l-adic Cohomology of the Dual Lubin-Tate Tower via the Exterior Power

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ABSTRACT

In this thesis, we study the l-adic cohomology of the dual Lubin-Tate tower by using the exterior power of a π_L-divisible O_L-module to relate it to the cohomology of the Lubin-Tate tower. By using a result of Harris-Taylor on the cohomology of the Lubin-Tate tower, we show that the supercuspidal part of the cohomology of the dual Lubin-Tate tower realizes the local Langlands and the Jacquet-Langlands correspondences, up to certain twists.

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Chapter 1

INTRODUCTION

Generalizing earlier work of Lubin-Tate ([LT66]) and Drinfeld ([Dri74]), Rapoport-Zink ([RZ96]) constructed moduli spaces of deformations of p-divisible groups for data of type EL and PEL. These spaces, which are now known as Rapoport-Zink spaces, are local models of Shimura varieties. Kottwitz (cf. [Rap95]) conjectured that the l-adic cohomology of basic Rapoport-Zink spaces should partly realize the local Langlands and the Jacquet-Langlands correspondences.

A particularly well-known example of a Rapoport-Zink space is the Lubin-Tate tower. Let p be an odd prime, L be a finite extension of $\mathbb{Q}_p$ with uniformizer π_L and residue field $k_L = \mathbb{F}_q$, and let $G_{1,h-1}$ be the isoclinic π_L-divisible O_L-module over $\overline{\mathbb{F}}_p$ of dimension 1 and height $h \geq 2$. Rapoport-Zink ([RZ96]) showed that, in this special case, the functor of deformations of $G_{1,h-1}$ with a quasi-isogeny is representable by a formal scheme LT_h, which is non-canonically isomorphic to $\coprod_{i \in \mathbb{Z}} \operatorname{Spf} O_{\breve{L}}[[t_1, \ldots, t_{h-1}]]$, where $O_{\breve{L}}$ is the ring of integers of $\breve{L} = L \cdot \widehat{\mathbb{Q}_p^{\mathrm{nr}}}$. We get a tower of étale covers over the rigid analytic generic fiber of LT_h by trivializing the Tate module of the universal π_L-divisible O_L-module. Let LT_h^n be the étale cover that trivializes the Tate module of the universal π_L-divisible O_L-module mod Γ_n, where Γ_n is the subgroup of $GL_h(O_L)$ consisting of matrices congruent to I_h mod π_L^n. The projective system $LT_h^\infty = \{LT_h^n\}_n$ is known as the Lubin-Tate tower.

Let C be the completion of an algebraic closure of $\breve{L}$. After base change to C, the Lubin-Tate tower has actions by the three groups $GL_h(L)$, $D^\times$ (where $D = \operatorname{End}(G_{1,h-1}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is the central division algebra over L of invariant $1/h$) and W_L (the Weil group of L). By using global methods involving Shimura varieties, Harris-Taylor ([HT01]) showed that the supercuspidal part of the cohomology of the Lubin-Tate tower realizes the local Langlands correspondence for GL_h and the Jacquet-Langlands correspondence for $D^\times$, up to some twists, thus verifying Kottwitz's conjecture in this particular case.

If, in the above moduli problem, we replace $G_{1,h-1}$ by $G_{h-1,1}$, the isoclinic π_L-divisible O_L-module $G_{h-1,1}$ of dimension $h-1$ and height h, we obtain another projective system of rigid analytic spaces $\{M_h^n\}_n$, and we call this the dual Lubin-Tate tower. Again, after base change to C, the dual Lubin-Tate tower M_h^∞ has actions by the groups $GL_h(L)$, $E^\times$ (where $E = \mathrm{End}(G_{h-1,1}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$) and W_L.

The aim of this paper is to understand the cohomology of the dual Lubin-Tate tower M_h^∞ by relating it to the cohomology of the Lubin-Tate tower LT_h^∞. We will show that the cohomology of the dual Lubin-Tate tower also realizes the Jacquet-Langlands and the local Langlands correspondences, up to certain twists.

For the rest of the introduction, we will slightly abuse notation, and not distinguish between the Lubin-Tate tower LT_h^∞, which is defined over $\breve{L}$, and its base change to C. We will also do the same for the dual Lubin-Tate tower M_h^∞. Although some of the statements still hold even without base change, for simplicity, the reader may assume (for the rest of the introduction) that we are always working with the base change to C.

As usual, let l be an odd prime different from p. We consider the functor

$$\begin{aligned} H_c^i(M_h^\infty) : \mathrm{Rep}\, E^\times &\to \mathrm{Rep}(GL_h(L) \times W_L) \\ \rho &\mapsto \mathrm{Hom}_{E^\times}(H_c^i(M_h^\infty, \overline{\mathbb{Q}}_l), \rho). \end{aligned}$$

The following are the main results:

Theorem A. *For $\pi \in \mathrm{Cusp}(GL_h(L))$,*

$$H_c^{h-1}(M_h^\infty)_{cusp}(\mathrm{JL}^{-1}(\pi)) = \pi \otimes \mathrm{rec}(\pi^\vee \otimes (\chi_\pi \circ \det) \otimes (|\cdot| \circ \det)^{\frac{h-1}{2}})(h-1),$$

where χ_π is the central character of π, $H_c^{h-1}(M_h^\infty)_{cusp}$ is the supercuspidal part of $H_c^{h-1}(M_h^\infty)$, rec *is the local Langlands correspondence and* JL *is the Jacquet-Langlands correspondence.*

Theorem A is a consequence of the following Theorem which expresses the cohomology of the dual Lubin-Tate tower M_h^∞ in terms of the cohomology of the Lubin-Tate tower LT_h^∞. For convenience, we will fix certain embeddings of the division algebras $D^\times$ and $E^\times$ into $GL_h(L_h)$ (see Section 4.1), where L_h is the unramified extension of degree h over L.

Theorem B. *Define*

$$\begin{aligned} \theta : D^\times &\to E^\times & \psi : GL_h(L) &\to GL_h(L) \\ \phi &\mapsto (\det \phi)(\phi^{-1})^T, & g &\mapsto (\det g)(g^{-1})^T. \end{aligned}$$

Let $\theta_* H^i_c(LT^\infty_h, \overline{\mathbb{Q}}_l)$ *be the pushforward of the representation* $H^i_c(LT^\infty_h, \overline{\mathbb{Q}}_l)$ *under the map*

$$\theta : D^\times \twoheadrightarrow \theta(D^\times).$$

Then for all $i \geq 0$*,*

$$\psi^* \left(\operatorname{Res}^{GL_h(L)}_{\psi(GL_h(L))} H^i_c(M^\infty_h, \overline{\mathbb{Q}}_l)\right) \cong \operatorname{Ind}^{E^\times}_{\theta(D^\times)} \theta_* \left(\frac{H^i_c(LT^\infty_h, \overline{\mathbb{Q}}_l)}{K}\right)$$

as $E^\times \times GL_h(L) \times W_L$ *representations, where* $K = \ker(\theta)$*.*

To prove Theorem B, we first define a map from LT^∞_h to M^∞_h.

Since the dual of $G_{1,h-1}$ is $G_{h-1,1}$, the Serre dual of a π_L-divisible O_L-module can be used to define a map

$$\vee : LT^\infty_h \to M^\infty_h.$$

However, this map is not W_L-equivariant. For example, the Tate module of the Serre dual of $(L/O_L)^h$ is $O^h_L(1)$, and not O^h_L. As such, while the duality map $\vee$ could be used to study the cohomology of M^∞_h as a $E^\times \times GL_h(L)$-representation, the action of W_L appears to be difficult to understand.

Instead, we will use the exterior power $\bigwedge^r$, which was introduced by Pink and studied by Hedayatzadeh in [Hed10], [Hed13] and [Hed14] for certain π_L-divisible O_L-modules. The idea of using the exterior power has previously been explored by Chen in [Che13] where she used the top exterior power to define a determinant map from Rapoport-Zink spaces to a tower of dimension 0 rigid analytic spaces. In this thesis, however, we will consider the $h-1$ exterior power, and not the top exterior power. As $\bigwedge^{h-1} G_{1,h-1} \approx G_{h-1,1}$, using the $h-1$ exterior power, we can define a map

$$\bigwedge^{h-1} : LT^\infty_h \to M^\infty_h.$$

Unlike the duality map $\vee$, the map $\bigwedge^{h-1}$ is W_L-equivariant, and the other group actions can be understood as well. Under the map $\bigwedge^{h-1}$, the action of $\phi \in D^\times$ on LT^∞_h corresponds to the action of $\theta(\phi)$ on M^∞_h, and the action of $g \in GL_h(L)$ on LT^∞_h corresponds to the action of $\psi(g)$ on M^∞_h.

Before proving Theorem B, it is useful to first look at the level 0 case. Using Grothendieck-Messing theory and the fact that at level 0, the duality map $\vee$ induces an isomorphism on the connected components, we can show that the same is true for the map $\bigwedge^{h-1}$. Let $(LT^n_h)_m$ be the subspace of LT^n_h corresponding to quasi-isogenies of height m, and define $(M_h^n)_m$ similarly. By the above, we can identify $(LT_h^0)_0$ with $(M_h^0)_0$ using the exterior power map $\bigwedge^{h-1}$.

For simplicity, we will first specialize to the case where $h-1$ is coprime to $p(q-1)$. Under this condition, we can check that $K = \ker(\theta)$ is trivial, the map θ gives a bijection from $\mathcal{O}_D^\times$ to $\mathcal{O}_E^\times$, and θ maps a uniformizer of $D^\times$ to an element of $E^\times$ with normalized valuation $h-1$. The fact that θ gives a bijection from $\mathcal{O}_D^\times$ to $\mathcal{O}_E^\times$ shows that $\bigwedge^{h-1}$ induces a bijection from the connected components of $(LT_h^n)_0$ to the connected components of $(M_h^n)_0$. Using this, and the fact that $(LT_h^n)_0$ and $(M_h^n)_0$ are both étale covers of $(LT_h^0)_0$ of the same degree, we deduce that $\bigwedge^{h-1}$ induces an isomorphism from $(LT_h^n)_0$ to $(M_h^n)_0$. By considering the group actions, it is then clear that $\bigwedge^{h-1}$ induces an isomorphism from $(LT_h^n)_m$ to $(M_h^n)_{(h-1)m}$. With the above knowledge of the geometry of $\bigwedge^{h-1} : LT_h^\infty \to M_h^\infty$, we can then prove Theorem B in this special case.

Now, let us look at the general case. Unlike in the previous case where $h-1$ is coprime to $p(q-1)$, the map $\bigwedge^{h-1}$ no longer induces a bijection from the connected components of $(LT_h^n)_0$ to the connected components of $(M_h^n)_0$. However, it is still true that each connected component of LT_h^n is mapped isomorphically onto some connected component of M_h^n. Let $d = (h-1, q-1)$ and $e = v_p(h-1)$. We will separately consider the cases where

(i) $d \neq 1$ and $e = 0$,

(ii) $d = 1$ and $e \neq 0$.

It should then be clear how to handle the general case.

First, we recall that $(LT_h^n)_0$ has $q^{n-1}(q-1)$ connected components and a result of Strauch ([Str06]) tells us that the action of $\phi \in \mathcal{O}_D^\times$ on the connected components depends on $\det(\phi) \bmod (1 + \pi_L^n \mathcal{O}_L)$. We would like to understand the map on the connected components induced by $\bigwedge^{h-1}$. To do so, we study the map $\theta_{0,n} : (\mathcal{O}_L/\pi_L^n\mathcal{O}_L)^\times \to (\mathcal{O}_L/\pi_L^n\mathcal{O}_L)^\times$ induced by $\theta : \mathcal{O}_D^\times \to \mathcal{O}_E^\times$ under the $\det \bmod (1 + \pi_L^n\mathcal{O}_L)$ map. In fact, $\theta_{0,n}$ is simply the map $x \mapsto x^{h-1}$.

In case (i), the kernel K of the map $\theta : \mathcal{O}_D^\times \to \mathcal{O}_E^\times$ is cyclic of order d and this maps isomorphically to $\ker(\theta_{0,n})$ under the $\det \bmod (1 + \pi_L^n\mathcal{O}_L)$ map. Hence, if we write Y_n for the image of the map $\bigwedge^{h-1} : LT_h^n \to M_h^n$, then we have

$$\psi^* H_c^i(Y_n, \overline{\mathbb{Q}}_l) \cong \theta_* \left(\frac{H_c^i(LT_h^n, \overline{\mathbb{Q}}_l)}{K} \right)$$

as $\theta(D^\times) \times GL_h(L) \times W_L$ representations.

Let E^θ be the subgroup of $E^\times$ generated by $\mathrm{Im}(\theta)$ and $\mathcal{O}_{L_h}^\times$, and let X_n be the subspace $\coprod_{m\in\mathbb{Z}}(M_h^n)_{(h-1)m}$ of M_h^n. It is easy to check that, in fact, $E^\theta = \{\varphi \in E^\times : (h-1)|\operatorname{val}(\varphi)\}$. Using that $E^\theta/\operatorname{Im}(\theta)$ is a finite abelian group, and that X_n is equal to the disjoint union

$$X_n = \coprod_{\varphi\in E^\theta/\operatorname{Im}(\theta)} \varphi(Y_n),$$

we can show that $H_c^i(X_n, \overline{\mathbb{Q}}_l) \cong \operatorname{Ind}_{\theta(D^\times)}^{E^\theta}(H_c^i(Y_n, \overline{\mathbb{Q}}_l))$. Similarly, $E^\times/E^\theta$ is finite abelian (in fact cyclic of order d), and

$$M_h^n = \coprod_{\varphi\in E^\times/E^\theta} \varphi(X_n),$$

so we have $H_c^i(M_h^n, \overline{\mathbb{Q}}_l) \cong \operatorname{Ind}_{E^\theta}^{E^\times}(H_c^i(X_n, \overline{\mathbb{Q}}_l))$. Putting these together gives us Theorem B for case (i).

We now look at case (ii). In this case, the kernel of the map $\theta : D^\times \to E^\times$ is $K = \boldsymbol{\mu}_{p^e}(L)$, the group of p^e roots of unity in L. Unlike in the previous case, K does not surject onto $\ker(\theta_{0,n})$ under the $\det \bmod (1+\pi_L^n\mathcal{O}_L)$ map. But if we let $K_n = 1 + \pi_L^{n-ee_L}\mathcal{O}_L \leq D^\times$, then the $\det \bmod (1+\pi_L^n\mathcal{O}_L)$ map gives a surjection of $K \times K_n$ onto $\ker(\theta_{0,n})$, and we can show that

$$\psi^* H_c^i(Y_n, \overline{\mathbb{Q}}_l) \cong \theta_*\left(\frac{H_c^i(LT_h^n, \overline{\mathbb{Q}}_l)}{K \times K_n}\right).$$

Now, by following the argument in case (i), we have

$$\psi^*\left(\operatorname{Res}^{GL_h(L)}_{\psi(GL_h(L))} H_c^i(M_h^n, \overline{\mathbb{Q}}_l)\right) \cong \operatorname{Ind}_{\theta(D^\times)}^{E^\theta} \theta_*\left(\frac{H_c^i(LT_h^n, \overline{\mathbb{Q}}_l)}{K \times K_n}\right).$$

The map $H_c^i(LT_h^{n-ee_L}, \overline{\mathbb{Q}}_l) \to H_c^i(LT_h^n, \overline{\mathbb{Q}}_l)$ factors through $\frac{H_c^i(LT_h^n, \overline{\mathbb{Q}}_l)}{K_n}$ since $K_n = 1 + \pi_L^{n-ee_L}\mathcal{O}_L$ acts trivially on $LT_h^{n-ee_L}$, so we have

$$\varinjlim_n \frac{H_c^i(LT_h^n, \overline{\mathbb{Q}}_l)}{K_n} = H_c^i(LT_h^\infty, \overline{\mathbb{Q}}_l).$$

Piecing together the above, we obtain Theorem B for case (ii).

It is not hard to see that the methods used to deal with cases (i) and (ii) can be combined to give a proof of Theorem B in the general case.

