A DECOMPOSITION THEORY FOR LATTICES WITHOUT CHAIN CONDITIONS

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ABSTRACT

In this thesis we are concerned with representing an element of a lattice as an irredundant meet of elements which are irreducible in the sense that they are not proper meets, and with certain arithmetical properties of these decompositions. A theory is developed for the class of compactly generated atomic lattices which extends the classical theory for finite dimensional lattices. The principal results are the following. Every element of an arbitrary compactly generated atomic lattice has an irredundant meet decomposition into irreducible elements. These decompositions are unique in distributive lattices. In a modular lattice the decompositions of an element have the Kurosh-Ore replacement property, that is, for any two decompositions of an element, each irreducible in the first decomposition can be replaced by a suitable irreducible in the second decomposition. Moreover, characterizations are obtained of those lattices having unique decompositions and those lattices having the replacement property.

1. INTRODUCTION

NOTATION AND TERMINOLOGY. Throughout this thesis, lattice join, meet, inclusion, and proper inclusion are denoted by the symbols $\cup$, $\cap$, $\leq$, and $<$, respectively. Set union and intersection are denoted by the darker symbols $\boldsymbol{\cup}$ and $\boldsymbol{\cap}$, respectively, with set inclusion and proper inclusion denoted by $\subseteq$ and $\subset$. If S and T are sets, then $S - T$ denotes the set $\{x : x \in S,\ x \notin T\}$; if T contains a single element t, then $S - T$ is also written $S - t$.

For every pair of elements a, b in a lattice L such that $b \geq a$, the quotient sublattice b/a is defined by

$$b/a = \{x \in L : b \geq x \geq a\}.$$

If a, b are elements of a lattice L, then b is said to <u>cover</u> a if $b > a$ and $b \geq x > a$ implies $b = x$ for all $x \in L$. This covering relation is denoted by $b \succ a$. A lattice L is called <u>atomic</u> if for every pair of elements $a, b \in L$ with $b > a$ there exists an element $p \in L$ such that $b \geq p \succ a$. In a complete atomic lattice L the element u_a is defined for each $a \in L$ by

$$u_a = \bigcup\{p \in L : p \succ a\}.$$

If c is an element of a complete lattice L, then c is <u>compact</u> if for every subset $S \subseteq L$ with $c \leq \bigcup S$ there exists a finite subset $S' \subseteq S$ such that $c \leq \bigcup S'$. A lattice L is called <u>compactly generated</u> if L is complete and for every element $x \in L$

$$x = \bigcup\{c : c \leq x,\ c \text{ compact in } L\}.$$

Distributive and modular lattices are defined as usual. A lattice L is called semimodular (more precisely upper-semimodular) if $x \succ x \cap y$ implies $x \cup y \succ y$ for every $x, y \in L$. A lattice L is lower-semimodular if $x \cup y \succ y$ implies $x \succ x \cap y$. If L is a complete atomic lattice, then L is locally distributive if the sublattice u_a/a is distributive for every $a \in L$. Local modularity is analogously defined.

An element r in a lattice is said to be irreducible if $r = x \cap y$ implies $r = x$ or $r = y$. An element q in a complete lattice L is completely irreducible if for every subset $S \subseteq L$, $q = \bigcap S$ implies $q \in S$. A representation of an element as a meet of completely irreducible elements is said to be a decomposition of the element. A decomposition $a = \bigcap Q$ of an element a is irredundant if $\bigcap(Q - q) \neq a$ for each $q \in Q$.

DISCUSSION AND SUMMARY OF RESULTS. A very natural problem that often arises in the study of an algebraic system is that of representing the elements of the system as images of some cononical subset under a specific operation of the system. Usually the cononical subset is taken to be the set of those elements which can not be further represented by means of the operation. The most elementary example of this is the representation of rational integers as products of primes.

Another fundamental problem is that of representing the algebraic system as a whole as a direct or subdirect product of simpler systems. This second problem is not unrelated to the first. For the congruence relations of any algebraic system form a lattice in a natural way, and the representations of the system as direct or subdirect products correspond with meet representations of the null element in the lattice of congruence relations. In particular, the representations of the system as a subdirect product of subdirectly irreducible systems correspond with the representations of the null congruence as a meet of completely irreducible congruences.

For these reasons, questions concerning the arithmetical properties of meet representations of an element in a lattice have been among the most fundamental lattice theoretic questions.

During the period from 1935 to 1945 a rather satisfactory theory was developed concerning the representation of an element in a lattice as a finite meet of irreducible elements. Briefly the main results are as follows. If a lattice L satisfies the ascending chain condition, then it is an immediate consequence of the chain condition that every element of L has a representation as a finite irredundant meet of irreducible elements. Birkhoff [1] showed that every element of a distributive lattice satisfying the ascending chain condition has a unique representation. Kurosh [10] and Ore [11] independently showed that in a modular lattice, the finite meet representations have the replacement property, that is, if an element has two representations as a finite meet of irreducibles, then each irreducible in one representation can be replaced by a suitable irreducible in the other representation. This implies that the number of irreducibles in a finite irredundant representation of an element in a modular lattice is unique. Stronger versions of the Kurosh-Ore Theorem were obtained by Dilworth [5]. He showed that if an element in a modular lattice has two representations as a finite irredundant meet of irreducibles, then each irreducible in the first representation can be replaced by and at the same time replace a suitable irreducible in the second representation. He further showed that the two representations can be put into a one-one correspondence such that each irreducible in the first representation can be replaced by the corresponding irreducible in the second representation. In a series of papers

[2, 3, 4] Dilworth took up the important problem of characterizing those lattices having unique and replaceable representations. Principally he obtained two fundamental results: every element of a finite dimensional lattice L has a unique representation as an irredundant meet of irreducibles if and only if L is locally distributive; and the conclusion of the Kurosh-Ore Theorem always holds in a finite dimensional semimodular lattice L if and only if L is locally modular.

As indicated above, the theory of finite meet representations depends heavily on very restrictive chain conditions. However, certain lattices appropriate for a representation theory, specifically lattices of congruence relations, do not satisfy chain conditions, and yet they suggest that some more general representation theory exists. For example, it is well known that an arbitrary abstract algebra A is a subdirect product of subdirectly irreducible algebras, and hence the null congruence of A is a meet of completely irreducible congruence relations. On the other hand, a representation theory for a general lattice is clearly impossible. An atomless Boolean algebra contains no irreducible elements, and hence a theory of meet representations for this lattice would really be meaningless. Some restrictions are needed, therefore, to provide an adaquate theory of meet representations. Moreover, it is desirable that the class of lattices satisfying these restrictions contains the finite dimensional lattices as well as the lattices of congruence relations. The compactly generated lattices form such a class.

Examples of compactly generated lattices abound in mathematics. If A is any abstract algebra, then the lattice of congruence relations of A is compactly generated; the compact congruences are those generated by identifying a finite number of pairs of elements of A. Similarly, the lattice of subalgebras of A is compactly generated, the compact elements being the finitely generated subalgebras. If L is any lattice, then the lattice of ideals of L is compactly generated; in this case the compact elements are the principal ideals. Thus if L satisfies the ascending chain condition, then every element of L is compact and L is necessarily compactly generated.

It is easily seen that every element of a compactly generated lattice can be represented as a meet of completely irreducible elements. But in general, since these representations are infinite, they are not irredundant. If in addition the lattice is assumed to be atomic, then irredundant decompositions exist for every element. Atomicity is not an unnatural condition if one expects to develop a theory of irredundant decompositions, for in a modular lattice the existence of irredundant decompositions and atomicity are equivalent conditions. Also, compactly generated atomic lattices generalize finite dimensional lattices, in that compact generation generalizes the ascending chain condition and atomicity generalizes the descending chain condition. Some important examples of compactly generated atomic lattices are the following: the lattice of subgroups of a torsion abelian group, the lattice of pure subgroups of a torsionfree abelian group, the lattice of subspaces of a vector space, the lattice of congruences of a weakly atomic modular lattice, and any exchange lattice.

In this thesis a theory of irredundant decompositions is developed for compactly generated atomic lattices which extends the theory for finite dimensional lattices. As mentioned in the preceding paragraph, one of the principal results is the proof that every element of an arbitrary compactly generated atomic lattice has an irredundant decomposition. Unlike the existence theorem in the finite dimensional case, this result is one of the most difficult hurdles of the theory. The analogues of Birkhoff's theorem and the Kurosh-Ore Theorem are shown to hold in the more general case. In connection with the latter result, a fundamental difference from the finite dimensional case is encountered. Since the decompositions are infinite, the replacement property does not imply that two irredundant decompositions have the same cardinality. In fact the decompositions of an element in a modular lattice fall naturally into equivalence classes, and the cardinalities of the decompositions as well as the stronger replacement properties analogous to those of Dilworth [5] depend on the classes involved. It is further shown that a compactly generated atomic lattice has unique irredundant decompositions if and only if the lattice is locally distributive, thus extending Dilworth's result directly. A condition is found which is necessary and sufficient for an arbitrary compactly generated atomic lattice to have the Kurosh-Ore replacement property. This condition is a modification of lower-semimodularity, and in the presence of (upper) semimodularity is easily seen to be equivalent to local modularity. Hence Dilworth's characterization of the Kurosh-Ore Theorem in finite dimensional semimodular lattices also carries over.

The problem of developing a decomposition theory for compactly generated atomic lattices was first conceived by Professor Dilworth about five years ago. In the course of the 1956-57 lattice theory seminar at Caltech, he presented three initial results, the existence of irredundant decompositions in semimodular lattices (essentially Theorem 5.3 below), the uniqueness of the decompositions in distributive lattices (corollary to Theorem 4.2), and the replacement property in modular lattices (a slightly less general form of Theorem 5.1). He also raised the question whether or not two irredundant decompositions of an element in a modular lattice have the same cardinality.

My interest in the problem began with an attempt to answer this question. Eventually the question was answered in the negative, and then most of the results of section 5 were developed in an attempt to salvage for the general case the primary results concerning the decompositions in finite dimensional modular lattices. Professor Dilworth and I then began working more or less together on the decomposition theory. Two results were discovered: that the existence of unique decompositions is equivalent to local distributivity (Theorem 4.2), and that the replacement property in a semimodular lattice is equivalent to local modularity (corollary to Theorem 6.2). At this point two fundamental questions remained open. The first was: does an arbitrary compactly generated atomic lattice have irredundant decompositions? And the second: what is a characterization of those (nonsemimodular) lattices having the replacement property? Subsequently these questions were resolved, and their solutions comprise a major part of this thesis. In section 3 the general existence theorem is presented, and in section 4 this result is applied to give a new and more succinct development of Dilworth's results on unique decompositions. It is further applied in section 6 to obtain the characterization of those lattices with the replacement property.

2. PRELIMINARIES

This section gathers together a few important elementary lemmas concerning compactly generated lattices and applies these lemmas to characterizations of modularity and semimodularity.

LEMMA 2.1. Let L be a compactly generated lattice. If $\{x_\alpha\}$ is a chain of L and $a \in L$, then $a \cap \bigcup_\alpha x_\alpha = \bigcup_\alpha a \cap x_\alpha$.

Since in any complete lattice $a \cap \bigcup_\alpha x_\alpha \geq \bigcup_\alpha a \cap x_\alpha$ holds trivially, it is enough to show that $a \cap \bigcup_\alpha x_\alpha \leq \bigcup_\alpha a \cap x_\alpha$. And since L is compactly generated, it suffices to show that if c is a compact element and $c \leq a \cap \bigcup_\alpha x_\alpha$, then $c \leq \bigcup_\alpha a \cap x_\alpha$. If c is such an element, then $c \leq a \cap \bigcup_\alpha x_\alpha$ implies $c \leq a$ and $c \leq \bigcup_\alpha x_\alpha$. However, c is compact, whence $c \leq x_{\alpha_1} \cup \cdots \cup x_{\alpha_k}$ for some finite subset $\{x_{\alpha_1}, \ldots, x_{\alpha_k}\} \subset \{x_\alpha\}$. But the elements $x_{\alpha_1}, \ldots, x_{\alpha_k}$ form a chain, and hence if x_{α_j} is the largest of these elements, then $c \leq x_{\alpha_j}$. Thus $c \leq a \cap x_{\alpha_j} \leq \bigcup_\alpha a \cap x_\alpha$, completing the proof of the lemma.

