Behavior of $O(\log n)$ local commuting hamiltonians

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Abstract

We study the variant of the k-local hamiltonian problem which is a natural generalization of k-CSPs, in which the hamiltonian terms all commute. More specifically, we consider a hamiltonian $H = \sum_i H_i$ over n qubits, where each H_i acts non-trivially on $O(\log n)$ qubits and all the terms H_i commute, and show the following -

1. We show that a specific case of $O(\log n)$ local commuting hamiltonians over the hypercube is in NP, using the Bravyi-Vyalyi Structure theorem from [7].
2. We give a simple proof of a generalized area law for commuting hamiltonians (which seems to be a folklore result) in all dimensions, and deduce the case for $O(\log n)$ local commuting hamiltonians.
3. We show that traversing the ground space of $O(\log n)$ local commuting hamiltonians is QCMA complete.

The first two behaviours seem to indicate that deciding whether the ground space energy of $O(\log n)$-local commuting hamiltonians is *low* or *high* might be in NP, or possibly QCMA, though the last behaviour seems to indicate that it may indeed be the case that $O(\log n)$-local commuting hamiltonians are QMA complete.

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1 Introduction

Since the definition and development of NP completeness in 1971 [8], various problems and their hardness have been intensively studied, including the hardness of finding approximate solutions. Among the vast variety of NP-complete problems studied over time, consider the following problem called k-constraint satisfaction problem (k-CSP) - Given a set of n variables each taking values in a finite set S, and given a set of constraints on the values that any specific k tuple of variables could take, is there an assignment to the variables such that all the constraints are satisfied? The k-CSP generalized many well-known NP-complete problems, and has been studied exhaustively. Consider a quantum variant of the k-CSP, called the k-local hamiltonian problem (k-LHP), defined as follows: Given some hamiltonian $H = \sum H_i$ acting over n qubits, such that each H_i acts non-trivially on at most k qubits, decide whether the smallest eigenvalue of H is less than some threshold α or more thanβ (where $\beta - \alpha \sim \frac{1}{\text{poly}(n)}$). The 5-local hamiltonian problem was proven to be complete for the complexity class QMA by Kitaev in 1999 ([15]), which was akin to the Cook-Levin theorem in classical complexity theory. Following this result, the k-local hamiltonian problem was extensively studied, and it was further shown that the 3-local hamiltonian problem is QMA complete [14], and even the 2-local hamiltonian problem is QMA complete [13]. After this, attention was put to not only reducing locality, but making the hamiltonians H_i *geometrically* local. Along this direction, it was shown in [18], that the 4-local hamiltonian problem, where the hamiltonians lie on a grid over qubits is QMA complete , and further, it was concluded that the 2-local hamiltonian problem on a line over qudits with dimension 12 is QMA complete [2]. Thus, the k-local hamiltonian problem has been well classified.

On the other hand, although the k-local hamiltonian problem generalizes k-constraint satisfaction problems, it is the case that converting constraints into local hamiltonians always creates terms H_i that are diagonal. This means that for all i and j, the terms H_i and H_j always commute, i.e. $H_i H_j = H_j H_i$. This observation leads us to a problem that very naturally lies between the k-CSP and k-LHP, of deciding the ground energy of commuting hamiltonians. More formally, the k-commuting local hamiltonian problem (k-CLHP) is defined as follows - Given some hamiltonian $H = \sum H_i$ acting over n qubits, such that each H_i acts non-trivially on at most k qubits, and each of the terms H_i and H_j commute, decide whether the smallest eigenvalue of H is less than some threshold α or more thanβ (where $\beta - \alpha \sim \frac{1}{\text{poly}(n)}$).

Apart from being a natural generalization of k-CSPs, the k-CLHP seems to naturally arise in many physical systems, that have only commuting constraints. Moreover, we may suspect that k-CLHP might be much easier than k-LHP due to the commutativity constraint, since we may intuitively think that the power of qauntum mechanics arises mainly due to non-commutativity. Thus, we might intuitively think that k-CLHP lies in NP. But since we have a classic example of a commuting system with highly entangled ground states, the toric code, it might be the case that commuting hamiltonians have ground states that are as general as those of non-commuting hamiltonians, and thus the problem might be hard.

The first result along understanding the complexity of k-CLHPs came in the work of Bravyi and Vyalyi [7]. In [7], it was shown that the 2-CLHP problem over qudits is in NP. Indeed, [7] shows something *stronger* - Not only is deciding the the threshold ground space energy in NP, but for 2-CLHP, the ground state is a product state, and can be efficiently found given a witness. The main tool used were techniques from C^* algebras that help to prove a structure theorem about commuting algebras, which is discussed more in section 3. Note that on the other hand, the 2-LHP is known to be QMA complete, and thus this seems to indicate that the k-CLHP is much easier than k-LHP. Going along these lines, it was shown in [2] that the 3-CLHP over qubits is in NP, which gives further evidence for the seeming easiness of k-CLHP. However, the techniques of [7] and [2] which explicitly find a ground state cannot work for k-CLHP for $k \geq 4$, since it is known that the toric code which is a case of 4-CLHP over a grid has highly entangled ground states that do not have a short straightforward classical description. However, it was shown in [19] that one can indeed verify the threshold ground space energy without explicitly finding the ground state for 4-CLHP over a grid of qubits in NP. Thus, the highly entangled states of the toric code do not seem to be a barrier to this. These 3 results are indeed the state of the art regarding the complexity of k-CLHP.

In this work, we consider the $O(\log n)$-CLHP. One natural reason for doing this is to find an explicit hardness result if there is one, i.e., to verify whether $O(\log n)$-CLHP is indeed QMA hard (since very trivally, n-CLHP is QMA-hard). Although such a result would be desirable, we do not prove something of the sort. Instead, we further give evidence that it might indeed be hard to decide whether $O(\log n)$-CLHP is QMA hard, or easy.

Towards a first result showing easiness, we observe that $O(\log n)$-CLHP is indeed in NP if the hamiltonians are on the vertices of a hypercube and qudits on the edges. Note that the system is highly symmetric but also geometrically local in this case. This seems to indicate that even the $O(\log n)$-CLHP might be easy.

Next, we focus our attention towards area laws for commuting hamiltonians. Area laws have a rich history, and due to a sequence of results in physics - Bekenstein's result that the entropy of a black hole is proportional to the surface of its horizon, the entanglement entropy in the ground (vacuum) states in models in quantum field theories proportional to the surface area (with logarithmic corrections), etc. - a similar conjecture has been state in hamiltonian complexity. The main conjecture says that given an instance of 2^D-LHP over a D dimensional grid (i.e. the hamiltonian terms act on hypercubes), and given some region A of the hypercube, the entanglement entropy of a ground state scales $O(|\partial A|)$, where $|\partial A|$ is the size of the boundary of A. The first proof for an area law in 1D came in the work of Hastings [12], and the parameters were further simplified and improved in [5] and [4]. The case for 2 and more dimensions remains open. But following the combinatorial proof of the area law in [5], it was shown in [17] that one can indeed *find* the ground state of gapped 1D hamiltonians in (random) polynomial time. This is again the first algorithm that works for *all* systems in 1D. Thus, if we had an area law for commuting hamiltonians, would it be possible to find the ground state, possibly given a witness and using a quantum algorithm? This was the main motivation to consider the area law for commuting hamiltonians. Area laws are known to hold for commuting hamiltonians as a forklore result, and we present a simple proof of a generalized area law (which is shown to be false for not-necesarily commuting hamiltonians [3]). Given the area law in all dimensions, including for $O(\log n)$ local hamiltonians, it again seems to indicate that the entanglement entropy of the ground states of commuting hamiltonians indeed scales as the surface area, and not as the volume of the region, and this seemingly indicates that it might be possible to find the ground state using further the fact that our hamiltonians commute and we have results like the Structure Theorem, possibly using a witness and a quantum algorithm. However, the ideas in [17] crucially seem to use the fact that we are working in 1D, and it does not seem that they can be directly applied. Thus, we only have the evidence of the area law here.

So far, it seems to indicate to us that $O(\log n)$-CLHP might not indeed be QMA-hard, and possibly might be in QCMA. We then change our focus yet again, and consider the problem of *traversing* the ground space of the local hamiltonians. Consider the following classical problem - Given 2 satisfiable assignments ϕ_1 and ϕ_2 for a 3-SAT instance, is it possible to move from ϕ_1 to ϕ_2 by simple operations (say, flipping at most 2 bits), such that all intermediate assignments satisfy the SAT instance or satisfy *most* of the constraints? The classical problem was characterized in [11], and different cases were shown in which the hardness of the problem was in P, or coNP or PSPACE (and whether complete for it). The quantum analogue of the problem - traversing the ground space of hamiltonians was studied in [10]. Once again, the problem is defined as follows - Given a local hamiltonian H on n qubits and two unitaries that efficiently prepare the starting and final states ψ and ϕ with energy close to the ground space of H, is there a sequence of poly(n) local unitaries such that the ground states of H can be traversed, i.e., we can move from ψ to a state close to ϕ while all intermediate states have energy close to the ground space energy? It was shown in [10], that traversing the ground space of 5-local hamiltonians, (when the number of intermediate unitaries are polynomial in the number of qubits) is QCMA complete. This was indeed the first problem shown to be complete for the class QCMA. We consider the commuting variant of the problem. But unlike the previous two observations, we indeed show that traversing the ground space of $O(\log n)$ local commuting hamiltonians is QCMA complete. We present a novel construction that helps us use the tools from [10] for the commuting case after certain modifications. Thus, in this case, it indeed seems the case that traversing the ground space of $O(\log n)$ local commuting hamiltonians is just as hard as traversing the ground space of 5-local hamiltonians.

Thus, we seem to have indications in both directions about the hardness of the $O(\log n)$-CLHP. The first two obervations seem to indicate that the problem is easier than QMA, while the last result seems to indicate that the problem might indeed be QMA-hard. One thing to note is that in case of the first results, the hamiltonians were geometrically local, while in our ground space traversal construction, the hamiltonians are highly *non-local*. Is it the case that for $O(\log n)$ local commuting hamiltonians, geometrically local hamiltonians are easier than the ones that are not-geometrically local? Such a fact would indeed be surprising, since it does not hold for the k-LHP. We state other such problems towards the end.

In section 2, we review certain background definitions and results. In section 3, we give an exposition of the Structure Theorem as in [7], and some of its applications. In section 4, we give a proof of a generalized area law for commuting hamiltonians in all dimensions. In section 5, we show that traversing the ground space of $O(\log n)$ local commuting hamiltonians is QCMA complete. In section 6 we conclude with some open problems.

2 Background

The reader is referred to [21] for a review of relevant linear algebra and to [6] for a review of complexity theory. We give some relevant definitions and lemmas here.

Singular Value Decomposition - Every operator (finite square matrices throughout for us) A can be written as $A = UDV$, where U and V are unitaries ($UU^* = I$) and D is a diagonal matrix containing non-negative reals called the singular values of A. A hermitian operator A is such that $A = A^*$, whose diagonal matrix is real, and $V = U^*$. A positive semidefinite operator is hermitian and has non-negative values in the diagonal matrix. An orthogonal projection is an operator P such that $P^2 = P$ and $P = P^*$.

For any operator A, the *operator norm* will be represented as $\|A\| = \max_{\|x\|=1} \|Ax\|$ which is the *largest singular value* of A. We can define the *hilbert schmidt inner product* between two operators, $\langle A, B\rangle = \text{Tr}(A^*B)$. For any hilbert space A, by $\mathcal{L}(A)$ we mean the operators that map vectors in A to A.

