Electricity Markets for the Smart Grid: Networks, Timescales, and Integration with Control

Thesis by

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This thesis is dedicated to

my soulmate Brenda,

who's love and encouragement made this thesis possible,

and my parents,

who have always supported me in everything I've attempted.

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Abstract

We are at the dawn of a significant transformation in the electric industry. Renewable generation and customer participation in grid operations and markets have been growing at tremendous rates in recent years and these trends are expected to continue. These trends are likely to be accompanied by both engineering and market integration challenges. Therefore, to incorporate these resources efficiently into the grid, it is important to deal with the inefficiencies in existing markets. The goal of this thesis is to contribute new insights towards improving the design of electricity markets.

This thesis makes three main contributions. First, we provide insights into how the economic dispatch mechanism could be designed to account for price-anticipating participants. We study this problem in the context of a networked Cournot competition with a market maker and we give an algorithm to find improved market clearing designs. Our findings illustrate the potential inefficiencies in existing markets and provides a framework for improving the design of the markets. Second, we provide insights into the strategic interactions between generation flexibility and forward markets. Our key insight is an observation that spot market capacity constraints can significantly impact the efficiency and existence of equilibrium in forward markets, as they give producers incentives to strategically withhold offers from the markets. Third, we provide insights into how optimization decomposition theory can guide optimal design of the architecture of power systems control. In particular, we illustrate a context where decomposition theory enables us to jointly design market and control mechanisms to allocate resources efficiently across both the economic dispatch and frequency regulation timescales.

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Chapter 1

Introduction

We are at the dawn of a significant transformation in the electric industry. Renewable generation and customer participation in grid operations and markets have been growing at tremendous rates in recent years. These trends are supported in part by federal law which requires grid modernization to increase dependency on variable and distributed energy resources [41]. Therefore, the electrical grid is likely to transition from a vertical structure with large predictable resources and centralized operations to a more horizontal structure with intermittent resources and distributed operations. This revolution presents both challenges and opportunities. There has never been a more important and exciting time for power researchers.

To date, 38 states have enacted renewable portfolio standards or goals that require 10% to 33% of energy delivered to customers to be generated from renewable resources by 2020 [3, 43]. Currently, renewable generation accounts for 17.27% of total generation capacity in the U.S. and 70% of total generation capacity installed in the first half of 2015 [1, 2]. The unpredictability of some of these resources, such as solar power, wind power, and tidal power, can lead to grid integration challenges, because the existing grid has been designed to operate primarily with dispatchable generation such as gas, coal, nuclear, or hydro plants [52, 7]. Renewable resources also have high fixed costs and close to zero variable costs – a completely different cost structure from the majority of dispatchable resources – which could lead to undesirable market behavior [117].

Customers are also increasingly active in grid operations and markets. Onsite generation such as rooftop solar photovoltaics and energy storage enable customers to supply excess energy to the grid. Demand response technologies such as remotely controllable electric vehicle chargers and smart appliances allow customers to provide flexible demand services to the market. These are new sources of flexibility that can be utilized to manage the grid. However, they also introduce new dynamics and feedback loops that could destabilize both the grid and markets, because the latter have been designed to operate with power flowing one way from generators to unresponsive demand [109].

To deal with these challenges, new operational and market architectures are needed. This thesis addresses some of the challenges in electricity markets.

1.1 Role of Markets in the Power Grid

Markets have important roles in the functioning of the power grid. Power grids in the U.S. are operated by an Independent Systems Operator (ISO), which is a regulated non-profit entity with the responsibility of ensuring efficient and reliable grid operations. To operate the grid efficiently, the ISO operates wholesale spot markets. In these markets, the ISO uses bids/offers from generators/retailers to solve a problem known as economic dispatch, which allocates resources on the transmission network to meet demand at least cost subject to operational constraints [34, 138, 35, 74, 14, 66, 114]. This problem is important because resources at different locations of the network could have different costs. The operational constraints include Kirchoff's laws that govern how power can flow, as well as safety constraints, such as capacity constraints, line limits, contingency constraints, and stability constraints. The economic dispatch problem is solved numerous times every day and as frequently as every 5 minutes. The ISO also calculates the payments/fees to the generators/retailers based on locational marginal prices which are derived from Lagrange multipliers of the economic dispatch problem.

To ensure that the grid performs reliably and provide appropriate incentives for investments in capacity, the ISO also operates other markets that ensure there are always adequate resources for meeting demand so that the probability of blackouts is low. This is important because of the non-storable nature of electricity, capacity and ramping limitations of generators, and the need to balance supply and demand at every instant and at every location. The ISO operates multiple forward markets ahead of time where power is procured in advance. These forward markets range from hour-ahead markets to day-ahead markets and beyond. The ISO also operates markets to procure reserve capacity for ancillary services such as frequency regulation or reactive power control. These markets provide resources for power systems controllers to use to maintain the stability and security of the grid.

Although electricity markets have been prevalent and widely studied since the deregulation in the 90s, market design issues continue to be relevant and important as market inefficiencies can be extremely costly. During 2000-01, market manipulations in California led to state-wide blackouts which were estimated to cost between \$40 to \$45 billion [125]. Even though California's electricity crisis is long past, examples of generators attempting to exploit this sort of market power are still common today, e.g., JPMorgan Chase was fined \$410 million for market manipulations in California and the Midwest from September 2010 to November 2012 [54].

1.2 Challenges Facing Electricity Markets Today

The growth of intermittent renewable resources and active customer participation in grid operations and markets could exacerbate existing market inefficiencies and undermine the efficiency and reliability of the grid. Among other impacts, these trends are likely to be accompanied by increased usage of fast-ramping resources and ancillary services [107, 47, 122]. Increased dependence on the latter resources could give those producers more opportunities to manipulate the market. Forward markets will have an increasingly important role in hedging against the variability of renewable resources but market inefficiencies will impede their ability to ensure there are adequate resources to balance unexpected loss of output from renewables [21, 29]. Ancillary service markets need to provide the right price signals to ancillary service providers to offer their capacity where and when it is needed most. Therefore, to reduce the costs of integrating intermittent renewable resources and customer participation in markets, it is important to first address the inefficiencies in markets today. This thesis focuses on three challenges.

1.2.1 How to allocate resources efficiently on the network

The economic dispatch problem and pricing mechanism is the basis by which resources are allocated on the power grid in real-time, and therefore, the ISO's design of the problem and pricing mechanism has a significant impact on the efficiency of the market and the incentives to the participants. Existing implementations typically seek to maximize some metric of social benefit and derive locational marginal prices from the Lagrange multipliers of the optimization problem. These designs have their roots in [34] and can be viewed as extensions of classical results in microeconomics to the power grid. However, these implementations are susceptible to strategic manipulations by market participants, as have been observed in both theoretical models and data [22, 81, 130, 83, 82, 112, 132, 53].

There are many reasons for why producers are able to strategically manipulate the market. The operating constraints of the network such as Kirchoff's laws and safety constraints could restrict the competition between different regions of the network. Moreover, dispatchable resources, such as gas, coal, or nuclear, are typically served by large generating plants that may individually produce a significant fraction of the total load. In addition, the grid is typically composed of a few firms, each owning multiple generators. These factors imply that producers could drive up market prices significantly when they withhold offers from the market.

Therefore, a key challenge is how to improve the efficiency of the economic dispatch mechanism against the strategic incentives of market participants. The pricing mechanism needs to provide incentives for generators to offer their resources to the market without distorting prices. The design must also respect the operating constraints of the system and be computationally efficient since economic dispatch operates as frequently as every 5 minutes.

1.2.2 How to ensure adequate resources to meet demand

Forward markets ranging from short term (1 to 24 hours) to medium term (1 to 3 years) and long term (3 years and beyond) provide multiple benefits such as helping to reduce risk, mitigating market power, and coordinating new investment in generation capacity [13]. Short term forward markets are typically operated by the ISO while the medium to long term markets are typically left to bilateral negotiations. Forward contracts cause producers to enter the real-time market with more balanced positions and reduce incentives for them to manipulate the market. They also provide needed long-term price signals to coordinate investment in capacity where and when they are most needed.

However, forward markets are also susceptible to strategic manipulation by market participants. Producers can anticipate the impact of their actions in the forward markets on the real-time market, and have incentive to withhold offers from the forward markets, if doing so would cause real-time prices to increase disproportionately and earn them more profits. Such sophisticated incentives could arise due to network constraints and the nature of dispatchable resources being typically owned by large firms. In addition, timescale constraints such as ramping limitations could also restrict generators' competitiveness in the markets.

Therefore, a key challenge is how to improve the efficiency of forward markets against the strategic anticipatory behavior of market participants. These questions include: How many markets should the ISO operate? How far in advance should the ISO operate these markets? What mechanisms should the ISO use to clear these markets? What is the impact of timescale constraints on the competitiveness of these markets?

1.2.3 How to integrate markets with control systems

Grid operations are divided into two timescales/layers. The timescale of 5 minutes and longer is focused on efficiency and comprises economic dispatch and forward markets. Balancing of supply and demand within 5-minute intervals is focused on stability and is handled by frequency regulation [64, 18, 46]. In the latter, the ISO seeks to restore the nominal frequency in the system by rescheduling fast ramping generators. It is currently implemented by a control mechanism known as Automatic Generation Control (AGC). In this mechanism, the ISO uses information on local frequency deviation to compute the aggregate generation that would rebalance power and restore the nominal frequency. Then, the ISO divides the needed generation among frequency regulation resources based on participation factors, which are computed from the outcome of the previous economic dispatch run [138]. The resources are also compensated based on the LMP in the previous economic dispatch run.

The economic dispatch and frequency regulation layers have largely been studied and designed independently. However, such a separation could be costly. Participation factors and prices from the previous economic dispatch run might not be the most efficient way to dispatch frequency regulation resources. Such inefficiencies could in turn distort the incentives of the providers of frequency regulation resources. It could also lead to over or (worse) under procurement of frequency regulation reserves.

Hence, a key challenge is how to jointly design the market and control mechanisms to achieve efficiency and reliability across both timescales. At the fast timescale, frequency regulation controllers need to utilize information on cost of resources to maintain stability in a cost-efficient manner. At the slow timescale, economic dispatch mechanisms also need to consider the cost and stability benefits of frequency regulation resources when allocating generation.

1.3 Contributions of This Thesis

This thesis contributes to each of the three challenges described above.

1.3.1 Strategic design of the economic dispatch objective

In Chapter 2, we study the problem of designing the economic dispatch objective to account for price-anticipating producers. We model the market using a networked Cournot competition with a market maker who clears the market by maximizing a parameterized objective function subject to DC power flow constraints. Hence, the market maker models an ISO who could choose from a class of market clearing designs. We make a few contributions.

The first contribution is the formulation of the parameterized objective function, which in itself is a novel formulation that includes the design of social welfare that is typically used in existing work. The second contribution is characterizing a subregion of the design space where the market has a unique Nash equilibrium that is also the solution to a convex program (Theorem 2.2). This result provides both analytical insights as well as a new computational tool for finding the Nash equilibrium of the market. This result is novel even in the restricted setting where the market maker maximizes social welfare, which is the setting that is commonly used in existing work [142]. The third contribution is in finding the most efficient design for the market maker. In general, this problem is not convex and hence it is challenging to solve. We exploit our characterization to propose an algorithm for finding an approximate solution and provide another algorithm to judge the quality of our solution. Although this work is restricted to a Cournot model, our findings illustrate the potential inefficiencies in existing markets and provide a framework for improving the design of the markets.

1.3.2 Impact of generation flexibility on forward markets

In Chapter 3, we study the strategic interaction between generation flexibility and forward markets in the context of a non-networked setting with two classes of generators with different flexilibities. Generators are typically classified into two types: baseloads and peakers [123, 61, 75]. Baseloads refer to generators that are suited for supplying base demand – they have long start-up times and slow ramping rates but are typically bigger units with lower marginal costs (e.g. nuclear ($11.6/MWh) and coal ($28.6/MWh)). In contrast, peakers refer to generators that are suited for supplying peak demand – they have short start-up times and fast ramping rates but are typically smaller units with higher marginal costs (e.g. gas turbine ($48.0-$79.9/MWh)).

We model the market by combining the classical Stackelberg and forward contracting models. We assume that there are two types of firms and two stages. Leaders choose productions in the first stage while followers choose productions in the second stage. In addition, followers have capacity constraints and they sell forward contracts in the first stage. Hence, leaders model less flexible generation such as baseloads while followers model more flexible generation such as peakers.

We derive closed-form expressions of all symmetric Nash equilibria of the market. Our results yield novel insights into the impact of generation constraints on strategic incentives, among which two key insights are (1) peakers' generation constraints may give producers incentive to withhold offers from the forward market which could lead to non-existence of market equilibria, and (2) peakers' generation constraints may decrease the efficiency of the forward market to the extent that removing the forward market could increase the efficiency of the equilibrium. Hence, our results illustrate that forward markets might not always be beneficial.

1.3.3 Joint design of economic dispatch and frequency regulation

In Chapter 4, we study the joint problem of economic dispatch and frequency regulation in the context of a DC power flow model and two classes of generators with different flexibilities. We show that the two-timescale problem can be decomposed into two sub-problems that correspond to the economic dispatch and frequency regulation timescales, without loss of optimality, as long as the ISO is able to estimate the difference between the frequency regulation and economic dispatch LMPs (Theorem 4.1). Using this result, we design a frequency control algorithm and a market mechanism for economic dispatch, in a way such that the two mechanisms jointly solve the two-timescale problem. Our results can be viewed as a first-principles justification for the separation of power systems control into economic dispatch and frequency regulation problems. More importantly, our results illustrate how optimization decomposition theory can guide optimal design of the architecture of power systems control.

1.4 Overview of This Thesis

This thesis is organized as follows.

1. In Chapter 2, we study a class of market clearing objectives for economic dispatch using a networked cournot framework. We characterize cases where market equilibrium can be computed by solving a convex program and exploit this to design an algorithm for finding an improved design for the market clearing objective. This work has been submitted to a journal and is under review.

2. In Chapter 3, we study a game between two types of generators with different flexibilities in a forward market. We provide insights into the strategic incentives and equilibria of the game. This work appeared in [32]. A journal submission of this work is forthcoming.

3. In Chapter 4, we study the joint design of economic dispatch and frequency regulation. We propose a market mechanism and a frequency control algorithm such that the combination solves the joint problem. This work appeared in [27]. This work has also been submitted to a journal and is under review.

Not included in this thesis are papers on other market issues, such as contracting wind power [29], optimal investment of renewable generation [33], and the impact of rooftop solar photovoltaic adoption on the utility death spiral [30, 8], which were also written during my time at Caltech.

Chapter 2

Network Cournot Competition with Market Maker

In this chapter, we study the problem of designing the economic dispatch objective. This work has wider applications beyond electricity to other modern networked marketplaces. Classical models of competition often feature multiple firms operating in a single, isolated market; however, power systems, the internet, transportation networks, infrastructure networks, and global supply chains are just a few of the places where varied and complex interconnections among participants are crucial to understanding and optimizing marketplaces. Consequently, the study of competition in networked markets has emerged as an area with both rich theoretical challenges and important practical applications.

At this point, a wide variety of models for competition in networked markets have emerged across economics, operations research, and computer science. The work in this literature focuses both on extensions of classical models of competition to networked settings, e.g. networked Bertrand competition [60, 12, 37, 6] and networked Cournot competition [20, 4, 65], and on models of specific applications where networked competition is fundamental, e.g. electricity markets [97, 17, 16, 140, 141, 142, 67].

Intermediaries, market makers, and transport

The complexity of networked marketplaces typically leads to (and often necessitates) the emergence of intermediaries. A prominent illustration of this is financial markets, where central core banks intermediate trade between smaller periphery banks. Similar examples are common in infrastructure networks: natural gas is traded through pipelines, which are managed by a Transmission System Operator (TSO), and transport in electricity markets is governed by an Independent System Operator (ISO). One can view platforms in the sharing economy, e.g., Uber, as intermediaries between service providers and customers, and supply chains can be regarded as a form of intermediation in networked markets.

Intermediaries can play many roles in networked markets, from aggregation to risk mitigation to informational and beyond. Our focus in this study is on the role intermediaries play with respect to transport and trade. In particular, in many networked marketplaces participating firms depend on an intermediary, a.k.a., market maker, to provide transport of their goods between geographically distinct markets.

Electricity markets is a particularly prominent example. In these markets, the ISO solves a centralized dispatch problem by utilizing the offers/bids from the generators/retailers. This problem seeks to maximize some metric of social benefit subject to the operational constraints of the grid. These operational constraints include physical laws that govern the flow of power in the network as well as safety constraints such as line capacity limits. The payments are calculated based on locational marginal prices (LMP). Therefore, the ISO plays a crucial role in matching the demand and supply of power within the confines of the grid and also defines payments to the market participants. As an independent regulated entity, it further designs rules to limit the possible exercise of market power by the suppliers.

Beyond electricity markets, natural gas markets, and more generally, supply chains often have a similar structure where a market maker manages transport between geographically distributed markets.

Clearly, the design of the market maker in such situations is crucial to the efficiency of the marketplace. By facilitating trade, the market maker is providing a crucial opportunity for increased efficiency. However, constraints inherent to the transport network can make it difficult to realize this potential. As an example, network constraints can give rise to hidden monopolies, where even a small firm can exhibit market dominance because of its position in the network.

The dangers of such hidden monopolies are especially salient (and the corresponding efficiency loss is especially large) in the case of electricity markets, since power flows cannot be controlled in an end-to-end manner due to Kirchhoff's laws. Even though California's electricity crisis is long past, examples of generators attempting to exploit this sort of market power are still common today, e.g., JP Morgan was fined $410 million for market manipulations in California and the midwest from September 2010 to November 2012 [54].

2.1 Our Contributions

Our goal in this study is to provide insight into the design (and regulation) of market makers that govern transport in networked marketplaces. In particular, we study a model of networked Cournot competition in which transport between geographically distinct markets is governed by a market maker (market operator) and subject to network flow constraints. Our results focus on the impact the design of the market maker has on the equilibrium outcomes of the game between firms and the market maker.

Our first contribution is the model itself. We introduce a general, parameterized model of a market maker (Section 3.3) in a centrally managed networked Cournot competition. In our model, each market contains multiple firms competing locally in a Cournot competition. The market maker acts as an intermediary between markets by buying from some markets and selling to other markets, using its network to transport the goods between markets, subject to the constraints of the network. The market maker clears the market by maximizing a payoff function that is parameterized by the tradeoff between the benefit to each of the three key parties – the consumer, the producer, and the market maker itself.

Our second contribution is the characterization of the equilibria structure as a function of the design parameters of the market maker (Section 2.4). Our main result (Theorem 2.2) highlights a wide variety of behaviors – depending on the design of the market maker, there may be a unique equilibrium, multiple equilibria, or no equilibria. Furthermore, when equilibria do exist, the game may form a weighted potential game or not depending on the design choice. Beyond characterizing the existence of equilibria in the case of linear costs, homogeneous demands, and an unconstrained network, we are able to explicitly characterize the unique equilibrium outcome as a function of the market maker design. This allows us to perform a more detailed study of the impact of the market maker. For example, the characterization highlights that the total production by all firms is independent of the design of the market maker (in this setting), but that the relative production of the firms may vary dramatically depending on the design of the market maker. Additionally, the characterization allows us to provide results highlighting the value of the trade provided by the market maker as well as the efficiency of the market maker (i.e., how close the outcomes of the game are to the outcomes of a single, aggregate Cournot market) as a function of the market maker design.

