(\omega \otimes \vdash)$, resp. $(\omega \otimes \perp)$) by $\binom{\omega}{\dashv}$ $\left(\text{resp. } \binom{\omega}{\vdash}, \text{ resp. } \binom{\omega}{\perp}\right)$ for $\omega \in \mathcal{V}$. Then $\bigcup_{\omega \in \mathcal{V}} \left\{\binom{\omega}{\dashv}, \binom{\omega}{\vdash}, \binom{\omega}{\perp}\right\}$ is a basis of $\mathrm{Tri}(V)$.

(b) Let τ be a labeled n-tree in $\mathfrak{T}(\mathcal{V})$ and let J be a subset of $\mathrm{L}in(\tau)$. The **triplicator** $\mathrm{Tri}_J(\tau)$ of τ with respect to J is a subset of $\mathcal{T}_{ns}(\mathrm{Tri}(\mathcal{V}))$ defined by induction on $|\mathrm{L}in(\tau)|$ as follows:

- $\mathrm{Tri}_J(\,|\,) = \{|\}$;
- assume that $\mathrm{Tri}_J(\tau)$ have been defined for τ with $|\mathrm{L}in(\tau)| \leq k$ for a $k \geq 1$. Then, for a

labeled $(k+1)$-tree $\tau \in \mathcal{T}(\mathcal{V})$ with decomposition $\tau = \tau_\ell \vee_\omega \tau_r$, we define

$$\mathrm{Tri}_J(\tau) = \mathrm{Tri}_J(\tau_\ell \vee_\omega \tau_r) = \mathrm{Tri}_{J\cap \mathrm{L}in(\tau_\ell)} \vee_{\binom{\omega}{(\tau,\, J)}} \mathrm{Tri}_{J\cap \mathrm{L}in(\tau_r)},$$

where

$$(\tau, J) = \begin{cases} \dashv, & J\cap \mathrm{L}in(\tau_\ell) \neq \varnothing, J\cap \mathrm{L}in(\tau_r) = \varnothing, \text{ that is, } J \subseteq \mathrm{L}in(\tau_r), \\ \vdash, & J\cap \mathrm{L}in(\tau_\ell) = \varnothing, J\cap \mathrm{L}in(\tau_r) \neq \varnothing, \text{ that is, } J \subseteq \mathrm{L}in(\tau_\ell), \\ \dagger := \{\dashv, \vdash, \perp\}, & J\cap \mathrm{L}in(\tau_\ell) = \varnothing, J\cap \mathrm{L}in(\tau_r) = \varnothing, \text{ that is, } J = \varnothing, \\ \perp, & J\cap \mathrm{L}in(\tau_\ell) \neq \varnothing, J\cap \mathrm{L}in(\tau_r) \neq \varnothing, \text{ that is, none of the above.} \end{cases}$$

Equivalently,

$$\mathrm{Tri}_J(\tau) = \begin{cases} \mathrm{Tri}_J(\tau_\ell) \vee_{\binom{\omega}{\dashv}} \mathrm{Tri}_\varnothing(\tau_r), & J \subseteq \mathrm{L}in(\tau_\ell), \\ \mathrm{Tri}_\varnothing(\tau_\ell) \vee_{\binom{\omega}{\vdash}} \mathrm{Tri}_J(\tau_r), & J \subseteq \mathrm{L}in(\tau_r), \\ \mathrm{Tri}_\varnothing(\tau_\ell) \vee_{\binom{\omega}{\dagger}} \mathrm{Tri}_\varnothing(\tau_r), & J = \varnothing, \\ \mathrm{Tri}_{J\cap \mathrm{L}in(\tau_\ell)}(\tau_\ell) \vee_{\binom{\omega}{\perp}} \mathrm{Tri}_{J\cap \mathrm{L}in(\tau_r)}(\tau_r), & \text{otherwise.} \end{cases}$$

We have the following explicit description of the triplicator that follows from an induction on $|\mathrm{L}in(\tau)|$.

Proposition 3.1.6. *Let V be a vector space with a basis $\mathcal{V}$, let τ be in $\mathcal{T}(\mathcal{V})$ and let J be a nonempty subset of $\mathrm{L}in(\tau)$. The triplicator $\mathrm{Tri}_J(\tau)$ is obtained by relabeling each vertex ω of $\mathrm{V}in(\tau)$ by the following rules:*

(a) *Suppose ω is on the paths from the root of τ to some (possibly multiple) x in J. Then*

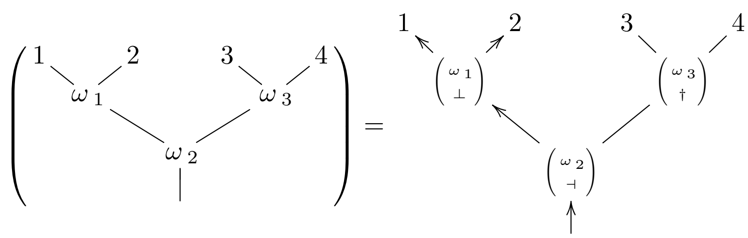
(i) replace ω by $\binom{\omega}{\dashv}$ if all of such paths turn left at ω;

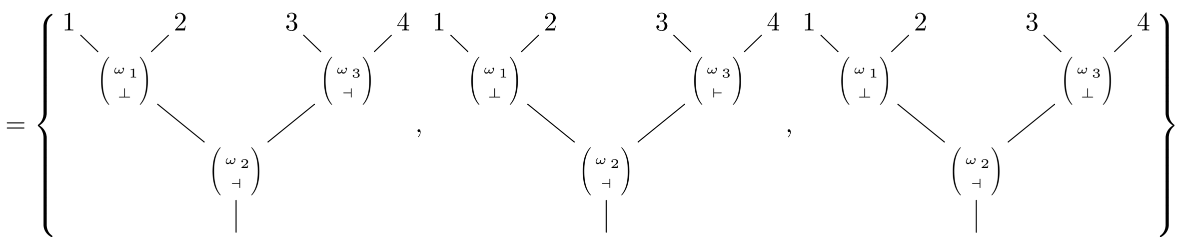
(ii) replace ω by $\binom{\omega}{\vdash}$ if all of such paths turn right at ω;

(iii) replace ω by $\binom{\omega}{\perp}$ if some of such paths turn left at ω and some of such paths turn right at ω.

(b) *Suppose ω is not on the path from the root of τ to any $x \in J$. Then replace ω by* $\binom{\omega}{\dagger} := \left\{ \binom{\omega}{\dashv}, \binom{\omega}{\vdash}, \binom{\omega}{\perp} \right\}$;

Example 3.1.2. $\mathrm{Tri}_{\{1,2\}}$





3.1.2 The replicators of a binary nonsymmetric operad

Definition 3.1.7. Let V be a vector space with a basis $\mathcal{V}$.

(a) An element

$$r := \sum_{i=1}^{r} c_i \tau_i, \quad c_i \in \mathbf{k}, \tau_i \in \mathfrak{T}(\mathcal{V}),$$

in $\mathcal{T}_{ns}(V)$ is called **homogeneous** if $\mathrm{L}in(\tau_i)$ are the same for $1 \leq i \leq r$. Then denote $\mathrm{L}in(r) = \mathrm{L}in(\tau_i)$ for any $1 \leq i \leq r$.

(b) A collection of elements

$$r_s := \sum_{i=1}^{r} c_{s,i} \tau_{s,i}, \quad c_{s,i} \in \mathbf{k}, \tau_{s,i} \in \mathfrak{T}(\mathcal{V}), 1 \leq s \leq k, k \geq 1,$$

in $\mathcal{T}_{ns}(V)$ is called **locally homogenous** if each element r_s, $1 \leq s \leq k$, is homogeneous.

Definition 3.1.8. Let $\mathcal{P} = \mathcal{T}_{ns}(V)/(R)$ be a binary nonsymmetric operad where V is a vector space with a basis $\mathcal{V}$ regarded as an arity graded vector space concentrated in arity two: $V = V_2$ and R

is a set consisting of locally homogeneous elements:

$$r_s = \sum_i c_{s,i}\tau_{s,i} \ \in \mathcal{T}_{ns}(V)\ ,\ \ c_{s,i} \in \mathbf{k},\ \tau_{s,i} \in \mathfrak{T}(\mathcal{V}),\ 1 \le s \le k.$$

(a) The **duplicator** of $\mathcal{P}$ is defined to be the binary nonsymmetric operad

$$\mathrm{Du}(\mathcal{P}) := \mathcal{T}_{ns}(\mathrm{Du}(V))/(\mathrm{Du}(R)).$$

Here $\mathrm{Du}(V) = V \otimes (\mathbf{k} \dashv \oplus \mathbf{k} \vdash)$ is regarded as an arity graded vector space concentrated in arity two and

$$\mathrm{Du}(R) := \bigcup_{s=1}^{k} \left(\bigcup_{x \in Lin(t_s)} \mathrm{Du}_x(r_s) \right),\ \ \text{where } \mathrm{Du}_x(r_s) := \sum_i c_{s,i}\mathrm{Du}_x(\tau_{s,i})),$$

with the notation in Eq. (3.1.3).

(b) The **triplicator** of $\mathcal{P}$ is defined to be the binary nonsymmetric operad

$$\mathrm{Tri}(\mathcal{P}) := \mathcal{T}_{ns}(\mathrm{Tri}(V))/(\mathrm{Tri}(R)).$$

Here $\mathrm{Tri}(V) = V \otimes (\mathbf{k} \dashv \oplus \mathbf{k} \vdash \oplus \mathbf{k} \perp)$ is regarded as an arity graded vector space concentrated in arity two and

$$\mathrm{Tri}(R) := \bigcup_{s=1}^{k} \left(\bigcup_{\varnothing \ne J \subseteq Lin(r_s)} \mathrm{Tri}_J(r_s) \right),\ \text{where } \mathrm{Tri}_J(r_s) := \sum_i c_{s,i}\mathrm{Tri}_J(\tau_{s,i}).$$

Proposition 3.1.9. *The duplicator (resp. triplicator) of a binary nonsymmetric operad* $\mathcal{P} = \mathcal{T}_{ns}(V)/(R)$ *does not depend on the choice of a basis* $\mathcal{V}$ *of* V*.*

Proof. It is straightforward to check from the linearity of the duplicator (resp. triplicator) and from the treewise tensor module structure on $\mathcal{T}_{ns}(V)$. □

We give some examples of duplicators and triplicators of nonsymmetric operads.

Example 3.1.3. Let *Ass* be the nonsymmetric operad of the associative algebra with product $\cdot$.

Using the abbreviations $\dashv := \binom{\cdot}{\dashv}$ and $\vdash := \binom{\cdot}{\vdash}$, we have

$$\begin{aligned}
\mathrm{Du}_y((x\cdot y)\cdot z - x\cdot(y\cdot z)) &= \{(x\vdash y)\dashv z - x\vdash(y\dashv z)\},\\
\mathrm{Du}_x((x\cdot y)\cdot z - x\cdot(y\cdot z)) &= \{(x\dashv y)\dashv z - x\dashv(y\dashv z), (x\dashv y)\dashv z - x\dashv(y\vdash z)\},\\
\mathrm{Du}_z((x\cdot y)\cdot z - x\cdot(y\cdot z)) &= \{(x\dashv y)\vdash z - x\vdash(y\vdash z), (x\vdash y)\vdash z - x\vdash(y\vdash z)\},
\end{aligned}$$

giving the five relations of the **diassociative algebra** of Loday (70). Therefore the duplicator of *Ass* is *Diass*.

Example 3.1.4. A similar computation shows that the triplicator of *Ass* is the operad *Trias* of the **triassociative algebra** of Loday and Ronco (75). For example,

$$\begin{aligned}
\mathrm{Tri}_{\{x\}}((xy)z - x(yz)) &= \{(x\dashv y)\dashv z - x\dashv(y\dashv z), (x\dashv y)\dashv z - x\dashv(y\vdash z), (x\dashv y)\dashv z - x\dashv(y\perp z)\},\\
\mathrm{Tri}_{\{x,y\}}((xy)z - x(yz)) &= \{(x\perp y)\dashv z - x\perp(y\dashv z)\},\\
\mathrm{Tri}_{\{x,y,z\}}((xy)z - x(yz)) &= \{(x\perp y)\perp z - x\perp(y\perp z)\}.
\end{aligned}$$

3.1.3 The replicators of a binary operad

When $V = V(2)$ is an $\mathbb{S}$-module concentrated in arity two with a linear basis $\mathcal{V}$. For any finite set $\mathcal{X}$ of cardinal n, define the coinvariant space

$$V(\mathcal{X}) := \left(\bigoplus_{f:\underline{n}\to\mathcal{X}} V(n)\right)_{\mathbb{S}_n},$$

where the sum is over all the bijections from $\underline{n} := \{1,\dots,n\}$ to $\mathcal{X}$ and where the symmetric group acts diagonally.

Let $\mathbb{T}$ denote the set of isomorphism classes of reduced binary trees (77, Appendix C). For $\mathtt{t}\in\mathbb{T}$, define the treewise tensor $\mathbb{S}$-module associated to $\mathtt{t}$, explicitly given by

$$\mathtt{t}[V] := \bigotimes_{v\in \mathrm{V}in(\mathtt{t})} V(\mathrm{In}(v))$$

(see (77, Section 5.5.1)). Then the free operad $\mathcal{T}(V)$ on an $\mathbb{S}$-module $V = V(2)$ is given by the

$\mathbb{S}$-module

$$\mathcal{T}(V) := \bigoplus_{\mathsf{t}\in\mathbb{T}} \mathsf{t}[V]\,.$$

Each tree t in $\mathbb{T}$ can be represented by a planar tree t in $\mathfrak{T}$ by choosing a total order on the set of inputs of each vertex of t. Further, $t[V] \cong \mathsf{t}[V]$ (56, Section 2.8). Fixing such a choice t for each $\mathsf{t} \in \mathbb{T}$ gives a subset $\mathfrak{R} \subseteq \mathfrak{T}$ with a bijection $\mathbb{T} \cong \mathfrak{R}$. Then we have

$$\mathcal{T}(V) \cong \bigoplus_{t\in\mathfrak{R}} t[V]\ ,$$

allowing us to use the notations in Section 3.1.2.