Schmidt Decomposition - Let $H = A \otimes B$ be a finite dimensional hilbert space. Then every operator K over H can be written as $K = \sum_i P_i \otimes Q_i$ such that $\{P_i\} \subseteq \mathcal{L}(A)$ and $\{Q_i\} \subseteq \mathcal{L}(B)$, and for $i \neq j$,$\langle P_i, P_j\rangle = 0$ and $\langle Q_i, Q_j\rangle = 0$. Similarly, any vector $v \in \mathcal{H}$ can be written as $v = \sum_i \lambda_i u_i \otimes w_i$ such that $\{u_i\} \in A$ and $\{w\} \in B$, for $i \neq j$, $\langle u_i, u_j\rangle = 0$, $\langle w_i, w_j\rangle = 0$, and $\langle u_i, u_i\rangle = 1$ and $\langle w_i, w_i\rangle = 1$, and the λ_i's are positive and non-increasing.

We will consider only the complex field $\mathbb{C}$. The hilbert space of n qudits will generally be $(\mathbb{C}^d)^{\otimes n}$, unless the dimension d is used for other purposes. Any k-local term H_i in a hamiltonian $H = \sum_i H_i$ will act on k out of n qubits, and $H_i \in \mathcal{L}(\mathbb{C}^{d^k} \times \mathbb{C}^{d^k})$.

An *algebra* A over a set S of operators, is a set that is closed under taking polynomials (over $\mathbb{C}$) of elements in S, and is closed under taking complex conjugates of elements. (This would be C^* algebra with the * operation is defined by complex conjugation). The *center* of an algebra A is $C(A) = \{P \in A : \forall D \in A, PD = DP\}$, i.e., the set of all operators *in* A that commute with all of the operators in A.

A hermitian operator $H \in \mathcal{L}(A)$ always preserves both a subspace and its complement. One key lemma is the following, which is easily shown by induction - A set of hermitian operators $\{A_i\}$ commute, i.e. for all i, j, $A_iA_j = A_jA_i$ *if and only if* there is some U such that for every i, $A_i = UD_iU^*$.

Most of the operators that we will look at will be hermitian, unless in certain cases, when we are looking at the operators appearing in a schmidt decomposition.

Regarding the notation, we may use the following phrases. By *ground space energy* of a hamiltonian H, we mean the *smallest eigenvalue* of H. By *ground state* of a hamiltonian H, we mean an *eigenvector* in the eigenspace of the smallest eigenvalue of H. We shall generally refer to vectors as v, ϕ,etc., and will use the bra-ket notation only when it helps to differentiate between various notation. We will use $\|\psi\|^2 = \langle\psi|\psi\rangle$, and $\langle\phi, H\psi\rangle = \langle\phi|H|\psi\rangle$. Other notation will be clear from context.

A language (subset of all set of strings over some alphabet) L is in the complexity class P if there is some deterministic machine M that always halts after some fixed polynomial number of steps on every input, such that

$$\begin{aligned} x \in L &\Rightarrow M(x) = 1 \\ x \notin L &\Rightarrow M(x) = 0 \end{aligned}$$

A language (subset of all set of strings over some alphabet) L is in the complexity class BPP if there is some deterministic polynomial time machine M such that

$$x \in L \quad \Rightarrow \quad \mathbb{E}_{r \leftarrow U_{\mathrm{poly}(|x|)}}(M(x,y,r)) \geq \frac{2}{3}$$
$$x \notin L \quad \Rightarrow \quad \mathbb{E}_{r \leftarrow U_{\mathrm{poly}(|x|)}}(M(x,y,r)) \leq \frac{1}{3}$$

A language (subset of all set of strings over some alphabet) L is in the complexity class NP if there is some deterministic polynomial time machine M such that

$$x \in L \quad \Rightarrow \quad \exists y, |y| \leq \mathrm{poly}(|x|), M(x,y) = 1$$
$$x \notin L \quad \Rightarrow \quad \forall y, |y| \leq \mathrm{poly}(|x|), M(x,y) = 0$$

A language (subset of all set of strings over some alphabet) L is in the complexity class MA if there is some deterministic polynomial time machine M such that

$$x \in L \quad \Rightarrow \quad \exists y, |y| \leq \mathrm{poly}(|x|), \mathbb{E}_{r \leftarrow U_{\mathrm{poly}(|x|)}}(M(x,y,r)) \geq \frac{2}{3}$$
$$x \notin L \quad \Rightarrow \quad \forall y, |y| \leq \mathrm{poly}(|x|), \mathbb{E}_{r \leftarrow U_{\mathrm{poly}(|x|)}}(M(x,y,r)) \leq \frac{1}{3}$$

A language L is in the complexity class QCMA if there is some quantum polynomial time machine M such that

$$x \in L \quad \Rightarrow \quad \exists y, |y| \leq \mathrm{poly}(|x|), \mathbb{E}(M(x,y)) \geq \frac{2}{3}$$
$$x \notin L \quad \Rightarrow \quad \forall y, |y| \leq \mathrm{poly}(|x|), \mathbb{E}(M(x,y)) \leq \frac{1}{3}$$

A language L is in the complexity class QMA if there is some quantum polynomial time machine M such that

$$x \in L \quad \Rightarrow \quad \exists v \in \mathbb{C}^{2^{\mathrm{poly}(|x|)}}, \mathbb{E}(M(x,v)) \geq \frac{2}{3}$$
$$x \notin L \quad \Rightarrow \quad \forall v \in \mathbb{C}^{2^{\mathrm{poly}(|x|)}}, \mathbb{E}(M(x,v)) \leq \frac{1}{3}$$

The constants $\frac{2}{3}$ and $\frac{1}{3}$ are arbitrary and can be modified/amplified for the probabilistic classes above to $(1 - 2^{-\mathrm{poly}(n)}, 2^{-\mathrm{poly}(n)})$. It is straightforward to show: $\mathsf{P} \subseteq \mathsf{NP} \subseteq \mathsf{MA} \subseteq \mathsf{QCMA} \subseteq \mathsf{QMA}$. It is conjectured that $\mathsf{P} = \mathsf{BPP}$ (and similarly $\mathsf{NP} = \mathsf{MA}$) and it would be highly unlikely for them not to hold.

3 Bravyi-Vyalyi Structure Theorem for Commuting Hamiltonians

Consider the problem of determining the ground state energy of local commuting hamiltonians. More specifically, assume we are given a hamiltonian $H = \sum_{i=1}^{m} H_i$ acting over n qudits, where each H_i is an orthogonal projection and acts non-trivially on at most k qudits and trivially on the rest. Further, we have that for all i, j, $H_i H_j = H_j H_i$, where each H_i is considered an operator over all the n qudits when considering commutativity. For simplicity, we consider the frustration free case (though the frustrated case follows in this case for commuting hamiltonians). We want to differentiate between two cases - where $\lambda_{\min}(H) = 0$ or $\lambda_{\min}(H) \neq 0$.

Note that the problem is exactly similar to the k-local hamiltonian problem, except that we have the extra condition of commutativity. The main question is whether the constraint of commutativity makes the problem any easier. Note that since the hamiltonians all commute, they are diagonalizable in a common basis, but this does not prevent the eigenvectors from being highly entangled, as is seen in the toric code. Historically, the problem seems to have been first considered in [7], where in it was shown that the problem is in NP over qudits for 2-local commuting hamiltonians. The main tool used in [7] for showing that the 2-local commuting hamiltonian case is in NP is a Structure Theorem from C^* algebras. This is indeed the key tool that is used in [7], [1], and [19] to show variants of the problem in NP.

In this section, we first present the structure theorem for commuting hamiltonians (abbrievated BVST) as shown in [7] for completeness. Our presentation shall follow the original paper [7] and the exposition in [9].

3.1 Key Idea

The main idea used in [7] is neat and remarkable, and uses a decomposition fact for C^* algebras. Before we present the idea, we state the following fact, and the reader is referred to [16] for a proof.

Fact 1. *Given an algebra A over a hilbert space H, it is possible to write $H = \bigoplus H_i$ where $H_i = H_{i1} \otimes H_{i2}$ such that $A = \bigoplus \mathcal{L}(H_{i1}) \otimes I$, and the center of A is generated by the orthogonal projections on H_i.*

We now prove the Structure Theorem in [7]. The main idea is to show that if two operators commute, then they can be made to commute trivially, i.e., it is possible to split the common hilbert space on which they act in a manner such that the operators act on different hilbert spaces.

Theorem 2. *(Structure Theorem) Let $A \otimes B \otimes C$ be some three hilbert spaces, and let $H \in \mathcal{L}(A \otimes B) \otimes I$ and $G \in I \otimes \mathcal{L}(B \otimes C)$. Let $H = \sum P_i \otimes Q_i$ and $G = \sum Q'_i \otimes R_i$. Let Q be the algebra generated by $\{Q_i\}$, and Q' the algebra generated by $\{Q'_i\}$. If $HG = GH$, then it is possible to write $B = \bigoplus_i B_i$ where $B_i = B_i^{Q} \otimes B_i^{Q'} \otimes B_i^{Q_r}$, such that H and G act invariantly on each of B_i, and restricted to the i'th slice, $H|_{B_i} \in \mathcal{L}(A \otimes B_i^{Q}) \otimes I$ and $G|_{B_i} \in I \otimes \mathcal{L}(B_i^{Q'} \otimes C)$.*

Proof. First, we show that if $HG = GH$ then the algebras Q and Q' commute. Note that

$$\begin{aligned} HG &= \sum_{i,j} P_i \otimes Q_i Q'_j \otimes R_j \\ GH &= \sum_{i,j} P_i \otimes Q'_j Q_i \otimes R_j \end{aligned}$$

Since the above expressions are equal, we have

$$HG - GH = \sum_{i,j} P_i \otimes (Q_i Q'_j - Q'_j Q_i) \otimes R_j = 0$$

Note that if any set of operators $\{E_1, ..., E_t\}$ are orthogonal under the hilbert schmidt inner product, then they are linearly independent, since if $E_1 = \sum_{i \geq 2} c_i E_i$, then

$$0 = \mathrm{Tr}(E_1^* E_j) = \sum_{i \geq 2} c_i^* \mathrm{Tr}(E_i^* E_j) = c_j^* \mathrm{Tr}(E_j^* E_j)$$

which implies $c_j = 0$ since $\mathrm{Tr}(E_j^* E_j) \neq 0$. Thus, for any $(i, j) \neq (i', j')$, the $(P_i \otimes R_j)$ and $(P_{i'} \otimes R_{j'})$ are linearly independent (by virtue of the schmidt decomposition), and thus, for all (i, j), $Q_i Q'_j = Q'_j Q_i$. Thus, the algebras Q and Q' commute.

Now, let $R \subseteq \mathcal{L}(B)$ be the algebra of all operators that commute with all the operators in Q and Q', i.e. $R = \{K \in \mathcal{L}(B) : \forall M \in Q \cup Q', KM = MK\}$. Thus, note that $C(Q) \subseteq R$ and $C(Q') \subseteq R$. But we can say something stronger - $C(Q) \subseteq C(R)$ and $C(Q') \subseteq C(R)$. For some operator M in $C(Q)$, and for any operator K in R, note that since K commutes with all operators in Q by definition, and M is in $C(Q)$, K commutes with M. Thus, M commutes with every operator K in R. Since $M \in R$, we get that $M \in C(R)$. Thus $C(Q) \subseteq C(R)$ and $C(Q') \subseteq C(R)$.