LEMMA 2.2. If a, b are elements of a compactly generated lattice L and if $a > b$, then there exist elements $p, q \in L$ such that $a \geq p \succ q \geq b$.

For since L is compactly generated, there is a compact element $c \in L$ such that $c \leq a$ but $c \nleq b$. Let $p = c \cup b$. Consider now the collection $T = \{x : p > x \geq b\}$. T is nonempty since $b \in T$. Let $\{x_\alpha\}$ be a chain of elements in T. Then clearly $p \geq \bigcup_\alpha x_\alpha$. If $p \leq \bigcup_\alpha x_\alpha$, then $c \leq \bigcup_\alpha x_\alpha$, and since c is compact and $\{x_\alpha\}$ is a chain, it follows that

$c \leq x_{\alpha_j}$ for some $x_{\alpha_j} \in \{x_\alpha\}$. But then $p = b \cup c \leq x_{\alpha_j}$, contrary to $x_{\alpha_j} \in T$. Hence $p > \bigcup_\alpha x_\alpha$, and thus $\bigcup_\alpha x_\alpha \in T$. It now follows by the Maximal Principle (Zorn's Lemma) that T contains a maximal element q. Because of the maximality of q we must have $p \succ q$, and hence the lemma is proved.

LEMMA 2.3. If a, b are elements of a compactly generated lattice L, then there exists a maximal element $m \in L$ such that $m \geq a$ and $m \cap b = a \cap b$.

Consider the collection $T = \{x : x \geq a, \ x \cap b = a \cap b\}$. T is nonempty since $a \in T$. If $\{x_\alpha\}$ is a chain of T, then by Lemma 2.1 we have $b \cap \bigcup_\alpha x_\alpha = \bigcup_\alpha b \cap x_\alpha = a \cap b$, and hence $\bigcup_\alpha x_\alpha \in T$. Therefore, by the Maximal Principle, T contains a maximal element m.

LEMMA 2.4. If a, b are elements of a compactly generated lattice and if $b > a$, then b/a is compactly generated. Moreover, if $b \geq p \succ a$, then p is compact in b/a.

If $x \in b/a$, then $x = \bigcup\{a \cup c : c \leq x, \ c \text{ compact}\}$. Suppose c is compact and $a \cup c \leq \bigcup S$ for some subset $S \subseteq b/a$. Then $c \leq \bigcup S$ and hence $c \leq \bigcup S'$ for some finite subset $S' \subseteq S$. Since $\bigcup S' \geq a$, we have $a \cup c \leq \bigcup S'$, whence $a \cup c$ is compact in b/a. Thus b/a is compactly generated. If $b \geq p \succ a$, then there is a compact element c such that $c \leq p$, $c \nleq a$. But then $p = a \cup c$, and thus p is compact in b/a.

We now proceed with the characterizations of modularity and semi-modularity.

It is a well known property of a modular lattice L that if $a, b \in L$, then the quotient sublattices $a \cup b/a$ and $b/a \cap b$ are isomorphic. The converse in general is not true, that is, there are simple examples of non-modular lattices in which $a \cup b/a$ and $b/a \cap b$ are always isomorphic. However, for lattices satisfying a chain condition, Ward [12] has shown that the isomorphism property is equivalent to modularity. The following theorem extends Ward's result to compactly generated lattices.

THEOREM 2.1. Let L be a compactly generated lattice. If $a \cup b/a$ and $b/a \cap b$ are isomorphic for all $a, b \in L$, then L is modular.

Proof. Suppose L is a compactly generated lattice satisfying the hypothesis of the theorem, but L is not modular. Then L contains a five-element sublattice $\{a, b, t, u, v\}$ such that $a > b$ and $t \cup a = t \cup b = u$, $t \cap a = t \cap b = v$. If a does not cover b, then by Lemma 2.2 there exist elements $p, q \in L$ such that $a \geq p \succ q \geq b$. Clearly $t \cup p = t \cup q = u$ and $t \cap p = t \cap q = v$. Thus the sublattice $\{p, q, t, u, v\}$ is a non-modular five-element sublattice in which $p \succ q$. Hence we may assume that the sublattice $\{a, b, t, u, v\}$ was originally picked in such a way that $a \succ b$.

Let T be the set of all ordered triples (x, y, z), $x, y, z \in L$, such that $x \geq a$, $y \geq b$, $z \geq v$, and such that the following relations hold:

(i) $t \cup y = u$,

(ii) $t \cap x = z$,

(iii) $x \succ y$,

(iv) $a \cap y = b$.

T is nonempty since the triple (a, b, v) is in T. Now partially order T by defining $(x, y, z) \leq (x', y', z')$ if and only if $x \leq x'$, $y \leq y'$, and $z \leq z'$. Suppose $\{(x_\alpha, y_\alpha, z_\alpha)\}$ is a chain of T. Let $\bar{x} = \bigcup_\alpha x_\alpha$, $\bar{y} = \bigcup_\alpha y_\alpha$, and $\bar{z} = \bigcup_\alpha z_\alpha$. By infinite associativity and Lemma 2.1 we have, $t \cup \bar{y} = t \cup \bigcup_\alpha y_\alpha = \bigcup_\alpha t \cup y_\alpha = u$, $t \cap \bar{x} = t \cap \bigcup_\alpha x_\alpha = \bigcup_\alpha t \cap x_\alpha = \bigcup_\alpha z_\alpha = \bar{z}$, and $a \cap \bar{y} = a \cap \bigcup_\alpha y_\alpha = \bigcup_\alpha a \cap y_\alpha = b$. Note that for each index α, $a \cup y_\alpha = x_\alpha$ since $x_\alpha \geq a$ and $x_\alpha \succ y_\alpha$. Thus $a \cup \bar{y} = a \cup \bigcup_\alpha y_\alpha = \bigcup_\alpha a \cup y_\alpha = \bigcup_\alpha x_\alpha = \bar{x}$. Hence $\bar{x}/\bar{y} = a \cup \bar{y}/\bar{y} \cong a/a \cap \bar{y} = a/b$ by hypothesis, and since $a \succ b$ we must have $\bar{x} \succ \bar{y}$. Thus the triple $(\bar{x}, \bar{y}, \bar{z}) \in T$, and every chain of T has un upper bound. By the Maximal Principle, T contains a maximal element (a_o, b_o, v_o).

The remainder of the proof uses the following lemma which is an immediate consequence of the fact that the lattice is both upper and lower semimodular.

LEMMA. Let L be a lattice such that $a \cup b/a \cong b/a \cap b$ for all $a, b \in L$. Then if $p, q, r \in L$ and $p \succ q$, either $r \cap p = r \cap q$ or $r \cap p \succ r \cap q$. The dual statement also holds.

Continuing with the proof of the theorem, notice that $u/b_o = t \cup b_o/b_o \cong t/t \cap b_o = t/v_o$. Thus, since $u > a_o \succ b_o$, there must exist an element $v_1 \in L$ such that $t > v_1 \succ v_o$. Let $a_1 = a_o \cup v_1$ and $b_1 = b_o \cup v_1$. We shall show that the triple (a_1, b_1, v_1) satisfies conditions (i) - (iv).

It is clear that $t \cup b_1 = u$. Now $a_o \ngeq v_1$ since $t \cap a_o = v_o$. Thus $a_1 = a_o \cup v_1 \neq a_o \cup v_o = a_o$, and hence by the lemma, $a_1 \succ a_o$ since $v_1 \succ v_o$. Computing further, $t \cap a_1 \geq v_1 \succ v_o = t \cap a_o$, and hence by the lemma, $t \cap a_1 \succ v_o$. Thus $t \cap a_1 = v_1$. Just as $a_1 \succ a_o$ it follows that $b_1 \succ b_o$. Hence $a_1 \neq b_1$, for otherwise $b_1 > a_o > b_o$, contrary to $b_1 \succ b_o$. Thus by the lemma and the fact that $a_o \succ b_o$ we must have $a_1 \succ b_1$. Finally, consider $a \cap b_1$. Since $a_o \geq a > b$ and $b_1 > b$, it follows that $a \cap b_1 \leq a_o \cap b_1 = b_o$. Hence $b \leq a \cap b_1 \leq a \cap b_o = b$; whence $a \cap b_1 = b$. Thus the triple (a_1, b_1, v_1) satisfies conditions (i) - (iv), and hence $(a_1, b_1, v_1) \in T$.

But (a_1, b_1, v_1) is properly bigger than (a_o, b_o, v_o), and since (a_o, b_o, v_o) was picked to be a maximal element of T, we have a contradiction. Thus L must be modular, and the proof of the theorem is complete.

In the proof of Theorem 2.1, every application of the assumption $a \cup b/b \cong b/a \cap b$, with one exception, actually requires only the weaker hypothesis that $a \cup b \succ a$ if and only if $b \succ a \cap b$. The exception occurs in proving the existence of v_1. However, if the lattice is atomic, then the existence of v_1 follows from atomicity. Hence we have the following corollary.

COROLLARY. If a compactly generated atomic lattice L is both upper and lower semimodular, then L is modular.

THEOREM 2.2. Let L be a compactly generated atomic lattice. If $x, y \succ x \cap y$ imply $x \cup y \succ x, y$, then L is semimodular.

Proof. Suppose $a, b \in L$ are such that $a \succ a \cap b$. Let $T = \{x : a \cap b \leq x \leq b,\ a \cup x \succ x\}$. Then $a \cap b \in T$. Let $\{x_\alpha\}$ be a chain of T. Then clearly $a \cap b \leq \bigcup_\alpha x_\alpha \leq b$. Suppose there is an element $y \in L$ with $a \cup \bigcup_\alpha x_\alpha > y \geq \bigcup_\alpha x_\alpha$. Since $y \ngeq a$ and $a \cup x_\alpha \succ x_\alpha$, we have

$$y = y \cap (a \cup \bigcup_\alpha x_\alpha) = y \cap \bigcup_\alpha a \cup x_\alpha = \bigcup_\alpha y \cap (a \cup x_\alpha) = \bigcup_\alpha x_\alpha ,$$

and hence $a \cup \bigcup_\alpha x_\alpha \succ \bigcup_\alpha x_\alpha$. Thus $\bigcup_\alpha x_\alpha \in T$, and by the Maximal Principle, T contains a maximal element m. If $m < b$, then by the atomicity of L, there is an element p with $b \geq p \succ m$. And since $p \cap (a \cup m) = m$ and $p, a \cup m \succ m$, it follows that $a \cup m \cup p = a \cup p \succ p$. But this contradicts the maximal choice of m. Hence $b = m$, and $a \cup b \succ b$. Thus L is semimodular.

COROLLARY. Every locally modular, compactly generated, atomic lattice is semimodular.

We close this section with two lemmas on independent sets of covering elements.

If a is an element of a compactly generated atomic lattice and J is a set of elements covering a, then J is said to be <u>independent</u> if $\bigcup(J - p) \ngeq p$ for each $p \in J$.

LEMMA 2.5. Let a be an element of a semimodular, compactly generated, atomic lattice and let J be an independent set of elements covering a. Then J is a maximal independent subset of the elements covering a if and only if $\bigcup J = u_a$.

