Constructing Self-Dual Automorphic Representations on General Linear Groups

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To my family, within and without

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Abstract

We prove a globalization theorem for self-dual representations of GL_N over a totally real number field F, which gives a positive existence criterion for self-dual cuspidal automorphic representations of $GL_N(\mathbf{A}_F)$ with prescribed local components at a finite set of finite places. A byproduct of our argument is that the automorphic representations that we construct are cohomological (equivalently, regular algebraic) and so fall into the class of automorphic representations on GL_N for which there is a well-established theory for how to attach Galois representations, using the étale cohomology of certain Shimura varieties. The primary motivation is to give a sort of "bare-handed" or "low tech" proof of a result that is implied by the philosophy of twisted endoscopy in the Langlands program. While we are guided by this overarching picture, in the argument itself, we obtain all our results by working directly on GL_N and the group obtained by twisting it under the "inverse-transpose" involution. In particular, we do not appeal to any general results on twisted endoscopic transfer or assume any big "black box" results like the (conjectured) stabilization of the twisted trace formula. Hence, such results are unconditional as stated, and we remark throughout on why the particular assumptions that we impose turn out to be necessary, indicating the (often substantial amount of) additional work required to generalize the stated results.

In an appendix, in stark contrast to our approach above, we give an abridged argument for proving a globalization theorem on GL_N in great generality, assuming a couple of major technical hypotheses (albeit, ones that are widely believed to be true) and yielding to Arthur's endoscopic classification of representations of symplectic and special orthogonal groups. Our hope is for such an argument to provide an outline for how we might ultimately prove results like generalizations of the globalization criterion above in the future.

Contents

Chapter 1

My Thesis

Bare-handed proofs of concrete arithmetic results are feasible and useful for applications of the Langlands program.

In some arithmetic applications of the Langlands program using the trace formula, it is enough to directly analyze the terms that arise, without having to assume any results "on faith" that rely on thousands of pages of technical proof.

1.1 Goal of this Work

This work is a systematic study of the self-dual automorphic representations on GL_N over a totally real number field F through the lens of twisted endoscopy in the Langlands program.

We summarize and interpret the general results known about the local components of such representations—the self-dual smooth admissible representations of $GL_N(F_v)$ for completions of F at a place v—and use this to determine which components can simultaneously arise as the local components of a single irreducible cuspidal automorphic representation.

1.2 What is Twisted Endoscopy?

One pillar of the Langlands program is Langlands functoriality, which predicts that if H and G are two reductive groups over number fields, any homomorphism between their L-groups (an L-

homomorphism for short)

$$\Phi : {}^L H \to {}^L G$$

leads to a corresponding transfer of automorphic representations on H to those on G. Many of the known correspondences between automorphic representations can be summarized according to this philosophy, including solvable base change, automorphic induction, symmetric square lifting, the Jacquet–Langlands correspondence, etc. While this provides an elegant unifying framework for the plethora of relations between automorphic representations on different groups, this philosophy has yet to materialize into a general theorem or proof technique. Instead, given the current state of knowledge, we can only use this to guide our investigations and establish conjectural statements that we prove using more concrete and familiar methods.

Endoscopy refers to some of the more accessible cases of Langlands functoriality; roughly speaking, it applies when the L-homomorphism $\Phi : {}^L H \to {}^L G$ is an inclusion. Endoscopic transfer results encompass all of the correspondences mentioned in the previous paragraph, but there are a number of correspondences that do not fall under this paradigm, such as symmetric nth power liftings for large enough n, but knowledge of non-endoscopic correspondences are scarce and hold challenges that currently seem out of reach in all but the most specialized of cases. Morally speaking, when such a Φ exists, we say that H is an endoscopic group of G. In practice, there is a precise mathematical definition that applies. Note that an endoscoipc group H is not generally a subgroup of G; the two groups are only weakly linked via a relation between their respective Langlands duals.

Twisted endoscopy essentially concerns cases where at least one of the reductive groups in the case of Langlands functoriality is a "connected reductive group that is twisted by a finite-order automorphism;" this yields a disconnected reductive group that allows for more interesting phenomena than if the L-homomorphisms were restricted to maps between connected groups. A particularly interesting simple case of twisted endoscopy applies to the functorial transfer of automorphic representations from symplectic or special orthogonal groups to general linear groups. While general linear groups do not have nontrivial endoscopic groups, their twists by certain involutions realize these classical groups as endoscopic groups, and this has led to a number of results being established in this vein; one of the most striking realizations of this idea is Arthur's recent work on the endoscopic classification of representations of classical groups [Art13]. The image of Arthur's transfer map is the set of self-dual automorphic representations on a general linear group, and it is this perspective that forms the starting point of our investigations.

1.3 Why Self-Dual Automorphic Representations?

Self-dual (a.k.a. self-contragredient) representations are objects of interest in the theory of group representations (especially Lie groups) that play a distinguished role in the theory of automorphic representations and their local components: namely, the smooth admissible representation theory of real, complex, and p-adic reductive groups. For example, the Langlands philosophy predicts that every self-dual automorphic representation on a general linear group is the transfer of an automorphic representation on a special orthogonal or symplectic group.

Another reason for studying self-dual representations is that the Galois representations that naturally arise in arithmetic questions are often self-dual. For example, every 2-dimensional Galois representation—such as those attached to a modular form or an elliptic curve over a number field—is self-dual up to a twist by a character. More generally, Galois representations that occur in the middle-dimensional étale cohomology of Shimura varieties are all self-dual in a certain strong sense with respect to the intersection pairing, so in particular, automorphic representations associated with such geometric Galois representations must necessary be self-dual. Indeed, a folklore conjecture predicts the converse: the only Galois representations realizable in the cohomology of Shimura varieties are those attached to self-dual automorphic representations; that is, geometric Galois representations attached to non-self-dual automorphic representations arise naturally from some currently unknown source. These phenomena are encapsulated in the fact that the local and global Langlands correspondences commute with taking duals, so self-dual representations on one side of the automorphic side of the correspondence correspond to self-dual representations on the Galois side of the correspondence, and vice versa. Indeed, most known cases of the global Langlands correspondence for $GL(n)$ (e.g. the special case for $GL(2)$ used by Wiles and Taylor to prove

Fermat's Last Theorem) are for self-dual (up to twist) Galois and automorphic representations.

The global Langlands correspondence for GL(n)[1]

$$\begin{array}{ccc} \{\text{Automorphic Representations of } GL(n)\} & \longleftrightarrow & \{n\text{-dimensional Galois Representations}\} \\ \uparrow & & \uparrow \\ \{\text{Self-Dual Aut. Rep. of } GL(n)\} & \longleftrightarrow & \{\text{Self-Dual } n\text{-dimensional Gal. Rep.}\} \end{array}$$

Despite their importance, however, constructing self-dual automorphic representations even on $GL(n)$ usually requires appealing to some major results, such as the existence of Langlands functorial transfers from classical groups to $GL(n)$, which often requires assuming some strong technical hypotheses, some of which are currently unproven. And even after admitting such hypotheses, the proofs of such results often rely on lengthy specialized technical arguments, which are rarely read or understood by practitioners, who largely take such results as a "black box" and thus rarely adapt such arguments or techniques to other problems. While results for general reductive groups over general number fields must necessarily involve such hypotheses, it is desirable to obtain a "bare-handed" result in the case of $GL(n)$ over $\mathbf{Q}$ (or a totally real number field), which is often the case of interest for many concrete arithmetic applications. Aside from the obvious benefit of having an alternate proof of a useful result, following this methodology also highlights some interesting analytic, arithmetic, or representation-theoretic phenomena that occur and indicate arguments that at least have some hope of being able to be applied outside of the specific context of the proof.

Thus, throughout this work, we try to keep the techniques as "low tech" as possible, in particular, taking care to not assume any unproven hypotheses or results whose proofs require thousands of pages of specialized technical arguments. In doing so, we hope to not only give an alternate proofs, but to exhibit the specific points at which the additional assumptions become necessary.

[1] Of course, a precise statement requires many more technical details; for instance only "algebraic" automorphic representations are expected to have an associated Galois representation, one should probably work with Weil groups, Weil–Deligne groups, or variants thereof on the Galois side, etc.

1.4 The Main Theorem

Our study of self-dual representations involved in the theory of automorphic representations on $GL(n)$ culminate in the following result: a criterion for "globalizing" a finite set of self-dual local representations on $G = GL(n)$ into an automorphic representation that "interpolates" these chosen local representations and is *itself* self-dual (in the global sense).

Theorem 1.1. *Let T be a finite set of pairs (v, π_v) where*

- *v is a finite place of a totally real number field F, and*
- *π_v is an irreducible admissible self-dual essentially discrete representation of $G(F_v)$ (and if n is even, are all of symplectic type).*

Then there exists a cohomological self-dual cuspidal automorphic representation $\Pi = \otimes'_v \Pi_v$ of $G(\boldsymbol{A}_F)$ such that for all $(v, \pi_v) \in T$, we have $\Pi_v \cong \pi_v \otimes \chi_v$, where χ_v is an unramified character of $G(F_v)$.

If we only cared about constructing *some* automorphic representation that globalizes some set of discrete local representations, such a result has been within reach for some time, using the standard theory of pseudocoefficients. What makes the problem difficult in our setting is to ensure that you produce an automorphic representation that is *self-dual*. To ensure that the automorphic representations we construct have this property, we work with the disconnected reductive group $G^+ \cong GL(n) \rtimes \mathbf{Z}/2$.

Even establishing a property as basic as self-duality for automorphic representations involves a number of subtleties. For example, the condition that all the chosen self-dual representations at finite places have *matching parity* (i.e. that their Langlands parameters all preserve Galois-invariant symmetric bilinear forms, or all preserve Galois-invariant alternating bilinear forms) turns out to be *necessary*. This phenomenon was observed, for instance, in the case where all the local self-dual representations are supercuspidal, by Prasad and Ramakrishnan [PR12]. This agrees with the aforementioned expectation that all self-dual automorphic representations on $GL(n)$ "come from" those of orthogonal or symplectic groups, and was one of the principles underlying Arthur's endoscopic classification of representations of these classical groups [Art13].

There are a number of ways to realize such a self-dual globalization result. In spirit, all the methods boil down to different ways of realizing the philosophy of self-dual automorphic representations coming from classical groups. For example, in the aforementioned work of Prasad and Ramakrishnan [PR12], they approach the problem in a different way, applying a number of correspondences between automorphic representations and their local components, such as the theta correspondence. Perhaps the shortest way to prove a globalization result—and undoubtedly the way to approach such a question once certain foundational results are established in the near future—is to construct the "preimage" of the desired representation on the desired endoscopic group and use results on twisted endoscopic transfer. Indeed, we have outlined such a method in the appendix (§8). However, the method that we follow in the main part of the work will use the trace formula on GL_N and its twisted counterpart on twisted GL_N, and will boil down to an explicit analysis of the orbital integrals that appear. Doing so allows us to work "entirely on GL_N" and allows us to avoid results on endoscopic transfer, which are still conditional in our setting. But as we will see, even in this deliberately simplified setting, the endoscopic *groups* still make their presence known, due to the way that they control the harmonic analytic properties of the twisted GL_N.

A result along these lines for $G = GL(2n)$ was obtained by Chenevier and Clozel, who were motivated by an application to a specific question in Galois theory [CC09]. We strengthen their result in the $GL(2n)$ case, but the main work is in proving the analogous result in the case of $GL(2n+1)$. While the overall approach remains the same—we use techniques inspired by the theory of twisted endoscopy to apply the Arthur–Selberg trace formula to a carefully chosen family of pseudocoefficients and use harmonic analysis techniques to show that orbital integrals on the geometric side of the resulting trace formula is nonzero—the details in extending the results to the $GL(2n+1)$ case can be intricate and involve modifications at almost every step of the argument.

One of the primary goals for our treatment of the problem is to extend this kind of strategy for constructing automorphic representations to its "naturally general" framework, and to precisely indicate what technical lemmas need to be improved in order to obtain the desired generalizations using this approach.

1.5 Strategy of Proof

The (global) self-duality of an automorphic representation is a delicate condition: self-dual automorphic representations must have self-dual local components, but self-duality of local components does not guarantee the self-duality of the automorphic representation. This issue and related analytic difficulties are the primary reasons why constructing such automorphic representations is a subtle and tricky procedure, and why relatively explicit methods like Poincaré series cannot be applied in this setting. Instead, to construct such self-dual automorphic representations, we compare the Arthur–Selberg trace formula on $G = GL(n)$ with a twisted version on the disconnected reductive group $G^+ = G \amalg G\theta$, where θ is the order-2 automorphism of G given by

$$\theta(g) = {}^t g^{-1},$$

that is, by taking the inverse transpose[2]. This gives us a distribution

$$J_{spec}(\cdot) = J_{geom}(\cdot)$$

on the space of smooth, compactly supported $\mathbf{C}$-valued functions $C_c^\infty(G(\mathbf{A}))$, where the former denotes the "spectral side," which will consist of (traces of) certain self-dual automorphic representations, and the latter denotes the "geometric side," consisting of certain (twisted) orbital integrals. We choose an appropriate test function $f = \otimes_v f_v \in C_c^\infty(G(\mathbf{A}_F))$ based on our initial data $\{(v, \pi_v)\}_{v \in T}$, choosing the corresponding pseudocoefficients of the chosen representations, trying when possible to find ones with the most well-behaved nonvanishing properties. Note that since most pseudocoefficients are defined in the context of *connected* reductive groups, which excludes the case of G^+, it is necessary to develop twisted analogues of such functions in the context of our problem.

To show that our desired self-dual cuspidal automorphic representation exists, we need to show

[2] For technical reasons, we actually use a variant of this map, composing the θ above with the conjugate of a certain antidiagonal matrix.

that the spectral side of the trace formula is nonzero, and to do so, it is enough to show that the geometric side of the trace formula is nonzero:

$$J_{spec}(f) = J_{geom}(f) \neq 0.$$

Away from the chosen places $v \in T$, the local components f_v of the test function f are just the characteristic functions of $G(\mathcal{O}_{F_v})$, where $\mathcal{O}_{F_v}$ denotes the ring of integers of F_v. At places $v \in T$, the test functions are pseudocoefficients that are chosen to have simple θ-twisted orbital integrals, and much of our work is in finding such functions and establishing such properties. At ∞, we essentially take coefficients of discrete series representations. Now, the group $GL_N(\mathbf{R})$ does not have discrete series for $N > 2$, but it does have θ-discrete series (a twisted analogue of the discrete series), and it is these representations that we prescribe at the archimedean place. Eventually, for the test functions $f = \otimes_v f_v$ that we construct, we will be able to apply a simplified version of the trace formula [Art88b], and reduce the analysis of the geometric side to twisted orbital integrals that correspond to conjugacy classes of a highly restricted sets of elements: the elliptic θ-semisimple elements.

However, producing nonvanishing results for such a test function f even with the simple version of the trace formula used here is still too difficult to tackle in general, but here we can exploit a key observation that was successfully developed and applied by Chenevier and Clozel: the asymptotic simplification of the geometric side of the trace formula "as the weight goes to infinity." Indications of such an idea can be found in the case of (untwisted) $GL(2)$ in the work of Serre on equidistribution results for Hecke eigenvalues [Ser97], and this Chenevier–Clozel observation itself can be seen as indicative of general equidistribution phenomena for automorphic representations, namely those of "Plancherel" type (i.e. equidistribution with respect to the Plancherel measure) that occur as we vary "in the weight aspect" (since we fix the behavior at the local places but vary the weight).

We briefly describe this observation and the subsequent technique. Given the rigidity of our problem, the only freedom that we have in choosing our test function f is to vary the component at infinity f_∞ among the pseudocoefficients of cohomological θ-discrete series representations. These representations are naturally parametrized by the highest weight λ of an irreducible representation V_λ of a compact group $H(\mathbf{R})$ (a compact form of the real points of an endoscopic group of the θ-twisted G). The insight is noticing that as the weight λ goes to infinity away from the walls of the Weyl chamber, the geometric side of the trace formula becomes asymptotically equivalent to (in other words, all the remaining terms are dominated by) a single orbital integral, called the "principal term." Up to a positive scalar, this is the twisted orbital integral $TO_{\gamma_0}(f)$ of f attached to a certain elliptic θ-semisimple element $\gamma_0 \in G(F)$ (the "principal element") whose twisted centralizer is an F-group whose $\mathbf{C}$-points yield the dual group of the endoscopic group H. If we can show that the principal term does not vanish, then the geometric side of the trace formula does not vanish.

In symbols, as $\lambda \to \infty$ away from the walls,

$$J_{spec}(f) = J_{geom}(f) \sim TO_{\gamma_0}(f) = C \cdot \dim(V_\lambda),$$

where C is an explicit nonzero constant that only depends on the components of f away from ∞ and $f_\infty = f_{\infty,\lambda}$. Since the nonvanishing of $TO_{\gamma_0}(f)$ is reduced to the simultaneous nonvanishing of its local components $TO_{\gamma_0}(f_v)$, we then only need to prove nonvanishing results for these local orbital integrals applied to our twisted pseudocoefficients at a single element γ_0. Once these analytic results are established, we conclude that the geometric side of the trace formula is nonzero, completing the proof.

1.6 Summary of the Contents

In §2, we introduce self-dual representations and present some general definitions and results.

In §3, we define the non-connected reductive group G^+. This is the fundamental object that we "work on" in order to prove our main theorem, and we recall the key structural results, culminating in a description of the twisted endoscopy "norm map," which is one major ingredient of our proof.

Even guided by overarching theory of twisted endoscopy, translating this into concrete mathematical results is an involved process. In §4, we describe some of the simplifications and reductions that we exploit in the course of the proof of the theorem.

In §5 and §6, we establish the analytic results upon which the theorem ultimately rests. This is the technical heart of the work.

In §7, we put all of the results together to give a proof of our main theorem. On an initial reading, we advise the reader to begin with this section and refer back to the previous sections as needed.