Now let us use the above fact with $A = R$ and $H = B$. This gives us that we can decompose $B = \bigoplus B_i$ such that $\{\Pi_i\}$ the orthogonal projections on B_i generate $C(R)$ and we can write $B_i = B_i^0 \otimes B_i^{\text{rest}}$ such that $R = \bigoplus \mathcal{L}(B_i^0) \otimes I$.

Consider the projector Π_i on B_i. Since the Π_i's are in $C(R)$, note that for any operator $M \in Q$, M commutes with Π_i (by definition of elements of R). Thus, note that M preserves the subspace B_i, since if for some $v \in B_i$, $Mv = u_1 + u_2$ where $u_1 \in B_i, u_2 \in \overline{B_i}$, then $u_1 = \Pi_i M v = M \Pi_i v = u_1 + u_2$ which implies $u_2 = 0$. Thus, the algebras Q and Q' preserve each of the subspaces B_i.

Now note that as observed above, since $C(Q) \subseteq C(R)$, we have that the algebras Q restricted to any B_i has a trivial center since R restricted to B_i has a trivial center. Let $Q|_i = \Pi_i Q \Pi_i$, i.e. the algebra Q restricted to the i'th subspace, and similarly $Q'|_i$ and $R|_i$. Note that $Q|_i$, $R|_i$ and Q'_i commute, since $\Pi_i \in C(R)$. Now note that since $R|_i = \mathcal{L}(B_i^0) \otimes I$, since $Q|_i$ commutes with $R|_i$, it must commute with all the operators on B_i^0, and thus, $Q|_i$ must act trivially on B_i^0 and non-trivially only on B_i^{rest}. Similarly for $Q'|_i$. Now let us apply the fact stated above for the space B_i^{rest} with $Q|_i$ as the algebra. Note that since the algebra has a trivial center, it simply decomposes $B_i^{\text{rest}} = B_i^Q \otimes B_i^{\text{rest}}$, such that $Q|_i = \mathcal{L}(B_i^Q) \otimes I$. And further, we can decompose B_i^{rest} so that $Q'|_i = \mathcal{L}(B_i^{Q'}) \otimes I$. Thus, we get that each of B_i can be decomposed as $B_i = B_i^0 \otimes B_i^Q \otimes B_i^{Q'} \otimes B_i^{\text{rest}}$, such that $R|_i = \mathcal{L}(B_i^0) \otimes I \otimes I \otimes I$, $Q|_i = I \otimes \mathcal{L}(B_i^Q) \otimes I \otimes I$, and $Q'|_i = I \otimes I \otimes \mathcal{L}(B_i^{Q'}) \otimes I$.

Thus, the claim follows. □

Thus, as was stated, the lemma above essentially says that if two hamiltonians commute, then they can be made to commute *trivially* (as if acting on different hilbert spaces) by slicing the common hilbert space into orthogonal subspaces, and further decomposing each subspace into smaller spaces on which the hamiltonians commute trivially. This lemma is extremely useful, can be applied in a variety of ways. We now present certain applications that use the Structure Theorem above, with proof sketches of the main ideas.

3.2 Applications of the Structure Lemma

Now we give some applications that use the structure lemma crucially.

Theorem 3. *([7]) Deciding the ground space of 2-local commuting hamiltonians over qudits is in* NP.

Proof. Consider the problem of deciding whether the ground energy of 2-local commuting hamiltonians over qudits is 0 or not. Let $H = \sum_i H_i$ be the hamiltonian. Note that since we have only

2-local hamiltonians, any two hamiltonians either interact on one qudit or not at all. This will help us apply the Structure Theorem stated above.

Consider any qudit q. Using the structure theorem, the hilbert space H_q of q can be decomposed so that $H_q = \bigoplus G_i$, and we shall refer to each G_i as a "slice". Further, every hamiltonian acting on q is invariant over each slice G_i, and inside each G_i, each of the hamiltonians act on different hilbert spaces. Thus, assume we have divided the space of every qudit into slices. Note that if there is a vector in the ground space of H, then since each of the H_i's are invariant over every slice of all the qubits, H itself is invariant over it. And if H is invariant over any spaces A and B, and there is a vector $v \in A \oplus B$ in the ground space of H, then we can rewrite $v = v_A + v_B$ such that both v_A and v_B are in the ground space of H. Thus, since H is invariant over each of the slices, if there is a vector in its ground space, there is a vector in one of the slices that is in its ground space.

This is the proof that we ask the prover to provide, i.e., we ask the prover to give us the description of the specific G_i for every qudit q in which to find a ground space vector. But note that once we have the specific slice or subspace for each of the qubits, we know that the hamiltonians acting on q act on different hilbert spaces. Note that since the local dimension d of each of the qudits is a constant, we can easily find the decomposition of every hamiltonian acting on q inside the slice of this qudit provided by the prover. Finally, we have a tensor product of hilbert spaces over all the qudits, such that all the hamiltonians act on different spaces, and we can easily verify whether it has a vector with eigenvalue 0 or not. Thus, the problem is in NP. □

Apart from the 2-local commuting hamiltonian problem over qudits, [7] also show the problem of deciding the ground space energy is in NP if all the hamiltonians are tensor products over all n individual qubits and commute, and we refer the reader to their paper for the proof.

After [7], the theorem above was extended to 3-local commuting hamiltonians, but over qubits in [1]. The main tool is the Structure Theorem, but now many different structures could arise due to the hamiltonians acting on 3 qubits instead of graphs with vertices and edges (as in the 2-local case), that doesnt allow us to apply the Structure Theorem directly. However, it is shown that we can characterize such structures in a manner that we can apply the structure theorem. We refer the reader to [1] for complete details.

Theorem 4. *([1]) Deciding the ground space of 3-local commuting hamiltonians over qubits is in* NP*.*

Both the above theorems for the 2 and 3 local cases showed that the ground state is a tensor product, since an explicit ground state was found in each case. But for the 4-local case, note that since the toric code has 4-local commuting hamiltonians and has a highly entangled ground state, the above methods of explicitly finding the ground state should not work. Nevertheless, it was shown in [19] that whether the ground energy of 4-local commuting hamiltonians over the grid is nonzero can be decided in NP, which includes the toric code, without finding the explicit ground state.

Theorem 5. *([19]) Deciding the ground space of 4-local commuting hamiltonians over qubits over a grid is in* NP*.*

Proof. Given a 4-local commuting hamiltonian $H' = \sum H'_i$, over qubits over the grid, assume that all terms in the Hamiltonian are projections onto the ground space of each hamiltonian, i.e. $H = \prod H_i$ where $H_i = I - H'_i$. Thus, assuming H' is frustration free and all H'_i are projections, we need to find whether there exists a vector v such that $Hv = v$.

First, note that over the grid, all the hamiltonians can be grouped into two sets A and B, each containing alternating terms like a chessboard, so that $H = AB$, and inside A and B, the ground state can be found using the structure theorem. Thus, we can ask first the prover to give us the slices at each of the qubits, so that we can verify if both A and B have a ground state or not. If not, we can immediately reject.

Further, $\mathrm{Tr}(AB) = 0$ if and only if $AB = 0$ since A and B commute and have a common eigenbasis. But $AB = 0$ implies there is no ground space vector. Thus, what we need to verify is the fact that $\mathrm{Tr}(AB) = 0$ or not. Now if we apply the structure theorem to each of the qubits, in two different ways so that both A and B slice the hilbert space at each qubit, we would need to verify $\mathrm{Tr}(\bigoplus_{i,j} A_i B_j) = \sum_{i,j} \mathrm{Tr}(A_i B_j) = 0$, where each A_i is an orthogonal projection of A to the i'th slice (which would be a specific slice of every qubit, as before) as is provided by the prover, and similarly B_j. Thus, note that since we are taking the trace of only 2 positive semidefinite operators A_i and B_j (that do not necessarily commute), the trace of each of the slices is always non-negative, i.e. $\mathrm{Tr}(A_i B_j) \geq 0$, and we have that $\mathrm{Tr}(AB) = 0$ only if the trace of each of each the slices is zero, i.e. $\mathrm{Tr}(A_i B_j) = 0$. This fact does not hold if we had 3 operators A_i, B_j, C_k, i.e. for any 3 operators that do not commute but are positive semidefinite, it is not the case that $\mathrm{Tr}(A_i B_j C_k) \geq 0$, and thus this method works only for 2 layers of operators.

Finally, we can ask the prover to tell us about the slice A_i and B_j such that $\mathrm{Tr}(A_i B_j) \neq 0$. Once the prover tells us the different ways in which to slice the space of each of the qubits, we note that we can slice the space only into 2 spaces of dimension 1 or not at all. This creates linear structures over which we would have to evaluate $\mathrm{Tr}(A_i B_j)$, which we can do in polynomial time. Note that in general, computing $\mathrm{Tr}(A_i B_j) = 0$ is not easy. The specific details of the resulting structures can be found in [19]. Thus we get that the problem is in NP. □

Since we will be looking at $O(\log n)$ local commuting hamiltonians throughout, we now observe that a particular case of $O(\log n)$ local commuting hamiltonians that is in NP, as an application of the Structure Theorem.

Lemma 6. *Given a hypercube containing hamiltonians on vertices and qudits on edges such that all the hamiltonians commute, computing whether its ground space energy is zero is in* NP

Proof. Let the number of qudits be n. Note that in a hypercube of dimension d, the number of edges is $d2^{d-1}$, which is a solution to $E(d) = 2E(d-1) + 2^{d-1}$. Thus, $2^{d-1} \leq n = d2^{d-1} \leq 2^d$, and $d \sim \log n$. Thus, note that each hamiltonian is $O(\log n)$ local. Although the locality is high, we note that any pair of hamiltonians act only one edge (one qudit).

In the 2 local commuting case where schmidt decomposing a hamiltonian on one qudit gave us the decomposition on another, and thus gave us the algebras on both the qubits. For the present case, consider a hamiltonian acting on the qudits $q_1, q_2, ..., q_{\log n}$. We will need to show that after decomposing H over q_1 and $q_2, ..., q_{\log n}$, taking the union of the algebras generated by further decomposing each of the terms of H over q_2 are the same as the algebra generated by decomposing H directly over q_2, i.e. after a hamiltonian has been decomposed into the subspace of a particular qudit, we can further decompose it into those of the remaining qudits without changing the algebra on it.

More formally, let $Y \otimes X \otimes Z$ be a hilbert space. Let $H = \sum_i A_i \otimes F_i = \sum_j B_i \otimes Q_i$ where $A_i \in \mathcal{L}(X)$ and $F_i \in \mathcal{L}(Y \otimes Z)$, and $B_i \in \mathcal{L}(Y)$ and $Q_i \in \mathcal{L}(X \otimes Z)$. Let $F_i = \sum_j C^i_j \otimes D^i_j$, where $C^i_j \in \mathcal{L}(Y)$ and $D^i_j \in \mathcal{L}(Z)$. Then the algebras generated by $\{B_i\}$ and $\{C^i_j\}$ are the same. To see this, let us first write H in 2 ways, first by decomposing it over X and then the decomposed terms over Y; and by directly decomposing H over Y. Thus, $H = \sum_{i,j} C^i_j \otimes A_i \otimes D^i_j = \sum_k B_k \otimes Q_k$. Note that $\langle A_i \otimes D^i_j, A_{i'} \otimes D^{i'}_{j'} \rangle = 0$, since for $i \neq i'$ the A_i's are orthogonal and for $i = i'$ and $j \neq j'$, D^i_j

and $D^i_{j'}$ are orthogonal. Let $C^i_j = C_{i,j}$ and $A_i \otimes D^i_j = R_{i,j}$. Note that all the sets of operators except $C_{i,j}$'s are orthogonal.