Our third contribution focuses on the design of the market maker. In particular, we show how to (approximately) optimally design the market maker payoff so as to maximize a desired social/regulatory objective, e.g., social welfare, (Section 2.5). Our primary tool is the characterization of the equilibria provided in Section 2.4. Then, we utilize the sum of squares (SOS) relaxation framework to judge the quality of our approximately optimal design choice. The results highlight the, perhaps counterintuitive, observation that if the market maker intends to optimize social welfare, it should not use social welfare as the objective in clearing the market. We further illustrate our proposed approach to market maker design on a stylized example that represents a caricature of the California electricity market. Our results underscore the importance of careful design.

2.2 Related Literature

Models of competition in networked settings have received considerable attention in recent years. These models come in various forms, including networked Bertrand competition, e.g., [60, 12, 37, 6], networked Cournot competition, e.g., [20, 4, 65], and various other non-cooperative bargaining games where agents can trade via bilateral contracts and a network determines the set of feasible trades, e.g., [51, 40, 96, 5, 88].

Our study fits into the emerging literature on networked Cournot competition; however our focus and model differ considerably from existing work. In particular, beginning with [25] and continuing through [65, 20, 4], the literature on networked Cournot competition has focused on models where the network structure emerges as a result of firms having a fixed, limited set of markets in which they can participate and participation in these markets is unconstrained and independent of the actions of other firms. In contrast, in our model the network constrains flows between markets, and so there are coupled participation constraints for the firms. Further, the literature on networked Cournot competition has focused on situations where firms operate independently, without governance, while we focus on situations where transport across markets is managed by a market maker.

The line of work that is most relevant to the questions studied in this study comes from the electricity market literature, where versions of Cournot competition subject to network constraints have been studied for nearly two decades, see [129] for a survey. In this setting Cournot models often provide good explanations for observed price variations [133], and so are quite popular. For example, Cournot models have been applied to perform detailed studies of electricity markets in the US [23], Scandinavia [11], Spain [9, 108], and New Zealand [116, 115], among others.

Due to the importance of the ISO in electricity markets, papers within this literature often include a model of a market maker, e.g., [134, 140, 92, 67, 62, 142, 24]. However, with rare exception, these papers focus on a market maker that is regulated to maximize social welfare, and thus do not explore the impact of differing market maker payoffs, nor how to design the market maker to optimize a particular social objective. Further, these papers focus exclusively on detailed models of power flows, and thus do not apply to more general network models, such as classical flow models, which are relevant to other applications. Our results, on the other hand, apply to networks with general linear constraints, including both linearized power flow constraints and classical network flow constraints.

To the best of our knowledge, this is the first work to focus on understanding the impact of, and how to optimally design a market maker that governs transport in a networked marketplace.

2.3 Model

Our focus is a marketplace where a constrained transport network, operated by a market maker, connects firms and markets. Specifically, we consider an economy dealing in a single commodity that is composed of a set of markets $\mathcal{M}$, a set of firms $\mathcal{F}$, and a market maker who facilitates transport of the commodity between the markets. Within this setting, we study Cournot competition over the networked markets, considering a static game of complete information among the firms and the market maker.

Each firm $f \in \mathcal{F}$ supplies to exactly one market, denoted by $\mathcal{M}(f)$. Let $\mathcal{F}(m)$ denote the set of firms that supply to market $m \in \mathcal{M}$. Denote the supply of firm $f \in \mathcal{F}$ to market $\mathcal{M}(f)$ by $q_f \in \mathbb{R}_+$, and let $\boldsymbol{q} := (q_f, f \in \mathcal{F}) \in \mathbb{R}_+^{|\mathcal{F}|}$ denote the vector of supplies of all firms in $\mathcal{F}$. Additionally, for each $f \in \mathcal{F}$, let $\boldsymbol{q}_{-f}$ denote the vector of supplies of all firms in $\mathcal{F}$, except f. The cost incurred by firm $f \in \mathcal{F}$ for producing $q_f \in \mathbb{R}_+$ is $c_f(q_f)$. Assume $c_f : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing, convex, differentiable, and $c_f(0) = 0$.

Crucially, the production of each firm in our model can be reallocated to other markets by a market maker that controls a constrained transport network. We consider a single market maker that facilitates transport of the commodity between markets. The market maker can procure supply from one market and transport it to a different market, subject to network constraints. Denote the quantity supplied by the market maker to market $m \in \mathcal{M}$ by r_m. Our convention is that $r_m \geq 0$ ($r_m < 0$) denotes a net supply (net demand) of the commodity by the market maker in market m. For convenience, let $\boldsymbol{r} := (r_m,\ m \in \mathcal{M}) \in \mathbb{R}^{|\mathcal{M}|}$ denote the vector of supplies by the market maker. Since the market maker only transports the commodity, the market maker neither consumes nor produces. So, we have $\mathbf{1}^\top \boldsymbol{r} = 0$, where $\mathbf{1}$ is a vector of ones with dimension $|\mathcal{M}|$.[1]

The reallocation of supply by the market maker, $\boldsymbol{r}$, is subject to the flow constraints of the network. We model these constraints by restricting $\boldsymbol{r}$ to a polyhedral set $\mathcal{P} := \left\{\boldsymbol{r} \in \mathbb{R}^{|\mathcal{M}|}\ :\ \boldsymbol{A}\boldsymbol{r} \leq \boldsymbol{b}\right\} \subseteq \mathbb{R}^{|\mathcal{M}|}$, where $\boldsymbol{A}$ and $\boldsymbol{b}$ define the half-spaces of $\mathcal{P}$. This formulation can capture constraints in traditional flow networks, as well as power flow constraints arising from linearized Kirchoff's laws and line limits.

The price at each market in the network is dependent on both the production of the firms and the reallocation performed by the market maker. As is traditional when studying Cournot competition, we focus on the case of linear inverse demand functions. In particular, assume that the price p_m, in each market $m \in \mathcal{M}$, has the form

$$p_m(d_m) := \alpha_m - \beta_m d_m, \tag{2.1}$$

[1]We recognize that, in some cases, the market maker may have an incentive to dispose of some of its purchases. We can model such behavior by replacing the constraint $\mathbf{1}^\top \boldsymbol{r} = 0$ with $\mathbf{1}^\top \boldsymbol{r} \leq 0$. However, our motivating application of electricity markets does not feature disposal; thus, we assume $\mathbf{1}^\top \boldsymbol{r} = 0$.

for some $\alpha_m, \beta_m > 0$. Here, d_m is the aggregate demand in market m. Importantly, the aggregate demand in each market is determined by both the actions of the firms and the market maker, i.e., $d_m = r_m + \sum_{f \in \mathcal{F}(m)} q_f$.

Given the prices in each market, p_m, we can write the payoff functions for the firms and the market maker. The payoff of firm $f \in \mathcal{F}$ is given by its profit, defined as

$$\pi_f(\boldsymbol{q}, \boldsymbol{r}) := q_f \cdot p_{\mathcal{M}(f)} \left(r_{\mathcal{M}(f)} + \sum_{f' \in \mathcal{F}(\mathcal{M}(f))} q_{f'} \right) - c_f(q_f). \tag{2.2}$$

Thus, firm f maximizes $\pi_f(\boldsymbol{q}, \boldsymbol{r})$ over $q_f \in \mathbb{R}_+$, given $(\boldsymbol{q}_{-f}, \boldsymbol{r})$.

For the market maker, the payoff function is a design choice. In many regulated settings, e.g., electricity markets, it is common for the market maker to optimize some metric of social benefit. Our goal in this study is to explore the impact of the market maker payoff functions, and so we focus on a broad, parameterized class of maker maker payoff functions defined as follows. Given $\boldsymbol{q}$, the market maker maximizes $\Pi(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ over $\boldsymbol{r} \in \mathcal{P}$ and $\mathbf{1}^\top \boldsymbol{r} = 0$, where

$$\Pi(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) := \sum_{m \in \mathcal{M}} \left[\theta_C \cdot \mathsf{CS}_m(\boldsymbol{q}, \boldsymbol{r}) + \theta_P \cdot \mathsf{PS}_m(\boldsymbol{q}, \boldsymbol{r}) + \theta_M \cdot \mathsf{MS}_m(\boldsymbol{q}, \boldsymbol{r}) \right]. \tag{2.3}$$

In $\Pi(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$, the design parameter $\boldsymbol{\theta} := (\theta_C, \theta_P, \theta_M)^\top \in \mathbb{R}^3_+$ allows the designer to weigh the importance of the following terms, for each $m \in \mathcal{M}$:[2]

$$\mathsf{CS}_m(\boldsymbol{q}, \boldsymbol{r}) := \int_0^{r_m + \sum_{f \in \mathcal{F}(m)} q_f} p_m(w_m)\, dw_m - \left(r_m + \sum_{f \in \mathcal{F}(m)} q_f \right) \cdot p_m \left(r_m + \sum_{f \in \mathcal{F}(m)} q_f \right);$$

$$\mathsf{PS}_m(\boldsymbol{q}, \boldsymbol{r}) := \left(\sum_{f \in \mathcal{F}(m)} q_f \right) \cdot p_m \left(r_m + \sum_{f \in \mathcal{F}(m)} q_f \right) - \sum_{f \in \mathcal{F}(m)} c_f(q_f);$$

$$\mathsf{MS}_m(\boldsymbol{q}, \boldsymbol{r}) := r_m \cdot p_m \left(r_m + \sum_{f \in \mathcal{F}(m)} q_f \right),$$

The quantities CS_m, PS_m, and MS_m admit natural interpretations. Namely, CS_m equals the consumer surplus in market m, PS_m equals the collective producer surplus of all firms supplying in market m, and MS_m equals the merchandizing surplus of the market maker by supplying in market m.

The parameterized class of market maker payoff functions defined in (2.3) encompasses a wide

[2]The notation $\boldsymbol{\theta} \in \mathbb{R}^3_+$ should be understood to mean that $\boldsymbol{\theta} \gneq \mathbf{0}$ since the market maker's payoff becomes zero trivially when $\boldsymbol{\theta} = \mathbf{0}$.

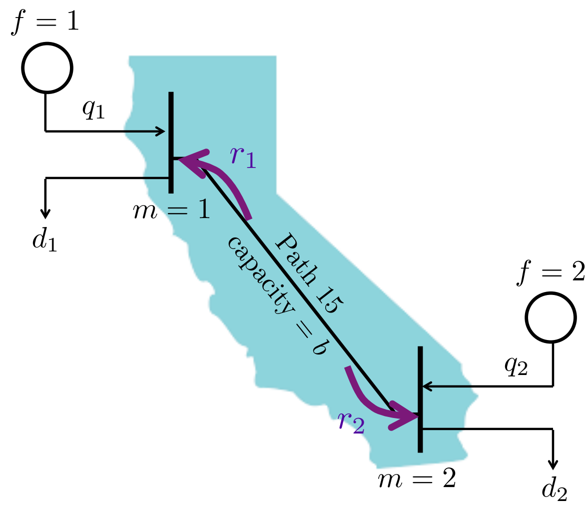


Figure 2.1: Example of a two-market two-firm networked marketplace. This example rerepresents a caricature of the wholesale electricity market in California. Here, northern and southern California are represented as two nodes connected by a transmission line – Path 15 – that is often congested (see [125]).

class of common objectives. To illustrate a few, consider the following definitions.

$$\boldsymbol{\theta}^{\mathsf{SW}} := (1, 1, 1), \quad \boldsymbol{\theta}^{\mathsf{CS}} := (1, 0, 0), \quad \boldsymbol{\theta}^{\mathsf{RSW}} := (1, 0, 1), \quad \boldsymbol{\theta}^{\mathsf{MS}} := (0, 0, 1). \tag{2.4}$$

The market maker's payoff function with $\boldsymbol{\theta}^{\mathsf{SW}}$ as the design parameter is the Walrasian social welfare that is widely used in many centrally managed networked marketplaces, including wholesale electricity markets. In the same vein, $\Pi\left(\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{\theta}^{\mathsf{CS}}\right)$ is the collective consumer surplus across all markets, and hence defines a pro-consumer design choice by the market maker. Another common choice is $\Pi\left(\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{\theta}^{\mathsf{RSW}}\right)$, the *residual social welfare*, which equals the Walrasian social welfare less the collective producer surplus of all firms. By maximizing the residual social welfare, one hopes that the market maker strikes a balance in optimizing the different components of the Walrasian social welfare by different players of the game. In contrast with $\boldsymbol{\theta}^{\mathsf{SW}}$, $\boldsymbol{\theta}^{\mathsf{CS}}$, and $\boldsymbol{\theta}^{\mathsf{RSW}}$, the choice of $\boldsymbol{\theta}^{\mathsf{MS}}$ as the design parameter defines a profit-maximizing market maker.

Note that the class of payoffs we consider does not account for any variable costs associated with transporting the commodity through the network. However, as long as the variable costs are convex in $\boldsymbol{r}$, most of our results continue to hold.

A motivating example: Many networked marketplaces with market makers that govern transport can be described by the model discussed above, but to provide a concrete motivating example for use throughout this study, we consider the case of *wholesale electricity markets*. We illustrate our results with this example in Section 2.5.3.

Most electricity markets in the US are managed by a regulatory entity known as an Independent System Operator (ISO). The role of the ISO is to facilitate efficient exchange of power between supply and demand while ensuring that power flows through the network satisfy the operating constraints of the grid. Thus, the ISO plays the role of the market maker in our model.

To illustrate the model, consider the two-node network in Figure 2.1. Here, northern and southern California are modeled as two nodes connected by a transmission line – Path 15. Assume, for simplicity, that there is one generator at each node and the transmission line has a capacity $b \in \mathbb{R}_+$. The California Independent System Operator (CAISO) serves as the market maker, governing transport, and seeks to maximize social welfare through reallocating generation.

We can model the strategic interactions in this simple example as a game $\mathcal{G}(\boldsymbol{\theta})$, where there are two markets $\mathcal{M} = \{1, 2\}$ with inverse linear demand functions $p_1(d_1) = \alpha_1 - \beta_1 d_1$ and $p_2(d_2) = \alpha_2 - \beta_2 d_2$, and two firms $\mathcal{F}(1) = \{1\}$ and $\mathcal{F}(2) = \{2\}$ with cost functions $c_1(q_1)$ and $c_2(q_2)$, respectively. The set of feasible reallocations by the market maker is $\mathcal{P} = \{\boldsymbol{r} \in \mathbb{R}^2 : |r_1| \leq b,\ |r_2| \leq b\}$. The market maker's payoff is the social welfare, i.e., the design parameter is $\boldsymbol{\theta}^{\mathsf{SW}}$.

Equilibrium definition: We conclude this section by formally describing the networked Cournot competition as a game, $\mathcal{G}(\boldsymbol{\theta})$, where the *players* include the collection of firms $\mathcal{F}$ and the market maker, the *strategy sets* are defined by $q_f \in \mathbb{R}_+$ for each $f \in \mathcal{F}$ and $\boldsymbol{r} \in \mathcal{P}$ and $\mathbf{1}^\top \boldsymbol{r} = 0$, and the *payoff functions* are π_f for each firm $f \in \mathcal{F}$ and Π for the market maker.

We focus our analysis on the Nash equilibria outcomes, which are defined as follows: $(\boldsymbol{q}, \boldsymbol{r}) \in \mathbb{R}_+^{|\mathcal{F}|} \times \mathcal{P}$, satisfying $\mathbf{1}^\top \boldsymbol{r} = 0$, comprises a *Nash equilibrium* of $\mathcal{G}(\boldsymbol{\theta})$, if

$$\begin{aligned}\pi_f(\boldsymbol{q}, \boldsymbol{r}) &\geq \pi_f(q_f', \boldsymbol{q}_{-f}, \boldsymbol{r}) \ \text{ for all } q_f' \in \mathbb{R}_+,\\ \Pi(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) &\geq \Pi(\boldsymbol{q}, \boldsymbol{r'}; \boldsymbol{\theta}), \ \text{ for all } \boldsymbol{r'} \in \mathcal{P},\ \mathbf{1}^\top \boldsymbol{r'} = 0.\end{aligned}$$

The effect of $\boldsymbol{\theta}$ on the Nash equilibria of $\mathcal{G}(\boldsymbol{\theta})$ represents the impact of market maker design, and is the focus of the current study.

2.4 Characterizing the Nash Equilibria

In this section, we describe our first set of results, which provide characterizations of the equilibria outcomes, and contrast the equilibrium in our networked Cournot marketplace to non-networked Cournot models. Then, in Section 2.5, we use the characterizations provided here in order to inform the design of the market maker.

2.4.1 Existence and uniqueness

Classical Cournot competition among a set of firms in a single market with inverse linear demand functions is known to be a potential game (see [121] and [94]) and, recently, this property has been shown to extend to a form of networked Cournot competition, as shown in [4]). These characterizations are powerful, since they allow results about equilibrium existence and uniqueness to be derived through analysis of the underlying potential function of the game. However, the results in [4] focus on a form of networked competition over bipartite graphs with no market maker; thus they do not apply to the model we consider here. But, given the results for these classical and networked Cournot models, an optimistic reader expects a similar conclusion for the model we consider. In the results that follow, we show that this is true in some situations – under some assumptions, we show that the model we consider yields a weighted potential game – however, the structure of the game is more complex in general.

Before stating our results, we begin by formally defining the notion of a weighted potential game. Consider a game $\mathcal{G}$ with players $1, \ldots, N$, actions $\boldsymbol{x}_i \in \mathbb{R}^{n_i}$ for $i = 1, \ldots, N$, where $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \in \mathcal{X} \subseteq \mathbb{R}^{n_1+\ldots+n_N}$, and payoff functions $\varphi_i : \mathbb{R}^{n_1+\ldots+n_N} \to \mathbb{R}$ for each player $i = 1, \ldots N$. The game $\mathcal{G}$ is said to be a *weighted potential game*, if there exists a vector of weights $w \in \mathbb{R}^N_{++}$ and a potential function $\Phi : \mathbb{R}^{n_1+\ldots+n_N} \to \mathbb{R}$ that satisfies

$$\Phi(\boldsymbol{x}_i, \boldsymbol{x}_{-i}) - \Phi(\boldsymbol{x}'_i, \boldsymbol{x}_{-i}) = w_i \cdot [\varphi_i(\boldsymbol{x}_i, \boldsymbol{x}_{-i}) - \varphi_i(\boldsymbol{x}'_i, \boldsymbol{x}_{-i})] \tag{2.5}$$

for each $(\boldsymbol{x}_i, \boldsymbol{x}_{-i}) \in \mathcal{X}$, and $i = 1, \ldots, N$.

Our first result highlights that, for some design parameters $\boldsymbol{\theta}$, the model of networked competition we consider is a weighted potential game with a potential function that can be represented as a perturbed version of the market maker payoff.