Using Theorem B, and applying Frobenius reciprocity, we see that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Rep}\, E^\times & \xrightarrow{\rho\mapsto\theta^*\left(\mathrm{Res}^{E^\times}_{\theta(D^\times)}\,\rho\right)} & \mathrm{Rep}\, D^\times \\
\Big\downarrow{\scriptstyle H^i_c(M_h^\infty)} & & \Big\downarrow{\scriptstyle H^i_c(LT_h^\infty)} \\
\mathrm{Rep}(GL_h(L)\times W_L) & \xrightarrow[\pi\otimes r\mapsto\psi^*\left(\mathrm{Res}^{GL_h(L)}_{\psi(GL_h(L))}\,\pi\right)\otimes r]{} & \mathrm{Rep}(GL_h(L)\times W_L)
\end{array}$$

Although the above diagram only gives us $H^i_c(M_h^\infty)[\rho]$ as a $\psi(GL_h(L))$-representation, we can use the duality map to understand the $GL_h(L)$-action. Alternatively, in the special case where $p(q-1)$ is coprime to $h-1$, the group $GL_h(L)$ is generated by the subgroup $\psi(GL_h(L))$ and the element $\pi_L \in GL_h(L)$, so we just need to understand the action of π_L, but the action of $\pi_L I_h \in GL_h(L)$ on M_h^∞ corresponds to the action of $\pi_L^{-1} \in E^\times$.

Finally, we observe that $\theta^*\left(\mathrm{Res}^{E^\times}_{\theta(D^\times)}\,\rho\right)$ and $\psi^*\left(\mathrm{Res}^{GL_h(L)}_{\psi(GL_h(L))}\,\pi\right)$ can be written as twists of ρ and π respectively by their central characters. This is useful since both the local Langlands and the Jacquet-Langlands correspondences are compatible with twists. Using the theorem of Harris-Taylor on the cohomology of the Lubin-Tate tower, we can then prove Theorem A.

We start by introducing some background material on π_L-divisible O_L-modules in Chapter 2. In particular, we recall the Serre dual of a π_L-divisible O_L-module and introduce the exterior power for π_L-divisible O_L-modules of dimension 1, whose existence has been proven in [Hed10].

In Chapter 3, we introduce moduli spaces of certain π_L-divisible O_L-modules, which includes, as special cases, the Lubin-Tate and the dual Lubin-Tate towers. We also state the well known result of Harris-Taylor on the cohomology of the Lubin-Tate tower.

We collect some useful results on the endomorphism algebras and on representation theory in Chapter 4. We will describe in this chapter embeddings of $D^\times$ and $E^\times$ into $GL_h(L_h)$, and study properties of the maps θ and ψ.

In Chapter 5, we use the exterior power introduced in Chapter 2 to define a map from the Lubin-Tate tower LT_h^∞ to the dual Lubin-Tate tower M_h^∞. We also study how the group actions behave under this map.

And finally, in Chapter 6, we study the geometry of the dual Lubin-Tate tower, and then derive results on its cohomology, including Theorems A and B.

Notation

p: odd prime

l: odd prime different from p

L: finite extension of $\mathbb{Q}_p$

π_L: uniformizer of L

q: order of k_L, the residue field of L

n_L: degree of L over $\mathbb{Q}_p$

e_L: ramification degree of L over $\mathbb{Q}_p$

k: perfect field of characteristic p containing $k_L = \mathbb{F}_q$

W_L: Weil group of L

$W_{O_L}(k)$: ramified Witt vectors of k

$D(G)$: Dieudonné O_L-module of the π_L-divisible O_L-module G

$G_{m,n}$: isoclinic π_L-divisible O_L-module over $\overline{\mathbb{F}}_p$ of dim m and height $m+n$

$D_{m,n}$: endomorphism algebra of $G_{m,n}$

$G^\vee$: Serre dual of the π_L-divisible O_L-module G

$\bigwedge^r G$: r-th exterior power of the π_L-divisible O_L-module G

D: endomorphism algebra of $G_{1,h-1}$

E: endomorphism algebra of $G_{h-1,1}$

LT_h: Lubin-Tate space, moduli space of deformations of $G_{1,h-1}$ with a quasi-isogeny

LT_h^n: étale cover over the generic fiber of LT_h parameterizing level n structure

LT_h^∞: Lubin-Tate tower, the projective system $\{LT_h^n\}_{n\in\mathbb{N}}$

M_h: dual Lubin-Tate space, moduli space of deformations of $G_{h-1,1}$ with a quasi-isogeny

M_h^n: étale cover over the generic fiber of M_h parameterizing level n structure

M_h^∞: dual Lubin-Tate tower, the projective system $\{M_h^n\}_{n\in\mathbb{N}}$

Nrd: reduced norm of a division algebra

Chapter 2

BACKGROUND ON π_L-DIVISIBLE $\mathcal{O}_L$-MODULES

2.1 Ramified Witt vectors, Dieudonné $\mathcal{O}_L$-modules and $\mathcal{O}_L$-displays

Let p be an odd prime, L be a finite extension of $\mathbb{Q}_p$ with uniformizer π_L and residue field $k_L = \mathbb{F}_q$. For an $\mathcal{O}_L$ algebra R, the ring of ramified Witt vectors $W_{\mathcal{O}_L}(R)$ (cf. [Haz80]) is $W_{\mathcal{O}_L}(R) = R^{\mathbb{N}}$ with the unique $\mathcal{O}_L$-algebra structure such that

1. the Witt polynomials
$$w_n : W_{\mathcal{O}_L}(R) \to R$$
$$(x_0, x_1, x_2, \ldots) \mapsto x_0^{q^n} + \pi_L x_1^{q^{n-1}} + \cdots + \pi_L^n x_n$$
are $\mathcal{O}_L$ algebra homomorphisms for all $n \geq 0$,

2. $\mathcal{O}_L$-algebra homomorphisms $R \to S$ induce $\mathcal{O}_L$-algebra homomorphisms $W_{\mathcal{O}_L}(R) \to W_{\mathcal{O}_L}(S)$.

There is an endomorphism of $W_{\mathcal{O}_L}(R)$, defined by $w_n(\sigma(x)) = w_{n+1}(x)$, called the Frobenius endomorphism. In fact, σ is a homomorphism of $\mathcal{O}_L$-algebras.

Let $k \supseteq k_L$ be a perfect field of characteristic p. Then $W_{\mathcal{O}_L}(k) \cong \mathcal{O}_L \widehat{\otimes}_{W(k_L)} W(k)$, where $W(k) = W_{\mathbb{Z}_p}(k)$ is the usual ring of Witt vectors of k. It is a complete discrete valuation ring with residue field k and uniformizer π_L. In particular, $W_{\mathcal{O}_L}(\overline{\mathbb{F}}_p)$ is isomorphic to the ring of integers $\mathcal{O}_{\breve{L}}$ of $\breve{L}$, where $\breve{L} = L \cdot \widehat{\mathbb{Q}_p^{\mathrm{nr}}}$ is the completion of the maximal unramified extension of L.

We now introduce Dieudonné $\mathcal{O}_L$-modules and $\mathcal{O}_L$-displays. A Dieudonné $\mathcal{O}_L$-module over k is a free $W_{\mathcal{O}_L}(k)$-module of finite rank, together with two maps F and V such that F is σ-linear, and V is σ^{-1}-linear, and $FV = \pi_L = VF$. There is an equivalence of categories between the category of π_L-divisible formal $\mathcal{O}_L$-modules over k and the category of Dieudonné $\mathcal{O}_L$-modules over k such that V is topologically nilpotent in the π_L-adic topology (cf. [Ahs11], Proposition 2.2.3 and Theorem 5.3.8).

For a π_L-divisible $\mathcal{O}_L$-module G, we will denote its covariant Dieudonné $\mathcal{O}_L$-module by $D(G)$. Under this equivalence of categories,
$$\mathrm{ht}(G) = \mathrm{rank}_{W_{\mathcal{O}_L}(k)} D(G), \qquad \dim(G) = \dim_k \frac{D(G)}{VD(G)}.$$

Generalizing earlier work of Zink ([Zin02]) and Lau ([Lau08]), Ahsendorf ([Ahs11]) introduced the notion of an $\mathcal{O}_L$-display. An $\mathcal{O}_L$-display over an $\mathcal{O}_L$-algebra R is a quadruple $\mathcal{P} = (P, Q, F, V^{-1})$ with P a finitely generated projective $W_{\mathcal{O}_L}(R)$-module, $Q \subset P$ a submodule and $F : P \to P$ and $V^{-1} : Q \to P$ Frobenius linear maps, which satisfy some additional conditions.

If π_L is nilpotent in R, then there is an equivalence of categories between the category of nilpotent $\mathcal{O}_L$-displays over R and the category of formal π_L-divisible $\mathcal{O}_L$-modules over R (cf. [Ahs11], Theorem 5.3.8).

2.2 Isoclinic π_L-divisible $\mathcal{O}_L$-modules and their endomorphism algebras

We now introduce certain isoclinic π_L-divisible $\mathcal{O}_L$-modules which will be useful later. Suppose m, n are coprime non-negative integers. We shall write $G_{m,n}$ for the π_L-divisible $\mathcal{O}_L$-module over $\overline{\mathbb{F}}_p$ with covariant Dieudonné $\mathcal{O}_L$-module $D(G_{m,n})$ having basis $\{b_0, b_1, \ldots, b_{m+n-1}\}$ such that

$$Fb_i = b_{i+n}, \quad Vb_i = b_{i+m} \quad \text{and} \quad b_{(m+n)+i} = \pi_L b_i.$$

$G_{m,n}$ is an isoclinic π_L-divisible $\mathcal{O}_L$-module of dimension m and height $m + n$.

In particular, $D(G_{1,h-1})$ has a basis $\{e_0, \ldots, e_{h-1}\}$ such that

$$Ve_i = e_{i+1} \quad (0 \le i \le h-2), \qquad Ve_{h-1} = \pi_L e_0,$$

and $D(G_{h-1,1})$ has a basis $\{f_0, \ldots, f_{h-1}\}$ such that

$$Ff_i = f_{i-1} \quad (1 \le i \le h-1), \qquad Ff_0 = \pi_L f_{h-1}.$$

Let $\mathcal{O}_{D_{m,n}} = \mathrm{End}(G_{m,n})$ be the endomorphism ring of $G_{m,n}$ and let $D_{m,n} = \mathcal{O}_{D_{m,n}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathrm{End}(G_{m,n}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ be the endomorphism algebra of $G_{m,n}$. The following description of the endomorphism ring $\mathcal{O}_{D_{m,n}}$ is a straightforward generalization of [JO99, Lemma 5.4]. We include a proof here for lack of a suitable reference.

Lemma 2.2.1. *Let $a, b \in \mathbb{Z}$ be such that $am + bn = 1$, and let $\mathcal{O}_{L_{m+n}} = W_{\mathcal{O}_L}(\mathbb{F}_{m+n})$ be the ring of integers of L_{m+n}, the unramified extension of degree $m + n$ over L. Then*

$$\mathcal{O}_{D_{m,n}} = \mathrm{End}(G_{m,n}) \cong \mathcal{O}_{L_{m+n}}[\pi_{m,n}],$$

where $\lambda \cdot \pi_{m,n} = \pi_{m,n} \cdot \sigma^{b-a}(\lambda)$ for $\lambda \in \mathcal{O}_{L_{m+n}}$, and σ is the Frobenius map. $\lambda \in \mathcal{O}_{L_{m+n}}$ acts on b_0 via multiplication by λ, and $\pi_{m,n}$ is a uniformizer of the discrete valuation ring $\mathrm{End}(G_{m,n})$.

Proof. Using the above basis $\{b_0, b_1, \ldots, b_{m+n-1}\}$ of $D(G_{m,n})$, we easily check that

$$\pi_{m,n}(b_i) = b_{i+1} \quad (0 \le i \le h-2), \qquad \pi_{m,n}(b_{h-1}) = \pi_L b_0,$$

is an element of $\mathrm{End}(G_{m,n})$.

Suppose $\phi \in \mathrm{End}(G_{m,n})$ is such that $\phi(b_0) = \lambda b_0$ for some $\lambda \in \mathcal{O}_{\breve{L}}$. Note that $(F^b V^a)^i(b_0) = b_i$, so

$$\phi(b_i) = \phi((F^b V^a)^i b_0) = (F^b V^a)^i \phi(b_0) = (F^b V^a)^i(\lambda b_0) = \sigma^{(b-a)i}(\lambda) b_i.$$

In particular,

$$\lambda b_0 = \phi(b_0) = \phi(\pi_L^{-1} b_{m+n}) = \sigma^{(b-a)(m+n)}(\lambda)\pi_L^{-1} b_{m+n} = \sigma^{(b-a)(m+n)}(\lambda) b_0,$$

which shows that $\sigma^{(b-a)(m+n)}(\lambda) = \lambda$. If we let $a' = a+n$ and $b' = b-m$, then $a'm + b'n = 1$, so similarly, we have $\sigma^{(b'-a')(m+n)}(\lambda) = \lambda$. But $(b-a, b'-a') = 1$ since $(n+a)(b-a) - a(b'-a') = am + bn = 1$, so $\sigma^{m+n}(\lambda) = \lambda$ and $\lambda \in \mathcal{O}_{L_{m+n}}$. The fact that such an element does indeed lie in $\mathrm{End}(G_{m,n})$ is again easy to check. We shall denote this element of $\mathrm{End}(G_{m+n})$ by λ.

Now, let ϕ be an arbitrary element of $\mathrm{End}(G_{m,n})$. Then $\phi(b_0) = \sum_{i=0}^{m+n-1} \alpha_i b_i$ for some $\alpha_i \in \mathcal{O}_{\breve{L}}$. A similar argument to that above shows, in fact, that $\alpha_i \in \mathcal{O}_{L_{m+n}}$. Clearly, $\phi(b_0) - \sum_{i=0}^{m+n-1} \alpha_i \pi_{m,n}^i(b_0) = 0$, from which it follows that $\phi = \sum_{i=0}^{m+n-1} \alpha_i \pi_{m,n}^i$. This proves that $\mathrm{End}(G_{m,n}) \cong \mathcal{O}_{L_{m+n}}[\pi_{m,n}]$.

To show that $\mathcal{O}_{L_{m+n}}[\pi_{m,n}]$ is a discrete valuation ring with uniformizer $\pi_{m,n}$, we first note that $\mathcal{O}_{L_{m+n}}[\pi_{m,n}]$ is a subring of the division ring $L_{m+n}(\pi_{m,n})$. Now, define a map $v : L_{m+n}(\pi_{m,n}) \to \mathbb{Z} \cup \{\infty\}$ by

$$\alpha_0 + \alpha_1 \pi_{m,n} + \cdots + \alpha_{m+n-1}\pi_{m,n}^{m+n-1} \mapsto \min\left\{(m+n)v_{L_{m+n}}(\alpha_i) + i : 0 \le i < m+n\right\}.$$

It is easy to check that v is a valuation on $L_{m+n}(\pi_{m,n})$. Since

$$\mathcal{O}_{L_{m+n}}[\pi_{m,n}] = \{\phi \in L_{m+n}(\pi_{m,n}) : v(\phi) \ge 0\},$$

and $v(\pi_{m,n}) = 1$, we have shown that $\mathrm{End}(G_{m,n}) = \mathcal{O}_{L_{m+n}}[\pi_{m,n}]$ is a discrete valuation ring with uniformizer $\pi_{m,n}$. □

Corollary 2.2.2. *$D_{m,n}$ is the central division algebra over L with invariant $n/(m+n)$.*

Let us introduce some notation for the cases $(m,n) = (1, h-1)$ and $(m,n) = (h-1, 1)$. We write

$$\mathcal{O}_D = \mathcal{O}_{D_{1,h-1}} = \operatorname{End}(G_{1,h-1}), \quad D = D_{1,h-1} = \operatorname{End}(G_{1,h-1}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$
$$\mathcal{O}_E = \mathcal{O}_{D_{h-1,1}} = \operatorname{End}(G_{h-1,1}), \quad E = D_{h-1,1} = \operatorname{End}(G_{h-1,1}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Also let ϖ (respectively ϖ') be the uniformizer of $\mathcal{O}_D$ (respectively $\mathcal{O}_E$) as in Lemma 2.2.1.

2.3 Serre dual and exterior powers of π_L-divisible $\mathcal{O}_L$-modules

Let G be a π_L-divisible $\mathcal{O}_L$-module over a scheme S. The dual π_L-divisible $\mathcal{O}_L$-module $G^\vee$ is

$$0 \to (G[\pi_L])^\vee \to (G[\pi_L^2])^\vee \to (G[\pi_L^3])^\vee \to \cdots$$

where $(G[\pi_L^i])^\vee = \operatorname{Hom}_S(G[\pi_L^i], \mathbb{G}_m)$ is the Cartier dual of $G[\pi_L^i]$. We have

$$\operatorname{ht}(G) = \dim(G) + \dim(G^\vee) = \operatorname{ht}(G^\vee).$$

If $S = \operatorname{Spec} k$ for some perfect field k of characteristic p, then $G^\vee$ has Dieudonné $\mathcal{O}_L$-module

$$D(G^\vee) \cong D(G)^\vee = \operatorname{Hom}_{W_{\mathcal{O}_L}(k)}(D(G), W_{\mathcal{O}_L}(k))$$

with $(Fl)(v) = \sigma(l(Vv))$, $(Vl)(v) = \sigma^{-1}(l(Fv))$, for $v \in D(G)$, $l \in D(G)^\vee$.