Extend the set of operators Q_k's to a basis, and let $R_{i,j} = \sum_l \alpha_l Q_l$. Then

$$H = \sum_{i,j} C_{i,j} \otimes R_{i,j} = \sum_{i,j,l} \alpha_l C_{i,j} \otimes Q_l = \sum_l (\alpha_l \sum_{i,j} C_{i,j}) \otimes Q_l = \sum_k B_k \otimes Q_k$$

Now note that since the Q_i's are orthogonal and hence linearly independent, we get that $B_k = \alpha_k \sum_{i,j} C_{i,j}$. Since the B_k's are a linear combination of the $C_{i,j}$'s, we get that the algebra generated by both are the same.

Now we simply apply the Structure Theorem, and proceed exactly as the 2-local case. Assume the prover gave us the slice for each qudit in which to look for the ground state. We consider the way in which each of the hamiltonians acts on the qudit and on which space inside the slice it acts, which as shown above, we can do since the algebra would be the same irrespective of how we decompose the hamiltonian, and we can directly decompose the hamiltonian over the qudit of interest. Since only 2 hamiltonians act on each qudit, we can find the way the hamiltonians act on each of the slices, and finally find the ground state. Note that since the hamiltonians are $O(\log n)$ local, all operations on them can be done in poly(n) time. Given the description of each of the slices, we can verify the ground energy and even find a ground state if there is one, and thus, the problem lies in NP. □

4 Area Laws

In this section, we change our focus, and consider area laws. We first review the main ideas in the combinatorial proof of [5] for showing an area law for Hamiltonians on a 1D chain. We then give a simple proof of a generalized area law for commuting hamiltonians in all dimensions.

4.1 Review of the combinatorial proof of the area law in 1D

The setup in [5] is as follows. We are given a local hamiltonian $H = \sum_{i=1}^{n} H_i$ on n qudits (dimension d) on a line. The assumptions are as follows:

1. Each H_i is 2-local (each $H_i \in (\mathbb{C}^d)^2$) and an orthogonal projection, i.e. $H_i^2 = H_i = H_i^*$
2. The hamiltonian H is frustration free, i.e. $\lambda_{min}(H) = 0$
3. There is a unique ground state v such that $Hv = 0$
4. The smallest non-zero eigenvalue of H is at least ϵ

Assumptions 1 and 3 are mild, in the sense that they can be achieved by scaling the hamiltonian terms appropriately (after assuming 2). Assumption 2 is strong, though in a later work [4], it was shown that the same ideas can be made to work for frustrated systems. Assumption 4 is strong too, since it requires ϵ to be at least a *constant* for the proof to work. The bound on the entanglement entropy of the ground state across the cut in [4] scales as $\log(\frac{1}{\epsilon})$, and becomes weaker if one considers ϵ to be inverse polynomial in n, though it is still stronger than a naive volume law.

We now state some definitions.

Definition 7. (Schmidt Rank) Given a system of n qudits, let $S \subseteq [n]$ be some set of qubits. Then for any vector (state) v, the schmidt rank of v, denoted $\mathrm{SR}_S(v)$ is the minimum integer D such that $v = \sum_{i=1}^{D} \lambda_i u_i \otimes w_i$. For the minimum D, the set $\{u_i\}$ are orthonormal vectors over S and $\{w_i\}$ are orthonormal vectors over $\overline{S}$. Further, the λ_i's are in non-decreasing order, and strictly greater than 0. Similarly, given any operator K, the schmidt rank of K, denoted $\mathrm{SR}_S(K)$ is the minimum integer D such that $K = \sum_{i=1}^{D} P_i \otimes Q_i$, where for the minimum D, the $\{P_i\}$ and $\{Q_i\}$ are orthogonal (with respect to the trace norm) operators acting on the sets of qudits S and $\overline{S}$ respectively.

We state a few simple properties of schmidt rank that will be useful for us.

Lemma 8. *(i)* $SR_S(u+v) \leq SR_S(u) + SR_S(v)$
(ii) for $c > 0$, $SR_S(cu) = SR_S(u)$
(iii) for any two operators A *and* B, $SR_S(AB) \leq SR_S(A) SR_S(B)$

The proof of the above lemma follows trivially from the definitions.

One main tool used in [4] is the use of Approximate Ground State Projectors (AGSPs), which we define next.

Definition 9. (AGSP) Let $H = \sum_{i=1}^{n} H_i$ be a local hamiltonian on a chain as described above, and let $(i, i+1)$ be the cut on the chain. An operator K is a (D, γ)-AGSP if,

1. It preserves the ground space, i.e. if $Hv = 0$, then $Kv = v$
2. It shrinks (and is invariant over) the space orthogonal to the ground space by a factor γ, i.e., for all w orthogonal to the ground space, Kw is orthogonal to the ground space, and $\|Kw\| \leq \lambda \|w\|$
3. It has bond dimension D, i.e., $SR_{\leq i}(K) \leq D$.

Given the definition of $AGSP$s, the proof in [4] follows 3 main steps.

1. Show that for any fixed cut $(i, i+1)$, the existence of a (D,γ)-AGSP such that $D\gamma \leq \frac{1}{2}$ implies the existence of a product state across the cut, $\mu = u \otimes w$ with considerable overlap with the ground state, i.e. $\langle v, \mu \rangle \geq \frac{1}{\sqrt{2D}}$.

2. If all conditions and implications of 1 hold, then the entanglement entropy of v, $S(v) \leq O(1)\frac{\log(1/\mu)}{\log(1/\gamma)} \log D$

3. Show the existence of a (D,γ)-AGSP such that $D\gamma \leq \frac{1}{2}$, and $\log D \leq O(\frac{\log d}{\epsilon})^4$ (where d is the local qudit dimension).

Note that 1,2,3 above imply an area law for Hamiltonians on a 1D chain with constant spectral gap. The lemmas 1 and 2 above are purely linear algebraic facts, and they follow from the definition of AGSPs. The key ingenuity is in showing 3 above, where a (D,γ)-AGSPs is explicitly constructed by a neat use of Chebyshev polynomials. The interested reader is referred to [4] or [20] for an exposition of each of the above.

4.2 Area law for commuting hamiltonians

In this section, we give a simple proof of an area law for commuting hamiltonians in all dimensions, which seems to be known though not written formally anywhere. We describe the setup and state our assumptions first.

Assume we are working over a system of n qudits, each of local dimension at most d (thus forming a hilbert space $\mathcal{H} \in (\mathbb{C}^d)^{\otimes n}$). We have a system of m hamiltonians, $H = \sum_{i=1}^{m} H_i$, and the following holds -

1. The H_i's are k local, i.e. each H_i acts non-trivially on at most k qudits, and trivially on the rest. Each H_i is an orthogonal projection, i.e. $H_i^2 = H_i = H_i^*$. Importantly, the system is commuting, i.e. for all i, j, $H_iH_j = H_jH_i$, where each H_i is considered a hamiltonian over the whole space.

2. The hamiltonian H is frustration free, i.e. $\lambda_{min}(H) = 0$

3. There is a unique ground state v such that $Hv = 0$

The first condition above is weaker (and hence more general than the condition stated previously). By condition 2, we assume frustration free hamiltonians, and we leave the frustrated case for future work. The condition 3 for commuting hamiltonians can easily be enforced by adding additional terms in the hamiltonian. We do not need to assume condition 4 since for commuting hamiltonians, the minimum spectral gap is always 1.

Definition 10. (Exact Ground Space Projector) We say that an operator K is a D-EGSP if it is a $(D, 0)$-AGSP across the particular cut.

Lemma 11. *Let $H = \sum_{i=1}^{m} H_i$ be a frustration free hamiltonian, where any pair of terms H_i and H_j commute and are orthogonal projections. Then there exists a D-EGSP for H for some D.*

Proof. Let $K = \prod_{i=1}^{m}(I - H_i)$. The main idea is that since the H_i's commute, there is a common basis in which each of them is diagonal. Thus, we can write each H_i as $H_i = UD_iU^*$, where D_i is a diagonal matrix containing only 0's and 1's since the H_i's are projections. Let the ground space of H be W, and spanned by vectors u_1 to u_r for U and the space orthogonal to the ground space be spanned by v_1 to v_t.

Let w be in the ground space of H. Since H is frustration free, it is in the ground space of each of H_i's, and thus $Kw = w$ since $H_iw = 0$ for all i. Now let w be orthogonal to the ground space of H. Let $w = \sum c_iv_i$. Note that the v_i's are eigenvectors of each of the H_i's. Thus, for each v_i, there is some H_j such that $H_jv_i = 1$, for if not, then v_i lies in the ground space of each of the H_i's, and thus is in the ground space of H, a contradiction. Thus, when we apply K to w, some term of the form $(I - H_j)$ will annihilate each of the v_i's, and thus $Kw = 0$. Thus, K is an exact ground state projector. □

Now note that bounding the schmidt rank of K will give us simple control over the entanglement entropy of any ground state of H. We do that next.

Lemma 12. *(Area Law for commuting hamiltonians) Let $H = \sum_{i=1}^{m} H_i$ be a frustration free k-local Hamiltonian over n qudits (local dimension d) made of commuting terms. Let v be the unique ground state of H. Let $S \subseteq [n]$ be a cut such that $|S| \le |\overline{S}|$. Let T be the set of Hamiltonians acting across the cut, i.e. $H_i \in T$ iff there are qubits $a \in S$ and $b \in \overline{S}$ on which H_i acts non-trivially. Let $t = |T|$. Then $S(v) \le \alpha \log d$ where $\alpha = \min\{kt, 2|S|\}$.*

Proof. Let $G = \prod_{H_i \in T}(I - H_i)$ and $R = \prod_{H_i \notin T}(I - H_i)$. Our EGSP as stated above will be $K = GR$. Note that since the hamiltonians in R do not act across the cut, they do not increase the schmidt rank across the cut. Let $v = \sum_i \lambda_i u_i \otimes w_i$ be the schmidt decomposition of v across the cut $(S, \overline{S})$. Note that each $\lambda_i > 0$. Let $\phi = u_1 \otimes v_1$. Since the u_i's and v_i's are orthonormal, we have that $\langle \phi, v \rangle = \lambda_1$. Let $\phi = \lambda_1 v' + \beta v^{\perp}$, where $v^{\perp}$ lies in the space orthogonal to v', note that we have absorbed the phase into v and written it as v'. Now note that $K\phi = \lambda_1 v' + 0 = \lambda_1 v'$ due to the lemma above.

$$
\begin{aligned}
SR(v) &= SR(v') \\
&= SR(\lambda_1 v') \\
&= SR(K\phi) \\
&= SR(GR\phi) \\
&= SR(RG\phi) \\
&= SR(G\phi) \\
&= SR(G)
\end{aligned}
$$

The second line follows since $\lambda_1 > 0$, the third line since $K\phi = \lambda_1 v'$, the fifth line follows due to commutativity, the sixth line follows since R does not act across the cut, and the last line follows since ϕ is a product state. Thus, we need to bound the schmidt rank of G.