Theorem 2.1. *If $\theta_M + \theta_P - \theta_C > 0$, then $\mathcal{G}(\boldsymbol{\theta})$ is a weighted potential game with the potential function $\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$, given by*

$$\begin{aligned}\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) := \Pi(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) - (\theta_M - \theta_P) \sum_{m \in \mathcal{M}} \frac{\beta_m}{2} \left(\sum_{f \in \mathcal{F}(m)} q_f \right)^2 \\ - \sum_{f \in \mathcal{F}} \left[(\theta_M + \theta_P - \theta_C) \frac{\beta_{\mathcal{M}(f)}}{2} q_f^2 + (\theta_C - \theta_M) \left(\alpha_{\mathcal{M}(f)} q_f - c_f(q_f) \right) \right].\end{aligned} \tag{2.6}$$

A proof of this result is provided in Appendix 2.A. The fact that $\mathcal{G}(\boldsymbol{\theta})$ is a weighted potential game highlights that the game has a number of favorable properties. In particular, maximizers of $\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ over the joint strategy set of $\mathcal{G}(\boldsymbol{\theta})$ are Nash equilibria, and this fact can be used to infer the existence of Nash equilbria. Furthermore, strict concavity of $\hat{\Pi}$ can be leveraged to conclude the uniqueness of Nash equilibrium, that is characterized as the solution to

$$\begin{aligned} \mathcal{C}(\boldsymbol{\theta}) \ : \underset{\boldsymbol{q}, \boldsymbol{r}}{\text{maximize}} \quad & \hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}), \\ \text{subject to} \quad & \boldsymbol{q} \in \mathbb{R}^{|\mathcal{F}|}_+, \boldsymbol{r} \in \mathcal{P},\ \boldsymbol{1}^\top \boldsymbol{r} = 0. \end{aligned} \tag{2.7}$$

In addition, if cost functions are increasing linear functions or convex quadratic functions, there exists a unique equilibrium that can be found efficiently. Finally, many natural learning dynamics are guaranteed to converge to an equilibrium in potential games. See [94] and more recent publications, e.g., [55, 144, 118, 89, 90], for a comprehensive discussion on the topic.

However, the characterization of existence provided by Theorem 2.1 is not complete. It turns out that, for many design parameters, the structure of the game is more complex and, in particular, the game is not a weighted potential game. Despite this, in many such cases a Nash equilibrium is still guaranteed to exist. The theorem below provides a more complete view of equilibrium existence and uniqueness.

Theorem 2.2. *Suppose $\mathcal{P}$ is compact, and let*

$$\gamma := 1 - \min_{m \in \mathcal{M}} \left(1 + \sum_{f \in \mathcal{F}(m)} \frac{\beta_m}{\beta_m + \inf_{q_f \geq 0} c''_f(q_f)} \right)^{-1}. \tag{2.8}$$

(a) If $2\theta_M - \theta_C \geq 0$ or $\theta_M + \theta_P - \theta_C > 0$, then $\mathcal{G}(\boldsymbol{\theta})$ has a Nash equilibrium.

(b) If $2\theta_M - \theta_C \geq \gamma \cdot (\theta_M + \theta_P - \theta_C) > 0$, then the set of Nash equilibria of $\mathcal{G}(\boldsymbol{\theta})$ is nonempty, and is identical to the set of optimizers of $\mathcal{C}(\boldsymbol{\theta})$. Furthermore, if the inequalities are strict, then $\mathcal{G}(\boldsymbol{\theta})$ has a unique Nash equilibrium.

The formal proof is deferred till Appendices 2.B and 2.C. The argument hinges on a result due to [110]. It relies on the market maker's payoff $\Pi(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ being continuous in $\boldsymbol{q}$ and concave in $\boldsymbol{r}$. In essence, $\mathcal{G}(\boldsymbol{\theta})$ has additional structure for design parameters even beyond where it is a potential game. Some insight for the form of the conditions in the theorem can be understood from the proof. In particular, the market maker's payoff function $\Pi(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ is concave in $\boldsymbol{r}$ if and only if $2\theta_M - \theta_C \geq 0$. Additionally, from Theorem 2.1, we know that $\mathcal{G}(\boldsymbol{\theta})$ is a potential game with $\hat{\Pi}$ as the potential function when $\theta_M + \theta_P - \theta_C > 0$. Finally, $2\theta_M - \theta_C \geq \gamma \cdot (\theta_M + \theta_P - \theta_C) > 0$ implies that $\hat{\Pi}$ is concave, and it is strictly concave when the inequality is strict. Finally, the concavity of $\hat{\Pi}$, together with Neyman's result [99], yields the equality between the sets of Nash equilibria of $\mathcal{G}(\boldsymbol{\theta})$ and the optimizers of $\mathcal{C}(\boldsymbol{\theta})$.

In Figure 2.2, we visualize the regions defined by the conditions in Theorem 2.2. Note that the Nash equilibria of $\mathcal{G}(\boldsymbol{\theta})$ is invariant under a positive scaling of $\boldsymbol{\theta}$; thus it suffices to consider $\boldsymbol{\theta} \in \mathbb{R}^3_+$ for which $\theta_C + \theta_P + \theta_M = 1$, i.e., $\boldsymbol{\theta}$ lies on the 3-dimensional simplex. Hereafter, denote by Δ the 3-dimensional simplex. Notice that the conditions on $\boldsymbol{\theta}$ in Theorem 2.2(b) depend on γ, which in turn is a function of the inverse linear demand functions in the markets and the cost functions of the firms. Since costs are convex, c''_f is nonnegative, and thus,

$$\gamma \leq \max_{m \in \mathcal{M}} \frac{|\mathcal{F}(m)|}{1 + |\mathcal{F}(m)|}.$$

The fraction on the right hand side of the above equation is the smallest when each market has one firm, and hence $\gamma \leq \frac{1}{2}$. In fact, $\gamma = \frac{1}{2}$ when each market has only one firm, and costs are increasing linear functions. For illustrative purposes, we choose $\gamma = \frac{1}{2}$ to portray the various regions of Δ in Figure 2.2, where $\mathcal{G}(\boldsymbol{\theta})$ has different properties.

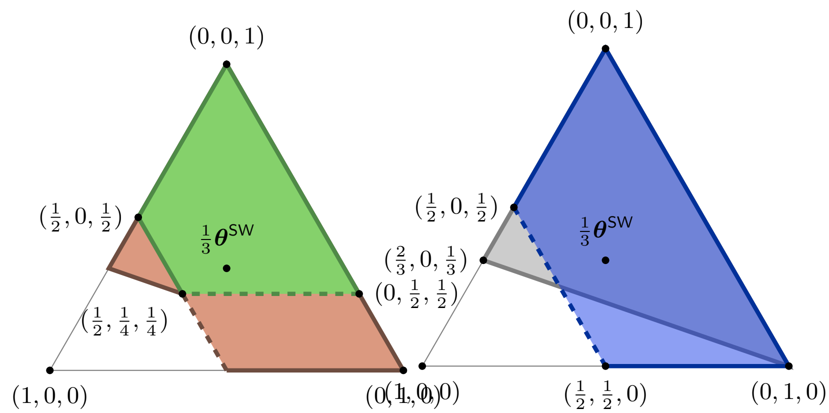


Figure 2.2: (a) An illustration of Theorem 2.2 for $\boldsymbol{\theta} \in \Delta$. A Nash equilibrium may not exist in the unshaded region, it exists but may not be unique in the brown region, and it is unique and is given by the unique optimizer of $\mathcal{C}(\boldsymbol{\theta})$ in the green region. (b) An illustration of Theorem 2.2(a) for $\boldsymbol{\theta} \in \Delta$. The grey region is defined by $2\theta_M - \theta_C \geq 0$, where a Nash equilibrium exists owing to a variant of $\mathcal{G}(\boldsymbol{\theta})$ being a concave game. The blue region is defined by $\theta_M + \theta_P - \theta_C > 0$, where a Nash equilibrium exists because $\mathcal{G}(\boldsymbol{\theta})$ is a potential game. Dotted line segments on the boundaries of various sets do not belong to the respective sets.

Theorems 2.1 and 2.2 provide sufficient conditions for equilibrium existence and uniqueness, but do not address the question of necessity or tightness. To provide some insight into necessity, we provide examples to highlight that each of the properties may fail to hold if the respective conditions are not met. The examples are all constructed using the simple two-market two-firm example in Section 2.3. Furthermore, they all focus only on the case when each firm has increasing linear costs and both markets have identical inverse linear demand functions.

The Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$ in this restricted setting can be explicitly computed for all $\boldsymbol{\theta} \in \mathbb{R}^3_+$. The results are derived in Lemmas 2.2, 2.3, and 2.4 in Appendix 2.F. Using these results, we construct examples of $\boldsymbol{\theta}$ in Appendix 2.F to illustrate the following:

1. When neither $\theta_M + \theta_P - \theta_C > 0$ nor $2\theta_M \geq \theta_C$ holds, a Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$ may not exist.

2. When $2\theta_M \geq \theta_C$, but not $\theta_M + \theta_P - \theta_C > 0$, a Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$ exists, but $\mathcal{G}(\boldsymbol{\theta})$ is not a weighted potential game.

3. When $2\theta_M - \theta_C \geq 0$, and $\theta_M + \theta_P - \theta_C > 0$, but $2\theta_M - \theta_C \geq \gamma \cdot (\theta_M + \theta_P - \theta_C)$ does not hold, a Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$ may exist that is not an optimizer of $\mathcal{C}(\boldsymbol{\theta})$.

4. When $2\theta_M - \theta_C = \gamma\left(\theta_M + \theta_P - \theta_C\right) > 0$, then $\mathcal{G}(\boldsymbol{\theta})$ may have a multitude of Nash equilibria, all of which are optimizers of $\mathcal{C}(\boldsymbol{\theta})$.

2.4.2 Example with linear costs and homogeneous demands

To this point our results have focused only on existence and uniqueness. We now provide a more detailed characterization of the equilibria. Specifically, our goal is to study the parametric dependence of the Nash equilibria on $\boldsymbol{\theta}$.

Without making stronger assumptions on the nature of the game, such a characterization is difficult. To allow interprebility of the results, we focus on a restricted setting where each market has a single firm with linear increasing cost and the markets have identical linear demand functions. Additionally, we focus on the case of an unconstrained network, i.e., $\mathcal{P} = \mathbb{R}^{|\mathcal{M}|}$. It is possible to provide a more general characterization at the expense of interprebility.

In this setting, we are able to offer explicit formulae for the unique Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$ under a subset of the design parameters in Proposition 2.1. Importantly, this characterization allows us to contrast the result of competition in the networked marketplace we consider with two cases of particular interest: (a) competition in a collection of non-networked markets, i.e., a setting without transport between markets, and (b) competition in an aggregated market, i.e., a setting where the markets are merged into a single aggregate marketplace without a market maker. The comparison with (a) provides insight into the efficiency of the network and the comparison with (b) provides insight into the efficiency of the market maker.

Consider $\mathcal{G}(\boldsymbol{\theta})$ with: (1) an unconstrained network, $\mathcal{P} := \mathbb{R}^{|\mathcal{M}|}$, (2) a collection of firms $\mathcal{F}$ having linear costs $c_f(q_f) := C_f q_f$ for each $f \in \mathcal{F}$, where $C_f > 0$, (3) a collection of markets $\mathcal{M}$, where a single firm supplies in each market, i.e., $|\mathcal{F}(m)| = 1$, for $m \in \mathcal{M}$, and (4) spatially homogeneous inverse linear demand functions, given by $p_m(d_m) = \alpha - \beta d_m$, for each $m \in \mathcal{M}$, for some $\alpha, \beta > 0$. Define $\boldsymbol{C} := (C_f, f \in \mathcal{F})$. Denote the mean and the standard deviation of the firms' marginal costs by

$$\bar{C} := \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} C_f, \quad \text{and} \quad \sigma_C := \sqrt{\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} (C_f - \bar{C})^2}, \tag{2.9}$$

respectively. Denote this family of games by $\mathcal{G}^u(\boldsymbol{\theta}; \boldsymbol{C}, \alpha, \beta)$, parameterized by the design parameter, firms' marginal costs, and the parameters defining the identical market demand functions. Then, we have the following result on $\mathcal{G}^u(\boldsymbol{\theta}; \boldsymbol{C}, \alpha, \beta)$.

Proposition 2.1. *Consider $\mathcal{G}^u(\boldsymbol{\theta}; \boldsymbol{C}, \alpha, \beta)$, where $\bar{C}$ and σ_C are as defined in* (2.9). *If $2\theta_M - \theta_C > \frac{1}{2}\left(\theta_M + \theta_P - \theta_C\right) > 0$ and $\alpha \geq (1 + \kappa(\boldsymbol{\theta})) \max_{f \in \mathcal{F}} C_f - \kappa(\boldsymbol{\theta})\bar{C}$, then $\mathcal{G}^u(\boldsymbol{\theta}; \boldsymbol{C}, \alpha, \beta)$ has a unique*

Nash equilibrium, given by

$$q_f = \frac{1}{2\beta}\left[\alpha - \bar{C} - (1+\kappa(\boldsymbol{\theta}))\left(C_f - \bar{C}\right)\right], \tag{2.10}$$

$$r_{\mathcal{M}(f)} = \frac{\kappa(\boldsymbol{\theta})}{\beta}\left(C_f - \bar{C}\right), \tag{2.11}$$

for each $f \in \mathcal{F}$*, where* $\kappa(\boldsymbol{\theta}) := \frac{\theta_M + \theta_P - \theta_C}{3\theta_M - \theta_P - \theta_C}$*. Moreover, the Walrasian social welfare at the unique Nash equilibrium is given by*

$$\sum_{m \in \mathcal{M}} \left[\mathsf{CS}_m(\boldsymbol{q}, \boldsymbol{r}) + \mathsf{PS}_m(\boldsymbol{q}, \boldsymbol{r}) + \mathsf{MS}_m(\boldsymbol{q}, \boldsymbol{r})\right] = \frac{3|\mathcal{F}|}{8\beta}\left[(\alpha - \bar{C})^2 + \sigma_C^2 + \frac{1}{3}\kappa(\boldsymbol{\theta})(6 - \kappa(\boldsymbol{\theta}))\sigma_C^2\right].$$

A proof is given in Appendix 2.D. Note that, because we consider an unconstrained network, our proof technique from Theorem 2.2(b) no longer applies. Thus, we take a different approach to analyzing $\mathcal{C}(\boldsymbol{\theta})$ focused on the Karush-Kuhn-Tucker (KKT) optimality conditions.

To obtain some insight from Proposition 2.1, note that equations (2.10) – (2.11) reveal that, the production of a firm q_f and the market-maker's supply in the market served by that firm $r_{\mathcal{M}(f)}$, both depend on the marginal cost of the firm C_f relative to the average marginal cost of all firms $\bar{C}$. Under the conditions of Proposition 2.1, one can show that $\kappa(\boldsymbol{\theta}) > 0$. Hence, the firms' productions are in fact ordered inversely by their marginal costs. Moreover, the market maker buys from markets having firms with lower marginal costs and supplies to markets having firms with higher marginal costs. The total production by all firms, however, is independent of $\boldsymbol{\theta}$, and is given by

$$\sum_{f \in \mathcal{F}} q_f = \frac{|\mathcal{F}|}{2\beta}\left(\alpha - \bar{C}\right). \tag{2.12}$$

The market maker's design choice only influences the relative production between the firms and the quantities supplied by the market maker to various markets.

Proposition 2.1 also lets us investigate the efficiency of the equilibria. By studying the effect of the design parameter on the social welfare at the unique Nash equilibrium, we unravel the impact of the design choice on the competitiveness of the market. As we remarked earlier, a popular choice of $\boldsymbol{\theta}$ for a regulated marketplace like the wholesale electricity markets is $\boldsymbol{\theta}^{\mathsf{SW}}$ defined in (2.4), i.e., the market maker optimizes the social welfare function. Then, $\kappa\left(\boldsymbol{\theta}^{\mathsf{SW}}\right) = 1$. We notice that the social welfare at the unique Nash equilibrium increases with $\kappa(\boldsymbol{\theta})$ over the interval $[1, 3]$. Moreover, it is easy to construct a $\boldsymbol{\theta}$ that satisfies the conditions in Proposition 2.1 with $1 < \kappa(\boldsymbol{\theta}) < 3$. So, if maximizing the social welfare at the unique Nash equilibrium is indeed the design goal, $\boldsymbol{\theta}^{\mathsf{SW}}$ is *not* the optimal design choice.

This motivates the question of how much efficiency is lost by naively choosing the design parameter $\boldsymbol{\theta}^{\mathsf{SW}}$. To address this, note that the social welfare at the Nash equilibrium with $\boldsymbol{\theta}^{\mathsf{SW}}$ is given by $(\alpha-\bar{C})^2+\frac{8}{3}\sigma_C^2$. Also, observe that $\kappa(\boldsymbol{\theta})(6-\kappa(\boldsymbol{\theta}))\leq 9$. Combining these allows us to derive a simple upper bound on the ratio of the largest attainable social welfare at a Nash equilibrium to that obtained with $\boldsymbol{\theta}^{\mathsf{SW}}$.

$$\frac{(\alpha-\bar{C})^2+\sigma_C^2+\frac{1}{3}\kappa(\boldsymbol{\theta})(6-\kappa(\boldsymbol{\theta}))\sigma_C^2}{(\alpha-\bar{C})^2+\frac{8}{3}\sigma_C^2}\leq\frac{1+4\left(\frac{\sigma_C}{\alpha-\bar{C}}\right)^2}{1+\frac{8}{3}\left(\frac{\sigma_C}{\alpha-\bar{C}}\right)^2}\leq\frac{3}{2}.$$

The last step follows from the fact that $h(x):=(1+4x)/\left(1+\frac{8}{3}x\right)$ is increasing in $x\geq 0$, and is bounded from above by $\lim_{x\to\infty}h(x)=\frac{3}{2}$.

When $\boldsymbol{\theta}$ is varied such that $\kappa(\boldsymbol{\theta})$ increases from one, we have already argued that the social welfare at the Nash equilibrium increases. Who stands to benefit from such an increase? Is it the consumers, the producers, or the market maker? Recall that a metric of consumer benefit is the aggregate consumer surplus $\sum_{m\in\mathcal{M}}\mathsf{CS}_m(\boldsymbol{q},\boldsymbol{r})$ at the Nash equilibrium. Similarly, the aggregate producer surplus $\sum_{m\in\mathcal{M}}\mathsf{PS}_m(\boldsymbol{q},\boldsymbol{r})$ and the merchandising surplus $\sum_{m\in\mathcal{M}}\mathsf{MS}_m(\boldsymbol{q},\boldsymbol{r})$ at the Nash equilibrium measures the benefits to the producers and the market maker, respectively. One can show that the aggregate consumer and producer surpluses both increase, when $\boldsymbol{\theta}$ is changed to increase $\kappa(\boldsymbol{\theta})$ from one. However, the merchandising surplus decreases. Thus, in the framework considered, a design choice that improves the efficiency of the market, does so to the benefit of the consumers and the producers, but at the expense of the market maker.

Comparison with non-networked Cournot

To study the role of the network, we next analyze the same setting, but remove the network. That is, $\mathcal{P}$ only contains the origin. Each firm then effectively competes as a monopoly in its own market, and the market maker plays no role. Define $\mathcal{G}^n(\boldsymbol{C},\alpha,\beta)$ as the non-networked Cournot competition among a collection of firms $\mathcal{F}$. Here, $\boldsymbol{C}$ again denotes the vector of firms' marginal costs, and identical market demand functions are identified by parameters α,β. We characterize the Nash equilibria of $\mathcal{G}^n(\boldsymbol{C},\alpha,\beta)$ in the following result.

Proposition 2.2. *Consider $\mathcal{G}^n(\boldsymbol{C},\alpha,\beta)$, where $\bar{C}$ and σ_C are as defined in* (2.9). *If $\alpha\geq\max_{f\in\mathcal{F}}C_f$, then $\mathcal{G}^n(\boldsymbol{C},\alpha,\beta)$ has a unique Nash equilibrium, given by*

$$q_f^n=\frac{1}{2\beta}\left(\alpha-C_f\right).$$

Moreover, the Walrasian social welfare at the unique Nash equilibrium is given by

$$\sum_{f\in\mathcal{F}}\left[\int_0^{q_f^n}p_{\mathcal{M}(f)}(w_f)\,dw_f-C_fq_f^n\right]=\frac{3|\mathcal{F}|}{8\beta}\left[\left(\alpha-\bar{C}\right)^2+\sigma_C^2\right].$$

The proof is straightforward and is omitted. To compare Propositions 2.1 and 2.2, assume that α satisfies the conditions required in both results.