Definition 3.1.10. Let $\mathcal{P} = \mathcal{T}(V)/(R)$ be a binary operad where the $\mathbb{S}$-module V is concentrated in arity 2: $V = V(2)$ with an $\mathbb{S}_2$-basis $\mathcal{V}$ and the space of relations is generated, as an $\mathbb{S}$-module, by a set R of locally homogeneous elements

$$r_s := \sum_i c_{s,i}\tau_{s,i},\ \ c_{s,i} \in \mathbf{k}, \tau_{s,i} \in \bigcup_{t\in\mathfrak{R}} t(\mathcal{V}),\ 1 \leq s \leq k. \tag{3.1.5}$$

(a) The **duplicator** of $\mathcal{P}$ is defined to be the binary operad

$$\mathrm{Du}(\mathcal{P}) = \mathcal{T}(\mathrm{Du}(V))/(\mathrm{Du}(R)),$$

where the $\mathbb{S}_2$-action on $\mathrm{Du}(V) = V \otimes (\mathbf{k}\dashv \oplus \mathbf{k}\vdash)$ is given by

$$\binom{\omega}{\dashv}^{(12)} := \binom{\omega^{(12)}}{\vdash},\quad \binom{\omega}{\vdash}^{(12)} := \binom{\omega^{(12)}}{\dashv},\ \ \omega \in V,$$

and the space of relations is generated, as an $\mathbb{S}$-module, by

$$\mathrm{Du}(R) := \bigcup_{s=1}^{k}\left(\bigcup_{x\in Lin(r_s)} \mathrm{Du}_x(r_s)\right)\ \text{with}\ \mathrm{Du}_x(r_s) := \sum_i c_{s,i}\mathrm{Du}_x(\tau_{s,i}). \tag{3.1.6}$$

(b) The **triplicator** of $\mathcal{P}$ is defined to be the binary operad

$$\mathrm{Tri}(\mathcal{P}) = \mathcal{T}(\mathrm{Tri}(V))/(\mathrm{Tri}(R)),$$

where the $\mathbb{S}_2$-action on $\mathrm{Tri}(V) = V \otimes (\mathbf{k} \dashv \oplus \mathbf{k} \vdash \oplus \mathbf{k} \perp)$ is given by

$$\binom{\omega}{\dashv}^{(12)} := \binom{\omega^{(12)}}{\vdash}, \quad \binom{\omega}{\vdash}^{(12)} := \binom{\omega^{(12)}}{\dashv}, \quad \binom{\omega}{\perp}^{(12)} := \binom{\omega^{(12)}}{\perp}, \ \omega \in V,$$

and the space of relations is generated, as an $\mathbb{S}$-module, by

$$\mathrm{Tri}(R) := \bigcup_{s=1}^{k} \left(\bigcup_{\varnothing \neq J \subseteq Lin(r_s)} \mathrm{Tri}_J(r_s) \right) \text{ with } \mathrm{Tri}_J(r_s) := \sum_i c_{s,i} \mathrm{Tri}_J(\tau_{s,i}).$$

See(48) for the closely related notions of the di-Var-algebra and tri-Var-algebra, and (64) for these notions for not necessarily binary operads. For later reference, we also recall the definitions of bisuccessors (9).

Definition 3.1.11. The **bisuccessor** (9) of a binary operad $\mathcal{P} = \mathcal{T}(V)/(R)$ is defined to be the binary operad $\mathrm{Su}(\mathcal{P}) = \mathcal{T}(\widetilde{V})/(\mathrm{Su}(R))$ where the $\mathbb{S}_2$-action on $\widetilde{V}$ is given by

$$\binom{\omega}{\prec}^{(12)} := \binom{\omega^{(12)}}{\succ}, \quad \binom{\omega}{\succ}^{(12)} := \binom{\omega^{(12)}}{\prec}, \ \omega \in V,$$

and the space of relations is generated, as an $\mathbb{S}$-module, by

$$\mathrm{Su}(R) := \left\{ \mathrm{Su}_x(r_s) := \sum_i c_{s,i} \mathrm{Su}_x(t_{s,i}) \;\middle|\; x \in \mathrm{L}in(r_s),\ 1 \leq s \leq k \right\}. \tag{3.1.7}$$

Here for $\tau \in \mathcal{T}(V)$ and a leaf $x \in \mathrm{L}in(\tau)$, $\mathrm{Su}_x(\tau)$ is defined by relabeling a vertex ω of $\mathrm{V}in(\tau)$ by

$$\begin{cases} \binom{\omega}{\prec}, & \text{the path from the root of } \tau \text{ to } x \text{ turns left at } \omega; \\ \binom{\omega}{\succ}, & \text{the path from the root of } \tau \text{ to } x \text{ turns right at } \omega; \\ \binom{\omega}{\star}, & \omega \text{ is not on the path from the root of } \tau \text{ to } x, \end{cases}$$

where $\binom{\omega}{\star} := \left\{ \binom{\omega}{\prec} + \binom{\omega}{\succ} \right\}$.

There is a similar notion of a trisuccessor splitting an operation into three pieces (9).

With an argument similar to the proof of Proposition 2.20 in (9), we see that the duplicator and triplicator of a binary algebraic operad $\mathcal{P} = \mathcal{T}(V)/(R)$ depends neither on the linear basis $\mathcal{V}$ of V

nor on the set $\mathfrak{R}$.

3.1.4 Examples of duplicators and triplicators

We give some examples of duplicators and triplicators of binary operads.

Let V be an $\mathbb{S}$-module concentrated in arity two. Then we have

$$\mathcal{T}(V)(3) = (V \otimes_{\mathbb{S}_2} (V \otimes \mathbf{k} \oplus \mathbf{k} \otimes V)) \otimes_{\mathbb{S}_2} \mathbf{k}[\mathbb{S}_3],$$

which can be identify with 3 copies of $V \otimes V$, denoted by $V \circ_{\mathrm{I}} V, V \circ_{\mathrm{II}} V$, and $V \circ_{\mathrm{III}} V$, following the convention in (93). Then, as an abelian group, $\mathcal{T}(V)(3)$ is generated by elements of the form

$$\omega \circ_{\mathrm{I}} \nu \, (\leftrightarrow (x\,\nu\,y)\,\omega\,z), \; \omega \circ_{\mathrm{II}} \nu \, (\leftrightarrow (y\,\nu\,z)\,\omega\,x), \; \omega \circ_{\mathrm{III}} \nu \, (\leftrightarrow (z\,\nu\,x)\,\omega\,y), \forall\, \omega\,,\, \nu \in V. \tag{3.1.8}$$

For an operad where the space of generators V is equal to $\mathbf{k}[\mathbb{S}_2] = \mu.\mathbf{k} \oplus \mu'.\mathbf{k}$ with $\mu.(12) = \mu'$, we will adopt the convention in (93, p. 129) and denote the 12 elements of $\mathcal{T}(V)(3)$ by $v_i, 1 \leq i \leq 12$, in the following table.

v_1	$\mu \circ_{\mathrm{I}} \mu \leftrightarrow (xy)z$	v_5	$\mu \circ_{\mathrm{III}} \mu \leftrightarrow (zx)y$	v_9	$\mu \circ_{\mathrm{II}} \mu \leftrightarrow (yz)x$
v_2	$\mu' \circ_{\mathrm{II}} \mu \leftrightarrow x(yz)$	v_6	$\mu' \circ_{\mathrm{I}} \mu \leftrightarrow z(xy)$	v_{10}	$\mu' \circ_{\mathrm{III}} \mu \leftrightarrow y(zx)$
v_3	$\mu' \circ_{\mathrm{II}} \mu' \leftrightarrow x(zy)$	v_7	$\mu' \circ_{\mathrm{I}} \mu' \leftrightarrow z(yx)$	v_{11}	$\mu' \circ_{\mathrm{III}} \mu' \leftrightarrow y(xz)$
v_4	$\mu \circ_{\mathrm{III}} \mu' \leftrightarrow (xz)y$	v_8	$\mu \circ_{\mathrm{II}} \mu' \leftrightarrow (zy)x$	v_{12}	$\mu \circ_{\mathrm{I}} \mu' \leftrightarrow (yx)z$

3.1.4.1 Examples of duplicators

Recall that a **(left) Leibniz algebra** (70) is defined by a bilinear operation $\{,\}$ and a relation

$$\{x, \{y, z\}\} = \{\{x, y\}, z\} + \{y, \{x, z\}\}.$$

Proposition 3.1.12. *The operad Leib of the Leibniz algebra is the duplicator of Lie, the operad of the Lie algebra.*

Proof. Let μ denote the operation of the operad *Lie*. The space of relations of *Lie* is generated as an $\mathbb{S}_3$-module by

$$v_1 + v_5 + v_9 = \mu \circ_{\mathrm{I}} \mu + \mu \circ_{\mathrm{II}} \mu + \mu \circ_{\mathrm{III}} \mu = (x\mu y)\mu z + (z\mu x)\mu y + (y\mu z)\mu x. \qquad (3.1.9)$$

Use the abbreviations $\dashv := \left(\begin{smallmatrix}\mu\\ \dashv\end{smallmatrix}\right)$ and $\vdash := \left(\begin{smallmatrix}\mu\\ \vdash\end{smallmatrix}\right)$. Then from $\left(\begin{smallmatrix}\mu\\ \dashv\end{smallmatrix}\right)^{(12)} = \left(\begin{smallmatrix}\mu^{(12)}\\ \vdash\end{smallmatrix}\right) = -\left(\begin{smallmatrix}\mu\\ \vdash\end{smallmatrix}\right)$, we have $\dashv^{(12)} = -\vdash$. Then we have

$$\begin{aligned}\mathrm{Du}_z(v_1 + v_5 + v_9) &= \{(x \vdash y) \vdash z + (y \vdash z) \dashv x + (z \dashv x) \dashv y, (x \dashv y) \vdash z + (y \vdash z) \dashv x + (z \dashv x) \dashv y\}\\ &= \{\underline{(x \vdash y) \vdash z - x \vdash (y \vdash z) + y \vdash (x \vdash z)}, y \vdash (x \vdash z) - (y \vdash x) \vdash z - x \vdash (y \vdash z)\},\end{aligned}$$

with similar computations for Du_x and Du_y. Replacing the operation $\vdash$ by $\{,\}$, we see that the underlined relation is precisely the relation of the Leibniz algebra while the other relations are obtained from this relation by a permutation of the variables. Therefore $\mathrm{Du}(\mathit{Lie}) = \mathit{Leib}$. ☐

Also recall that a **(left) permutative algebra** (25) (also called commutative diassociative algebra) is defined by one bilinear operation $\cdot$ and the relations

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z = (y \cdot x) \cdot z.$$

Proposition 3.1.13. *The operad Perm of the permutative algebra is the duplicator of Comm, the operad of the commutative associative algebra.*

Proof. Let ω denote the operation of the operad Comm. Setting $\dashv := \left(\begin{smallmatrix}\omega\\ \dashv\end{smallmatrix}\right)$ and $\vdash := \left(\begin{smallmatrix}\omega\\ \vdash\end{smallmatrix}\right)$, then from $\left(\begin{smallmatrix}\omega\\ \vdash\end{smallmatrix}\right)^{(12)} = \left(\begin{smallmatrix}\omega^{(12)}\\ \vdash\end{smallmatrix}\right) = \left(\begin{smallmatrix}\omega\\ \vdash\end{smallmatrix}\right)$ we have $\dashv^{(12)} = \vdash$. The space of relations of Comm is generated as an $\mathbb{S}_3$-module by

$$v_1 - v_9 = \omega \circ_{\mathrm{I}} \omega - \omega \circ_{\mathrm{II}} \omega = (x\,\omega\,y)\,\omega\,z - (y\,\omega\,z)\,\omega\,x.$$

Then we have

$$\begin{aligned}\mathrm{Du}_z(v_1 - v_9) &= \{(x \dashv y) \vdash z - (y \vdash z) \dashv x, (x \vdash y) \vdash z - (y \vdash z) \dashv x\}\\ &= \{\underline{(y \vdash x) \vdash z - x \vdash (y \vdash z), (x \vdash y) \vdash z - x \vdash (y \vdash z)}\},\end{aligned}$$

with similar computations for Du_x and Du_y. Replacing the operation $\vdash$ by $\cdot$ and following the same proof as in Proposition 3.1.12, we get $\mathrm{Du}(\mathit{Comm}) = \mathit{Perm}$. ☐

A **(left) Poisson algebra** is defined to be a **k**-vectors space with two bilinear operations $\{,\}$ and $\circ$ such that $\{,\}$ is the Lie bracket and $\circ$ is the product of commutative associative algebra, and they are compatible in the sense that

$$\{x, y \circ z\} = \{x, y\} \circ z + y \circ \{x, z\}.$$

A **dual (left) pre-Poisson algebra** (1) is defined to be a **k**-vector space with two bilinear operations $\{\,,\}$ and $\circ$ such that $\{\,,\}$ is a Leibniz bracket and $\circ$ is a product of permutative algebra, and they are compatible in the sense that

$$\{x, y \circ z\} = \{x, y\} \circ z + y \circ \{x, z\},\ \{x \circ y, z\} = x \circ \{y, z\} + y \circ \{x, z\},\ \{x, y\} \circ z = -\{y, x\} \circ z.$$

By a similar argument as in Proposition 3.1.12, we obtain

Proposition 3.1.14. *The duplicator of Pois, the operad of the Poisson algebra, is DualPrePois, the operad of the dual pre-Poisson algebra.*

We next consider the duplicator of the the operad *preLie* of **(left) pre-Lie** algebra (also called left-symmetric algebra). A pre-Lie algebra is defined by a bilinear operation $\{\ ,\ \}$ that satisfies

$$R_{preLie} := \{\{x, y\}, z\} - \{x, \{y, z\}\} - \{\{y, x\}, z\} + \{y, \{x, z\}\} = 0.$$

By Definition 3.1.10 and the abbreviations $\dashv := \binom{\omega}{\dashv}, \vdash := \binom{\omega}{\vdash}$, we have

$$\begin{aligned} \mathrm{Du}(R_{preLie}) \quad = \quad & \Big\{ \underline{x \dashv (y \dashv z) - x \dashv (y \vdash z)},\ y \dashv (x \dashv z) - y \dashv (x \vdash z), \\ & \underline{(x \vdash y) \vdash z - (x \dashv y) \vdash z},\ (y \vdash x) \vdash z - (y \dashv x) \vdash z, \\ & \underline{x \dashv (y \dashv z) - (x \dashv y) \dashv z - y \vdash (x \dashv z) + (y \vdash x) \dashv z}, \\ & x \vdash (y \dashv z) - (x \vdash y) \dashv z - y \dashv (x \dashv z) + (y \dashv x) \dashv z, \\ & \underline{x \vdash (y \vdash z) - (x \vdash y) \vdash z - y \vdash (x \vdash z) + (y \vdash x) \vdash z} \Big\}. \end{aligned}$$

These underline relations coincide with the axioms of preLie dialgebra (left-symmetric dialgebra) defined in (39), and the other relations are obtained from this relation by a permutation of the variables. Then we have

Proposition 3.1.15. *The duplicator of preLie, the operad of the pre-Lie algebra, is DipreLie, the operad of the pre-Lie dialgebra.*

3.1.4.2 Examples of triplicators

We similarly have the following examples of triplicators of operads.