If $\bigcup J = u_a$, then J is clearly a maximal independent subset of the elements covering a. Suppose then that J is maximal and that $p \succ a$. Then either $\bigcup J \geq p$ or there exists $p' \in J$ such that $\bigcup(J - p') \cup p \geq p'$. In the second case $\bigcup(J - p') \ngeq p$, since otherwise $\bigcup(J - p') \geq p'$ contrary to the independence of J. Thus we have $\bigcup(J - p') \cup p \succ \bigcup(J - p')$ by semimodularity. But $\bigcup(J - p') \cup p \geq \bigcup(J - p') \cup p' = \bigcup J \succ \bigcup(J - p')$. Thus $\bigcup J = \bigcup(J - p') \cup p \geq p$. Hence $\bigcup J \geq p$ in either case, and it follows that $\bigcup J \geq u_a$.

LEMMA 2.6. Let a be an element of a semimodular, compactly generated, atomic lattice and let J be an independent set of elements covering a. Then the elements of u_a/a which are joins of subsets of J form a complete sublattice of u_a/a which is isomorphic with the Boolean algebra of all subsets of J.

Consider the mapping $S \rightarrow \bigcup S$ as S ranges over the subsets of J. The independence of J implies that this mapping is one-one. Let $\{S_\alpha\}$ be a collection of subsets of J. Clearly $\bigcup_\alpha(\bigcup S_\alpha) = \bigcup(\boldsymbol{\cup}_\alpha S_\alpha)$, and hence the mapping preserves joins. Let $b = \bigcup(\boldsymbol{\cap}_\alpha S_\alpha)$. If $\bigcap_\alpha(\bigcup S_\alpha) > b$, then there is an element r such that $\bigcup S_\alpha \geq r \succ b$ for all the indices α. Consider a particular index β. Then $\bigcup S_\beta = b \cup \bigcup(S_\beta - \bigcap_\alpha S_\alpha)$, and since r is compact in u_b/b there is a minimal finite subset $\{p_1, \cdots, p_k\} \subseteq S_\beta - \bigcap_\alpha S_\alpha$ such that $b \cup p_1 \cup \cdots \cup p_k \geq r$. Now $b \cup p_1 \cup \cdots \cup p_{k-1} \ngeq r$, and hence by semimodularity $b \cup p_1 \cup \cdots \cup p_{k-1} \cup r \succ b \cup p_1 \cup \cdots \cup p_{k-1}$. Similarly $b \cup p_1 \cup \cdots \cup p_k \succ b \cup p_1 \cup \cdots \cup p_{k-1}$, and since $b \cup p_1 \cup \cdots \cup p_k \geq b \cup p_1 \cup \cdots \cup p_{k-1} \cup r$, it follows that $b \cup p_1 \cup \cdots \cup p_{k-1} \cup r = b \cup p_1 \cup \cdots \cup p_k \geq p_k$. Since $p_k \notin \bigcap_\alpha S_\alpha$, there is a set S_γ with $p_k \notin S_\gamma$. But then $p_k \leq b \cup p_1 \cup \cdots \cup p_{k-1} \cup r \cup \bigcup S_\gamma = p_1 \cup \cdots \cup p_{k-1} \cup \bigcup S_\gamma \leq \bigcup(J - p_k)$. This contradicts the independence of J, whence $\bigcap_\alpha(\bigcup S_\alpha) = \bigcup(\bigcap_\alpha S_\alpha)$. The mapping therefore preserves meets, and the lemma follows.

3. THE EXISTENCE OF IRREDUNDANT DECOMPOSITIONS

The aim of this section is to prove the existence of irredundant decompositions in any compactly generated atomic lattice. First, two lemmas are needed.

LEMMA 3.1. An element q in a complete atomic lattice L is completely irreducible if and only if at most one element of L covers q.

For if q is covered by only one element s, then since L is atomic, $x > q$ implies $x \geq s$. Hence $\bigcap\{x : x > q\} = s > q$, and it follows that q is completely irreducible. On the other hand, if q is covered by two distinct elements s and s', then $q = s \cap s'$, and q is not irreducible.

This shows that in a complete atomic lattice every irreducible element is completely irreducible. Therefore we may refer to completely irreducible elements in these lattices simply as "irreducibles".

LEMMA 3.2. If a, b are elements of a compactly generated lattice L and $a \ngeq b$, then there exists a completely irreducible element $q \in L$ such that $q \geq a$ and $q \ngeq b$.

Since $a \ngeq b$ and L is compactly generated, there is a compact element $c \in L$ such that $c \leq b$ and $c \nleq a$. Let $\{x_\alpha\}$ be a chain of elements in L such that $x_\alpha \geq a$ and $x_\alpha \ngeq c$. If $\bigcup_\alpha x_\alpha \geq c$, then since c is compact and $\{x_\alpha\}$ is a chain, there is an element $x_{\alpha_j} \in \{x_\alpha\}$ such that $x_{\alpha_j} \geq c$. This contradiction shows that $\bigcup_\alpha x_\alpha \ngeq c$. By the Maximal Principle there exists a maximal element $q \geq a$ such that $q \ngeq c$. Suppose $q = \bigcap S$. Then for some $s \in S$ we must have $s \ngeq c$. But then by the maximality of q it follows that $q = s$. Hence q is completely irreducible, and the lemma follows.

The existence theorem is now the following.

THEOREM 3.1. Every element of a compactly generated atomic lattice has an irredundant decomposition into completely irreducible elements.

Proof. Let L be a compactly generated atomic lattice, and let a be an element of L. If $p \succ a$, then by Lemma 3.2 there exists a completely irreducible element $q \in L$ such that $q \geq a$ and $q \ngeq p$. Hence $a = q \cap p$, and thus to prove the theorem it is sufficient to prove the following lemma.

LEMMA 3.3. Let L be a compactly generated lattice. Then every element of L has an irredundant decomposition if and only if for each $a \in L$ (a distinct from the unit element of L) there exist a completely irreducible element q and an element $x > a$ such that $a = q \cap x$.

The necessity is clear. To prove the sufficiency, let $a \in L$ be an element which is not completely irreducible.

Let W be the collection of all ordered pairs (R, x) such that R is a set of completely irreducible elements of L, $x \in L$, and such that the following conditions are satisfied:

(i) $x \cap \bigcap R = a$,

(ii) $\bigcap R > a$, and $x \cap \bigcap(R - q) > a$ for all $q \in R$.

Partially order W by defining $(R, x) \geq (R', x')$ if and only if the following two conditions hold:

(iii) $R \supseteq R'$,

(iv) $x \cap \bigcap(R - R') \geq x'$.

Now since a is not completely irreducible, by the assumption of the lemma there exist an element $x_o \in L$ and a completely irreducible element

$q_o \in L$ such that $x_o, q_o > a$ and $a = x_o \cap q_o$. Then with $R = \{q_o\}$, the ordered pair (R, x_o) is a member of W, and hence W is nonempty. By the Maximal Principle, W contains a maximal chain $\{(R_\alpha, x_\alpha)\}$.

Define

(v) $$Q = \bigcup_\alpha R_\alpha,$$

(vi) $$y = \bigcup_\alpha x_\alpha.$$

Suppose $y \cap \bigcap Q > a$. Then since L is compactly generated, there exists a compact element $c \in L$ such that $y \cap \bigcap Q \geq c \cup a > a$. Therefore, $q \geq c \cup a$ for all $q \in Q$, and $y \geq c \cup a \geq c$. But since $\{x_\alpha\}$ forms a chain by (iv) and c is compact, $y = \bigcup_\alpha x_\alpha \geq c$ implies that for some index α we must have $x_\alpha \geq c$. This implies that $x_\alpha \cap \bigcap R_\alpha \geq c \cup a > a$, contrary to (i). Thus $y \cap \bigcap Q = a$. Now let q be any element of Q. Then for some index α we have $q \in R_\alpha$. By conditions (iii), (v), and (iv), if q' is any other element of Q, then either $q' \in R_\alpha$ or $q' \geq x_\alpha$. Since clearly $y \geq x_\alpha$, it follows that $y \cap \bigcap (Q - R_\alpha) \geq x_\alpha$. Thus

$$y \cap \bigcap (Q - q) = [y \cap \bigcap (Q - R_\alpha)] \cap \bigcap (R_\alpha - q) \geq x_\alpha \cap \bigcap (R_\alpha - q) > a.$$

In particular, $\bigcap (Q - q) > a$ for every $q \in Q$.

If $\bigcap Q = a$, then a has an irredundant decomposition and the lemma is proved. We may suppose, therefore, that $\bigcap Q > a$. Since $y \cap \bigcap Q = a$, by Lemma 2.3 there exists a maximal element $z \in L$ such that $z \geq y$ and $z \cap \bigcap Q = a$. Suppose z is not a completely irreducible element of L. Then by the assumption of the lemma there exist an element $t > z$ and a completely irreducible element $r > z$ such that $t \cap r = z$. Let $Q_1 = Q \cup \{r\}$. We then have

$$t \cap \bigcap Q_1 = t \cap r \cap \bigcap Q = z \cap \bigcap Q = a.$$

If q is any element of Q, then

$$t \cap \bigcap (Q_1 - q) = z \cap \bigcap (Q - q) \geq y \cap \bigcap (Q - q) > a.$$

Since $t, r > z$, the maximal property of z implies that

$$t \cap \bigcap (Q_1 - r) = t \cap \bigcap Q > a, \quad \text{and} \quad \bigcap Q_1 = r \cap \bigcap Q > a.$$

Hence $(Q_1, t) \in W$. Furthermore, if (R_α, x_α) is any element of the chain $\{(R_\alpha, x_\alpha)\}$, then $Q_1 \supset Q \supseteq R_\alpha$, and

$$t \cap \bigcap (Q_1 - R_\alpha) = z \cap \bigcap (Q - R_\alpha) \geq y \cap \bigcap (Q - R_\alpha) \geq x_\alpha .$$

But this implies that $\{(R_\alpha, x_\alpha)\} \cup \{(Q_1, t)\}$ is a chain of W properly containing the maximal chain $\{(R_\alpha, x_\alpha)\}$. Since this is impossible, it follows that z must be completely irreducible. Hence $a = z \cap \bigcap Q$ is an irredundant decomposition of a into completely irreducible elements, and the proof is complete.

From the proof of Theorem 3.1 we have the following corollary.

COROLLARY. If a is an element of a compactly generated atomic lattice and q is an irreducible with $q \geq a$, then q appears in at least one irredundant decomposition of a if and only if $q \not\geq u_a$.

In the proof of the preceding theorem the Axiom of Choice (in the form of the Maximal Principle) was used several times. It is perhaps interesting that

Theorem 3.1 is equivalent to the Axiom of Choice.

To prove this assertion, let P be an arbitrary partially ordered set. Let L denote the set of all chains of P together with the null set and

P itself, partially ordered by set inclusion. Since L has a largest element and is closed under set intersection, L is a complete lattice. Suppose S is a subset of L. If the set sum of the elements of S is a chain X, then the lattice join $\cup S = X$. If the set sum is not a chain, that is, if the set sum of S contains two noncomparable elements, then $\cup S = P$. It follows that every element of L is the join of one-element chains. Thus to show that L is compactly generated, it suffices to show that each one-element chain is compact. Let $p \in P$ and let $S \subseteq L$ be such that $\{p\} \subseteq \cup S$. If $\cup S$ is a chain, then it follows that $\{p\} \subseteq X$ for some chain $X \in S$. If $\cup S$ is not a chain, then there are two chains $X, Y \in S$ whose set sum is not a chain, so that $\{p\} \subseteq P = X \cup Y$. Thus $\{p\}$ is compact, and hence L is compactly generated. If X, Y are chains in L and $X \supset Y$, then with $p \in X - Y$ we have $X \supseteq Y \cup p \succ Y$, and hence L is atomic. Now suppose Theorem 3.1 holds. Then there is a completely irreducible element $Z \in L$ such that $Z \neq P$. By the atomicity of L there is an element $Y \in L$ such that $Y \succ Z$. If $Y = P$, then Z is a maximal chain. Suppose $Y \neq P$. If Y is not maximal, then there is a chain X such that $X \supset Y \supset Z$. With $p \in Y - Z$, $q \in X - Y$, and $U = Z \cup p$, $V = Z \cup q$, it follows that $U, V \neq Z$ and $Z = U \cap V$. But this contradicts the irreducibility of Z, whence Y must be maximal. Thus P contains a maximal chain, and hence the Axiom of Choice follows from Theorem 3.1.