We will now look at the exterior power of π_L-divisible $\mathcal{O}_L$-modules, which has been shown to exist by Hedayatzadeh in [Hed10] in certain situations. For G and H π_L-divisible $\mathcal{O}_L$-modules over a base scheme S, we will write $\operatorname{Alt}_S^{\mathcal{O}_L}(G^r, H)$ for the group of alternating $\mathcal{O}_L$-multilinear morphisms from G^r to H.

Definition 2.3.1. [Hed10, Definition 5.3.3] Let G, G' be π_L-divisible $\mathcal{O}_L$-modules over a scheme S, with an alternating $\mathcal{O}_L$-multilinear morphism $\lambda : G^r \to G'$ such that for all π_L-divisible $\mathcal{O}_L$-modules H over S, the induced morphism

$$\lambda^* : \operatorname{Hom}_S(G', H) \to \operatorname{Alt}_S^{\mathcal{O}_L}(G^r, H)$$
$$\psi \mapsto \psi \circ \lambda$$

is an isomorphism. Then G' (or, more precisely, $\lambda : G^r \to G'$) is called the r-th exterior power of G, and we denote it by $\bigwedge^r G$.

Theorem 2.3.2. *[Hed10, Theorem 9.2.36] Let S be a locally Noetherian O_L-scheme and G be a π_L-divisible O_L-module of height h and dimension at most* 1*. Then there exists a π_L-divisible O_L-module $\bigwedge^r G$ over S of height $\binom{h}{r}$, and an alternating morphism $\lambda : G^r \to \bigwedge^r G$ such that for every morphism $f : S' \to S$ and every π_L-divisible O_L-module H over S', the homomorphism*

$$\mathrm{Hom}_{S'}^{O_L}(f^* \bigwedge^r G, H) \to \mathrm{Alt}_{S'}^{O_L}((f^*G)^r, H)$$

induced by $f^*\lambda$ is an isomorphism. In other words, $\bigwedge^r G$ is the r-th exterior power of G over S, and taking the exterior power commutes with arbitrary base change. Moreover, the dimension of $\bigwedge^r G$ is a locally constant function

$$\dim : S \to \left\{0, \binom{h-1}{r-1}\right\}$$
$$s \mapsto \begin{cases} 0 & \text{if } G_s \text{ is étale,} \\ \binom{h-1}{r-1} & \text{otherwise.} \end{cases}$$

In the cases that S is the spectrum of a perfect field of char $p > 2$, or the spectrum of a local Artin O_L-algebra with residue char $p > 2$, we can write down the exterior power in terms of its Dieudonné O_L-module or its O_L-display.

Theorem 2.3.3. *[Hed10, Construction 6.3.1, Lemma 6.3.2, Proposition 6.3.3, Section 9.1] Let $\mathcal{P} = (P, Q, F, V^{-1})$ be an O_L-display of height h with tangent module of rank one. Fix a normal decomposition*

$$P = L \oplus T, \quad Q = L \oplus I_R T.$$

Then

$$\bigwedge^r \mathcal{P} = \left(\bigwedge^r P, \bigwedge^r Q, \bigwedge^{r-1} V^{-1} \wedge F, \bigwedge^r V^{-1} \right)$$

is an O_L-display of height $\binom{h}{r}$ and rank $\binom{h-1}{r-1}$. The map

$$\lambda : P^r \to \bigwedge^r P$$
$$(x_1, \dots, x_r) \mapsto x_1 \wedge \cdots \wedge x_r$$

is an alternating morphism of O_L-displays and satisfies the universal property that the map

$$\mathrm{Hom}(\bigwedge^r \mathcal{P}, \mathcal{P}') \to \mathrm{Alt}(\mathcal{P}^r, \mathcal{P}')$$

induced by λ is an isomorphism.

Theorem 2.3.4. *[Hed10, Proposition 9.2.24] Let R be a local Artin $\mathcal{O}_L$-algebra, and G a π_L-divisible $\mathcal{O}_L$-module over R with special fiber that is a connected π_L-divisible $\mathcal{O}_L$-module of dim 1. Let $\mathcal{P}$ be the $\mathcal{O}_L$-display of the π_L-divisible $\mathcal{O}_L$-module G. Then the $\mathcal{O}_L$-display of $\bigwedge^r G$ is $\bigwedge^r \mathcal{P}$.*

As an immediate corollary, for G a π_L-divisible $\mathcal{O}_L$-module of dimension 1 over a perfect field of characteristic p, the Dieudonné $\mathcal{O}_L$-module of $\bigwedge^r G$ is just $\bigwedge^r D(G)$, where F and V are as in Theorem 2.3.3.

So using the above description of $D(G_{1,h-1})$, we see that the Dieudonné $\mathcal{O}_L$-module of $\bigwedge^{h-1} G_{1,h-1}$ has a basis $\tilde{e}_0, \ldots, \tilde{e}_{h-1}$, where

$$\tilde{e}_i = (-1)^i e_0 \wedge e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_{h-1},$$

and

$$\begin{aligned} F\tilde{e}_i &= (-1)^{h-1}\tilde{e}_{i-1} \quad (1 \leq i \leq h-1), & F\tilde{e}_0 &= (-1)^{h-1}\pi_L\tilde{e}_{h-1}, \\ V\tilde{e}_i &= (-1)^{h-1}\pi_L\tilde{e}_{i+1} \quad (0 \leq i \leq h-2), & V\tilde{e}_{h-1} &= (-1)^{h-1}\tilde{e}_0. \end{aligned}$$

Comparing this with the above description of $D(G_{h-1,1})$, we see that $\bigwedge^{h-1} G_{1,h-1}$ is isomorphic to $G_{h-1,1}$.

Chapter 3

MODULI SPACES OF π_L-DIVISIBLE $\mathcal{O}_L$-MODULES

3.1 Basic Rapoport-Zink spaces

Let $\mathcal{C}$ be the category of schemes S over $\operatorname{Spec}\mathcal{O}_{\breve{L}}$ such that p is locally nilpotent on S. Let $\bar{S}$ be the closed subscheme of S defined by the ideal sheaf $p\mathcal{O}_S$.

We consider a special case of the moduli spaces constructed by Rapoport-Zink in [RZ96].

Theorem 3.1.1. *[RZ96, Theorem 3.25 and Proposition 3.79] Let $h, d \in \mathbb{Z}^+$ coprime, and G be an isoclinic π_L-divisible $\mathcal{O}_L$-module over $\overline{\mathbb{F}}_p$ of height h and dim d. Let F be the functor*

$$\begin{aligned} \mathcal{C} &\longrightarrow \textbf{Set} \\ S &\longmapsto \left\{ (X,\beta) : \begin{array}{c} X \text{ is a connected } \pi_L\text{-divisible } \mathcal{O}_L\text{-module}/S \\ \text{and } \beta : G_{\bar{S}} \to X_{\bar{S}} \text{ a quasi-isogeny} \end{array} \right\} \Big/ \sim . \end{aligned}$$

Then F is representable by a formal scheme $\mathcal{M}_{\frac{d}{h}}$, which is formally locally of finite type over $\operatorname{Spf}\mathcal{O}_{\breve{L}}$. For $d = 1$, the formal scheme $\mathcal{M}_{\frac{1}{h}}$ is (non-canonically) isomorphic to $\coprod_{i\in\mathbb{Z}} \operatorname{Spf}\mathcal{O}_{\breve{L}}[[t_1,\dots,t_{h-1}]]$.

Let $\mathcal{M}^0_{\frac{d}{h}}$ be the rigid analytic generic fiber of $\mathcal{M}_{\frac{d}{h}}$. We consider a special case of the tower of rigid-analytic coverings of $\mathcal{M}^0_{\frac{d}{h}}$ introduced in [RZ96, 5.34]. We write $\mathcal{G}$ for the universal π_L-divisible $\mathcal{O}_L$-module on $\mathcal{M}_{\frac{d}{h}}$, and $T_{\pi_L}\mathcal{G}$ for its Tate module. Let Γ_n be the subgroup of $GL_h(\mathcal{O}_L)$ consisting of matrices congruent to I_h mod π_L^n. Then we have a finite étale cover $\mathcal{M}^n_{\frac{d}{h}}$ of $\mathcal{M}^0_{\frac{d}{h}}$ that parameterizes trivializations

$$\alpha_n : \mathcal{O}_L^h \xrightarrow{\cong} T_{\pi_L}\mathcal{G} \mod \Gamma_n.$$

Define $\mathcal{M}^\infty_{\frac{d}{h}}$ to be the projective system $(\mathcal{M}^n_{\frac{d}{h}})_{n\in\mathbb{N}}$ with maps

$$\begin{aligned} f_{mn} : \mathcal{M}^n_{\frac{d}{h}} &\to \mathcal{M}^m_{\frac{d}{h}} \quad (m \le n) \\ (X,\beta,\alpha_n) &\mapsto (X,\beta,\alpha_n \bmod \Gamma_m). \end{aligned}$$

Recall from Section 2.2 that $G_{d,h-d}$ is an isoclinic π_L-divisible $\mathcal{O}_L$-module of height h and dim d. Any two π_L-divisible $\mathcal{O}_L$-modules with the same Newton polygon are isogenous, so in the definition of $\mathcal{M}^n_{\frac{d}{h}}$ above, we might as well take $G = G_{d,h-d}$.

The groups $D_{d,h-d} = \mathrm{End}(G_{d,h-d}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $GL_h(L)$ act on $\mathcal{M}^\infty_{\frac{d}{h}}$. Let C be the completion of an algebraic closure of $\breve{L}$. We also have an action of the Weil group W_L on $\mathcal{M}^\infty_{\frac{d}{h}}$ after base change to C. We describe the actions here.

1. *Action of $D_{d,h-d}$*: $\gamma \in D_{d,h-d}$ acts on $\mathcal{M}^n_{\frac{d}{h}}$ by

$$(X, \beta, \alpha_n) \mapsto (X, \beta \circ \gamma^{-1}, \alpha_n).$$

2. *Action of $GL_h(L)$*: Any element $g' \in GL_h(L)$ can be written as $g' = \pi_L^k g$ with $k \in \mathbb{Z}$ and $g \in GL_h(L) \cap \mathrm{Mat}_h(\mathcal{O}_L)$. Hence it is sufficient to describe the action of such elements.

   So let $g \in GL_h(L) \cap \mathrm{Mat}_h(\mathcal{O}_L)$ and $S \to \mathcal{M}_{\frac{d}{h}}$ be a formal scheme. We have an isomorphism $\alpha^{\mathrm{rig}} : (L/\mathcal{O}_L)^h \xrightarrow{\cong} X^{\mathrm{rig}}$. Let $\ker(g)$ be the kernel of the map $g : (L/\mathcal{O}_L)^h \to (L/\mathcal{O}_L)^h$, and Y^{rig} be the π_L-divisible $\mathcal{O}_L$-module $\frac{X^{\mathrm{rig}}}{\alpha^{\mathrm{rig}}(\ker(g))}$ over S^{rig}. In other words,

$$\begin{array}{ccc}
\ker(g) & \xrightarrow[\cong]{\alpha^{\mathrm{rig}}} & \alpha^{\mathrm{rig}}(\ker(g)) \\
\downarrow & & \downarrow \\
(L/\mathcal{O}_L)^h & \xrightarrow[\cong]{\alpha^{\mathrm{rig}}} & X^{\mathrm{rig}} \\
\downarrow g & & \downarrow q^{\mathrm{rig}} \\
(L/\mathcal{O}_L)^h & \xrightarrow[\cong]{\overline{\alpha^{\mathrm{rig}}}} & Y^{\mathrm{rig}} = \frac{X^{\mathrm{rig}}}{\alpha^{\mathrm{rig}}(\ker(g))}
\end{array}$$

   Then g acts on $\mathcal{M}^\infty_{\frac{d}{h}}$ by

$$(X, \beta, \alpha) \mapsto (Y, q \circ \beta, \overline{\alpha})$$

   where

   - Y is a π_L-divisible $\mathcal{O}_L$-module over some admissible blow-up of S with rigid analytic generic fiber Y^{rig},
   - q is the special fiber of the quotient map $X \to Y$ which has generic fiber $X^{\mathrm{rig}} \to \frac{X^{\mathrm{rig}}}{\alpha^{\mathrm{rig}}(\ker(g))} = Y^{\mathrm{rig}}$,

- $\overline{\alpha} : O_L^h \xrightarrow{\cong} T_{\pi_L} Y$ has generic fiber $\overline{\alpha}^{\mathrm{rig}} : O_L^h \xrightarrow{\cong} T_{\pi_L} Y^{\mathrm{rig}}$ which corresponds to $\overline{\alpha^{\mathrm{rig}}} : (L/O_L)^h \xrightarrow{\cong} Y^{\mathrm{rig}}$.

3. *Action of W_L*: Suppose $w \in W_L \mapsto \sigma^n \in \mathrm{Gal}(\overline{k_L}/k_L) = \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_q) \subseteq \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ under the reduction map, where $\sigma = \{x \mapsto x^p\} \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ is the arithmetic Frobenius. Then the action of w on $\mathcal{M}^\infty_{\frac{d}{h}} \times_{\breve{L}} C$ is

$$w : (X, \beta, \alpha) \mapsto (X^w, \beta^w \circ F^n_{G/\overline{\mathbb{F}}_p}, \alpha^w)$$

where

- $X^w = X \times_S S^w, \quad G^{(p^n)} = X \times_{\overline{\mathbb{F}}_p, \sigma^n} \overline{\mathbb{F}}_p,$
- $\beta^w : G^{(p^n)}_{\bar{S}} \to X^w_{\bar{S}}$ is induced by $\beta : G_{\bar{S}} \to X_{\bar{S}}$,
- α^w is the map
$$O_L^h = (O_L^h)^w \xrightarrow[\cong]{\alpha^w} T_{\pi_L} X^w.$$

Let us introduce some notation for the cases $d = 1$ and $d = h - 1$. For $d = 1$, we will write LT_h^n for $\mathcal{M}^n_{\frac{1}{h}}$ and LT_h^∞ for the Lubin-Tate tower $\mathcal{M}^\infty_{\frac{1}{h}}$.
And for $d = h-1$, we write M_h^n for $\mathcal{M}^n_{\frac{h-1}{h}}$, M_h^∞ for the dual Lubin-Tate tower $\mathcal{M}^\infty_{\frac{h-1}{h}}$.

3.2 Results on the Lubin-Tate tower

A well known result of Harris-Taylor tells us that the l-adic cohomology (cf. [Ber93]) of $LT_h^\infty \times_{\breve{L}} C$ realizes the local Langlands and the Jacquet-Langlands correspondence. To be precise, let us consider the functor

$$H_c^i(LT_h^\infty) : \mathrm{Rep}\, D^\times \to \mathrm{Rep}(GL_h(L) \times W_L)$$
$$\rho \mapsto \mathrm{Hom}_{D^\times}(H_c^i(LT_h^\infty \times_{\breve{L}} C, \overline{\mathbb{Q}}_l), \rho),$$

and let $[H_c(LT_h^\infty)(\rho)]$ denote the virtual representation $(-1)^{h-1}\sum_{i=0}^{h-1}(-1)^i[H_c^i(LT_h^\infty)(\rho)]$.

Theorem 3.2.1. *[HT01, Theorem VII.1.3] Let π be an irreducible supercuspidal representation of $GL_h(L)$. Then*

$$[H_c(LT_h^\infty)(\mathrm{JL}^{-1}(\pi))] = [\pi \otimes \mathrm{rec}(\pi^\vee \otimes (|\cdot| \circ \det)^{\frac{h-1}{2}})(h-1)]$$

in the Grothendieck group, where rec is the local Langlands correspondence and JL is the Jacquet-Langlands correspondence.

Theorem 3.2.2. *[Fal02, Section 6] For $i \neq h-1$,*

$$H_c^i(LT_h^\infty)_{cusp}(\rho) = 0$$

for any $\rho \in \mathrm{Rep}\, D^\times$, where $H_c^i(LT_h^\infty)_{cusp}$ is the supercuspidal part of $H_c^i(LT_h^\infty)$.