A **commutative trialgebra** (75) is a vector space A equipped with a product $\star$ and a commutative product $\bullet$ satisfying the following equations:

$$(x \star y) \star z = \star(y \star z), x \star (y \star z) = x \star (y \bullet z), x \bullet (y \star z) = (x \bullet y) \star z, (x \bullet y) \bullet z = x \bullet (y \bullet z).$$

Proposition 3.1.16. *The operad ComTrias of the commutative trialgebra is the triplicator of Comm.*

Proof. Let ω be the operation of the operad *Comm*. Set $\dashv := \binom{\omega}{\dashv}$, $\vdash := \binom{\omega}{\vdash}$ and $\perp := \binom{\omega}{\perp}$. Since $\binom{\omega}{\dashv}^{(12)} = \binom{\omega^{(12)}}{\vdash} = \binom{\omega}{\vdash}$ and $\binom{\omega}{\perp}^{(12)} = \binom{\omega^{(12)}}{\perp} = \binom{\omega}{\perp}$, we have $\dashv^{(12)} = \vdash$ and $\perp^{(12)} = \perp$.The space of relations of *Comm* is generated as an $\mathbb{S}_3$-module by

$$v_1 - v_9 = \omega \circ_{\mathrm{I}} \omega - \omega \circ_{\mathrm{II}} \omega = (x\,\omega\,y)\,\omega\,z - (y\,\omega\,z)\,\omega\,x.$$

Then we have, for example,

$$\begin{aligned}
\mathrm{Tri}_x(v_1 - v_9) &= \{(x \dashv y) \dashv z - (y \dashv z) \vdash x, (x \dashv y) \dashv z - (y \vdash z) \vdash x, (x \dashv y) \dashv z - (y \perp z) \vdash x\} \\
&= \{\underline{(x \dashv y) \dashv z - x \dashv (y \dashv z), (x \dashv y) \dashv z - x \dashv (z \dashv y), (x \dashv y) \dashv z - x \dashv (y \perp z)}\}; \\
\mathrm{Tri}_{\{x,y\}}(v_1 - v_9) &= \{\underline{(x \perp y) \dashv z - (y \dashv z) \perp x}\}; \\
\mathrm{Tri}_{\{x,y,z\}}(v_1 - v_9) &= \{(x \perp y) \perp z - (y \perp z) \perp x\} = \{\underline{(x \perp y) \perp z - x \perp (y \perp z)}\}.
\end{aligned}$$

Replacing the operation $\dashv$ by $\star$ and $\perp$ by $\bullet$, we see that the underlined relations are equivalent to the relations of the commutative trialgebra. The other relations can be obtained from these relations by a permutation of the variables and the commutativity of $\perp$. Thus we get Tri(*Comm*) = *ComTrias*. □

We next consider the triplicator of *Lie*. Let μ be the operation of the operad *Lie*. Set $\dashv := \binom{\mu}{\dashv}$,

$\vdash := \binom{\mu}{\vdash}$ and $\perp := \binom{\mu}{\perp}$. Since $\binom{\mu}{\dashv}^{(12)} = \binom{\mu^{(12)}}{\vdash} = -\binom{\mu}{\vdash}$ and $\binom{\mu}{\perp}^{(12)} = \binom{\mu^{(12)}}{\perp} = -\binom{\mu}{\perp}$, we have $\dashv^{(12)} = -\vdash$ and $\perp^{(12)} = -\perp$. The space of relations of *Lie* is generated as an $\mathbb{S}_3$-module by

$$v_1 + v_5 + v_9 = \mu \circ_{\mathrm{I}} \mu + \mu \circ_{\mathrm{II}} \mu + \mu \circ_{\mathrm{III}} \mu = (x\mu y)\mu z + (z\mu x)\mu y + (y\mu z)\mu x.$$

Then we compute

$$\begin{aligned}
\mathrm{Tri}_{\{x\}}(v_1 + v_5 + v_9) &= \{(x \dashv y) \dashv z + (z \vdash x) \dashv y + (y \dashv z) \vdash x, (x \dashv y) \dashv z + (z \vdash x) \dashv y \\
&\quad +(y \vdash z) \vdash x, (x \dashv y) \dashv z + (z \vdash x) \dashv y + (y \perp z) \vdash x\} \\
&= \{\underline{(x \dashv y) \dashv z - (x \dashv z) \dashv y - x \dashv (y \dashv z)}, (x \dashv y) \dashv z - (x \dashv z) \dashv y \\
&\quad +x \dashv (z \dashv y), \underline{(x \dashv y) \dashv z - (x \dashv z) \dashv y - x \dashv (y \perp z)}\}; \\
\mathrm{Tri}_{\{x,y\}}(v_1 + v_5 + v_9) &= \{\underline{(x \perp y) \dashv z + (z \vdash x) \perp y + (y \dashv z) \perp x}\}; \\
\mathrm{Tri}_{\{x,y,z\}}(v_1 + v_5 + v_9) &= \{\underline{(x \perp y) \perp z + (z \perp x) \perp y + (y \perp z) \perp x}\},
\end{aligned}$$

and other computations yield the same relations up to permutations.

Replacing the operation $\dashv$ by $\diamond$ and $\perp$ by $[,]$, then $[\ ,\ ]$ is skew-symmetric and the underlined relations are

$$\begin{aligned}
[x,[y,z]] + [y,[z,x]] + [z,[x,y]] &= 0, \\
x \diamond [y,z] &= x \diamond (y \diamond z), \\
[x,y] \diamond z &= [x \diamond z, y] + [x, y \diamond z], \\
(x \diamond y) \diamond z &= x \diamond (y \diamond z) + (x \diamond z) \diamond y.
\end{aligned} \tag{3.1.10}$$

Then in particular $(A, \diamond)$ is a right Leibniz algebra. Since the duplicator of *Lie* is *Leib*, the operad of the Leibniz algebra, we tentatively call the new algebra **triLeibniz algebra**. In summary, we obtain

Proposition 3.1.17. *The triplicator of Lie is TriLeib, the operad of the triLeibniz algebra.*

As we will see in Section 3.2.1, *TriLeib* is precisely the Koszul dual of the operad $CTD = ComTriDend$ of the commutative tridendriform algebra, namely the **Dual CTD algebra** in (97).

We next show that *TriLeib* plays the same role for the triassociative algebra as the role of the Leibniz algebra for the diassociative algebra (42; 75).

Proposition 3.1.18. *Let* $(A, \dashv, \vdash, \perp)$ *be an associative trialgebra. Define new binary operations by*

$$x \diamond y := x \dashv y - y \vdash x, \quad [x, y] := x \perp y - y \perp x.$$

Then $(A, \diamond, [,])$ *becomes a Leibniz trialgebra.*

Proof. By definition, for any $x, y, z \in A$, we have

$$[x, y] \diamond z = [x, y] \dashv z - z \vdash [x, y] = (x \perp y - y \perp x) \dashv z - z \vdash (x \perp y - y \perp x)$$

and

$$\begin{aligned} & [x \diamond z, y] + [x, y \diamond z] \\ = \; & [x \dashv z - z \vdash x, y] + [x, y \dashv z - z \vdash y] \\ = \; & (x \dashv z - z \vdash x) \perp y - y \perp (x \dashv z - z \vdash x) + x \perp (y \dashv z - z \vdash y) - (y \dashv z - z \vdash y) \perp x. \end{aligned}$$

Since

$$(x \perp y) \dashv z = x \perp (y \dashv z),\ (y \perp x) \dashv z = y \perp (x \dashv z),\ z \vdash (x \perp y) = (z \vdash x) \perp y,$$

$$z \vdash (y \perp x) = (z \vdash y) \perp x,\ (x \dashv z) \perp y = x \perp (z \vdash y),\ y \perp (z \vdash x) = (y \dashv z) \perp x$$

in a triassociative algebra, we have $[x, y] \diamond z = [x \diamond z, y] + [x, y \diamond z]$. The other defining equations of the triLeibniz algebra can be proved in the same way. □

Moreover, we have the following commuting diagram:

$$\begin{array}{ccc} \mathit{Leib} & \xleftarrow{\ -\ } & \mathit{Diass} \\ {\scriptstyle \diamond \to (\diamond, 0)} \downarrow & & \downarrow {\scriptstyle (\vdash, \dashv) \to (\vdash, \dashv, 0)} \\ \mathit{TriLeib} & \xleftarrow{\ -\ } & \mathit{Triass} \end{array}$$

It would be interesting to consider the left adjoint of the functor defined in the bottom line of the above diagram, which could be called **the universal envelope algebra of a triLeibniz algebra** just as in (70).

3.1.5 Operads, their duplicators and triplicators

In this section, we study the relationship among a binary operad, its duplicator, and its triplicator.

3.1.5.1 Operads and their duplicators and triplicators

For a given $\mathbb{S}$-module V concentrated in ariry 2: $V = V(2)$. Let $i_V : V \to \mathcal{T}(V)$ denote the natural embedding to the free operad $\mathcal{T}(V)$. Let $\mathcal{P} := \mathcal{T}(V)/(R)$ be a binary operad and let $j_V : V \to \mathcal{P}$ be $p_V \circ i_V$, where $p_V : \mathcal{T}(V) \to \mathcal{P}$ is the operad projection. Similarly define the maps $i_{\mathrm{Du}(V)} : \mathrm{Du}(V) \to \mathcal{T}(\mathrm{Du}(V))$ and operad morphism $p_{\mathrm{Du}(V)} : \mathcal{T}(\mathrm{Du}(V)) \to \mathrm{Du}(\mathcal{P})$ and $j_{\mathrm{Du}(V)} := p_{\mathrm{Du}(V)} \circ i_{\mathrm{Du}(V)}$, as well as the corresponding map and operad morphisms for $\mathrm{Tri}(V)$.

Proposition 3.1.19. *Let $\mathcal{P} = \mathcal{T}(V)/(R)$ be a binary operad.*

(a) *The linear map*

$$\eta : \mathrm{Du}(V) \to V, \quad \binom{\omega}{u} \longmapsto \omega \textit{ for all } \binom{\omega}{u} \in \mathrm{Du}(V), u \in \{\dashv, \vdash\} \tag{3.1.11}$$

induces a unique operad morphism

$$\tilde{\eta} : \mathrm{Du}(\mathcal{P}) \to \mathcal{P}$$

such that $\tilde{\eta} \circ j_{\mathrm{Du}(V)} = j_V \circ \eta$.

(b) *The linear map*

$$\zeta : \mathrm{Tri}(V) \to V, \quad \binom{\omega}{u} \longmapsto \omega \textit{ for all } \binom{\omega}{u} \in \mathrm{Tri}(V), u \in \{\dashv, \vdash, \perp\} \tag{3.1.12}$$

induces a unique operad morphism

$$\tilde{\zeta} : \mathrm{Tri}(\mathcal{P}) \to \mathcal{P}$$

such that $\tilde{\zeta} \circ j_{\mathrm{Tri}(V)} = j_V \circ \zeta$.

(c) *There is a morphism* $\rho : \mathrm{Tri}(\mathcal{P}) \to \mathcal{P}$ *of operads that extends the linear map from* $\mathrm{Tri}(V)$ *to* V *defined by*

$$\binom{\omega}{\perp} \longmapsto \omega, \quad \binom{\omega}{u} \longmapsto 0, \quad \textit{where } u \in \{\dashv, \vdash\}. \tag{3.1.13}$$

Proof. Let R be the set of locally homogeneous elements

$$r_s := \sum_i c_{s,i}\tau_{s,i},\ c_{s,i} \in \mathbf{k}, \tau_{s,i} \in \bigcup_{t \in \mathfrak{R}} t(\mathcal{V}),\ 1 \leq s \leq k,$$

as given in Eq.(3.1.5).

(a) By the universal property of the free operad $\mathcal{T}(\mathrm{Du}(V))$ on the $\mathbb{S}$-module $\mathrm{Du}(V)$, the $\mathbb{S}$-module morphism $i_V \circ \eta : \mathrm{Du}(V) \to \mathcal{T}(V)$ induces a unique operad morphism $\bar{\eta} : \mathcal{T}(\mathrm{Du}(V)) \to \mathcal{T}(V)$ such that $i_{\mathrm{Du}(V)} \circ \bar{\eta} = i_V \circ \eta$.

For any $x \in \mathrm{L}in(r_s)$ and $1 \leq s \leq k$, by the description of $\mathrm{Du}_x(\tau_{s,i}))$ in Proposition 3.1.4 and the definition of η in Eq. (3.1.11), the element $\eta(\mathrm{Du}(\tau_{s,i}))$ is obtained by replacing each decoration $\binom{\omega}{u}$ of the vertices of $\mathrm{Du}(\tau_{s,i})$ by ω, where $\omega \in V$ and $u \in \{\dashv, \vdash\}$. Thus $\bar{\eta}(\mathrm{Du}(\tau_{s,i})) = \tau_{s,i}$. Then we have

$$\bar{f}\left(\sum_i c_{s,i}\mathrm{Du}_x(\tau_{s,i})\right) = \sum_i c_{s,i}\tau_{s,i} \equiv 0 \mod (R).$$

By Eq. (3.1.6), we see that $(\mathrm{Du}(R)) \subseteq \ker(\eta)$. Thus there is a unique operad morphism $\tilde{\eta}$: $\mathrm{Du}(\mathcal{P}) := \mathcal{T}(\mathrm{Du}(V))/(\mathrm{Du}(R)) \to \mathcal{P} := \mathcal{T}(V)/(R)$ such that $\tilde{\eta} \circ p_{\mathrm{Du}(V)} = p_V \circ \bar{\eta}$. We then have $\tilde{\eta} \circ j_{\mathrm{Du}(V)} = j_V \circ \eta$. In summary, we have the following diagram in which each square commutes.

$$\begin{array}{ccccc}
\mathrm{Du}(V) & \xrightarrow{i_{\mathrm{Du}(V)}} & \mathcal{T}(\mathrm{Du}(V)) & \xrightarrow{p_{\mathrm{Du}(V)}} & \mathrm{Du}(\mathcal{P}) \\
\downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \bar{\eta}} & & \downarrow{\scriptstyle \tilde{\eta}} \\
V & \xrightarrow{i_V} & \mathcal{T}(V) & \xrightarrow{p_V} & \mathcal{P}
\end{array}$$

Suppose $\tilde{\eta}' : \mathrm{Du}(\mathcal{P}) \to \mathcal{P}$ be another operad morphism such that $\tilde{\eta}' \circ j_{\mathrm{Du}(V)} = j_V \circ \eta$. Then we have $\tilde{\eta}' \circ j_{\mathrm{Du}(V)} = \tilde{\eta}' \circ p_{\mathrm{Du}(V)} \circ i_{\mathrm{Du}(V)}$ and $j_V \circ \eta = p_V \circ j_V \circ \eta = p_V \circ \bar{\eta} \circ i_{\mathrm{Du}(V)}$. By the universal property of the free operad $\mathcal{T}(\mathrm{Du}(V))$, we obtain $\tilde{\eta}' \circ p_{\mathrm{Du}(V)} = p_V \circ \bar{\eta} = \tilde{\eta} \circ p_{\mathrm{Du}(V)}$. Since $p_{\mathrm{Du}(V)}$ is surjective, we obtain $\tilde{\eta}' = \tilde{\eta}$. This proves the uniqueness of $\tilde{\eta}$.

(b) The proof is similar to the proof of Item (a).

(c) By the description of $\mathrm{Tri}_{\{x\}}(\tau_{s,i})$ in Proposition 3.1.6, $\rho(\mathrm{Tri}_{\{x\}}(\tau_{s,i}))$ is obtained by replacing $\binom{\omega}{u}$ by $\rho(\binom{\omega}{u})$. Since $\rho(\binom{\omega}{\dashv})=0$, $\rho(\binom{\omega}{\vdash})=0$ and $\rho(\binom{\omega}{\perp})=\omega$, it is easy to see that if $J\neq Lin(\tau)$, then $\rho(\sum_i c_{s,i}\mathrm{Tri}_J(\tau_{s,i}))=\sum_i c_{s,i}\tau_{s,i}=0$, and, if $J=Lin(\tau)$, then $\rho(\sum_i c_{s,i}\mathrm{Tri}_{Lin(\tau)}(\tau_{s,i}))=\sum_i c_{s,i}\tau_{s,i}\equiv 0 \mod (R)$. Thus $\rho(\mathrm{Tri}(R))\subseteq R$ and ρ induces the desired operad morphism. □

3.1.5.2 Relationship between duplicators and triplicators of a binary operad

The following result relates the duplicator and the triplicator of a binary algebraic operad.