Now let us bound the schmidt rank of G in two ways. Note that since $SR(AB) \le SR(A)SR(B)$, we can write $SR(G) \le (SR(H_1))^t$ for some $H_i \in T$. Since H_1 is k local, let H_1 act on a qudits in S and $k - a$ qudits in $\overline{S}$. Thus, we can write $H_1 = \sum_{i=1}^{\min\{d^{2a}, d^{2(k-a)}\}} \lambda_i P_i \otimes Q_i$ which is maximized for $a = \frac{k}{2}$. Thus, $SR(H_1) \le d^k$. This gives $SR(G) \le d^{kt}$.

Now we can count in a different manner. Consider the product $G = \prod_{i=1}^{t} P_i$ (where $P_i = 1 - H_i$, and we've assumed without loss of generality that the hamiltonians H_1 to H_t are in T). Note that the expansion of each of the terms P_i across the cut, and their products will create too many terms. But note that since all terms will be across the two sets S and $\overline{S}$, the maximum number of terms in the schmidt decomposition of G (or any operator across these two spaces) will be at most $d^{2|S|}$. Thus, $SR(G) \le d^{2|S|}$ (such a volume law applies for any hamiltonian).

Note that since entropy of any distribution is bounded by the log of its support, and any state with schmidt rank at most D is supported on D elements, we get that

$$S(v) \le \log SR(v) = \log SR(G) \le \min\{kt, 2|S|\} \log d$$

as required. □

Remark 13. The key thing to note in the above lemma is that the entanglement entropy is a function only of the number of hamiltonians acting across the cut. Further, this holds for *any* configuration of hamiltonians, not necessary hamiltonians that are geometrically local, which is surprising, since it was shown in [3] that for general hamiltonians (not necessarily commuting) that are not geometrically local, there exist states with very high entanglement entropy across the cut. The above lemma is essentially the area law for commuting hamiltonians.

Corollary 14. *Consider a L dimensional hypergrid of side length n consisting of n^L qudits each of dimension d. Let $H = \sum_{i=1}^{n^L} H_i$ be a system of frustration free commuting 2^L-local hamiltonians with a unique ground state v, where each H_i acts over a hypercube of 2^L qudits in a natural way. Let A be any contiguous region of qudits. Then the number of hamiltonians acting from A to $\overline{A}$ is exactly $|\partial A|$ (where $|\partial A|$ denotes the boundary of A). Thus, from the lemma above, $S(v) \leq 2^L|\partial A| \log d$, which for constant L and d gives $S(v) \leq O(|\partial A|)$. Further, note that if the hamiltonians have locality that is logarithm of the total number of qubits, then $2^L = L \log n$, and we would have (assuming d constant),*

$$S(v) \leq O(L \log n |\partial A|)$$

In general for n qudits (constant d) and Hamiltonians with locality $O(\log n)$,

$$S(v) \leq O(\log n |\partial A|)$$

which is much stronger than volume law, and is infact an area law for $O(\log n)$ local commuting hamiltonians.

Given the above area law, and tools like the structure theorem, it may be possible to design algorithms for commuting hamiltonians, possibly in QCMA, though that is left as an open problem.

5 Hardness of Traversing the Ground Space

In this section, we finally consider the problem of traversing the ground space of $O(\log n)$ local commuting hamiltonians. Informally, as stated in the introduction, we want to understand the hardness of going from one ground state of commuting hamiltonians to another by applying only 2-qubit unitaries a polynomial (in system size) number of times, and be close to the ground space at each point. Towards this end, we will show that traversing the ground space of $O(\log n)$ local commuting hamiltonians is QCMA complete. The proof that the problem is in QCMA follows almost verbatim from the proof for general local hamiltonians as in [10], and we will refer the reader to the paper for a proof. We shall provide a proof of QCMA hardness of traversing the ground space of $O(\log n)$ local commuting hamiltonians.

Our proof consists of two parts. For the first part, we show shall show that given any instance of the local hamiltonian problem, we can convert it to another instance such that the instance has only 2 layers of hamiltonians (where hamiltonians inside any layer commute), and the completeness and soundness parameters are almost the same. Next, we show how to construct an instance of $O(\log n)$ local commuting hamiltonians from a QCMA verifier circuit, such that traversing the ground space is possible only if the QCMA circuit accepts some state with high probability.

5.1 Definitions and Background

The formal definition of the problem is as follows (adapted from [10]):

Definition 15. GSCON for commuting Hamiltonians - GSCONCH $(H, k, l, m, c, s, U_\psi, U_\phi)$

Input: Input parameter n, the number of qubits on which the hamiltonian H acts. A k local hamiltonian $H = \sum_{i=1}^{q} H_i$ acting on n qubits, where each $H_i \in Herm((C^2)^{\otimes k})$ acts nontrivially on a subset of at most k qubits (and trivially on the rest). For each i, $\|H_i\| \leq 1$. Further, for all i, j, $H_i H_j = H_j H_i$, where each H_i is interpretated as an operator acting on 2^n dimensional vector space. U_ψ and U_ϕ are unitaries that generate the initial and final states $|\psi\rangle$ and $|\phi\rangle$. Real numbers c, s such that $s - c \in O(\frac{1}{\text{poly}(n)})$.

Output:

1. *Yes*, if there exists *some* sequence of l local unitaries U_1 to U_m, such that $\||\phi\rangle - U_m ... U_1|\psi\rangle\| \leq c$, *and,* for all $1 \leq i \leq m$, $\langle \psi_i, H\psi_i \rangle \leq c$, where $|\psi_i\rangle = U_i ... U_1|\psi\rangle$.

2. *No*, if for *any* sequence of l local unitaries U_1 to U_m, either $\||\phi\rangle - U_m ... U_1|\psi\rangle\|_2 \geq s$ *or,* for some i, $\langle \psi_i, H\psi_i \rangle \geq s$.

Note that the two conditions that are checked, are precisely if the movement is indeed through the ground space, i.e. if every intermediate state is close to the ground state and has low energy, and the final state is close to the expected state $|\phi\rangle$.

For the case of general hamiltonians, it was shown in [10] that the problem is QCMA hard, as long as m or the number of unitaries is polynomial in n. Our proof shall use many of the tools that their proof uses. Indeed, our key contribution is a construction that is amenable to most of the techniques in [10] after simple changes.

5.2 Hardness of 2-layered $O(\log n)$ local Hamiltonians

Theorem 16. *([15]) The k local hamiltonian problem for $k \geq 2$ is* QMA *complete, i.e., given a system of k local hamiltonians $H = \sum_{i=1}^{m} H_i$ where each H_i acts on a system of k out of n qubits, it is* QMA *hard to decide between the two cases (given c, s such that $s - c \in O(\frac{1}{poly(n)})$):*

1. $\exists|\psi\rangle$ *such that* $\langle\psi, H\psi\rangle \leq c$
2. $\forall|\psi\rangle$*,* $\langle\psi, H\psi\rangle \geq s$

The reader may consult [15] or [20] for an exposition of the proof.

One key tool used in the proof of QMA hardness of 2-local hamiltonian was the Projection Lemma in [13]. Although we do not need the full power of the projection lemma, we now state another lemma that is similar to the projection lemma in spirit. Intuitively, the projection lemma helps to chop of subspaces without much affecting the ground space. Let W be the subspace (of the Hilbert Space H) in which we want our ground space vector of some hamiltonian H_1 to lie. Then we can *create* a hamiltonian (projection) H_2, that has eigenvalue 0 over the subspace W, and gives an enormous penalty $J \gg \|H_1\|$ to the space $W^{\perp}$. The projection lemma says that the lowest eigenvalue of $H_1 + H_2$ is close to the lowest eigenvalue of the operator H_1 restricted to W. Similar to the projection lemma, we group all the hamiltonians into one hamiltonian H_1 after making their spaces orthogonal to each other, and then create another hamiltonian H_2 that applies a penalty in case the ground state tries to annihilate any of the terms in H_1 by not being in the orthogonal subspace of any particular term. More specifically,

Theorem 17. *(Ground Space Preserving Layers) Let* $H = \sum_{i=1}^{m} H_i$ *be a system of 3 local hamiltonians acting on a system of n qubits where* $\|H_i\| \leq 1$*. Let c and s be two reals such that* $s - c \geq n^{-b'} = m^{-b}$ *(for some* b, b'*). Consider a system of* $n' = n + 2\log m$ *qubits. For* $1 \leq i \leq m$*, let* $G_i = H_i \otimes |i\rangle\langle i|$ *where* $|i\rangle$ *is the* $2\log m$ *bit representation of i in binary. Let* $G = \sum_{i=1}^{m} G_i$*. Let* $G_{pen} = m^{\gamma} I \otimes (I - |\phi\rangle\langle\phi|)$ *where* $|\phi\rangle = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} |i\rangle$*. Then the following hold:*

1. If $\exists|\psi\rangle \in C^{2^n}$ *such that* $\langle\psi, H\psi\rangle \leq c$*, then* $\exists|\psi'\rangle \in C^{2^{n'}}$*, such that* $\langle\psi', (G + G_{pen})\psi'\rangle \leq \frac{c}{m}$*.*
2. If $\forall|\psi\rangle \in C^{2^n}$*,* $\langle\psi, H\psi\rangle \geq s$*, then* $\forall|\psi'\rangle \in C^{2^{n'}}$*,* $\langle\psi', (G + G_{pen})\psi'\rangle \geq \frac{s}{m} - \frac{1}{m^{b+1}}$*.*

Proof. 1. *(Completeness)* Let $|\psi\rangle$ be such that $\langle\psi|H|\psi\rangle \leq c$. Define $|\psi'\rangle = |\psi\rangle \otimes |\phi\rangle$. Then by straightforward calculation,

$$\begin{aligned}
\langle\psi'|G + G_{pen}|\psi'\rangle &= \langle\psi| \otimes \langle\phi|G|\psi\rangle \otimes |\phi\rangle + \langle\psi| \otimes \langle\phi|G_{pen}|\psi\rangle \otimes |\phi\rangle \\
&= \sum_{i=1}^{m} \langle\psi| \otimes \langle\phi|G_i|\psi\rangle \otimes |\phi\rangle + \langle\psi| \otimes \langle\phi|G_{pen}|\psi\rangle \otimes |\phi\rangle \\
&= \sum_{i=1}^{m} \langle\psi|H_i|\psi\rangle |\langle\phi|i\rangle|^2 + m^{\gamma}\langle\psi|I|\psi\rangle\langle\phi|(I - |\phi\rangle\langle\phi|)|\phi\rangle \\
&= \frac{1}{m}\sum_{i=1}^{m} \langle\psi|H_i|\psi\rangle + 0 \\
&\leq \frac{c}{m}
\end{aligned}$$

2. *(Soundness)* It is worthwhile to note that the term G_{pen} has been created to give a high penalty to the subspace that is orthogonal to the space spanned by the uniform superposition $|\phi\rangle = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} |i\rangle$. We do this since we want the ground space of the new system to span the space spanned by $|\phi\rangle$ in the second register, for if not, then it would be possible to create simple states that annhilate the hamiltonian G easily.

Notice that G_{pen} is a projection with only two eigenvalues - 0 and m^{γ}, i.e. the smallest nonzero eigenvalue of G_{pen} is m^{γ}. Let W be the subspace spanned by $|\phi\rangle$. Thus, for all $|\mu\rangle \in W^{\perp}$, we have that $\langle\mu, G_{pen}\mu\rangle = m^{\gamma}$.