Finally in the Appendix (§8), we give a sort of "dream proof" of a general globalization result like that of our main theorem, assuming certain technical hypotheses.

Chapter 2

Self-Dual Representations

Self-dual representations are easy to define and naturally arise in the representation theory of various groups, but it is less common to study them exclusively as a central characteristic rather than as an auxiliary property of a specific representation. Many representation-theoretic results are exclusive to self-dual representations, so in this chapter we recall some of these results, oriented towards those that will be useful in the course of our proof.

While the definitions are crucial and stated carefully, we do not include complete proofs of results that are not used our main argument.

2.1 General Self-Dual Representations

We collect some basic results on general self-dual group representations. In this section, G denotes a general group.

Definition 2.1. Let $\pi : G \to GL(V)$ be a smooth representation over the complex numbers. The **dual** (or **contragredient**) representation is defined to be $(\pi^\vee, V^\vee)$, where $V^\vee$ is the complex vector space of all linear functionals $\ell : V \to \mathbf{C}$ such that

$$\ell(\pi(k).v) = \ell(v)$$

for all k in some open compact subgroup $K \subset G$ and $v \in V$; and where the G-action of $\pi^\vee$ is given by

$$\pi^\vee(g).\ell(v) := \ell(\pi(g^{-1}).v)$$

for all $\ell \in V^\vee$, $v \in V$, and $g \in G$.

We have a canonical bilinear form $V \times V^\vee \to \mathbf{C}$ defined by

$$\langle v, \ell \rangle := \ell(v)$$

for $\ell \in V^\vee$ and $v \in V$. It is G-invariant in the sense that

$$\langle \pi(g).v, \pi^\vee(g), \ell \rangle = \langle v, \ell \rangle \,,$$

and if (π, V) is irreducible, then any other representation (π', V') inducing a non-zero invariant bilinear form $V \times V' \to \mathbf{C}$ is isomorphic to $(\pi^\vee, V^\vee)$.

Proposition 2.2. *Let (π, V) and (π', V') be two admissible representations of $GL_n(F)$ for a local nonarchimedean field F. Suppose that there exists a nondegenerate bilinear form*

$$\langle , \rangle : V_1 \times V_2 \to \boldsymbol{C}$$

that is G-invariant in the sense that

$$\langle \pi(g).v, \pi'(g).v' \rangle = \langle v, v' \rangle$$

for $v \in V$, $v' \in V'$, and $g \in G$. Here, nondegenerate means that for any fixed $v' \in V$, we have $\langle v, v' \rangle \neq 0$ for some $v \in V$ and vice versa. Then $(\pi', V) \cong (\pi^\vee, V^\vee)$.

Proof. Define a map

$$L : V' \to V^\vee$$

$$v' \mapsto \ell_{v'} = \langle -, v' \rangle \,.$$

This is an intertwining map. If a nonzero $v' \in \mathrm{Ker}(L)$, then $\langle v, v' \rangle = 0$ for all $v \in V$ and $\langle , \rangle$ is degenerate. Similarly, if $\mathrm{Im}(L)$ is a proper subspace of $V^\vee$, then by admissibility of V and V' there exists $v \in V$ such that $\langle v, v' \rangle = 0$ for all $v' \in V'$, and so $\langle , \rangle$ is degenerate. Thus, the intertwining map L must be an isomorphism. □

Let (π, V) be an irreducible admissible self-dual (complex) representation of a p-adic group G, then there exists a nondegenerate G-invariant bilinear form $\langle , \rangle : V \times V \to \mathbf{C}$. This form is unique up to scalars by Schur's lemma. Such a form is either symmetric or skew-symmetric, and it is useful to distinguish between these two cases.

Definition 2.3. An irreducible admissible self-dual representation (π, V) is **orthogonal** if $\langle , \rangle$ is symmetric and **symplectic** if $\langle , \rangle$ is skew-symmetric.

For the case we will eventually consider, we can deduce this property by looking at poles of the appropriate L-functions, which relies on the following fact.

Lemma 2.4. *If π is an irreducible representation, then the self-dual representation $\pi \otimes \pi^\vee$ is reducible and contains exactly one copy of the trivial representation* 1 *as a factor.*

Proof. For any two irreducible representations π, π' of G, we have

$$\begin{aligned}
\mathrm{Hom}(1, \pi^\vee \otimes \pi') &\cong \mathrm{Hom}(1, \mathrm{Hom}(\pi, \pi')) \\
&\cong \mathrm{Hom}(1 \otimes \pi, \pi') \\
&\cong \mathrm{Hom}(\pi, \pi') \\
&\cong \begin{cases} \mathbf{C}, & \pi \cong \pi' \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

by the tensor-hom adjunction and Schur's lemma. □

For a representation (π, V), we have $\pi \otimes \pi^\vee \cong \mathrm{Sym}(V) \oplus \Lambda(V)$ and so the trivial representation must either lie in $\mathrm{Sym}(V)$ in which case $\pi \otimes \pi^\vee$ is orthogonal, or in $\Lambda(V)$ in which case $\pi \otimes \pi^\vee$ is symplectic.

2.2 Self-Dual Representations of Orthogonal and Symplectic Type

Under the Langlands classification of (complex) representations of a (connected) reductive group G over a local field F, every smooth admissible representation π of $G(F)$ corresponds to a Langlands parameter (a.k.a. L-parameter)

$$\sigma_\pi : \mathcal{L}_F \to^L G,$$

where ${}^L G$ is the Langlands dual group of G over F and $\mathcal{L}_F$ is, say, a Weil group, a Galois group, a Weil–Deligne group, or some variant thereof (there is usually a bijective correspondence between the isomorphism classes of representations of each such group). For a general group, multiple representations of G can correspond to the same Langlands parameter (such representations are said to be in the same "L-packet," corresponding to σ_π), but for GL_N, it is known that there is a one-to-one correspondence between irreducible, smooth, admissible representations of $GL_N(F)$

and Langlands parameters (that is, the L-packets for GL_N are singletons). Such a correspondence between irreeducible smooth, admissible representations and Langlands parameters is called a (local) Langlands correspondence.

The Langlands correspondence respects certain natural operations on representations. For one, it respects the taking of duals, in that the L-parameter of the dual representation $\pi^\vee$ is the dual of the L-parameter of π. In particular, it maps self-dual irreducible smooth admissible representations of $G(F)$ to self-dual representations of $\mathcal{L}_F$ on ${}^LG$.

Self-dual representations of reductive groups over local fields come in two (non-mutually exclusive) flavors, according to properties of their Langlands parameters.

Definition 2.5. A smooth, admissible representation of π of $G(F)$ is said to be **of orthogonal type** if its Langlands parameter is orthogonal as in Definition 2.3. Similarly, π is said to be **of symplectic type** if its Langlands parameter is symplectic as in Definition 2.3.

Note that this notion is different from the representation π itself being orthogonal or symplectic. It is possible that these two notions of being "orthogonal" or being "symplectic" coincide, but there are cases in which they differ. A particularly striking case occurs for inner forms of GL_N: if π is an irreducible self-dual representation of $D^\times$ for a division algebra D of invariant $1/n$ for n even, then π is orthogonal if and only if its Langlands parameter (associated to π under the Jacquet–Langlands and local Langlands correspondences) is symplectic; in other words, π is orthogonal if and only if it is of symplectic type [PR12, Cor. B].

Warning 2.6. It is important to be aware that a self-dual smooth admissible local representation can be both of orthogonal type and of symplectic type (or neither!). For example, certain Eisenstein series fall into this category, as well as some reducible representations that we can construct relatively explicitly (e.g. the direct sum of two irreducible representations of orthogonal type). One concrete example is if ω is a self-dual supercuspidal representation such that no unramified twist yields its dual representation $\omega^\vee$, then the representation of $G = GL_{2n}(F)$ obtained by the induction $\mathrm{Ind}_P^G(\omega \otimes \omega^\vee)$ (where P is the usual (n, n) parabolic in G) is a self-dual local representation that is of both orthogonal and symplectic type. This representation plays an important role in the solution to the Galois theory problem that was the initial motivation for the work of Chenevier–Clozel [CC09, §5].

However, the situation is not completely hopeless. By Lemma 2.4, we see that every *irreducible* self-dual smooth, admissible representation of $G(F)$ must be either of orthogonal type of or symplectic type.

2.3 Self-Dual Automorphic Representations

We recall some basic properties of self-dual automorphic representations of GL_N.

The following result can be summarized as "global self-duality implies local self-duality."

Proposition 2.7. *Let $\pi = \otimes_v \pi_v$ be a self-dual automorphic representation of $GL_N(\boldsymbol{A})$. For all places v, the local component π_v is self-dual.*

Proof. We can prove this, say, by looking at the local and global L-functions corresponding to such a representation and its properties, in particular, its functional equation. □

But to show that an automorphic representation with local components that are all self-dual is itself self-dual (as an automorphic representation) is subtle, and it is this for this reason that we work with the twisted group G^+ in the first place, to ensure the global self-duality.

In general, "most" self-dual automorphic representations of GL_N are not self-dual. There are many ways to see this and in which this property manifests itself, but here is one particular realization that has the quality of being relatively quantitative.

Proposition 2.8. *Let N be a natural number and assume that $N \neq 2$. Given a real number $\lambda > 0$, let $\mathcal{N}_{cusp}(\lambda)$ denote the number of cuspidal automorphic representations of $GL_N(\boldsymbol{A})$ whose Laplacian eigenvalues are at most λ. As $\lambda \to \infty$, we have*

$$\mathcal{N}_{cusp}(\lambda) \sim c\lambda^{(N^2+N-2)/2}.$$

Proof. Weyl's law. □

However, there is a single setting where there are proportionally more self-dual cusp forms than others.

Proposition 2.9. *Self-dual cusp forms have positive density among the cusp forms of GL_2.*

Proof. Due to the accidental isomorphism $SO_3 \cong PGL_2$, transfers from SO_3 to GL_2 give us a positive proportion of self-dual representations. □

This leads to some phenomena that occur for self-dual automorphic representations in the $GL(2)$ case that make it dramatically different from the other cases. We refer to §**??** for a more in-depth discussion.

Chapter 3

The Twisted Group $\mathbf{G}^+ = GL(n) \rtimes \mathbf{Z}/2$.

We recall some facts about the non-connected reductive group $G^+ = GL(n) \rtimes \langle\theta\rangle \cong GL(n) \rtimes \mathbf{Z}/2$ where θ is an involution that acts on $G = GL(n)$ via $g \mapsto {}^t(g^{-1})$.

3.1 Definition of $\mathbf{G}^+$

Let $\mathbf{G} = GL_N$ for a positive integer N and $G = \mathbf{G}(F)$ the set of F-points for a field F. We have an automorphism

$$\theta : \mathbf{G} \to \mathbf{G}$$
$$g \mapsto {}^t g^{-1},$$

sending an element of g to its inverse transpose, noting that the inverse and transpose operations commute with each other, so it does not matter in which order they are taken. The map θ^2 is the identity homomorphism, so θ is of order 2; that is, it is an involution.

We consider the group $\mathbf{G}^+ = \mathbf{G} \rtimes \langle\theta\rangle$, which we call **$\mathbf{G}$ twisted by θ** or the **θ-twisted $\mathbf{G}$**, characterized by the relations

$$\theta^2 = 1, \quad \theta g \theta^{-1} = \theta(g)$$

for all $g \in \mathbf{G}$. It is a non-connected reductive group whose identity connected component is $\mathbf{G}$ and its component group is $\mathbf{G}^+/\mathbf{G} \simeq \langle\theta\rangle \simeq \mathbf{Z}/2\mathbf{Z}$.

We denote the non-neutral connected component of $\mathbf{G}^+$ by $\widetilde{\mathbf{G}} = \mathbf{G}\theta$. Note that $\widetilde{\mathbf{G}}$ is an algebraic variety that is isomorphic to $\mathbf{G}$ under the map

$$\delta_\theta : g \mapsto g\theta$$

and admits a transitive $\mathbf{G}$-action on both the left and the right. We have a decomposition

$$\mathbf{G}^+ = \mathbf{G} \amalg \widetilde{\mathbf{G}}.$$

On F-points, we write $G^+ := \mathbf{G}^+(F) = G \amalg \widetilde{G}$.

3.2 Realizing $\mathbf{G}^+$ Inside a General Linear Group

The twisted group $\mathbf{G}^+$ is linear algebraic and so we should be able to realize it in a linear group GL_k for a certain $k \in \mathbf{N}$. In this section, we describe such a realization.

Consider the embedding $i : \mathbf{G}^+ \to GL_{2N}$ given by

$$i(g) = \begin{bmatrix} g & 0 \\ 0 & \theta(g) \end{bmatrix} \quad \text{for } g \in \mathbf{G}, \text{ and} \quad i(\theta) = \begin{bmatrix} 0 & I_N \\ I_N & 0 \end{bmatrix}.$$

We can then describe the non-neutral component as

$$i(\widetilde{\mathbf{G}}) = \left\{ \begin{bmatrix} 0 & g \\ \theta(g) & 0 \end{bmatrix}, \quad g \in \mathbf{G} \right\}.$$

Using this linear realization, it is easy to prove the following elementary proposition.

Proposition 3.1. *Let $g \in G$. Then $g\theta \in \widetilde{G}$ is semisimple if and only if $g\theta(g) \in G$ is semisimple. Furthermore, $g\theta$ is strongly regular semisimple if and only if $g\theta(g)$ is strongly regular semisimple.*

Proof. First, we note that $g \in G$ is semisimple (respectively, nilpotent) if and only if $\theta(g)$ is.

Consequently, $g \in G$ is semisimple if and only if $i(g)$ is. Since $(g\theta)^2 = g\theta(g)$, it follows that if $g\theta$ is semisimple, then $g\theta(g)$ is as well.

Conversely, suppose that $g\theta(g)$ is semisimple. We have the Jordan decomposition

$$g\theta = g_{ss}g_u$$

such that the elements $g_{ss}, g_u \in G^+$ commute, where $i(g_{ss})$ is semisimple and $i(g_u)$ is unipotent. Then

$$g\theta(g) = g_{ss}^2 g_u^2.$$

But by the uniqueness of the Jordan decomposition, we must have $g_u^2 = 1$ and so $g_u = 1$.

It remains to prove the statement about strong regularity. The characteristic polynomials $P(X)$ of $g\theta(g)$ and $i(g\theta)$ are related through the equality

$$P_{i(g\theta)}(X) = P_{g\theta(g)}(X^2)$$

(to see this, calculate the determinant in blocks). In particular, $P_{g\theta(g)}$ splits into a product of simple roots if and only if $P_{i(g\theta)}$ does (note that 0 is not a root). This concludes the proof. □

This result indicates that the natural notion of semisimplicity for elements of the form $g\theta \in \widetilde{\mathbf{G}} \subset \mathbf{G}^+$ is to check whether $g\theta(g)$ is semisimple in the usual sense. We say that $g \in \mathbf{G}$ is **θ-semisimple** if $g\theta(g)$ is semisimple.

3.3 Smooth Representations of $\mathbf{G}^+$ and θ-stable Representations of G

Let F be a p-adic field (i.e. a nonarchimedean local field of characteristic zero) and let $G = \mathbf{G}(F)$ and $G^+ = \mathbf{G}^+(F)$ be the sets of F-points of the respective groups. The groups G^+ and G are totally discontinuous and locally compact.

We write $\mathrm{Rep}(G^+)$ and $\mathrm{Rep}(G)$ for the categories of smooth complex representations of the p-adic groups G^+ and G respectively. Similarly, we write $\mathcal{H}(G^+)$ and $\mathcal{H}(G)$ for the corresponding Hecke algebras and $\mathcal{H}(\widetilde{G})$ for the subspace of functions with support in the non-neutral component $\widetilde{G}$ of G^+. We have a natural injection

$$\mathcal{H}(G) \hookrightarrow \mathcal{H}(G^+),$$

which equips $\mathcal{H}(G^+)$ with the structure of an $\mathcal{H}(G)$-module on the left and right and under which we can view $\mathcal{H}(\widetilde{G})$ as a submodule. Thus, as a $\mathcal{H}(G)$-bimodule, we have the decomposition

$$\mathcal{H}(G^+) = \mathcal{H}(G) \oplus \mathcal{H}(\widetilde{G}).$$

The map $f \mapsto f * \delta_\theta$ gives a bijection between $\mathcal{H}(\widetilde{G})$ and $\mathcal{H}(G)$ that shows that $\mathcal{H}(\widetilde{G})$ is isomorphic to $\mathcal{H}(G)$ as a right module, and the action on the right of a function f is given on $\mathcal{H}(G)$ by the multiplication by (that is, convolution product with) $f \circ \theta$.

We say that a representation $(\pi, V) \in \mathrm{Rep}(G)$ is **θ-stable** if there exists a G^+-isomorphism between (π, V) and $(\pi \circ \theta, V)$. We write $\mathrm{Rep}(G)^\theta$ for the full subcategory of $\mathrm{Rep}(G)$ that consists of θ-stable representations.

For $(\pi, V) \in \mathrm{Rep}(G^+)$, consider the restrictions

$$\pi_0 = \pi|_G$$

$$\widetilde{\pi_0} = \pi|_{\widetilde{G}}.$$

(Note that the latter is not a representation per se, because $\widetilde{G}$ is not a group.) Note that the data encoded by $\widetilde{\pi_0}$ can be extracted if we know how G acts on V and how θ acts on V. Let's state this result formally and give a proof. This result seems to be well-known, but we were unable to find an appropriate reference, and so we give details.

Theorem 3.2. *A smooth representation (π, V) of G^+ is entirely determined by the triplet $(V, \pi|_G, \pi(\theta))$.*

We break this down into a couple of simple lemmas.