An immediate corollary of Theorems 3.2.1 and 3.2.2 is:

Corollary 3.2.3. *Let π be an irreducible supercuspidal representation of $GL_h(L)$. Then*

$$H_c^{h-1}(LT_h^\infty)_{cusp}(\mathrm{JL}^{-1}(\pi)) = \pi \otimes \mathrm{rec}(\pi^\vee \otimes (|\cdot| \circ \det)^{\frac{h-1}{2}})(h-1).$$

Let $(LT_h^n)_m$ be the subspace of LT_h^n corresponding to quasi-isogenies of height m. The following result of Strauch describes the action of $\mathcal{O}_D^\times \times GL_h(\mathcal{O}_L)$ on the connected components of $(LT_h^n)_0 \times_{\breve{L}} C$.

Theorem 3.2.4. *[Str06, Theorem 4.4(i)] There exists a bijection*

$$\pi_0\left((LT_h^n)_0 \times_{\breve{L}} C\right) \to (\mathcal{O}_L/\pi_L^n \mathcal{O}_L)^\times$$

which is $\mathcal{O}_D^\times \times GL_h(\mathcal{O}_L)$-equivariant if we let $(\phi, g) \in \mathcal{O}_D^\times \times GL_h(\mathcal{O}_L)$ act on $(\mathcal{O}_L/\pi_L^n \mathcal{O}_L)^\times$ by $\mathrm{Nrd}(\phi)\det(g)^{-1} \bmod (1 + \pi_L^n \mathcal{O}_L)$, where Nrd is the reduced norm.

Chapter 4

THE ENDOMORPHISM ALGEBRAS, GL_H AND REPRESENTATION THEORY

4.1 The endomorphism algebras and GL_h

Recall from Lemma 2.2.1 that

$$\mathcal{O}_{D_{m,n}} = \mathrm{End}(G_{m,n}) \cong \mathcal{O}_{L_{m+n}}[\pi_{m,n}],$$

where $\pi_{m,n}$ is a uniformizer of the discrete valuation ring $\mathcal{O}_{D_{m,n}}$.

Using the basis $\{e_0, e_1, \ldots, e_{h-1}\}$ of $D(G_{1,h-1})$ introduced in Section 2.2, the proof of Lemma 2.2.1 shows that we get the following embedding of $D^\times \cong L_h(\varpi)\backslash\{0\}$ into $GL_h(L_h)$:

$$\begin{aligned}
\iota_D : D^\times &\hookrightarrow GL_h(L_h) \\
\lambda &\mapsto \begin{pmatrix} \lambda & & & \\ & \sigma^{-1}(\lambda) & & \\ & & \ddots & \\ & & & \sigma^{-(h-1)}(\lambda) \end{pmatrix} \\
\varpi &\mapsto \begin{pmatrix} 0 & 0 & \ldots & 0 & \pi_L \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.
\end{aligned}$$

Similarly, using the basis $\{f_0, f_1, \ldots, f_{h-1}\}$ of $D(G_{h-1,1})$ in Section 2.2, we get the embedding

$$\begin{aligned} \iota_E : E^\times &\hookrightarrow GL_h(L_h) \\ \lambda &\mapsto \begin{pmatrix} \lambda & & & \\ & \sigma^{-1}(\lambda) & & \\ & & \ddots & \\ & & & \sigma^{-(h-1)}(\lambda) \end{pmatrix} \\ \varpi' &\mapsto \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \pi_L & 0 & 0 & \dots & 0 \end{pmatrix} \end{aligned}$$

of $E^\times \cong L_h(\varpi')\backslash\{0\}$ into $GL_h(L_h)$.

Lemma 4.1.1. *(a) For any $\phi \in D^\times$,*

$$\mathrm{Nrd}(\phi) = \det(\iota_D(\phi))$$

where Nrd *is the reduced norm of the central division algebra D over L. In particular, we have* $\det(\iota_D(\phi)) \in L^\times$*. Furthermore,*

$$\mathrm{val}(\phi) = v_L(\det(\iota_D(\phi)))$$

where val *is the normalized valuation on the division algebra D, and v_L is the usual normalized π_L-adic valuation on L.*

(b) The result in (a) holds with D replaced by E, and ι_D replaced by ι_E.

Proof. We shall prove (a). The proof of (b) is almost identical. The map

$$\begin{aligned} D \times L_h &\to M_h(L_h) \\ (\phi, \mu) &\mapsto \mu \cdot \iota_D(\phi) \end{aligned}$$

is L-bilinear, so we have an L-linear map

$$\begin{aligned} f : D \otimes_L L_h &\to M_h(L_h) \\ \phi \otimes \mu &\mapsto \mu \cdot \iota_D(\phi). \end{aligned}$$

It is clear that this map respects multiplication, so it is in fact a ring homomorphism.

Let us show that f is injective. Suppose $f(\phi \otimes \mu) = I_h$. Then $\mu \cdot \iota_D(\phi) = I_h$, so $\iota_D(\phi) = \mu^{-1} I_h$. But the only scalar matrices in $\mathrm{Im}(\iota_D)$ are of the form μI_h where $\mu \in L$. Hence, $\phi \otimes \mu = \mu^{-1} \otimes \mu = 1$, proving injectivity.

Since the dimensions of $D \otimes L_h$ and $M_h(L_h)$ over L_h are both equal to h^2, f is an isomorphism. Hence

$$\mathrm{Nrd}(\phi) = \det(f(\phi \otimes 1)) = \det(\iota_D(\phi)),$$

proving the first assertion.

The second part follows immediately since $\mathrm{val}(\phi) = v_L(\mathrm{Nrd}(\phi))$. □

We shall view $D^\times$ and $E^\times$ as subgroups of $GL_h(L_h)$ using the embeddings ι_D and ι_E. Define the maps

$$\begin{aligned} \theta : D^\times &\to E^\times \\ \phi &\mapsto (\det \phi)(\phi^{-1})^T, \end{aligned} \qquad \begin{aligned} \psi : GL_h(L) &\to GL_h(L) \\ g &\mapsto (\det g)(g^{-1})^T. \end{aligned}$$

Lemma 4.1.2. *Let $d = (h-1, q-1)$ and $e = v_p(h-1)$. The map*

$$\begin{aligned} \theta_0 : \mathcal{O}_D^\times &\to \mathcal{O}_E^\times \\ \phi &\mapsto (\det \phi)(\phi^{-1})^T \end{aligned}$$

(a) has kernel

$$\ker(\theta_0) = \boldsymbol{\mu}_{h-1}(\mathcal{O}_L) = \boldsymbol{\mu}_{p^c d}(\mathcal{O}_L) = \{\zeta \in \mathcal{O}_L : \zeta^{p^c d} = 1\}$$

of order $p^c d$, where c is the maximum integer $\leq e$ such that $\mathcal{O}_L$ contains the p^c roots of unity. If L is unramified over $\mathbb{Q}_p$, then $c = 0$ and $\ker(\theta_0) = \boldsymbol{\mu}_d(\mathcal{O}_L)$ has order d.

(b) image

$$\mathrm{Im}(\theta_0) = \{\varphi \in \mathcal{O}_E^\times : \det(\varphi) \in (\mathcal{O}_L^\times)^{h-1}\}$$

which is a normal subgroup of $\mathcal{O}_E^\times$.

Proof.

(a) Let $\phi \in \ker(\theta_0)$. Since

$$1 = \det(\theta_0(\phi)) = (\det \phi)^{h-1},$$

$\det\phi$ must be a $(h-1)$ root of unity in $\mathcal{O}_L$. By Hensel's lemma, any root of $x^{\frac{h-1}{p^e}} - 1 = 0$ in $\mathbb{F}_q$ lifts uniquely to a root in $\mathcal{O}_L$. Since $\mathbb{F}_q^\times$ is cyclic of order $q-1$, and $(h-1, q-1) = d$, the map $x \mapsto x^{\frac{h-1}{p^e d}}$ is an automorphism of $\mathbb{F}_q^\times$, so $\boldsymbol{\mu}_{\frac{h-1}{p^e}}(\mathbb{F}_p) = \boldsymbol{\mu}_d(\mathbb{F}_p)$, which shows that $\boldsymbol{\mu}_{h-1}(\mathcal{O}_L) = \boldsymbol{\mu}_{p^e d}(\mathcal{O}_L) = \boldsymbol{\mu}_{p^c d}(\mathcal{O}_L)$. So

$$\phi \in \ker(\theta_0) \Rightarrow \det\phi \in \boldsymbol{\mu}_{p^c d}(\mathcal{O}_L) \Rightarrow (\phi^{-1})^T \in \boldsymbol{\mu}_{p^c d}(\mathcal{O}_L) \Rightarrow \phi \in \boldsymbol{\mu}_{p^c d}(\mathcal{O}_L).$$

If L is unramified over $\mathbb{Q}_p$, then L does not contain any non-trivial p-th roots of unity, so $c = 0$ and $\boldsymbol{\mu}_{h-1}(\mathcal{O}_L) = \boldsymbol{\mu}_d(\mathcal{O}_L)$.

(b) Since $\det(\theta_0(\phi)) = (\det\phi)^{h-1}$, and $\det\phi \in \mathcal{O}_L^\times$, the inclusion

$$\operatorname{Im}(\theta_0) \subseteq \{\varphi \in \mathcal{O}_E^\times : \det(\varphi) \in (\mathcal{O}_L^\times)^{h-1}\}$$

is clear. To show the reverse inclusion, we need to show that any φ satisfying $\det(\varphi) \in (\mathcal{O}_L^\times)^{h-1}$ lies in $\operatorname{Im}(\theta_0)$. Let $\gamma \in \mathcal{O}_L^\times$ be a $(h-1)$ root of $\det(\varphi)$. Then

$$\theta(\gamma(\varphi^T)^{-1}) = \det(\gamma(\varphi^T)^{-1})\gamma^{-1}\varphi = \gamma^{h-1}(\det\varphi)^{-1}\varphi = \varphi.$$

This proves the reverse inclusion. □

Proposition 4.1.3. *Let c, d, e be as in Lemma 4.1.2. The map*

$$\begin{aligned} \theta : D^\times &\to E^\times \\ \phi &\mapsto (\det\phi)(\phi^{-1})^T \end{aligned}$$

has

(a) kernel

$$\ker(\theta) = \boldsymbol{\mu}_{p^c d}(\mathcal{O}_L) = \{\zeta \in \mathcal{O}_L : \zeta^{p^c d} = 1\}$$

of order $p^c d$, and

(b) image

$$\operatorname{Im}(\theta) = \{\varphi \in E^\times : (h-1) \mid \operatorname{val}(\varphi),\ \det(\varphi) \in (L^\times)^{h-1}\}.$$

Let E^θ be the subgroup of $E^\times$ generated by the subgroups $\mathcal{O}_{L_h}^\times$ and $\operatorname{Im}(\theta)$, i.e. $E^\theta = \langle \mathcal{O}_{L_h}^\times, \operatorname{Im}(\theta)\rangle$. Then

(c) $E^\theta = \{\varphi \in E^\times : (h-1) \mid \operatorname{val}(\varphi)\}$,

(d) the cokernel

$$\operatorname{coker}(\theta) = \frac{E^\times}{\operatorname{Im}(\theta)} \cong \frac{E^\theta}{\operatorname{Im}(\theta)} \times \frac{E^\times}{E^\theta}$$

with

$$\frac{E^\theta}{\operatorname{Im}(\theta)} \cong \frac{\mathcal{O}_E^\times}{\operatorname{Im}(\theta_0)} \quad \text{and} \quad \frac{E^\times}{E^\theta} \cong \frac{\mathbb{Z}}{(h-1)\mathbb{Z}}.$$

Proof. Parts (a) and (b) follow almost immediately from Lemma 4.1.2.

(c) It is clear that $E^\theta \subseteq \{\varphi \in E^\times : (h-1) | \operatorname{val}(\varphi)\}$ since

$$\det(\theta(\phi)) = (\det \phi)^{h-1}$$

and $\mathcal{O}_{L_h}^\times \subseteq \mathcal{O}_E^\times$.

Let us prove the reverse inclusion. It suffices to show that $\mathcal{O}_E^\times \subseteq E^\theta$. An arbitrary element of $\mathcal{O}_E^\times$ is of the form

$$\varphi = a_0 + a_1\varpi' + a_2(\varpi')^2 + \cdots + a_{h-1}(\varpi')^{h-1},$$

with $a_0 \in \mathcal{O}_{L_h}^\times$ and $a_i \in \mathcal{O}_{L_h}$ for all i.

We observe that the determinant map on $D^\times$, when restricted to the subgroup $\mathcal{O}_{L_h}^\times \leq D^\times$, is equal to the norm map $N_{L_h/L} : \mathcal{O}_{L_h}^\times \to \mathcal{O}_L^\times$. Since L_h is an unramified extension of L, the map $N_{L_h/L} : \mathcal{O}_{L_h}^\times \to \mathcal{O}_L^\times$ is surjective.

Now, given any $\varphi \in \mathcal{O}_E^\times$, since $\det \varphi \in L^\times$ by Lemma 4.1.1, we must have $\det \varphi \in \mathcal{O}_L^\times$. By surjectivity of the map $N_{L_h/L} : \mathcal{O}_{L_h}^\times \to \mathcal{O}_L^\times$, we can find $u \in \mathcal{O}_{L_h}^\times \leq D^\times$ such that $\det(u) = \det(\varphi)$. Then

$$\theta(u(\varphi^T)^{-1}) = \det(u(\varphi^T)^{-1})u^{-1}\varphi = u^{-1}\varphi,$$

so $u^{-1}\varphi \in E^\theta$ and hence $\varphi \in E^\theta$. This proves the reverse inclusion.

(d) Define

$$\frac{E^\times}{\operatorname{Im}(\theta)} \to \frac{E^\theta}{\operatorname{Im}(\theta)} \times \frac{E^\times}{E^\theta}$$
$$\varphi \operatorname{Im}(\theta) \mapsto (\varphi\varpi^{-b} \operatorname{Im}(\theta), \varpi^b E^\theta)$$

where b is any integer congruent to $\operatorname{val}(\varphi) \bmod (h-1)$. Since $\varpi^{h-1} \in \operatorname{Im}(\theta) \subseteq E^\theta$, it is clear that this is well defined and is an homomorphism. By (iii), $E^\theta = \{\varphi \in E^\times : (h-1) | \operatorname{val}(\varphi)\}$, so the valuation map gives an isomorphism

$$\frac{E^\times}{E^\theta} \cong \frac{\mathbb{Z}}{(h-1)\mathbb{Z}},$$

from which surjectivity of the above map follows easily. For injectivity, we note that

$$\varpi^b \in E^\theta = \{\varphi \in E^\times : (h-1) | \operatorname{val}(\varphi)\} \Rightarrow (h-1)|b \Rightarrow \varpi^b \in \operatorname{Im}(\theta).$$

Hence

$$\begin{aligned} \varphi\varpi^{-b} \in \operatorname{Im}(\theta) \text{ and } \varpi^b \in E^\theta &\Rightarrow \varphi\varpi^{-b} \in \operatorname{Im}(\theta) \text{ and } \varpi^b \in \operatorname{Im}(\theta) \\ &\Rightarrow \varphi \in \operatorname{Im}(\theta). \end{aligned}$$

Now we consider the map

$$\begin{aligned} E^\theta &\twoheadrightarrow \frac{\mathcal{O}_E^\times}{\operatorname{Im}(\theta_0)} \\ \varphi &\mapsto \varphi\varpi^{-\operatorname{val}(\varphi)} \operatorname{Im}(\theta_0) \end{aligned}$$

Then

$$\begin{aligned} &\varphi \in E^\theta \text{ lies in the kernel} \\ \Leftrightarrow\ &\varphi\varpi^{-\operatorname{val}(\varphi)} \in \operatorname{Im}(\theta_0) \\ \Leftrightarrow\ &\det(\varphi\varpi^{-\operatorname{val}(\varphi)}) \in (\mathcal{O}_L^\times)^{h-1} \quad (\text{since } (h-1)|\operatorname{val}(\varphi) \Rightarrow \det(\varpi^{-\operatorname{val}(\varphi)}) \in (L^\times)^{h-1}) \\ \Leftrightarrow\ &\det(\varphi) \in (L^\times)^{h-1} \\ \Leftrightarrow\ &\varphi \in \operatorname{Im}(\theta), \end{aligned}$$

so

$$\frac{E^\theta}{\operatorname{Im}(\theta)} \cong \frac{\mathcal{O}_E^\times}{\operatorname{Im}(\theta_0)}.$$