We now prove by contradiction. Let $|\psi'\rangle$ be a vector such that $\langle\psi', (G+G_{pen})\psi'\rangle \leq \frac{s}{m} - \frac{1}{m^{b+1}}$. Then we can rewrite $|\psi'\rangle = c_0|v\rangle \otimes |\phi\rangle + c_1|\phi^\perp\rangle$, where $|c_0|^2 + |c_1|^2 = 1$. Note that since both G and G_{pen} positive semidefinite, we have that $|\langle\psi'|G|\psi'\rangle| = \langle\psi'|G|\psi'\rangle$ and $|\langle\psi'|G_{pen}|\psi'\rangle| = \langle\psi'|G_{pen}|\psi'\rangle$.

Then we have that

$$\langle\psi'|G_{pen}|\psi'\rangle = \langle\psi'|G_{pen}(c_0|v\rangle \otimes |\phi\rangle + c_1|\phi^\perp\rangle) = |c_1|^2 m^\gamma$$

Further,

$$\begin{aligned}
|\langle\psi'|G|\psi'\rangle| &= |\langle\psi'|G(c_0|v\rangle \otimes |\phi\rangle + c_1|\phi^\perp\rangle)| \\
&= \Big| |c_0|^2 \langle v| \otimes \langle\phi|G|v\rangle \otimes |\phi\rangle + c_0^* c_1 \langle v| \otimes \langle\phi|G|\phi^\perp\rangle \\
&\quad + c_0 c_1^* \langle\phi^\perp|G|v\rangle \otimes |\phi\rangle + |c_1|^2 \langle\phi^\perp|G|\phi^\perp\rangle \Big| \\
&\geq |c_0|^2 |\langle v| \otimes \langle\phi|G|v\rangle \otimes |\phi\rangle| - |c_0||c_1||\langle v| \otimes \langle\phi|G|\phi^\perp\rangle| \\
&\quad - |c_0||c_1^*||\langle\phi^\perp|G|v\rangle \otimes |\phi\rangle| - |c_1|^2 |\langle\phi^\perp|G|\phi^\perp\rangle| \\
&\geq |c_0|^2 |\langle v| \otimes \langle\phi|G|v\rangle \otimes |\phi\rangle| - 3|c_1|\|G\| \\
&\geq \frac{|c_0|^2}{m} |\langle v|H|v\rangle| - 3|c_1|m \\
&\geq \frac{s|c_0|^2}{m} - 3|c_1|m
\end{aligned}$$

where the 3rd line follows by triangle inequality, the fourth line uses Cauchy-Schwarz inequality, and the fifth line uses the fact that $\|G\| \leq m$.

Now we choose γ to get the required bounds. Let $\gamma = b + 5$.

Thus, we have that

$$\begin{aligned}
\langle\psi'|G + G_{pen}|\psi'\rangle &= |\langle\psi'|G + G_{pen}|\psi'\rangle| \\
&\geq \frac{s|c_0|^2}{m} - 3|c_1|m + |c_1|^2 m^\gamma \\
&= \frac{s}{m}(1 - |c_1|^2) - 3|c_1|m + |c_1|^2 m^\gamma \\
&\geq \frac{s}{m} - \frac{1.5^2 m^2}{m^\gamma - \frac{s}{m}} \\
&\geq \frac{s}{m} - \frac{1.5^2 m^2}{m^\gamma - \frac{s}{m}} \\
&\geq \frac{s}{m} - \frac{1}{m^{b+2}}
\end{aligned}$$

which is a contradiction since we had assumed that $\langle\psi'|G + G_{pen}|\psi'\rangle \leq \frac{s}{m} - \frac{1}{m^{b+1}}$.

The expression in the third line is minimized for $|c_1| = \frac{1.5m}{m^\gamma - \frac{s}{m}}$ which gave the expression in the next line. □

Proposition 18. *The Hamiltonians G_i and G_j inside G in the theorem above trivially commute, since they act on orthogonal spaces in the second register. Further, each G_i is $O(\log m) = O(\log n)$ local. Thus, we have two layers, G and G_{pen}, the hamiltonians inside each of which commute, and G and G_{pen} faithfully preserve the ground space and the eigenvalue gap of H.*

Proposition 19. *The state $|\phi\rangle$ acting on $O(\log n)$ qubits can be efficiently prepared by a polynomial sized circuit, since any state on $O(\log n)$ qubits can be prepared in poly(n) time, and more specifically, we can apply a hadamard gate to each of the qubits.*

5.3 A Modified Traversal Lemma

Now we discuss one of the main tools used in [10], the traversal lemma. Consider any two states $|\psi\rangle$ and $|\phi\rangle$ with a special property - for any k-local unitary U, $\langle\psi, U\phi\rangle = 0$, i.e., no k-local unitary can map the state $|\psi\rangle$ to $|\phi\rangle$. Such states will be called k-orthogonal states. For example, the states $|000\rangle$ and $|111\rangle$ are 2-orthogonal.

Now first consider the following simple problem. Assume we have a local hamiltonian H acting on n qubits, and assume its ground state can be prepared efficiently by applying a unitary U (made only of universal 2-local quantum gates) to the all zeroes state, in case a ground state exists. Now consider two registers, one consisting of n qubits and another consisting of 3 qubits (often referred to as GO qubits). Assume we are given the state $|0...0\rangle \otimes |000\rangle$ and we want to reach the state $|0...0\rangle \otimes |111\rangle$, and always remain in the ground space of the hamiltonian $H' = H \otimes (I - |000\rangle\langle 000| - |111\rangle\langle 111|)$. Note that in case H has a ground state $|\psi\rangle$ that can be prepared efficiently, we can move from $|0...0\rangle \otimes |000\rangle$ to $|\psi\rangle \otimes |000\rangle$, to $|\psi\rangle \otimes |111\rangle$, to $|0...0\rangle \otimes |111\rangle$ using only a polynomial (in n) number of 2-qubit unitaries. Further, note that during the first and the last conversions, we will be in the ground space of H'. Only during the conversion of $|\psi\rangle \otimes |000\rangle$ to $|\psi\rangle \otimes |111\rangle$, we shall be above the ground energy by $\langle\psi, H\psi\rangle$. Thus, if the energy of $|\psi\rangle$ is low, we can always stay in the low energy space of H and go from $|0...0\rangle \otimes |000\rangle$ to $|0...0\rangle \otimes |111\rangle$.

But what if H has no ground state that can be prepared efficiently? That is, what if every efficiently preparable state $|\psi\rangle$ has energy at least β? Now if we move from $|0...0\rangle \otimes |000\rangle$ to $|0...0\rangle \otimes |111\rangle$, since we would have to go into the $+1$ eigenspace of $(I - |000\rangle\langle 000| - |111\rangle\langle 111|)$ at some point in going from $|000\rangle$ to $|111\rangle$ in the second register, and at that point, we would have energy atleast β for the state on the first register with respect to H. Thus, intuitively, we may think that that if every efficiently preparable state $|\psi\rangle$ has energy at least β for H, if we go from the 2-orthogonal states $|0...0\rangle \otimes |000\rangle$ to $|0...0\rangle \otimes |111\rangle$ in m steps by using only 2 local unitaries, there has to be at least one step in which we make an energy jump of $\sim \frac{\beta}{m}$. The traversal lemma in [10] makes this intuition precise. We will tune the traversal lemma of [10] to suit our needs.

Before we proceed we will need the Gentle Measurement Lemma of Winter, restated here.

Lemma 20. *(Gentle Measurement Lemma [22]) Let ρ be a density operator, and Π a measurement operator $0 \leq \Pi \leq I$, such that $Tr(\Pi\rho) \geq 1 - \epsilon$. Then $\|\rho - \sqrt{\Pi}\rho\sqrt{\Pi}\|_{tr} \leq 2\sqrt{\epsilon}$.*

We now formally state the definition of k-orthogonal subspaces as given in [10]. Intuitively, two vectors $|\psi\rangle$ and $|\phi\rangle$ are k-orthogonal if for any k-qudit unitary U, $\langle\psi|U|\phi\rangle = 0$. More formally,

Definition 21. (k-orthogonality) For $k \geq 1$, a pair of states $|\psi\rangle$,$|\phi\rangle \in (\mathbb{C}^d)^{\otimes n}$ is k-orthogonal if for all k-qudit unitaries U, we have $\langle\psi, U\phi\rangle = 0$. Further, any two subspaces $S, T \subseteq (\mathbb{C}^d)^{\otimes n}$ are k-orthogonal if every pair of states $|\psi\rangle$,$|\phi\rangle$ in S and T respectively are k-orthogonal.

We shall now work with 3 registers. The first register contains n qubits and the second and the third registers contain 3 qubits each. Let the following be shorthands -
$P_0 = |000\rangle\langle 000|$,
$P_1 = |111\rangle\langle 111|$,
$\Pi = I - P_0 - P_1$,
$P_{01} = |000\rangle\langle 111|$,

$P_{10} = |111\rangle\langle 000|$,
$P_+ = \frac{1}{2}(|000\rangle + |111\rangle)(\langle 000| + \langle 111|)$,
$P_- = \frac{1}{2}(|000\rangle - |111\rangle)(\langle 000| - \langle 111|)$

Lemma 22. *(Small Projection Lemma) Let S and T be 2-orthogonal subspaces spanned by $I \otimes I \otimes |000\rangle$ and $I \otimes I \otimes |111\rangle$ respectively. Let $v_0 = u \otimes |000\rangle$ where u spans the first two registers. Let U_1 to U_m be any sequence of 2-local unitaries, and let $v_i = U_i v_{i-1}$ for $1 \le i \le m$. Further, assume we have that for every i, $\|I \otimes I \otimes \Pi v_i\| \le \gamma$. Then, for every i, $\|(I \otimes I \otimes P_1)v_i\| \le i\gamma$ and $\|(I \otimes I \otimes P_0)v_i\| \ge 1 - (i+1)\gamma$*

Proof. First, note that since each U_i is a 2-local unitary, for any vector v, $(I \otimes I \otimes P_1)U_i(I \otimes I \otimes P_0)v = 0$ and $(I \otimes I \otimes P_0)U_i(I \otimes I \otimes P_1)v = 0$. We now inductively show that for every i, $\|(I \otimes I \otimes P_1)v_i\| \le i\gamma$. It is trivially true for $i = 0$. Let it be true for $i - 1$. Then for i,

$$\begin{aligned}
\|(I \otimes I \otimes P_1)v_i\| &= \|(I \otimes I \otimes P_1)U_i v_{i-1}\| \\
&= \|(I \otimes I \otimes P_1)U_i((I \otimes I \otimes P_0) + (I \otimes I \otimes P_1) + (I \otimes I \otimes \Pi))v_{i-1}\| \\
&\le \|(I \otimes I \otimes P_1)U_i(I \otimes I \otimes \Pi)v_{i-1}\| + \|(I \otimes I \otimes P_1)U_i(I \otimes I \otimes P_1)v_{i-1}\| \\
&\le \|(I \otimes I \otimes \Pi)v_{i-1}\| + \|(I \otimes I \otimes P_1)v_{i-1}\| \\
&\le \gamma + (i-1)\gamma \\
&= i\gamma
\end{aligned}$$

The second line used the fact that $I = (I \otimes I \otimes P_0) + (I \otimes I \otimes P_1) + (I \otimes I \otimes \Pi)$, the third line follows from triangle inequality and the fact that $(I \otimes I \otimes P_1)U_i(I \otimes I \otimes P_0)v = 0$ since U_i's are 2-local and the subspaces 3 orthogonal, the fourth line follows from the fact that $\|P\| \le 1$ for any orthogonal projection P, and the fifth line uses the given condition and the inductive hypothesis.