Like in the networked marketplace, the production quantities of the firms are ordered inversely by their marginal costs. Also, the total production of the firms at the Nash equilibrium is given by

$$\sum_{f\in\mathcal{F}} q_f^n = \frac{|\mathcal{F}|}{2\beta}\left(\alpha - \bar{C}\right),$$

which, due to (2.12), happens to be identical to that in the networked marketplace. Thus, the network does not impact the total production of the firms. Instead, the value of the network is reflected in the social welfare at the Nash equilibrium.

It is straightforward to conclude from Propositions 2.1 and 2.2 that the social welfare at the Nash equilibrium is *higher* for the networked marketplace for any design choice $\boldsymbol{\theta}$. This aligns with the intuition that a network available for trade improves the efficiency of the marketplace. Recall that in the networked setting, the social welfare at the unique Nash equilibrium is given by $(\alpha-\bar{C})^2+\sigma_C^2+\frac{1}{3}\kappa(\boldsymbol{\theta})(6-\kappa(\boldsymbol{\theta}))\sigma_C^2$. Again, leveraging the fact that $\kappa(\boldsymbol{\theta})(6-\kappa(\boldsymbol{\theta}))\leq 9$, we obtain the following bound on the ratio of the social welfares in the networked and the non-networked case.

$$\frac{(\alpha-\bar{C})^2+\sigma_C^2+\frac{1}{3}\kappa(\boldsymbol{\theta})(6-\kappa(\boldsymbol{\theta}))\sigma_C^2}{(\alpha-\bar{C})^2+\sigma_C^2} \leq \frac{1+4\left(\frac{\sigma_C}{\alpha-\bar{C}}\right)^2}{1+\left(\frac{\sigma_C}{\alpha-\bar{C}}\right)^2} \leq 4.$$

The last step follows from the fact that $h(x) := (1+4x)/(1+x)$ is increasing in $x \geq 0$, and is bounded from above by $\lim_{x\to\infty} h(x) = 4$.

Comparison with aggregated Cournot

To study the efficiency of the market maker, we next analyze the same setting, but where the firms are aggregated into a single Cournot market. This comparison is motivated by the fact that one may hope an efficient market maker can facilitate trade in order to allow the networked marketplace to behave like a single market – especially when the network is unconstrained.

Recall that in our example, we considered $|\mathcal{M}|$ markets with identical inverse linear demand functions $p_m(d_m) = \alpha - \beta d_m$ for each $m \in \mathcal{M}$. Then, an aggregation of these markets with a collective demand d admits an inverse linear demand function $p(d) = \alpha - \frac{\beta}{|\mathcal{F}|}d$. Denote the aggregated Cournot competition by $\mathcal{G}^a(\boldsymbol{C},\alpha,\beta)$, where $\boldsymbol{C}$ denotes the vector of firms' marginal costs. The following result then characterizes the unique Nash equilibrium of $\mathcal{G}^a(\boldsymbol{C},\alpha,\beta)$.

Proposition 2.3. *Consider $\mathcal{G}^a(\boldsymbol{C},\alpha,\beta)$, where $\bar{C}$ and σ_C are as defined in* (2.9)*. If*

$\alpha \geq (1+|\mathcal{F}|)\max_{f\in\mathcal{F}} C_f - |\mathcal{F}|\bar{C}$, then $\mathcal{G}^a(\boldsymbol{C},\alpha,\beta)$ has a unique Nash equilibrium, given by

$$q_f^a = \frac{|\mathcal{F}|}{(1+|\mathcal{F}|)\beta}\left[\alpha - \bar{C} - (1+|\mathcal{F}|)\left(C_f - \bar{C}\right)\right].$$

Moreover, the Walrasian social welfare at the unique Nash equilibrium is given by

$$\int_0^{\sum_{f\in\mathcal{F}} q_f^a} p(w)\,dw - \sum_{f\in\mathcal{F}} C_f q_f^a = \frac{|\mathcal{F}|^2(2+|\mathcal{F}|)}{2(1+|\mathcal{F}|)^2\beta}\left[(\alpha-\bar{C})^2 + \frac{2(1+|\mathcal{F}|)^2}{2+|\mathcal{F}|}\sigma_C^2\right].$$

A proof can be found in [80], and is omitted for brevity. When comparing the results obtained in Proposition 2.3 to 2.1 or 2.2, assume α satisfies the conditions delineated in each result.

As in each case before, the firms' productions in the aggregated Cournot competition are ordered inversely by their marginal costs. However, in this case the total production quantity is different. In particular, we have

$$\sum_{f\in\mathcal{F}} q_f^a = \frac{|\mathcal{F}|^2}{(1+|\mathcal{F}|)\beta}\left(\alpha - \bar{C}\right).$$

Since $\frac{|\mathcal{F}|^2}{1+|\mathcal{F}|} \geq \frac{|\mathcal{F}|}{2}$, it follows from (2.12) that the total production quantity in the aggregated Cournot competition is no less than that in the networked marketplace with an unconstrained network. Furthermore, the inequality is strict when $|\mathcal{F}| \geq 2$.

Given increased production, it is natural to expect that the social welfare will be larger in the aggregated Cournot market as well. This turns out to be true. Towards comparing the social welfare of the aggregated Cournot case to our networked marketplace with an unconstrained network, notice that (i) $\frac{|\mathcal{F}|^2(2+|\mathcal{F}|)}{2(1+|\mathcal{F}|)^2} \geq \frac{3|\mathcal{F}|}{8}$ for all $|\mathcal{F}| \geq 1$, (ii) $\frac{2(1+|\mathcal{F}|)^2}{2+|\mathcal{F}|} \geq 4$ for all $|\mathcal{F}| \geq 2$, and (iii) $|\mathcal{F}| = 1$ imply $\sigma_C = 0$. These observations, together with $\kappa(\boldsymbol{\theta})\,(6-\kappa(\boldsymbol{\theta})) \leq 9$, yield

$$\frac{|\mathcal{F}|^2(2+|\mathcal{F}|)}{2\beta(1+|\mathcal{F}|)^2}\left[(\alpha-\bar{C})^2 + \frac{2(1+|\mathcal{F}|)^2}{2+|\mathcal{F}|}\sigma_C^2\right] \geq \frac{3|\mathcal{F}|}{8\beta}\left[(\alpha-\bar{C})^2 + \sigma_C^2 + \frac{\kappa(\boldsymbol{\theta})(6-\kappa(\boldsymbol{\theta}))}{3}\sigma_C^2\right].$$

As a result, the social welfare in the aggregate Cournot model is no less than that in the networked Cournot model for all possible choices of the design parameter. The inequality is strict when $|\mathcal{F}| \geq 2$. Also, $\frac{2(1+|\mathcal{F}|)^2}{2+|\mathcal{F}|} \to \infty$ as $|\mathcal{F}| \to \infty$. Thus, the ratio of equilibrium social welfares in the aggregated market and the unconstrained networked marketplace (with any choice of $\boldsymbol{\theta}$) grows without bound as the number of firms increases. In a sense, the higher the number of firms, the larger the need for transport, leading to a higher efficiency loss due to the market maker's transport.

2.5 Market Maker Design

The characterization results from the previous section provide the foundation for us to approach the question of market maker design. That is, to engineer the 'right' design parameter $\boldsymbol{\theta}$, when the market maker has a certain design objective. The example considered in Section 2.4.2 highlights the importance of this task – even in simple settings, using $\boldsymbol{\theta}^{\mathsf{SW}}$ yields suboptimal outcomes if the goal is to optimize social welfare.

Concretely, the contribution of this section is to find an approximation to the optimal design parameter, and leverage a *sum of squares* (SOS) framework to bound the suboptimality of that choice. We illustrate the efficacy of our approach to market maker design on our two-market two-firm example in Figure 2.1.

2.5.1 Formulating the market maker design problem

Assume that the design objective of the market maker is given by a polynomial function $g : \mathbb{R}^{|\mathcal{F}|} \times \mathbb{R}^{|\mathcal{M}|} \to \mathbb{R}$. That is, if the market maker owned and operated the firms, it would maximize $g(\boldsymbol{q}, \boldsymbol{r})$ over the joint strategy set, defined by $\boldsymbol{q} \in \mathbb{R}_+^{|\mathcal{F}|}, \boldsymbol{r} \in \mathcal{P}, \mathbf{1}^\top \boldsymbol{r} = 0$. When playing the game with a collection of strategic firms, the market maker would ideally seek a design parameter $\boldsymbol{\theta}$ that maximizes g at the Nash equilibrium outcome of $\mathcal{G}(\boldsymbol{\theta})$. If there are multiple Nash equilibria, one can modify the goal to maximize the *worst case* g over all Nash equilibria of $\mathcal{G}(\boldsymbol{\theta})$.

Recall that we can restrict $\boldsymbol{\theta}$ to the 3-dimensional simplex Δ without loss of optimality. However, optimizing $\boldsymbol{\theta}$ over Δ is challenging. A difficulty arises from having to minimize $g(\boldsymbol{q}, \boldsymbol{r})$ over all Nash equilibria $(\boldsymbol{q}, \boldsymbol{r})$ of $\mathcal{G}(\boldsymbol{\theta})$ for any candidate $\boldsymbol{\theta}$. For instance, if $\mathcal{G}(\boldsymbol{\theta})$ has multiple isolated Nash equilibria, such a minimization amounts to solving a combinatorial problem. However, even if $\mathcal{G}(\boldsymbol{\theta})$ has a unique Nash equilibrium, describing said equilibrium is challenging. For example, if the market maker's payoff function is not a concave function of its action, then its optimal strategy cannot be described by first-order conditions alone. Even if it is concave, computing a Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$ – and hence computing g for any candidate $\boldsymbol{\theta}$ – is generally hard.

In light of these challenges, we restrict the search space for $\boldsymbol{\theta}$ to Θ_ε, described by:

$$\theta_C, \theta_P, \theta_M \geq 0,\ \theta_C + \theta_P + \theta_M = 1,\ 2\theta_M - \theta_C \geq \varepsilon + \gamma \cdot (\theta_M + \theta_P - \theta_C) \geq (1 + \gamma) \cdot \varepsilon,$$

where $\epsilon > 0$ is sufficiently small. Theorem 2.2(b) implies that $\mathcal{G}(\boldsymbol{\theta})$ has a unique Nash equilibrium for each $\boldsymbol{\theta} \in \Theta_\varepsilon$ that also equals the unique optimizer of the convex program $\mathcal{C}(\boldsymbol{\theta})$ in (2.7). Hence, the unique Nash equilibrium is exactly characterized by the Karush-Kuhn-Tucker conditions for $\mathcal{C}(\boldsymbol{\theta})$. Thus, we can formulate the following market design problem over Θ_ε.

$$
\begin{aligned}
\underset{\boldsymbol{q},\boldsymbol{r},\boldsymbol{\mu},\boldsymbol{\lambda},\boldsymbol{\theta}}{\text{maximize}} \quad & g(\boldsymbol{q},\boldsymbol{r}),\\
\text{subject to} \quad & \boldsymbol{q} \in \mathbb{R}_+^{|\mathcal{F}|}, \boldsymbol{A}'\boldsymbol{r} \leq \boldsymbol{b}',\\
& \boldsymbol{\mu} \in \mathbb{R}_+^{|\mathcal{F}|}, \boldsymbol{\lambda} \in \mathbb{R}_+^{\dim(\boldsymbol{b}')}, \boldsymbol{\theta} = (\theta_C, \theta_P, \theta_M) \in \Delta,\\
& \nabla_{\boldsymbol{r}} \left[\hat{\Pi}(\boldsymbol{q},\boldsymbol{r};\boldsymbol{\theta}) + \boldsymbol{\lambda}^\top \left(\boldsymbol{b}' - \boldsymbol{A}'\boldsymbol{r}\right)\right] = 0, \qquad (2.13)\\
& \nabla_{\boldsymbol{q}} \left[\hat{\Pi}(\boldsymbol{q},\boldsymbol{r};\boldsymbol{\theta}) + \boldsymbol{\mu}^\top \boldsymbol{q}\right] = 0,\\
& \boldsymbol{\mu}^\top \boldsymbol{q} = 0,\ \boldsymbol{\lambda}^\top \left(\boldsymbol{b}' - \boldsymbol{A}'\boldsymbol{r}\right) = 0,\\
& 2\theta_M - \theta_C \geq \varepsilon + \gamma \cdot (\theta_M + \theta_P - \theta_C) \geq (1+\gamma)\cdot\varepsilon,
\end{aligned}
$$

where $\boldsymbol{A}' = \left(\boldsymbol{A}^\top, \mathbf{1}, -\mathbf{1}\right)^\top$, $\boldsymbol{b}' = \left(\boldsymbol{b}^\top, 0, 0\right)^\top$, and $\dim(\boldsymbol{b}')$ denotes the dimension of $\boldsymbol{b}'$. For an arbitrary function $h : \mathbb{R}^{m_x+m_y} \to \mathbb{R}$, define $\nabla_{\boldsymbol{x}} h(\boldsymbol{x},\boldsymbol{y})$ as the gradient of h with respect to $\boldsymbol{x}$. In an optimization problem, if the search space is not closed, an optimizer may not exist. To avoid such technical difficulties, we choose to optimize over the closed subset Θ_ε of the design space where $\mathcal{G}(\boldsymbol{\theta})$ admits a unique Nash equilibrium. Denote by $\boldsymbol{\theta}^*$ an optimizer of (2.13) that defines an optimal design choice. As we illustrated in Section 2.4.2 through an example, even if $g(\boldsymbol{q},\boldsymbol{r}) = \Pi(\boldsymbol{q},\boldsymbol{r};\boldsymbol{\theta}^0)$ for some $\boldsymbol{\theta}^0 \in \Theta_\varepsilon$, the design choice $\boldsymbol{\theta}^0$ may not be optimal, that is, it may not be an optimizer of (2.13).

2.5.2 Approximately solving the market maker design problem

The market maker design problem in (2.13) is a so-called Mathematical Program with Equilibrium Constraints (MPEC). Such problems are nonconvex and hard to solve efficiently in general (see [86, 106, 102]). Many heuristic searches have been applied to MPECs, but they often do not come with any optimality guarantees. Instead of using such a heuristic, we provide a scheme to find an approximate solution by exploiting our characterization results, and further bound the approximation quality.

Assume henceforth that the cost functions $c_f, f \in \mathcal{F}$ are polynomial functions for which $\mathcal{C}(\boldsymbol{\theta})$ can be cast as a convex program that is solvable in polynomial time. For example, when said costs are quadratic, $\mathcal{C}(\boldsymbol{\theta})$ can be solved as a convex quadratic program. For each $\boldsymbol{\theta} \in \Theta_\varepsilon$, one can efficiently compute the unique Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$ by solving $\mathcal{C}(\boldsymbol{\theta})$. Denote the unique Nash equilibrium by $(\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{r}(\boldsymbol{\theta}))$. Any metaheuristic (e.g., grid search, simulated annealing, Monte-Carlo sampling) can be used to explore the space Θ_ε for the largest $g(\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{r}(\boldsymbol{\theta}))$ to obtain an approximate solution of (2.13). For this exposition, we choose a finite and uniform discretization of Θ_ε. If $g(\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{r}(\boldsymbol{\theta}))$ attains its maximum at $\boldsymbol{\theta}_{\max}$ over this finite set, we have

$$
g(\boldsymbol{q}(\boldsymbol{\theta}_{\max}), \boldsymbol{r}(\boldsymbol{\theta}_{\max})) \leq g(\boldsymbol{q}(\boldsymbol{\theta}^*), \boldsymbol{r}(\boldsymbol{\theta}^*)),
$$

where $\boldsymbol{\theta}^*$ is an optimizer of (2.13). The difference between the two expressions in the above inequality is a measure of the optimality gap of $\boldsymbol{\theta}_{\max}$.

We next present a hierarchy of successively tighter upper bounds for $g(\boldsymbol{q}(\boldsymbol{\theta}^*), \boldsymbol{r}(\boldsymbol{\theta}^*))$. The upper bound at any level of the hierarchy can be computed in polynomial time, and yields a bound on the optimality gap of $\boldsymbol{\theta}_{\max}$. In presenting this hierarchy, we need the following technical result that allows us to restrict the feasible set of (2.13) to a compact basic semi-algebraic set. The proof is deferred till Appendix 2.E.

Lemma 2.1. *Suppose $\mathcal{P}$ is compact. Consider the optimization problem* (2.13)*, defined in the variables $\boldsymbol{z} := (\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{\mu}, \boldsymbol{\lambda}, \boldsymbol{\theta})$. There exists a compact set $\mathcal{Z} := \{\boldsymbol{z} : h_i(\boldsymbol{z}) \geq 0, i = 1, \ldots, I\}$, where $h_i, i = 1, \ldots, I-1$ are polynomial functions, and $h_I(\boldsymbol{z}) = \bar{Z} - \|\boldsymbol{z}\|_2$ for some constant $\bar{Z} \in \mathbb{R}_+$, such that the feasible set of* (2.13) *can be restricted to $\mathcal{Z}$ without loss of optimality.*

With a slight abuse of notation, we use $g(\boldsymbol{z})$ to denote $g(\boldsymbol{q}, \boldsymbol{r})$. Equipped with Lemma 2.1, (2.13) can be reformulated as a polynomial optimization problem that seeks to minimize $t \in \mathbb{R}$, subject to $g(\boldsymbol{z}) \leq t$ and $\boldsymbol{z} \in \mathcal{Z}$. Our upper bounds are then given by the so-called *Lasserre hierarchy* to this polynomial optimization problem (see [78, Chapter 4], [113, 42]).

A polynomial is said to be *sum of squares*, denoted SOS, if it can be expressed as a sum of other squared polynomials. For a positive integer d such that $2d \geq \max_{i=1,\ldots,I} \deg(h_i)$, consider the following optimization problem and its optimal value.

$$
\begin{aligned}
v_d^* := \underset{t, \sigma_0, \ldots, \sigma_I}{\text{minimize}} \quad & t, \\
\text{subject to} \quad & t = g + \sigma_0 + \sigma_1 h_1 + \ldots, \sigma_I h_I, \\
& \deg(\sigma_0) \leq 2d, \ \deg(\sigma_i h_i) \leq 2d, \ i = 1, \ldots, I, \\
& t \in \mathbb{R}, \ \sigma_0, \ldots, \sigma_I \text{ are SOS.}
\end{aligned}
\tag{2.14}
$$

The convergence of the Lasserre hierarchy, as given by [78, Theorem 4.1], yields $v_d^* \downarrow g(\boldsymbol{q}(\boldsymbol{\theta}^*), \boldsymbol{r}(\boldsymbol{\theta}^*))$. That is, v_d^* approaches $g(\boldsymbol{q}(\boldsymbol{\theta}^*), \boldsymbol{r}(\boldsymbol{\theta}^*))$ monotonically from above. Furthermore, the determination of whether a polynomial with degree $\leq 2d$ is SOS can be written as a linear matrix inequality in the coefficients of that polynomial (see [113]). Therefore, (2.14) can be reformulated as a semidefinite program that is solvable in polynomial time. If $\boldsymbol{\theta}_{\max}$ is a candidate approximate solution for (2.13), then $v_d^* - g(\boldsymbol{q}(\boldsymbol{\theta}_{\max}), \boldsymbol{r}(\boldsymbol{\theta}_{\max}))$ defines a bound on the optimality gap of $\boldsymbol{\theta}_{\max}$.