The existence of irredundant decompositions and atomicity appear to be closely related. In a modular lattice these conditions are equivalent.

THEOREM 3.2. If every element of a modular compactly generated lattice L has an irredundant decomposition, then L is atomic.

Proof. Let a be an element of L distinct from the unit element.

Since every element of L has an irredundant decomposition, it follows from Lemma 3.3 that there exist a completely irreducible element q and an element $x > a$ such that $a = q \cap x$. Since q is completely irreducible, $\bigcap\{y : y > q\} = s \succ q$, and hence $q \cup x \geq s$. Applying modularity,

$$q \cup (s \cap x) = s \cap (q \cup x) = s \succ q,$$

whence $s \cap x \succ q \cap s \cap x = q \cap x = a$. Thus every element of L is covered by at least one element.

Suppose $b > a$. From the first paragraph of the proof, $p \in L$ exists such that $p \succ a$. If $b \not\geq p$, then $b \cap p = a$, and by Lemma 2.3 there is a maximal element $m \geq p$ such that $m \cap b = a$. Again there is an element s such that $s \succ m$, and the maximality of m implies that $b \cap s > a$. Hence $(b \cap s) \cup m = s \succ m$, whence $b \cap s \succ b \cap s \cap m = a$. Thus, under any circumstances, there exists $r \in L$ such that $b \geq r \succ a$, and hence L is atomic.

Without modularity the existence of irredundant decompositions no longer implies atomicity. The following is an example of a nonatomic semimodular lattice satisfying the ascending chain condition in which each element is uniquely the meet of at most two completely irreducible elements.

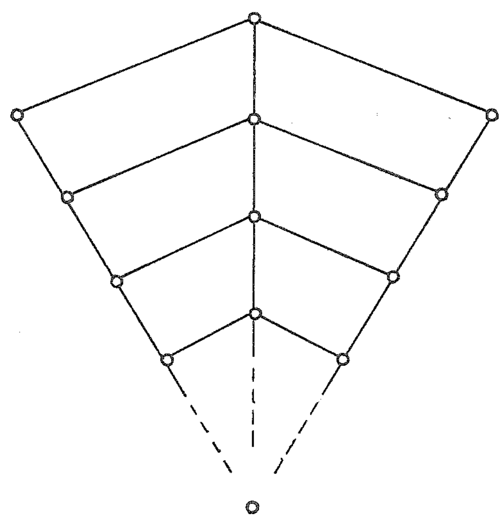

Fig. 1.

4. UNIQUE DECOMPOSITIONS AND THE DISTRIBUTIVE CASE

Here we are concerned with the existence of unique decompositions and the characterization of those lattices having unique decompositions. In particular, it will follow that every element of a distributive, compactly generated, atomic lattice has a unique irredundant decomposition.

We begin with a theorem describing those irreducibles which appear in every decomposition of an element.

THEOREM 4.1. Let a be an element of a compactly generated atomic lattice. Then an irreducible $q \geq a$ appears in every irredundant decomposition of a if and only if $q \cap p = a$ for some element $p \succ a$ such that (1) $x \cap p = a$ implies $x \cup p \succ x$, and (2) $p \cap (x \cup y) = (p \cap x) \cup (p \cap y)$, for all $x, y \geq a$.

Proof. Suppose q is an irreducible for which such an element $p \succ a$ exists. Let q' be any other irreducible such that $q' \cap p = a$. Then since q and q' are completely irreducible, there exist unique elements s, s' such that $s \succ q$ and $s' \succ q'$. Moreover, by (1) we have $p \cup q = s$ and $p \cup q' = s'$. Now if $q \neq q'$, then either $q \cup q' \geq s$ or $q \cup q' \geq s'$. In either case $q \cup q' \geq p$, contrary to $p \cap (q \cup q') = (p \cap q) \cup (p \cap q') = a$. Thus $q = q'$, and it follows that q appears in every decomposition of a.

Let q now be an irreducible that appears in every irredundant decomposition of a. Let p be any element covering a such that $q \ngeq p$. We shall show that p satisfies (1) and (2).

Suppose there is an element x such that $x \cap p = a$ but $x \cup p$ does not cover x. Let y be such that $x \cup p > y > x$. By Lemma 3.2 there exist irreducibles q_1 and q_2 such that $q_1 \geq x$, $q_1 \ngeq y$, and $q_2 \geq y$, $q_2 \ngeq x \cup p$. Observe that $q_1 \cap p = q_2 \cap p = a$. Hence there exist maximal elements $m_1, m_2 \geq p$ such that $q_1 \cap m_1 = q_2 \cap m_2 = a$. If $m_1 = \bigcap R_1$ and $m_2 = \bigcap R_2$ are irredundant decompositions of the elements m_1, m_2, then the maximality of these elements implies that $a = q_1 \cap \bigcap R_1 = q_2 \cap \bigcap R_2$ are two irredundant decompositions of a. Since $\bigcap R_1$, $\bigcap R_2 \geq p$ we must have $q \notin R_1$ and $q \notin R_2$. And since $q_1 \neq q_2$, it follows that at least one of the decompositions $a = q_1 \cap \bigcap R_1 = q_2 \cap \bigcap R_2$ does not contain q. This is contrary to the assumption that q appears in every irredundant decomposition of a, whence $x \cup p \succ x$, and (1) holds.

To show (2) it suffices to show that if $x, y \ngeq p$, then $x \cup y \ngeq p$. Let x, y be elements containing a but not p. By Lemma 3.2 there exist irreducibles q_x and q_y such that $q_x \geq x$, $q_y \geq y$, and $q_x, q_y \ngeq p$. Now $q_x \cap p = q_y \cap p = a$. Hence if $m_1, m_2 \geq p$ are maximal elements with $q_x \cap m_1 = q_y \cap m_2 = a$, and $m_1 = \bigcap R_1$, $m_2 = \bigcap R_2$ are irredundant decompositions of m_1 and m_2, then $a = q_x \cap \bigcap R_1 = q_y \cap \bigcap R_2$ are irredundant decompositions of a. But q appears in every such decomposition and $q \ngeq p$, whence $q = q_x = q_y$. Thus $q \geq x \cup y$, and hence $x \cup y \ngeq p$. (2) therefore holds, and the proof is complete.

We now turn to the characterization of those lattices with unique decompositions. The following preliminary lemma is needed.

LEMMA 4.1. If a, b are elements of a locally modular, compactly generated, atomic lattice and $p_1, p_2 \succ a$, then $b \cap (p_1 \cup p_2) = a$ and $p_1 \cup b = p_2 \cup b$ imply $p_1 = p_2$.

Suppose the conditions of the lemma hold, but $p_1 \neq p_2$. Let $T = \{x : x \cup p_1 \ngeq p_2,\ b \cap (p_1 \cup p_2 \cup x) = x\}$. T is nonempty since $a \in T$. If

$\{x_\alpha\}$ is a chain of T, then by Lemma 2.1, $b \cap (p_1 \cup p_2 \cup \bigcup_\alpha x_\alpha) = \bigcup_\alpha b \cap (p_1 \cup p_2 \cup x_\alpha) = \bigcup_\alpha x_\alpha$, and the compactness of p_2 in b/a implies that $p_1 \cup \bigcup_\alpha x_\alpha \not\geq p_2$. Hence $\bigcup_\alpha x_\alpha \in T$, and therefore T contains a maximal element m. Since by the corollary to Theorem 2.2 the lattice is semi-modular, $m \not\geq p_1, p_2$ implies that $m \cup p_1, m \cup p_2 \succ m$, and $m \cup p_1 \neq m \cup p_2$. Thus $m < b$, and hence there is an element p such that $b \geq p \succ m$. Since $p_1 \cup p_2 \cup m \not\geq p$, it follows that $p_1 \cup p_2 \cup p \succ p_1 \cup p_2 \cup m \succ p_1 \cup m, p_2 \cup m \succ m$. Now since u_m/m is modular by assumption, it follows that $p \cup p_1 \not\geq p_2$. Because of the modularity of $p_1 \cup p_2 \cup p/m$, either $p_1 \cup p_2 \cup p \succ b \cap (p_1 \cup p_2 \cup p)$ or $b \cap (p_1 \cup p_2 \cup p) = p$. If the first alternative holds, then the modularity of $p_1 \cup p_2 \cup p/m$ implies that $b \cap (p_1 \cup p_2 \cup p) \cap (p_1 \cup p_2 \cup m) = b \cap (p_1 \cup p_2 \cup m) \succ m$, contrary to $m \in T$. The second alternative contradicts the maximal choice of m. Hence we must have $p_1 = p_2$, and the lemma is proved.

Notice that the lemma shows the following: if a is an element of a locally modular, compactly generated, atomic lattice and q is an irreducible with $q \geq a$, then $q \not\geq u_a$ implies $u_a \succ q \cap u_a$. For if u_a does not cover $q \cap u_a$, then there are two distinct elements p_1, p_2 such that $u_a \geq p_1, p_2 \succ q \cap u_a$, and since q is irreducible and hence covered by a unique element, $p_1 \cup q = p_2 \cup q \succ q$, contrary to the lemma.

THEOREM 4.2. Every element of a compactly generated atomic lattice L has a unique irredundant decomposition into irreducibles if and only if L is locally distributive.

Proof. Suppose L has unique irredundant decompositions. Then it follows from the proof of Theorem 4.1 that conditions (1) and (2) of Theorem 4.1 hold for every $a \in L$ and every $p \succ a$. Consider a particular element $a \in L$. Let $x \in u_a/a$, and set $b = \bigcup \{p : p \succ a, \ p \leq x\}$. If $x > b$, then there is an element r such that $x \geq r \succ b$. By condition (1), if $p \succ a$ and $p \nleq b$, then $p \cup b \succ b$. Moreover, $r \neq p \cup b$ for every $p \succ a$. Now $\bigcup \{p \cup b : p \succ a,\ p \nleq p\} = u_a \geq r$, and since r is compact in u_a/b it follows that $r \leq p_1 \cup \cdots \cup p_k \cup b$ for some finite subset $\{p_1, \ldots , p_k\} \subseteq \{p : p \succ a,\ p \nleq b\}$. But then by (2),

$$r = r \cap [(p_1 \cup b) \cup \cdots \cup (p_k \cup b)] = [r \cap (p_1 \cup b)] \cup \cdots \cup [r \cap (p_k \cup b)] = b,$$

a contradiction. Hence $x = b$, and every element of u_a/a is a join of elements covering a. Thus, since the elements covering a are independent by (2), it follows from Lemma 2.6 that u_a/a is a Boolean algebra.