□

Proposition 4.1.4. *Let c, d, e be as in Lemma 4.1.2, and let e_L be the ramification degree of L over $\mathbb{Q}_p$. $\theta_0 : \mathcal{O}_D^\times \to \mathcal{O}_E^\times$ induces a map $\theta_{0,n} : (\mathcal{O}_L/\pi_L^n\mathcal{O}_L)^\times \to (\mathcal{O}_L/\pi_L^n\mathcal{O}_L)^\times$ making the following diagram*

$$\begin{array}{ccccc} \mathcal{O}_D^\times & \xrightarrow{\det} & \mathcal{O}_L^\times & \twoheadrightarrow & \left(\frac{\mathcal{O}_L}{\pi_L^n\mathcal{O}_L}\right)^\times \\ \downarrow{\scriptstyle \theta_0} & & \downarrow & & \downarrow{\scriptstyle \theta_{0,n}} \\ \mathcal{O}_E^\times & \xrightarrow[\det]{} & \mathcal{O}_L^\times & \twoheadrightarrow & \left(\frac{\mathcal{O}_L}{\pi_L^n\mathcal{O}_L}\right)^\times \end{array}$$

commute. $\theta_{0,n}$ is the map

$$\begin{aligned} \theta_{0,n} : (\mathcal{O}_L/\pi_L^n\mathcal{O}_L)^\times &\to (\mathcal{O}_L/\pi_L^n\mathcal{O}_L)^\times \\ x &\mapsto x^{h-1}. \end{aligned}$$

For $n > 2ee_L$,

(a) $\mathrm{Im}(\theta_0)$ *is the preimage of* $\mathrm{Im}(\theta_{0,n})$ *under the* $\det \bmod (1+\pi_L^n\mathcal{O}_L)$ *map,*

(b) the $\det \bmod (1+\pi_L^n\mathcal{O}_L)$ *map induces an isomorphism*

$$\mathrm{coker}(\theta_0) = \frac{\mathcal{O}_E^\times}{\mathrm{Im}(\theta_0)} \xrightarrow{\cong} \frac{(\mathcal{O}_L/\pi_L^n\mathcal{O}_L)^\times}{\mathrm{Im}(\theta_{0,n})} = \mathrm{coker}(\theta_{0,n}),$$

(c) $\ker(\theta_{0,n})$ *can be written as a direct product* $\ker(\theta_{0,n}) = K'_{n,1} \times K'_{n,2}$*, where* $K'_{n,2} = (1+\pi_L^{n-ee_L}\mathcal{O}_L)/(1+\pi_L^n\mathcal{O}_L)$*, and the* $\det \bmod (1+\pi_L^n\mathcal{O}_L)$ *map restricts to an isomorphism*

$$\ker(\theta_0) \xrightarrow[\cong]{\det \bmod (1+\pi_L^n\mathcal{O}_L)} K'_{n,1}.$$

In particular, if $e=0$*, then the* $\det \bmod (1+\pi_L^n\mathcal{O}_L)$ *map restricts to an isomorphism from* $\ker(\theta_0)$ *to* $\ker(\theta_{0,n})$.

Proof. To show that $\theta_0 : \mathcal{O}_D^\times \to \mathcal{O}_E^\times$ induces a map

$$\theta_{0,n} : \left(\mathcal{O}_L/\pi_L^n\mathcal{O}_L\right)^\times \to \left(\mathcal{O}_L/\pi_L^n\mathcal{O}_L\right)^\times$$

making the above diagram commute, we need to check that if $\phi \in \mathcal{O}_D^\times$ satisfies

$$\det\phi \equiv 1 \mod (1+\pi_L^n\mathcal{O}_L),$$

then

$$\det\theta_0(\phi) \equiv 1 \mod (1+\pi_L^n\mathcal{O}_L).$$

But this is clear since $\det(\theta_0(\phi)) = (\det\phi)^{h-1}$. The fact that $\theta_{0,n}$ is the $(h-1)$ power map is also immediate from this.

For $n_1 \geq n_2$, consider the map

$$\mathrm{coker}(\theta_{0,n_1}) = \frac{\left(\mathcal{O}_L/\pi_L^{n_1}\mathcal{O}_L\right)^\times}{\left((\mathcal{O}_L/\pi_L^{n_1}\mathcal{O}_L)^\times\right)^{h-1}} \twoheadrightarrow \frac{\left(\mathcal{O}_L/\pi_L^{n_2}\mathcal{O}_L\right)^\times}{\left((\mathcal{O}_L/\pi_L^{n_2}\mathcal{O}_L)^\times\right)^{h-1}} = \mathrm{coker}(\theta_{0,n_2}).$$

Suppose $a \in \mathcal{O}_L^\times$ is a $(h-1)$ power $\bmod\, \pi_L^{n_2}$. By Hensel's lemma, for $n_2 > 2ee_L$, a is also a $(h-1)$ power $\bmod\, \pi_L^{n_1}$, so the above map is a bijection for all $n_1 \geq n_2 > 2ee_L$. So, for all $n_1, n_2 > 2ee_L$, we have $|\mathrm{coker}(\theta_{0,n_1})| = |\mathrm{coker}(\theta_{0,n_2})|$, and hence $|\ker(\theta_{0,n_1})| = |\ker(\theta_{0,n_2})|$.

Now, let us assume that $n > 2ee_L$. Since $n > 2ee_L \Leftrightarrow 2(n-ee_L) > n$, it is clear that

$$\ker(\theta_{0,n}) \supseteq (1+\pi_L^{n-ee_L}\mathcal{O}_L)/(1+\pi_L^n\mathcal{O}_L) = K'_{n,2}.$$

We note that

$$
\begin{aligned}
&\left|\ker(\theta_{0,n'})\right| = \left|\ker(\theta_{0,n})\right| \text{ for all } n' \geq n \\
\Rightarrow &\left|\frac{\ker(\theta_{0,n'})}{1+\pi_L^{n'-ee_L}O_L}\right| = \left|\frac{\ker(\theta_{0,n})}{1+\pi_L^{n-ee_L}O_L}\right| \text{ for all } n' \geq n.
\end{aligned}
$$

Hence for any $a_0 + \cdots + a_{n-ee_L-1}\pi_L^{n-ee_L-1} \in \ker(\theta_{0,n})$, there exists a unique $a_{n-ee_L} \in O_L/\pi_L$ such that $a_0+\cdots+a_{n-ee_L-1}\pi_L^{n-ee_L-1}+a_{n-ee_L}\pi_L^{n-ee_L} \in \ker(\theta_{0,n+1})$. Therefore, the set of elements $a_0 + \cdots + a_{n-ee_L-1}\pi_L^{n-ee_L-1} \in \ker(\theta_{0,n})$ are given precisely by truncations of elements in $\ker(\theta_0)$. Let $K'_{n,1} = \ker(\theta_0)/(1+\pi_L^n O_L) \leq \ker(\theta_{0,n})$. We have shown that $\ker(\theta_{0,n}) = K'_{n,1}K'_{n,2}$. As $K'_{n,1} \cap K'_{n,2} = \{1\}$, this is a direct product, proving (c).

To prove (b), we first show that

$$
\operatorname{coker}(\theta_0) \xrightarrow[\cong]{\det} \frac{O_L^\times}{(O_L^\times)^{h-1}}.
$$

Consider the commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & K & \longrightarrow & \operatorname{Im}(\theta_0) & \xrightarrow{\det} & (O_L^\times)^{h-1} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & K & \longrightarrow & O_E^\times & \xrightarrow{\det} & O_L^\times & \longrightarrow & 1
\end{array}
$$

with exact rows, where $K = \{\varphi \in O_E^\times : \det\varphi = 1\}$. By the snake lemma,

$$
\operatorname{coker}(\theta_0) = \frac{O_E^\times}{\operatorname{Im}\theta_0} \xrightarrow[\cong]{\det} \frac{O_L^\times}{(O_L^\times)^{h-1}}.
$$

We have an isomorphism

$$
O_L^\times \cong \boldsymbol{\mu}_{q-1} \times \boldsymbol{\mu}_{p^a} \times \mathbb{Z}_p^{n_L}
$$

where $n_L = [L : \mathbb{Q}_p]$ and a is such that the group of p-power roots of unity in $O_L^\times$ is $\boldsymbol{\mu}_{p^a}$. So

$$
\begin{aligned}
\operatorname{coker}(\theta_0) \cong \frac{O_L^\times}{(O_L^\times)^{h-1}} &\cong \frac{\boldsymbol{\mu}_{q-1}}{(\boldsymbol{\mu}_{q-1})^{h-1}} \times \frac{\boldsymbol{\mu}_{p^a}}{(\boldsymbol{\mu}_{p^a})^{h-1}} \times \left(\frac{\mathbb{Z}_p}{(h-1)\mathbb{Z}_p}\right)^{n_L} \\
&\cong \frac{\boldsymbol{\mu}_{q-1}}{(\boldsymbol{\mu}_{q-1})^{d}} \times \frac{\boldsymbol{\mu}_{p^a}}{(\boldsymbol{\mu}_{p^a})^{p^e}} \times \left(\frac{\mathbb{Z}}{p^e\mathbb{Z}}\right)^{n_L}.
\end{aligned}
$$

Clearly, the kernel of the map

$$\mathcal{O}_L^\times \xrightarrow{\text{mod }(1+\pi_L^n\mathcal{O}_L)} \frac{\left(\mathcal{O}_L/\pi_L^n\mathcal{O}_L\right)^\times}{\left((\mathcal{O}_L/\pi_L^n\mathcal{O}_L)^\times\right)^{h-1}} = \operatorname{coker}(\theta_{0,n})$$

contains $(\mathcal{O}_L^\times)^{h-1}$, hence is equal to $(\mathcal{O}_L^\times)^{h-1}$ for $n > 2ee_L$, since

$$\left|\frac{\mathcal{O}_L^\times}{(\mathcal{O}_L^\times)^{h-1}}\right| = p^{en_L+c}d = |\ker(\theta_{0,n})| = |\operatorname{coker}(\theta_{0,n})|\,.$$

Therefore

$$\operatorname{coker}(\theta_0) \underset{\cong}{\xrightarrow{\det}} \frac{(\mathcal{O}_L^\times)}{(\mathcal{O}_L^\times)^{h-1}} \underset{\cong}{\xrightarrow{\text{mod }(1+\pi_L^n\mathcal{O}_L)}} \operatorname{coker}(\theta_{0,n}).$$

(a) then follows immediately from this. □

Results analogous to those in Lemma 4.1.2 and Propositions 4.1.3, 4.1.4 also hold for the maps

$$\begin{aligned} \psi_0 : GL_h(\mathcal{O}_L) &\to GL_h(\mathcal{O}_L) \\ g &\mapsto (\det g)(g^{-1})^T, \end{aligned} \qquad \begin{aligned} \psi : GL_h(L) &\to GL_h(L) \\ g &\mapsto (\det g)(g^{-1})^T. \end{aligned}$$

4.2 Representation theory

We start this section with a useful result that expresses the cohomology of certain rigid analytic spaces in terms of the cohomology of a subspace.

Proposition 4.2.1. *Let X be a rigid analytic space with an action of the group G, and let Y be a subspace of X on which the subgroup H of G acts. Suppose*

(i) H is normal in G,

(ii) $\frac{G}{H}$ is a finite abelian group,

(iii) X is the disjoint union

$$X = \coprod_{g\in\frac{G}{H}} g(Y).$$

Then

$$H_c^i(X, \overline{\mathbb{Q}}_l) \cong \operatorname{Ind}_H^G H_c^i(Y, \overline{\mathbb{Q}}_l)$$

as G-representations.

Proof. We first prove the result in the case that $\frac{G}{H}$ is cyclic of order m, generated by $g_0 \in G$. In this case, (iii) becomes

$$X = Y \sqcup g_0(Y) \sqcup \cdots \sqcup g_0^{m-1}(Y).$$

So we have

$$H_c^i(X, \overline{\mathbb{Q}}_l) \cong \bigoplus_{j=0}^{m-1} H_c^i(Y, \overline{\mathbb{Q}}_l)$$

as $\overline{\mathbb{Q}}_l$-vector spaces. Let x_k be the element of $\bigoplus_{j=0}^{m-1} H_c^i(Y, \overline{\mathbb{Q}}_l)$ with k-th coordinate equal to x and which is 0 everywhere else. We pick the above isomorphism so that

$$g_0 \cdot x_k = x_{k+1}, \quad 0 \leq k < m-1.$$

Define

$$f : \operatorname{Ind}_H^G H_c^i(Y, \overline{\mathbb{Q}}_l) \to H_c^i(X, \overline{\mathbb{Q}}_l) \cong \bigoplus_{j=0}^{m-1} H_c^i(Y, \overline{\mathbb{Q}}_l)$$

$$g_o^k x \mapsto x_k.$$

We will show that this is an isomorphism of representations.

Let $g \in G$. Then we can write

$$g = hg_0^l \quad \text{with } h \in H,\ 0 \leq l \leq m-1.$$

So it suffices to show that f is compatible with both the actions of $h \in H$ and of g_0.

For $0 \leq k < m-1$,

$$g_0 \cdot (g_0^k x) = g_0^{k+1} x \overset{f}{\longmapsto} x_{k+1} = g_0 \cdot x_k = g_0 \cdot f(g_0^k x)$$

and since $g_0^m \in H$,

$$g_0 \cdot (g_0^{m-1} x) = g_0^0 (g_0^m x) \overset{f}{\longmapsto} (g_0^m \cdot x)_0 = g_0^m \cdot x_0 = g_0 \cdot x_{m-1} = g_0 \cdot f(g_0^{m-1} x).$$

And for $h \in H$, we have $hg_0^k = g_0^k h'$ for some $h' \in H$ since H is normal in G, so

$$h \cdot (g_0^k x) = g_0^k (h' \cdot x) \overset{f}{\longmapsto} (h' \cdot x)_k = g_0^k \cdot (h' \cdot x)_0 = (g_0^k h') \cdot x_0 = (hg_0^k) \cdot x_0 = h \cdot x_k = h \cdot f(g_0^k x).$$

So f is an isomorphism of G-representations.

Now, for the general case where $\frac{G}{H}$ is finite abelian, we can write $\frac{G}{H}$ as a direct product $\frac{G}{H} = \frac{G_1}{H} \times \frac{G_2}{H} \times \cdots \times \frac{G_n}{H}$, where each G_j is a subgroup of G containing H and each $\frac{G_j}{H}$ is finite cyclic. Let $H_j = G_1 \cdots G_j$. Since $\frac{G}{H}$ is abelian, $g_{j_1} G_{j_2} = G_{j_2} g_{j_1}$ for all $g_{j_1} \in G_{j_1}$, $1 \leq j_1, j_2 \leq n$, so H_j is a subgroup of G and H_{j-1} is normal in H_j. Furthermore, we have $H_n = G$ and $\frac{H_j}{H_{j-1}} = \frac{G_1 \cdots G_j}{G_1 \cdots G_{j-1}} \cong \frac{G_j}{(G_1 \cdots G_{j-1}) \cap G_j} = \frac{G_j}{H}$. Define X_j recursively by $X_0 = Y$ and

$$X_j = \coprod_{g_j \in \frac{H_j}{H_{j-1}}} g_j(X_{j-1}).$$

(iii) implies that the above is indeed a disjoint union for all $1 \leq j \leq n$, and that $X = X_n$. Applying the previous case repeatedly shows that

$$H^i_c(X, \overline{\mathbb{Q}}_l) \cong \operatorname{Ind}^{H_n}_{H_{n-1}} \operatorname{Ind}^{H_{n-1}}_{H_{n-2}} \cdots \operatorname{Ind}^{H_1}_{H_0} H^i_c(Y, \overline{\mathbb{Q}}_l) \cong \operatorname{Ind}^G_H H^i_c(Y, \overline{\mathbb{Q}}_l).$$

□

Next, we include a result that will be needed later in Section 6.3 to show that the supercuspidal part of the cohomology of the dual Lubin-Tate tower realizes the local Langlands and the Jacquet-Langlands correspondences. This result is useful as local Langlands and Jacquet-Langlands both behave well with respect to twists.