Further,

$$\begin{aligned}
\|(I \otimes I \otimes P_0)v_i\| &= \|v_i - (I \otimes I \otimes P_1)v_i - (I \otimes I \otimes \Pi)v_i\| \\
&\ge \|v_i\| - \|(I \otimes I \otimes P_1)v_i\| - \|(I \otimes I \otimes \Pi)v_i\| \\
&\ge 1 - i\gamma - \gamma \\
&= 1 - (i+1)\gamma \\
&\ge 1 - (m+1)\gamma
\end{aligned}$$

Again, the second line uses the triangle inequality, and the third line uses the above proven claim and the given condition. □

We are now ready to prove a modified version of the Traversal Lemma from [10] that we will require in the next section.

Lemma 23. *(Modified Traversal Lemma) Let $\Pi_S = I \otimes P_0 \otimes P_0$ and S be the +1 eigenspace of Π_S, $\Pi_T = I \otimes P_1 \otimes P_0$ and T be the +1 eigenspace of Π_T. Define $P = I \otimes \Pi \otimes P_0$, $Q = I \otimes (P_0 + P_1) \otimes P_0$. Let $v_0 \in S$ and $w \in T$, and $\|w - v_m\| \le \epsilon$. If $U_1, ..., U_m$ is a sequence of 2-local unitaries that map v_0 to v_m, with $v_i = U_i v_{i-1}$ and $\|I \otimes I \otimes \Pi v_i\| \le \gamma$ for $1 \le i \le m$, then there exists an i such that $\|Pv_i\|^2 \ge \frac{1}{2}\left(\frac{1-2\epsilon}{2m}\right)^2$, provided $\gamma \le \frac{1}{4(m+1)}\left(\frac{1-2\epsilon}{2m}\right)^2$.*

Proof. We prove by contradiction. Let $\delta = \frac{1}{2}\left(\frac{1-2\epsilon}{2m}\right)^2$. Assume that for all $1 \le i \le m$, $\|Pv_i\|^2 < \delta$. Define the following sequences as in [10]. For all $1 \le i \le m$, $v'_i = Qv_i$, and $v''_i = QU_i v''_{i-1}$ where

$v''_1 = v'_1$. Note that the vectors v'_i and v''_i may not necessarily be unit vectors. Also note that we have for every i, $\langle v_i, Pv_i\rangle < \delta$ (P is a projection).

Using the lemma proven above, we have that $\|(I \otimes I \otimes P_0)v_i\| \geq 1-(m+1)\gamma$, and thus

$$\|(I \otimes I \otimes P_0)v_i\|^2 = \langle v_i, (I \otimes I \otimes P_0)v_i\rangle \geq 1-2(m+1)\gamma = \mu$$

Thus, we get that

$$\mathrm{Tr}(Q|v_i\rangle\langle v_i|) = \langle v_i|Q|v_i\rangle = \langle v_i|I \otimes I \otimes P_0|v_i\rangle - \langle v_i|P|v_i\rangle \geq \mu - \delta = 1-(1+\delta-\mu)$$

We now use the gentle measurement lemma stated above for the projection Q and the density matrix $|v_i\rangle\langle v_i|$. Thus we get that for every i,$\||v_i\rangle\langle v_i| - |v'_i\rangle\langle v'_i|\|_{\mathrm{tr}} < 2\sqrt{1+\delta-\mu}$. We now use induction to show that $\||v_i\rangle\langle v_i| - |v''_i\rangle\langle v''_i|\|_{\mathrm{tr}} < 2i\sqrt{1+\delta-\mu}$.

For the base case, note that $\||v_1\rangle\langle v_1| - |v''_1\rangle\langle v''_1|\|_{\mathrm{tr}} = \||v_1\rangle\langle v_1| - |v'_1\rangle\langle v'_1|\|_{\mathrm{tr}} \leq 2\sqrt{1+\delta-\mu}$ as shown above. Now using induction on i, we get that

$$\begin{aligned}
\||v_i\rangle\langle v_i| - |v''_i\rangle\langle v''_i|\|_{\mathrm{tr}} &\leq \||v_i\rangle\langle v_i| - |v'_i\rangle\langle v'_i|\|_{\mathrm{tr}} + \||v'_i\rangle\langle v'_i| - |v''_i\rangle\langle v''_i|\|_{\mathrm{tr}} \\
&\leq 2\sqrt{1+\delta-\mu} + \|QU_i(|v_{i-1}\rangle\langle v_{i-1}| - |v''_{i-1}\rangle\langle v''_{i-1}|)U_i^*Q\|_{\mathrm{tr}} \\
&\leq 2\sqrt{1+\delta-\mu} + \||v_{i-1}\rangle\langle v_{i-1}| - |v''_{i-1}\rangle\langle v''_{i-1}|\|_{\mathrm{tr}} \\
&\leq 2\sqrt{1+\delta-\mu} + 2(i-1)\sqrt{1+\delta-\mu} \\
&\leq 2i\sqrt{1+\delta-\mu}
\end{aligned}$$

as required.

The first statement is by triangle inequality, the second by the gentle measurement lemma as shown above and the definitions, the third follows from the fact that $\|ABC\|_{\mathrm{tr}} \leq \|A\|_\infty \|B\|_{\mathrm{tr}} \|C\|_\infty$, and the fourth follows by the induction hypothesis.

Now note that

$$\begin{aligned}
\||v''_m\rangle\langle v''_m| - |w\rangle\langle w|\|_{\mathrm{tr}} &\leq \||v''_m\rangle\langle v''_m| - |v_m\rangle\langle v_m|\|_{\mathrm{tr}} + \||v_m\rangle\langle v_m| - |w\rangle\langle w|\|_{\mathrm{tr}} \\
&< 2m\sqrt{1+\delta-\mu} + 2\||v_m\rangle\langle v_m| - |w\rangle\langle w|\| \\
&< 2m\sqrt{1+\delta-\mu} + 2\epsilon \\
&< 1
\end{aligned}$$

where the first line was the triangle inequality, the second line used the fact that $\|A\|_{\mathrm{tr}} \leq \|A\|$, and the fourth line follows based on our condition on γ.

But note that $|v''_m\rangle$ always lies in S. This is because inductively, $|v_0\rangle$ lies in S, and if $|v''_{i-1}\rangle$ lies in S, then applying any two local unitary U_i will not move any component of $|v''_{i-1}\rangle$ to T, and projecting on the space $S \cup T$ using Q will just project the rotation onto S. Since $|w\rangle$ lies in T, we have that $\langle v''_m|w\rangle = 0$, and

$$\||v''_m\rangle\langle v''_m| - |w\rangle\langle w|\|_{\mathrm{tr}} = 1 + \|v''_m\| \geq 1$$

which is a contradiction. □

5.4 Traversing the ground space of 2 nonlocal Hamiltonians

We now give the main construction that will help us comment on the ground space of 2 nonlocal hamiltonians as constructed before, by traversing the ground space of another set of *commuting* Hamiltonians.

5.4.1 Construction of commuting local hamiltonians

Let A and B be two hamiltonians, not necessarily local, that act on n qubits, and $\|A\| \leq 1$ and $\|B\| \leq 1$. Further, we have the following promise - Either there exists v such that $\langle v, (A+B)v \rangle \leq \alpha$, or for all v, $\langle v, (A+B)v \rangle \geq \beta$. We shall now construct commuting Hamiltonians, traversing whose ground space would help us comment on the ground state energy of $A+B$.

Define the three new Hamiltonians as follows, acting on three registers, first of n qubits and the remaining two of 3 qubits each:

$$H_A = A \otimes \Pi \otimes P_+$$
$$H_B = B \otimes \Pi \otimes P_-$$
$$G = I \otimes I \otimes \Pi$$

The initial and final states will be $v_0 = |0\rangle \otimes |000\rangle \otimes |000\rangle$ and $w = |0\rangle \otimes |111\rangle \otimes |000\rangle$. The main idea in defining the hamiltonians as above is that because they are in a superposition on $|000\rangle$ and $|111\rangle$, the starting state on the third register does not make the energy on the states 0. Further, since you'll always stay close to the $|000\rangle$ state as shown in lemma A above, H_A and H_B will always have high energy due to the third register, unless it is cancelled by the second or first register. The reason we need the third register is to make the three hamiltonians above commute. Note that $H_A H_B = 0$, and since the hamiltonian terms on the third register are all orthogonal projections (hermitian), they commute simply as $H_A G = G H_A$ and $H_B G = G H_B$. Moreover, we cannot use less than 3 qubits in the third register, since we do not want any unitary to go from $|000\rangle$ to $|111\rangle$ in 1 step on the third register, in which case it can traverse the ground space of $H_A + H_B + G$ without traversing the ground space of $A+B$.

We now proceed to show that the construction above is both complete and sound.

5.4.2 Completeness

Assume there is some classical v such that $\langle v, (A+B)v \rangle \leq \alpha$, then the following sequence of unitaries help traverse the ground space of $H_A + H_B + G$. The starting state is $v_0 = |0\rangle \otimes |000\rangle \otimes |000\rangle$.

1. Prepare the classical string v in the first register using Pauli X gates
2. Flip the bits of the second register from 000 to 111, again using Pauli X gates.
3. Unprepare the string v back to the all zeroes string in the first register.

Note that during the first and the third stage, we will remain in the ground state of H_A and H_B by the virtue of the state in the second register, and in the ground state of G due to the state in the third register.

During the second step, let second register be in the state $|a\rangle = |001\rangle$ (or $|100\rangle$ or $|010\rangle$). Note that the first register is in the state v, and third is in the state $|000\rangle$ and thus we are still in the ground state of G. Then we have that

$$\begin{aligned}
\langle v|\langle a|\langle 000|(H_A+H_B+G)|v\rangle|a\rangle|000\rangle &= \langle v|\langle a|\langle 000|(H_A+H_B)|v\rangle|a\rangle|000\rangle \\
&\quad +\langle v|\langle a|\langle 000|G|v\rangle|a\rangle|000\rangle \\
&= \langle v|\langle a|\langle 000|(A\otimes\Pi\otimes P_+ + B\otimes\Pi\otimes P_-)|v\rangle|a\rangle|000\rangle \\
&\quad +\langle v|\langle a|\langle 000|I\otimes I\otimes\Pi|v\rangle|a\rangle|000\rangle \\
&= \langle v|A|v\rangle\langle a|\Pi|a\rangle\langle 000|P_+|000\rangle \\
&\quad +\langle v|B|v\rangle\langle a|\Pi|a\rangle\langle 000|P_-|000\rangle + 0 \\
&= \frac{1}{2}\langle v|A|v\rangle + \frac{1}{2}\langle v|B|v\rangle \\
&= \frac{1}{2}\langle v|A+B|v\rangle \\
&\leq \frac{1}{2}\alpha
\end{aligned}$$

Thus, at any stage, the energy of any state is at most $\frac{1}{2}\alpha$, and thus we always remain close to the ground space.

5.4.3 Soundness

To show the soundness, assume that for all classical v, $\langle v,(A+B)v\rangle \geq \beta$. Assume some sequence of 2 qubit unitaries maps v_0 to w and $\|v_m - w\| \leq \epsilon$ for some ϵ that will be set to a constant. Further, assume that $\langle v_i, Gv_i\rangle \leq \gamma^2$, because if not, then the energy will be atleast γ^2. Thus, note that for every i, we have that $\|v_i, I\otimes I\otimes \Pi v_i\| \leq \gamma$.