Suppose now that L is locally distributive. Then L is semimodular, and hence (1) holds throughout L. Suppose $a, p \in L$ with $p \succ a$, and suppose that $p \cap (x \cup y) \neq (p \cap x) \cup (p \cap y)$ for some $x, y \geq a$. By Lemma 3.2 there is an irreducible $q \geq x$ such that $q \ngeq p$. Note that $p \cup (q \cap y) \succ q \cap y$. Furthermore, if p' is such that $y \geq p' \succ q \cap y$, then $q \ngeq p'$, and $p \cup (q \cap y) \neq p'$. If we set $z = q \cap u_{q \cap y}$, then by the remark following the proof of Lemma 4.1 we have $u_{q \cap y} \succ z$. But then $z \cap p' = z \cap [p \cup (q \cap y)] = q \cap y$, and $z \cup p' = z \cup [p \cup (q \cap y)] = u_{q \cap y}$, contrary to the distributivity of $u_{q \cap y}/q \cap y$. Thus (2) holds for every $a \in L$ and every $p \succ a$, and hence by Theorem 4.1 every element of L has a unique irredundant decomposition.

COROLLARY. Every element of a distributive, compactly generated, atomic lattice has a unique irredundant decomposition.

5. THE MODULAR CASE

Even though modular lattices in general do not have unique decompositions, they enjoy a weak form of uniqueness, the Kurosh-Ore replacement property. Moreover, the decompositions of a particular element fall naturally into equivalence classes, and the cardinalities of the decompositions as well as stronger replacement properties depend upon the classes involved.

The fundamental replacement property is illustrated by the following.

THEOREM 5.1. If a is an element of a complete modular lattice and $a = \bigcap Q = \bigcap Q'$ are two decompositions of a, then for each $q \in Q$ there exists $q' \in Q'$ such that $a = q' \cap \bigcap (Q - q)$. If the decomposition $a = \bigcap Q$ is irredundant, then the decomposition $a = q' \cap \bigcap (Q - q)$ is also irredundant.

Proof. Let $q \in Q$. For each $q' \in Q'$, define $r_{q'}$ by

$$r_{q'} = q' \cap \bigcap (Q - q).$$

Then $a = \bigcap_{q' \in Q} r_{q'}$, and $a \leq r_{q'} \leq \bigcap (Q - q)$ for each $q' \in Q'$. By modularity, the quotient sublattices $q \cup \bigcap (Q - q)/q$ and $\bigcap (Q - q)/a = \bigcap (Q - q)/q \cap \bigcap (Q - q)$ are isomorphic. q is completely irreducible in the lattice, and hence it is also completely irreducible in the sublattice $q \cup \bigcap (Q - q)/q$. Thus a is completely irreducible in the sublattice $\bigcap (Q - q)/a$. Since $a = \bigcap_{q'} r_{q'}$ is a representation of a as a meet of elements of $\bigcap (Q - q)/a$, it follows that $a = r_{q'} = q' \cap \bigcap (Q - q)$ for some $q' \in Q'$.

Suppose the decomposition $a = \bigcap Q$ is irredundant. Then $\bigcap(Q - q) \neq a$, and therefore if $a = q' \cap \bigcap(Q - q)$ is a redundant decomposition, there is an element $q_1 \in Q - q$ such that $a = \bigcap(Q - \{q, q_1\}) \cap q'$. Now in this decomposition, q' can be replaced by some $q_2 \in Q$ giving a decomposition of a. But then either $\bigcap(Q - q) = a$ or $\bigcap(Q - q_1) = a$, contrary to the irredundance of the decomposition $a = \bigcap Q$. Hence the decomposition $a = q' \cap \bigcap(Q - q)$ is also irredundant.

The following example shows that in spite of Theorem 5.1 two irredundant decompositions need not have the same cardinality.

For each integer i, let A_i be a group isomorphic with the additive group of integers modulo a fixed prime p, and let G be the complete direct sum of the groups A_i, that is, the set of all functions f on the integers such that $f(i) \in A_i$, with addition defined componentwise. Then G is an (additive) abelian group every element of which has order p. Let L be the lattice of subgroups of G. L is then a modular, compactly generated, atomic lattice. For each i, let Q_i be that subgroup of G consisting of all functions $f \in G$ for which $f(i)$ is the zero element of A_i. Then G/Q_i is isomorphic with A_i, and hence Q_i is a maximal subgroup of G. In particular Q_i is a completely irreducible element of L for each i. Since G is the complete direct sum of the groups A_i, it follows that $0 = \bigcap_i Q_i$, where 0 denotes the zero subgroup of G. Moreover, it is clear that $\bigcap_{j \neq i} Q_j \neq 0$ for every i, and hence the decomposition $0 = \bigcap_i Q_i$ is irredundant. Now G can be considered as a vector space over the field of integers modulo p, and accordingly G has a basis $\{f_\alpha\}$. Since G has cardinality $2^{\aleph_0}$, the number of basis elements f_α must also be $2^{\aleph_0}$. For each index α, let Q'_α be that subgroup of G generated by the set $\{f_\beta : \beta \neq \alpha\}$. Then each Q'_α is a maximal subgroup of G, and just as above, it follows that $0 = \bigcap_\alpha Q'_\alpha$ is an irredundant decomposition of 0. Thus $0 \in L$ has two irredundant decompositions with different cardinalities.

This example indicates that an element in a modular lattice may have different kinds of irredundant decompositions. The remainder of this section explores this idea more completely. <u>Throughout the section, L will denote a modular, compactly generated, atomic lattice.</u>

LEMMA 5.1. For each $a \in L$, the sublattice u_a/a is complemented, and every element of u_a/a is a join of elements covering a.

Let $x \in u_a/a$. If S is a maximal independent subset of $\{p : p \succ a, p \leq x\}$, then by the Maximal Principle S can be extended to a maximal independent subset T of the elements covering a. Set $b = \bigcup S$, $d = \bigcup(T - S)$. Then by Lemma 2.5, $u_a = b \cup d = x \cup d$, and by Lemma 2.6, $b \cap d = a$. Since $x \geq p$, $p \succ a$ imply $b \geq p$, it follows that $x \cap d = a$. Thus by modularity $x = b$. Hence every element of u_a/a has a complement and is a join of elements covering a.

LEMMA 5.2. Let $a \in L$ and let p, p' be two distinct elements covering a. If q is an irreducible such that $q \geq a$ and $q \ngeq p, p'$, then $q \cap (p \cup p') \succ a$.

For since $q \ngeq p, p'$ we have $p \cap q = p' \cap q = a$, and hence $p \cup q, p' \cup q \succ q$. Since q is covered by a unique element, $p \cup q = p' \cup q = p \cup p' \cup q$. Thus $(p \cup p') \cup q \succ q$, and hence $p \cup p' \succ q \cap (p \cup p')$. Now $p \cup p' \succ p \succ a$, whence it follows that $q \cap (p \cup p') \succ a$.

Suppose now that $a = \bigcap Q$ is an irredundant decomposition of $a \in L$. Then for each $q \in Q$ we have $\bigcap(Q - q) > a$, and hence there is an element

$p \succ a$ such that $\bigcap(Q - q) \geq p$. Note that p is unique; for if $\bigcap(Q - q) \geq p' \succ a$ and $p' \neq p$, then $a = \bigcap Q = q \cap \bigcap(Q - q) \geq q \cap (p \cup p') \geq a$, contrary to Lemma 5.2. Let H_Q be the collection of all such elements p. Pick a particular element $p \in H_Q$, and let $\bigcap(Q - q) \geq p$. If $p' \in H_Q - p$ and $\bigcap(Q - q') \geq p'$, then $q \neq q'$, and hence $q \geq \bigcap(Q - q') \geq p'$. Thus $q \geq \bigcup(H_Q - p)$, and since $q \not\geq p$, it follows that $\bigcup(H_Q - p) \not\geq p$. Hence H_Q is independent.

Let us then define for each $a \in L$ and each irredundant decomposition $a = \bigcap Q$ the set H_Q and the element h_Q by

$$H_Q = \{p \succ a : \bigcap(Q - q) \geq p \text{ some } q \in Q\},$$

$$h_Q = \bigcup H_Q.$$

Summarizing the remarks above we therefore have

THEOREM 5.2. If $a \in L$ and $a = \bigcap Q$ is an irredundant decomposition of a, then H_Q is an independent set of elements covering a, and for each $q \in Q$ there is precisely one element $p \in H_Q$ such that $\bigcap(Q - q) \geq p$.

We may now prove two stronger versions of Theorem 3.1.

THEOREM 5.3. If $a \in L$ and J is a maximal independent set of elements covering a, then there exists an irredundant decomposition $a = \bigcap Q$ such that $H_Q = J$.

Proof. For each $p \in J$, let $s_p = \bigcup(J - p)$. Then by Lemma 2.6 it follows that $\bigcap_{p \in J} s_p = a$. By Lemma 2.5, $\bigcup J = u_a$. Hence $s_p \not\geq p$ implies $p \cup s_p = \bigcup J = u_a \succ s_p$ for each $p \in J$. Therefore by Lemma 3.2 there exists an irreducible q_p such that $q_p \cap u_a = s_p$ for each $p \in J$. Let $Q = \{q_p : p \in J\}$. Then since $u_a \cap \bigcap Q = \bigcap_{p \in J} u_a \cap q_p = \bigcap_{p \in J} s_p = a$, the atomicity of L implies that $\bigcap Q = a$. Since $\bigcap(Q - q_p) \geq p$, the decomposition $a = \bigcap Q$ is irredundant, and $H_Q = J$.

THEOREM 5.4. Let $a \in L$ and $a = \bigcap Q$ be an irredundant decomposition of a. If J is an independent set of elements covering a with $J \supseteq H_Q$, then there exists an irredundant decomposition $a = \bigcap Q'$ such that $J = H_{Q'}$.

Proof. By the Maximal Principle J can be extended to a maximal independent subset M of the elements covering a. Let $J_1 = J - H_Q$ and $M_1 = M - J_1$. If we set $b = \bigcup J_1$ and $c = \bigcup M_1$, then by Lemma 2.5 we have $b \cup c = \bigcup M = u_a$, and by Lemma 2.6 we have $b \cap c = a$. For each $p \in H_Q$ let q_p be the unique element of Q such that $\bigcap(Q - q_p) \geq p$. Then $q_p \ngeq p$, and hence $q_p \cup p \succ q_p$. Since q_p is irreducible and therefore covered by a unique element, $q_p \cup p' = q_p \cup p$ for any other $p' \in M_1$ such that $q_p \ngeq p'$. Thus $q_p \cup c = q_p \cup \bigcup M_1 = q_p \cup p \succ q_p$, and hence by modularity we have $c \succ c \cap q_p$. Now $b \cup (c \cap q_p) \neq u_a$, since otherwise

$$c = c \cap u_a = c \cap (b \cup (c \cap q_p)) = (c \cap b) \cup (c \cap q_p) = c \cap q_p,$$

contrary to $c \succ c \cap q_p$. Thus $u_a = b \cup c \succ b \cup (c \cap q_p)$. Let us set $s_p = b \cup (c \cap q_p)$. s_p is then a maximal element of u_a/a for each $p \in H_Q$.

Now for each $p \in J_1$ let us set $s_p = \bigcup(J_1 - p) \cup c$. Clearly $u_a \succ s_p$ for each $p \in J_1$. Furthermore, since J_1 is independent, it follows from Lemma 2.6 that $\bigcap_{p \in J_1} \bigcup (J_1 - p) = a$. Since L is modular, the mapping $x \longrightarrow x \cup c$ maps the sublattice $b/a = b/b \cap c$ isomorphically onto the sublattice $b \cup c/c = u_a/c$. Hence $\bigcap_{p \in J_1} s_p = c$. Thus we have

$$\bigcap_{p \in J} s_p = \bigcap_{p \in J_1} s_p \cap \bigcap_{p \in H_Q} s_p = c \cap \bigcap_{p \in H_Q} b \cup (c \cap q_p) = \bigcap_{p \in H_Q} c \cap (b \cup (c \cap q_p))$$

$$= \bigcap_{p \in H_Q} (c \cap b) \cup (c \cap q_p) = \bigcap_{p \in H_Q} c \cap q_p = c \cap \bigcap Q = a.$$

It follows immediately that if $p \in J_1$ then $s_p \geq c \geq \bigcup H_Q$, and if $p \in H_Q$ then $s_p \geq b \geq \bigcup J_1$. On the other hand, if $p \in J_1$ then $s_p \geq \bigcup(J_1 - p)$, and if $p \in H_Q$ then $s_p \geq c \cap q_p \geq \bigcup(H_Q - p)$. Thus for each $p \in J$ we have $s_{p'} \geq p$ all $p' \neq p$, and hence $\bigcap\{s_{p'} : p' \in J - p\} \geq p$. By Lemma 3.2 there exists an irreducible q'_p such that $q'_p \cap u_a = s_p$, for every $p \in J$. Let $Q' = \{q'_p : p \in J\}$. Then it follows as in the proof of the preceding theorem that $a = \bigcap Q'$ is an irredundant decomposition and $H_{Q'} = J$.