2.5.3 Returning to our motivating example

Consider again the two-market two-firm example discussed in Section 2.3 that is motivated by the California electricity market. Assume that the design objective is social welfare, i.e., $g(\boldsymbol{q}, \boldsymbol{r}) = \Pi\left(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}^0\right)$, where $\boldsymbol{\theta}^0 := \frac{1}{3}\boldsymbol{\theta}^{\mathsf{SW}}$. In Section 2.4.2, we argued that $\boldsymbol{\theta}^0$ is not the optimal choice for such a design objective. In the following, we apply our approximation scheme towards choosing the design parameter to maximize $g(\boldsymbol{q}, \boldsymbol{r})$ at the equilibrium.

Consider a particular setting where the two markets have identical linear inverse demand functions $p_m(d_m) := 1 - d_m$ for each $m \in \{1, 2\}$, the firms have linear costs $c_1(q_1) := \frac{1}{2}q_f$ and $c_2(q_2) := \frac{1}{4}q_2$, and the capacity of the line is $b = \frac{1}{2}$. As a benchmark for comparison, consider $\mathcal{G}(\boldsymbol{\theta})$ with $\boldsymbol{\theta} = \boldsymbol{\theta}^0$, which represents the current practice in electricity markets. Our results in Appendix 2.F indicate that $q_1 = \frac{3}{16}$, $q_2 = \frac{7}{16}$, $r_1 = -r_2 = \frac{1}{8}$ defines the unique Nash equilibrium of $\mathcal{G}\left(\boldsymbol{\theta}^0\right)$ with a social welfare of $\frac{83}{256} \approx 0.324$.

Now, let us design $\boldsymbol{\theta}$ using our approximation scheme. Assume $\varepsilon = 0.001$ and discretize the set Θ_ε as follows. Tile the triangle in Fig. 2.2 by squares (aligning with the base) with sides that are one-tenth the length of the base. Solve $\mathcal{C}(\boldsymbol{\theta})$ as a convex quadratic program at the vertices of the square tiles that satisfy $\boldsymbol{\theta} \in \Theta_\varepsilon$. Upon maximizing $g(\boldsymbol{q}, \boldsymbol{r}) = \Pi\left(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}^0\right)$ at the solutions of $\mathcal{C}(\boldsymbol{\theta})$ over this discrete set, we obtain $\boldsymbol{\theta}_{\max} = (0.027, 0.627, 0.346)^\top$, and the social welfare at the unique Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta}_{\max})$ is 0.339, which is higher than 0.324 obtained at that of $\mathcal{G}\left(\boldsymbol{\theta}^0\right)$. One can show that, while the total production from the two firms is identical for both design choices, the cheaper firm produces a larger share with $\boldsymbol{\theta}_{\max}$ than with $\boldsymbol{\theta}^0$.

To gauge the suboptimality of our design choice $\boldsymbol{\theta}_{\max}$, we obtain an upper bound $v_1^* = 0.340$ on the maximum attainable social welfare at a Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$ over Θ_ε. The upper bound is remarkably close to the social welfare obtained with $\boldsymbol{\theta}_{\max}$. Hence, our design choice $\boldsymbol{\theta}_{\max}$ has a provably good approximation quality.

Appendices

In the following Appendices, we prove Theorems 2.1, 2.2, Proposition 2.1, and Lemma 2.1. We begin by defining some notation important in the sequel. For a symmetric matrix $\boldsymbol{X}$, let $\boldsymbol{X} \preceq 0$ (resp. $\boldsymbol{X} \prec 0$) denote that $\boldsymbol{X}$ is negative semidefinite (resp. negative definite). For a finite index set $\mathcal{I}$, let $(x_i, i \in \mathcal{I})$ define a vertical vector concatenation of x_i's, for which $i \in \mathcal{I}$. For an arbitrary function $h : \mathbb{R}^m \to \mathbb{R}$, define $\frac{\partial}{\partial x_i} h\Big|_{\boldsymbol{x}=\boldsymbol{x}_0}$ as the partial derivative of h with respect to x_i at $\boldsymbol{x}_0 \in \mathbb{R}^m$ for $i = 1, \ldots, m$. Further, let $\frac{\partial^2}{\partial x_i \partial x_j} h\Big|_{\boldsymbol{x}=\boldsymbol{x}_0} := \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} h\Big|_{\boldsymbol{x}=\boldsymbol{x}_0}$. For an arbitrary function $h : \mathbb{R}^{m_x + m_y} \to \mathbb{R}$, define $\nabla_{\boldsymbol{x}} h(\boldsymbol{x}, \boldsymbol{y})$ as the gradient of h with respect to $\boldsymbol{x}$. The sub-level and super-level sets of h are given by $\{\boldsymbol{x} \in \mathbb{R}^m : h(\boldsymbol{x}) \leq \eta\}$ and $\{\boldsymbol{x} \in \mathbb{R}^m : h(\boldsymbol{x}) \geq \eta\}$, respectively, as η varies over $\mathbb{R}$. Let $\mathbf{0}$ denote a vector of zeros of appropriate dimension.

Also, define

$$\mathcal{P}' := \{\boldsymbol{r} \in \mathbb{R}^{|\mathcal{M}|} \ : \ \boldsymbol{A}\boldsymbol{r} \leq \boldsymbol{b}, \ \mathbf{1}^\top \boldsymbol{r} = 0\},$$

and hence $\boldsymbol{r} \in \mathcal{P}$ and $\mathbf{1}^\top \boldsymbol{r} = 0$ is succinctly represented as $\boldsymbol{r} \in \mathcal{P}'$, henceforth.

Our proofs shall make use of two known results in the literature. We describe them briefly. Consider a game $\mathcal{G}$ with (a) players $1, \ldots, N$, (b) actions $\boldsymbol{x}_i \in \mathbb{R}^{n_i}$ for $i = 1, \ldots, N$, where $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \in \mathcal{X} \subseteq \mathbb{R}^{n_1 + \cdots + n_N}$, and (c) payoff functions $\varphi_i : \mathbb{R}^{n_1 + \cdots + n_N} \to \mathbb{R}$ for each player $i = 1, \ldots N$.

- Game $\mathcal{G}$ is said to be *concave*, if $\mathcal{X}$ is a compact convex set, and φ_i is concave in $\boldsymbol{x}_i$ for each $i = 1, \ldots, N$. Then, Theorem 1 in [110] states that a Nash equilibrium always exists for a concave game.

- Recall that game $\mathcal{G}$ is said to be a *weighted potential game*, if there exists a vector of weights $w \in \mathbb{R}^N_{++}$ and a potential function $\Phi : \mathbb{R}^{n_1 + \cdots + n_N} \to \mathbb{R}$ that satisfies (2.5). Then, Theorem 1 in [99] implies that if $\mathcal{G}$ is a weighted potential game with Φ as the potential function, and Φ is concave and continuously differentiable, then any Nash equilibrium of $\mathcal{G}$ is an optimizer of Φ over $\mathcal{X}$.[3]

[3] Neyman's result guarantees that if a game with finitely many players admits a concave and continuously differentiable potential function, then any correlated equilibrium of the game is a pure strategy Nash equilibrium, and is given by a global optimizer of the potential function of the game.

2.A Proof of Theorem 2.1

Proof. From the definition of $\hat{\Pi}$, it follows that $\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) - \Pi(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ does not depend on $\boldsymbol{r}$. Hence, for every $\boldsymbol{q} \in \mathbb{R}_+^{|\mathcal{F}|}$, we have

$$\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) - \hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}'; \boldsymbol{\theta}) = \Pi(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) - \Pi(\boldsymbol{q}, \boldsymbol{r}'; \boldsymbol{\theta}) \tag{2.15}$$

for each $\boldsymbol{r}, \boldsymbol{r}' \in \mathcal{P}'$. Expanding $\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$, we obtain

$$\begin{aligned}\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) = (\theta_M + \theta_P - \theta_C) \sum_{m \in \mathcal{M}} &\left[\sum_{f \in \mathcal{F}(m)} ((\alpha_m - \beta_m r_m) q_f - c_f(q_f)) - \frac{\beta_m}{2} \sum_{f, f' \in \mathcal{F}(m)} q_f q_{f'} \right] \\ + (\theta_C - 2\theta_M) \sum_{m \in \mathcal{M}} \frac{\beta_m}{2} r_m^2 + \theta_M \sum_{m \in \mathcal{M}} \alpha_m r_m.&\end{aligned}$$

Then, for every $f \in \mathcal{F}$ and $\boldsymbol{r} \in \mathcal{P}'$, it follows that

$$\begin{aligned}&\hat{\Pi}(q_f, \boldsymbol{q}_{-f}, \boldsymbol{r}; \boldsymbol{\theta}) - \hat{\Pi}(q'_f, \boldsymbol{q}_{-f}, \boldsymbol{r}; \boldsymbol{\theta}) \\ &= (\theta_M + \theta_P - \theta_C) \Bigg[\left(\alpha_{\mathcal{M}(f)} - \beta_{\mathcal{M}(f)} r_{\mathcal{M}(f)}\right) \left(q_f - q'_f\right) - \beta_{\mathcal{M}(f)} \left(\sum_{f' \in \mathcal{F}(\mathcal{M}(f))} q_{f'} \right) q_f \\ &\qquad\qquad + \beta_{\mathcal{M}(f)} \left(q'_f + \sum_{f' \in \mathcal{F}(\mathcal{M}(f)) \setminus \{f\}} q_{f'} \right) q'_f - \left(c_f(q_f) - c_f(q'_f)\right) \Bigg] \\ &= (\theta_M + \theta_P - \theta_C) \Bigg[\left(\alpha_{\mathcal{M}(f)} - \beta_{\mathcal{M}(f)} \left(r_{\mathcal{M}(f)} + \sum_{f' \in \mathcal{F}(\mathcal{M}(f))} q_{f'} \right) \right) q_f - c_f(q_f) \\ &\qquad\qquad - \left(\alpha_{\mathcal{M}(f)} - \beta_{\mathcal{M}(f)} \left(r_{\mathcal{M}(f)} + q'_f + \sum_{f' \in \mathcal{F}(\mathcal{M}(f)) \setminus \{f\}} q_{f'} \right) \right) q'_f + c_f(q'_f) \Bigg] \\ &= (\theta_M + \theta_P - \theta_C) \left[\pi_f(q_f, \boldsymbol{q}_{-f}, \boldsymbol{r}) - \pi_f(q'_f, \boldsymbol{q}_{-f}, \boldsymbol{r}) \right]\end{aligned}$$

for each $q_f, q'_f \in \mathbb{R}_+$. Hence, if $\theta_M + \theta_P - \theta_C > 0$, then $\mathcal{G}(\boldsymbol{\theta})$ is a weighted potential game with $\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ as the potential function. □

2.B Proof of Theorem 2.2(a)

Proof. We prove the existence of a Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$ for $2\theta_M - \theta_C \geq 0$ and $\theta_M + \theta_P - \theta_C > 0$ separately.

When $2\theta_M - \theta_C \geq 0$. We leverage Rosen's result to show that a Nash equilibrium exists. Note that $\mathcal{G}(\boldsymbol{\theta})$ is not a concave game since its joint strategy set is unbounded. Our key idea is to define another game $\hat{\mathcal{G}}(\boldsymbol{\theta})$ with a bounded joint strategy set such that any Nash equilibrium of $\hat{\mathcal{G}}(\boldsymbol{\theta})$ is also a Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$. Then, we utilize Rosen's result to guarantee the existence of a Nash equilibrium of $\hat{\mathcal{G}}(\boldsymbol{\theta})$, and by extension, to that of $\mathcal{G}(\boldsymbol{\theta})$.

Recall that $\boldsymbol{r} \in \mathcal{P}'$. Since $\mathcal{P}'$ is compact, there exists $\bar{r} \in \mathbb{R}_+$ such that $|r_m| \leq \bar{r}$ for every $m \in \mathcal{M}$. Let $\bar{q} := \max_{f \in \mathcal{F}} (1/2) \left(\alpha_{\mathcal{M}(f)}/\beta_{\mathcal{M}(f)} + \bar{r}\right)$. Consider a game $\hat{\mathcal{G}}(\boldsymbol{\theta})$ that is identical to $\mathcal{G}(\boldsymbol{\theta})$ except that each firm $f \in \mathcal{F}$ has a strategy set $[0, \bar{q}]$. Then $(\boldsymbol{q}, \boldsymbol{r}) \in [0, \bar{q}]^{|\mathcal{F}|} \times \mathcal{P}'$ is a Nash equilibrium of $\hat{\mathcal{G}}(\boldsymbol{\theta})$ if

$$\begin{aligned} \pi_f(\boldsymbol{q}, \boldsymbol{r}) &\geq \pi_f(q'_f, \boldsymbol{q}_{-f}, \boldsymbol{r}), \quad \text{for all } q'_f \in [0, \bar{q}]; \\ \Pi(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) &\geq \Pi(\boldsymbol{q}, \boldsymbol{r'}; \boldsymbol{\theta}), \qquad \text{for all } \boldsymbol{r'} \in \mathcal{P}'. \end{aligned}$$

Now, π_f is continuous in its arguments. It is also concave in q_f because

$$\frac{\partial^2}{\partial q_f^2} \pi_f(\boldsymbol{q}, \boldsymbol{r}) = -2\beta_{\mathcal{M}(f)} - c''_f(q_f) < 0.$$

Moreover, we have

$$\begin{aligned} \frac{\partial}{\partial q_f} \pi_f(\boldsymbol{q}, \boldsymbol{r}) \bigg|_{q_f = \bar{q}} &= \alpha_{\mathcal{M}(f)} - \beta_{\mathcal{M}(f)} r_{\mathcal{M}(f)} - 2\beta_{\mathcal{M}(f)} \bar{q} - c'_f(\bar{q}) \\ &\leq \alpha_{\mathcal{M}(f)} + \beta_{\mathcal{M}(f)} \bar{r} - 2\beta_{\mathcal{M}(f)} \bar{q} \\ &\leq 0. \end{aligned}$$

The first inequality holds since c_f is nondecreasing and $|r_{\mathcal{M}(f)}| \leq \bar{r}$. The second inequality follows from the definition of $\bar{q}$. Hence, we infer that $\pi_f(\boldsymbol{q}, \boldsymbol{r})$ is decreasing over $q_f \geq \bar{q}$. In turn, it implies that if $(\boldsymbol{q}, \boldsymbol{r})$ is a Nash equilibrium of $\hat{\mathcal{G}}(\boldsymbol{\theta})$, then

$$\pi_f(\boldsymbol{q}, \boldsymbol{r}) \geq \pi_f(q'_f, \boldsymbol{q}_{-f}, \boldsymbol{r})$$

for all $q'_f \in \mathbb{R}_+$. Hence, any Nash equilibrium of $\hat{\mathcal{G}}(\boldsymbol{\theta})$ is also a Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$.

Next, we argue that $\hat{\mathcal{G}}(\boldsymbol{\theta})$ is a concave game. The joint strategy set of $\hat{\mathcal{G}}(\boldsymbol{\theta})$ is given by the compact set $[0, \bar{q}]^{|\mathcal{F}|} \times \mathcal{P}'$. The payoff function of firm f, i.e., π_f, is continuous in all its arguments and concave in q_f. The market maker's payoff function $\Pi(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ is continuous in all its arguments, and we have

$$\frac{\partial^2}{\partial r_m \partial r_{m'}} \Pi(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) = \begin{cases} -(2\theta_M - \theta_C)\beta_m, & \text{if } m = m', \\ 0, & \text{otherwise.} \end{cases} \tag{2.16}$$

Since $-(2\theta_M - \theta_C)\beta_m \leq 0$, the Hessian of $\Pi(\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{\theta})$ with respect to $\boldsymbol{r}$ is then negative semidefinite, and hence $\Pi(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ is concave in $\boldsymbol{r}$. We conclude that $\hat{\mathcal{G}}(\boldsymbol{\theta})$ is a concave game, and therefore has a

Nash equilibrium. Hence, $\mathcal{G}(\boldsymbol{\theta})$ has a Nash equilibrium.

When $\theta_M + \theta_P - \theta_C > 0$. For such a design parameter, Theorem 2.1 implies that $\mathcal{G}(\boldsymbol{\theta})$ is a weighted potential game. Then, if $\mathcal{C}(\boldsymbol{\theta})$ has a finite optimizer, this optimizer is a Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$, and therefore a Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$ exists. We show that the super-level sets of $\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ are compact, and thus that $\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ has a finite optimizer. Rewrite $\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ as

$$\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) = -(\theta_M + \theta_P - \theta_C) h_1(\boldsymbol{q}, \boldsymbol{r}) + h_2(\boldsymbol{r}),$$

where $h_1(\boldsymbol{q}, \boldsymbol{r})$ and $h_2(\boldsymbol{r})$ are defined as

$$\begin{aligned} h_1(\boldsymbol{q}, \boldsymbol{r}) &:= \sum_{m \in \mathcal{M}} \Bigg[\frac{\beta_m}{2} \Bigg(\bigg(\sum_{f \in \mathcal{F}(m)} q_f \bigg)^2 + \sum_{f \in \mathcal{F}(m)} q_f^2 \Bigg) \\ &\qquad - \sum_{f \in \mathcal{F}(m)} (\alpha_m q_f - c_f(q_f)) + \beta_m r_m \sum_{f \in \mathcal{F}(m)} q_f \Bigg], \\ h_2(\boldsymbol{r}) &:= \left(\frac{\theta_C}{2} - \theta_M \right) \sum_{m \in \mathcal{M}} \beta_m r_m^2 + \sum_{m \in \mathcal{M}} (\theta_M \alpha_m + \theta_C \beta_m) r_m. \end{aligned}$$

Recall that $\mathcal{P}'$ is compact, and hence there exists $\bar{r} \in \mathbb{R}_+$ such that $|r_m| \leq \bar{r}$ for all $\boldsymbol{r} \in \mathcal{P}'$. It is easy to verify from the definition of h_1 that

$$h_1(\boldsymbol{q}, \boldsymbol{r}) \leq h_1(\boldsymbol{q}, \bar{r}\mathbf{1})$$

for all $\boldsymbol{q} \in \mathbb{R}_+^{|\mathcal{F}|}$. Continuity of h_1 and h_2 implies that the super-level set of $\hat{\Pi}$ is closed. Furthermore, since c_f is convex and non-decreasing, we have that $\lim_{\|\boldsymbol{q}\| \to \infty} h_1(\boldsymbol{q}, \bar{r}\mathbf{1}) \to \infty$. Hence, its sub-level sets are bounded. In turn, it implies that the sub-level sets of $h_1(\boldsymbol{q}, \boldsymbol{r})$ are bounded. Also, $h_2(\boldsymbol{r})$ solely depends on $\boldsymbol{r}$, which varies over a compact set $\mathcal{P}'$. Then, h_2 takes values over a bounded set in $\mathbb{R}$. Putting the arguments together then implies that the super-level sets of $\hat{\Pi}$ are compact. □

2.C Proof of Theorem 2.2(b)