Proposition 3.1.20. *Let $\mathcal{P}=\mathcal{T}(V)/(R)$ be a binary algebraic operad. There is a morphism of operads from* $\mathrm{Tri}(\mathcal{P})$ *to* $\mathrm{Du}(\mathcal{P})$ *that extends the linear map defined by*

$$\binom{\omega}{\dashv}\to\binom{\omega}{\dashv},\quad \binom{\omega}{\vdash}\to\binom{\omega}{\vdash},\quad \binom{\omega}{\perp}\to 0,\quad \omega\in V. \tag{3.1.14}$$

Proof. The linear map $\phi:\mathrm{Tri}(V)\to\mathrm{Du}(V)$ defined by Eq.(3.1.14) is $\mathbb{S}_2$-equivariant. Thus it induces a morphism of the free operads $\phi:\mathcal{T}(\mathrm{Tri}(V))\to\mathcal{T}(\mathrm{Du}(V))$ which, by composing with the quotient map, induces the morphism of operads

$$\phi:\mathcal{T}(\mathrm{Tri}(V))\to\mathrm{Du}(\mathcal{P})=\mathcal{T}(\mathrm{Du}(V))/(\mathrm{Du}(R)).$$

Let $\mathrm{Tri}_J(r)\in\mathrm{Tri}(R)$ be one of the generators of $(\mathrm{Tri}(R))$ with $r=\sum_i c_i\tau_i\in R$ in Eq. (3.1.5) and $\varnothing\neq J\subseteq \mathrm{L}in(r)$. If J is the singleton $\{x\}$ for some $x\in \mathrm{L}in(r)$, then by the description of $\mathrm{Tri}_{\{x\}}(\tau_i)$ in Proposition 3.1.6, $\phi(\mathrm{Tri}_{\{x\}}(\tau))$ is obtained by keeping all the $\binom{\omega}{\dashv}$ and $\binom{\omega}{\vdash}$, and by replacing all $\binom{\omega}{\perp}$, $\omega\in V$ by zero. Thus in Case (b) of Proposition 3.1.6 we have $\phi(\mathrm{Tri}_{\{x\}}(\tau_i))=\mathrm{Du}_x(\tau_i)$. Also Case (a)(iii) cannot occur for the singleton $\{x\}$. Thus in Case (a) of Proposition 3.1.6, we also have $\phi(\mathrm{Tri}_{\{x\}}(\tau_i))=\mathrm{Du}_x(\tau_i)$. Thus $\phi(\mathrm{Tri}_{\{x\}}(r))=\mathrm{Du}_x(r)$ and hence is in $\mathrm{Du}(R)$.
If J contains more than one element, then at least one of the vertices of $\mathrm{Tri}_J(\tau_i)$ is $\binom{\omega}{\perp}$ and hence the corresponding vertex of $\phi(\mathrm{TSu}_J(\tau_i))$ is zero. Thus we have $\phi(\mathrm{Tri}_J(\tau_i))=0$, $\phi(\mathrm{Tri}_J(r))=0$ and hence $\phi(\mathrm{Tri}_J(R))=0$. Thus, for any $J\neq\varnothing$ and $r\in R$, we have $\phi(\mathrm{Tri}_J(r))\in\mathrm{Du}(R)$ and hence $\phi(\mathrm{Tri}(R))$ is a subset of $\mathrm{Du}(R)$.
In summary, we have $\phi((\mathrm{Tri}(R))\subseteq\mathrm{Du}(R)$. Thus the morphism $\phi:\mathcal{T}(\mathrm{Tri}(V))\to\mathrm{Du}(\mathcal{P})$ induces a morphism $\phi:\mathrm{Tri}(\mathcal{P})\to\mathrm{Du}(\mathcal{P})$. □

If we take $\mathcal{P}$ to be the operad of the associative algebra, then we obtain the following result of Loday and Ronco (75):

Corollary 3.1.5. *Let* $(A, \dashv, \vdash)$ *be an associative dialgebra. Then* $(A, \dashv, \vdash, 0)$ *is an associative trialgebra, where 0 denotes the trivial product.*

3.2 Duality of replicators with successors and Manin products

The similarity between the definitions of the replicators and successors (9) suggests that there is a close relationship between the two constructions. We show that this is indeed the case. More precisely, taking the replicator of a binary quadratic operad is in Koszul dual with taking the successor of the dual operad. This in particular allows us to identify the duplicator (resp. triplicator) of a binary quadratic operad $\mathcal{P}$ with the Manin white product of *Perm* (resp. *ComTrias*) with $\mathcal{P}$, providing an easy way to compute these white products. Since it is shown in (48) that taking di-Var and tri-Var is also isomorphic to taking these Manin products, taking duplicator (resp. triplicator) is isomorphic to taking di-Var (resp. tri-Var) other than the case of free operads.

3.2.1 The duality of replicators with successors

Let $\mathcal{P} = \mathcal{T}(V)/(R)$ be a binary quadratic operad. Then with the notations in Section 3.1.4, we have $\mathcal{T}(V)(3) = 3V \otimes V = \bigoplus_{u \in \{\mathrm{I},\mathrm{II},\mathrm{III}\}} V \circ_u V$.

Proposition 3.2.1. *Let* $\mathbf{k}$ *be an infinite field. Let* W *be a nonzero* $\mathbb{S}$*-submodule of* $3V \otimes V$*. Then there is a basis* $\{e_1, \cdots, e_n\}$ *of* V *such that the restriction to* W *of the coordinate projections*

$$p_{i,j,u} : 3V \otimes V = \bigoplus_{1 \le k, \ell \le n, v \in \{\mathrm{I},\mathrm{II},\mathrm{III}\}} \mathbf{k}\, e_k \circ_v e_\ell \to \mathbf{k}\, e_i \circ_u e_j,$$

are nonzero and hence surjective for all $1 \le i, j \le n$ *and* $u \in \{\mathrm{I},\mathrm{II},\mathrm{III}\}$*.*

Proof. Fix a $0 \neq w \in W$ and write $w = w_{\mathrm{I}} + w_{\mathrm{II}} + w_{\mathrm{III}}$ with $w_u \in V \circ_u V, u \in \{\mathrm{I},\mathrm{II},\mathrm{III}\}$. Then at least one of the three terms is nonzero. Since W is an $\mathbb{S}$-module and $(w_u)^{(123)} = w_{u+I}$ (where III+I is taken to be I), we might assume that $w \in W$ is chosen so that $w_{\mathrm{I}} \neq 0$. Fix a basis $\{v_1, \cdots, v_n\}$ of V. Then there are $c_{ij} \in \mathbf{k}, 1 \leq i, j \leq n$, that are not all zero such that $w_{\mathrm{I}} = \sum_{1\leq i,j\leq n} c_{ij} u_i \circ_{\mathrm{I}} u_j$. Consider the set of polynomials

$$f_{k\ell}(x_{rs}) := f_{k\ell}(\{x_{rs}\}) := \sum_{1\leq i,j\leq n} c_{ij} x_{ik} x_{j\ell} \in \mathbf{k}[x_{rs} \,|\, 1 \leq r, s \leq n], \quad 1 \leq k, \ell \leq n.$$

Then the polynomial $\prod_{1\leq k,\ell\leq n} f_{k\ell}(x_{rs})$ is nonzero since at least one of c_{ij} is nonzero, giving a monomial $\prod_{1\leq r,s\leq n} c_{ij} x_{ir} x_{js}$ in the product with nonzero coefficient. Thus the product

$$f(x_{rs}) := \det(x_{rs}) \prod_{1\leq k,\ell\leq n} f_{k\ell}(x_{rs})$$

is nonzero since $\det(x_{rs}) := \prod_{\sigma\in\mathbb{S}_n} x_{1\sigma(1)} \cdots x_{n\sigma(n)}$ is also a nonzero polynomial. Thus, by our assumption that $\mathbf{k}$ is an infinite field, there are $d_{rs} \in \mathbf{k}, 1 \leq r, s \leq n$, such that $f(d_{rs}) \neq 0$. Thus $D := (d_{rs}) \in M_{n\times n}(\mathbf{k})$ is invertible and $f_{k\ell}(d_{rs}) \neq 0, 1 \leq k, \ell \leq n$.

Fix such a matrix $D = (d_{rs})$ and define

$$(e_1, \cdots, e_n)^T := D^{-1}(v_1, \cdots, v_n)^T.$$

Then $\{e_1, \cdots, e_n\}$ is a basis of V and $v_i = \sum_{k=1}^{n} d_{ik} e_k$. Further

$$w_{\mathrm{I}} = \sum_{1\leq i,j\leq n} c_{ij} v_i \circ_{\mathrm{I}} v_j = \sum_{1\leq i,j\leq n} c_{ij} \left(\sum_{1\leq k,\ell n} d_{ik} d_{j\ell} e_k \circ_{\mathrm{I}} e_\ell \right) = \sum_{1\leq k,\ell\leq n} \left(\sum_{1\leq i,j\leq n} c_{ij} d_{ik} d_{j\ell} \right) e_k \circ_{\mathrm{I}} e_\ell.$$

The coefficients are $f_{k\ell}(d_{rs})$ and are nonzero by the choice of D. Thus $p_{i,j,\mathrm{I}}(w) = p_{i,j,\mathrm{I}}(w_{\mathrm{I}})$ is nonzero and hence $p_{i,j,\mathrm{I}}(W)$ is onto for all $1 \leq i, j \leq n$.

Since W is an $\mathbb{S}$-module, we have $w^{(123)} \in W$ and $(w^{(123)})_{\mathrm{II}} = (w_{\mathrm{I}})^{(123)}$. Thus $p_{i,j,\mathrm{II}}(w^{(123)}) = p_{i,j,\mathrm{II}}((w_{\mathrm{I}})^{(123)})$ is nonzero and hence $p_{i,j,\mathrm{II}}(W)$ is onto for all $1 \leq i, j \leq n$. By the same argument, $p_{i,j,\mathrm{III}}(W)$ is onto for all $1 \leq i, j \leq n$, completing the proof. □

Lemma 3.2.1. *Let W be a nonzero $\mathbb{S}$-submodule of $3V \otimes V$ and let $\{e_1, \cdots, e_n\}$ be a basis as* *chosen in Proposition 3.2.1. Let* $\{r_1, \cdots, r_m\}$ *be a basis of* U *and write*

$$r_k = \sum_{1\le i,j\le n} c^{\ell}_{iju} e_i \circ_u e_j, \quad c^k_{iju} \in \mathbf{k}, 1 \le i,j \le n, u \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}, 1 \le k \le m.$$

Then for each $1 \le i,j \le n$ *and* $u \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$*, there is* $1 \le k \le m$*, such that* c^{ℓ}_{iju} *is not zero.*

Proof. Suppose there is $1 \le i,j \le n$ and $u \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$ such that $c^k_{iju} = 0$ for all $1 \le k \le m$. Then $p_{iju}(r_k) = 0$ and hence $p_{iju}(W) = 0$. This contradicts Proposition 3.2.1. □

Let $\mathcal{P} = \mathcal{T}(V)/(R)$ be a binary quadratic operad. Fix a $\mathbf{k}$-basis $\{e_1, e_2, \cdots, e_n\}$ for (R). The space $\mathcal{T}(V)(3)$ is spanned by the basis $\{e_i \circ_u e_j \mid 1 \le i,j \le n, u \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}\}$. Thus if $f \in \mathcal{T}(V)(3)$, we have

$$f = \sum_{i,j} a_{i,j} e_i \circ_{\mathrm{I}} e_j + \sum_{i,j} b_{i,j} e_i \circ_{\mathrm{II}} e_j + \sum_{i,j} c_{i,j} e_i \circ_{\mathrm{III}} e_j.$$

Then we can take the relation space $(R) \subset \mathcal{T}(V)(3)$ to be generated by m linearly independent relations

$$R = \left\{ f_k = \sum_{i,j} a^k_{i,j} e_i \circ_{\mathrm{I}} e_j + \sum_{i,j} b^k_{i,j} e_i \circ_{\mathrm{II}} e_j + \sum_{i,j} c^k_{i,j} e_i \circ_{\mathrm{III}} e_j \;\middle|\; 1 \le k \le m \right\}. \tag{3.2.1}$$

We state the following easy fact for later applications.

Lemma 3.2.2. *Let* $f_i, 1 \le i \le m$*, be a basis of* (R)*. Then* $\{\mathrm{BSu}_x(f_i) \,|\, x \in \mathrm{L}in(f_i), 1 \le i \le m\}$ *is a linear spanning set of* $(\mathrm{BSu}(R))$ *and* $\{\mathrm{Du}_x(f_i) \,|\, x \in \mathrm{L}in(f_i), 1 \le i \le m\}$ *is a linear spanning set of* $(\mathrm{Du}(R))$*.*

Proof. Let L be the linear span of $\{\mathrm{BSu}_x(f_i) \,|\, x \in \mathrm{L}in(f_i), 1 \le i \le m\}$. Then from $\mathrm{BSu}_x(f_i) \in (\mathrm{BSu}(R))$ we obtain $L \subseteq (\mathrm{BSu}(R))$. On the other hand, by (9, Lemma 2.6), L is already an $\mathbb{S}$-submodule. Thus from $\mathrm{BSu}(R) \subseteq L$ we obtain $(\mathrm{BSu}(R))_{\mathbb{S}} \subseteq L$. The proof for $(\mathrm{Du}(R))$ is the same. □

For the finite dimensional $\mathbb{S}_2$-module V, we define its Czech dual $V^\vee = V^* \otimes sgn_2$. There is a natural pairing with respect to this duality given by:

$$\langle , \rangle : \mathcal{T}(V^\vee)(3) \otimes \mathcal{T}(V)(3) \longrightarrow \mathbf{k},$$

$$\langle e_i^\vee \circ_u e_j^\vee, e_k \circ_v e_\ell \rangle = \delta_{(i,k)}\delta_{(j,\ell)}\delta_{(u,v)} \in \mathbf{k}.$$

We denote by $R^\perp$ the annihilator of R with respect to this pairing. Given relations as in Eq. (3.2.1), we can express a basis of $(R^\perp)$ as

$$R^\perp = \left\{ g_\ell = \sum_{i,j} \alpha_{i,j}^\ell e_i^\vee \circ_{\mathrm{I}} e_j^\vee + \sum_{i,j} \beta_{i,j}^\ell e_i^\vee \circ_{\mathrm{II}} e_j^\vee + \sum_{i,j} \gamma_{i,j}^\ell e_i^\vee \circ_{\mathrm{III}} e_j^\vee \mid 1 \leq \ell \leq 3n^2 - m \right\}, \quad (3.2.2)$$

where, for all k and ℓ, we have

$$\sum_{i,j} a_{i,j}^k \alpha_{i,j}^\ell + \sum_{i,j} b_{i,j}^k \beta_{i,j}^\ell + \sum_{i,j} c_{i,j}^k \gamma_{i,j}^\ell = 0. \quad (3.2.3)$$

Further for any $(x_{i,j}, y_{i,j}, z_{i,j}) \in \mathbf{k}^3, 1 \leq i,j \leq n$, if

$$\sum_{i,j} a_{i,j}^k x_{i,j} + \sum_{i,j} b_{i,j}^k y_{i,j} + \sum_{i,j} c_{i,j}^k z_{i,j} = 0 \ \text{ for all } 1 \leq k \leq m,$$

then $\sum_{i,j} x_{i,j} e_i^\vee \circ_{\mathrm{I}} e_j^\vee + \sum_{i,j} y_{i,j} e_i^\vee \circ_{\mathrm{II}} e_j^\vee + \sum_{i,j} z_{i,j} e_i^\vee \circ_{\mathrm{III}} e_j^\vee$ is in $R^\perp$ and hence is of the form $\sum\limits_{\ell=1}^{3n^2-m} d_\ell g_\ell$ for some $d_\ell \in \mathbf{k}$. Thus

$$(x_{i,j}, y_{i,j}, z_{i,j}) = \sum_{\ell=1}^{3n^2-m} d_\ell \left(\alpha_{i,j}^\ell, \beta_{i,j}^\ell, \gamma_{i,j}^\ell \right). \quad (3.2.4)$$