For finite irredundant decompositions it is easily verified that $\bigcup_{q\in Q} \bigcap(Q - q) \geq u_a$. This property no longer holds for general decompositions. We shall show that $u_a \cap \bigcup_{q\in Q} \bigcap(Q - q) = h_Q$.

LEMMA 5.3. Let $a \in L$ and $a = \bigcap Q$ be an irredundant decomposition of a. Let $S = \{q_1, \dots, q_n\}$ be a finite subset of Q, and let $\bigcap(Q - q_i) \geq p_i \succ a$ for each $i = 1, \dots, n$. Then $\bigcap(Q - S) \geq p \succ a$ implies that $p_1 \cup \cdots \cup p_n \geq p$.

For $n = 1$ the lemma follows from Theorem 5.2. Suppose then that the lemma is true for $n = k - 1$, and suppose that for some $p \succ a$, $\bigcap(Q - \{q_1, \dots, q_k\}) \geq p$. If $q_k \geq p$, then $\bigcap(Q - \{q_1, \dots, q_{k-1}\}) \geq p$, and hence by induction we have $p_1 \cup \cdots \cup p_k \geq p_1 \cup \cdots \cup p_{k-1} \geq p$. If $q_k \ngeq p$ and $p \neq p_k$, then since $q_k \ngeq p_k$ it follows from Lemma 5.2 that $q_k \cap (p \cup p_k) \succ a$. Now $\bigcap(Q - \{q_1, \dots, q_{k-1}\}) \geq q_k \cap (p \cup p_k)$, and hence by induction we have $p_1 \cup \cdots \cup p_{k-1} \geq q_k \cap (p \cup p_k)$. Thus, since $p \cup q_k = p_k \cup q_k \succ q_k$,

$$p_1 \cup \cdots \cup p_k \geq p_k \cup (q_k \cap (p \cup p_k)) = (p_k \cup q_k) \cap (p \cup p_k) = (p \cup q_k) \cap (p \cup p_k) \geq p,$$

and hence the lemma follows.

THEOREM 5.5. If $a \in L$ and $a = \bigcap Q$ is an irredundant decomposition of a, then $u_a \cap \bigcup_{q\in Q} \bigcap(Q - q) = h_Q$.

Proof. Suppose $\bigcup_{q \in Q} \bigcap (Q - q) \geq p \succ a$. Then since p is compact in u_a/a, there exists a finite subset $\{q_1, \ldots, q_n\} \subseteq Q$ such that $\bigcup_{i=1}^{n} \bigcap (Q - q_i) \geq p$. Thus $\bigcap (Q - \{q_1, \ldots, q_n\}) \geq \bigcup_i \bigcap (Q - q_i) \geq p$. If $\bigcap (Q - q_i) \geq p_i \succ a$, then $p_i \in H_Q$, and it follows from Lemma 5.3 that $p_1 \cup \cdots \cup p_n \geq p$. Hence $h_Q = \cup H_Q \geq p$ for all p such that $\bigcup_{q \in Q} \bigcap (Q - q) \geq p \succ a$. But then by Lemma 5.1, $h_Q \geq \bigcup_{q \in Q} \bigcap (Q - q) \cap u_a$. Since $u_a \cap \bigcup_{q \in Q} \bigcap (Q - q) \geq h_Q$ holds trivially, the theorem follows.

In the succeeding theorems it will be shown that the replacement properties of irredundant decompositions are determined by the order properties of the elements h_Q. The first theorem shows that h_Q is invariant under a finite number of replacements.

THEOREM 5.6. Let $a \in L$ and $a = \bigcap Q$ be an irredundant decomposition of a. If $a = \bigcap Q'$ where Q' is obtained from Q by replacing an element $q \in Q$ by an irreducible q', then $h_{Q'} = h_Q$.

Proof. By Theorem 5.1 the decomposition $a = \bigcap Q'$ is irredundant. Let p be the unique element of H_Q such that $\bigcap (Q - q) \geq p$. For each $p^* \in H_Q - p$ let q^* be that element of Q such that $\bigcap (Q - q^*) \geq p^*$, and let $\bar{p}^*$ be the unique element of $H_{Q'}$ such that $\bigcap (Q' - q^*) \geq \bar{p}^*$. The correspondence $p^* \longrightarrow \bar{p}^*$ is then a one-one mapping of $H_Q - p$ onto $H_{Q'} - p$. If $q' \geq p^*$, then $\bigcap (Q' - q^*) = q' \cap \bigcap (Q - \{q, q^*\}) \geq p^*$, and hence $p^* = \bar{p}^*$. Thus $p^* \neq \bar{p}^*$ implies that $q' \not\geq p^*$. Now $q' \not\geq p$, and thus if $p^* \neq \bar{p}^*$ it follows from Lemma 5.2 that $q' \cap (p \cup p^*) \succ a$. But then $\bigcap (Q' - q^*) = \bigcap (Q - \{q, q^*\}) \cap q' \geq q' \cap (p \cup p^*)$, and thus $\bar{p}^* = q' \cap (p \cup p^*)$. Hence if $p^* \neq \bar{p}^*$ we have

$$p \cup \bar{p}^* = p \cup (q' \cap (p \cup p^*)) = (p \cup q') \cap (p \cup p^*) = p \cup p^*.$$

Since $p \cup \bar{p}^* = p \cup p^*$ holds trivially when $p^* = \bar{p}^*$, it follows that

$$h_{Q'} = \cup H_{Q'} = \bigcup_{p^*} (p \cup \bar{p}^*) = \bigcup_{p^*} (p \cup p^*) = \cup H_Q = h_Q.$$

This completes the proof.

THEOREM 5.7. If $a \in L$ and $a = \cap Q = \cap Q'$ are two irredundant decompositions of a with $h_Q \geq h_{Q'}$, then for each $q' \in Q'$ there exists $q \in Q$ such that $a = q' \cap \bigcap(Q - q) = q \cap \bigcap(Q' - q')$.

Proof. Let $q' \in Q'$, and let $p' \in H_{Q'}$ be such that $\bigcap(Q' - q') \geq p'$. Since $\cup H_Q \geq \cup H_{Q'} \geq p'$, there is a minimal finite subset $\{p_1, \ldots, p_n\} \subseteq H_Q$ such that $p_1 \cup \cdots \cup p_n \geq p'$. If $q_i \in Q$ is such that $\bigcap(Q - q_i) \geq p_i$ $(i = 1, \ldots, n)$, then $\bigcap(Q - \{q_1, \ldots, q_n\}) \geq p_1 \cup \cdots \cup p_n \geq p'$. If $q_j \geq p'$ then $\bigcap(Q - \{q_1, \ldots, q_{j-1}, q_{j+1}, \ldots, q_n\}) \geq p'$, and hence by Lemma 5.3 it follows that $p_1 \cup \cdots \cup p_{j-1} \cup p_{j+1} \cup \cdots \cup p_n \geq p'$, contrary to the minimality of $\{p_1, \ldots, p_n\}$. Thus $q_i \ngeq p'$ for each $i = 1, \ldots, n$. Now $q' \ngeq p'$, and hence there exists $p_k \in \{p_1, \ldots, p_n\}$ such that $q' \ngeq p_k$. Thus it follows from Theorem 5.2 and the atomicity of L that $a = q' \cap \bigcap(Q - q_k) = q_k \cap \bigcap(Q' - q')$, completing the proof.

COROLLARY. Let $a \in L$ and $a = \cap Q$ be an irredundant decomposition of a. Then for every irredundant decomposition $a = \cap Q'$ and every $q' \in Q'$ there exists $q \in Q$ such that $a = q' \cap \bigcap(Q - q)$ if and only if $h_Q = u_a$.

Proof. If $h_Q = u_a$, then for any irredundant decomposition $a = \cap Q'$ we have $h_Q = u_a \geq h_{Q'}$, and hence Theorem 5.7 holds for the decompositions $a = \cap Q = \cap Q'$. Suppose $h_Q < u_a$. Then H_Q is not a maximal independent set of elements covering a, and hence there is an independent subset J of the elements covering a such that $J \supset H_Q$. By Theorem 5.4 there exists an irredundant decomposition $a = \cap Q'$ such that $H_{Q'} = J$. Let $p \in J - H_Q$, and let $q' \in Q'$ be such that $\cap(Q' - q') \geq p$. Then $q' \geq \cup H_Q$, and hence $q' \cap \cap(Q - q) > a$ for all $q \in Q$. Therefore q' can replace none of the irreducibles in the decomposition $a = \cap Q$.

Let $a = \cap Q$ be an irredundant decomposition of $a \in L$, and let S be a subset of Q. A set T of irreducibles of L is said to be Q-<u>equivalent to</u> S if there is a one-one mapping f of S onto T such that $a = f(q) \cap \cap(Q - q)$ for each $q \in S$.

THEOREM 5.8. Let $a \in L$ and $a = \cap Q = \cap Q'$ be two irredundant decompositions of a. Then for each finite subset $S \subseteq Q$ there exists a subset $T \subseteq Q'$ such that T is Q-equivalent to S. If $h_Q \geq h_{Q'}$, then for each finite subset $T \subseteq Q'$ there exists a subset $S \subseteq Q$ such that T is Q-equivalent to S.

Proof. Let $\{q_1, \ldots, q_n\}$ be a finite subset of Q. For each $i = 1, \ldots, n$ let $S_i = \{q' \in Q' : a = q' \cap \cap(Q - q_i)\}$. Suppose for some k-element subset $\{q_{i_1}, \ldots, q_{i_k}\} \subseteq \{q_1, \ldots, q_n\}$, $S_{i_1} \cup \cdots \cup S_{i_k}$ contains $m < k$ elements, say $q_1', \ldots, q_m'$. Let $p_{i_j} \in H_Q$ be such that $\cap(Q - q_{i_j}) \geq p_{i_j}$ $(j = 1, \ldots, k)$, and let $p_j' \in H_{Q'}$ be such that $\cap(Q' - q_j') \geq p_j'$ $(j = 1, \ldots, m)$. Then for each $q' \in Q' - \{q_1', \ldots, q_m'\}$, $q' \cap \cap(Q - q_i) > a$, and hence $q' \geq p_i$ for each $i = 1, \ldots, k$, since $\cap(Q - q_i)$ contains a unique element covering a. Thus $\cap(Q' - \{q_1', \ldots, q_m'\}) \geq p_1 \cup \cdots \cup p_k$, and it follows from Lemma 5.3 that $p_1' \cup \cdots \cup p_m' \geq p_1 \cup \cdots \cup p_k$. But this is impossible since the dimension m of the sublattice $p_1' \cup \cdots \cup p_m'/a$ is less that the dimension k of the sublattice $p_1 \cup \cdots \cup p_k/a$. Hence for every k-element subset $\{q_{i_1}, \ldots, q_{i_k}\} \subseteq \{q_1, \ldots, q_n\}$ we have that $S_{i_1} \cup \cdots \cup S_{i_k}$ contains at least k distinct elements. It now follows from the Hall theorem [9] on representatives of subsets that there are distinct elements $q_1', \dots, q_n' \in Q'$ such that $q_i' \in S_i$ for each $i = 1, \dots, n$, completing the proof of the first part.