Proof. When $2\theta_M - \theta_C \geq \gamma \cdot (\theta_M + \theta_P - \theta_C) \geq 0$, part (a) implies that $\mathcal{G}(\boldsymbol{\theta})$ has at least one Nash equilibrium. To equate the set of Nash equilibria of $\mathcal{G}(\boldsymbol{\theta})$ to the set of optimizers of $\mathcal{C}(\boldsymbol{\theta})$, we first establish that the potential function $\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ is jointly concave in $(\boldsymbol{q}, \boldsymbol{r})$. The Hessian of $\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ with respect to $(\boldsymbol{q}, \boldsymbol{r})$ can be shown to be block diagonal with $|\mathcal{M}|$ block matrices. Denote said block matrices by $\boldsymbol{H}_m, m \in \mathcal{M}$, where

$$\boldsymbol{H}_m = -\begin{pmatrix} (\theta_M + \theta_P - \theta_C)\left(\beta_m \mathbf{1}\mathbf{1}^\top + \operatorname{diag}(\boldsymbol{d}_m)\right) & (\theta_M + \theta_P - \theta_C)\,\beta_m \mathbf{1} \\ (\theta_M + \theta_P - \theta_C)\,\beta_m \mathbf{1}^\top & (2\theta_M - \theta_C)\,\beta_m \end{pmatrix},$$

and $\boldsymbol{d}_m := \left(\beta_m + c_f''(q_f), f \in \mathcal{F}(m)\right) \in \mathbb{R}_+^{|\mathcal{F}(m)|}$. It suffices to show that $\boldsymbol{H}_m \preceq 0$ for each $m \in \mathcal{M}$. By the Sherman-Woodbury matrix identity (see [63] for example), we have

$$\begin{aligned}&\left(\beta_m \mathbf{1}\mathbf{1}^\top + \operatorname{diag}(\boldsymbol{d}_m)\right)^{-1} \\ &= \operatorname{diag}(\boldsymbol{d}_m)^{-1} - \left(\frac{1}{1/\beta_m + \mathbf{1}^\top \operatorname{diag}(\boldsymbol{d}_m)^{-1}\mathbf{1}}\right) \operatorname{diag}(\boldsymbol{d}_m)^{-1}\mathbf{1}\mathbf{1}^\top \operatorname{diag}(\boldsymbol{d}_m)^{-1}.\end{aligned}$$

Using Schur complements, we obtain

$$\begin{aligned}\boldsymbol{H}_m \preceq 0 &\iff (2\theta_M - \theta_C)\beta_m - (\theta_M + \theta_P - \theta_C)\beta_m \mathbf{1}^\top(\beta_m \mathbf{1}\mathbf{1}^\top + \operatorname{diag}(\boldsymbol{d}_m))^{-1}\beta_m \mathbf{1} \geq 0 \\ &\iff (2\theta_M - \theta_C) - (\theta_M + \theta_P - \theta_C)\left(1 - \frac{1}{1 + \beta_m \mathbf{1}^\top \operatorname{diag}(\boldsymbol{d}_m)^{-1}\mathbf{1}}\right) \geq 0. \qquad (2.17)\end{aligned}$$

And $\boldsymbol{H}_m \prec 0$ follows from the observation that

$$1 - \frac{1}{1 + \beta_m \mathbf{1}^\top \operatorname{diag}(\boldsymbol{d}_m)^{-1}\mathbf{1}} \leq 1 - \left(1 + \sum_{f \in \mathcal{F}(m)} \frac{\beta_m}{\beta_m + \inf_{q_f \geq 0} c_f''(q_f)}\right)^{-1} \leq \gamma.$$

Also, $\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ is continuously differentiable. Thus, $\mathcal{G}(\boldsymbol{\theta})$ is a potential game with a concave and continuously differentiable potential function. Neyman's result then implies that the set of Nash equilibria of $\mathcal{G}(\boldsymbol{\theta})$ is identical to the set of optimizers of $\mathcal{C}(\boldsymbol{\theta})$.

When $2\theta_M - \theta_C > \gamma \cdot (\theta_M + \theta_P - \theta_C) \geq 0$, it follows from (2.17) that $\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ is strictly concave. Then, $\mathcal{C}(\boldsymbol{\theta})$ has at most one optimizer. Finally, the desired result follows from the equality of the sets of Nash equilibria of $\mathcal{G}(\boldsymbol{\theta})$ and the optimizers of $\mathcal{C}(\boldsymbol{\theta})$, and the fact that $\mathcal{G}(\boldsymbol{\theta})$ has at least one Nash equilibrium. □

2.D Proof of Proposition 2.1

Proof. Since $\theta_M + \theta_P - \theta_C > 0$ and $2\theta_M - \theta_C > \gamma \cdot (\theta_M + \theta_P - \theta_C)$, we infer from Theorem 2.1 that $\mathcal{G}(\boldsymbol{\theta})$ is a weighted potential game with a strictly concave and continuously differentiable potential function $\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$. Hence, the set of Nash equilibria of $\mathcal{G}(\boldsymbol{\theta})$ is identical to the set of optimizers of $\mathcal{C}(\boldsymbol{\theta})$. Moreover, since $\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ is strictly concave, $\mathcal{C}(\boldsymbol{\theta})$ has at most one finite optimizer. When $\mathcal{P} = \mathbb{R}^{|\mathcal{M}|}$, then $\mathcal{C}(\boldsymbol{\theta})$ seeks to maximize $\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ subject to $\boldsymbol{q} \in \mathbb{R}_+^{|\mathcal{F}|}$ and $\mathbf{1}^\top \boldsymbol{r} = 0$. Now, $\mathcal{C}(\boldsymbol{\theta})$

being equivalent to a convex optimization problem with linear constraints, the KKT conditions imply that $(\boldsymbol{q}, \boldsymbol{r})$ solves $\mathcal{C}(\boldsymbol{\theta})$ if and only if $\boldsymbol{q} \in \mathbb{R}_+^{|\mathcal{F}|}$, $\mathbf{1}^\top \boldsymbol{r} = 0$, and there exists $\boldsymbol{\mu} \in \mathbb{R}_+^{|\mathcal{F}|}$ and $\lambda \in \mathbb{R}$ that satisfy

$$\nabla_{\boldsymbol{r}} \left[\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) - \lambda \mathbf{1}^\top \boldsymbol{r}\right] = \mathbf{0}, \tag{2.18a}$$

$$\nabla_{\boldsymbol{q}} \left[\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) + \boldsymbol{\mu}^\top \boldsymbol{q}\right] = \mathbf{0}, \tag{2.18b}$$

$$\boldsymbol{\mu}^\top \boldsymbol{q} = 0. \tag{2.18c}$$

Let $\boldsymbol{\mu} \in \mathbb{R}_+^{|\mathcal{F}|}$ be the all-zero vector, and

$$\lambda := \frac{1}{2} \left(\theta_P + \theta_M - \theta_C\right) \left[\frac{1}{|\mathcal{F}|} \sum_{f'} C_{f'} - \alpha\right] + \theta_M \alpha. \tag{2.19}$$

In what follows, we show that $\boldsymbol{q}, \boldsymbol{r}$ defined in (2.10) – (2.11) and $\boldsymbol{\mu}, \lambda$ defined above together satisfy the Karush-Kuhn-Tucker optimality conditions.

Using the lower bound on α, we infer that $\boldsymbol{q} \in \mathbb{R}_+^{|\mathcal{F}|}$. Also, it is easy to verify that $\mathbf{1}^\top \boldsymbol{r} = 0$. Substituting the values of $\boldsymbol{q}, \boldsymbol{r}$ into the left hand side of (2.18a), we get

$$\begin{aligned} \frac{\partial}{\partial r_{\mathcal{M}(f)}} \left[\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) - \lambda \mathbf{1}^\top \boldsymbol{r}\right] &= -\left(2\theta_M - \theta_C\right) \beta r_{\mathcal{M}(f)} - \left(\theta_P + \theta_M - \theta_C\right) \beta q_f + \theta_M \alpha - \lambda \\ &= 0, \end{aligned}$$

where the last step follows from (2.19). Similarly, substituting the values of $\boldsymbol{q}, \boldsymbol{r}$ into the left hand side of (2.18b) gives

$$\frac{\partial}{\partial q_f} \left[\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) + \boldsymbol{\mu}^\top \boldsymbol{q}\right] = \alpha - \beta \left(r_{\mathcal{M}(f)} + 2q_f\right) - C_f + \mu_f = 0.$$

Finally, (2.18c) trivially holds, since $\boldsymbol{\mu}$ is the all-zero vector. Hence, we conclude that $(\boldsymbol{q}, \boldsymbol{r})$, as defined in (2.10) – (2.11), defines the unique Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$.

Towards computing the social welfare at the unique Nash equilibrium, we obtain

$$\begin{aligned}
\sum_{m\in\mathcal{M}} \mathsf{CS}_m(\boldsymbol{q},\boldsymbol{r}) &= \frac{\beta}{2}\sum_{f\in\mathcal{F}}\left(q_f + r_{\mathcal{M}(f)}\right)^2 \\
&= \frac{1}{8\beta}\left[|\mathcal{F}|(\alpha-\bar{C})^2 + (\kappa(\boldsymbol{\theta})-1)^2\sum_{f\in\mathcal{F}}(C_f-\bar{C})^2\right]; \\
\sum_{m\in\mathcal{M}} \mathsf{PS}_m(\boldsymbol{q},\boldsymbol{r}) &= \sum_{f\in\mathcal{F}}\left[\alpha - C_f - \beta\left(q_f + r_{\mathcal{M}(f)}\right)\right]\cdot q_f \\
&= \frac{1}{4\beta}\left[|\mathcal{F}|(\alpha-\bar{C})^2 + (\kappa(\boldsymbol{\theta})+1)^2\sum_{f\in\mathcal{F}}(C_f-\bar{C})^2\right]; \\
\sum_{m\in\mathcal{M}} \mathsf{MS}_m(\boldsymbol{q},\boldsymbol{r}) &= -\beta\sum_{f\in\mathcal{F}}\left(q_f + r_{\mathcal{M}(f)}\right)\cdot r_{\mathcal{M}(f)} \\
&= -\frac{1}{2\beta}\kappa(\boldsymbol{\theta})(\kappa(\boldsymbol{\theta})-1)\sum_{f\in\mathcal{F}}(C_f-\bar{C})^2.
\end{aligned}$$

The details are omitted for space constraints. Then the social welfare at the unique Nash equilibrium is given by

$$\begin{aligned}
&\sum_{m\in\mathcal{M}}\left[\mathsf{CS}_m(\boldsymbol{q},\boldsymbol{r}) + \mathsf{PS}_m(\boldsymbol{q},\boldsymbol{r}) + \mathsf{MS}_m(\boldsymbol{q},\boldsymbol{r})\right] \\
&= \frac{3|\mathcal{F}|}{8\beta}\left(\alpha-\bar{C}\right)^2 + \frac{1}{8\beta}\left[(\kappa(\boldsymbol{\theta})-1)^2 + 2(\kappa(\boldsymbol{\theta})+1)^2 - 4\kappa(\boldsymbol{\theta})(\kappa(\boldsymbol{\theta})-1)\right]\sum_{f\in\mathcal{F}}\left(C_f-\bar{C}\right)^2 \\
&= \frac{3|\mathcal{F}|}{8\beta}\left(\alpha-\bar{C}\right)^2 + \frac{1}{8\beta}\left(-\kappa(\boldsymbol{\theta})^2 + 6\kappa(\boldsymbol{\theta}) + 3\right)\sum_{f\in\mathcal{F}}\left(C_f-\bar{C}\right)^2.
\end{aligned}$$

Finally, utilizing the definition of σ_c^2 in the above equation yields the desired result. □

2.E Proof of Lemma 2.1

Proof. $\mathcal{C}(\boldsymbol{\theta})$ is a multiparametric convex program, parameterized by $\boldsymbol{\theta}$. When $\boldsymbol{\theta}\in\Theta_\varepsilon$, $\mathcal{C}(\boldsymbol{\theta})$ admits a unique primal optimal solution for each $\boldsymbol{\theta}\in\Theta_\varepsilon$; call it $(\boldsymbol{q}(\boldsymbol{\theta}),\boldsymbol{r}(\boldsymbol{\theta}))$. Let $(\lambda(\boldsymbol{\theta}),\mu(\boldsymbol{\theta}))$ denote a dual optimal solution of $\mathcal{C}(\boldsymbol{\theta})$. Dual optimal solutions may not be unique. Since Θ_ε is compact, it suffices to argue that there exists positive constants $\bar{q}$, $\bar{r}$, $\bar{\lambda}$, and $\bar{\mu}$, such that $\mathcal{C}(\boldsymbol{\theta})$ admits a primal/dual pair of optimal solutions that satisfies

$$\|\boldsymbol{q}(\boldsymbol{\theta})\|_2\le\bar{q},\ \|\boldsymbol{r}(\boldsymbol{\theta})\|_2\le\bar{r},\ \|\boldsymbol{\lambda}(\boldsymbol{\theta})\|_2\le\bar{\lambda},\ \|\boldsymbol{\mu}(\boldsymbol{\theta})\|_2\le\bar{\mu}$$

for each $\boldsymbol{\theta}\in\Theta_\varepsilon$.

If $\boldsymbol{r}$ is feasible in $\mathcal{C}(\boldsymbol{\theta})$, then $\boldsymbol{r}\in\mathcal{P}'$, where $\mathcal{P}'$ is compact. That yields a uniform bound on

$\|\boldsymbol{r}(\boldsymbol{\theta})\|_2$ for all $\boldsymbol{\theta} \in \Theta_\varepsilon$. Call that bound $\bar{r}$.

Recall that $(\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{r}(\boldsymbol{\theta}))$ is also the unique Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$. Therefore, $q_f(\boldsymbol{\theta})$ maximizes $\pi_f\left(q_f, \boldsymbol{q}_{-f}(\boldsymbol{\theta}), \boldsymbol{r}(\boldsymbol{\theta})\right)$ over $q_f \geq 0$. However, we also have

$$\begin{aligned}\frac{\partial}{\partial q_f}\pi_f\left(q_f, \boldsymbol{q}_{-f}(\boldsymbol{\theta}), \boldsymbol{r}(\boldsymbol{\theta})\right) &= \alpha_{\mathcal{M}(f)} - \beta_{\mathcal{M}(f)} r_{\mathcal{M}(f)}(\boldsymbol{\theta}) - 2\beta_{\mathcal{M}(f)} q_f - c_f'(q_f) \\ &\leq \alpha_{\mathcal{M}(f)} + \beta_{\mathcal{M}(f)}\bar{r} - 2\beta_{\mathcal{M}(f)} q_f,\end{aligned}$$

which implies that $\pi_f\left(q_f, \boldsymbol{q}_{-f}(\boldsymbol{\theta}), \boldsymbol{r}(\boldsymbol{\theta})\right)$ decreases with q_f for $q_f > \frac{\bar{r}}{2} + \max_{m\in\mathcal{M}} \frac{\alpha_m}{2\beta_m}$. In turn, we conclude

$$\|\boldsymbol{q}(\boldsymbol{\theta})\|_2 \leq \bar{q} := |\mathcal{F}|\left(\frac{\bar{r}}{2} + \max_{m\in\mathcal{M}} \frac{\alpha_m}{2\beta_m}\right).$$

For each $f \in \mathcal{F}$, we have assumed c_f to be continuously differentiable, implying $\nabla_{\boldsymbol{q}}\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ is continuous in $(\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{\theta})$. Therefore, $\nabla_{\boldsymbol{q}}\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ remains bounded over $\{\boldsymbol{q} : \|\boldsymbol{q}\|_2 \leq \bar{q}\} \times \mathcal{P}' \times \Theta_\varepsilon$. As a result, $\mu(\boldsymbol{\theta}) = -\nabla_{\boldsymbol{q}}\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})\Big|_{\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{r}(\boldsymbol{\theta})}$ admits a uniform bound $\bar{\mu}$.

Let $\Lambda(\boldsymbol{\theta})$ denote the set of optimal Lagrange multipliers for the constraint $\boldsymbol{A}'^\top \boldsymbol{r} \leq \boldsymbol{b}'$ in $\mathcal{C}(\boldsymbol{\theta})$. We conclude the proof by showing that $\inf_{\boldsymbol{\lambda}(\boldsymbol{\theta})\in\Lambda(\boldsymbol{\theta})} \|\boldsymbol{\lambda}(\boldsymbol{\theta})\|_2$ is uniformly bounded over $\boldsymbol{\theta} \in \Theta_\varepsilon$. To that end, suppose the rows of $\boldsymbol{A}'$ are given by $\boldsymbol{a}_1^\top, \ldots, \boldsymbol{a}_{\dim(\boldsymbol{b}')}^\top$, where $\boldsymbol{a}_i \in \mathbb{R}^{|\mathcal{M}|}$. Denote by $\mathcal{S}(\boldsymbol{\theta})$, the set of active constraints at optimality of $\mathcal{C}(\boldsymbol{\theta})$. That is, $\boldsymbol{a}_i^\top \boldsymbol{r}(\boldsymbol{\theta}) = \boldsymbol{b}_i'$ for each $i \in \mathcal{S}(\boldsymbol{\theta}) \subseteq \{1, \ldots, \dim(\boldsymbol{b}')\}$, and $\boldsymbol{a}_i^\top \boldsymbol{r}(\boldsymbol{\theta}) < \boldsymbol{b}_i'$ for $i \in \mathcal{S}^c(\boldsymbol{\theta}) := \{1, \ldots, \dim(\boldsymbol{b}')\} \setminus \mathcal{S}(\boldsymbol{\theta})$. Then, $\Lambda(\boldsymbol{\theta})$ is given by

$$\Lambda(\boldsymbol{\theta}) = \left\{\boldsymbol{\lambda} \in \mathbb{R}^{\dim(\boldsymbol{b}')} : \lambda_i = 0 \text{ for } i \in \mathcal{S}^c(\boldsymbol{\theta}), \sum_{i\in\mathcal{S}(\boldsymbol{\theta})} \lambda_i a_i = -\nabla_{\boldsymbol{r}}\, \hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})\Big|_{\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{r}(\boldsymbol{\theta})}\right\}. \tag{2.20}$$

If $\boldsymbol{A}'(\boldsymbol{\theta})$ denotes the $|\mathcal{S}(\boldsymbol{\theta})| \times |\mathcal{M}|$ matrix with rows $\boldsymbol{a}_i^\top$ for $i \in \mathcal{S}(\boldsymbol{\theta})$, we conclude from (2.20) that

$$\inf_{\boldsymbol{\lambda}(\boldsymbol{\theta})\in\Lambda(\boldsymbol{\theta})} \|\boldsymbol{\lambda}(\boldsymbol{\theta})\|_2 = \left\|[\boldsymbol{A}'^\top(\boldsymbol{\theta})]^\dagger \nabla_{\boldsymbol{r}}\, \hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})\Big|_{\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{r}(\boldsymbol{\theta})}\right\|_2,$$

where $[\boldsymbol{A}'^\top(\boldsymbol{\theta})]^\dagger$ denotes the Moore-Penrose inverse of $\boldsymbol{A}'^\top(\boldsymbol{\theta})$. Using the continuous differentiability of $c_f, f \in \mathcal{F}$, one can argue that $\nabla_{\boldsymbol{r}}\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ remains bounded over $\{\boldsymbol{q} : \|\boldsymbol{q}\|_2 \leq \bar{q}\} \times \mathcal{P}' \times \Theta_\varepsilon$. And the rest follows from the fact that $\boldsymbol{A}'(\boldsymbol{\theta})$ has finitely many possibilities for $\boldsymbol{\theta} \in \Theta_\varepsilon$. □

2.F Analyzing the Two-Market Two-Firm Example in Fig. 2.1

This section is devoted to deriving all Nash equilibria of $\mathcal{G}(\boldsymbol{\theta})$ for all $\boldsymbol{\theta} \in \mathbb{R}_+^3$ in a two-market two-firm example, portrayed in Figure 2.1. Our formulae let us gain insights into the parametric dependence of the Nash equilibria on the design parameter.

Consider $\mathcal{G}(\boldsymbol{\theta})$, where $\mathcal{M} = \{1,2\}$, $\mathcal{F} = \{1,2\}$, and $\mathcal{F}(1) = \{1\}$, $\mathcal{F}(2) = \{2\}$. Each firm has an increasing linear cost, given by $c_f(q_f) = C_f q_f$. Assume that the markets are spatially homogeneous, having inverse linear demand functions $p_m(d_m) = \alpha - \beta d_m$ for $m \in \mathcal{M}$, where $\alpha, \beta > 0$. The network constraint is given by $\mathcal{P} := \{\boldsymbol{r} = (r_1, r_2)^\top : |r_1| \leq b,\ |r_2| \leq b\}$, where b denotes the capacity of the link between the two markets. Hence, the markets only differ in the marginal costs of the firms supplying in each market.