By Proposition 3.1.4, we have

$$\begin{aligned}
&\mathrm{Du}_x(e_i \circ_{\mathrm{I}} e_j) = \left\{ \binom{e_i}{\dashv} \circ_{\mathrm{I}} \binom{e_j}{\dashv} \right\}, \mathrm{Du}_x(e_i \circ_{\mathrm{II}} e_j) = \left\{ \binom{e_i}{\vdash} \circ_{\mathrm{II}} \binom{e_j}{\dagger} \right\}, \mathrm{Du}_x(e_i \circ_{\mathrm{III}} e_j) = \left\{ \binom{e_i}{\dashv} \circ_{\mathrm{III}} \binom{e_j}{\vdash} \right\}, \\
&\mathrm{Du}_y(e_i \circ_{\mathrm{I}} e_j) = \left\{ \binom{e_i}{\dashv} \circ_{\mathrm{I}} \binom{e_j}{\vdash} \right\}, \mathrm{Du}_y(e_i \circ_{\mathrm{II}} e_j) = \left\{ \binom{e_i}{\dashv} \circ_{\mathrm{II}} \binom{e_j}{\dashv} \right\}, \mathrm{Du}_y(e_i \circ_{\mathrm{III}} e_j) = \left\{ \binom{e_i}{\vdash} \circ_{\mathrm{III}} \binom{e_j}{\dagger} \right\}, \quad (3.2.5) \\
&\mathrm{Du}_z(e_i \circ_{\mathrm{I}} e_j) = \left\{ \binom{e_i}{\vdash} \circ_{\mathrm{I}} \binom{e_j}{\dagger} \right\}, \mathrm{Du}_z(e_i \circ_{\mathrm{II}} e_j) = \left\{ \binom{e_i}{\dashv} \circ_{\mathrm{II}} \binom{e_j}{\vdash} \right\}, \mathrm{Du}_z(e_i \circ_{\mathrm{III}} e_j) = \left\{ \binom{e_i}{\dashv} \circ_{\mathrm{III}} \binom{e_j}{\dashv} \right\},
\end{aligned}$$

where $\binom{e_i}{\vdash} \circ_{\mathrm{u}} \binom{e_j}{\dagger} := \left\{ \binom{e_i}{\vdash} \circ_{\mathrm{u}} \binom{e_j}{\vdash}, \binom{e_i}{\vdash} \circ_{\mathrm{u}} \binom{e_j}{\dashv} \right\}, u \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$.
Let $\mathrm{BSu}(\mathcal{P}^!)$ be the bisuccessor of the dual operad $\mathcal{P}^!$ recalled in Definition 3.1.11. Then we also

have

$$
\begin{aligned}
&\mathrm{BSu}_x(e_i^\vee \circ_{\mathrm{I}} e_j^\vee) = \left\{\binom{e_i^\vee}{\prec} \circ_{\mathrm{I}} \binom{e_j^\vee}{\prec}\right\}, \mathrm{BSu}_x(e_i^\vee \circ_{\mathrm{II}} e_j^\vee) = \left\{\binom{e_i^\vee}{\succ} \circ_{\mathrm{II}} \binom{e_j^\vee}{\star}\right\}, \mathrm{BSu}_x(e_i^\vee \circ_{\mathrm{III}} e_j^\vee) = \left\{\binom{e_i^\vee}{\prec} \circ_{\mathrm{III}} \binom{e_j^\vee}{\succ}\right\},\\
&\mathrm{BSu}_y(e_i^\vee \circ_{\mathrm{I}} e_j^\vee) = \left\{\binom{e_i^\vee}{\prec} \circ_{\mathrm{I}} \binom{e_j^\vee}{\succ}\right\}, \mathrm{BSu}_y(e_i^\vee \circ_{\mathrm{II}} e_j^\vee) = \left\{\binom{e_i^\vee}{\prec} \circ_{\mathrm{II}} \binom{e_j^\vee}{\prec}\right\}, \mathrm{BSu}_y(e_i^\vee \circ_{\mathrm{III}} e_j^\vee) = \left\{\binom{e_i^\vee}{\succ} \circ_{\mathrm{III}} \binom{e_j^\vee}{\star}\right\},\\
&\mathrm{BSu}_z(e_i^\vee \circ_{\mathrm{I}} e_j^\vee) = \left\{\binom{e_i^\vee}{\succ} \circ_{\mathrm{I}} \binom{e_j^\vee}{\star}\right\}, \mathrm{BSu}_z(e_i^\vee \circ_{\mathrm{II}} e_j^\vee) = \left\{\binom{e_i^\vee}{\prec} \circ_{\mathrm{II}} \binom{e_j^\vee}{\succ}\right\}, \mathrm{BSu}_z(e_i^\vee \circ_{\mathrm{III}} e_j^\vee) = \left\{\binom{e_i^\vee}{\prec} \circ_{\mathrm{III}} \binom{e_j^\vee}{\prec}\right\},
\end{aligned}
\tag{3.2.6}
$$

where $\star = \prec + \succ$.

Theorem 3.2.3. *Let* $\mathbf{k}$ *be an infinite field. Let* $\mathcal{P} = \mathcal{T}(V)/(R)$ *be a binary quadratic operad. Then*

$$\mathrm{Du}(\mathcal{P})^! = \mathrm{BSu}(\mathcal{P}^!)$$

if and only if $R \neq 0$.

Proof. For the if part, let $\mathcal{P} = \mathcal{T}(V)/(R)$ be a binary quadratic operad with $R \neq 0$. Take $W = (R)$ in Proposition 3.2.1 and fix a $\mathbf{k}$-basis $\{e_1, e_2, \cdots, e_n\}$ of V as in the proposition. Let $f_k, 1 \leq k \leq m$, be the basis of (R) as defined in Eq. (3.2.1).

By Eq. (3.2.5), we have

$$\mathrm{Du}_x(f_k) = \left\{\sum_{i,j} a_{i,j}^k \binom{e_i}{\dashv} \circ_{\mathrm{I}} \binom{e_j}{\dashv} + \sum_{i,j} b_{i,j}^k \binom{e_i}{\vdash} \circ_{\mathrm{II}} \binom{e_j}{\dagger} + \sum_{i,j} c_{i,j}^k \binom{e_i}{\dashv} \circ_{\mathrm{III}} \binom{e_j}{\vdash}\right\},$$

$$\mathrm{Du}_y(f_k) = \left\{\sum_{i,j} a_{i,j}^k \binom{e_i}{\dashv} \circ_{\mathrm{I}} \binom{e_j}{\vdash} + \sum_{i,j} b_{i,j}^k \binom{e_i}{\dashv} \circ_{\mathrm{II}} \binom{e_j}{\dashv} + \sum_{i,j} c_{i,j}^k \binom{e_i}{\vdash} \circ_{\mathrm{III}} \binom{e_j}{\dagger}\right\},$$

$$\mathrm{Du}_z(f_k) = \left\{\sum_{i,j} a_{i,j}^k \binom{e_i}{\vdash} \circ_{\mathrm{I}} \binom{e_j}{\dagger} + \sum_{i,j} b_{i,j}^k \binom{e_i}{\dashv} \circ_{\mathrm{II}} \binom{e_j}{\vdash} + \sum_{i,j} c_{i,j}^k \binom{e_i}{\dashv} \circ_{\mathrm{III}} \binom{e_j}{\dashv}\right\}.$$

From Eq. (3.2.6), we similarly obtain

$$\mathrm{BSu}_x(g_\ell) = \left\{\sum_{i,j} \alpha_{i,j}^\ell \binom{e_i^\vee}{\prec} \circ_{\mathrm{I}} \binom{e_j^\vee}{\prec} + \sum_{i,j} \beta_{i,j}^\ell \binom{e_i^\vee}{\succ} \circ_{\mathrm{II}} \binom{e_j^\vee}{\star} + \sum_{i,j} \gamma_{i,j}^\ell \binom{e_i^\vee}{\prec} \circ_{\mathrm{III}} \binom{e_j^\vee}{\succ}\right\},$$

$$\mathrm{BSu}_y(g_\ell) = \left\{ \sum_{i,j} \alpha^\ell_{i,j} \binom{e_i^\vee}{\prec} \circ_{\mathrm{I}} \binom{e_j^\vee}{\succ} + \sum_{i,j} \beta^\ell_{i,j} \binom{e_i^\vee}{\prec} \circ_{\mathrm{II}} \binom{e_j^\vee}{\prec} + \sum_{i,j} \gamma^\ell_{i,j} \binom{e_i^\vee}{\succ} \circ_{\mathrm{III}} \binom{e_j^\vee}{\star} \right\},$$

$$\mathrm{BSu}_z(g_\ell) = \left\{ \sum_{i,j} \alpha^\ell_{i,j} \binom{e_i^\vee}{\succ} \circ_{\mathrm{I}} \binom{e_j^\vee}{\star} + \sum_{i,j} \beta^\ell_{i,j} \binom{e_i^\vee}{\prec} \circ_{\mathrm{II}} \binom{e_j^\vee}{\succ} + \sum_{i,j} \gamma^\ell_{i,j} \binom{e_i^\vee}{\prec} \circ_{\mathrm{III}} \binom{e_j^\vee}{\prec} \right\}.$$

By Lemma 3.2.2, we have

$$(\mathrm{Du}(R)) = \sum_{k=1}^{m} \mathbf{k}\mathrm{Du}(f_k) = \sum_k \left(\mathbf{k}\mathrm{Du}_x(f_k) + \mathbf{k}\mathrm{Du}_y(f_k) + \mathbf{k}\mathrm{Du}_z(f_k) \right),$$

$$\mathrm{BSu}(R^\perp) = \sum_{\ell=1}^{3n^2-m} \mathbf{k}\mathrm{BSu}(g_\ell) = \sum_{\ell=1}^{3n^2-m} \left(\mathbf{k}(\mathrm{BSu}_x(g_\ell) + \mathbf{k}\mathrm{BSu}_y(g_\ell) + \mathbf{k}\mathrm{BSu}_z(g_\ell) \right).$$

To reach our conclusion, it suffices to show the equality $(\mathrm{Du}(R)^\perp) = (\mathrm{BSu}(R^\perp))$ of $\mathbb{S}$-modules under the condition $R \neq 0$. For all $1 \leq k \leq m$ and $1 \leq \ell \leq 3n^2 - m$, by Eq. (3.2.3), we have

$$\langle \mathrm{BSu}_p(g_\ell), \mathrm{Du}_q(f_k) \rangle = 0, \text{ where } p, q \in \{x, y, z\}.$$

Thus $\langle \mathrm{BSu}(g_\ell), \mathrm{Du}(f_k) \rangle = 0$ and hence $\mathrm{BSu}(R^\perp) \subset \mathrm{Du}(R)^\perp$, implying that $(\mathrm{BSu}(R^\perp)) \subseteq (\mathrm{Du}(R)^\perp)$. On the other hand, if

$$h = \sum_{i,j,u,v} x_{i,j,u,v} \binom{e_i^\vee}{u} \circ_{\mathrm{I}} \binom{e_j^\vee}{v} + \sum_{i,j,u,v} y_{i,j,u,v} \binom{e_i^\vee}{u} \circ_{\mathrm{II}} \binom{e_j^\vee}{v} + \sum_{i,j,u,v} z_{i,j,u,v} \binom{e_i^\vee}{u} \circ_{\mathrm{III}} \binom{e_j^\vee}{v}$$

is in $\mathrm{Du}(R)^\perp$, where $u, v \in \{\prec, \succ\}$. Then for all $1 \leq k \leq m$, we have

$$\langle h, \mathrm{Du}_x(f_k) \rangle = 0, \langle h, \mathrm{Du}_y(f_k) \rangle = 0, \langle h, \mathrm{Du}_z(f_k) \rangle = 0.$$

Since $R \neq 0$, by Proposition 3.2.1, for any fixed $i_0, j_0 \in \{1, 2, \cdots, n\}$, there exists $1 \leq k_0 \leq m$, such that $b^{k_0}_{i_0,j_0} \neq 0$. Then, for any k, by the definition of Du_x we see that the relations

$$F_1 := \sum_{i,j} a^k_{i,j} \binom{e_i}{\dashv} \circ_{\mathrm{I}} \binom{e_j}{\dashv} + b^k_{i_0,j_0} \binom{e_i}{\vdash} \circ_{\mathrm{II}} \binom{e_j}{\dashv} + \sum_{i\neq i_0, j\neq j_0} b^k_{i,j} \binom{e_i}{\vdash} \circ_{\mathrm{II}} \binom{e_j}{\dashv} + \sum_{i,j} c^k_{i,j} \binom{e_i}{\dashv} \circ_{\mathrm{III}} \binom{e_j}{\vdash},$$
$$F_2 := \sum_{i,j} a^k_{i,j} \binom{e_i}{\dashv} \circ_{\mathrm{I}} \binom{e_j}{\dashv} + b^k_{i_0,j_0} \binom{e_i}{\vdash} \circ_{\mathrm{II}} \binom{e_j}{\vdash} + \sum_{i\neq i_0, j\neq j_0} b^k_{i,j} \binom{e_i}{\vdash} \circ_{\mathrm{II}} \binom{e_j}{\dashv} + \sum_{i,j} c^k_{i,j} \binom{e_i}{\dashv} \circ_{\mathrm{III}} \binom{e_j}{\vdash},$$

are in $\mathrm{Du}_x(f_k)$. Thus, for $1 \leq k \leq m$, we obtain

$$\sum_{i,j} a^k_{i,j} x_{i,j,\prec,\prec} + b^k_{i_0,j_0} y_{i_0,j_0,\succ,\prec} + \sum_{i\neq i_0, j\neq j_0} b^k_{i,j} y_{i,j,\succ,\prec} + \sum_{i,j} c^k_{i,j} z_{i,j,\prec,\succ} = \langle h, F_1 \rangle = 0,$$
$$\sum_{i,j} a^k_{i,j} x_{i,j,\prec,\prec} + b^k_{i_0,j_0} y_{i_0,j_0,\succ,\succ} + \sum_{i\neq i_0, j\neq j_0} b^k_{i,j} y_{i,j,\succ,\prec} + \sum_{i,j} c^k_{i,j} z_{i,j,\prec,\succ} = \langle h, F_2 \rangle = 0.$$

Comparing the two equations and applying $b^{k_0}_{i_0,j_0} \neq 0$, we obtain $y_{i_0,j_0,\succ,\succ} = y_{i_0,j_0,\succ,\prec}$ for all $1 \leq i_0, j_0 \leq n$. From the second equation and Eq. (3.2.4), we also have

$$(x_{i,j,\prec,\prec}, y_{i,j,\succ,\prec}, z_{i,j,\prec,\succ}) = \sum_{\ell=1}^{3n^2-m} d_\ell (\alpha^\ell_{i,j}, \beta^\ell_{i,j}, \gamma^\ell_{i,j}),$$

for some $d_\ell \in \mathbf{k}$. Thus we obtain

$$\begin{aligned}
h_x \quad := \quad & \sum_{i,j} x_{i,j,\prec,\prec} \binom{e_i^\vee}{\prec} \circ_{\mathrm{I}} \binom{e_j^\vee}{\prec} + \sum_{i,j} y_{i,j,\succ,\prec} \binom{e_i^\vee}{\succ} \circ_{\mathrm{II}} \binom{e_j^\vee}{v} + \sum_{i,j} y_{i,j,\succ,\succ} \binom{e_i^\vee}{\succ} \circ_{\mathrm{II}} \binom{e_j^\vee}{v} + \sum_{i,j} z_{i,j} \binom{e_i^\vee}{\prec} \circ_{\mathrm{III}} \binom{e_j^\vee}{\succ} \\
= \quad & \sum_{i,j} x_{i,j,\prec,\prec} \binom{e_i^\vee}{\prec} \circ_{\mathrm{I}} \binom{e_j^\vee}{\prec} + \sum_{i,j} y_{i,j,\succ,\prec} \left(\binom{e_i^\vee}{\succ} \circ_{\mathrm{II}} \binom{e_j^\vee}{v} + \binom{e_i^\vee}{\succ} \circ_{\mathrm{II}} \binom{e_j^\vee}{v} \right) + \sum_{i,j} z_{i,j} \binom{e_i^\vee}{\prec} \circ_{\mathrm{III}} \binom{e_j^\vee}{\succ} \\
= \quad & \sum_{\ell=1}^{3n^2-m} d_\ell \left(\sum_{i,j} \alpha^\ell_{i,j} \binom{e_i^\vee}{\prec} \circ_{\mathrm{I}} \binom{e_j^\vee}{\prec} + \sum_{i,j} \beta^\ell_{i,j} \left(\binom{e_i^\vee}{\succ} \circ_{\mathrm{II}} \binom{e_j^\vee}{\prec} + \binom{e_i^\vee}{\succ} \circ_{\mathrm{II}} \binom{e_j^\vee}{\succ} \right) + \sum_{i,j} \gamma^\ell_{i,j} \binom{e_i^\vee}{\prec} \circ_{\mathrm{III}} \binom{e_j^\vee}{\succ} \right).
\end{aligned}$$