Lemma 3.3. *A smooth representation of G^+ is given by a θ-stable representation of G and a choice of an isomorphism $A \in \mathrm{Hom}_G(\pi, \pi \circ \theta)$, which is of order 2 and is also an automorphism of V.*

Proof of Lemma. If (π, V) is a smooth representation of G^+, then $\pi(\theta)$ is an automorphism of order 2 on V that intertwines π and $\pi \circ \theta$, so the representation $\pi|_G$ is θ-stable.

Conversely, suppose that we have a triple (V, π, A) where (π, V) is a smooth θ-stable representation of G and A is an automorphism of order 2 of V that intertwines π and $\pi \circ \theta$, then we can construct a smooth representation of G^+ by setting $\pi(\theta) = A$. □

Lemma 3.4. *Up to G^+-isomorphism, the choice of operator A is unique up to sign.*

Proof of Lemma. Given any two representations $\pi, \pi' \in \mathrm{Rep}(G^+)$, their restrictions to G are isomorphic if and only if $\pi' \simeq \pi \otimes \chi$ for χ a character of $\langle \theta \rangle$; that is, if we realize π and π' in the same vector space, we must have $\pi'(\theta) = \pm\pi(\theta)$. □

In particular, the previous result implies that if $(\pi, V) \in \mathrm{Rep}(G)$ and π^+ is an extension to G^+, then the restriction $\mathrm{Tr}_{\widetilde{G}}(\pi^+)$ to $\widetilde{G}$ of the character of π^+ is determined by π up to sign.

Lemma 3.5. *All irreducible θ-stable representations of G are extendable to a representation of G^+.*

Proof of Lemma. Suppose that (π, V) is an irreducible θ-stable representation of G and $A \in \mathrm{Hom}_G(\pi, \pi \circ \theta)$ an arbitrary isomorphism. Since $A^2 \in \mathrm{Hom}_G(\pi, \pi)$, Schur's lemma implies that there exists a nonzero $\lambda \in \mathbf{C}$ such that $A^2 = \lambda \cdot \mathrm{Id}_V$. Thus, given a square root μ of λ, it follows that $\frac{A}{\mu} \in \mathrm{Hom}_G(\pi, \pi \circ \theta)$ is of order 2, which allows us to extend π to G^+. □

We combine these results to give a proof of our theorem.

Proof of Theorem. A smooth representation (π, V) of G^+ certainly determines a triple $(V, \pi|_G, \pi(\theta))$. It remains to show that we can recover the representation of G^+ from this data.

Let $(V, \pi|_G, \pi(\theta))$ be such a triple. Then (π, V) is a θ-stable representation of G and $\pi(\theta) \in \mathrm{Hom}_G(\pi, \pi \circ \theta)$ is an order-two automorphism of V. By the previous lemmas, these determine a unique representation of G^+. □

3.4 Twisting G by a Conjugate of θ

For technical reasons, it is sometimes useful to consider $\mathbf{G}$ when it is twisted by another involution instead of θ. For example, at a crucial part of their argument [CC09, §4], Chenevier and Clozel have to use an automorphism θ_0 that is a conjugate of the involution they use in the remainder of their paper. In this section, we will show that the choice of conjugate of θ that we twist $\mathbf{G}$ by in the construction of $\mathbf{G}^+$ is of little consequence, and indicate precisely what minor adjustments need to be made.

3.4.1 The involution θ_0 of Chenevier–Clozel

For the moment, assume that we are working with $\mathbf{G} = GL_N$ where N is *even*, and explain the relation between our "bare" involution $\theta : g \mapsto {}^t g^{-1}$ and the θ_0 that Chenevier and Clozel apply at certain points of their argument.

Consider the matrix

$$J_0 = \begin{bmatrix} & & & & 1 \\ & & & -1 & \\ & & \cdots & & \\ & 1 & & & \\ -1 & & & & \end{bmatrix} \in \mathbf{G}.$$

Note that $J_0^2 = -I_N$ and $\theta(J_0) = J_0$. We define an automorphism

$$\begin{aligned} \theta_0 : \mathbf{G} &\to \mathbf{G} \\ g &\mapsto J_0 {}^t g^{-1} J_0^{-1} = \theta(J_0 g J_0^{-1}), \end{aligned}$$

given by applying θ and then applying conjugation by J_0. The automorphisms θ and θ_0 are both involutions, that is, they are both of order 2.

We defined the twisted group $\mathbf{G}^+$ above in terms of the involution θ. But what if we considered a different involution, such as θ_0? We will show in this section that we can replace the involution θ

by any conjugate and in particular, θ_0. For example, since θ and θ_0 are congruent modulo the inner automorphisms of $\mathbf{G}$, we have

$$\mathbf{G} \rtimes \langle \theta_0 \rangle = \mathbf{G} \rtimes \langle \theta \rangle .$$

For the embedding i giving the realization in the general linear group (cf. §3.2), we define

$$i(\theta_0) = \begin{bmatrix} 0 & J_0 \\ J_0 & 0 \end{bmatrix}.$$

The same reasoning of §3.3 applies when we replace θ by θ_0, since we did not use any property of θ other than the fact that it is an involution. Furthermore, as the two automorphisms are conjugate, a representation is θ-stable if and only if it is θ_0-stable. To see this, just note if $(\pi, V) \in \mathrm{Rep}(G)$, then $\pi \circ \theta$ and $\pi \circ \theta_0$ are always isomorphic via the map $\pi(J_0) : V \to V$.

Finally, note that for $\pi \in \mathrm{Rep}(G)$ is irreducible, we always have $\pi \circ \theta_0 \simeq \pi^\vee$, and so an irreducible representation is θ_0-stable if and only if it is self-dual, just like the case of θ.

3.4.2 The involution θ of Waldspurger

We return to the case of $\mathbf{G} = GL_N$ for N an arbitrary positive integer. In his work on twisted $\mathbf{G}$ over p-adic fields [Wal07], Waldspurger uses an involution θ of $\mathbf{G}$ the form

$$\theta(g) = J\,{}^t g^{-1} J,$$

where

$$J = \begin{bmatrix} & & 1 \\ & \cdots & \\ 1 & & \end{bmatrix}.$$

The same arguments that we delineated in the last section for the involution θ_0 also apply to the above involution θ.

Remark 3.6. In case the notational abuse has not made this clear, this isomorphism "à la Waldspurger" will be the involution that we ourselves use most of the time in the proofs of our results—at least, outside of this preliminary section (§3) where we talk about general context in which we can prove such results—so we can apply the results of [Wal07] directly.

3.5 Conjugacy and Stable Conjugacy in $\mathbf{G}^+$

We return to letting θ denote the "raw" inverse-transpose $\theta(g) = {}^t g^{-1}$.

We write $\mathrm{Ad} : \mathbf{G}^+ \to \mathrm{Aut}(\mathbf{G}^+)$ for the action of $\mathbf{G}^+$ on itself by conjugation. The components of $\mathbf{G}^+$ are stable under this action.

If $g \in \mathbf{G}$, then $\widetilde{\mathbf{G}}$ is stable by $\mathrm{Ad}(g)$, and we write

$$\mathrm{Ad}_\theta(g) := \delta_\theta \circ \mathrm{Ad}(g)|_{\widetilde{\mathbf{G}}} \circ \delta_\theta^{-1}$$

for the action on $\mathbf{G}$ deduced via $\delta_\theta : g \mapsto g\theta$, that is,

$$\mathrm{Ad}_\theta(g)(h) = hg\theta(h)^{-1} = hg\,{}^t h_0.$$

This action is called **θ-twisted conjugation**. We define the θ_0-twisted conjugation Ad_{θ_0} in an analogous manner by replacing θ with θ_0, and same for the θ "of Waldspurger."

Two semisimple elements $x, y \in G^+ = \mathbf{G}^+(F)$ are said to be **stably conjugate** if there exists a $g \in \mathbf{G}^+(\overline{F})$ such that

$$x = gyg^{-1},$$

and for all $\sigma \in \mathrm{Gal}(\overline{F}/F)$, we have

$$g^{-1}\sigma(g) \in Z(\mathbf{G})^\theta \mathbf{G}_y,$$

where $\mathbf{G}_y^0$ denotes the neutral component of the centralizer of y in $\mathbf{G}$.

Remark 3.7. This additional latter Galois condition for stable conjugacy is required since $\mathbf{G}^+$ is not a connected reductive group with simply connected derived group.

Note that conjugation in $\mathbf{G}(F)$ implies stable conjugation, which itself implies conjugation in $\mathbf{G}(\overline{F})$.

3.5.1 Semisimple conjugacy classes of $\widetilde{G}$

For our calculations later involving the twisted endoscopy norm map, it is useful to know what the semisimple conjugacy classes of the non-neutral connected component $\widetilde{G}$ of G^+ are. Given the realization of G^+ and thus $\widetilde{G}$ in a general linear group (cf. §3.2), this boils down to some elementary linear algebra calculations. We refer the reader to [Wal07, §I.3] for further details. In this section, the θ denotes the involution "à la Waldspurger" (cf. §3.4.2).

The goal of this section is to produce explicit representatives of each θ-semisimple conjugacy class in $\widetilde{G}$. While it does involve establishing a somewhat intimidating amount of notation, the ideas and the deductions are simple and elementary.

Let $\overline{F}$ denote an algebraic closure of a field F and $\mathrm{Gal}(\overline{F}/F)$ the absolute Galois group of F. For any finite set I and a subset $I^* \subset I$, we pick the following series of objects:

- For $i \in I$, pick an $a_i \in \overline{F}^\times$;
  - if $i \notin I^*$, we set $F_i' = F[a_i]$, and denote the degree of its extension by $f_i = [F_i' : F]$; we assume that a_i is not conjugate to a_i^{-1} by the Galois group $\mathrm{Gal}(F_i'/F)$;
  - if $i \in I^*$, we set $F_i = F[a_i]$, we assume that F_i is the quadratic extension of a subextension F_i' of F, and we set $f_i = [F_i' : F]$, we assume that $a_i\tau_i(a_i) = 1$ where τ_i is the unique nontrivial element of $\mathrm{Gal}(F_i/F_i')$;
- For $i \in I$, pick an integer $d_i \geq 1$;
- For $i \in I^*$, let V_i be a vector space of dimension d_i over F_i, equipped with a nondegenerate sesquilinear form $q_i : V_i \times V_i \to F_i$, where

$$q_i(zv, z'v') = \tau_i(z)z'q_i(v, v')$$

for $z, z' \in F_i$ and satisfying the relation

$$q_i(v', v) = a_i \tau_i(q_i(v, v')).$$

Remark 3.8. For $i \in I$, fix $b_i \in F_i^\times$ such that $a_i b_i \tau(b_i)^{-1} = 1$. The symmetry condition imposed on q_i is equivalent to $b_i q_i$ being Hermitian. It implies that the group of isometries $\mathbf{U}(q_i)$ of the form q_i is the usual unitary group.

We also define two other vector spaces:

- V_+ is a vector space over F equipped with a nondegenerate quadratic form q_+; we write d_+ for its dimension;
- V_- is a vector space over F equipped with a nondegenerate symplectic form q_-; we write d_- for its dimension.

We will eventually be able to take V_+ or V_- to be zero-dimensional.

We assume that:

$$N = d_+ + d_- + 2\sum_{i \in I} d_i f_i,$$

and that for $i, j \in I$ with $i \neq j$, there is no F-linear isomorphism $F[a_i] \to F[a_j]$ that sends a_i to a_j or a_j^{-1}.

The above choices of data:

$$(I, I^*, \{a_i\}_{i \in I}, \{d_i\}_{i \in I}, \{V_i\}_{i \in I^*}, V_+, V_-),$$

determines a conjugacy class in $\widetilde{G}$. We write V_i^* for the dual of V_i when we consider it as a space over F and denote by $\mathrm{Isom}(V_i, V_i^*)$ the set of F-linear isomorphisms of V_i in V_i^*. We establish analogous notation for V_+^* and V_-^*.

For $i \in I\backslash I^*$, we set

$$V_i' = F_i'^{d_i}$$

$$V_i'' = \mathrm{Hom}_{F_i'}(V_i', F_i')$$

and so define

$$V_i = V_i' \oplus V_i''.$$

For such an $i \notin I^*$, we define $\sigma_i \in \mathrm{Isom}(V_i, V_i^*)$ by the equality:

$$\langle x' + x'', \sigma_i(y' + t'') \rangle = \mathrm{Tr}_{F_i'/F}(\langle x', y'' \rangle + a_i \langle y', x'' \rangle)$$

for $x', y' \in V_i'$ and $x'', y'' \in V_i''$.

For $i \in I^*$, we define $\sigma_i \in \mathrm{Isom}(V_i, V_i^*)$ by the equality:

$$\langle x, \sigma_i(y) \rangle = \mathrm{Tr}_{F_i/F}(q_i(x, y)).$$

For $\zeta = \pm 1$, we define $\sigma_\zeta \in \mathrm{Isom}(V_\zeta, V_\zeta^*)$ by the equality:

$$\langle x, \sigma_\zeta(y) \rangle = q_\zeta(x, y).$$

We identify an N-dimensional F-vector space V with

$$V = V_+ \oplus V_- \oplus (\oplus_{i \in I} V_i),$$

and the collection $(\sigma_+, \sigma_-, (\sigma_i)_{i \in I})$ defines an element $\sigma \in \mathrm{Isom}(V, V^*)$. From this we obtain an element

$$s = \widetilde{\sigma} \in \widetilde{G}.$$

It's a semisimple element with a well-defined conjugacy class. Every semisimple conjugacy class of $\widetilde{G}$ can be realized in this way. (Consider a realization as in §3.2 to make this obvious.)

Note that the following elementary modifications do not change the conjugacy class of $s \in \widetilde{G}$:

- changing I and I^* to other sets with the same number of elements;
- replacing a_i with its conjugation under an element of $\mathrm{Gal}(\overline{F}/F)$;
- replacing a_i with a_i^{-1};
- replacing the forms q_i, q_+ or q_- by equivalent forms.

Up to these elementary modifications, we thus obtain a classification of conjugacy classes of semisimple elements of $\widetilde{G}$.

The commutant $\mathbf{Z_G}(s)$ in $\mathbf{G}$ (that is, the connected component, not $\mathbf{G}^+$!) of the element $s \in \widetilde{G}$ constructed above is equal to:

$$\mathbf{O}(q_+) \times \mathbf{Sp}(q_-) \times \left(\prod_{i \in I \setminus I^*} \mathbf{GL}_{d_i/F_i'} \right) \times \left(\prod_{i \in I^*} \mathbf{U}(q_i)_{/F_i'} \right),$$

where $\mathbf{O}(q_+)$ is the orthogonal group of q_+, $\mathbf{Sp}(q_-)$ is the symplectic group of q_-, and for example, for $i \in I^*$, $\mathbf{U}(q_i)_{/F_i'}$ is the restriction of F_i' to F of the group of automorphisms of F_i' of the form q_i.

3.5.2 θ-twisted conjugacy classes of a skew-symmetric matrix (N even)

A result that we need to use at one point of the proof is that the set of $\gamma\theta \in \widetilde{G}$ such that γ is skew-symmetric forms a stable conjugacy class.

Let us return to letting θ denote the involution $g \mapsto {}^t g^{-1}$, and $\mathbf{G} = GL_{2n}$. A key property of the principal element γ_0 (c.f. §4.3) in the proof of the theorem is that the twisted centralizer of γ_0 is a symplectic group. We will show here that this property is not affected by replacing θ with a conjugate (i.e. applying a subsequent inner automorphism).

Let $\mathrm{Skew}_{2n}(F)$ be the set of $2n \times 2n$ matrices with coefficients in a field F that are skew-symmetric and invertible:

$$\mathrm{Skew}_{2n}(F) = \{\gamma = [\gamma_{ij}] \in GL_{2n}(F) \mid {}^t\gamma = -\gamma\}.$$

From the definition, we can see that $\mathrm{Skew}_{2n}(F)$ is closed in $G = \mathbf{G}(F)$. Moreover, G acts on $\mathrm{Skew}_{2n}(F)$ by θ-conjugation, that is

$$\mathrm{Ad}_\theta(g).\gamma = g\gamma\, {}^t g$$

for $\gamma \in \mathrm{Skew}_{2n}(F)$ and $g \in G$.

Define a matrix $J_{2n} \in \mathrm{Skew}_{2n}(F)$ by

$$J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

where I_n denotes the identity matrix of size n.

The symplectic group can be defined as the stabilizer of J_n under the θ-conjugacy action:

$$\mathrm{Stab}_{\mathrm{Ad}_\theta}(J_n) = \{g \in G \mid gJ_n\, {}^t g = J_n\} = Sp_{2n}(F).$$

For all $\gamma \in \mathrm{Skew}_{2n}(F)$ and $g \in G$, we have

$$\begin{aligned} \mathrm{Stab}_{\mathrm{Ad}_\theta}(\mathrm{Ad}_\theta(g).\gamma) &= \{h \in G \mid h(\mathrm{Ad}_\theta(g).\gamma)\, {}^t h = \mathrm{Ad}_\theta(g).\gamma\} \\ &= \{h \in G \mid hg\gamma\, {}^t g\, {}^t h = g\gamma\, {}^t g\} \\ &= \{h \in G \mid g^{-1}hg\gamma\, {}^t g\, {}^t h\, {}^t g^{-1} = \gamma\} \\ &= g\{x \in G \mid x\gamma\, {}^t x = \gamma\}g^{-1} \\ &= g\, \mathrm{Stab}_{\mathrm{Ad}_\theta}(\gamma)g^{-1} \end{aligned}$$

so this action is transitive. This property is holds over arbitrary fields (at least those of characteristic zero, which we are only ones we're interested in), and so $\mathrm{Skew}_n(F)$ is precisely the stable θ-twisted conjugacy class of any invertible skew-symmetric matrix. Moreover, all the stabilizers of such matrices are thus conjugate with each other, and in particular conjugate to $Sp_{2n}(F)$.