Hence, applying the Modified Traversal Lemma, we would have that there is some i for which $\langle v_i, I\otimes\Pi\otimes P_0 v_i\rangle \geq \delta$. Then we have that

$$\begin{aligned}
\langle v_i,(H_A+H_B+G)v_i\rangle &\geq \langle v_i,(H_A+H_B)v_i\rangle = \langle v_i,(A\otimes\Pi\otimes P_+ + B\otimes\Pi\otimes P_-)v_i\rangle \\
&= \langle v_i,(A+B)\otimes\Pi\otimes P_0 v_i\rangle + E \\
&\geq \beta\delta + E
\end{aligned}$$

Now we bound the error term. Rewrite $v_i = (I\otimes I\otimes P_0)v_i + (I\otimes I\otimes P_1)v_i + (I\otimes I\otimes \Pi)v_i$. Thus,

$$\begin{aligned}
E &= \langle v_i,(A-B)\otimes\Pi\otimes P_{01}v_i\rangle + \langle v_i,(A-B)\otimes\Pi\otimes P_{10}v_i\rangle + \langle v_i,(A+B)\otimes\Pi\otimes P_1 v_i\rangle \\
&= \langle (I\otimes I\otimes P_0)v_i,(A-B)\otimes\Pi\otimes P_{01}P_1 v_i\rangle + \langle (I\otimes I\otimes P_1)v_i,(A-B)\otimes\Pi\otimes P_{10}P_0 v_i\rangle \\
&\quad +\langle (I\otimes I\otimes P_1)v_i,(A+B)\otimes\Pi\otimes P_1 v_i\rangle
\end{aligned}$$

Thus, we get that,

$$
\begin{aligned}
|E| &\leq |\langle (I\otimes I\otimes P_0)v_i, (A-B)\otimes\Pi\otimes P_{01}P_1v_i\rangle| + |\langle (I\otimes I\otimes P_1)v_i, (A-B)\otimes\Pi\otimes P_{10}P_0v_i\rangle| \\
&\quad +|\langle (I\otimes I\otimes P_1)v_i, (A+B)\otimes\Pi\otimes P_1v_i\rangle| \\
&\leq \|(I\otimes I\otimes P_0)v_i\|\|A-B\|\|\Pi\|\|(I\otimes I\otimes P_{01})\|\|(I\otimes I\otimes P_1)v_i\| + \\
&\quad \|(I\otimes I\otimes P_1)v_i\|\|A-B\|\|\Pi\|\|(I\otimes I\otimes P_{10})\|\|(I\otimes I\otimes P_0)v_i\| + \\
&\quad \|(I\otimes I\otimes P_1)v_i\|\|A+B\|\|\Pi\|\|(I\otimes I\otimes P_1)v_i\| \\
&\leq 4\|(I\otimes I\otimes P_1)v_i\| \\
&\leq 4i\gamma \\
&\leq 4m\gamma
\end{aligned}
$$

where we got the first inequality using the Cauchy-Schwarz inequality, the second inequality used the fact that the norm of a projection is less than 1 and that $\|A\| \leq 1$ and $\|B\| \leq 1$, and the third inequality used the Small Projection Lemma.

Now we'll set the number of steps $m = 3n$, which would be sufficient for completeness. Thus $m = O(n)$, and we'll treat m and n interchangeably. Now let $\gamma = \frac{\beta\delta}{Cm}$, where C is a large constant, say 1000. Then we have

$$
\begin{aligned}
\langle v_i, (H_A + H_B + G)v_i\rangle &\geq \beta\delta + E \\
&\geq \beta\delta - 4m\gamma \\
&\geq \frac{1}{2}\beta\gamma
\end{aligned}
$$

Also, this also satisfies our condition for γ that allows us to use the Modified Traversal Lemma, i.e.

$$
\begin{aligned}
\gamma &= \frac{\beta\delta}{Cm} \\
&\leq \frac{1}{1000m}\frac{1}{2}\left(\frac{1-2\epsilon}{2m}\right)^2 \\
&< \frac{1}{4(m+1)}\left(\frac{1-2\epsilon}{2m}\right)^2
\end{aligned}
$$

as was required.

Now note that the energy lower bound for soundness would be

$$
\min\{\frac{1}{2}\beta\gamma, \gamma^2\} = O(\min\{\frac{\beta^2}{m^3}, \frac{\beta^2}{m^6}\}) = O(n^{-6}\beta^2)
$$

Thus, if the energy bounds for $(A+B)$ are $\leq \alpha$ and $\geq \beta$, then the bounds for the $(H_A + H_B + G)$ system would be $\leq \frac{\alpha}{2}$ and $\geq n^{-6}\beta^2$. Observe that the $O(n^{-6}\beta - \alpha)$ gap is inverse polynomial only if α is exponentially small, and $\beta - \alpha$ gap is inverse polynomial.

5.5 QCMA **hardness of traversing the Ground Space of $O(\log n)$ local Commuting Hamiltonians**

In this section, we finally show the QCMA hardness of traversing the ground space of $O(\log n)$ local commuting hamiltonians, using all the machinery that we have seen so far. The containment in QCMA follows directly from the result of [10] section 5 and we omit the details.

Theorem 24. *Let $H = \sum_{i=1}^{q} H_i$ be a local hamiltonian acting on n qubits, where each H_i is a hamiltonian acting on at most $C \log n$ qubits (for any fixed C), and for every i, j, the hamiltonians commute, i.e. $H_i H_j = H_j H_i$. Then, traversing the ground space of H is* QCMA*-hard.*

Proof. Let V be a QCMA verifier circuit acting on a total of n qubits, and let $H = \sum_{i=1}^{m} H_i$ be the 3-local hamiltonian obtained from it by the Kempe-Regev construction [14] having $\|H_i\| \leq 1$, where α and β are the upper and lower bounds on the ground energy of H respectively in the accepting and rejecting cases. There is a specific reason why we use this particular construction, instead of the 2-local [13] or the 5 local [15] construction, which we state next. First, we can do the standard error reduction on V, and assume that V decides some language L in QCMA, so that

$$x \in L \Rightarrow \exists y,\ \mathbb{E}V(x,y) \geq 1 - \epsilon$$
$$x \notin L \Rightarrow \forall y,\ \mathbb{E}V(x,y) \leq \epsilon$$

where $\epsilon \leq 2^{-\text{poly}(n)}$. Let the circuit V contain the unitaries U_r to U_1, where $r = \text{poly}(n)$. Now the Kempe-Regev construction [14] guarantees that

$$x \in L \Rightarrow \exists v,\ \langle v, Hv \rangle \leq \frac{\epsilon}{r}$$
$$x \notin L \Rightarrow \exists v,\ \langle v, Hv \rangle \geq \frac{1}{r^3}$$

which gives us $\alpha \leq 2^{-\text{poly}(n)}$ and $\beta \geq \frac{1}{\text{poly}(n)}$. This exponentially small value of α is crucial for us, since as noted at the end of the previous section, our new bounds as α and β change will scale inverse polynomially only if α is exponentially small. Let $m^b = \beta - \alpha$

Now let $G_i = H_i \otimes |i\rangle\langle i|$ for all $1 \leq i \leq m$, and $G' = \sum G_i$, and $G_{pen} = m^{b+5} I \otimes (I - |\phi\rangle\langle\phi|)$ where $|\phi\rangle = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} |i\rangle$. Then as shown in the Ground Space Preserving Layers theorem, $G' + G_{pen}$ is a $(\frac{\alpha}{m}, \frac{\beta}{m} - \frac{1}{m^{b+1}})$ hamiltonians, or in other words, it is still a $(2^{-\text{poly}(n)}, \frac{1}{\text{poly}(n)})$ hamiltonian.

Define $A = G'/\|G' + G_{pen}\|$ and $B = G_{pen}/\|G' + G_{pen}\|$. Note that since we have merely scaled both hamiltonians by the same number, and since $\|G' + G_{pen}\| \leq \text{poly}(n)$, we still have the eigenvalue gap of the system $A + B$ as $(2^{-\text{poly}(n)}, \frac{1}{\text{poly}(n)})$, and further, $\|A\| \leq 1$, and $\|B\| \leq 1$.

Now we construct the system $H_A + H_B + G$, which satisfies all our constraints. This is essentially creating the following system, if we write H_A, H_B and G in terms of the original hamiltonians -

$$\begin{aligned} 1 \leq i \leq m : \quad & c_1(H_i \otimes |i\rangle\langle i| \otimes \Pi \otimes P_+) \\ & c_2(I \otimes (I - |\phi\rangle\langle\phi|) \otimes \Pi \otimes P_-) \\ & 1.(I \otimes I \otimes I \otimes \Pi) \end{aligned}$$

where the constants c_1 and c_2 are both inverse polynomial in n. Note that the above system is $O(\log n)$ local (mainly due to the second register above), and each term in the above system commutes with any other. As we had shown the Completeness and the Soundness of the above system before, we would have that the system $H_A + H_B + G$ has energy gap $(\frac{1}{2}\alpha, n^{-6}\beta)$ which is again $(2^{-\text{poly}(n)}, \frac{1}{n^6 \text{poly}(n)}) = (2^{-\text{poly}(n)}, \frac{1}{\text{poly}(n)})$. Thus, we get an instance of GSCONCH with the hamiltonian being $H_A + H_B + G$, the hamiltonian terms commute and are $O(\log n)$ local, and there are atmost $O(n)$ 2 local unitaries which would be required to traverse the ground space. Further, the initial state is $|0_n\rangle \otimes |\phi\rangle \otimes |000\rangle \otimes |000\rangle$ and the final state is $|0_n\rangle \otimes |\phi\rangle \otimes |111\rangle \otimes$

$|000\rangle$ which can be easily prepared by polynomial sized circuits since $|\phi\rangle$ can be prepared by a polynomial sized circuit (Proposition 19). The completeness parameter is $2^{-\text{poly}(n)}$ and soundness parameter is $\frac{1}{\text{poly}(n)}$. □

Finally, we can conclude,

Theorem 25. *The problem of traversing the ground space of commuting $O(\log n)$ local Hamiltonians which have an inverse polynomial gap between the ground and first excited states is* QCMA *complete.*

6 Conclusions and Open Problems

We have shown that the $O(\log n)$-commuting local hamiltonian problem seems to exhibit behaviours that makes it difficult to comment on its hardness. From section 3, we see that a very symmetric case of $O(\log n)$-CLHP is indeed in NP, and further from section 4, unlike the non-commuting case, a generalized area law holds for all commuting hamiltonians, which seems to suggest a certain limit on entanglement of ground states of commuting hamiltonians, possibly suggesting that the k-CLHP is in QCMA. However, since traversing the hardness of $O(\log n)$ local hamiltonians is as hard as k-local hamiltonians, it might be the case that $O(\log n)$ local commuting hamiltonians (not necessarily geometrically local) are indeed QMA-hard. However, it is difficult to make any specific comments, and we leave the following as open problems:

Open Problem 1: Is traversing the ground space of k-local commuting hamiltonians QCMA hard for a constant k?

Open Problem 2: Is $O(\log n)$-CLHP QMA complete?

Open Problem 3: Using the generalized area law, is it possible to given an algorithm in QCMA for k-CLHP over a D-dimensional grid?

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