This is in $\sum\limits_{\ell=1}^{3n^2-m} \mathbf{k}\mathrm{BSu}_x(g_\ell)$. By the same argument, we find that

$$h_y := \sum_{i,j} x_{i,j,\prec,\succ} \binom{e_i^\vee}{\prec} \circ_{\mathrm{I}} \binom{e_j^\vee}{\succ} + \sum_{i,j} y_{i,j,\prec,\prec} \binom{e_i^\vee}{\prec} \circ_{\mathrm{II}} \binom{e_j^\vee}{\prec} + \sum_{i,j,v} z_{i,j,\succ,\prec} \binom{e_i^\vee}{\succ} \circ_{\mathrm{III}} \binom{e_j^\vee}{\prec} + \sum_{i,j,\succ,\succ} z_{i,j,\succ,\succ} \binom{e_i^\vee}{\succ} \circ_{\mathrm{III}} \binom{e_j^\vee}{\succ}$$

is in $\sum\limits_{\ell=1}^{3n^2-m} \mathbf{k}\mathrm{BSu}_y(g_\ell)$ and

$$h_z := \sum_{i,j,\succ,\prec} x_{i,j,\succ,v} \binom{e_i^\vee}{\succ} \circ_{\mathrm{I}} \binom{e_j^\vee}{\prec} + \sum_{i,j,\succ,\succ} x_{i,j,\succ,\succ} \binom{e_i^\vee}{\succ} \circ_{\mathrm{I}} \binom{e_j^\vee}{\succ} + \sum_{i,j} y_{i,j,\prec,\succ} \binom{e_i^\vee}{\prec} \circ_{\mathrm{II}} \binom{e_j^\vee}{\succ} + \sum_{i,j} z_{i,j,\prec,\prec} \binom{e_i^\vee}{\prec} \circ_{\mathrm{III}} \binom{e_j^\vee}{\prec}$$

is in $\sum\limits_{\ell=1}^{3n^2-m} \mathbf{k}\mathrm{BSu}_z(g_\ell)$. Note that $h = h_x + h_y + h_z$. Thus in summary, we find that h is in

$$\sum_\ell \mathbf{k}\mathrm{BSu}_x(g_\ell) + \sum_\ell \mathbf{k}\mathrm{BSu}_y(g_\ell) + \sum_\ell \mathbf{k}\mathrm{BSu}_z(g_\ell)$$

and hence is in the $\mathbb{S}$-module generated by $\mathrm{BSu}(R^\perp)$. Thus we have the equality $(\mathrm{Du}(R)^\perp) = (\mathrm{BSu}(R^\perp))$ of $\mathbb{S}$-modules. Therefore

$$\mathrm{Du}(\mathcal{P})^! = \mathrm{BSu}(\mathcal{P}^!) \text{ and } \mathrm{Du}(\mathcal{P}) = \mathrm{BSu}(\mathcal{P}^!)^!.$$

To prove the "only if" part, suppose $R = 0$. Then we have $\mathrm{Du}(R) = 0 \subseteq \mathcal{T}(\mathrm{Du}(V))$ and hence $\mathrm{Du}(R)^\perp = \mathcal{T}(\mathrm{BSu}(V^\vee))(3)$. On the other hand, $R^\perp = \mathcal{T}(V^\vee)(3)$ which has a basis $e_i^\vee \circ_u e_j^\vee, 1 \leq i, j \leq n, u \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$. Then a linear spanning set of $\mathrm{BSu}(\mathcal{T}(V^\vee)(3))$ is given by $\mathrm{BSu}_v(e_i^\vee \circ_u e_j^\vee), 1 \leq i, j \leq n, v \in \{x, y, z\}, u \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$ in Eq. (3.2.6). Thus the dimension of $\mathrm{BSu}(\mathcal{T}(V^\vee)(3))$ is at most $9n^2$, while the dimension of $\mathcal{T}(\mathrm{BSu}(V^\vee))(3)$ is

$$3\dim(\mathrm{BSu}(V^\vee)^{\otimes 2}) = 3(2n)^2 = 12n^2.$$

Thus $\mathrm{BSu}(\mathcal{T}(V^\vee)(3))$ is a proper subspace of $\mathcal{T}(\mathrm{BSu}(V^\vee))(3)$ and thus $\mathrm{Du}(\mathcal{P})^! \neq \mathrm{BSu}(\mathcal{P}^!)$. □

Theorem 3.2.4. *Let* $\mathbf{k}$ *be an infinite field. Let* $\mathcal{P} = \mathcal{T}(V)/(R)$ *be a binary quadratic operad. Then*

$$\mathrm{Tri}(\mathcal{P})^! = \mathrm{TSu}(\mathcal{P}^!)$$

if and only if $R \neq 0$.

Proof. The proof is similar to Theorem 3.2.3. □

Taking $\mathcal{P}$ to be the operad of associative algebra in Theorem 3.2.4, we get the result of Loday and Ronco (75, theorem 3.1) that the triassociative algebra and the tridendriform algebra are in Koszul dual to each other.

More generally, Theorem 3.2.3 and Theorem 3.2.4 make it straightforward to compute the generating and relation spaces of the Koszul duals of the operads of some existing algebras. We give the following examples as illustrations.

(a) The operad $DualCTD$ (97) is defined to be the Koszul dual of the operad CTD of the commutative tridendriform algebra. Since the latter operad is $\mathrm{TSu}(Comm)$ (9), we have

$$DualCTD = \mathrm{TSu}(Comm)^! = \mathrm{Tri}(Comm^!) = \mathrm{Tri}(Lie),$$

which is precise is *TriLeib*, the operad of the triLeibniz algebra in Proposition 3.1.17. Thus we easily obtain the relations of $DualCTD$. See Eq. (3.1.10)

(b) The operad of the commutative quadri-algebra is the Kozul dual of $\mathrm{BSu}(Zinb)$ and hence is $\mathrm{Du}(Leib)$. Thus its relations can be easily computed.

(c) The Kozul dual of $\mathrm{BSu}(PreLie)$, the operad of the L-dendriform algebra, is $\mathrm{Du}(Perm)$ and hence can be easily computed.

(d) The operad *L-quad* (10) of the L-quadri-algebra is shown to be $\mathrm{BSu}(L\text{-}dend) = \mathrm{BSu}(\mathrm{BSu}(Lie))$ in (9). Thus the dual of *L-quad* is $\mathrm{Du}(\mathrm{Du}(Perm))$ and can be easily computed.

3.2.2 Replicators and Manin white products

As a preparation for later discussions, we recall concepts and notations on Manin white product, most following (93).

Ginzburg and Kapranov defined in (43) a morphism of operads $\Phi : \mathcal{T}(V \otimes W) \rightarrowtail \mathcal{T}(V) \otimes \mathcal{T}(W)$. Let $\mathcal{P} = \mathcal{T}(V)/(R)$ and $\mathcal{Q} = \mathcal{T}(W)/(S)$ be two binary quadratic operads with finite-dimensional generating spaces. Consider the composition of morphisms of operads

$$\mathcal{T}(V \otimes W) \xrightarrow{\Phi} \mathcal{T}(V) \otimes \mathcal{T}(W) \xrightarrow{\pi_{\mathcal{P}} \otimes \pi_{\mathcal{Q}}} \mathcal{P} \otimes \mathcal{Q},$$

where $\pi_{\mathcal{P}} : \mathcal{T}(V) \to \mathcal{P}$ and $\pi_{\mathcal{Q}} : \mathcal{T}(W) \to \mathcal{Q}$ are the natural projections. Its kernel is $(\Phi^{-1}(R \otimes \mathcal{T}(W) + \mathcal{T}(V) \otimes S))$, the ideal generated by $\Phi^{-1}(R \otimes \mathcal{T}(W) + \mathcal{T}(V) \otimes S)$.

Definition 3.2.2. *((43; 93))* Let $\mathcal{P} = \mathcal{T}(V)/(R)$ and $\mathcal{Q} = \mathcal{T}(W)/(S)$ be two binary quadratic operads with finite-dimensional generating spaces. The **Manin white product** of $\mathcal{P}$ and $\mathcal{Q}$ is defined by

$$\mathcal{P} \bigcirc \mathcal{Q} := \mathcal{T}(V \otimes W)/(\Phi^{-1}(R \otimes \mathcal{T}(W) + \mathcal{T}(V) \otimes S)).$$

In general, the white Manin product difficult to compute when the operads are given in terms of generators and relations. Theorem 3.2.5 provides a convenient way to compute the white Manin product of a binary quadratic operad with the operad *Perm* or *ComTrias* by relating them to the duplicator and triplicator.

Theorem 3.2.5. *Let $\mathcal{P} = \mathcal{T}(V)/(R)$ be a binary quadratic operad with $R \neq 0$. We have the isomorphism of operads*

$$\mathrm{Du}(\mathcal{P}) \cong \mathit{Perm} \bigcirc \mathcal{P}, \quad \mathrm{Tri}(\mathcal{P}) \cong \mathit{ComTrias} \bigcirc \mathcal{P}.$$

Proof. By (9), we have the isomorphisms of operads

$$\mathrm{BSu}(\mathcal{P}^!) \cong \mathit{PreLie} \bullet \mathcal{P}^!, \quad \mathrm{TSu}(\mathcal{P}^!) \cong \mathit{PostLie} \bullet \mathcal{P}^!.$$

Since $\mathit{PreLie}^! \cong \mathit{Perm}$, $\mathit{PostLie}^! = \mathit{ComTrias}$ and $(\mathcal{P} \bullet \mathcal{Q})^! \cong \mathcal{P}^! \bigcirc \mathcal{Q}^!$, we obtain

$$\mathrm{Du}(\mathcal{P}) \cong (\mathrm{BSu}(P^!))^! \cong (\mathit{PreLie} \bullet \mathcal{P}^!)^! \cong \mathit{Perm} \bigcirc \mathcal{P}.$$

Similarly $\mathrm{Tri}(\mathcal{P}) \cong \mathit{ComTrias} \bigcirc \mathcal{P}$. □

By taking replicators of suitable operads $\mathcal{P}$, we immediately get

Corollary 3.2.6. (a) *((93)) $\mathit{Perm} \bigcirc \mathit{Lie} = \mathit{Leib}$ and $\mathit{Perm} \bigcirc \mathit{Ass} = \mathit{Diass}$.*

(b) *((92)) $\mathit{Perm} \bigcirc \mathit{Pois} = \mathit{DualPrePois}$.*

(c) *$\mathit{ComTriass} \bigcirc \mathit{Ass} = \mathit{Triass}$.*

By a similar argument as for Theorem 3.2.5 we obtain

Proposition 3.2.3. *Let $\mathcal{P} = \mathcal{T}_{ns}(V)/(R)$ be a binary quadratic nonsymmetric operad with $R \neq 0$. There is an isomorphism of nonsymmetric operads*

$$\mathrm{Du}(\mathcal{P}) \cong \mathit{Dias} \;\square\; \mathcal{P}\ , \quad \mathrm{Tri}(\mathcal{P}) \cong \mathit{Trias} \;\square\; \mathcal{P}\ ,$$

where $\square$ denotes the white square product (93) while Dias and Trias denote the nonsymmetric operads for the diassociative and triassociative algebras.

3.3 Replicators and average operators on operads

In this section we establish the relationship between the duplicator and triplicator of an operad on one hand and the actions of the di-average and tri-average operators on the operad on the other hand. We will work with symmetric operads, but all the results also hold for nonsymmetric operads.

3.3.1 Duplicators and di-average operators

Averaging operators have been studied for associative algebras since 1960 by Rota and for other algebraic structures more recently (1; 22; 89; 92).

Definition 3.3.1. Let $(A, \cdot)$ be a $\mathbf{k}$-module A with a binary operation $\cdot$.

(a) A **di-average operator** on A is a $\mathbf{k}$-linear map $P : A \longrightarrow A$ such that

$$P(x \cdot P(y)) \;=\; P(x) \cdot P(y) = P(P(x) \cdot y), \quad \text{for all } x, y \in A. \tag{3.3.1}$$

(b) Let $\lambda \in \mathbf{k}$. A tri-average operator of weight λ on A is a $\mathbf{k}$-linear map $P : A \longrightarrow A$ such that Eq. (3.3.1) holds and

$$P(x) \cdot P(y) \;=\; \lambda P(xy), \quad \text{for all } x, y \in A. \tag{3.3.2}$$

We note that a tri-average operator of weight zero is not a di-average operator. So we cannot give a uniform definition of the average operators as in the case of Rota-Baxter algebras of weight λ. We next consider the operation of average operators on the level of operads.

Definition 3.3.2. Let $V = V(2)$ be an $\mathbb{S}$-module concentrated in arity 2.

(a) Let V_P denote the $\mathbb{S}$-module concentrated in arity 1 and arity 2 with $V_P(2) = V$ and $V_P(1) = \mathbf{k}\,P$, where P is a symbol. Let $\mathcal{T}(V_P)$ be the free operad generated by binary operations V and an unary operation $P \neq \mathrm{id}$.

(b) Define $\mathrm{Du}(V) = V \otimes (\mathbf{k} \dashv \oplus \mathbf{k} \vdash)$ as in Eq. (3.1.1), regarded as an $\mathbb{S}$-module concentrated in arity 2. Define a linear map of graded vector spaces from $\mathrm{Du}(V)$ to V_P by the following correspondence:

$$\xi : \quad \binom{\omega}{\dashv} \mapsto \omega \circ (\mathrm{id} \otimes P), \quad \binom{\omega}{\vdash} \mapsto \omega \circ (P \otimes \mathrm{id}), \text{ for all } \omega \in V,$$

where $\circ$ is the operadic composition. By the universality of the free operad, ξ induces a homomorphism of operads that we still denote by ξ:

$$\xi : \mathcal{T}(\mathrm{Du}(V)) \to \mathcal{T}(V_P).$$

(c) Let $\mathcal{P} = \mathcal{T}(V)/(R_{\mathcal{P}})$ be a binary operad defined by generating operations V and relations $R_{\mathcal{P}}$. Let

$$\mathrm{DA}_{\mathcal{P}} := \{\omega \circ (P \otimes P) - P \circ \omega \circ (P \otimes \mathrm{id}), \omega \circ (P \otimes P) - P \circ \omega \circ (\mathrm{id} \otimes P) \mid \omega \in V\}.$$

Define the **operad of di-average $\mathcal{P}$-algebras** by

$$\mathrm{DA}(\mathcal{P}) := \mathcal{T}(V_P)/(\mathrm{R}_{\mathcal{P}}, \mathrm{D}A_{\mathcal{P}}).$$

Let $p_1 : \mathcal{T}(V_P) \to \mathrm{DA}(\mathcal{P})$ denote the operadic projection.

Theorem 3.3.1. (a) *Let $\mathcal{P}$ be a binary operad. There is a morphism of operads*

$$\mathrm{Du}(\mathcal{P}) \longrightarrow \mathrm{DA}(\mathcal{P}),$$

which extends the map ξ given in Definition 3.3.2.