Notice that $\mathbf{1}^\top \boldsymbol{r} = 0$ for our example implies $r_1 = -r_2$. Defining $r := r_1 = -r_2$, the market maker's strategy set can be described by $\{r \in \mathbb{R} : |r| \leq b\}$. And, the market maker's payoff (with a slight abuse of notation) is given by

$$\begin{aligned}\Pi(q_1, q_2, r; \boldsymbol{\theta}) = &-(\theta_M + \theta_P - \theta_C)(q_1 - q_2)\beta r - (2\theta_M - \theta_C)\beta r^2 \\ &+ \theta_P\left((\alpha - C_1) q_1 + (\alpha - C_2) q_2\right) + \frac{1}{2}(\theta_C - 2\theta_P)\beta\left(q_1^2 + q_2^2\right).\end{aligned}$$

Restrict attention to the case where

$$0 \leq b \leq \frac{\alpha - \max\{C_1, C_2\}}{\beta}.$$

Then, (q_1, q_2, r), where $q_1, q_2 \geq 0$ and $|r| \leq b$, constitutes a Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$ if and only if

1. $\Pi(q_1, q_2, r; \boldsymbol{\theta}) \geq \Pi(q_1, q_2, r'; \boldsymbol{\theta})$ for any r' such that $|r'| \leq b$, and
2. the production quantities satisfy

$$q_1 = \frac{1}{2}\left(\frac{\alpha - C_1}{\beta} - r\right), \quad q_2 = \frac{1}{2}\left(\frac{\alpha - C_2}{\beta} + r\right). \tag{2.21}$$

Let $\mathcal{R}(\boldsymbol{\theta})$ denote the set of all r's that comprise a Nash equilibrium for the game, when the design parameter is $\boldsymbol{\theta}$. Then, for each $r \in \mathcal{R}(\boldsymbol{\theta})$, the production quantities at the Nash equilibrium are uniquely identified by (2.21). We provide $\mathcal{R}(\boldsymbol{\theta})$ for all $\boldsymbol{\theta} \in \mathbb{R}^3_+$ in Table 2.F.1 by summarizing the results of Lemmas 2.2 – 2.4. We state and prove Lemmas 2.2 – 2.4 at the end of this section. Presenting our results requires the following additional notation. For any $x \in \mathbb{R}$, define

$$[x]_\ell^u := \begin{cases} x, & \text{if } \ell \leq x \leq u, \\ \ell, & \text{if } x < \ell, \\ u, & \text{otherwise.} \end{cases}.$$

Let sgn (x) denote the sign of $x \in \mathbb{R}$. We denote the null set by $\varnothing$. For convenience, define

$$\Delta C := C_1 - C_2, \quad \text{and} \quad \kappa(\boldsymbol{\theta}) := \frac{\theta_P + \theta_M - \theta_C}{3\theta_M - \theta_C - \theta_P}.$$

Table 2.F.1: Summary of $\mathcal{R}(\boldsymbol{\theta})$ for the two-market two-firm example.

Conditions on $\boldsymbol{\theta}$		$\mathcal{R}(\boldsymbol{\theta})$
$2\theta_M - \theta_C > 0$	$3\theta_M - \theta_C - \theta_P > 0$	$\left\{ \left[\frac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta} \right]_{-b}^{+b} \right\}$
	$3\theta_M - \theta_C - \theta_P = 0$	$[-b, +b]$, if $\Delta C = 0$, $\{b \cdot \operatorname{sgn}(\Delta C)\}$, otherwise.
	$3\theta_M - \theta_C - \theta_P < 0$	$\left\{ \pm b, \dfrac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta} \right\}$, if $\left\lvert \dfrac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta} \right\rvert \le b$, $\{b \cdot \operatorname{sgn}(\Delta C)\}$, otherwise.
$2\theta_M - \theta_C = 0$	$\theta_P + \theta_M - \theta_C < 0$	$\left\{ \left[-\frac{\Delta C}{2\beta} \right]_{-b}^{+b} \right\}$
	$\theta_P + \theta_M - \theta_C = 0$	$[-b, +b]$
	$\theta_P + \theta_M - \theta_C > 0$	$\left\{ \pm b, -\dfrac{\Delta C}{2\beta} \right\}$, if $\left\lvert \dfrac{\Delta C}{2\beta} \right\rvert \le b$, $\{b \cdot \operatorname{sgn}(\Delta C)\}$, otherwise.
$2\theta_M - \theta_C < 0$	$\theta_P + \theta_M - \theta_C < 0$	$\{-b \cdot \operatorname{sgn}(\Delta C)\}$, if $\left\lvert \dfrac{\Delta C}{2\beta} \right\rvert \ge b$, $\varnothing$, otherwise.
	$\theta_P + \theta_M - \theta_C = 0$	$\{\pm b\}$
	$\theta_P + \theta_M - \theta_C > 0$	$\{\pm b\}$, if $\left\lvert \dfrac{\Delta C}{2\beta} \right\rvert \le b$, $\{b \cdot \operatorname{sgn}(\Delta C)\}$, otherwise.

Recall that Theorems 2.1 and 2.2 provide sufficient conditions on $\boldsymbol{\theta}$ for $\mathcal{G}(\boldsymbol{\theta})$ to exhibit certain properties. Though we do not address the question of necessity or tightness, we use the results in Table 2.F.1 for the two-market two-firm example to illustrate that each of the properties may fail to hold if the respective conditions are not satisfied.

1. When neither $\theta_M + \theta_P - \theta_C > 0$ nor $2\theta_M \ge \theta_C$ holds, a Nash equilibrium may not exist. Consider $\boldsymbol{\theta}$ that satisfies the above conditions, such that $\theta_M + \theta_P - \theta_C < 0$, and let $\left\lvert \frac{\Delta C}{2\beta} \right\rvert < b$ in our example. Then, Table 2.F.1 implies that $\mathcal{R}(\boldsymbol{\theta}) = \varnothing$.

2. When $2\theta_M \ge \theta_C$, but not $\theta_M + \theta_P - \theta_C > 0$, a Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$ exists, but $\mathcal{G}(\boldsymbol{\theta})$ is not a weighted potential game. Consider our example, where $2\theta_M - \theta_C = 0$, $\theta_M + \theta_P - \theta_C < 0$ and $C_1 = C_2$. From Table 2.F.1, the game admits a unique Nash equilibrium with $\mathcal{R}(\boldsymbol{\theta}) = \{0\}$. However, it can be shown that the actions of the players under best response dynamics exhibit the following cycle:

$$
\begin{array}{ccc}
r & q_1 & q_2 \\
+b & \frac{1}{2}\left(\frac{\alpha - C_1}{\beta} - b\right) & \frac{1}{2}\left(\frac{\alpha - C_2}{\beta} + b\right) \\
-b & \frac{1}{2}\left(\frac{\alpha - C_1}{\beta} + b\right) & \frac{1}{2}\left(\frac{\alpha - C_2}{\beta} - b\right) \\
+b & \frac{1}{2}\left(\frac{\alpha - C_1}{\beta} - b\right) & \frac{1}{2}\left(\frac{\alpha - C_2}{\beta} + b\right) \\
\vdots & \vdots & \vdots
\end{array}
$$

implying that the game under consideration is not a potential game.

3. When $2\theta_M - \theta_C \geq 0$, and $\theta_M + \theta_P - \theta_C > 0$, but $2\theta_M - \theta_C \geq \gamma \cdot (\theta_M + \theta_P - \theta_C)$ does not hold, there may exist a Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$ that is not an optimizer of $\mathcal{C}(\boldsymbol{\theta})$. In our example, $\gamma = \frac{1}{2}$, and hence

$$2\theta_M - \theta_C - \gamma \cdot (\theta_M + \theta_P - \theta_C) = \frac{1}{2}(3\theta_M - \theta_C - \theta_P).$$

   Consider $\boldsymbol{\theta}$ that satisfies $2\theta_M - \theta_C > 0$, $3\theta_M - \theta_C - \theta_P < 0$, and additionally, $\left|\frac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta}\right| < b$. Then, Table 2.F.1 reveals that $\mathcal{R}(\boldsymbol{\theta}) = \left\{\pm b, \frac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta}\right\}$, and hence, the game has three distinct Nash equilibria.

   Now, $\mathcal{C}(\boldsymbol{\theta})$ maximizes $\hat{\Pi}(q_1, q_2, r; \boldsymbol{\theta})$ over $q_1, q_2 \geq 0$ and $|r| \leq b$, where

$$\begin{aligned}\hat{\Pi}(q_1, q_2, r; \boldsymbol{\theta}) =& (\theta_M + \theta_P - \theta_C)\left((\alpha - C_1)q_1 + (\alpha - C_2)q_2 - \beta(q_1^2 + q_2^2) - (q_1 - q_2)\beta r\right) \\ & - (2\theta_M - \theta_C)\beta r^2.\end{aligned}$$

   One can argue that if q_1, q_2, r solves $\mathcal{C}(\boldsymbol{\theta})$, then, q_1, q_2 are related to r as in equation (2.21). In turn, solving $\mathcal{C}(\boldsymbol{\theta})$ then reduces to maximizing

$$\frac{1}{2}(\theta_M + \theta_P - \theta_C)\left[r \cdot \Delta C + \frac{1}{2\beta}(\alpha - C_1)^2 + \frac{1}{2\beta}(\alpha - C_2)^2\right] - \frac{1}{2}\left(3\theta_M - \theta_C - \theta_P\right)\beta r^2,$$

   subject to $|r| \leq b$. The above function being a strictly convex function in r attains its maximum at the boundary of the feasible set, i.e., at $\pm b$. As a result, there does *not* exist an optimizer of $\mathcal{C}(\boldsymbol{\theta})$ with $r = \frac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta}$.

4. When $2\theta_M - \theta_C = \gamma\left(\theta_M + \theta_P - \theta_C\right) > 0$, then $\mathcal{G}(\boldsymbol{\theta})$ may have a multitude of Nash equilibria, all of which are optimizers of $\mathcal{C}(\boldsymbol{\theta})$. This is observed in our example, where $\mathcal{R}(\boldsymbol{\theta}) = [-b, +b]$, when $2\theta_M - \theta_C > 0$, $3\theta_M - \theta_C - \theta_P = 0$, and $C_1 = C_2$.

In what follows, we formally characterize $\mathcal{R}(\boldsymbol{\theta})$ for all $\boldsymbol{\theta} \in \mathbb{R}_+^3$ for our example.

Lemma 2.2. *Suppose $\theta_P + \theta_M - \theta_C = 0$. Then, $\mathcal{R}(\boldsymbol{\theta})$ is given by*

$$\mathcal{R}(\boldsymbol{\theta}) = \begin{cases} \{0\}, & \text{if } 2\theta_M - \theta_C > 0, \\ [-b, +b], & \text{if } 2\theta_M - \theta_C = 0, \\ \{\pm b\}, & \text{otherwise.} \end{cases}$$

Proof. When $\theta_P + \theta_M - \theta_C = 0$, the maximizer of $\Pi(q_1, q_2, r; \boldsymbol{\theta})$ over r is independent of q_1 and q_2. Further, if $2\theta_M - \theta_C > 0$, then $\Pi(q_1, q_2, r; \boldsymbol{\theta})$ is a concave quadratic even function of r, and hence $r = 0$ is its unique maximizer. On the other hand, if $2\theta_M - \theta_C = 0$, then $\Pi(q_1, q_2, r; \boldsymbol{\theta})$ is independent of r, implying each $r \in [-b, b]$ constitutes a maximizer of $\Pi(q_1, q_2, r; \boldsymbol{\theta})$. Finally, if $2\theta_M - \theta_C < 0$, then $\Pi(q_1, q_2, r; \boldsymbol{\theta})$ is a convex quadratic even function of r, that attains its maximum at the boundaries of the feasible set, i.e., at $r = \pm b$. $\square$

Lemma 2.3. *Suppose $\theta_P + \theta_M - \theta_C < 0$. Then, $\mathcal{R}(\boldsymbol{\theta})$ is given by*

$$\mathcal{R}(\boldsymbol{\theta}) = \begin{cases} \left\{ \left[\frac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta}\right]_{-b}^{+b} \right\}, & \textit{if } 2\theta_M - \theta_C > 0, \\ \left\{ \left[-\frac{\Delta C}{2\beta}\right]_{-b}^{+b} \right\}, & \textit{if } 2\theta_M - \theta_C = 0, \\ \{-b \cdot \mathit{sgn}\ (\Delta C)\}, & \textit{if } 2\theta_M - \theta_C < 0, \ \textit{and } \left|\frac{\Delta C}{2\beta}\right| \geq b, \\ \varnothing, & \textit{otherwise.} \end{cases}$$

Proof. Suppose $\theta_P + \theta_M - \theta_C < 0$. Define the expressions in (2.21) as $q_1(r)$ and $q_2(r)$, respectively, to make explicit its dependence on r. When $2\theta_M - \theta_C > 0$, the function $\Pi(q_1, q_2, r; \boldsymbol{\theta})$ is strictly concave in r, and hence $q_1(r), q_2(r), r$ constitutes a Nash equilibrium of the game, if and only if one of three cases arise:

$$\rho(r, \boldsymbol{\theta}) = 0, \ \text{and } |r| \leq b, \tag{2.22a}$$

$$\text{or } \rho(r, \boldsymbol{\theta}) \leq 0, \ \text{and } r = -b, \tag{2.22b}$$

$$\text{or } \rho(r, \boldsymbol{\theta}) \geq 0, \ \text{and } r = +b. \tag{2.22c}$$

where $\rho(r, \boldsymbol{\theta})$ is the derivative of $\Pi(q_1, q_2, r; \boldsymbol{\theta})$ with respect to r, evaluated at $q_1(r), q_2(r), r$, given by

$$\rho(r, \boldsymbol{\theta}) = \frac{\Delta C}{2}(\theta_P + \theta_M - \theta_C) - (3\theta_M - \theta_C - \theta_P)\beta r. \tag{2.23}$$

Now, $\theta_P + \theta_M - \theta_C < 0$ and $2\theta_M - \theta_C > 0$ together imply $3\theta_M - \theta_C - \theta_P > 0$. Using the relations in (2.22a) – (2.22c), it is then straightforward to conclude that

$$\mathcal{R}(\boldsymbol{\theta}) = \left\{ \left[\frac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta}\right]_{-b}^{+b} \right\},$$

when $\theta_P + \theta_M - \theta_C < 0$ and $2\theta_M - \theta_C > 0$.

Next, consider the case, when $2\theta_M - \theta_C = 0$. Then, $\Pi(q_1, q_2, r; \boldsymbol{\theta})$ is a linear function of r with slope $\beta(q_1 - q_2)$. Thus, $q_1(r), q_2(r), r$ constitutes a Nash equilibrium of the game if and only if one of three cases arise: (i) $q_1(r) = q_2(r)$, and $|r| \leq b$, or (ii) $q_1(r) > q_2(r)$ and $r = +b$, or (iii) $q_1(r) < q_2(r)$

and $r = -b$. Substituting the values of $q_1(r)$ and $q_2(r)$ from (2.21), and rearranging, we get

$$\mathcal{R}(\boldsymbol{\theta}) = \left\{ \left[\frac{\Delta C}{2\beta} \right]_{-b}^{+b} \right\}.$$

Finally, consider the case when $2\theta_M - \theta_C < 0$. Then, $\Pi(q_1, q_2, r; \boldsymbol{\theta})$ is strictly convex in r, and is maximized at either $r = -b$ or $r = +b$ or both. Notice that

$$\Pi(q_1, q_2, +b; \boldsymbol{\theta}) - \Pi(q_1, q_2, -b; \boldsymbol{\theta}) = -2(\theta_M + \theta_P - \theta_C)(q_1 - q_2)\beta b,$$

implying (i) $+b$ is an optimizer if $q_1(+b) \geq q_2(+b)$, and (ii) $-b$ is an optimizer if $q_1(-b) \leq q_2(-b)$. Upon simplifying these conditions, we conclude,

$$\mathcal{R}(\boldsymbol{\theta}) = \begin{cases} \{-b \cdot \text{sgn } (\Delta C)\}, & \text{if} \left| \frac{\Delta C}{2\beta} \right| \geq b, \\ \varnothing, & \text{otherwise.} \end{cases}$$

The rest follows from combining the values of $\mathcal{R}(\boldsymbol{\theta})$ under different cases. $\square$

Lemma 2.4. *Suppose $\theta_P + \theta_M - \theta_C > 0$. Then, $\mathcal{R}(\boldsymbol{\theta})$ is given by*

$$\mathcal{R}(\boldsymbol{\theta}) = \begin{cases} \left\{ \left[\frac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta} \right]_{-b}^{+b} \right\}, & \textit{if } 2\theta_M - \theta_C > 0, \textit{ and } 3\theta_M - \theta_C - \theta_P > 0, \\ [-b, +b], & \textit{if } 2\theta_M - \theta_C > 0, 3\theta_M - \theta_C - \theta_P = 0, \textit{ and } \Delta C = 0, \\ \left\{ \pm b, \frac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta} \right\}, & \textit{if } 2\theta_M - \theta_C > 0, 3\theta_M - \theta_C - \theta_P < 0, \textit{ and } \left| \frac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta} \right| \leq b, \\ \left\{ \pm b, -\frac{\Delta C}{2\beta} \right\}, & \textit{if } 2\theta_M - \theta_C = 0, \textit{ and } \left| \frac{\Delta C}{2\beta} \right| \leq b, \\ \{\pm b\}, & \textit{if } 2\theta_M - \theta_C < 0, \textit{ and } \left| \frac{\Delta C}{2\beta} \right| \leq b, \\ \{b \cdot \textit{sgn } (\Delta C)\}, & \textit{otherwise.} \end{cases}$$

Proof. Suppose $\theta_P + \theta_M - \theta_C > 0$. First, consider the case, when $2\theta_M - \theta_C > 0$. Then, $\Pi(q_1, q_2, r; \boldsymbol{\theta})$ is strictly concave in r. Similar to our analysis in the proof of 2.3, it follows that $q_1(r), q_2(r), r$ constitute a Nash equilibrium of the game, if and only if one of three cases, given by (2.22a) – (2.22c), arises. We tackle three further cases separately, depending on the sign of $3\theta_M - \theta_C - \theta_P$.