Suppose now that $h_Q \geq h_{Q'}$. Let $\{q_1', \dots, q_n'\}$ be a finite n-element subset of Q', and let $T_i = \{q \in Q : a = q_i' \cap \bigcap (Q - q)\}$. In view of the preceding paragraph, to prove the second part of the theorem it suffices to show that $T_1 \cup \dots \cup T_n$ contains at least n distinct elements. Suppose $T_1 \cup \dots \cup T_n$ contains $m < n$ elements, say $q_1, \dots, q_m$. For each $i = 1, \dots, m$, let $p_i \in H_Q$ be such that $\bigcap (Q - q_i) \geq p_i$. Then if $p \in H_Q - \{p_1, \dots, p_m\}$, $q_i' \geq p$ for each $i = 1, \dots, n$, and hence $q_1' \cap \dots \cap q_n' \geq \bigcup (H_Q - \{p_1, \dots, p_m\})$. Since $h_Q \geq \bigcup H_{Q'}$ and q_i' is covered by a unique element, $h_Q \cup q_i' \succ q_i'$, and hence $h_Q \succ h_Q \cap q_i'$. Furthermore, if $k < n$, then $(h_Q \cap q_1' \cap \dots \cap q_k') \cup q_{k+1}' \succ q_{k+1}'$, and hence $h_Q \cap q_1' \cap \dots \cap q_k' \succ h_Q \cap q_1' \cap \dots \cap q_{k+1}'$. Thus the sublattice $h_Q / h_Q \cap q_1' \cap \dots \cap q_n'$ is of dimension n. But now we have a contradiction since the sublattice $h_Q / \bigcup (H_Q - \{p_1, \dots, p_m\})$ is of dimension $m < n$. Hence $T_1 \cup \dots \cup T_n$ contains at least n distinct elements, and the proof is complete.

COROLLARY. If $a \in L$ and $a = \bigcap Q = \bigcap Q'$ are two irredundant decompositions of a with $h_Q \leq h_{Q'}$, then there exists a subset $T \subseteq Q'$ such that T is Q-equivalent to Q.

Proof. For each $q \in Q$ let $S_q = \{q' \in Q' : a = q' \cap \bigcap (Q - q)\}$. Then from the theorem above it follows that for any finite subset $\{q_1, \dots, q_n\} \subseteq Q$, $S_{q1} \cup \dots \cup S_{qn}$ contains at least n distinct elements. Now suppose that $q \in Q$ and $p \in H_Q$ is such that $\bigcap (Q - q) \geq p$. Then $p \leq h_Q \leq h_{Q'} = \bigcup H_{Q'}$, and hence $p \leq p_1' \cup \dots \cup p_k'$ for some finite subset $\{p_1', \dots, p_k'\} \subseteq H_{Q'}$. If q_i' is such that $\bigcap (Q' - q_i') \geq p_i'$ $(i = 1, \dots, k)$, then

$\cap(Q' - \{q_1', \ldots, q_k'\}) \geq p$, and hence $\{q_1', \ldots, q_k'\} \supseteq S_q$. Thus S_q is finite for each $q \in Q$. The corollary now follows from the Hall theorem [9].

COROLLARY. If $a \in L$ and $a = \cap Q = \cap Q'$ are two irredundant decompositions of a with $h_Q = h_{Q'}$, then Q and Q' have the same cardinality.

6. LATTICES WITH REPLACEABLE DECOMPOSITIONS

Aside from uniqueness, the most fundamental arithmetical property of irredundant decompositions is the replacement property. Therefore, if L is a compactly generated atomic lattice and $a \in L$, we shall say that a has <u>replaceable irredundant decompositions</u> if for every pair of irredundant decompositions $a = \bigcap Q = \bigcap Q'$ and each $q \in Q$ there exists an element $q' \in Q'$ such that $a = q' \cap \bigcap (Q - q)$, and this decomposition is irredundant. The principal theorem of this section characterizes those lattices L which have replaceable irredundant decompositions.

For every pair of elements $x, y \in L$ define the element $u_{x/y}$ by

$$u_{x/y} = \bigcup \{p : p \leq x, \ p \succ y\}.$$

THEOREM 6.1. Every element of a compactly generated atomic lattice L has replaceable irredundant decompositions if and only if L satisfies the following condition:

(ρ) If $x, y \in L$, then $u_{x \cup y/x} \succ x$ implies $u_{y/x \cap y} \succ x \cap y$.

Notice that condition (ρ) may be stated alternatively as follows. If x, y are any two elements of L, and the quotient sublattice $x \cup y/x$ contains a unique element covering x, then the sublattice $y/x \cap y$ contains a unique element covering $x \cap y$.

Proof of Theorem 6.1. The proof of the theorem is based on the following lemmas.

LEMMA 6.1. An element $a \in L$ has replaceable irredundant decompositions if and only if $q \cap (p_1 \cup p_2) = a$ implies $p_1 = p_2$ for every irreducible $q \geq a$ and every pair of atoms $p_1, p_2 \succ a$.

Let $a \in L$ be an element for which the condition of Lemma 6.1 is satisfied. Let $a = \bigcap Q = \bigcap Q'$ be two irredundant decompositions of a, and let $q \in Q$. Since $a = q \cap \bigcap(Q - q)$, it follows from the condition of the lemma that there is a unique element $p \succ a$ such that $\bigcap(Q - q) \geq p$. Because $\bigcap Q' = a$, there is an element $q' \in Q'$ such that $q' \ngeq p$, and hence it follows from the atomicity of L that $q' \cap \bigcap(Q - q) = a$. Suppose this decomposition is redundant. Then there exists an element $q_1 \in Q - q$ such that $a = q' \cap \bigcap(Q - \{q, q_1\})$. Again since $q' \cap \bigcap(Q - \{q, q_1\}) = a$ and a satisfies the condition of the lemma, it follows that there is a unique element $p_1 \succ a$ such that $\bigcap(Q - \{q, q_1\}) \geq p_1$. But since $\bigcap(Q - \{q, q_1\}) \geq \bigcap(Q - q) \geq p$, we must have $p = p_1$. This implies that $q \cap \bigcap(Q - \{q, q_1\}) = \bigcap(Q - q_1) = a$, contrary to the irredundance of the decomposition $a = \bigcap Q$. Hence the decomposition $a = q' \cap \bigcap(Q - q)$ is irredundant, and a has replaceable irredundant decompositions.

Suppose now that $a \in L$ has replaceable irredundant decompositions. Let $q \geq a$ be an irreducible and $p_1, p_2 \succ a$ be elements such that $a = q \cap (p_1 \cup p_2)$. By Lemma 2.3 there exists a maximal element $m \geq p_1 \cup p_2$ such that $q \cap m = a$. Let $m = \bigcap R$ be an irredundant decomposition of m. Then because of the maximality of m it follows that $a = q \cap \bigcap R$ is an irredundant decomposition of a. If $p_1 \neq p_2$, then by Lemma 3.2 there exists an irreducible $q_1 \geq p_1$ such that $q_1 \ngeq p_2$, and hence $q_1 \cap p_2 = a$. Let $m_1 \geq p_2$ be a maximal element such that $q_1 \cap m_1 = a$, and let $m_1 = \bigcap R_1$ be an irredundant decomposition of m_1. Then again it follows from the maximality of m_1 that $a = q_1 \cap \bigcap R_1$ is an irredundant decomposition of a. But now it follows that $q_1 \cap \bigcap R \geq p_1 > a$, and $q' \cap \bigcap R \geq p_2 > a$ for every $q' \in R_1$. Since this is contrary to the assumption that a has replaceable irredundant decompositions we must have $p_1 = p_2$, and hence the lemma follows.

LEMMA 6.2. If an element $a \in L$ has replaceable irredundant decompositions in L, then a has replaceable irredundant decompositions in the sublattice x/a for every $x \geq a$.

For suppose r is a completely irreducible element of the quotient sublattice x/a and $r \cap (p_1 \cup p_2) = a$ for elements p_1, p_2 such that $x \geq p_1, p_2 \succ a$. If $r = \bigcap Q$ is a decomposition of r into elements which are completely irreducible in L, then

$$r = x \cap \bigcap Q = \bigcap_{q \in Q} x \cap q,$$

and since r is completely irreducible in x/a it follows that $r = x \cap q$ for some irreducible $q \in Q$. Hence

$$q \cap (p_1 \cup p_2) = q \cap x \cap (p_1 \cup p_2) = r \cap (p_1 \cup p_2) = a,$$

and since a has replaceable irredundant decompositions in L it follows from Lemma 6.1 that $p_1 = p_2$. Thus a has replaceable irredundant decompositions in the sublattice x/a.

Proceeding now with the proof of the theorem, let L be a compactly generated atomic lattice satisfying condition (ρ). Let $a \in L$, $q \geq a$ be an irreducible of L, and $p_1, p_2 \succ a$ be elements such that $q \cap (p_1 \cup p_2) = a$. Since q is completely irreducible in L, there is a unique element s covering q. Thus $u_{p_1 \cup p_2 \cup q/q} = s \succ q$, and hence by condition (ρ) it follows that $u_{p_1 \cup p_2/a} \succ a$. This implies that $p_1 = p_2$, and hence by Lemma 6.1 every element of L has replaceable irredundant decompositions.

Suppose every element of L has replaceable irredundant decompositions.

Let $x, y \in L$ be such that $u_{x\cup y/x} \succ x$. If $u_{y/x\cap y}$ does not cover $x \cap y$, then there are two distinct elements p_1, p_2 such that $y \geq p_1, p_2 \succ x \cap y$. Since $u_{x\cup y/x} \succ x$ it follows that x is completely irreducible in the quotient sublattice $x \cup y/x \cap y$. And since $x \cap (p_1 \cup p_2) = x \cap y$ for two distinct elements $p_1, p_2 \succ x \cap y$, it follows from Lemma 6.1 that $x \cap y$ does not have replaceable irredundant decompositions in the sublattice $x \cup y/x \cap y$. But then by Lemma 6.2 it follows that $x \cap y$ does not have replaceable decompositions in L. This contradiction implies that $u_{y/x\cap y} \succ x \cap y$. Thus L satisfies (ρ), and the proof of Theorem 6.1 is complete.

It is clear that if L is a <u>point lattice</u>, that is, if x is the join of elements covering y for every pair of elements $x > y$ in L, then condition (ρ) is equivalent to lower-semimodularity. Therefore a compactly generated point lattice has replaceable irredundant decompositions if and only if the lattice is lower-semimodular.

THEOREM 6.2. If L is a semimodular, compactly generated, atomic lattice, then L satisfies condition (ρ) if and only if L is locally modular.