(b) *Let A be a $\mathcal{P}$-algebra. Let $P : A \to A$ be a di-average operator. Then the following operations* *make A into a* Du($\mathcal{P}$)*-algebra:*

$$x \dashv_j y := x \circ_j P(y), \quad x \vdash_j y := P(x) \circ_j y, \quad \forall \circ_j \in \mathcal{P}(2), \text{ for all } x, y \in A.$$

The proof is parallel to the case of triplicators in Theorem 3.3.4 for which we will prove in full detail.

When we take $\mathcal{P}$ be the operad of the associative algebra, Lie algebra, or Poisson algebra, we obtain the following results of Aguiar (1).

Corollary 3.3.2. (a) *Let $(A, \cdot)$ be an associative algebra and $P : A \longrightarrow A$ be a di-averaging operator. Define two new operations on A by $x \vdash y = P(x) \cdot y$ and $x \dashv y = x \cdot P(y)$. Then $(A, \vdash, \dashv)$ is an associative dialgebra.*

(b) *Let $(A, [,])$ be a Lie algebra and $P : A \longrightarrow A$ be a di-averaging operator. Define a new operation on A by $\{x, y\} = [P(x), y]$. Then $(A, \{,\})$ is a left Leibniz algebra.*

(c) *Let $(A, \cdot, [,])$ be a Poisson algebra and let $P : A \to A$ be a di-averaging operator. Define two new products on A by $x \circ y := P(x) \cdot y$, and $\{x, y\} := [P(x), y]$. Then $(A, \circ, \{,\})$ is a dual left prePoisson algebra.*

Combining Theorem 3.3.1 with Theorem 3.2.5, we obtain the following relation between the Manin white product and the action of the di-average operator. It can be regarded as the interpretation of (92, Theorem 3.2) at the level of operads.

Proposition 3.3.3. *For any binary quadratic operad $\mathcal{P} = \mathcal{T}(V)/(R)$, there is a morphism of operads*

$$Perm \bigcirc \mathcal{P} \longrightarrow \mathrm{DA}(\mathcal{P}),$$

defined by the following map:

$$\begin{aligned} Perm(2) \bigcirc \mathcal{P}(2) &\longrightarrow \mathrm{DA}(\mathcal{P}), \\ \mu \otimes \omega &\longmapsto \omega \circ (id \otimes P), \\ \mu' \otimes \omega &\longmapsto \omega \circ (P \otimes id), \quad \omega \in \mathcal{P}(2), \end{aligned}$$

where μ denotes the generating operation of the operad Perm.

3.3.2 Triplicators and tri-average operators

In this section, we establish the relationship between the triplicator of an operad and the action of the tri-average operator with a nonzero weight on the operad. For simplicity, we assume that the weight of the tri-average operator is one.

Definition 3.3.4. Let $V = V(2), V_P$ and $\mathcal{T}(V_P)$ as defined in Definition 3.3.2.

(a) Let $\mathrm{Tri}(V) = V \otimes (\mathbf{k} \dashv \oplus \mathbf{k} \vdash \oplus \mathbf{k} \perp)$ in Eq. (3.1.4), seen as an $\mathbb{S}$-module concentrated in arity 2. Define a linear map of graded vector spaces from $\mathrm{Tri}(V)$ to V_P by the correspondence

$$\eta: \quad \binom{\omega}{\dashv} \mapsto \omega \circ (\mathrm{id} \otimes P), \quad \binom{\omega}{\vdash} \mapsto \omega \circ (P \otimes \mathrm{id}), \quad \binom{\omega}{\cdot} \mapsto \omega\,,$$

where $\circ$ is the operadic composition. By the universality of the free operad, η induces a homomorphism of operads:

$$\eta: \mathcal{T}(\mathrm{Tri}(V)) \to \mathcal{T}(V_P).$$

(b) Let $\mathcal{P} = \mathcal{T}(V)/(R_{\mathcal{P}})$ be a binary operad defined by generating operations V and relations $R_{\mathcal{P}}$. Let

$$\begin{aligned} \mathrm{T}A_{\mathcal{P}} \quad := \quad & \{\omega \circ (P \otimes P) - P \circ \omega \circ (P \otimes \mathrm{id}), \omega \circ (P \otimes P) - P \circ \omega \circ (\mathrm{id} \otimes P), \\ & \omega \circ (P \otimes P) - P \circ \omega \mid \omega \in V\}. \end{aligned}$$

Define the **operad of tri-average $\mathcal{P}$-algebras of weight one** by

$$\mathrm{T}A(\mathcal{P}) := \mathcal{T}(V_P)/(\mathrm{R}_{\mathcal{P}}, \mathrm{T}A_{\mathcal{P}}).$$

Let $p_1 : \mathcal{T}(V_P) \to \mathrm{T}A(\mathcal{P})$ denote the operadic projection.

We first prove a lemma relating triplicators and tri-average operators.

Lemma 3.3.3. *Let $\mathcal{P} = \mathcal{T}(V)/(R_{\mathcal{P}})$ be a binary operad and let $\tau \in \mathfrak{T}(V)$ with $\mathrm{Lin}(\tau)$.*

(a) *For each $\bar{\tau} \in \mathrm{Tri}(\tau)$, we have*

$$P \circ \eta(\ \bar{\tau}\ ) \equiv \tau \circ P^{\otimes n} \quad \mathrm{mod}\ (\mathrm{R}_{\mathcal{P}}, \mathrm{T}A_{\mathcal{P}}). \tag{3.3.3}$$

(b) *For $\varnothing \neq J \subseteq \mathrm{L}in(\tau)$, let $P^{\otimes n,J}$ denote the n-th tensor power of P but with the component from J replaced by the identity map. So, for example, for the two inputs x_1 and x_2 of $P^{\otimes 2}$, we have $P^{\otimes 2,\{x_1\}} = P \otimes \mathrm{id}$ and $P^{\otimes 2,\{x_1,x_2\}} = \mathrm{id} \otimes \mathrm{id}$. Then for each $\bar{\tau}_J \in \mathrm{Tri}_J(\tau)$, we have*

$$\eta(\bar{\tau}_J) \equiv \tau \circ (P^{\otimes n,J}) \mod (\mathrm{R}_{\mathcal{P}}, \mathrm{T}A_{\mathcal{P}})\,. \tag{3.3.4}$$

Proof. (a). We prove by induction on $|\mathrm{L}in(\tau)| \geq 1$. When $|\mathrm{L}in(\tau)| = 1$, τ is the tree with one leaf standing for the identity map. Then we have $\eta(\ \mathrm{Tri}(\tau)\ ) = \tau$, $P \circ \eta(\ \mathrm{Tri}(\tau)\ ) = P = \tau \circ P$. Assume the claim has been proved for τ with $|\mathrm{L}in(\tau)| = k$ and consider a τ with $|\mathrm{L}in(\tau)| = k+1$. Then from the decomposition $\tau = \tau_\ell \vee_\omega \tau_r$, we have $\mathrm{Tri}(\tau) = \mathrm{Tri}(\tau_\ell) \vee_{\binom{\omega}{\dagger}} \mathrm{Tri}(\tau_r)$. Recall that $\mathrm{Tri}(\tau)$ is a set of labeled trees. For each $\bar{\tau} \in \mathrm{Tri}(\tau)$, there exist $\bar{\tau}_\ell \in \mathrm{Tri}(\tau)_\ell$ and $\bar{\tau}_r \in \mathrm{Tri}(\tau_r)$ such that

$$\bar{\tau} \in \left\{ \bar{\tau}_\ell \vee_{\binom{\omega}{\vdash}} \bar{\tau}_r, \bar{\tau}_\ell \vee_{\binom{\omega}{\dashv}} \bar{\tau}_r, \bar{\tau}_\ell \vee_{\binom{\omega}{\cdot}} \bar{\tau}_r \right\}.$$

If $\bar{\tau} = \bar{\tau}_\ell \vee_{\binom{\omega}{\vdash}} \bar{\tau}_r$, then we have

$$\begin{aligned}
P \circ \eta(\bar{\tau}) &= P \circ \eta(\bar{\tau}_\ell \vee_{\binom{\omega}{\vdash}} \bar{\tau}_r) \\
&= P \circ \omega \circ ((P \circ \eta(\bar{\tau}_\ell)) \otimes \eta(\bar{\tau}_r)) \\
&\equiv \omega \circ ((P \circ \eta(\bar{\tau}_\ell)) \otimes (P \circ \eta(\bar{\tau}_r))) \mod (\mathrm{R}_{\mathcal{P}}, \mathrm{T}A_{\mathcal{P}}) \\
&= \omega \circ ((\bar{\tau}_\ell \circ P^{\otimes |\mathrm{L}in(\tau_\ell)|}) \otimes (\bar{\tau}_r \circ P^{\otimes |\mathrm{L}in(\tau_r)|})) \quad \text{(by induction hypothesis)} \\
&= \omega \circ (\bar{\tau}_\ell \otimes \bar{\tau}_r) \circ P^{\otimes (k+1)} \\
&= (\bar{\tau}_\ell \vee_{\binom{\omega}{\vdash}} \bar{\tau}_r) \circ P^{\otimes (k+1)} \\
&= \bar{\tau} \circ P^{\otimes (k+1)}.
\end{aligned}$$

Similarly, we have

$$P \circ \eta(\ \bar{\tau}_\ell \vee_{\binom{\omega}{\dashv}} \bar{\tau}_r\ ) \equiv \bar{\tau} \circ P^{\otimes(k+1)} \quad \operatorname{mod}\ (\mathrm{R}_{\mathcal{P}}, \mathrm{T}A_{\mathcal{P}}),$$
$$P \circ \eta(\ \bar{\tau}_\ell \vee_{\binom{\omega}{\cdot}} \bar{\tau}_r\ ) \equiv \bar{\tau} \circ P^{\otimes(k+1)} \quad \operatorname{mod}\ (\mathrm{R}_{\mathcal{P}}, \mathrm{T}A_{\mathcal{P}}).$$

(b). We again prove by induction on $|\mathrm{L}in(\tau)|$. When $|\mathrm{L}in(\tau)| = 1$, then x is the only leaf label of τ and $|\mathrm{Tri}_x(\tau)| = 1$. Thus we have

$$\eta(\bar{\tau}_x) = \eta(x) = x = \tau \circ (P^{\otimes 1, x}).$$

Assume that the claim has been proved for all τ with $|\mathrm{L}in(\tau)| = k$ and consider τ with $|\mathrm{L}in(\tau)| = k+1$. Write $\tau = \tau_\ell \vee_\omega \tau_r$. Let J be a nonempty subset of $\mathrm{L}in(\tau)$. If $J \subseteq \mathrm{L}in(\tau_\ell)$, then by the definition of $\mathrm{Tri}_J(\tau)$, for each $\bar{\tau}_J \in \mathrm{Tri}_J(\tau)$, there exist $\bar{\tau}_{J,\ell} \in \mathrm{Tri}_J \tau_\ell$ and $\bar{\tau}_{J,r} \in \mathrm{Tri}_\varnothing \tau_r$ such that $\bar{\tau}_J = \bar{\tau}_{J,\ell} \vee_{\binom{\omega}{\dashv}} \bar{\tau}_{J,r}$. Then we have

$$\begin{aligned}
\eta(\bar{\tau}_J) &= \eta(\bar{\tau}_{J,\ell} \vee_{\binom{\omega}{\dashv}} \bar{\tau}_{J,r}) \\
&= \omega \circ (\eta(\bar{\tau}_{J,\ell} \otimes P \circ \eta(\bar{\tau}_{J,r})) \\
&\equiv \omega \circ \Big((\tau_\ell \circ P^{\otimes |\mathrm{L}in(\tau_\ell)|, J}) \otimes (\tau_r \circ P^{\otimes |\mathrm{L}in(\tau_r)|}) \Big) \quad \operatorname{mod}\ (\mathrm{R}_{\mathcal{P}}, \mathrm{T}A_{\mathcal{P}}) \\
&\qquad \text{(by induction hypothesis and Item (a))} \\
&= \tau \circ P^{\otimes(k+1), J}.
\end{aligned}$$

When $J \subseteq \mathrm{L}in(\tau_r)$, the proof is the same. When $J \not\subseteq \mathrm{L}in(\tau_\ell)$ and $J \not\subseteq \mathrm{L}in(\tau_r)$, for each $\bar{\tau}_J \in \mathrm{Tri}_J(\tau)$, there exist $\bar{\tau}_{J,\ell} \in \mathrm{Tri}_{J \cap \mathrm{L}in(\tau_\ell)} \tau_\ell$ and $\bar{\tau}_{J,r} \in \mathrm{Tri}_{J \cap \mathrm{L}in(\tau_r)} \tau_r$ such that $\bar{\tau}_J = \bar{\tau}_{J,\ell} \vee_{\binom{\omega}{\cdot}} \bar{\tau}_{J,r}$. Then by the same argument we have

$$\begin{aligned}
\eta(\bar{\tau}_J) &\equiv \omega \circ \Big((\tau_\ell \circ P^{\otimes |\mathrm{L}in(\tau_\ell)|, J \cap \mathrm{L}in(\tau_\ell)}) \otimes (\tau_r \circ P^{\otimes |\mathrm{L}in(\tau_r)|, J \cap \mathrm{L}in(\tau_r)}) \Big) \quad \operatorname{mod}\ (\mathrm{R}_{\mathcal{P}}, \mathrm{T}A_{\mathcal{P}}) \\
&= \tau \circ P^{\otimes(k+1), J}.
\end{aligned}$$

This completes the induction.

□

Theorem 3.3.4. *Let $\mathcal{P}$ be a binary operad.*

(a) *There is a morphism of operads*

$$\mathrm{Tri}(\mathcal{P}) \longrightarrow \mathrm{T}A(\mathcal{P}),$$

which extends the map η given in Definition 3.3.4.

(b) *Let A be a $\mathcal{P}$-algebra. Let $P : A \longrightarrow A$ be a tri-average operator of weight one. Then the following operations make A into a* $\mathrm{Tri}(\mathcal{P})$*-algebra:*

$$x \dashv_j y = x \circ_j P(y), \quad x \vdash_j y = P(x) \circ_j y, \quad x \cdot_j y = x \circ_j y, \quad \textit{for all } \circ_j \in \mathcal{P}(2).$$

Proof. The second statement is just the interpretation of the first statement on the level of algebras. So we just need to prove the first statement. Let $R_{\mathrm{Tri}(\mathcal{P})}$ be the relation space of $\mathrm{Tri}(\mathcal{P})$. By definition, the relations of $\mathrm{Tri}(\mathcal{P})$ are generated by $\mathrm{Tri}_J(r)$ for locally homogeneous $r = \sum_i c_i \tau_i \in R_{\mathcal{P}}$, where $\varnothing \neq J \subseteq \mathrm{L}in(\tau_i)$. By Eqs.(3.3.3) and (3.3.4), we have

$$\eta\left(\sum_i c_i (\bar{\tau_i})_J\right) = \sum_i c_i \eta((\bar{\tau_i})_J) \equiv \sum_i c_i \tau_i \circ P^{\otimes n, J} \equiv \left(\sum_i c_i \tau_i\right) \circ P^{\otimes n, J} \mod (\mathrm{R}_{\mathcal{P}}, \mathrm{T}A_{\mathcal{P}}).$$

Thus $\eta(R_{\mathrm{Tri}(\mathcal{P})}) \subseteq (\mathrm{R}_{\mathcal{P}}, \mathrm{T}A_{\mathcal{P}})$ and η induces a morphism of operads

$$\bar{\eta} : \mathrm{Tri}(\mathcal{P}) \longrightarrow \mathrm{T}A(\mathcal{P}).$$

This proves the first statement. ☐

Corollary 3.3.5. (a) *Let A be an associative algebra and let $P : A \longrightarrow A$ be a tri-average operator on A. Then the new operations defined in Theorem 3.3.4(b) makes it into an associative trialgebra.*

(b) *Let L be a Lie algebra and let $P : L \longrightarrow L$ be a tri-average operator on L. Then the operations defined in Theorem 3.3.4(b) make it into a triLeibniz algebra.*

Bibliography

[1] M. Aguiar, *Pre-Poisson algebras*, Lett. Math. Phys. **54** (2000) 263-277.