Lemma 4.2.2. *(a) Let (ρ, V) be an irreducible representation of $E^\times$, and let $\iota : D^\times \to E^\times$ be the isomorphism*

$$\begin{array}{ccc} D^\times & \xrightarrow[\cong]{\iota} & E^\times \\ \big\downarrow & & \big\downarrow \\ GL_h(L_h) & \xrightarrow[\phi \mapsto (\phi^T)^{-1}]{} & GL_h(L_h). \end{array}$$

Then $\theta^*\left(\operatorname{Res}^{E^\times}_{\theta(D^\times)} \rho\right) \cong \iota^*\rho \otimes (\chi_{\iota^*\rho} \circ \mathrm{Nrd})^\vee$, where $\chi_{\iota^*\rho}$ is the central character of $\iota^*\rho$, and Nrd *: $D \to L$ is the reduced norm map of the division algebra D.*

(b) Let π be an irreducible representation of $GL_h(L)$, and let j be the isomorphism

$$\begin{aligned} j : GL_h(L) &\xrightarrow{\cong} GL_h(L) \\ g &\mapsto (g^T)^{-1}. \end{aligned}$$

Then $\psi^*\left(\operatorname{Res}^{GL_h(L)}_{\psi(GL_h(L))} \pi\right) \cong j^*\pi \otimes (\chi_{j^*\pi} \circ \det)^\vee$, where $\chi_{j^*\pi}$ is the central character of $j^*\pi$.

Proof. We will prove (a). The proof of (b) is almost identical.

First, note that, by Lemma 4.1.1, for our embedding of $D^\times$ into $GL_h(L_h)$, the determinant map on $GL_h(L_h)$, when restricted to $D^\times$, has image in L, and is in fact the reduced norm map $\mathrm{Nrd} : D^\times \to L$. In other words, the diagram

$$\begin{array}{ccc} D^\times & \xrightarrow{\mathrm{Nrd}} & L \\ \big\downarrow & & \big\downarrow \\ GL_h(L_h) & \xrightarrow[\det]{} & L_h \end{array}$$

commutes.

For $\phi \in D^\times$, $v \in V$,

$$((\theta^* \operatorname{Res} \rho)(\phi))(v) = (\rho((\det \phi)(\phi^T)^{-1}))(v) = (\iota^* \rho((\det \phi)^{-1}\phi))(v).$$

Let $\iota^*\rho \otimes (\chi_{\iota^*\rho} \circ \mathrm{Nrd})^\vee$ act on the vector space $V \otimes_{\overline{\mathbb{Q}}_l} \mathrm{Hom}_{\overline{\mathbb{Q}}_l}(\overline{\mathbb{Q}}_l, \overline{\mathbb{Q}}_l)$. Consider the isomorphism

$$\begin{aligned} V &\to V \otimes_{\overline{\mathbb{Q}}_l} \mathrm{Hom}_{\overline{\mathbb{Q}}_l}(\overline{\mathbb{Q}}_l, \overline{\mathbb{Q}}_l) \\ \alpha v &\mapsto v \otimes f_\alpha \quad (\text{where } f_\alpha : c \mapsto \alpha c). \end{aligned}$$

For $\phi \in D^\times$,

$$\begin{aligned} (((\chi_{\iota^*\rho} \circ \mathrm{Nrd})^\vee(\phi))(f_\alpha))(c) &= f_\alpha((\chi_{\iota^*\rho} \circ \mathrm{Nrd})(\phi^{-1})(c)) \\ &= f_\alpha(\iota^*\rho(\det \phi^{-1})(c)) \\ &= \alpha(\iota^*\rho(\det \phi^{-1}))(c), \end{aligned}$$

so $((\chi_{\iota^*\rho} \circ \mathrm{Nrd})^\vee(\phi))(f_\alpha) = f_{\alpha(\iota^*\rho(\det \phi^{-1}))}$, and

$$\begin{aligned} &(\iota^*\rho(\phi))(v) \otimes ((\chi_{\iota^*\rho} \circ \mathrm{Nrd})^\vee(\phi))(f_1) = (\iota^*\rho(\phi))(v) \otimes f_{\iota^*\rho(\det \phi^{-1})} \\ &\mapsto (\iota^*\rho((\det \phi)^{-1}\phi))(v), \end{aligned}$$

as required. □

Chapter 5

THE DUALITY MAP AND THE EXTERIOR POWER MAP ON MODULI SPACES

5.1 Definitions of the duality map and the exterior power map

In this section, we will use the Serre dual and the exterior power of a π_L-divisible $\mathcal{O}_L$-module to define maps from LT_h^∞ to M_h^∞.

We start with the Serre dual. Note that $G_{m,n}^\vee \cong G_{m,n}$, so $G_{1,h-1}^\vee \cong G_{h-1,1}$ has dimension $h-1$ and height h. Let us fix such an isomorphism. The dual of a π_L-divisible $\mathcal{O}_L$-module X has Tate module $(T_{\pi_L}X)^\vee(1)$. In particular, the Tate module of the dual of $(\mathcal{O}_L/\pi_L^n\mathcal{O}_L)^h$ is equal to $\mathcal{O}_L^h(1)$. Over $\breve{L}(\zeta_{p^\infty})$, we have $\mathcal{O}_L^h(1) \cong \mathcal{O}_L^h$ since $\breve{L}(\zeta_{p^\infty})$ contains the p^∞ roots of unity.

Definition 5.1.1. Define $\vee : LT_h^\infty \times_{\breve{L}} \breve{L}(\zeta_{p^\infty}) \to M_h^\infty \times_{\breve{L}} \breve{L}(\zeta_{p^\infty})$ by

$$(X, \beta, \alpha) \mapsto (X^\vee, (\beta^\vee)^{-1}, (\alpha_{\zeta_{p^n}}^\vee)^{-1})$$

where $\alpha_{\zeta_{p^n}}^\vee$ is given by the composition

$$T_{\pi_L}X^\vee \xrightarrow{\cong} (T_{\pi_L}X)^\vee(1) \xrightarrow{\cong} \mathcal{O}_L^h(1) \xrightarrow{\cong} \mathcal{O}_L^h.$$

Let us now look at the exterior power. Recall that $\bigwedge^{h-1} G_{1,h-1} \cong G_{h-1,1}$. We will fix such an isomorphism.

Definition 5.1.2. Define $\bigwedge^{h-1} : LT_h^\infty \to M_h^\infty$ by

$$(X, \beta, \alpha) \mapsto \left(\bigwedge^{h-1} X, \bigwedge^{h-1} \beta, \bigwedge^{h-1} \alpha\right).$$

In the above, by a slight abuse of notation, $\bigwedge^{h-1} \alpha$ is the level structure on $\bigwedge^{h-1} X$ given by

$$\mathcal{O}_L^h \xrightarrow[\cong]{\phi} \bigwedge^{h-1} \mathcal{O}_L^h \xrightarrow[\cong]{\bigwedge^{h-1}\alpha} \bigwedge^{h-1} T_pX \xrightarrow[\cong]{} T_p\left(\bigwedge^{h-1} X\right),$$

where ϕ is the isomorphism

$$v_i \mapsto (-1)^i v_1 \wedge v_2 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_h,$$

$v_1, \ldots, v_h$ being the standard basis vectors for $\mathcal{O}_L^h$.

5.2 Properties of the exterior power map and group actions

In order to study the cohomology of the dual Lubin-Tate tower using the exterior power map $\bigwedge^{h-1}$, it is necessary to first understand how the group actions behave with respect to $\bigwedge^{h-1}$.

Recall from Section 4.1 that we have embeddings of $D^\times$ and $E^\times$ in $GL_h(L_h)$. We shall view $D^\times$ and $E^\times$ as subgroups of $GL_h(L_h)$ using these embeddings. We also recall the maps

$$\begin{aligned} \theta : D^\times &\to E^\times & \psi : GL_h(L) &\to GL_h(L) \\ \phi &\mapsto (\det \phi)(\phi^{-1})^T, & g &\mapsto (\det g)(g^{-1})^T. \end{aligned}$$

Proposition 5.2.1. *The map* $\bigwedge^{h-1} : LT_h^\infty \to M_h^\infty$ *is*

(a) $D^\times$*-equivariant if we let* $D^\times$ *act on* M_h^∞ *via* θ*,*

(b) $GL_h(L)$*-equivariant if we let* $GL_h(L)$ *act on* M_h^∞ *via* ψ*,*

(c) W_L*-equivariant.*

Proof.

(a) Suppose $\phi \in D^\times$ is given by

$$\phi e_i = \sum_{j=0}^{h-1} \alpha_{ji} e_j.$$

Then

$$\begin{aligned} \left(\bigwedge^{h-1} \phi\right) \tilde{e}_i &= (-1)^i \phi e_0 \wedge \phi e_1 \wedge \cdots \wedge \phi e_{i-1} \wedge \phi e_{i+1} \wedge \cdots \wedge \phi e_{h-1} \\ &= (-1)^i \sum_{j=0}^{h-1} \alpha_{j0} e_j \wedge \cdots \wedge \sum_{j=0}^{h-1} \alpha_{j,i-1} e_j \wedge \sum_{j=0}^{h-1} \alpha_{j,i+1} e_j \wedge \cdots \wedge \sum_{j=0}^{h-1} \alpha_{j,h-1} e_j \\ &= \sum_{k=0}^{h-1} \beta_k (e_0 \wedge e_1 \wedge \cdots \wedge e_{k-1} \wedge e_{k+1} \wedge \cdots \wedge e_{h-1}) \end{aligned}$$

for some $\beta_k \in L_h$. To write down an expression for β_k, it is helpful to first introduce the order-preserving bijections

$$\begin{aligned} b_l : \{0, 1, \ldots, l-1, l+1, \ldots, h-1\} &\to \{1, 2, \ldots, h-1\} \\ j &\mapsto \begin{cases} j+1 & \text{if } j < l, \\ j & \text{if } j > l. \end{cases} \end{aligned}$$

For $\tau \in S_{h-1}$, let $\tau_{ik} = b_k^{-1} \circ \tau \circ b_i$. Then

$$\beta_k = (-1)^i \sum_{\tau \in S_{h-1}} \operatorname{sgn}(\tau) \alpha_{\tau_{ik}(0),0} \cdots \alpha_{\tau_{ik}(i-1),i-1} \alpha_{\tau_{ik}(i+1),i+1} \cdots \alpha_{\tau_{ik}(h-1),h-1}.$$

So the coefficient of $\tilde{e}_k$ in $\left(\bigwedge^{h-1} \phi\right) \tilde{e}_i$ is given by

$$\begin{aligned} &(-1)^{i+k} \det(\phi \text{ with row and column containing } \alpha_{k,i} \text{ deleted}) \\ =\; & C_{k,i}, \text{ the cofactor of the element } a_{k,i} \text{ of } \phi. \end{aligned}$$

So

$$\bigwedge^{h-1} \phi = \begin{pmatrix} C_{0,0} & C_{0,1} & \cdots & C_{0,h-1} \\ C_{1,0} & C_{1,1} & \cdots & C_{1,h-1} \\ \vdots & \vdots & \ddots & \vdots \\ C_{h-1,0} & C_{h-1,1} & \cdots & C_{h-1,h-1} \end{pmatrix} = (\det \phi)(\phi^{-1})^T.$$

(b) It suffices to prove the proposition for $g \in GL_h(L) \cap \mathrm{Mat}_h(\mathcal{O}_L)$. For such g, it acts on LT_h^∞ by

$$(X, \beta, \alpha) \mapsto (Y, q \circ \beta, \overline{\alpha})$$

where $Y, q, \overline{\alpha}$ are as defined in Section 3.1. Recall that Y^{rig}, q^{rig} and $\overline{\alpha^{\mathrm{rig}}}$ (which corresponds to $\overline{\alpha}^{\mathrm{rig}} : \mathcal{O}_L^h \xrightarrow{\cong} T_{\pi_L} Y^{\mathrm{rig}}$) satisfy the commutative diagram

$$\begin{array}{ccc} (L/\mathcal{O}_L)^h & \xrightarrow[\cong]{\alpha^{\mathrm{rig}}} & X^{\mathrm{rig}} \\ \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle q^{\mathrm{rig}}} \\ (L/\mathcal{O}_L)^h & \xrightarrow[\cong]{\overline{\alpha^{\mathrm{rig}}}} & Y^{\mathrm{rig}} = \dfrac{X^{\mathrm{rig}}}{\alpha^{\mathrm{rig}}(\ker(g))}. \end{array}$$

Applying the functor $\bigwedge^{h-1}$ to the above diagram gives

$$\begin{array}{ccc} \bigwedge^{h-1}(L/\mathcal{O}_L)^h & \xrightarrow[\cong]{\bigwedge^{h-1} \alpha^{\mathrm{rig}}} & \bigwedge^{h-1} X^{\mathrm{rig}} \\ \Big\downarrow{\scriptstyle \bigwedge^{h-1} g} & & \Big\downarrow{\scriptstyle \bigwedge^{h-1} q^{\mathrm{rig}}} \\ \bigwedge^{h-1}(L/\mathcal{O}_L)^h & \xrightarrow[\cong]{\bigwedge^{h-1} \overline{\alpha^{\mathrm{rig}}}} & \bigwedge^{h-1} Y^{\mathrm{rig}}. \end{array}$$

By an argument similar to that in (a), the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}_L^h & \xrightarrow{\cong} & \bigwedge^{h-1} \mathcal{O}_L^h \\ \Big\downarrow{\scriptstyle \psi(g) = (\det g)(g^{-1})^T} & & \Big\downarrow{\scriptstyle \bigwedge^{h-1} g} \\ \mathcal{O}_L^h & \xrightarrow[\cong]{} & \bigwedge^{h-1} \mathcal{O}_L^h. \end{array}$$

Since $\bigwedge^{h-1} \overline{\alpha^{\mathrm{rig}}}$ corresponds to $(\bigwedge^{h-1} \overline{\alpha})^{\mathrm{rig}}$, using the above 2 diagrams, we get the commutative diagram

$$\begin{array}{ccc} O_L^h & \xrightarrow[\cong]{\bigwedge^{h-1} \alpha} & T_p\left(\bigwedge^{h-1} X\right) \\ \downarrow{\scriptstyle \psi(g)} & & \downarrow{\scriptstyle \bigwedge^{h-1} q} \\ O_L^h & \xrightarrow[\cong]{\bigwedge^{h-1} \overline{\alpha}} & T_p\left(\bigwedge^{h-1} Y\right). \end{array}$$

Hence $\psi(g)$ acts on M_h^∞ by

$$\left(\bigwedge^{h-1} X, \bigwedge^{h-1} \beta, \bigwedge^{h-1} \alpha\right) \mapsto \left(\bigwedge^{h-1} Y, \bigwedge^{h-1} q \circ \bigwedge^{h-1} \beta, \bigwedge^{h-1} \overline{\alpha}\right),$$

as required.

(c) Let $w \in W_L$. By the universal property of the exterior power, we have

$$F_{(\bigwedge^{h-1} G_{1,h-1})/\overline{\mathbb{F}}_p} = \bigwedge^{h-1} F_{G_{1,h-1}/\overline{\mathbb{F}}_p},$$

and

$$T_p\left(\bigwedge^{h-1} X\right) = \bigwedge^{h-1} T_p X.$$

The first equality shows that $(\bigwedge^{h-1} \beta)^w = \bigwedge^{h-1} \beta^w$, and the second gives $(\bigwedge^{h-1} \alpha)^w = \bigwedge^{h-1} \alpha^w$, so the map $\bigwedge^{h-1} : LT_h^\infty \to M_h^\infty$ is W_L-equivariant. $\square$

Proposition 5.2.1 tells us that the exterior power map $\bigwedge^{h-1}$ is W_L-equivariant. Unlike $\bigwedge^{h-1}$ however, the duality map $\vee$ is not W_L-equivariant, one of the reasons being that the Tate module of the dual of $(L/O_L)^h$ is $O_L^h(1)$. As such, it is difficult to understand the W_L-action on M_h^∞ by considering the duality map. However, the duality map $\vee$ can still be used to study the cohomology of M_h^∞ as a $E^\times \times GL_h(L)$-representation, and hence can be used to show that the supercuspidal part of the cohomology of M_h^∞ in the middle degree realizes the Jacquet-Langlands correspondence up to certain twists.

Chapter 6

THE GEOMETRY AND COHOMOLOGY OF THE DUAL LUBIN-TATE TOWER

6.1 Geometry of the dual Lubin-Tate tower

Before looking at the tower, it is helpful to first understand the level 0 situation. Let us write LT_h for the formal scheme $\mathcal{M}_{\frac{1}{h}}$, and M_h for $\mathcal{M}_{\frac{h-1}{h}}$.