Proof. If L is locally modular, then (ρ) follows immediately from Lemma 4.1. Suppose then that L satisfies condition (ρ). If $a \in L$ and every element of u_a/a is a join of elements covering a, then u_a/a is a point lattice. For if $x > y$ in u_a/a, then $x = \bigcup \{p \cup y : p \succ a,\ p \leq x,\ p \not\leq y\}$, and since $p \not\leq y$ implies $p \cup y \succ y$, the assertion follows. Hence, in view of the corollary to Theorem 2.1 and the remark following the proof of Theorem 6.1, to show that L is locally modular it suffices to show that every element of u_a/a is a join of elements covering a for each $a \in L$. Since u_a/a is compactly generated we need only show that each compact element is a join of elements covering a. If c is a compact element of u_a/a, then there is a minimal finite number of elements $p_1, \dots, p_k \succ a$ such that $p_1 \cup \cdots \cup p_k \geq c$. In a semimodular lattice two finite maximal chains between a pair of elements of the lattice have the same length. Thus, if $p_1 \cup \cdots \cup p_k > c$ and t is such that $p_1 \cup \cdots \cup p_k \succ t \geq c$, then t is not a join of elements covering a. Let $t_1 = \bigcup\{p \succ a : p \leq t\}$. Then since $p_1 \cup \cdots \cup p_k/t_1$ has dimension at least two, there are two distinct elements $p, p' \succ a$ such that $p_1 \cup \cdots \cup p_k \geq p, p'$, $t_1 \not\geq p, p'$, and $t_1 \cup p \not\geq p'$. Now $p \cup p' \cup t = p_1 \cup \cdots \cup p_k \succ t$, and hence by condition (ρ) it follows that $t \cap (p \cup p') > a$. Since every chain in $p \cup p'/a$ has length at most two, we must have $t \cap (p \cup p') \succ a$. But since $p \cup (t \cap (p \cup p')) = p \cup p' \geq p'$, it follows that $t_1 \not\geq t \cap (p \cup p')$, contrary to the definition of t_1. Hence $c = p_1 \cup \cdots \cup p_k$, and Theorem 6.2 follows.

COROLLARY. Every element of a semimodular, compactly generated, atomic lattice L has replaceable irredundant decompositions if and only if L is locally modular.

Observe that a compactly generated atomic lattice L has unique irredundant decompositions if and only if u_a/a has this property for each $a \in L$. Similarly if L is semimodular, then L has replaceable irredundant decompositions if and only if every u_a/a has replaceable decompositions. Therefore the uniqueness of decompositions in a general lattice and the replacement property in a semimodular lattice are "local" properties in the sense that they are determined by the sublattices u_a/a.

In passing to the question of replaceable decompositions in a general compactly generated atomic lattice L a different situation is encountered. If L is finite dimensional, then it follows almost trivially that L

satisfies condition (ρ) if and only if u_a/a satisfies (ρ) for each $a \in L$. If finite dimensionality is dropped, however, then L need not satisfy (ρ) even though u_a/a satisfies (ρ) for every $a \in L$. Thus, unlike that of unique decompositions, the property of replaceable decompositions is fundamentally different in the general case than in the finite dimensional case. The following example illustrates this.

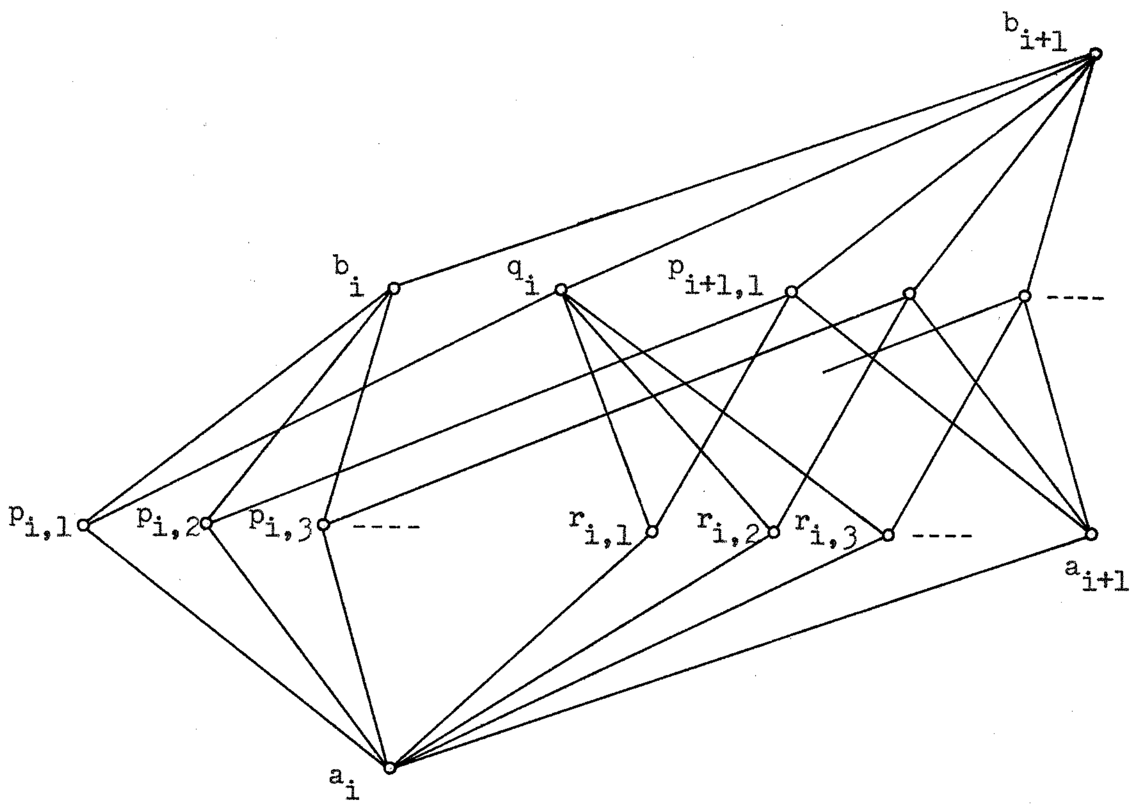

Fig. 2.

Let L_1 be that lattice comprised of two infinite chains $a_1 < a_2 < \cdots < a_i < \cdots$ and $b_1 < b_2 < \cdots < b_i < \cdots$, such that $b_i > a_i$, and such that the sublattice b_{i+1}/a_i is isomorphic with the lattice of Fig. 2. for each $i = 1, 2, \ldots$. Notice that the lattice of Fig. 2. has the property that $p_{i,1} \cup a_{i+1} = b_{i+1}$, and hence $p_{1,i} \cup a_i = b_i$ and $r_{1,i} \cup a_{i+1} = b_{i+1}$. Now let L be the lattice of ideals of L_1. Every nonprincipal ideal of L_1 necessarily contains the ideal $A = (a_1, a_2, a_3, \ldots)$. Suppose B is an ideal of L_1 with $B > A$. Then B must contain one of the elements $p_{1,k}$ or $r_{1,k}$ which cover a_1. Thus B contains either $p_{1,k} \cup a_k = b_k$ or $r_{1,k} \cup a_{k+1} = b_{k+1}$, and hence $B \geq (b_1, b_2, b_3, \ldots) = L_1$. Thus A and L_1 are the only two nonprincipal ideals, whence L is compactly generated and atomic. Now the only elements covered by more than two elements are the elements a_i. Thus if $x \neq a_i$ $(i = 1, 2, \ldots)$, then u_x/x satisfies (ρ). On the other hand, $u_{a_i}/a_i = b_{i+1}/a_i$, and it is apparent from Fig. 2. that u_{a_i}/a_i also satisfies (ρ). But L does not satisfy (ρ), since $u_{b_1 \cup A/A} \succ A$ and $u_{b_1/b_1 \cap A} = u_{b_1/a_1}$ does not cover a_1.

We may also consider the stronger replacement property described in Theorem 5.7. With this in mind, let us define an element a in a compactly generated atomic lattice L to have <u>simultaneously replaceable irredundant decompositions</u> if for every pair of irredundant decompositions $a = \bigcap Q = \bigcap Q'$ and each $q \in Q$ there exists $q' \in Q'$ such that $a = q' \cap \bigcap(Q - q) = q \cap \bigcap(Q' - q')$.

THEOREM 6.3. Every element of a semimodular, compactly generated, atomic lattice L has simultaneously replaceable irredundant decompositions if and only if u_a/a is a direct product of finite dimensional modular lattices for every $a \in L$.

Proof. If u_a/a is a direct product of finite dimensional modular lattices, then it is easily checked that $h_Q = u_a$ for every irredundant decomposition $a = \bigcap Q$. Hence the sufficiency follows from Theorem 5.1.

The proof of the necessity depends on the following well known lemma essentially due to Frink [8].

LEMMA. If K is a complete, atomic, complemented, modular lattice with a null element z, then K is a direct product of quotient sublattices e_α/z, where each sublattice e_α/z has the property that the join of any pair of distinct elements covering a contains a third distinct element covering a.

Now suppose every element of a semimodular, compactly generated, atomic lattice L has simultaneously replaceable decompositions. Then by the corollary to Theorem 6.2 it follows that L is locally modular. Let $a \in L$. Since u_a/a is complemented by Lemma 5.1, it follows by the preceding lemma that u_a/a is a direct product of sublattices e_α/a each of which has the property expressed in the lemma. Suppose one of the sublattices, say e_β/a, is not finite dimensional. Then e_β/a contains an infinite independent set J of elements covering a. We may assume that J is a maximal independent subset of $\{p \succ a : p \leq e_\beta\}$. Pick $p_o \in J$. Then for each $p \in J - p_o$ there is an element $\bar{p} \succ a$ such that $p_o \cup p \geq \bar{p}$ and $\bar{p} \neq p_o, p$. Define

$$s_p = \bar{p} \cup \bigcup(J - \{p_o, p\})$$

for each $p \in J - p_o$, and define $S_\beta = \{s_p : p \in J - p_o\}$. Then $e_\beta \succ s_p$ and $s_p \not\geq p_o, p$ for each $p \in J - p_o$. Suppose $\bigcap S_\beta > a$. Then $\bigcap S_\beta \geq p'$ for some $p' \succ a$. Since $\bigcup J = e_\beta \geq p'$, there is a minimal finite subset $J' \subseteq J$ such that $\bigcup J' \geq p'$. If $p_o \in J'$, then as in the proof of Lemma 2.7, $p' \leq \bigcup J'$ implies $p_o \leq p' \cup \bigcup(J' - p_o)$. Since J is infinite, there exists $p \in J - J'$, whence

$$p_o \leq p' \cup \bigcup(J' - p_o) \leq p' \cup \bigcup(J - \{p_o, p\}) \leq s_p,$$

a contradiction. If $p_o \notin J'$, then with $p \in J'$ we again have

$$p \leq p' \cup \bigcup(J' - p) \leq p' \cup \bigcup(J - \{p_o, p\}) \leq s_p,$$

a contradiction. Hence $\bigcap S_\beta = a$, and clearly $\bigcap(S_\beta - s_p) \geq p$ for each $p \in J - p_o$.

For every $\alpha \neq \beta$, let S_α be a collection of maximal elements of e_α/a such that $\bigcap S_\alpha = a$, and $\bigcap(S_\alpha - s) > a$ for all $s \in S_\alpha$. If α is any index (including $\alpha = \beta$) and $s \in S_\alpha$, then $u_a \succ s \cup \bigcup_{\gamma \neq \alpha} e_\gamma$, and hence there is an irreducible q_s such that $q_s \cap u_a = s \cup \bigcup_{\gamma \neq \alpha} e_\gamma$. Let $Q = \{q_s : s \in \bigcup_\alpha S_\alpha\}$. It then follows that $a = \bigcap Q$ is an irredundant decomposition of a.

If $p_o \leq h_Q$, then there is a minimal finite subset $\{p_1, \dots, p_k\} \subseteq H_Q$ such that $p_1 \cup \dots \cup p_k \geq p_o$. Observe that if $s \in S_\alpha$ where $\alpha \neq \beta$, then $q_s \geq p_o$. Hence if $\bigcap(Q - q_{s_i}) \geq p_i$, then $\{s_1, \dots, s_k\} \subseteq S_\beta$. This implies that $\{p_1, \dots, p_k\} \subseteq J - p_o$, contrary to the independence of J. Hence $p_o \nleq h_Q$. But now it follows from the corollary to Theorem 5.7 that a cannot have simultaneously replaceable decompositions, contrary to assumption. Thus each e_α/a is finite dimensional, and the proof is complete.

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