[2] M. Aguiar and J.-L. Loday, *Quadri-algebras*, J. Pure Appl. Algebra **191** (2004) 205-221.

[3] A.G. Aleksandrov, *Nonisolated hypersurface singularities*, in "Theory of Singularities and its Applications" (V.I. Arnold, Ed.) Advances in Soviet Mathematics, Vol. 1, American Mathematical Society, 1990, pp. 211–246.

[4] A.G. Aleksandrov, *Multidimensional residue theory and the logarithmic de Rham complex*, J. of Singularities, Vol. 5 (2012) 1–18.

[5] P. Aluffi, *Grothendieck classes and Chern classes of hyperplane arrangements*, Int. Math. Res. Not. IMRN (2013) no. 8, 1873?1900.

[6] P. Aluffi, M. Marcolli, *Parametric Feynman integrals and determinant hypersurfaces*, Adv. Theor. Math. Phys. 14 (2010), no. 3, 911–963.

[7] Y. André, *Une introduction aux motifs*, Société Mathématique de France, 2004.

[8] Y. André, *Motifs de dimension finie*, Séminaire Bourbaki, 2003-04, Exp. N. 929 viii, Astérisque N.299 (2005) 115–145.

[9] C. Bai, O. Bellier, L. Guo and X. Ni, *Splitting of operations, Manin products and Rota-Baxter operators*, IMRN **3** (2013) 485-524.

[10] C. Bai, L. Liu and X. Ni, *Some results on L-dendriform algebras*, J. Geom. Phys. **60** (2010) 940-950.

[11] A.A. Beilinson, A.B. Goncharov, V.V. Schechtman, A.N. Varchenko, *Aomoto dilogarithms, mixed Hodge structures and motivic cohomology of pairs of triangles on the plane*, in "The Grothendieck Festschrift", Vol. I, pp.135–172, Progr. Math., Vol.86, Birkhäuser, 1990.

[12] P. Belkale, P. Brosnan, *Matroids, motives, and a conjecture of Kontsevich*, Duke Math. J. 116 (2003) No. 1, 147–188.

[13] A. Bialynicki-Birula, *Some theorems on actions of algebraic groups*, Annals of Math. 98 (1973), 480–497.

[14] S. Bloch, H. Esnault, D. Kreimer, *On motives associated to graph polynomials*, Comm. Math. Phys. 267 (2006), No. 1, 181–225.

[15] S. Bloch, D. Kreimer, *Mixed Hodge structures and renormalization in physics*, Commun. Number Theory Phys. Vol.2 (2008) No.4, 637–718.

[16] M. Brion, *Spherical varieties*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zurich, 1994), 753–760, Birkhauser, Basel, 1995.

[17] M. Brion, D. Luna, *Sur la structure locale des variétés sphériques*, Bull. Soc. Math. France 115 (1987) 211–226.

[18] D. Broadhurst, D. Kreimer, *Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops*, Phys. Lett. B 393 (1997) No.3–4, 403–412.

[19] F. Brown, *Mixed Tate motives over* $\mathbb{Z}$, Ann. of Math. (2) 175 (2012), No. 2, 949–976.

[20] F. Brown, D. Doryn, *Framings for graph hypersurfaces*, arXiv:1301.3056.

[21] F. Brown, O. Schnetz, *A K3 in* ϕ^4, Duke Math. J. 161 (2012), No. 10, 1817–1862.

[22] W. Cao, *An algebraic study of averaging operators*, Ph.D. Thesis, Rutgers University at Newark, 2000.

[23] F.J. Castro-Jiménez, L. Narváez-Macarro, D. Mond, *Cohomology of the complement of a free divisor*, Trans. Amer. Math. Soc., Vol.348 (1996) N.8, 3037–3049.

[24] O. Ceyhan, M. Marcolli, *Algebraic renormalization and Feynman integrals in configuration spaces*, arXiv:1308.5687.

[25] F. Chapoton, *Un endofoncteur de la catégorie des opérades*, in: Dialgebras and related operads, Lect. Notes Math. **1763**, Springer-Verlag, 2001.

[26] A. Connes, D. Kreimer, *Renormalization in quantum field theory and the Riemann-Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem*, Comm. Math. Phys. 210 (2000), no. 1, 249–273.

[27] A. Connes, M. Marcolli, *Noncommutative Geometry, quantum fields and motives*, Colloquium Publications, Vol.55, American Mathematical Society, 2008.

[28] C. De Concini, C. Procesi, *Complete symmetric varieties*, in "Invariant theory" (Montecatini, 1982), pp. 1–44, Lecture Notes in Math. 996, Springer, 1983.

[29] P. Deligne, *Théorie de Hodge. II*, Inst. Hautes Études Sci. Publ. Math. No. 40 (1971), 5–57.

[30] P. Deligne, *Equations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, 1970.

[31] P. Deligne, A. Dimca, *Filtrations de Hodge et par l'ordre du pôle pour les hypersurfaces singulières*, Ann. Sci. École Norm. Sup. (4) 23 (1990), no. 4, 645–656.

[32] P. Deligne, A.B. Goncharov, *Groupes fondamentaux motiviques de Tate mixte*, Ann. Scient. Éc. Norm. Sup., Vol.38 (2005) 1–56.

[33] D. Doryn, *On one example and one counterexample in counting rational points on graph hypersurfaces*, Lett. Math. Phys. 97 (2011), no. 3, 303–315.

[34] C. Dupont, *Periods of hyperplane arrangements and motivic coproduct*, Ph.D. Thesis, Université Pierre et Marie Curie - Paris VI, 2014. `https://tel.archives-ouvertes.fr/tel-01083524/document`

[35] K. Ebrahimi-Fard and L. Guo, *On products and duality of binary, quadratic, regular operads*, J. Pure. Appl. Algebra, **200**, (2005), 293-317.

[36] K. Ebrahimi-Fard and L. Guo, *Unit actions on operads and Hopf algebras*, Theory and Applications of Categories **18** (2007), 348-371.

[37] K. Ebrahimi-Fard, L. Guo, D. Kreimer, *Spitzer's Identity and the Algebriac Birkhoff Decomposition in pQFT*, J. Phys. A: Math. Gen., 37, (2004) 11037-11052.

[38] K. Ebrahimi-Fard, L. Guo, D. Kreimer, *Integrable renormalization. II. The general case*, Ann. Henri Poincaré 6 (2005), no. 2, 369–395.

[39] R. Felipe, *A brief foundation of the left-symmetric dialgebras*, Comunicación del CIMAT, NO I, (2011).

[40] L. Foissy, *Les algebres de Hopf des arbes enracines decores* II, Bull. Sci. Math., 126, (2002), 249-288.

[41] A. Frabetti, *Dialgebra homology of associative algebras*, C. R. Acad. Sci. Paris, 325, (1997), 135-140.

[42] A. Frabetti, *Leibniz homology of dialgebras of matrices*, J. Pure Appl. Alg., 129, (1998), 121-141.

[43] V. Ginzburg and M. M. Kapranov, *Koszul duality for operads*, Duke Math. J. **76** (1995) 203-272.

[44] A.B. Goncharov, *Multiple polylogarithms and mixed Tate motives*, arXiv:math/0103059.

[45] A.B. Goncharov, Yu.I. Manin, *Multiple ζ-motives and moduli spaces $\overline{\mathcal{M}}_{0,n}$*, Compositio Math., Vol.140 (2004) 1–14.

[46] P. Griffiths, W. Schmid, *Recent developments in Hodge theory: a discussion of techniques and results*, in "Discrete subgroups of Lie groups and applicatons to moduli" (Internat. Colloq., Bombay, 1973), pp. 31–127. Oxford Univ. Press, Bombay, 1975.

[47] A. Grothendieck, *On the de Rham cohomology of algebraic varieties*, Pub. Math. I.H.E.S. 29 (1966) 95–105.

[48] V. Yu. Gubarev and P. S. Kolesnikov, *On embedding of dendriform algebras into Rota-Baxter algebras*, Cent. Eur. Jour. Math **11** (2013), 226-245. arXiv:1107.6021.

[49] L. Guo, *An Introduction to Rota-Baxter Algebra*, International Press and High Education Press, 2012.

[50] L. Guo, *Algebraic Birkhoff decompositiona and its applications*. In: Automorphic Forms and the Langlands Program. International Press, 2008, pp. 283-323.

[51] S. Habibi, *The motive of a bundle*, PhD Thesis, Università degli Studi di Milano, 2012.

[52] S. Habibi, M.E.A. Rad, *The motive of G-bundles*, arXiv:1112.4110v3.

[53] M.P. Holland, D. Mond *Logarithmic differential forms and the cohomology of the complement of a divisor*, Math. Scand. 83 (1998), No.2, 235–254.

[54] M. Huruguen, *Compactification d'espaces homogènes sphériques sur un corps quelconque*, PhD thesis, Université de Grenoble, 2011.

[55] D. Hou, X. Ni and C. Bai, *Pre-Jordan algebras*, Math. Scand. **112** (1). pp. 19-48.

[56] E. Hoffbeck, *A Poincaré-Birkhoff-Witt criterion for Koszul operads*, Manuscripta Math. **131** (2010) 87-110.

[57] R. Holtkamp, *Comparision of Hopf algebras on trees*, Archiv der Mathematik, Vol. 80, (2003), 368-383.

[58] B. Kahn, *Motivic Cohomology of smooth geometrically cellular varieties*, in "Algebraic K-theory, Seattle, 1997", pp. 149–174, Proceedings of Symposia in Pure Mathematics, vol. 67, American Mathematical Society, 1999.

[59] N. Karpenko, *Cohomology of relative cellular spaces and isotropic flag varieties*, St. Petersburg Math. J., Vol.12 (2001) N. 1, 1–50.

[60] I. Kausz, *A modular compactification of the general linear group*, Documenta Math. 5 (2000) 553–594.

[61] M. Kisin, G. Lehrer, *Eigenvalues of Frobenius and Hodge Numbers*, Pure and Applied Math. Quarterly, Vol.2 (2006) N.2, 199–219.

[62] B. Köck, *Chow motif and higher Chow theory of G/P*, Manuscripta Math. 70 (1991), no. 4, 363–372.

[63] P. Kolesnikov, *Varieties of dialgebras and conformal algebras* (Russian), Sib. Mat. Zh. **49** (2008) 323C340.

[64] P. Kolesnikov and V. Yu. Voronin, *On special identities for dialgebras*, arXiv:1201.5179.

[65] P. Leroux, *Construction of Nijenhuis operators and dendriform trialgebras*, Int. J. Math. Sci. (2004) 2595-2615.

[66] P. Leroux, *On some remarkable operads constructed from Baxter operators*, arXiv: math. QA/0311214.

[67] P. Leroux, *Ennea-algebras*, J. Algebra, **281**, (2004), 287-302.

[68] L. Liu, X. Ni and C. Bai, *L-quadri-algebras* (in Chinese), Sci. Sin. Math. **42** (2011) 105-124.

[69] J.-L. Loday, *Une version non commutative des algèbres de Lie: les algèbres de Leibniz*, Enseign. Math. **39** (1993) 269C293.

[70] J.-L. Loday, *Dialgebras*, in Dialgebras and related operads, Lecture Notes in Math. 1763 (2002) 7-66.

[71] J.-L. Loday, *Arithmetree*, J. Algebra **258** (2002), 275-309.

[72] J.-L. Loday, *Completing the operadic butterfly*, Georgian Math. J. **13** (2006)741749.

[73] J.-L. Loday, *On the algebra of quasi-shuffles*, Manuscripta Math. **123** (2007) 79-93.

[74] J.-L. Loday, *Scindement d'associativté et algébres de Hopf*. in the proceedings of conference in honor of Jean Leray, Nantes (2002), Séminaire et Congrés (SMF) **9** (2004), 155-172.

[75] J.-L. Loday and M. Ronco, *Trialgebras and families of polytopes*, in "Homotopy Theory: Relations with Algebraic Geometry, Group Cohomology, and Algebraic K-theory", Comtep. Math. **346** (2004) 369-398.

[76] J.-L. Loday and M. Ronco, *Algébre de Hopf colibres*, C. R. Acad. Sci. Paris **337** (2003), 153-158.

[77] J.-L. Loday and B. Vallette, *Algebraic Operads*, Grundlehren Math. Wiss. **346**, Springer, Heidelberg, 2012.

[78] Yu.I. Manin, *Correspondences, motifs and monoidal transformations*, Mat. Sb. (N.S.) 77 (119) 1968 475–507.

[79] M. Marcolli, *Feynman motives*, World Scientific, 2010.

[80] M. Marcolli, X. Ni, *Rota-Baxter algebras, singular hypersurfaces, and renormalization on Kausz compactifications*, arXiv:1408.3754.

[81] J. Martens, M. Thaddeus, *Compactifications of reductive groups as moduli stacks of bundles*, arXiv:1105.4830.

[82] B. Mohar, C. Thomassen, *Graphs on surfaces*, Johns Hopkins University Press, 2001.

[83] D. Mond, *Notes on logarithmic vector fields, logarithmic differential forms and free divisors*, preprint, 2012.

[84] X. Ni and C. Bai, *Pre-alternative algebras and pre-alternative bialgebras*, Pacific J. Math., **248** (2010) 355-391.

[85] J. Pei, C. Bai, L. Guo and X. Ni, *Replicating of binary operads, Koszul duality, Manin products and average operators*, arXiv:1212.0177.

[86] C. Pompeyo-Gutiérrez, *Chow motive of a locally trivial fibration*, arXiv:1205.4287.

[87] J. Rammer, *Quantum field theory of non-equilibrium states*, Cambridge University Press, 2007.

[88] M. R. Ronco, *Eulerian idempotents and Milnor-Moore theorem for certain non-commutative Hopf algebras*, J. Algebra **254** (2002), 152-172.

[89] G.-C. Rota, *Baxter operators, an intoduction*, in Gian-Carlo Rota on Combinatics: Introductory papers and commentaries (Joseph P.S. Kung, Ed), Birkhäuser, Boston (1995).

[90] K. Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 27 (1980), no. 2, 265–291.

[91] O. Schnetz, *Quantum field theory over* $\mathbb{F}_q$, Electron. J. Comb.18 (2011) N.1, P102.

[92] K. Uchino, *Derived bracket construction and Manin products*, Lett. Math. Phys. **93** (2010) 37-53.

[93] B. Vallette, *Manin products, Koszul duality, Loday algebras and Deligne conjecture*, J. Reine Angew. Math. **620** (2008) 105-164.

[94] I. Vainsencher, *Complete collineations and blowing up determinantal ideals*, Math. Ann. 267 (1984), no. 3, 417–432.

[95] V. Voevodsky, *Triangulated categories of motives over a field*, in "Cycles, transfer and motivic homology theories", Annals of Mathematical Studies, Vol. 143, Princeton Univ. Press, 2000, pp. 188–238.

[96] C. Voisin, *Hodge theory and complex algebraic geometry. I*, Cambridge Studies in Advanced Mathematics, Vol.76, Cambridge University Press, 2007.

[97] G.W. Zinbiel, *Encyclopedia of types of algebras 2010*, in: "Operads and Universal Algebra", Nankai Series in Pure, Applied Mathematics and Theoretical Physics **9**, 217-296, World Scientific, Singapore, 2011.