Proposition 6.1.1. *The map* $\bigwedge^{h-1} : LT_h \to M_h$ *induces an isomorphism*

$$\bigwedge^{h-1} : (LT_h)_0 \to (M_h)_0.$$

Proof. By Theorem 3.1.1, we have a non-canonical isomorphism

$$(LT_h)_0 \approx \operatorname{Spf} \mathcal{O}_{\breve{L}}[[t_1, \ldots, t_{h-1}]],$$

which shows, by considering the duality map $\vee$, that

$$(M_h)_0 \approx \operatorname{Spf} \mathcal{O}_{\breve{L}}[[T_1, \ldots, T_{h-1}]].$$

Let $\mathcal{P} = (P, Q, F, V^{-1})$ be the $\mathcal{O}_L$-display corresponding to the π_L-divisible $\mathcal{O}_L$-module $G_{1,h-1}$, so that $P = D(G_{1,h-1})$ and $Q = VD(G_{1,h-1})$ with F and V^{-1} as given in the definition of $D(G_{m,n})$. Let $S \in \mathcal{C}$, and let $\operatorname{Spec} A \subseteq S$ be an open affine subset of S. By Grothendieck-Messing theory (cf. [Mes72]), π_L-divisible $\mathcal{O}_L$-modules over A lifting $G_{1,h-1}$ correspond to s.e.s.

$$0 \longrightarrow M \longrightarrow P \otimes_{\mathcal{O}_{\breve{L}}} A \longrightarrow N \longrightarrow 0$$

of A-modules lifting the Hodge filtration

$$0 \longrightarrow \frac{Q}{\pi_L P} \longrightarrow \frac{P}{\pi_L P} \longrightarrow \frac{P}{Q} \longrightarrow 0.$$

Since

$$\frac{Q}{\pi_L P} = \frac{VD(G_{1,h-1})}{\pi_L D(G_{1,h-1})} = \frac{\langle Ve_0, \ldots, Ve_{h-1}\rangle}{\langle \pi_L e_0, \ldots, \pi_L e_{h-1}\rangle} = \frac{\langle \pi_L e_0, e_1, \ldots, e_{h-1}\rangle}{\langle \pi_L e_0, \ldots, \pi_L e_{h-1}\rangle},$$

the above condition is equivalent to

$$\frac{M}{m_A M} = \frac{Q}{\pi_L P} = \frac{\langle e_1, \ldots, e_{h-1}\rangle}{\langle \pi_L e_1, \ldots, \pi_L e_{h-1}\rangle}.$$

Applying Nakayama's lemma, we see that M must be of the form

$$\langle e_1 + t_1\pi_L e_0, \dots, e_{h-1} + t_{h-1}\pi_L e_0\rangle$$

for some $t_1, \dots, t_{h-1} \in A$.

Applying $\bigwedge^{h-1}$ to the s.e.s.

$$0 \longrightarrow \langle e_1 + t_1\pi_L e_0, \dots, e_{h-1} + t_{h-1}\pi_L e_0\rangle \longrightarrow \langle e_0, \dots, e_{h-1}\rangle \longrightarrow \langle \overline{e_0}\rangle \longrightarrow 0, \quad (6.1)$$

we get the s.e.s.

$$0 \longrightarrow \langle (e_1+t_1\pi_L e_0)\wedge\cdots\wedge(e_{h-1}+t_{h-1}\pi_L e_0)\rangle \longrightarrow \langle \tilde{e}_0, \dots, \tilde{e}_{h-1}\rangle \longrightarrow \langle \overline{\tilde{e}_1}, \dots, \overline{\tilde{e}_{h-1}}\rangle \longrightarrow 0$$

where

$$(e_1 + t_1\pi_L e_0) \wedge \cdots \wedge (e_{h-1} + t_{h-1}\pi_L e_0) = \tilde{e}_0 - t_1\pi_L\tilde{e}_1 - t_2\pi_L\tilde{e}_2 - \cdots - t_{h-1}\pi_L\tilde{e}_{h-1}.$$

Let us rewrite (6.1) as follows:

$$\begin{array}{ccccccc} 0 \longrightarrow \langle f_1, \dots, f_{h-1}\rangle & \xrightarrow{\rho} & \langle e_0, \dots, e_{h-1}\rangle & \xrightarrow{\tau} & \langle g_0\rangle \longrightarrow 0 & & \\ f_i & \mapsto & e_i + t_i\pi_L e_0 & & & & \\ & & e_0 & \mapsto & g_0 & & \\ & & e_i & \mapsto & -t_i\pi_L g_i, & & i \neq 0. \end{array}$$

Applying $\vee$, we get the s.e.s.

$$0 \longrightarrow \langle g_0^\vee\rangle \xrightarrow{\tau^\vee} \langle e_0^\vee, \dots, e_{h-1}^\vee\rangle \xrightarrow{\rho^\vee} \langle f_1^\vee, \dots, f_{h-1}^\vee\rangle \to 0$$

where

$$(\tau^\vee g_0^\vee)(e_i) = g_0^\vee(\tau e_i) = \begin{cases} g_0^\vee(g_0) = 1, & \text{if } i = 0, \\ g_0^\vee(-t_i\pi_L g_0) = -t_i\pi_L, & \text{if } i \neq 0, \end{cases}$$

$$(\rho^\vee e_i^\vee)(f_j) = e_i^\vee(\rho f_j) = e_i^\vee(e_j + t_j\pi_L e_0) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

So the above s.e.s. is the same as

$$0 \longrightarrow \langle e_0^\vee - t_1\pi_L e_1^\vee - \cdots - t_{h-1}\pi_L e_{h-1}^\vee\rangle \longrightarrow \langle e_0^\vee, \dots, e_{h-1}^\vee\rangle \longrightarrow \langle \overline{e_1^\vee}, \dots, \overline{e_{h-1}^\vee}\rangle \longrightarrow 0.$$

Therefore we have a commuting diagram

$$\begin{array}{ccc} & & (M_h)_0 \approx \operatorname{Spf} \mathcal{O}_{\breve{L}}[[T_1, \dots, T_{h-1}]] \\ & \nearrow^{\bigwedge^{h-1}} & \cong \Big\uparrow \begin{smallmatrix} T_i \\ \downarrow \\ S_i \end{smallmatrix} \\ (LT_h)_0 \approx \operatorname{Spf} \mathcal{O}_{\breve{L}}[[t_1, \dots, t_{h-1}]] & & \\ & \searrow_{\vee,\ \cong} & \\ & & (M_h)_0 \approx \operatorname{Spf} \mathcal{O}_{\breve{L}}[[S_1, \dots, S_{h-1}]] \end{array}$$

which shows that $\bigwedge^{h-1}$ induces an isomorphism from $(LT_h)_0$ to $(M_h)_0$ since the duality map $\vee : LT_h \to M_h$ clearly does. □

Corollary 6.1.2. *The map $\bigwedge^{h-1} : LT_h \to M_h$ induces an isomorphism*

$$\bigwedge^{h-1} : (LT_h)_m \to (M_h)_{(h-1)m}$$

for any $m \in \mathbb{Z}$.

Proof. We recall the definition of the map $\theta : D^\times \to E^\times$. Using the above embeddings of $D^\times$ and $E^\times$ into $GL_h(L_h)$, we have $\theta(\phi) = (\det \phi)(\phi^{-1})^T$.

Let $m \in \mathbb{Z}$. Fix some $\phi \in D^\times$ with $\operatorname{val}(\phi) = m$, then

$$\begin{aligned} \operatorname{val}(\theta(\phi)) = v_L(\det(\theta(\phi))) &= v_L\left(\det\left((\det \phi)(\phi^{-1})^T\right)\right) \\ &= v_L((\det \phi)^{h-1}) = (h-1)v_L(\det \phi) = (h-1)m. \end{aligned}$$

So the action of ϕ induces an isomorphism from $(LT_h^0)_0$ to $(LT_h^0)_m$, and the action of $\theta(\phi)$ induces an isomorphism from $(M_h^0)_0$ to $(M_h^0)_{(h-1)m}$.

By Proposition 5.2.1, we have the following commutative diagram:

$$\begin{array}{ccc} (LT_h^0)_0 & \xrightarrow[\cong]{\bigwedge^{h-1}} & (M_h^0)_0 \\ {\scriptstyle \phi}\Big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\Big\downarrow{\scriptstyle \theta(\phi)} \\ (LT_h^0)_m & \xrightarrow[\bigwedge^{h-1}]{} & (M_h^0)_{(h-1)m} \end{array}$$

where the top map is an isomorphism by Proposition 6.1.1. Hence $\bigwedge^{h-1}$ induces an isomorphism from $(LT_h^0)_m$ to $(M_h^0)_{(h-1)m}$, as desired. □

Proposition 6.1.3. *For any $n \geq 0$, $m, n \in \mathbb{Z}$, each connected component of $(LT_h^n)_m$ is mapped isomorphically onto some connected component of $(M_h^n)_{(h-1)m}$ under the map $\bigwedge^{h-1} : LT_h^\infty \to M_h^\infty$.*

Proof. By Proposition 6.1.1, $\bigwedge^{h-1}$ induces an isomorphism on the formal schemes $\bigwedge^{h-1} : (LT_h)_0 \to (M_h)_0$, hence an isomorphism on their generic fibers.

By considering the action of $O_D^\times$ on the connected components of $(LT_h^n)_0$, we see that each connected component of $(LT_h^n)_0$ is finite étale over (LT_h^0) of the same degree. Using duality, we see that each connected component of $(M_h^n)_0$ is also finite étale over $(M_h^0)_0$ of the same degree. Since

$$\begin{array}{ccc} (LT_h^n)_0 & \xrightarrow{\bigwedge^{h-1}} & (M_h^n)_0 \\ \text{finite étale}\downarrow & & \downarrow\text{finite étale} \\ (LT_h^0)_0 & \xrightarrow[\bigwedge^{h-1}]{\cong} & (M_h^0)_0 \end{array}$$

commutes, this shows that $\bigwedge^{h-1}$ induces an isomorphism from each connected component of $(LT_h^n)_0$ to some connected component of $(M_h^n)_0$.

Now, by considering the action of the groups $D^\times$ and $E^\times$, we see that, for any $m \in \mathbb{Z}$, the map $\bigwedge^{h-1}$ induces an isomorphism from each connected component of $(LT_h^n)_m$ to some connected component of $(M_h^n)_{(h-1)m}$. □

Corollary 6.1.4. *Suppose* $(p(q-1), h-1) = 1$. *Then* $\bigwedge^{h-1} : LT_h^\infty \to M_h^\infty$ *induces an isomorphism*

$$(LT_h^n)_m \to (M_h^n)_{(h-1)m}$$

for any $n \geq 0$, $m, n \in \mathbb{Z}$.

Proof. We first consider the case $m = 0$. Since $(p(q-1), h-1) = 1$, by Proposition 4.1.4, the map $\theta_{0,n}$ is bijective. By Theorem 3.2.4, this means that the map induced by $\bigwedge^{h-1}$ on the connected components is a bijection. Together with Proposition 6.1.3, this shows that $\bigwedge^{h-1}$ induces an isomorphism from $(LT_h^n)_0$ to $(M_h^n)_0$. The result then follows for all $m \in \mathbb{Z}$ by considering the action of the groups $D^\times$ and $E^\times$. □

6.2 Cohomology of the dual Lubin-Tate tower

In this section, we will reinterpret the results of the previous section in terms of the cohomology of the dual Lubin-Tate tower. In order to avoid unnecessarily cumbersome notation, from here on, all cohomology is understood to mean l-adic cohomology with $\overline{\mathbb{Q}}_l$ coefficients, where $l \neq p$ is an odd prime. We will also slightly abuse notation, and whenever the Lubin-Tate tower or the dual Lubin-Tate tower appears, we actually mean its change base from $\check{L}$ to C.

Lemma 6.2.1. *The kernel* $\ker(\theta)$ *acts trivially on* LT_h^0*, so the action of* $D^\times$ *on* LT_h^0 *induces an action of* $\theta(D^\times) = \mathrm{Im}(\theta) \leq E^\times$ *on* LT_h^0*. This action can be extended to an action of* $E^\theta = \langle \mathcal{O}_{L_h}^\times, \mathrm{Im}(\theta)\rangle$ *by letting* $\mathcal{O}_{L_h}^\times$ *act trivially on* LT_h^0*.*

Proof. Suppose $k \in \ker(\theta)$. The action of k on LT_h^0 is given by

$$(X, \beta) \mapsto (X, \beta \circ k^{-1}).$$

But $k \in \ker(\theta) = \boldsymbol{\mu}_{h-1}(\mathcal{O}_L) \subset \mathcal{O}_L^\times$ and X has multiplication by $\mathcal{O}_{L_h}^\times$, so $(X, \beta) = (X, \beta \circ k^{-1})$. So k acts trivially on LT_h^0, and we get an action of $\mathrm{Im}(\theta)$ on LT_h^0 induced by the action of $D^\times$.

To see that we can extend the above action of $\mathrm{Im}(\theta)$ to an action of $E^\theta = \langle \mathcal{O}_{L_h}^\times, \mathrm{Im}(\theta)\rangle$ where $\mathcal{O}_{L_h}^\times$ acts trivially, we just need to check that $\mathcal{O}_{L_h}^\times \cap \mathrm{Im}(\theta) \subseteq \mathrm{Im}(\theta)$ acts trivially on LT_h^0, but this is clear. □

Proposition 6.2.2. *Let* $E^\theta \leq E^\times$ *act on* LT_h^0 *as in Lemma 6.2.1. Then for all* $i \geq 0$*,*

$$H_c^i(M_h^0) \cong \mathrm{Ind}_{E^\theta}^{E^\times} H_c^i(LT_h^0)$$

as $E^\times \times W_L$ *representations.*

Proof. The proposition is clear for $i > 0$ since $H_c^i(LT_h^0) = 0 = H_c^i(M_h^0)$ for $i > 0$. Consider $i = 0$. Let Y_0 be the image of

$$\bigwedge^{h-1} : LT_h^0 \to M_h^0.$$

By Proposition 5.2.1, and the fact that $\mathcal{O}_{L_h}^\times$ acts trivially on M_h^0, we have

$$H_c^0(Y_0) \cong H_c^0(LT_h^0)$$

as $E^\theta \times W_L$-representations, where the action of E^θ on LT_h^0 is as described in Lemma 6.2.1.

By Proposition 4.1.3, $E^\theta = \{\varphi \in E^\times : (h-1) \mid \mathrm{val}(\varphi)\}$ and a full set of representatives for $E^\times/E^\theta$ is given by $(\varpi')^k$ for $k \in \{0, 1, \ldots, h-2\}$. It is clear that

$$M_h^0 = Y_0 \sqcup \varpi'(Y_0) \sqcup \cdots \sqcup \varpi'^{h-2}(Y_0).$$

So by Proposition 4.2.1,

$$H_c^i(M_h^0) \cong \mathrm{Ind}_{E^\theta}^{E^\times} H_c^i(LT_h^0)$$

as $E^\times \times W_L$ representations, as desired. □

We now prove an analogous result for $H_c^i(M_h^\infty)$.

Theorem 6.2.3. *Let $d = (q-1, h-1)$, $e = v_p(h-1)$ and c be the maximum integer $\leq e$ such that $\mathcal{O}_L$ contains the p^c roots of unity. Let $\theta_* H_c^i(LT_h^\infty)$ be the pushforward of the representation $H_c^i(LT_h^\infty)$ under the map*

$$\theta : D^\times \twoheadrightarrow \theta(D^\times).$$

Then for all $i \geq 0$,

$$\psi^* \left(\mathrm{Res}^{GL_h(L)}_{\psi(GL_h(L))} H_c^i(M_h^\infty) \right) \cong \mathrm{Ind}^{E^\times}_{\theta(D^\times)} \theta_* H_c^i(LT_h^\infty) \cong \mathrm{Ind}^{E^\times}_{\theta(D^\times)} \theta_* \left(\frac{H_c^i(LT_h^\infty)}{K} \right)$$
$$\cong \mathrm{Ind}^{E^\times}_{\theta(D^\times)} \theta_* H_c^i(LT_h^\infty)^K$$

as $E^\times \times GL_h(L) \times W_L$ representations, where $K = \ker(\theta_0) = \boldsymbol{\mu}_{p^c d}(\mathcal{O}_L)$ and $H_c^i(LT_h^\infty)^K$ is the subrepresentation of $H_c^i(LT_h^\infty)$ fixed by K.