In addition, for all $\gamma \in \mathrm{Skew}_{2n}(F)$, we thus have a surjection

$$\cdot\gamma : G \to \mathrm{Skew}_{2n}(F)$$

$$g \mapsto \mathrm{Ad}_\theta(g).\gamma$$

which induces a bijection

$$G/\operatorname{Stab}_{\mathrm{Ad}_\theta}(\gamma) = \mathrm{Skew}_{2n}(F).$$

In particular, for $\gamma = J_n$, this gives a bijection

$$GL_{2n}(F)/Sp_{2n}(F) = \mathrm{Skew}_{2n}(F).$$

To summarize the above deductions, we have (1) shown in this part that the θ-twisted centralizer of a matrix of $\mathrm{Skew}_{2n}(F)$ is conjugate to a symplectic group, and (2) that the quotient of GL_{2n} by this group (and in particular, GL_{2n}/Sp_{2n}) is in bijection with $\mathrm{Skew}_{2n}(F)$.

3.5.3 Twisted orbits and twisted orbital integrals

The goal of this section is to relate the twisted orbits and twisted orbital integrals for different conjugates of the involution θ.

On the quotient $GL_{2n}(F)/\operatorname{Stab}_{\mathrm{Ad}_\theta}(\gamma)$, choose a measure on it that is invariant under left translation; this exists and is unique up to constant. Equip $\mathrm{Skew}_{2n}(F)$ with the measure induced from that of $GL_{2n}(F)/\operatorname{Stab}_{\mathrm{Ad}_\theta}(\gamma)$ under the bijection above, this measure is thus invariant by the action of G under θ-conjugation, and up to constant is the only possible one with this property.

First, we want to show that θ_0-twisted orbits can be obtained from the θ-twisted orbits. Let $G.[g_0\theta]$ and $G.[g\theta_0]$ denote the θ-twisted orbits and θ_0-twisted orbits, respectively, of an element

$g \in G$. These two orbits are related as follows:

$$G.[\gamma\theta_0] = G.[(\gamma J_0)\theta].J_0^{-1}.$$

Similarly, the θ-twisted centralizer of γ is the θ_0-twisted centralizer of γJ_0. Moreover, the θ_0-twisted orbital integrals can be deduced from the θ-twisted orbital integrals via

$$J_{G\theta_0}(\gamma, f) = J_{G\theta}(\gamma J_0, \lambda(J_0).f),$$

where $J_{G^0\theta_0}(\gamma, f)$ denotes the orbital integral of f on the θ_0-twisted conjugacy class of γ (similarly with θ), and $\lambda(J_0).f$ is the function $\lambda(J_0).f : g \mapsto f(gJ_0^{-1})$.

Thus, up to some minor adaptations, we can thus easily pass from results twisting by θ to results twisting by θ_0 or any other conjugate involution. In particular, we can reinterpret the results of papers like [Sha92] and [Wal07] which use θ (§3.4.2), with those of [CC09] which uses θ_0.

3.6 The Twisted Endoscopy Norm Map

In this section, following the conventions of Waldspurger [Wal07, §III.2] (namely, we use his choice of involution θ, see §3.4.2), we recall the properties of the twisted endoscopy norm map (or more precisely, one direction of the norm correspondence) between twisted conjugacy classes in $\widetilde{G} = G\theta$ (the non-neutral connected component of the twisted group $G^+ = GL_N \rtimes \langle\theta\rangle$) and conjugacy classes in an endoscopic group $H(\mathbf{R})$ of G^+, which is defined to be

$$H = \begin{cases} Sp_{2n}, & \text{if } N = 2n+1 \text{ (odd case)} \\ SO_{2n+1}, & \text{if } N = 2n \text{ (even case).} \end{cases}$$

General results on twisted endoscopy can be found in the monograph of Kottwitz and Shelstad [KS99]; we specialize the results to our setting: namely, twisted endoscopy corresponding to the triple $(G = GL_N, \theta : g \mapsto J\,{}^tg^{-1}J, \omega = \mathrm{id})$.

A priori, the existence of such a miraculous correspondence seems unlikely to have come out of nowhere, so let us provide some context for the result. Through the lens of the Langlands program and Arthur's conjectures, if $\mathbf{H}$ is an endoscopic group of $\mathbf{G}^+$, it means, roughly speaking, that $\mathbf{H}$ is the group that controls the stably invariant distributions on the non-identity connected component $\widetilde{G}$. One consequence of this property is that if Π^H is an L-packet of tempered irreducible admissible representations of H, then it should be possible to attach to Π^H a tempered irreducible admissible representation π of G that is invariant under the automorphism θ, so that for a suitable extension π^+ of π to a representation of G^+, the distribution $\mathrm{Tr}_{\widetilde{G}}(\pi)$ is "a transfer of" the distribution $\mathrm{Tr}(\Pi^H) = \sum_{\tau\in\Pi^H} \mathrm{Tr}_H(\tau)$ on H, in a precise sense.

The use of the norm map is fundamental to our approach. It allows us to reduce the study of the representations we impose at the archimedean places to the corresponding representations on (a compact form of) the endoscopic group, which are parametrized by their highest weight and which we can vary to obtain a number of important simplifications "asymptotically." We will explain this in more detail in §4.

Since this is probably the most important section of the chapter, we recall the important notions again in an attempt to clarify at the expense of possible repetition.

Let G^+ be the semidirect product of $\{1, \theta\} \simeq \mathbf{Z}/2\mathbf{Z}$ by G, where θ operates by $g \mapsto J\,{}^t g^{-1} J$. We have $G^+ = G \coprod \theta G$.

Definition 3.9. If $g, h \in G$, we say that g and h are **θ-conjugate** if

$$g = x^{-1} h x^{\theta}$$

for an $x \in G$. This is equivalent to saying that θg and θh are conjugate under $G \subset G^+$.

The group G^+ is a non-connected reductive group and such groups admit a natural notion of semisimplicity: $\widetilde{g} = \theta g$ for $g \in G$ is semisimple if and only if $\widetilde{g}^2 = (g^{\theta})g$ is semisimple (cf. Prop. 3.1). We say such elements $g \in G$ are **θ-semisimple**.

If $g \in \theta G$ is semisimple, its centralizer $Z_G(g)$ in G is reductive. We say that g is **strongly**

regular if $Z_G(g)$ is a torus. If g and h are strongly regular, we say that they are **stably conjugate** if there exists $x \in G(\mathbf{C})$ such that

$$xhx^{-1} = h.$$

All these notions can be naturally defined over global fields and their completions (see, e.g. [Wal07] for such a description over the p-adics), but we will only apply them over the reals. In particular, we can define "stable θ-conjugate" and "strongly θ-regular" on $G(\mathbf{R})$. We can define notions of strongly regular elements, and thus stable conjugacy, on $H(\mathbf{R})$ as well.

Let $g \in G(\mathbf{R})$ be a strongly θ-regular element, and let $\Lambda(g)$ be the set of (complex) eigenvalues of $g^\theta \cdot g$; they are necessarily distinct because of the strong regularity property. Note that if N is odd, then $\Lambda(g)$ contains 1. Let $h \in H(\mathbf{R})$ be a strongly regular element, and let $\Lambda(h)$ be its set of eigenvalues (all which are distinct by strong regularity). If N is even, then $\Lambda(h)$ contains 1. Then the norm of the stable conjugacy class of g is equal to the stable conjugacy class of h if and only if:

- If N is even, then

$$\Lambda(h) = \{-x \mid x \in \Lambda(g)\} \cup \{1\}.$$

- If N is odd, then

$$\Lambda(h) \cup \{1\} = \Lambda(g).$$

Remark 3.10. The reason that we need to take the negatives of the eigenvalues in the case of even N is because, in this case, our automorphism θ does not fix the pinning.

This gives us a bijection called the twisted endoscopy norm correspondence.

Proposition 3.11. *([KS99, Thm 3.3A] in general, [Wal07, §III.2] for our G^+ and H.) The above correspondence defines a bijection between*

- *stable θ-conjugacy classes of strongly θ-regular elements in G (that is, "strongly regular (semisimple)" elements of $\widetilde{G}$), and*
- *stable conjugacy classes of strongly regular elements in H.*

For $g \in G(\mathbf{R})$ that is strongly θ-regular and $h \in H(\mathbf{R})$ that is strongly regular, we denote (one direction of) the correspondence above by

$$\mathcal{N} g = h$$

and say that h is the **norm** of g, and call the induced map the twisted endoscopy **norm map**.

Chapter 4

Key Notions

Here we describe the ideas and techniques that allow us simplify our proof of the theorem in §7.

4.1 The Asymptotic Simplification of the Geometric Side of the Trace Formula

The principal observation that simplifies the analysis of the non-vanishing of the geometric side of the trace formula is the *asymptotic simplification of the geometric side of the trace formula*. This phenomenon can be viewed as a sort of Plancherel equidistribution result and falls into the realm of general "Sato–Tate type" phenomena that we observe for automorphic representations that vary "in families" in different ways.

The result is described and proven for reductive groups over $\mathbf{Q}$ that satisfy certain hypotheses by Chenevier and Clozel [CC09, §1]. In the proof of our theorem, we only need the result for the compact forms of the endoscopic groups over $\mathbf{R}$, so we specialize to that setting. We emphasize that unlike results in, say, the theory of semisimple conjugacy classes, passing from the semisimple case to the reductive case (e.g., from a group to an isogenous group with a given center) involves dealing with some nontrivial issues with respect to the results that we describe here.

Let $G = G(\mathbf{R})$ be a connected compact Lie group and T a maximal torus. Assume that $G(\mathbf{R})$ has discrete series, so G has a real inner form G^* that is anisotropic mod center. We can parametrize discrete series representations δ by their highest weight $\lambda \in X^*(T)$. If an element γ lies in the center

$Z(G)$ of G, the character of δ is given by

$$\Theta_\delta(\gamma) = \deg(\delta)\omega(\gamma),$$

where ω is the central character of δ.

Write $\delta(\lambda)$ for the representation associated with $\lambda \in X^*(T)$. Then $\deg\delta(\lambda)$ is given by the Weyl polynomial

$$P(\lambda) = \prod_\alpha \frac{\langle \alpha, \lambda + \rho \rangle}{\langle \alpha, \rho \rangle},$$

where the product runs over a set of positive roots of which ρ is the half-sum.

For a dominant $\lambda \in X^*(T)$, let Θ_λ be the character of the representation V_λ of G with highest weight λ.

Proposition 4.1. *Let $\gamma \in T \subset G$. For a dominant λ,*

$$\Theta_\lambda(\gamma) = \sum_i E_i(\gamma, \lambda) P_i(\lambda),$$

where:

- *The sum is finite.*
- *The $E_i(\gamma, \lambda)$ are rational functions on the γ^χ's, where χ runs over a basis of $X^*(T)$. The degrees of such functions depend on λ. The denominators of $E_i(\gamma, \lambda)$ are nonzero on γ and independent of λ.*
- *$E_i(\gamma, \lambda)$ is uniformly bounded as λ varies.*

Furthermore, if γ is not central, then $P_i(\lambda)$ is a polynomial such that

$$\deg P_i(\lambda) < \deg P(\lambda).$$

Proof. If γ is regular or central, the proposition follows from the Weyl character formula and the

Weyl degree formula. If the centralizer of γ is a Levi subgroup, then we can use Kostant's formula [Kos59]. For the general case, we can imitate the proofs of the above, see [CC09, §1] for details. ☐

Proposition 4.2. *[CC09, Cor. 1.12] If $\gamma \in G$ is not central, then*

$$\frac{\Theta_\lambda(\gamma)}{\dim(V_\lambda)} \to 0$$

as $\lambda \in X^*(T) \otimes \boldsymbol{R}$ goes to infinity away from the walls of the Weyl chambers.

Proof. We can assume without loss of generality that $\gamma \in G$ has connected centralizer (this follows when the derived group of G is simply connected). Let $M = Z_G(\gamma)$, and let $R^+(G,T)$ and $R^+(M,T)$ denote a sets of positive roots for G and M respectively. Up to taking a cover, we can assume that $R_+(M,T) = R_+(G,T) \cap R(M,T)$, where $R(M,T)$ is the set of roots.

Let W be the Weyl group of (G,T) and let W_M the Weyl group of (M,T). Define

$$W^M = \{w \in W \mid w^{-1}\alpha \in R_+(G,T) \text{ for all } \alpha \in \Delta_M\}$$

for a choice of basis $\Delta_M \subset R_+(M,T)$. Any $w \in W$ admits a unique decomposition

$$w = w_s w_u$$

where $w_s \in W_M$ and $w_u \in W^M$.

An easy consequence of the definition of W^M is that if $\lambda \in X^*(T)$ is dominant for G and $w_u \in W^M$, then $w_u(\lambda+\rho)-\rho_M$ is dominant for M. Then by applying the previous proposition, using the generalization of the Weyl character formula or Kostant's formula if necessary, and rearranging the terms, we arrive at the expression

$$\Theta_\lambda(\gamma) = \frac{\gamma^{\rho_M-\rho}}{\prod_{\alpha\in R^+(G,T)\setminus R^+(M,T)}(1-\gamma^{-\alpha}} \sum_{w_u\in W^M} \epsilon(w_u)\gamma^{w_u(\lambda+\rho)-\rho_M} P_M(\lambda)$$

where ϵ is the sign character on W and P_M is the Weyl polynomial for M.

We can assume that $\gamma \in T$. Since we have

$$\frac{P_M(\lambda_u)}{P(\lambda)} = \frac{\prod_{\alpha \in R^+(G,T)} \langle \rho, \alpha \rangle}{\prod_{\alpha \in R^+(M,T)} \langle \rho_M, \alpha \rangle} \left(\prod_{\alpha \in R^+(G,T) \setminus w_u^{-1} R^+(M,T)} \langle \lambda + \rho, \alpha \rangle \right)^{-1},$$

the term vanishes as $\lambda \to \infty$ if γ is not central. □

4.2 θ-discrete Series Representations

One way to simplify the analysis of terms appearing in the trace formula when constructing automorphic representations is to impose discrete series representations at the archimedean places. However, not all real groups have such representations. In particular, $GL_N(\mathbf{R})$ only has discrete series representations for $N = 1$ or $N = 2$, so we cannot use this technique to solve our problem in general.

But it turns out that our involution θ can bring us into a situation where we can find a workaround. Namely, it turns out that $G(\mathbf{R})$ always has certain "θ-discrete" representations; roughly speaking, these are representations that are isolated among the tempered θ-invariant representations of $G(\mathbf{R})$. It turns out that some of the techniques that can be applied to discrete series can be adapted to θ-discrete representations.

In this section, we describe the representations in question. These are the representations that we will put in at the archimedean places of our desired self-dual cuspidal automorphic representation.

4.2.1 $GL(2n+1)$ case

In case of $G = GL_N$ where $N = 2n+1$ is odd, the representations at the archimedean places that are suitable for our approach are cohomological representations of a particular form that come from the endoscopic group Sp_{2n}. We describe their construction and verify their key properties.

Consider a pure weight μ with purity 0 given by

$$\mu = (\mu_1 > \mu_2 > \cdots > \mu_n > 0 > -\mu_n > \cdots > -\mu_2 > -\mu_1).$$

Let $H = Sp_{2n}$, defined so that the upper-triangular subgroup B_H of H is a Borel subgroup. The connected component of the L-group of H is ${}^L H^\circ = SO_{2n+1}(\mathbf{C})$. The maximal compact subgroup K_H of $H(\mathbf{R}) = Sp_{2n}(\mathbf{R})$ is isomorphic to $U(n)$. Define a dominant integral weight μ_H for H, which is given by

$$\mu_H := (\mu_1, \mu_2, \ldots, \mu_n) = \sum_{i=1}^{n} \mu_i e_i,$$

where e_i gives the i-th coordinate of a diagonal matrix. Let ρ_H be the half-sum of positive roots for H, written as

$$\rho_H = \sum_{j=1}^{n} (n+1-j) e_j = (n, n-1, \ldots, 1).$$

Define

$$w_H := \mu_H + \rho_H = (\mu_1 + n, \mu_2 + n - 1, \ldots, \mu_{n-1} + 1, \mu_n).$$

Then w_H is a regular weight and by Harish-Chandra's classification of discrete series representations (cf. [Kna01]), there exists a discrete series representation $\pi_H = \pi_{w_H}$ of $H(\mathbf{R})$ whose infinitesimal character is χ_{w_H}. Let V_{μ_H} be the irreducible algebraic representation $H(\mathbf{C})$. By some standard results on the cohomology of discrete series representations ([BW80, Theorem II.5.3]), we know that π_H is cohomological with respect to the coefficient system V_{μ_H} of H, that is, the relative Lie algebra cohomology $H^\bullet(\mathfrak{h}_\infty, K_H; \pi_H \otimes V_{\mu_H}) \neq 0$; indeed, it is nonzero only in the middle degree $\bullet = \frac{1}{2} \dim(H(\mathbf{R}))/\dim(K_H)$.

We can deduce the shape of the Langlands parameter σ_{w_H} attached to π_{w_H} [Bor79, Example 10.5]:

$$\sigma_{w_H} = \mathrm{Ind}_{\mathbf{C}^\times}^{W_\mathbf{R}}(\chi_{\ell_1}) \oplus \mathrm{Ind}_{\mathbf{C}^\times}^{W_\mathbf{R}}(\chi_{\ell_2}) \oplus \cdots \oplus \mathrm{Ind}_{\mathbf{C}^\times}^{W_\mathbf{R}}(\chi_{\ell_n}) \oplus \mathrm{sgn}^n,$$

where $\ell_1, \ldots, \ell_n$ are positive integers and the first n summands are irreducible 2-dimensional representations of the Weil group $W_\mathbf{R}$ of $\mathbf{R}$ induced from characters of $\mathbf{C}^\times$ with the character χ_{ℓ_j} sending $z \in \mathbf{C}^\times$ to $(z/\overline{z})^{\ell_j/2}$ where each summand being of orthogonal type forces ℓ_j to be even. Since the determinant of the parameter must be 1 (to land in $SO_{2n+1}(\mathbf{C})$), this forces the last summand to be sgn^n. The relation between the integers ℓ_j and the weight w_H is given through

$$\sigma_{w_H}|_{\mathbf{C}^\times} = z^{w_H}\overline{z}^{-w_H},$$

where we have implicitly used the fact that w_H, a character of a maximal torus T_H of H is also a cocharacter of the dual ${}^LT^\circ \subset {}^LH^\circ$. Thus, writing $\ell = (\ell_1, \ldots, \ell_n)$, we have $\ell = 2w_H$, that is,

$$(\ell_1, \ldots, \ell_n) = (2\mu_1 + 2n, 2\mu_2 + 2n - 2, \ldots, 2\mu_{n-1} + 2, 2\mu_n).$$

Let π_μ be the Langlands transfer of π_H to an irreducible representation of $GL_{2n+1}(\mathbf{R})$, noting that the Langlands parameter of π_H is that of π_μ via the standard embedding ${}^LH^\circ = SO_{2n+1}(\mathbf{C}) \subset {}^LG^\circ = GL_{2n+1}(\mathbf{C})$. By the local Langlands correspondence (cf. for example, [Kna94]), we can deduce that

$$\pi_\mu = \mathrm{Ind}^G_{P(2,2,\ldots,2,1)}(D_{\ell_1} \otimes D_{\ell_2} \otimes \cdots \otimes D_{\ell_n} \otimes \mathrm{sgn}^n), \tag{4.1}$$

where D_ℓ denotes the discrete series representation of $GL_2(\mathbf{R})$ whose lowest non-negative K-type is the character

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \mapsto e^{i(\ell+1)\theta}$$

with central character $a \mapsto \mathrm{sgn}(a)^{\ell+1}$ (e.g. so the representation at ∞ of a holomorphic elliptic modular cusp form of weight k is D_{k-1}). Let V_μ be the irreducible algebraic representation of $G(\mathbf{C})$ with highest weight μ. Noting that V_μ is self-dual, a result of Clozel [Clo90, Lemme 3.14] implies that

$$H^\bullet(\mathfrak{g}, \mathbf{R}_+^\times SO_{2n+1}; \pi_\mu \otimes V_\mu) \neq 0.$$

The π_μ of the form (4.1) will be the representations that we put at the archimedean places of our automorphic representation.

4.2.2 $GL(2n)$ case

We argue just like in the case above. Since the arguments are nearly identical, we are a little briefer in our remarks. Here, we take $H = SO_{2n+1,F}$ to be the split orthogonal group in $2n+1$ variables. We have $H(\mathbf{R}) = SO(n, n+1)$. The maximal compact subgroup $K_H \subset H(\mathbf{R})$ is isomorphic to $S(O(n) \times O(n+1))$. The connected component of the L-group is ${}^L H^\circ = Sp(2n, \mathbf{C})$. Fix a real place v of F. The constructions below will depend on all depend on this choice of v, but we will omit this dependence from the notation.

Consider the dominant integral weight

$$\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq -\mu_n \geq \cdots \geq -\mu_2 \geq -\mu_1),$$

which is pure of weight 0. By arguing as in the odd case above, set

$$\begin{aligned} \mu' &= (\mu_1, \mu_2, \ldots, \mu_n) \\ \Lambda' &= \mu' + \rho' = \left(\mu_1 + n - \frac{1}{2}, \mu_2 + n - \frac{3}{2}, \ldots, \mu_{n-1} + \frac{3}{2}, \mu_n + \frac{1}{2}\right). \end{aligned}$$

Consider the discrete series representation $\pi' = \pi_{\Lambda'}$ with infinitesimal character given by Λ'. The Langlands parameter $\sigma_{\Lambda'}$ of $\pi_{\Lambda'}$ is of the form

$$\sigma_{\Lambda'} = \mathrm{Ind}_{\mathbf{C}^\times}^{W_\mathbf{R}}(\chi_{\ell_1}) \oplus \mathrm{Ind}_{\mathbf{C}^\times}^{W_\mathbf{R}}(\chi_{\ell_2}) \oplus \cdots \oplus \mathrm{Ind}_{\mathbf{C}^\times}^{W_\mathbf{R}}(\chi_{\ell_n}),$$

where all the ℓ_j are odd positive integers. The infinitesimal character of the discrete series, seen in terms of the exponents of the parameter restricted to $\mathbf{C}^\times$, gives us $\ell = 2\Lambda'$, that is,

$$(\ell_1, \ldots, \ell_n) = (2\mu_1 + 2n - 1, 2\mu_2 + 2n - 3, \ldots, 2\mu_{n-1} + 3, 2\mu_n + 1).$$

Via the local Langlands correspondence for $GL_{2n}(\mathbf{R})$, we see that π' transfers to π_μ, given by

$$\pi_\mu = \mathrm{Ind}_{P(2,2,\dots,2)}^{G}(D_{\ell_1} \otimes D_{\ell_2} \otimes \cdots \otimes D_{\ell_n}),$$

which has the property that

$$H^\bullet(\mathfrak{gl}_N, \mathbf{R}_+^\times SO(N); \pi \otimes V_\mu) \neq 0.$$

4.2.3 Twisted characters at infinity and their stability

Let π be a regular algebraic, tempered, self-dual representation of $G(\mathbf{R})$. The choice of an involutive intertwining operator A between (the associated infinite-dimensional vector spaces of) π and $\pi \circ \theta$ allows us to extend π to a representation π^+ of $G^+(\mathbf{R})$. We write $\Theta_{\pi,\theta}$ for the character of π^+ on $\widetilde{G} = \theta G(\mathbf{R})$:

$$\Theta_{\pi,\theta}(g) := \Theta_{\pi^+}(\theta g) \quad (g \in G(\mathbf{R})).$$

A priori, this is just a distribution, but a twisted generalization of a theorem of Harish-Chandra implies that it is in fact an analytic function on strongly θ-regular elements [Bou87, Thm. 2.1.1].

Theorem 4.3. *For suitable choice of A, we have*

$$\Theta_{\pi,\theta}(g) = \Theta_{\pi_H}(\mathcal{N}g)$$

for all $g \in G(\boldsymbol{R})$ whose norm (3.11) is strongly regular and elliptic.

In particular, $\Theta_{\pi,\theta}$ is invariant under stable (twisted) conjugation on elements of elliptic norm.

Proof. The original proof is in [Bou87], but it was developed in a context that is very different from that of our present problem. This proof is reinterpreted in language compatible with our presentation in [CC09, §2.5]. □

4.3 The Principal Element γ_0

Let $f = \otimes_v f_v \in C_c^\infty(G(\mathbf{A}))$ be a test function that we plug into the trace formula. Recall that we have phrased our problem so that we are only prescribing local representations of our desired automorphic representations at finite places. Suppose that we let the components f_v at finite places v correspond to (twisted) pseudocoefficients that trace out our chosen representations in the main theorem. We then have the freedom to choose any function f_∞ at the archimedean places.

A key observation of the work of Chenevier and Clozel is that if we choose $f_\infty = f_\infty^\lambda$ to be the twisted pseudocoefficient of a cohomological θ-discrete series representation—each of which is parametrized by the highest weight λ of an irreducible representation V_λ of the compact real form of the endoscopic group $H(\mathbf{R})$—then as the weight λ goes to infinity away from the walls of the Weyl chamber, the geometric side of the trace formula becomes asymptotically equivalent to a single orbital integral called the **principal term**. Up to a positive scalar, this is the twisted orbital integral $TO_{\gamma_0}(f)$ of our test function f attached to a certain elliptic θ-semisimple element $\gamma_0 \in G(F)$ (**the principal element**) whose twisted centralizer is a group whose $\mathbf{C}$-points yield the dual group $\widehat{H}$ of the endoscopic group H. In symbols, as $\lambda \to \infty$ away from the walls,

$$J_{spec}(f) = J_{geom}(f) \sim TO_{\gamma_0}(f) = C \cdot \dim(V_\lambda),$$

where C is an explicit nonzero constant that only depends on the components of f away from ∞. In other words, we reduce our problem of showing $J_{spec}(f) \neq 0$ to showing that a single orbital integral $TO_{\gamma_0}(f)$ does not vanish.

In this section, we analyze this principal element. Its importance is due to the special role the element plays under the twisted endoscopy norm correspondence.

4.3.1 $GL(2n)$ case

Let I_n denote the identity element of GL_n. In the case of even $N = 2n$, the principal element is

$$\gamma_0 = \begin{bmatrix} I_n & \\ & -I_n \end{bmatrix} \in GL(2n).$$

We will see that Theorem 4.2 will imply that in order to that our desired automorphic representation exists, it is enough to prove nonvanishing results for orbital integrals on the particular element γ_0.

While it seems innocent at first glance, γ_0 is quite special and satisfies a number of critical properties. For any positive integer m, we write the "identity" antidiagonal matrix (a.k.a. exchange matrix) of size m as

$$J_m = \begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & \dots & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix} \in GL_m,$$

and follow the convention that $J = J_{2n}$.

Proposition 4.4.

(i) The element γ_0 is, up to θ-conjugation, the unique elliptic θ-semisimple element of $G(\boldsymbol{Q})$ such that $\gamma_0\theta(\gamma_0) = -1$. This corresponds to having central (i.e. trivial) norm.

(ii) Its twisted centralizer is the symplectic subgroup of G with respect to $\gamma_0 J$.

(iii) The stable θ-conjugacy class of γ_0 coincides with its θ-conjugacy class.

Proof. We have $\gamma_0\theta(\gamma_0) = -I_N$ by an easy matrix multiplication calculation, and applying the norm correspondence (cf. §3.11) gives us the desired result.

The twisted centralizer of γ_0 is the group

$$\begin{aligned} I_{\gamma_0} &= \{g \in G \mid g^{-1}\gamma_0\theta(g) = \gamma_0\} \\ &= \{g \in G \mid g^{-1}\gamma_0 J\, {}^t g^{-1} J = \gamma_0\} \\ &= \{g \in G \mid g^{-1}\gamma_0 J\, {}^t g^{-1} = \gamma_0 J\} \\ &= \{g \in G \mid g\gamma_0 J\, {}^t g = \gamma_0 J\}, \end{aligned}$$

and since

$$\gamma_0 J = \begin{bmatrix} & J_n \\ -J_n & \end{bmatrix}$$

is a $2n \times 2n$ nonsingular skew-symmetric matrix, I_{γ_0} is the set of symplectic matrices Sp_{2n} with respect to the form corresponding to $\gamma_0 J$. Thus, γ_0 is elliptic θ-semisimple.

The invertible antisymmetric matrices are all congruent to $\gamma_0 J$ in $GL_{2n}(F)$ (§3.5.2). The θ-conjugacy class of γ_0 coincides exactly with the set of elements γ such that $\gamma\theta(\gamma) = -1$.

For the same reason, the stable θ-conjugacy class of γ_0 consists solely of its θ-conjugacy class (or directly, $H^1(F, Sp_{2n}) = 0$, by [Ste65, Thm. 1.8], for example). □

4.3.2 $GL(2n+1)$ case

We now establish the analogous results in the odd case. Here the principal element is

$$\gamma_0 = \pm \begin{bmatrix} -1 & & \\ & 0 & I_n \\ & I_n & 0 \end{bmatrix} \in GL(2n+1), \tag{4.2}$$

where the sign in the front is taken so that $\det(\gamma_0) = 1$, and I_n denotes the $n \times n$ identity matrix.

Proposition 4.5.

(i) The element γ_0 is, up to θ-conjugation, the unique elliptic θ-semisimple element of $G(F)$ such that $\gamma_0\theta(\gamma_0) = 1$.

(ii) Its twisted centralizer is the orthogonal subgroup of G.

(iii) The stable θ-conjugacy class of γ_0 coincides with its θ-conjugacy class.

Proof. We have $\gamma_0\theta(\gamma_0) = I_N$, and applying the norm correspondence (cf. §3.11) gives us the desired result.

The twisted centralizer of γ_0 is the group

$$\begin{aligned} I_{\gamma_0} &= \{g \in G \mid g^{-1}\gamma_0\theta(g) = \gamma_0\} \\ &= \{g \in G \mid g^{-1}\gamma_0 J\, {}^t g^{-1} J = \gamma_0\} \\ &= \{g \in G \mid g^{-1}\gamma_0 J\, {}^t g^{-1} = \gamma_0 J\} \\ &= \{g \in G \mid g\gamma_0 J\, {}^t g = \gamma_0 J\}, \\ &= \{g \in G \mid g\, {}^t g = I_N\}. \end{aligned}$$

Thus, γ_0 is elliptic θ-semisimple.

The θ-conjugacy class of γ_0 coincides exactly with the set of elements γ such that $\gamma\theta(\gamma) = 1$.

For the same reason, the stable θ-conjugacy class of γ_0 is reduced to its θ-conjugacy class (or directly, $H^1(\mathbf{Q}, SO_{2n+1}) = 0$). □

Chapter 5

Twisted Pseudocoefficients and their Properties at Archimedean Places

5.1 Existence of the Twisted Pseudocoefficient f_π

For this section, let $G = G(\mathbf{R})$. The representation π (4.1) of G remains fixed. This representation π is θ-discrete, that is, isolated among the tempered θ-invariant representations of G.

Proposition 5.1. *For π as in (4.1), there exists a function $f_\pi \in C_c^\infty(G)$ that is K_∞-finite for a maximal compact subgroup $K_\infty \subset G$ such that*

$$\mathrm{Tr}(\pi(f_\pi)A) = 1 \tag{5.1}$$

and

$$\mathrm{Tr}(\rho(f_\pi)A_\rho) = 0 \tag{5.2}$$

for all irreducible tempered θ-invariant $\rho \neq \pi$ of G, where A is the intertwining operator between π and $\pi \circ \theta$, normalized with respect to Theorem 4.3, and A_ρ is a nonzero intertwining operator between ρ and $\rho \circ \theta$.

Proof. Such an f_π exists by the twisted trace Paley–Wiener theorem of Mezo [Mez04]. □

5.2 Twisted Orbits and Twisted Orbital Integrals

Let $\gamma \in G$ be a θ-semisimple element (cf. Prop. 3.1). Its **twisted centralizer** $I = I_\gamma$ is the neutral component of

$$I' = I'_\gamma = \{g \in G : g^{-1}\gamma g^\theta = \gamma\}$$

and is thus reductive. We consider the twisted orbital integral (for arbitrary Haar measures dg and di)

$$TO_\gamma(f) = \int_{I\backslash G} f(g^{-1}\gamma g^\theta)\,\frac{dg}{di}$$

for $f \in C_c^\infty(G)$. If γ is strongly regular, then $I = I'$ is a torus.

Let $P = MN \subset G$ be a θ-stable parabolic and let $\gamma \in M$ be a strongly θ-regular element. Then γ has a similar property relative to M, and its twisted centralizer is a torus of M. If $f \in C_c^\infty(G)$, let

$$\overline{f}(x) = \int_{K_\infty} f(k^{-1}xk^\theta)\,dk$$

(for the normalized Haar measure). Then

$$TO_\gamma(f) = \int_{I\backslash MN} \overline{f}(n^{-1}m^{-1}\gamma m^\theta n^\theta)\,\frac{dm\,dn}{di}.$$

If $h \in C_c^\infty(P)$ and $m \in M$,

$$\int_N h(n^{-1}mn^\theta m^{-1})\,dn = D(m)^{-1}\int_N h(n)\,dn,$$

where $D(m) = |\det(1 - \mathrm{Ad}(m)\circ\theta)|_{\mathfrak{n}}$ and $\mathfrak{n} = \mathrm{Lie}(N)$; $D(m)$ is nonzero if m is θ-regular. Thus

$$TO_\gamma(f) = D(\gamma)^{-1}\int_{I\backslash M} \overline{f}^{(P)}(m^{-1}\gamma m^\theta)\,\frac{dm}{di} = D(\gamma)^{-1}TO_\gamma^M(\overline{f}^{(P)}) \tag{5.3}$$

where the twisted orbital integral is taken in M and $f^{(P)}$ is defined according to Harish-Chandra by

$$f^{(P)}(m) = \int_N f(nm)\, dn.$$

The following lemma is a twisted analogue of the fact that any regular semisimple element is an elliptic element of some Levi subgroup.

Lemma 5.2. *Let γ be a non-elliptic strongly θ-regular element of $G(\boldsymbol{R})$. Then γ is θ-conjugate to a (strongly θ-regular) element in the Levi component M of a θ-stable proper parabolic subgroup of $G(\boldsymbol{R})$.*

Proof. In the even case, this lemma is proven in [CC09, Lem. 2.8]. It remains to establish it in the odd case.

If $G = GL_1(\mathbf{R})$, there is nothing to prove, so assume that $G = GL_{2n+1}(\mathbf{R})$ where $n > 0$.

The element is $\delta = \gamma^\theta\gamma$. By our hypothesis, this is a regular element of $G(\mathbf{R})$ whose set of (complex, distinct) eigenvalues is self-dual; one of the eigenvalues must be 1. There is at least one eigenvalue $\lambda \in \mathbf{C}^\times$ that is not of modulus 1. Setting

$$i = i(\lambda) := \begin{cases} 1, & \lambda \in \mathbf{R} \\ 2, & \text{otherwise,} \end{cases}$$

consider the standard upper parabolic of type $(i, 2n-2i, i, 1)$ of G; it is θ-stable. Up to conjugacy of δ (and thus up to θ-conjugacy of γ), we can assume that δ is an element of the standard Levi subgroup M of this parabolic of the form

$$\delta = \begin{bmatrix} \lambda & & & \\ & * & & \\ & & \lambda^{-1} & \\ & & & 1 \end{bmatrix}$$

(where if $\lambda \in \mathbf{C}$ and thus $i = 2$, we choose an embedding $\mathbf{C} \hookrightarrow M_2(\mathbf{R})$). Since $\delta^\theta \cdot \gamma = \gamma\delta$ and δ is regular, we can assume that $\gamma \in M$. □

5.3 Twisted Orbital Integrals of f_π

Lemma 5.3. *If γ is strongly θ-regular and non-elliptic,*

$$TO_\gamma(f_\pi) = 0.$$

Proof. By (5.3), this orbital integral is calculated in M, where $P = MN$ is a proper θ-stable parabolic and $\gamma \in M$. By [KR00] and [Mez04], a function h on M has vanishing twisted orbital integrals if $\mathrm{Tr}(\pi_M(h)A) = 0$ for all tempered θ-stable representations π_M of M, where $A \neq 0$ is an intertwining operator between π_M and $\pi_M \circ \theta$. For $h = \overline{f}^{(P)}$, the (twisted) Harish-Chandra lemma gives

$$\mathrm{Tr}(\pi_M(h)A) = \mathrm{Tr}(\pi_G(f)A_G),$$

where π_G is induced from π_M and A_G is is the induced intertwining operator. However, the right side of the equation vanishes by (5.2). □

Definition 5.4. We say that a θ-semisimple element $\gamma \in G$ is **θ-elliptic** if the split component of its twisted centralizer is just the neutral component.