Proof. Let Y_n be the image of

$$\bigwedge^{h-1} : LT_h^n \to M_h^n.$$

By Lemma 4.1.2 (a) and Proposition 4.1.4 (c), for $n > 2ee_L$, the kernel of $\theta_{0,n}$ can be written as a direct product

$$\ker(\theta_{0,n}) = K'_{n,1} \times K'_{n,2},$$

and the det mod $(1 + \pi_L^n \mathcal{O}_L)$ map gives an isomorphism

$$K = \ker(\theta_0) = \boldsymbol{\mu}_{p^c d}(\mathcal{O}_L) \xrightarrow[\cong]{\det \bmod (1+\pi_L^n \mathcal{O}_L)} K'_{n,1}.$$

Let $K_n = 1 + \pi_L^{n-ee_L}\mathcal{O}_L \leq D^\times$. Proposition 4.1.4 (c) tells us that we have an exact sequence

$$1 \longrightarrow 1 + \pi_L^n \mathcal{O}_L \longrightarrow K_n \xrightarrow{\det \bmod (1+\pi_L^n \mathcal{O}_L)} K'_{n,2} \longrightarrow 1.$$

Recall from Proposition 6.1.3 that $\bigwedge^{h-1}$ maps each connected component of $(LT_h^n)_m$ isomorphically onto some connected component of $(M_h^n)_{(h-1)m}$. Since

$$K \times K_n \xrightarrow{\det \bmod (1+\pi_L^n \mathcal{O}_L)} K'_{n,1} \times K'_{n,2} = \ker(\theta_{0,n}),$$

by Theorem 3.2.4, two connected components of LT_h^n will each be mapped isomorphically to the same connected component of M_h^n under the $\bigwedge^{h-1}$ map if and only if they are in the same orbit of $K \times K_n$. Furthermore, $1 + \pi_L^n \mathcal{O}_L \leq D^\times$ acts trivially on LT_h^n, so

$$\psi^* H_c^i(Y_n) \cong \theta_* \left(\frac{H_c^i(LT_h^n)}{K \times K_n} \right)$$

as $\theta(D^\times) \times GL_h(L) \times W_L$-representations.

Claim:

$$H_c^i(M_h^n) \cong \operatorname{Ind}_{E^\theta}^{E^\times} \operatorname{Ind}_{\theta(D^\times)}^{E^\theta} H_c^i(Y_n) \cong \operatorname{Ind}_{\theta(D^\times)}^{E^\times} H_c^i(Y_n).$$

Proof of claim: Let X_n be given by

$$X_n = \coprod_{m \in \mathbb{Z}} (M_h^n)_{(h-1)m}.$$

Note that X_n contains Y_n as a subspace. By Propositions 4.1.3 (d) and 4.1.4 (b),

$$\frac{E^\theta}{\operatorname{Im}\theta} \cong \operatorname{coker}\theta_0 = \frac{\mathcal{O}_E^\times}{\operatorname{Im}\theta_0} \xrightarrow[\cong]{\det \bmod (1+\pi_L^n \mathcal{O}_L)} \frac{(\mathcal{O}_L/\pi_L^n \mathcal{O}_L)^\times}{\operatorname{Im}\theta_{0,n}} = \operatorname{coker}\theta_{0,n}$$

is a finite abelian group.

$D^\times$ acts transitively on the connected components of LT_h^n, so for any $\varphi \notin \operatorname{Im}(\theta)$, $\varphi(Y_n)$ will be disjoint from Y_n. Furthermore, the orbit of Y_n under $\mathcal{O}_E^\times$ is X_n since $\mathcal{O}_E^\times$ acts transitively on the connected components of $(M_h^n)_{(h-1)m}$ for any $m \in \mathbb{Z}$. This shows that

$$X_n = \coprod_{\varphi \in \frac{E^\theta}{\operatorname{Im}(\theta)}} \varphi(Y_n).$$

By Proposition 4.2.1, we have

$$H_c^i(X_n) \cong \operatorname{Ind}_{\theta(D^\times)}^{E^\theta} H_c^i(Y_n).$$

Recall from Proposition 4.1.3 (d) that the valuation map gives an isomorphism

$$\frac{E^\times}{E^\theta} \cong \frac{\mathbb{Z}}{(h-1)\mathbb{Z}}.$$

So a full set of representatives for $\frac{E^\times}{E^\theta}$ is $\{1, \varpi', \varpi'^2, \ldots, \varpi'^{h-2}\}$, and it is clear that

$$M_h^n = X_n \sqcup \varpi'(X_n) \sqcup \cdots \sqcup \varpi'^{h-2}(X_n).$$

So, again, by Proposition 4.2.1,

$$H_c^i(M_h^n) \cong \operatorname{Ind}_{E^\theta}^{E^\times} H_c^i(X_n).$$

This proves the claim.

The map

$$H_c^i(LT_h^{n-ee_L}) \to H_c^i(LT_h^n)$$

factors through $\frac{H_c^i(LT_h^n)}{K_n}$ since $K_n = 1 + \pi_L^{n-ee_L}O_L$ acts trivially on $LT_h^{n-ee_L}$. So $\varinjlim_n \frac{H_c^i(LT_h^n)}{K_n} = H_c^i(LT_h^\infty)$, and

$$\varinjlim_n \frac{H_c^i(LT_h^n)}{K \times K_n} = \frac{H_c^i(LT_h^\infty)}{K}.$$

It remains to show that $\frac{H_c^i(LT_h^\infty)}{K} \cong H_c^i(LT_h^\infty)^K$. For m sufficiently large, the determinant of the elements of K are distinct mod $(1 + \pi_L^m O_L)$, so the orbits of the induced action of K on the connected components of LT_h^m each have size d. We define a subspace Z_m of LT_h^m by taking one connected component from each orbit. Let k be a generator of K. Then LT_h^m is the disjoint union

$$LT_h^m = Z_m \sqcup k(Z_m) \sqcup \cdots \sqcup k^{d-1}(Z_m),$$

so we have an identification

$$H_c^i(LT_h^m) \cong \bigoplus_{j=0}^{d-1} H_c^i(Z_m)$$

of $\overline{\mathbb{Q}}_l$-vector spaces such that the action of K on $\bigoplus_{j=0}^{d-1} H_c^i(Z_m)$ is given by $k \cdot (z_0, z_1, \ldots, z_{d-1}) = (z_{d-1}, z_0, \ldots, z_{d-2})$. Define an isomorphism of vector spaces

$$\frac{H_c^i(LT_h^m)}{K} \xrightarrow{\cong} H_c^i(LT_h^m)^K$$
$$[(z_0, \ldots, z_{d-1})] \longmapsto (z, \ldots, z),$$

where $z = \frac{1}{d}\sum_{j=0}^{d-1} z_j$. Using the fact that K is in the center of $D^\times$, an easy computation shows that the above is also an isomorphism of representations.

Putting these together, we have

$$\psi^* \left(\mathrm{Res}_{\psi(GL_h(L))}^{GL_h(L)} H_c^i(M_h^n)\right) \cong \mathrm{Ind}_{\theta(D^\times)}^{E^\times} \psi^* H_c^i(Y_n) \cong \mathrm{Ind}_{\theta(D^\times)}^{E^\times} \theta_* \left(\frac{H_c^i(LT_h^n)}{K \times K_n}\right),$$

and

$$\psi^* \left(\mathrm{Res}_{\psi(GL_h(L))}^{GL_h(L)} H_c^i(M_h^\infty)\right) \cong \mathrm{Ind}_{\theta(D^\times)}^{E^\times} \theta_* \left(\varinjlim_n \frac{H_c^i(LT_h^n)}{K \times K_n}\right) \cong \mathrm{Ind}_{\theta(D^\times)}^{E^\times} \theta_* \left(\frac{H_c^i(LT_h^\infty)}{K}\right)$$
$$\cong \mathrm{Ind}_{\theta(D^\times)}^{E^\times} \theta_* H_c^i(LT_h^\infty)^K,$$

as desired. □

Corollary 6.2.4. *Suppose $(q-1, h-1) = 1$. There is an action of $\theta(D^\times) \subset E^\times$ on $H^i_c(LT_h^\infty)$ induced from the action of $D^\times$. Then for all $i \geq 0$,*

$$\psi^* \left(\mathrm{Res}^{GL_h(L)}_{\psi(GL_h(L))} H^i_c(M_h^\infty)\right) \cong \mathrm{Ind}^{E^\times}_{\theta(D^\times)} H^i_c(LT_h^\infty)$$

as $E^\times \times GL_h(L) \times W_L$ representations.

Just like for LT_h^∞, we will consider the functor

$$\begin{aligned} H^i_c(M_h^\infty) : \mathrm{Rep}\, E^\times &\to \mathrm{Rep}(GL_h(L) \times W_L) \\ \rho &\mapsto \mathrm{Hom}_{E^\times}(H^i_c(M_h^\infty), \rho). \end{aligned}$$

The following is an easy corollary of Theorem 6.2.3.

Corollary 6.2.5. *For all $i \geq 0$,*

$$\psi^* \left(\mathrm{Res}^{GL_h(L)}_{\psi(GL_h(L))} H^i_c(M_h^\infty)[\rho]\right) = H^i_c(LT_h^\infty)\left[\theta^*\left(\mathrm{Res}^{E^\times}_{\theta(D^\times)} \rho\right)\right].$$

Proof. By Theorem 6.2.3 and Frobenius reciprocity,

$$\begin{aligned} \psi^* \left(\mathrm{Res}^{GL_h(L)}_{\psi(GL_h(L))} H^i_c(M_h^\infty)[\rho]\right) &= \mathrm{Hom}_{E^\times}\left(\psi^*\left(\mathrm{Res}^{GL_h(L)}_{\psi(GL_h(L))} H^i_c(M_h^\infty)\right),\ \rho\right) \\ &= \mathrm{Hom}_{E^\times}\left(\mathrm{Ind}^{E^\times}_{\theta(D^\times)} \theta_*\left(\frac{H^i_c(LT_h^\infty)}{K}\right),\ \rho\right) \\ &= \mathrm{Hom}_{\theta(D^\times)}\left(\theta_*\left(\frac{H^i_c(LT_h^\infty)}{K}\right),\ \mathrm{Res}^{E^\times}_{\theta(D^\times)} \rho\right) \\ &= \mathrm{Hom}_{D^\times}\left(\frac{H^i_c(LT_h^\infty)}{K},\ \theta^*\left(\mathrm{Res}^{E^\times}_{\theta(D^\times)} \rho\right)\right). \end{aligned}$$

Since K acts trivially on $\theta^*\left(\mathrm{Res}^{E^\times}_{\theta(D^\times)} \rho\right)$, the above is equal to

$$\mathrm{Hom}_{D^\times}\left(H^i_c(LT_h^\infty),\ \theta^*\left(\mathrm{Res}^{E^\times}_{\theta(D^\times)} \rho\right)\right) = H^i_c(LT_h^\infty)\left[\theta^*\left(\mathrm{Res}^{E^\times}_{\theta(D^\times)} \rho\right)\right].$$

□

6.3 Geometric realization of local correspondences and vanishing results

In this section, we will use Corollary 6.2.5 and results on the cohomology of the Lubin-Tate tower to deduce results for the cohomology of the dual Lubin-Tate tower. In particular, we will show that the supercuspidal part of the cohomology of the dual Lubin-Tate tower in the middle degree realizes the local Langlands and the Jacquet-Langlands correspondences (up to appropriate twists).

Theorem 6.3.1. *For* $\pi \in \mathrm{Cusp}(GL_h(L))$,

$$H_c^{h-1}(M_h^\infty)_{cusp}(\mathrm{JL}^{-1}(\pi)) = \pi \otimes \mathrm{rec}(\pi^\vee \otimes (\chi_\pi \circ \det) \otimes (|\cdot| \circ \det)^{\frac{h-1}{2}})(h-1),$$

where χ_π *is the central character of* π *and* $H_c^{h-1}(M_h^\infty)_{cusp}$ *is the supercuspidal part of* $H_c^{h-1}(M_h^\infty)$.

Proof. By Corollary 6.2.5, we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Irr}'\, E^\times & \xrightarrow{\rho \mapsto \theta^*\left(\mathrm{Res}^{E^\times}_{\theta(D^\times)} \rho\right)} & \mathrm{Irr}'\, D^\times \\ \downarrow{\scriptstyle H_c^i(M_h^\infty)_{\mathrm{cusp}}} & & \downarrow{\scriptstyle H_c^i(LT_h^\infty)_{\mathrm{cusp}}} \\ \mathrm{Cusp}(GL_h(L)) \times \mathrm{Irr}(W_L) & \xrightarrow[\pi \otimes r \mapsto \psi^*\left(\mathrm{Res}^{GL_h(L)}_{\psi(GL_h(L))} \pi\right) \otimes r]{} & \mathrm{Cusp}(GL_h(L)) \times \mathrm{Irr}(W_L) \end{array}$$

where $\mathrm{Irr}'\, D^\times \subset \mathrm{Irr}\, D^\times$ is the subset consisting of representations of the form $\mathrm{JL}^{-1}(\pi)$ for some $\pi \in \mathrm{Cusp}(GL_h(L))$ (similarly for $\mathrm{Irr}'\, E^\times$).

Note that

- $\theta^*\left(\mathrm{Res}^{E^\times}_{\theta(D^\times)} \rho\right) \in \mathrm{Irr}'\, D^\times$ since

$$\begin{aligned} \rho &= \mathrm{JL}^{-1}(\pi) \text{ with } \pi \in \mathrm{Cusp}(GL_h(L)) \\ \Rightarrow \rho \otimes (\chi_\pi \circ \mathrm{Nrd})^\vee &= \mathrm{JL}^{-1}(\pi \otimes (\chi_\pi \circ \det)^\vee), \end{aligned}$$

- $\pi \in \mathrm{Cusp}(GL_h(L))$ since

$$\psi^*\left(\mathrm{Res}^{GL_h(L)}_{\psi(GL_h(L))} \pi\right) = \pi \otimes (\chi_\pi \circ \det)^\vee \in \mathrm{Cusp}(GL_h(L)).$$

The commutative diagram only gives us $H_c^i(M_h^\infty)_{\mathrm{cusp}}[\rho]$ as a $\psi(GL_h(L)) \times W_L$-representation. But by considering the duality map $\vee : LT_h^\infty \to M_h^\infty$, we know that $H_c^i(M_h^\infty)_{\mathrm{cusp}}$ realizes the Jacquet-Langlands correspondence, so we already understand $H_c^i(M_h^\infty)_{\mathrm{cusp}}[\rho]$ as a $GL_h(L)$ representation.

Let us define

$$\begin{aligned} F : \mathrm{Irr}'\, E^\times &\to \mathrm{Cusp}(GL_h(L)) \times \mathrm{Irr}(W_L) \\ \mathrm{JL}^{-1}(\pi) &\mapsto \pi \otimes \mathrm{rec}(\pi^\vee \otimes (\chi_\pi \circ \det) \otimes (|\cdot| \circ \det)^{\frac{h-1}{2}})(h-1). \end{aligned}$$

By the above, it will suffice to check that F makes the diagram commute. Applying Theorem 3.2.1, and using the fact that JL and rec are compatible with twists, we see that $H_c^{h-1}(LT_h^\infty)_{\mathrm{cusp}}\left[\rho \otimes (\chi_\rho \circ \mathrm{Nrd})^\vee\right]$ and $\psi^*\left(\mathrm{Res}_{\psi(GL_h(L))}^{GL_h(L)} F(\rho)\right)$ are both equal to

$$(\mathrm{JL}(\rho) \otimes (\chi_{\mathrm{JL}(\rho)} \circ \det)^\vee) \otimes \mathrm{rec}(\mathrm{JL}(\rho)^\vee \otimes (\chi_{\mathrm{JL}(\rho)} \circ \det) \otimes (|\cdot| \circ \det)^{\frac{h-1}{2}})(h-1),$$

as desired. $\square$

We can also use Theorem 3.2.2 to deduce that

Theorem 6.3.2. *For $i \neq h-1$, $\rho \in \mathrm{Rep}\, E^\times$,*

$$H_c^i(M_h^\infty)_{cusp}(\rho) = 0.$$

Proof. This is immediate from Theorem 3.2.2 and the commutative diagram

$$\begin{CD}
\mathrm{Rep}\, E^\times @>{\rho \mapsto \theta^*\left(\mathrm{Res}_{\theta(D^\times)}^{E^\times} \rho\right)}>> \mathrm{Rep}\, D^\times \\
@V{H_c^i(M_h^\infty)_{\mathrm{cusp}}}VV @VV{H_c^i(LT_h^\infty)_{\mathrm{cusp}}}V \\
\mathrm{Rep}(GL_h(L) \times W_L) @>>{\pi \otimes r \mapsto \psi^*\left(\mathrm{Res}_{\psi(GL_h(L))}^{GL_h(L)} \pi\right) \otimes r}> \mathrm{Rep}(GL_h(L) \times W_L).
\end{CD}$$

$\square$

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