In the odd case, the element $\delta = \gamma^\theta\gamma$ of $G(\mathbf{R})$ is conjugate to a diagonal element of $G(\mathbf{C})$ of the form

$$(x_1, \ldots, x_n, , x_n^{-1}, \ldots, x_1^{-1}, 1).$$

Therefore, it defines a conjugacy class $\mathcal{N}(\gamma)$ in $Sp(2n, \mathbf{C})$ by its spectrum

$$\Lambda(\mathcal{N}\gamma) = \{x_1, \ldots, x_n, x_n^{-1}, \ldots, x_1^{-1}\}.$$

In the even case, the element $\delta = \gamma^\theta\gamma$ of $G(\mathbf{R})$ is conjugate to a diagonal element of $G(\mathbf{C})$ of the form

$$(x_1, \ldots, x_n, x_n^{-1}, \ldots, x_1^{-1}).$$

Therefore, it defines a conjugacy class $\mathcal{N}\gamma$ in $SO(2n+1, \mathbf{C})$ by its spectrum

$$\Lambda(\mathcal{N}\gamma) = \{-x_1, \ldots, -x^{-1}\} \cup \{1\}.$$

Lemma 5.5. *The element $\mathcal{N}\gamma$ (always conjugate to an element of the $H(\boldsymbol{R})$) is conjugate to an element (which is unique up to conjugation) in the compact $H_c(\boldsymbol{R})$ if and only if γ is θ-elliptic.*

Proof. Set $\delta = \gamma^\theta\gamma$. Conjugation by $\theta\gamma$ induces an involution on G_δ (a Levi subgroup of G) such that the subgroup of fixed points is the twisted centralizer G_γ of γ. The structure theorem for anti-involutions of complex semisimple algebras implies that the center Z of G_γ coincides with the subgroup of $\theta\gamma$-invariants of the center Z' of G_δ. Otherwise, $Z' = \mathbf{G}_m D$, where D is the Zariski-closure of a subgroup generated by δ that is central and fixed by $\theta\gamma$. Since $Z = Z'^{\gamma\theta} = \mathbf{G}_m^{\theta\gamma} D = \{\pm 1\}D$, we obtain our result. □

The following is the θ-twisted analogue of the fact that orbital integrals of pseudocoefficients of square-integrable representations of general linear groups vanish at non-elliptic semisimple elements (e.g. [HT01, Lem. I.3.1]). The standard proof of this appeals to the Shalika germ expansion, Harish-Chandra homogeneity, and knowledge that the germ is nonzero at the trivial unipotent conjugacy class. That is the spirit of the argument behind the following proof, which is slightly more elementary.

Lemma 5.6. *If $\gamma \in G(\boldsymbol{R})$ is θ-semisimple but not θ-elliptic, then $TO_\gamma(f_\pi) = 0$.*

Proof. Let I denote the twisted centralizer of γ. For any function $f \in C_c^\infty(G)$, we can find a function $h \in C_c^\infty(I)$ such that in a neighborhood of 1,

$$TO_{x\gamma}(f) = O_h^I(h),$$

for all $x \in I$, where the right element is an ordinary orbital integral in I. If $x\gamma$ is strongly regular, then x is regular in I; for x very close to 1, the neutral component of the centralizer of x and the twisted centralizer of $x\gamma$ coincide [Lab99, Cor 3.1.5.]. Since the split component of the center of I is nontrivial, the orbital integrals $O^I_x(h)$ are thus zero (in a neighborhood of 0) on the regular elements, and so, under a suitable normalization of measures, we have $h(1) = TO_\gamma(f)$, which vanishes. □

For $\gamma \in G$ that are θ-semisimple, the stable twisted orbital integral $STO_\gamma(f)$ of f at γ is defined by Labesse [Lab99, §2.7].

We can now prove the main result of this chapter.

Theorem 5.7. *Let γ be a θ-semisimple element of $G(\boldsymbol{R})$.*

(i) If γ is not θ-elliptic, then $TO_\gamma(f_\pi) = STO_\gamma(f_\pi) = 0$.

(ii) Let γ be θ-elliptic and $I = I_\gamma$. For a suitable choice of positive measures on G and I_γ, we have

$$TO_\gamma(f_\pi) = e(\gamma)\Theta_{\pi_H}(\mathcal{N}\gamma),$$

where π_H is the finite-dimensional representation of $H_c(\boldsymbol{R})$ associated with π, and $e(\gamma) = \pm 1$ is a sign independent of π. In particular, the orbital integrals are stable.

Proof. For the normalization of measures, we follow the conventions of [Lab99, §A.1]. We note in particular that once γ is fixed, such a normalization doesn't depend on π.

Part (i) follows from Lemma 5.6. We devote the rest of our efforts in this section to proving (ii).

First, consider the case where π_0 is the unique tempered representation of $G(\mathbf{R})$ lying in the cohomology with trivial coefficients. For this, Labesse has given a construction of f_{π_0} using cohomology, and he uses this to calculate the twisted orbital integrals [Lab99, Thm. A.1.1], which implies the theorem in this case. Here, the corresponding representation $\pi_{0,H}$ of H is just the trivial representation.

We now proceed with the general case. Given such a representation π, let (ρ, V) be the algebraic (θ-stable) representation such that the cohomology of π with coefficients in V is nonzero; we similarly define (ρ_H, V_H) for the corresponding representation on H, noting that ρ_H is identified with π_H in this case. By the Borel–Weil theorem (e.g. [Kna01, Thm. 5.29]), we can realize V in the cohomology of $G(\mathbf{C})/B(\mathbf{C})$ with coefficients in the line bundles $\mathbb{L}_m$ where $m = m(\pi)$. Note that m is invariant under θ, and so we can thus obtain a natural extension to a representation of $G^+(\mathbf{C})$. For a θ-regular element $g \in G(\mathbf{C})$, we can calculate $\mathrm{Tr}(g \times \theta \mid V)$ using the Atiyah–Bott fixed point theorem [AB67]: the fixed points are parametrized by the centralizer of θ in the Weyl group $W(G(\mathbf{C})) \cong \mathfrak{S}_{2n+1}$, which is isomorphic to W_H. Therefore,

$$\mathrm{Tr}(g \times \theta \mid V) = \mathrm{Tr}(\mathcal{N}g \mid V_H). \tag{5.4}$$

Let $\Theta_{\rho,\theta}$ be the twisted character of ρ (for the choice of intertwining operator), and let

$$g_\pi = \Theta_{\rho,\theta} f_0,$$

where $f_0 = f_{\pi_0}$.

We want to show that g_π has the same twisted orbital integrals as f_π. To do this, it is enough to show that for all θ-stable tempered representations τ (and associated intertwining operators A_θ)

$$\mathrm{Tr}(\tau(f_\pi)A_\theta) = \mathrm{Tr}(\tau(g_\pi)A_\theta) \tag{5.5}$$

by the density theorem of Kottwitz and Rogawski [KR00].

Recall that such a representation τ is **θ-discrete** if it is not induced from a θ-stable representation from a θ-stable proper parabolic. In this case, the twisted character is supported on the non-θ-elliptic elements. Thus, if τ is θ-discrete, (5.5) implies that the corresponding twisted orbital integrals vanish.

Lemma 5.8. *The θ-discrete representations are either of the form*

$$\tau = \mathrm{Ind}(\delta_1, \ldots, \delta_n, \epsilon), \tag{5.6}$$

where δ_i is a representation of $GL(2, \boldsymbol{R})$ in the discrete series associated to the representation of

$W_{\boldsymbol{R}}$ induced from a character $z \mapsto z^{p_i}(\overline{z})^{-p_i}$ of $W_{\boldsymbol{C}} = \boldsymbol{C}^{\times}$, with $p_i \in \frac{1}{2}\boldsymbol{Z}$ and the p_i's being distinct; or else of the form

$$\tau = \mathrm{Ind}(\delta_1, \ldots, \delta_{n-1}, \delta_n) \tag{5.7}$$

with the δ_i's as before, and ϵ a character of order 2 of $\boldsymbol{R}^{\times}$.

If the p_i's belong to $\frac{1}{2} + \boldsymbol{Z}$ (and thus τ is cohomological), we have

$$\mathrm{Tr}(\tau(f_\pi)A_\theta^\tau) = \delta(\tau, \pi),$$

where δ is the Kronecker delta and f_π is normalized by A_0^π. Furthermore

$$\mathrm{Tr}(\tau(g_\pi)A_\theta^\tau) = \int_G \Theta_{\tau,\theta}(g)\Theta_{\rho,\theta}(g)f_0(g)\, dg. \tag{5.8}$$

However, the twisted orbital integrals of f_0 are killed for g that are not θ-elliptic. If g is θ-elliptic regular (and thus has twisted centralizer $U(1)^n = T$) and if the measure on T is suitably normalized, we have [Lab99, Thm. A. 1.1.]

$$TO_g(f_0) = 1$$

(f_0 is, of course, the pseudocoefficient associated to a measure dg that is defining the orbital integral). By (5.8), we have

$$\int_G \Theta_{\tau,\theta}(g)\Theta_{\rho,\theta}(g)f_0(g)\, dg = \frac{1}{|W|}\int_T \left\{ \sum_{\mathcal{N}\delta=\gamma} \Theta_{\tau,\theta}(\delta)\Theta_{\rho,\theta}(\delta)\Delta(\gamma) \right\} d\gamma,$$

where $\Delta(\gamma)$ is a Weyl denominator (for the Weyl integration formula relative to twisted conjugation) that we check is equal up to a factor 2^n (the number δ of norm γ) to the Weyl denominator for H. Then Theorem 4.3 together with the identity (5.4) imply that

$$\mathrm{Tr}(\tau(g_\pi)A_\theta^\tau) = \delta(\tau, \pi)$$

by the orthogonality relations on $\widehat{H}$.

Finally, consider the other representation τ of type (5.6) or (5.7). We have

$$\begin{aligned}
\mathrm{Tr}(\tau(g_\pi)A_\theta^\tau) &= \mathrm{Tr}\left(\int_G \tau(x)g_\pi(x)A_\theta^\tau \, dx\right) \\
&= \mathrm{Tr}\left(\int_G \tau(x)f_0(x)\,\mathrm{Tr}(\rho(x)A_\theta^\rho)A_\theta^\tau \, dx\right) \\
&= \mathrm{Tr}\left(\int_G f_0(x)(\tau(x)\otimes\rho(x))A_\theta^\tau \otimes A_\theta^\rho \, dx\right).
\end{aligned}$$

If τ is of the types above and non-cohomological, its infinitesimal character λ (that is, the sum of the p_i's and $-p_i$'s with $p = 0$ for the characters 1 and ϵ) does not belong to $(\frac{1}{2} + \mathbf{Z})^{2n+1}$. Then the infinitesimal characters of subquotients of $\tau \otimes \rho$ are of the form $\lambda + \mu$ where μ is an (integral) weight of ρ. It thus has the same property; the trace of f_0 in $\tau \otimes \rho$ is thus zero, which completes our proof. $\square$

Chapter 6

Twisted Pseudocoefficients and their Properties at Finite Places

In this chapter, we explain the analytic results about the existence of the twisted pseudocoefficients in question and the vanishing and nonvanishing of their orbital integrals at the principal element (§4.3).

For p-adic groups, a primary tool for establishing the existence of twisted pseudocoefficients is Rogawski's twisted trace Paley–Wiener theorem [Rog88]. While this abstract result holds in great generality, since pseudocoefficients are highly non-unique, it is usually desirable to find such a function via certain "geometric" constructions, since the analytic properties of functions constructed in such a way are often "good," or at least more amenable to study. The ideas that come to mind should be analogues of results like the Borel–Weil theorem that we used in the archimedean setting or the Borel–Weil–Bott theorem for constructing holomorphic representations of a given complex semisimple group. Fortunately for us, for the discrete representations that we aim to prescribe at finite places, such geometric constructions exist.

However, a natural generalization of the theorem would be to not prescribe a specific (inertia class of) a discrete representation, but to, say, only allow the the local representation to lie in a specified Bernstein component or ideally to trace out a specific unitary representation without any restrictions, but for this, it seems as if one cannot yield to these geometric methods and must instead deal specifically with the analytic difficulties that arise.

Remark 6.1. Many of proofs in the general theory of pseudocoefficients at finite places can be simplified by assuming that we are only concerned with tracing out the desired representation among the class of tempered representations at each place. Indeed, the general convention for pseudocoefficients is to assume they trace out the representation in the tempered spectrum, due to the concomitant analytic complications that arise in non-tempered contexts and the belief in the validity of the Ramanujan conjecture for $GL(n)$. Since we are ultimately concerned with constructing cohomological self-dual automorphic representations on GL_n over a totally real field, purity for these representations is known (by looking at the corresponding Galois representations), so at the finite places of any such automorphic representations over totally real field, the local components are all tempered (the result is due to a number of people, but the general statement of the theorem in our setting can be found in, e.g. [Clo13] for unramified places, and [Car12] at ramified places). So strictly speaking, in the context of the proof of our theorem, we could assume this and remove the hypothesis of being "essentially tempered" in the statements of some of the results. However, to ensure that our results here are useful by themselves and to avoid possible confusion or imprecision caused by making such a large implicit assumption throughout this section, we do not wish to do this and state the results in full.

6.1 For Steinberg Representations in the θ-twisted Setting.

The existence of pseudocoefficients for Steinberg representations in the untwisted case is due to Kottwitz [Kot88, §2], using the theory of Bruhat–Tits buildings. The construction was extended to general connected reductive groups that are twisted by an F-rational automorphism of finite order by Chenevier and Clozel [CC09, §3.4], which, of course, includes the case of our θ-twisted group G^+. We summarize the properties needed for our proof.

Let F be a non-archimedean field of characteristic zero. Fix a Haar measure on $G^+(F)$. Let B be a minimal parabolic of G defined over F. Let I_B be the space of smooth complex-valued functions on $B(F)\backslash G(F)$. It is a space of a representation of $G(F)$ under right translation and its unique irreducible quotient is the Steinberg representation of $G(F)$, which we denote by St.

The main result on the existence of twisted pseudocoefficients for the Steinberg representation is the following statement.

Proposition 6.2.

(i) There exists a function $f_{EP} \in C_c^\infty(G(F))$ *that is a pseudo-coefficient of the Steinberg representation* St*, that is,*

$$\mathrm{Tr}(\mathrm{St}(f_{EP})) = 1$$

and if π *is irreducible and essentially tempered such that* $\pi|_{G(F)} \neq \mathrm{St}$*, then* $\mathrm{Tr}(\pi(f_{EP})) = 0$*.*

(ii) Let $\gamma \in \theta G(F)$ *be a semisimple element and let* I_γ *be the neutral component of the centralizer of* γ *in* G*. Choose a* $G(F)$*-invariant measure* $\overline{\mu}$ *on* $I_\gamma(F)\backslash G(F)$*. Then the "twisted" orbital integral*

$$O_\gamma(f_{EP}) := \int_{I_\gamma(F)\backslash G(F)} f_{EP}(g^{-1}\gamma g)\,\overline{\mu}$$

is nonzero if and only if $I_\gamma(F)$ *has compact center.*

Proof. This is a consequence of the general result in [CC09, Prop. 3.8]. □

Note that (ii) is an extremely strong vanishing condition for orbital integrals of the pseudocoefficient. It is hard to emphasize how dramatically this simplifies the ultimate analysis of the geometric side of the trace formula. It it all the more surprising, because it is due to the Steinberg representations not being integrable that causes the standard method of using Poincaré series to construct automorphic representations to fail for our specific problem.

Remark 6.3. In the untwisted case, the functions f_{EP} were introduced by Kottwitz [Kot88, §2], under the name of "Euler–Poincaré" functions, whence the notation. Some of the results above (in the twisted setting) can also be deduced from the work Borel, Labesse, and Schwermer [BLS96].

While we could have used Rogawski's Paley–Wiener theorem to prove Prop. 6.2 (i), the advantage of using the (generalized) Euler–Poincaré functions is that they provide an explicit function f_{EP} in terms of the Bruhat–Tits building of G, which allows for an simpler proof of Prop. 6.2 (ii).

Remark 6.4. Another advantage of using Euler–Poincaré functions for when π is Steinberg is that in that case, the pseudocoefficient $f_\pi = f_{EP}$ is "very cuspidal" (in the sense of Laumon) and in particular, the orbital integral of is nonzero *only on* elliptic semisimple elements. A priori, orbital integrals of the pseudocoefficients of discrete representations can be nonzero outside of the regular semisimple elements.

6.2 For other Discrete Representations in the θ-twisted Setting

We now show that similar nonvanishing results hold for functions f_v that trace out other discrete representations. Let F be a p-adic field.

The most important case is that of supercuspidal representations. Recall that in the untwisted setting, if π_v is a supercuspidal representation, we can simply take f_v to be a matrix coefficient of π_v such that $f_v(1) \neq 0$. Such a function is also very cuspidal, in the sense of Laumon, and in particular, satisfies strong vanishing properties for its orbital integrals on non-elliptic orbits. More precisely, it satisfies the the following condition: for any proper parabolic $P = MN$ and a special maximal compact subgroup K in good relative position with respect to P,

$$f_v^P(m) =:= \delta_P(m)^{1/2} \int_{N(F)} \int_K f_v(k^{-1}mnk)\, dk\, dn = 0$$

as a function on $M(F)$. Essentially the idea of this section is just finding results that establish this exact procedure in this case of twisted groups, so if we believe in the existence of the twisted analogue of this result, we can safely proceed with the proof. But so that we nail down all the details and have results that apply to general discrete representations, we carefully extract the necessary results from the literature.

It remains to find a corresponding twisted pseudocoefficient in the twisted setting. Once again, the main result of [Rog88] applies to all θ-discrete representations, so we could appeal to that result to obtain a pseudocoefficient f_p that corresponds to a supercuspidal representation π_p. However, in this setting, we also have an geometric realization of such a function. Namely, an alternative proof of the existence of f_p can be obtained using the Waldspurger's θ-twisted generalization [Wal07] of Schneider–Stuhler pseudocoefficients [SS97], with the additional benefit that they exhibit nice vanishing properties, due to their method of construction.

Unfortunately, the language to do so is different from that of the Chenevier–Clozel result that we use in the last section and also different from that which we used in the archimedean setting, so we must establish a significant amount of notation that we do not use anywhere else.

Let $\widetilde{G}_{reg}$ be the subset of strongly regular elements of $\widetilde{G}$, so if $g \in \widetilde{G}_{reg}$, then the centralizer $Z_G(g)$ is commutative and its neutral component is a torus. Let $\mathbf{A}_g$ be the maximal torus split in such a $Z_G(g)$.

For any $f \in C_c^\infty(\widetilde{G})$ and $\gamma \in \widetilde{G}_{reg}$, we have the orbital integral

$$O_\gamma^G(f) = \Delta(g)^{-1/2} \int_{A_g \backslash G} f(x^{-1}\gamma x)\, dx,$$

where we have a fixed a suitable Haar measure and the modulus $\Delta(g)$ denotes the absolute value of the determinant of $\mathrm{Ad}(g) - 1$ acting on $\mathfrak{g}/\mathfrak{z}_G(g)$.

Proposition 6.5. *(i) For any irreducible θ-twisted supercuspidal representation π of $G(F)$, There exists a function $f_v \in C_c^\infty(G(F))$ that is a pseudocoefficient for π.*

(ii) We have $O_\gamma^G(f_v) = 0$ for all non-elliptic $g \in \widetilde{G}_{reg}$.

(iii) For all $\gamma \in \widetilde{G}_{ell}$, we have

$$\mathrm{Tr}_{\widetilde{G}}\, \pi^+(\gamma) = \Delta(g)^{-1/2} O_\gamma^G(\widetilde{f}_v).$$

Proof. These results correspond to the Corollary of [Wal07, §2.2]. □

We end our discussion of twisted pseudocoefficients at finite places with a general result that says that the restriction to tempered representations is not too restrictive, as the following lemma indicates.

Lemma 6.6. *Let π be a θ-discrete series representation of G with pseudocoefficient f_π. If σ is an irreducible representation of G such that $\operatorname{Tr}\sigma(f_\pi) \neq 0$, then σ and τ have the same supercuspidal support.*

Proof. If $\sigma \ncong \pi$, then σ is non-tempered and can be written as a finite $\mathbf{Z}$-linear combination of induced modules, all of whose irreducible subquotients have the same supercuspidal support. Since the trace of f_π vanishes on any representation induced from a proper parabolic and on any tempered representation different from π, one of these induced modules must be π. Thus, σ must have the same supercuspidal support as π. $\square$

Chapter 7

Proof of the Theorem

The proof of the theorem naturally divides into two cases for GL_N, according to the parity of N, that is,

(i) where $N = 2n + 1$ is odd; or

(ii) where $N = 2n$ is even and the self-dual representations are all of symplectic type.

The main distinction between the two cases is that the endoscopic groups involved are different, and this necessitates a number of modifications at each step of the argument.

Note that if N is odd, all irreducible self-dual representations of GL_N are of orthogonal type, so we do not need to address the "odd, symplectic type" case. In the remaining case—where $N = 2n$ is even and the self-dual representations are all of orthogonal type—a globalizing automorphic representation cannot be constructed using our method, because if an automorphic representation $\Pi = \otimes_v \Pi_v$ of GL_{2n} over a totally real field F is (i) self-dual, (ii) essentially square-integrable in at least one place, and (iii) cohomological at all archimedean places, then for all places v of F, the Langlands parameter of Π_v must preserve a nondegenerate symplectic bilinear form, that is, Π_v must be of symplectic type ([CC09, Thm. F], which was proven assuming a harmonic analysis result that was later proven in [CR10]).

7.1 Statement of the Theorem and the Initial Setup

We finally collect the results and give a proof of our main result. We restate the theorem for our convenience. Let $G = GL_{N,F}$ where $N \geq 1$ and F is a totally real number field of degree d. Let S_∞ denote the archimedean places of F. Write $\mathbf{A} = \mathbf{A}_F$ for its ring of adeles.

Theorem. *Let T be a finite set of pairs (v, π_v) where*

- *v is a finite place of a totally real number field F, and*
- *π_v is an irreducible admissible self-dual essentially discrete representation of $G(F_v)$ (and if n is even, are all of symplectic type).*

Then there exists a cohomological self-dual cuspidal automorphic representation $\Pi = \otimes'_v \Pi_v$ of $G(\boldsymbol{A}_F)$ such that for all $(v, \pi_v) \in T$, we have $\Pi_v \cong \pi_v \otimes \chi_v$, where χ_v is an unramified character of $G(F_v)$.

Let $G^+ = G \rtimes \langle\theta\rangle = G \amalg G\theta$ denote the group G twisted by the involution θ (§3.4.2). Let $A = (\mathbf{R}_+^\times)$ be the topological neutral component of the center of $G(\mathbf{R})$, and equip the homogeneous space $A \cdot G(F) \backslash G(\mathbf{A})$ with a (finite) Haar measure that is right $G(\mathbf{A})$-invariant. The unitary (right) regular representation R of $G(\mathbf{A})$ is given by right-translation on the space of cuspidal functions

$$L^2_{cusp}(A \cdot G(F) \backslash G(\mathbf{A})),$$

which extends to a unitary representation of $G^+(\mathbf{A})$, by letting θ act via the operator $I_\theta(\phi)(x) = \phi(\theta(x))$. If we choose a test function $f = \bigotimes_{\infty \in S_\infty} f_\infty \otimes f^{S_\infty} \in C_c^\infty(G(\mathbf{A}))$ such that each $f_\infty \in C_c^\infty(G(\mathbf{R}))$ is SO_N-finite, then $R(f)I_\theta$ is of trace class and

$$\mathrm{Tr}(R(f)I_\theta) = \sum_\Pi \mathrm{Tr}(R(f)I_\theta, \Pi),$$

where the sum runs over the irreducible self-dual cuspidal automorphic representations of $G(\mathbf{A})$. The traces all depend on a choice of adelic Haar measure $dg_\mathbf{A}$ on $G(\mathbf{A})$ that we fix for the remainder of the argument.

7.2 Choosing the Test Function

We want to choose our test function $f = \otimes_v f_v \in C_c^\infty(G(\mathbf{A}))$ to simplify the analysis of the geometric side of the trace formula as much as possible, while still tracing out our desired automorphic representation. Indeed, we will eventually apply a simple version of the trace formula, by ensuring that, under our hypotheses, local twisted orbital integrals of certain components f_v of f have special vanishing properties.

Let T be the set of pairs (v, π_v) in the hypotheses of our theorem, so v is a finite place of F and π_v is an irreducible self-dual representation of $G(F_v)$ that is (essentially) discrete. Let $T_{St} \subseteq T$ be the subset of Steinberg representations at the prescribed places and let $T' \subseteq T$ be the other ones, so $T = T_{St} \amalg T'$. We define the test function to be

$$f = \bigotimes_{\infty \in S_\infty} f_\infty \otimes \bigotimes_{v \in T} f_v \otimes f^{\infty,T}, \tag{7.1}$$

where

- f_∞ is the twisted pseudocoefficient of a cohomological θ-discrete series representation (cf. §4.2, Prop. 5.1);
- if $(v, \pi_v) \in T_{St}$, then f_v is taken to be an Euler–Poincaré function fixed by the automorphism θ of $G(F_v)$ (cf. Prop. 6.2(i));
- if $(v, \pi_v) \in T'$, then f_v is the twisted Schneider–Stuhler coefficient corresponding to π_v (cf. Prop. 6.5);
- $f^{\infty,T}$ is the characteristic function of $\prod_{v \notin T \cup S_\infty} GL_{2n}((\mathcal{O}_F)_v)$.

7.3 Analysis of the Geometric Side of the Trace Formula

Recall that a θ-semisimple element $\gamma \in G(F)$ is **elliptic** if the split component of the center of the twisted centralizer is trivial. Write $\{G(F)\}_{ell}$ for the set of θ-conjugacy classes of θ-semisimple elliptic elements. For such an element $\gamma \in G(F)$, we choose an adelic Haar measure $di_{\mathbf{A}}$ on the twisted centralizer $I_\gamma(\mathbf{A})$, denote the corresponding volume by

$$v_\gamma = \mu(I_\gamma(F)\backslash I_\gamma(\mathbf{A})) > 0$$

and set

$$TO_\gamma(f) := \int_{I_\gamma(\mathbf{A})\backslash G(\mathbf{A})} f(g^{-1}\gamma\theta(g))\, di_{\mathbf{A}}\backslash dg_{\mathbf{A}}.$$

We will also consider the local versions of the above twisted orbital integrals, replacing the adeles $\mathbf{A}$ with the local field F_v.

The Arthur–Selberg trace formula can be applied to any test function, but analyzing the terms that arise is complicated in general. However, these are a number of "simple" trace formulas that can be derived from the general form, by restricting the class of test functions to which one can apply the trace formula. Since the orbital integrals of the pseudocoefficients we have chosen have very specific and strong vanishing properties, we can apply one of these simple trace formulas, due to Arthur.

Proposition 7.1. *For f of the form (7.1), the geometric side of the trace formula is*

$$\mathrm{Tr}(R(f)I_\theta) = \sum_{\gamma \in \{G(\boldsymbol{Q})\}_{ell}} v_\gamma TO_\gamma(f),$$

noting that the sum runs over the finite subset of classes that only depend on the compact set of $G(\boldsymbol{A})$ that contain the support of f.

Proof. We apply the Arthur's invariant trace formula to the connected component $G\theta$ [Art88b]. These results rely on two hypotheses [Art88a, p.330] [Art88b, p.528] (a) a Galois cohomology argument and (b) the validity of a Paley–Wiener theorem for $G(\mathbf{R})\theta$. These have since been resolved: hypothesis (a) is now proven in full generality by Kottwitz and Rogawski [KR00] and hypothesis (b) was proven in our setting by Mezo [Mez04].

Suppose that we have a given Steinberg representation at the place v. Since the twisted Euler–

Poincaré function f_v vanishes outside of the elliptic (semisimple) orbits (Prop. 6.2 (ii)), f is cuspidal at v, in the sense of Arthur. The pseudocoefficients for the other discrete local representations at finite places exhibit the same property (Prop. 6.5 (ii)). Since f_∞ also vanishes outside of the elliptic orbits, f is also cuspidal at ∞. Thus, since we have cuspidal functions in at least two places, the formula of [Art88b, Cor. 7.4] applies. This corollary identifies the terms on the right-hand side of the trace of $R(g)I_\theta$ (as a representation of $G^+(\mathbf{A})$ on $L^2_{disc}(A_G G(F)\backslash G(\mathbf{A}))$. There, Arthur considers an expansion of this trace, where a sum runs over the possible norms t of the infinitesimal characters of Π_∞. Since f_∞ only traces out representations with the same infinitesimal character, only one of these terms in the sum is relevant.

Finally, it remains to show that if a discrete irreducible representation Π of $G^+(\mathbf{A})$ is not cuspidal, then

$$\mathrm{Tr}(\Pi(f)I_\theta) = 0.$$

If we have a supercuspidal representation, then our desired automorphic representation (provided it exists) must lie in the cuspidal spectrum and we are done. Otherwise, since we can assume that the infinitesimal character of the θ-discrete series attached to f_∞ is sufficiently regular, the statement follows. Alternatively, if T does not contain any supercuspidal representations and we do not mind losing control at one auxiliary finite place v (e.g. if we only cared about proving the stated result in our main theorem, instead of having control of the ramification of our automorphic representation at all places), we can simply impose a supercuspidal representation at v to ensure that the constructed automorphic representation is cuspidal. □

Recall that a function f_∞ (5.1) depends on the choice of a cohomological θ-discrete series of $G(\mathbf{R})$ (4.1), which is indexed by an irreducible representation of the compact endoscopic group $H(\mathbf{R})$. Choose a maximal torus $T \subset H(\mathbf{R})$ and let V_λ denote the irreducible representation of highest weight $\lambda \in X^*(T)$. For all such λ, fix a pseudocoefficient $f_\infty = f_\lambda$ of the associated θ-discrete series π_λ such that the support of all the f_∞'s for varying λ are contained in the support of a single compact set of $G(\mathbf{R})$ (such a thing is possible by appealing to the work of [CD90, Thm. 1] or [Lab91]).

For the $|S_\infty|$-tuple of highest weights $\vec{\lambda} = (\lambda_1, \dots, \lambda_{|S_\infty|})$, let $f^{\vec{\lambda}}$ denote the function of the form (7.1) where

$$\bigotimes_{\infty \in S_\infty} f_\infty := f_{\lambda_1} \otimes f_{\lambda_2} \otimes \cdots \otimes f_{\lambda_{|S_\infty|}}.$$

Suppose that there exists a cuspidal representation Π such that

$$\mathrm{Tr}(\Pi(f^{\vec{\lambda}})I_\theta) \neq 0.$$

Then for each $\infty \in S_\infty$, the representation Π_∞ is generic (in the sense that it has a Whittaker model) and has the same infinitesimal character as π_{λ_i}, so $\Pi_\infty \cong \pi_{\lambda_i}$. To prove our theorem, it is enough to show that we can choose a $\lambda \in X^*(T)$ for each archimedean place such that the corresponding terms on the right-hand side ("geometric terms") of Prop. 7.1 are nonzero.

These geometric terms are supported in a finite set

$$\Sigma \subset \{G(\mathbf{Q})\}_{ell}$$

that is independent of λ. We want to show that as λ tends to infinity away from the walls of the Weyl chambers, the geometric terms are only supported on the principal element γ_0 (cf. §4.3).

We recall the key properties of the principal element γ_0, proven in §4.3.

Lemma 7.2. *Up to θ-conjugation, the element γ_0 is the unique θ-semisimple elliptic element of $G(F)$ such that $\gamma_0\theta(\gamma_0)$ corresponds to the central element of the endoscopic group H under the twisted endoscopy norm map. The twisted centralizer of γ_0 is (a form of) a group whose base change to $\boldsymbol{C}$ is the dual group of H. The stable θ-conjugacy class of γ_0 coincides with its θ-conjugacy class.*

We first confirm the nonvanishing of the twisted orbital integral of the test function f, at least for the components $f^{\vec{\lambda},\infty}$ away from infinity that remain fixed as we vary the components f^{λ}_∞ at the archimedean places.

Lemma 7.3. *The twisted orbital integral $TO_{\gamma_0}(f^{\vec{\lambda},\infty})$ is a nonzero constant.*

Proof. We have the decomposition

$$TO_{\gamma_0}(f^{\lambda,\infty}) = \prod_{v\in T} TO_{\gamma_0}(f_v)\cdot TO_{\gamma_0}(f^{\infty,T}),$$

so it is enough to show that the local orbital integrals $TO_{\gamma_0}(f_v)$ do not vanish. Since $f^{\infty,T}$ is the characteristic function of $\prod_{v\notin T\cup S_\infty} GL_{2k}((\mathcal{O}_F)_v)$, we have $\prod_{v\notin T\cup S_\infty} TO_{\gamma_0}(f_v)\neq 0$. It remains to show that

$$\prod_{v\in T=T_{St}\cup T'} TO_{\gamma_0}(f_v) = \prod_{v\in T_{St}} TO_{\gamma_0}(f_v)\prod_{v\in T'} TO_{\gamma_0}(f_v)\neq 0.$$

By Proposition 6.2, $TO_{\gamma_0}(f_v)\neq 0$ for places v such that $(v,\pi_v)\in T_{St}$. By Lemma 6.5, $TO_{\gamma_0}(f_v)\neq 0$ for places v such that $(v,\pi_v)\in T'$. This concludes the proof. ☐

Given a θ-semisimple element $\gamma\in G(F)$, we can view it as an element of $G(\mathbf{R})$ and consider its norm $\mathcal{N}\gamma\in H(\mathbf{R})$ (§3.11). By definition, $\mathcal{N}\gamma$ is an element whose conjugacy class in $H(\mathbf{R})$ only depends on the θ-conjugacy class of γ in $G(\mathbf{R})$. By Theorem 5.7, we know that for a suitable normalization of measures, we have for all λ,

$$TO_\gamma(f_\lambda) = \pm\operatorname{Tr}(\mathcal{N}\gamma, V_\lambda). \tag{7.2}$$

Now, we have

$$|TO_{\gamma_0}(f^{\vec{\lambda}})| = c\cdot\prod_{\lambda\in\vec{\lambda}}\dim(V_\lambda)\neq 0$$

for a certain constant $c>0$ by (7.2) and Lemma 7.3. By Proposition 4.2, as a λ tends towards infinity in $X^*(T)$ away from the walls, we have

$$\frac{TO_\gamma(f^\lambda)}{\dim(V_\lambda)}\to 0$$

for all $\gamma\in\Sigma\backslash\{\gamma_0\}$, because the twisted conjugacy class of $\gamma_0\theta(\gamma_0)$ is the unique one of central norm in $H(\mathbf{R})$ by Lemma 7.2.

Hence,

$$|\operatorname{Tr}(R(f^{\vec{\lambda}})I_\theta)| \sim v_{\gamma_0} \cdot c \cdot \prod_{\lambda \in \vec{\lambda}} \dim(V_\lambda) \neq 0$$

as desired. This completes the proof.

Chapter 8

Appendix: Constructing Self-Dual Representations on GL_n via Arthur's Endoscopic Classification

The chapter is independent from all of the previous chapters and is drastically different in scope. Here, we construct self-dual automorphic representations on GL_N over a *general* number field F with prescribed local components by ultimately appealing to Arthur's endoscopic classification of representations of classical (symplectic and special orthogonal) groups [Art13]. In particular, we no longer impose the condition that our base field F be totally real, and do not restrict ourselves to constructing automorphic representations that are cohomological (equivalently, regular algebraic).

However, as mentioned before, Arthur's results are conditional on the stabilization of the twisted trace formula. At a key point in the argument, we will also need to assume a certain globalization theorem for semisimple groups that only seems to be currently known under additional hypotheses.

Our goal in this appendix is to give an outline for a sort of "ideal" way to construct self-dual automorphic representations on GL_n, once certain technical hypotheses are resolved.

8.1 The General Strategy and Setup

We aim to prove a theorem of the following form. Let $G = GL_N$ over a number field F.

Theorem 8.1. *Let S be a finite set of places of F. For every $v \in S$, choose an "allowable" subset $U_v^\wedge$ of the unitary dual of $G(F_v)$. Then there exists a self-dual cuspidal automorphic representation*

$\Pi = \otimes_v \Pi_v$ *of* $G(\boldsymbol{A}_F)$ *such that each local component* $\Pi_v \in U_v^\wedge$ *for all* $v \in S$.

This will turn out, for instance, to imply a precise version of our main globalization theorem above for general number fields, where we can prescribe Π_v to not just lie in an inertia class of the prescribed essentially discrete local component (that is, be an unramified twist of the local component), but to be precisely the prescribed component. It will be evident from the method of proof that many more specific versions of the theorem will follow from slight variations of the argument.

There are three main steps to the strategy.

0. Translate the local conditions on GL_N into those of the semisimple group from which we expect a twisted endoscopic transfer.

1. Construct a corresponding automorphic representation on either $H = Sp_{2n}$ or $H = SO_{2n+1}$ depending on whether N is or odd ($N = 2n+1$) or even ($N = 2n$), respectively.

2. Transfer the automorphic representation to $G = GL_N$.

Step 0 is fairly straightforward, but is guided by the expected instances of endoscopic transfer.

Since we are in the setting of semisimple groups, there are a number of ways to resolve Step 1. For this step, we will use a variation of a theorem of Shin [Shi12] to globalize the representations. For the result over general number fields, it is here that we need to assume a stronger version of a key globalization theorem that does not seem to be currently available in the literature.

For Step 2, we yield to the work of Arthur [Art13], in particular, the existence of twisted endoscopic transfer from H to G.

8.2 Plancherel Measures and Prescribable Subsets

We begin by recalling some general facts about Plancherel measures on p-adic groups. Let $\mathbf{G}$ be a reductive group over a nonarchimedean local field K of characteristic zero. Let $G = \mathbf{G}(K)$ be its set of K-points and let $G^\wedge$ denote the unitary dual of the topological group G, that is, the set of all irreducible unitary representations of G, equipped with the Fell topology.

Let $X(G)$ denote the set of all *unitary, unramified* characters of G. Harish-Chandra proved that there is a natural Borel measure $\widehat{\mu}^{pl}$ on $G^\wedge$, called the **Plancherel measure**, such that for all $\phi \in C_c^\infty(G)$, we have

$$\phi(1) = \int_{G^\wedge} \widehat{\phi}(\pi)\, \widehat{\mu}^{pl}(\pi),$$

where $\widehat{\phi}$ is defined to be

$$\widehat{\phi}(\pi) = \operatorname{Tr} \pi(\phi).$$

for $\pi \in G^\wedge$ (see, e.g. [Wal03]).

Let $\Theta(G)$ denote the Bernstein variety, which is a (generally infinite) disjoint union of complex affine algebraic varieties. Identify $\Theta(G)$ with its $\mathbf{C}$-points, and equip it with the analytic topology. Then the map that assigns each irreducible representation to its supercuspidal support

$$\nu : G^\wedge \to \Theta(G)$$

is continuous [Tad88, Thm. 2.2].

Let $\mathbf{L}$ be a Levi subgroup of $\mathbf{G}$ and let σ be a discrete series of $L = \mathbf{L}(K)$. Let P be a parabolic associated with L. Consider the function on the unitary unramified characters $X(L)$ of L defined by

$$\Phi_{L,\sigma} : X(L) \to \mathbf{R}$$

$$\chi \mapsto \#\{\text{irreducible subquotients of (normalized) } \operatorname{Ind}_P^G(\sigma \otimes \chi) \text{ lying in } U^\wedge \text{ (counted with multiplicity)}\}.$$

Following [AdRDSW15], we introduce a bit of non-standard terminology that encompasses the kinds of local conditions that we wish to impose on our automorphic representations.

Definition 8.2. A subset $U^\wedge$ of the unitary dual $G^\wedge$ is said to be **prescribable** if it satisfies all the following conditions:

- The subset $U^\wedge$ is a Borel set that is $\widehat{\mu}^{pl}$-measurable of finite positive volume.
- The image of $U^\wedge$ under the map ν is contained in a compact subset of $\Theta(G)$.
- For each Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ and each discrete series σ of $L = \mathbf{L}(K)$, the set of points of discontinuity of the map $\Phi_{L,\sigma}$ is measure zero.

These conditions have been concocted so that the characteristic function of a prescribable subset $U^\wedge$ belongs to the class of functions for which we can apply the Sauvageot density principle [Sau97, Thm. 7.3.]. While this definition may seem technical and unmotivated given our presentation, it turns out to encompass a number of common conditions that we might want to impose at local places of our automorphic representation.

Example 8.3. The subset of unramified representations of $G^\wedge$ is prescribable.

Example 8.4. The set of all $\tau \in G^\wedge$ in a fixed Bernstein component (that is, for τ with the same supercuspidal support, up to a twist by an unramified character) is prescribable.

Example 8.5. If τ is a unitary discrete series representation (that is, an irreducible representation whose matrix coefficients are square-integrable modulo center), the set $\{\tau \otimes \chi \mid \chi \in X(G)\}$ is prescribable. In particular, note that if $\mathbf{G}$ is anisotropic over K (e.g. if $\mathbf{G}$ is semisimple), then $X(G)$ is trivial and so this set consists of a single element.

Note that this last example shows the difficulty in making our globalization theorem for GL_n over a totally real number field more precise, that is, to get the prescribed local component "on the nose" instead of just landing the prescribed inertia class. It means that such a refinement would not follow by yielding to equidistribution results like Sauvageot's density principle, and must instead be tackled by itself, which is a much more involved procedure than that of the globalization theorem (which in a sense, also appeals to equidistribution results at archimedean places).

Thus, it seems that to answer the globalization question for GL_N in greater generality or to obtain a more precise result, we must assume the results of some great body of work. Indeed, to construct the self-dual automorphic representations using the method given here, it is *necessary* work to on the relevant semisimple group and then yield to Arthur's results to transfer the representation on the semisimple group to a self-dual automorphic representation on GL_N.

8.3 A Globalization Theorem for Semisimple Groups

There are a number of ways to solve the globalization problem for cuspidal automorphic representations on semisimple groups. Here we give one such result, due to Shin, based on the principle that the local components of automorphic representations at a fixed prime are equidistributed in the unitary dual [Shi12]. The interested reader can look at the introduction of this paper for more references and a general discussion of the approach.

Let G be a connected reductive group over a totally real number field F such that

(i) G has trivial center and

(ii) $G(F_w)$ contains an $\mathbf{R}$-elliptic maximal torus for every real place w of F.

Remark 8.6. If we fix the central character in the trace formula argument of [Shi12], it would be possible to relax condition (i). In general, trace formula arguments with fixed central character can be derived from the non-fixed central character methods using some elementary Fourier analysis. However, the author does not know where to find the results in the literature at the necessary level of generality, so we simply impose the condition above, which is more or less harmless for our eventual application.

We will first recall the unconditional results, but we will eventually need to make some assumptions to obtain a globalization theorem that is strong enough to apply to prove our general result.

Let S be a finite set of finite place of F. Let $\widehat{\mu}_v^{pl}$ denote the Plancherel measure on $\mathbf{G}(F_v)^\wedge$ for $v \in S$. Let $U_v^\wedge \subset \mathbf{G}(F_v)^\wedge$ be a prescribable subset for each $v \in S$.

Proposition 8.7. *[Shi12, Thm 5.8.] There exists a cuspidal automorphic representation τ of $G(\boldsymbol{A}_F)$ for F totally real such that:*

(i) $\tau_V \in U_V^\wedge$ *for all* $v \in S$*;*

(ii) τ *is unramified at all finite places away from* S*; and*

(iii) τ_w *is a discrete series whose infinitesimal character is sufficiently regular for every infinte place* w*.*

The regularity condition needs to be explained. Fix a maximal torus T and a Borel subgroup B containing T in $\mathbf{G}$ over $\mathbf{C}$ (the base change of $\mathbf{G}$ to $\mathbf{C}$ via $w : F \hookrightarrow \mathbf{C}$). Let W denote the Weyl group of T in G. The infinitesimal character χ_w of τ_w can be viewed as a element of $X^*(T) \otimes_{\mathbf{Z}} \mathbf{Q}$ and we say that it is **sufficiently regular** if there exists a $\sigma \in W$ such that

$$\langle \sigma\chi_w, \alpha^\vee \rangle \geq C$$

for every B-positive coroot $\alpha^\vee$ of T in G, where C is a large constant that (only) depends on G, S, and $\{U_v^\wedge\}_{v\in S}$. Note that this condition is, in particular, independent of the choice of T and B.

From this point on, we assume one of our main hypotheses.

Assumption. The analogue of Proposition 8.7 above is true over a general number field F, even in the case where $G(F_w)$ does not have discrete series for infinite places w.

While this is an assumption, it is not an absurd one, for it is not far from currently existing results. A proof for this weaker unconditional result can be found in [Shi12, §4.3], if we allow ourselves to impose extra conditions at two auxiliary finite places (in order to yield to an argument that uses a simple trace formula). It is undoubtedly possible to remove these restrictions by using the full trace formula, but such a result does not seem to exist in the literature yet.

8.4 Results from Arthur's Endoscopic Classification

We recall the main results that we need from the monograph of Arthur [Art13].

8.4.1 For SO_{2n+1}

We recall some relevant facts for representations on odd orthogonal groups.

Let F be a number field. Let SO_{2n+1} be the split special orthogonal group over F. The dual group of SO_{2n+1} is is $Sp_{2n}(\mathbf{C})$. We have the standard embedding

$$\xi : Sp_{2n}(\mathbf{C}) \hookrightarrow GL_{2n}(\mathbf{C})$$

and for a place v of F, set

$$\mathcal{L}_{F_v} = \begin{cases} W_{F_v} \times SL_2(\mathbf{C}), & v \text{ infinite} \\ W_{F_v}, & v \text{ finite} \end{cases}$$

where W_{F_v} is the Weil group of F_v. A local Langlands parameter

$$\phi_v : \mathcal{L}_{F_v} \to GL_{2n}(\mathbf{C})$$

is said to be **symplectic** (equivalently, correspond to a local representation **of symplectic type**) if it preserves a suitable symplectic form on the ambient $2n$-dimensional vector space; this is equivalent to the condition that ϕ_v factors through ξ (after possibly conjugating by an element of $GL_{2n}(\mathbf{C})$):

$$\begin{array}{ccc} \mathcal{L}_{F_v} & \xrightarrow{\phi_v} & GL_{2n}(\mathbf{C}) \\ & \searrow \quad \nearrow_{\xi} & \\ & Sp_{2n}(\mathbf{C}) & \end{array}$$

For a place v of F and any positive integer r, we have the (unitarily normalized) local Langlands correspondence for GL_n, that is, the bijection

$$\mathrm{rec}_v : \{\text{irreducible representations of } GL_r(F_v)\} \to \{L\text{-parameters } \mathcal{L}_{F_v} \to GL_r(\mathbf{C})\}.$$

When v is finite, we have a one-to-one correspondence between local L-parameters for GL_r and

r-dimensional Frobenius-semisimple Weil–Deligne representations of W_{F_v}, in a bijective manner.

To each local L-parameter

$$\phi_v : \mathcal{L}_{F_v} \to Sp_{2n}(\mathbf{C}),$$

or local L-parameter for GL_{2n} of symplectic type, Arthur attaches an L-packet Π_{ϕ_v}, which is a finite set of irreducible representations of $SO_{2n+1}(F_v)$. Up to equivalence, every irreducible representation of SO_{2n+1} belongs to a unique such L-packet. If ϕ_v has a finite centralizer group in $Sp_{2n}(\mathbf{C})$ so that it is a discrete parameter, then the L-packet Π_{ϕ_v} only contains discrete series representations. If v is an infinite place of F, a similar construction was known earlier by Langlands, based on Harish-Chandra's results on real reductive groups.

Let τ be a discrete automorphic representation of $SO_{2n+1}(\mathbf{A}_F)$, where $\mathbf{A}_F$ denotes the adeles of the number field F. Arthur shows that there exists a self-dual isobaric automorphic representation π of $GL_{2n}(\mathbf{A}_F)$ which is a functorial transfer of τ along the standard embedding ξ. For representations that are generic in the sense of Arthur (that is, when the SL_2-factor in the global Arthur parameter corresponding to the representation τ has trivial image), this translates to the condition

$$\mathrm{rec}_v(\pi_v) \simeq \xi \circ \phi_v$$

for the unique ϕ_v such that $\tau_v \in \Pi_{\phi_v}$.

8.4.2 For Sp_{2n}

Analogous results hold for Sp_{2n} over a number field F. Here the dual group is $SO_{2n+1}(\mathbf{C})$, and we have the standard embedding

$$\xi : SO_{2n+1}(\mathbf{C}) \hookrightarrow GL_{2n+1}(\mathbf{C}).$$

A Langlands parameter

$$\phi_v : \mathcal{L}_{F_v} \to GL_{2n+1}(\mathbf{C})$$

is of **orthogonal type** if it preserves a suitable orthogonal (i.e. symmetric bilinear) form on the ambient $2n+1$-dimensional vector space, and this is equivalent to the condition that ϕ_v factors through ξ (up to conjugation by an element of $GL_{2n+1}(\mathbf{C})$):

$$\begin{array}{ccc} \mathcal{L}_{F_v} & \xrightarrow{\phi_v} & GL_{2n+1}(\mathbf{C}) \\ & \searrow \quad \nearrow_{\xi} & \\ & SO_{2n+1}(\mathbf{C}) & \end{array}$$

Note that all irreducible self-dual representations into $GL_{2n+1}(\mathbf{C})$ must be of orthogonal type, since the existence of a symplectic form on the space would imply that the ambient space is even-dimensional.

To each local L-parameter

$$\phi_v : \mathcal{L}_{F_v} \to SO_{2n+1}(\mathbf{C})$$

or local L-parameter for GL_{2n+1} of symplectic type, Arthur attaches an L-packet Π_{ϕ_v}, which consists of finitely many irreducible representations of $Sp_{2n}(F_v)$; we have the analogous results on L-packets and discrete parameters. For a discrete automorphic representation τ of $Sp_{2n}(\mathbf{A}_F)$, Arthur shows that there exists a self-dual isobaric automorphic representation π of $GL_{2n+1}(\mathbf{A}_F)$ which is a functorial transfer of τ along the standard embedding ξ, with the analogous correspondence condition for representations that are generic in the sense of Arthur.

8.5 Existence of Self-Dual Representations on GL_N

We now construct the self-dual automorphic representations with prescribed local conditions using the globalization theorem on the endoscopic group and using Arthur's results on functorial transfer from the endoscopic group to GL_N.

Theorem 8.8. *Let S be a finite set of places of a number field F. At each place $v \in S$, impose a condition that corresponds to a prescribable subset U_v on the endoscopic group H (e.g. lying in a specific Bernstein component of $G(F_v)$ for v finite, or being a specific essentially discrete represen-*

tation of $G(F_v)$). Then there exists a cuspidal automorphic representation of $GL_N(\boldsymbol{A}_F)$ satisfying those local conditions such that:

(i) π is unramified away from $v \in S$; and

(ii) $\pi \simeq \pi^\vee$, that is, π is self-dual

Proof. Here, as in the proof of the globalization theorem above, our case divides into the even and odd cases; if $N = 2n$ is even let $H = SO_{2n+1}$ and if $N = 2n+1$ is odd, let $H = Sp_{2n}$.

Apply the generalized version of Proposition 8.7 with our S and where each $U_v^\wedge$ is prescribable for $v \in S$. Thus, there exists a cuspidal automorphic representation τ of $H(\mathbf{A}_F)$ such that

1. τ_v is unramified away from S;
2. $\tau_v \in U_v^\wedge$ for all $v \in S$;
3. either τ_∞ is a discrete series whose infinitesimal character is sufficiently regular, or

3'. τ_∞ is any discrete series, but we lose control of the prescribed representation at two auxiliary primes.

The functorial transfer π of τ then has the desired properties. For example, to see that π is cuspidal, by condition (3) or (3'), then π is generic in the sense of Arthur. To see that the condition on the central character holds, note that the central character is trivial at almost all finite places. Indeed, the central character corresponds to the determinant of the Langlands parameter for π at each place via local class feld, but the determinant is trivial since the parameter factors through $\widehat{H}$. □

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