Case (i): $3\theta_M - \theta_C - \theta_P > 0$. This case is identical to the case when $\theta_P + \theta_M - \theta_C < 0$ and $3\theta_M - \theta_C - \theta_P > 0$, and we obtain

$$\mathcal{R}(\boldsymbol{\theta}) = \left\{ \left[\frac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta} \right]_{-b}^{+b} \right\}.$$

Case (ii): $3\theta_M - \theta_C - \theta_P = 0$. Since $\theta_P + \theta_M - \theta_C > 0$, the sign of $\rho(r, \boldsymbol{\theta})$, defined in (2.23), is given by the sign of ΔC. Then, (2.22a) – (2.22c) implies that

$$\mathcal{R}(\boldsymbol{\theta}) = \begin{cases} [-b, +b], & \text{if } \Delta C = 0, \\ \{b \cdot \text{sgn}\,(\Delta C), & \text{otherwise.} \end{cases}$$

Case (iii): $3\theta_M - \theta_C - \theta_P < 0$. We solve for r by setting $\rho(r, \boldsymbol{\theta}) = 0$, and further utilize (2.22a) – (2.22c) to obtain

$$\mathcal{R}(\boldsymbol{\theta}) = \begin{cases} \left\{\pm b, \frac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta}\right\}, & \text{if } \left|\frac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta}\right| \leq b, \\ \{b \cdot \text{sgn}\,(\Delta C)\}, & \text{otherwise.} \end{cases}$$

Second, consider that case when $2\theta_M - \theta_C = 0$. Then, $\Pi(q_1, q_2, r; \boldsymbol{\theta})$ is linear in r with slope $\beta(q_1 - q_2)$. The analysis is similar to the case when $2\theta_M - \theta_C = 0$, but with $\theta_P + \theta_M - \theta_C < 0$. Proceeding as in the proof of Lemma 2.3, we obtain

$$\mathcal{R}(\boldsymbol{\theta}) = \begin{cases} \left\{\pm b, -\frac{\Delta C}{2\beta}\right\}, & \text{if} \left|\frac{\Delta C}{2\beta}\right| \leq b, \\ \{b \cdot \text{sgn}\,(\Delta C)\}, & \text{otherwise.} \end{cases}$$

Finally, consider the case when $2\theta_M - \theta_C < 0$. Then, $\Pi(q_1, q_2, r; \boldsymbol{\theta})$ is strictly convex in r, and is maximized at either $r = -b$ or $r = +b$ or both. Again, the analysis mirrors the argument in the proof of Lemma 2.3, and yields

$$\mathcal{R}(\boldsymbol{\theta}) = \begin{cases} \{\pm b\}, & \text{if} \left|\frac{\Delta C}{2\beta}\right| \leq b, \\ \{b \cdot \text{sgn}\,(\Delta C)\}, & \text{otherwise.} \end{cases}$$

The rest follows from combining the values of $\mathcal{R}(\boldsymbol{\theta})$ under different cases. $\square$

Chapter 3

Inefficiency of Forward Markets in Leader-Follower Competition

In this chapter, we study the strategic interaction between generation flexibility and forward markets. This work has wider applications beyond electricity to other settings such as manufacturing where firms could have heterogeneous production lead times and forward contracts are commonly used to lock in business in advance. The significance of forward trading in many industries have motivated numerous studies into forward markets. Among the literature, one of the most important issues has been the relationship between forward contracts and market power. In this area, the seminal paper is [10] which showed using a two-stage model that long forward positions mitigate market power. The intuition is that any individual firm has a strategic incentive to sell forward. This creates a prisoner's dilemma, and therefore, in equilibrium, all firms produce more. Subsequently, this phenomenon was also developed further and tested empirically by others [72, 57, 59, 128, 136, 135, 26]. Some studies, on the other hand, sought to investigate the robustness of those findings, and found different results with other models [56, 98, 58, 124, 85, 79].

In this work, we study the impact of forward contracting on markets where there is a group of leader firms that choose their productions before a group of follower firms. This game is also known as a leader-follower (or Stackelberg) competition [120]. This classical setting arises, for example, when new entrants to an industry such as gas and telecommunication must decide whether to invest in capacities [44, 48]. The capacity expansion process is time consuming so the new entrants (leaders) must decide in advance the quantities they will supply to the market. The incumbents (followers) already have capacity and need only decide how much goods/services to provide. Another prominent example is electricity, where generators have significantly different startup times and ramp rates. Generators with longer startup times and slower ramp rates (leaders) must decide on the amount of power they will supply to the market before the more flexible generators (followers). In both examples, although follower firms choose their productions later than leader firms, they may still sell a fraction of their outputs in advance. In fact, gas, telecommunications, and electricity, are all industries where forward contracts form a significant portion of the total output. These contracts are traded through bilateral negotiations or in centralised exchanges. Hence, the study of how forward contracting impacts leader-follower competition is a crucial area with important practical applications.

We analyze a model that combines key elements from the classical forward contracting model by [10] and the classical leader-follower model by [120]. In particular, we consider a setting in which there are two types of firms – *leaders* and *followers* – that choose production levels at different stages. Leaders choose production levels before followers. However, followers are allowed to sell forward contracts when leaders are choosing their productions. We assume that followers have capacity constraints while leaders are unconstrained. Imposing capacity constraints on leaders does not change the analysis significantly and the same insights still hold.

Both classical models are appealing due to their tractability and each model has been extensively studied and further developed. However, to our knowledge, the combined setting has not been investigated before. Due to the prisoner's dilemma effect, the natural inference is that allowing followers to trade forward contracts would increase their productions. Moreover, because forward contracts mitigate market power, another expectation is that the total production would also increase. In this work, we show that this intuition is not always true, and the impact of forward contracting is ambiguous. The market power mitigation property of forward contracting might, in fact, create opportunities for leaders to manipulate the market by exploiting followers' capacity constraints.

3.1 Our Contributions

Our goal in this study is to provide insight into the strategic interactions between leaders and followers in forward markets. We contribute to the existing literature in the following ways.

Our first contribution is the insight that forward contracting may decrease the efficiency of the market. The reason is that forward contracting may create opportunities for leaders to exploit the capacity constraints of the followers. Forward contracts provide incentives for followers to produce more. However, if this results in followers becoming capacity constrained, then leaders would be able to profit by withholding their productions disproportionately, and the net effect is a decrease in total production. *Informally, the increased competition due to forward contracting is offset by the decreased competition faced by the leaders due to followers being capacity constrained.* Therefore, this is a phenomenon where capacity constraints and forward markets create opportunities for market manipulation.

Our second contribution is the insight that non-existence of symmetric equilibrium could be attributed to exploitation of capacity constraints. In particular, we show that symmetric equilibria do not exist precisely when followers are operating close to capacity. Our analyses shows that, when any follower operates close to capacity, other followers have a strategic incentive to exploit the fact that this follower is now *less flexible* by reducing their forward positions. However, if all firms were to reduce their forward positions simultaneously, the high prices would create incentives for them to increase their forward positions. Therefore, there is no symmetric equilibria. This insight is related to the observation by [95] that equilibria may not exist. However, the argument in [95] was based on showing that profit functions are not convex and no explicit insights into strategic incentives or the circumstances under which equilibria do not exist were provided. On the other hand, we provide explicit conditions under which symmetric equilibria do not exist, and our analyses reveal the strategic interactions that lead to no symmetric equilibria.

Our third contribution is a complete characterization of symmetric equilibria for a model that is technically challenging to analyze. As observed by [95], capacity constraints may cause profit functions to be non-convex. Hence, standard techniques used to show existence and uniqueness no longer apply. Nevertheless, we provide closed-form expressions of equilibria as a function of the parameters, including the number of leaders, number of followers, their marginal costs, and the capacity of the followers. Our explicit characterizations enable us to infer tradeoffs between the parameters as well as obtain the asymptotic behavior of the system as the numbers of leaders and followers increase. Our characterizations also show that market equilibria are especially interesting – they may not exist or may not be unique – at the transition between interior equilibria and full capacity utilization due to opportunities for market power exploitation. Hence, our work motivates further analysis on capacity constrained games. Moreover, since capacity constraints may be due to lack of flexibility in adjusting production levels or long-run production capacities, both operational and long-term market power analyses are important.

3.2 Related Literature

Our model, being a combination of the classical forward contracting and leader-follower models, has not been studied before. However, our study fits into the extensive literature on forward contracting and leader-follower competition. We review the literature on these and explain how our work contributes to them.

Forward markets

[10] was the first to provide and analyze a model showing that strategic forward contracting mitigates market power. Later studies by [56, 98, 58, 124, 85, 79] reaffirmed or invalidated their findings under other assumptions. As these are not directly relevant to our work, we do not discuss their details here (see [95] for a survey). However, the general conclusion is that the original findings do not always hold when the assumptions change.

The domain of electricity markets has seen the most application of the model from [10]. This may be attributed to the fact that the bulk of trade in electricity are through forward contracts and market power was a significant issue in wholesale electricity markets after their deregulation. However, capacity constraints is an important feature in electricity markets, and this feature was not present in their model. Therefore, there have been numerous extensions in this direction. [73] and [141] added network constraints and price caps. However, due to the complexity of the problem, only numerical solutions were provided. [101, 71] proposed the idea that forward contracts may increase capacity investment. This idea was then investigated analytically by [95] by adding an endogenous capacity investment stage. The authors made the interesting finding that forward contracts may not mitigate market power when capacities are endogenous.

To our knowledge, our work is the first to study the robustness of the findings by [10] in the classical leader-follower setting with capacity-constrained followers. In addition, our work supplements existing results on the impact of capacity constraints on existence of equilibria, by providing explicit characterizations under which symmetric equilibria exists and vice versa.

Leader-follower competition

The first extension of Stackelberg's framework to multiple leaders and followers was provided by [120, 119]. The author also gave conditions for existence and uniqueness of equilibria. Subsequently, there has been significant interest in relaxing the assumptions of the model. However, most studies focus on the technical conditions required for existence and uniqueness, and neglect to study the underlying strategic behavior. [49] showed that equilibrium is no longer unique if one removes Sherali's assumption that identical producers make identical decisions. [44, 139, 48] generalized some of Sherali's existence and uniqueness results to the setting with uncertainty. There are also other efforts by [105, 77] that provide conditions for existence using variational inequality techniques.

We are not aware of any work that add capacity constraints to Sherali's model. The closest related work was by [100] but the authors were investigating price competition (while we focus on quantity competition). [48] might appear to have included capacity limits in their analyses. However, the authors used the capacity limits as a technical condition for their proof, since it was defined by the point where marginal cost exceeds price. Hence, their capacity constraints are never binding, and firms in their model do not strategically withhold productions unlike in our model.

To our knowledge, our work is the first to extend Sherali's model with capacity constraints on followers and allowing them to sell forward contracts. Similar to Sherali's work, we restrict ourselves to symmetric equilibria in the sense that leaders have equal productions and followers have equal forward positions. We characterize all symmetric equilibria and provide insights into strategic behavior. Note that [49] showed that equilibrium is no longer unique if Sherali's symmetry assumptions are relaxed. However, his findings are technically different from ours. His results are attributed to non-smoothness due to the non-negativity constraints on quantities, while our results are attributed to non-smoothness due to the capacity constraints. Hence, symmetric equilibria always exists in [49] but may not exist in our model.

3.3 Model

Our goal is to understand whether forward contracting mitigates market power when firms have capacity constraints and heterogeneous production lead times. To this end, we formulate a model that combines key elements from the classical forward contracting model proposed in [10] as well as the classical leader-follower model proposed in [120].

We assume that there are two types of firms – *leaders* and *followers* – that choose production quantities at two different times. Leaders, who have longer lead times than followers, choose production quantities in the first stage while followers choose production quantities in the second stage. However, followers sell forward contracts in the first stage. Hence, we also refer to the first stage as the *forward market* and the second stage as the *spot market*.

3.3.1 Forward contracting

Our model for forward contracting is based on the classical model from [10]. This model is commonly used in many studies of forward markets [98, 58, 73, 85, 79, 141, 95]. In the forward market, firms sign contracts to deliver a certain quantity of good at a price p_f. These contracts are binding and observable pre-commitments. Then, in the spot market, firms sell the good at a price $P(q)$ which is a function of the total quantity q of the good sold in both the forward and spot markets. We assume a linear demand model given by

$$P(q) = \alpha - \beta q,$$

where the constants $\alpha, \beta > 0$. This is a common model for demand [10, 95] and implies that buyers' aggregate utility is quasilinear in money and quadratic in the quantity of the good consumed.

We assume that there is perfect foresight. That is, in the first stage, both leaders and followers know the demand in the second stage. Equilibrium then requires that the forward and spot prices are aligned:

$$p_f = P(q).$$

That is, no arbitrage is possible. This assumption was also used in both the classical forward contracting model [10] and the classical leader-follower model [120]. An extension to the case of uncertain demand is definitely relevant and interesting. But our results show that the model with certain demand is rich enough to capture interesting strategic interactions between leaders and followers. The case of uncertain demand is left to future work.

3.3.2 Production lead times

Our model for leader-follower competition is based on the classical model from [120]. We assume that there are M leaders and N followers, with marginal costs C and c, respectively, where $c \geq C > 0$. We also abuse notation and use M and N to denote the set of leaders and followers, respectively. The assumption that $c \geq C$ is motivated by the expectation that there is typically a cost to flexibility, e.g. in electricity markets more flexible generators typically have higher operating costs than less flexible generators.

Each leader $i \in M$ chooses its production quantity x_i in the forward market. Each follower $j \in N$ chooses its production quantity y_j in the spot market and also sells a forward contract of quantity f_j in the forward market. We assume that leaders sell forward contracts in the forward market equal to their committed productions. It is possible to show that allowing leaders to sell forward contracts that differ from their committed productions does not change the analyses.

We assume that each follower has a production capacity $k > 0$ but leaders are not capacity constrained. In practice, followers might only be able to adjust productions within a limited range around operating points. Hence, a more sophisticated model would have followers choose set points in the forward market and impose constraints on deviations from those set points. Our model for followers can be interpreted as them having zero set points and being allowed to ramp productions to a maximum of k. Similarly, our model for leaders can be interpreted as them choosing their operating points in the forward market and not being allowed to deviate from them.

3.3.3 Competitive model

We adopt the following equilibria concept for the market. Let the vectors $\mathbf{x} = (x_1, \ldots, x_M)$, $\mathbf{y} = (y_1, \ldots, y_N)$, and $\mathbf{f} = (f_1, \ldots, f_N)$ denote the leaders' productions, followers' productions, and followers' forward contracts, respectively. We also use the notation $\mathbf{f}_{-j} = (f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_N)$ to denote the forward contracts of all followers other than i. Similarly, we use the notations $\mathbf{x}_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_M)$ and $\mathbf{y}_{-j} = (y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_N)$.

Spot market (followers): We define the spot market equilibrium as follows. Only followers compete in the spot market. Follower j's profit from the spot market is:

$$\phi_j^{(s)}(y_j; \mathbf{y}_{-j}) = P\left(\sum_{i'=1}^{M} x_{i'} + \sum_{j'=1}^{N} y_{j'}\right) \cdot (y_j - f_j) - cy_j.$$

Given $\mathbf{y}_{-j}$, follower j chooses a production y_j to maximize its profit subject to its capacity constraint. Hence, a Nash equilibrium of the spot market is a vector $\mathbf{y}$ such that for all j:

$$\phi_j^{(s)}\left(y_j; \mathbf{y}_{-j}\right) \geq \phi_j^{(s)}\left(\bar{y}_j; \mathbf{y}_{-j}\right), \text{ for all } \bar{y}_j \in [0, k].$$

Theorem 5 of [67] implies that there always exists a unique spot equilibrium given any leader productions and follower forward positions $(\mathbf{x}, \mathbf{f})$. We denote this unique equilibrium by $\mathbf{y}(\mathbf{f}, \mathbf{x}) = (y_1(\mathbf{f}, \mathbf{x}), \ldots, y_N(\mathbf{f}, \mathbf{x}))$.

Forward market: The forward market equilibrium depends on behaviors of both followers and leaders. Their profits depend on the outcome of the spot market. In particular, follower j's profit is given by:

$$\begin{aligned}\phi_j\left(f_j; \mathbf{f}_{-j}, \mathbf{x}\right) &= P\left(\sum_{i'=1}^{M} x_{i'} + \sum_{j'=1}^{N} y_{j'}(\mathbf{f}, \mathbf{x})\right) \cdot f_j + \phi_j^{(s)}\left(\mathbf{y}(\mathbf{f}, \mathbf{x})\right) \\ &= \left(P\left(\sum_{i'=1}^{M} x_{i'} + \sum_{j'=1}^{N} y_{j'}(\mathbf{f}, \mathbf{x})\right) - c\right) \cdot y_j(\mathbf{f}, \mathbf{x}),\end{aligned}$$

where the second equality follows by substituting for $\phi_j^{(s)}(\mathbf{f}, \mathbf{x})$. Note that follower j anticipates the impact of the actions in the forward market on the spot market. Given $(\mathbf{f}_{-j}, \mathbf{x})$, follower j chooses its forward contract f_j to maximize its profit. This is an unconstrained maximization as followers can take positive or negative positions in the forward market. Next, leader i's profit is given by:

$$\psi_i\left(x_i; \mathbf{x}_{-i}, \mathbf{f}\right) = \left(P\left(\sum_{i'=1}^{M} x_{i'} + \sum_{j'=1}^{N} y_{j'}(\mathbf{f}, \mathbf{x})\right) - C\right) \cdot x_i.$$

Given $(\mathbf{x}_{-i}, \mathbf{f})$, leader i chooses a production $x_i \in \mathbb{R}_+$ to maximize its profit.

Hence, a subgame perfect Nash equilibrium of the forward market is a tuple $(\mathbf{f}, \mathbf{x})$ such that for all i:

$$\psi_i\left(x_i; \mathbf{x}_{-i}, \mathbf{f}\right) \geq \psi_i\left(\bar{x}_i; \mathbf{x}_{-i}, \mathbf{f}\right), \text{ for all } \bar{x}_i \in \mathbb{R}_+, \tag{3.1}$$

and for all j:

$$\phi_j\left(f_j; \mathbf{f}_{-j}, \mathbf{x}\right) \geq \phi_j\left(\bar{f}_j; \mathbf{f}_{-j}, \mathbf{x}\right), \text{ for all } \bar{f}_j \in \mathbb{R}. \tag{3.2}$$

It is this equilibrium that is the focus of this study. To capture the key strategic interactions between followers and leaders, we focus on equilibria in which leaders have symmetric productions and followers have symmetric forward positions. This symmetric case already offers many insights.

3.4 Summary of Main Insights

Our model differs from previous studies on forward contracting in two important aspects. First, in our model, firms may choose productions (not just forward contracts) in the forward market, which is a feature that has not been investigated before. Second, in our model, firms have capacity constraints, while the majority of forward market studies assume unconstrained productions. The most in-depth analysis on capacity-constrained forward markets, provided by [95], did not give explicit conditions under which equilibria do not exist; while our work provides explicit expressions for equilibria and hence offer stronger insights into the strategic interactions. We provide two macro insights into the strategic interactions of leader-follower competition with forward contracting.

Our first insight is that forward contracting may decrease the efficiency of the market. The intuition is that forward contracting creates opportunities for leaders to exploit the capacity constraints of followers. Conventional intuition on forward contracting suggests that followers will increase their productions with the introduction of forward contracting [10]. However, this may cause followers to operate closer to capacity. Leaders, in order to exploit the fact that followers are now *less flexible*, reduce productions to the extent that the equilibrium total market production decreases. Informally, the increased competition in the forward market due to forward contracting is offset by the decreased competition in the spot market due to followers operating closer to capacity. Hence, the capacity constraints create opportunities for market manipulation. Note that, while [95] also showed that forward contracting may decrease the efficiency of the market, that was a different phenomenon caused by the impact of forward contracting on capacity investment.

Our second insight is that non-existence of symmetric equilibria may be attributed to exploitation of followers' capacity constraints. The intuition is that each firm's individual incentive to exploit the capacity constraints of followers could lead to disequilibrium in the forward market. When any follower operates close to capacity, other firms have incentive to exploit the fact that the follower is now *less flexible* by withholding offers from the forward market. However, if all firms were to reduce their offers, the high prices would create incentives for them to increase their offers. Hence, there is no equilibrium. Note that, while [95] also observed that equilibria may not exist when there are capacity constraints, their argument was based on showing that profit functions are not convex and no explicit conditions or strategic insights were provided. In contrast, we give explicit conditions under which symmetric equilibria do not exist and hence are able to attribute non-existence of symmetric equilibria to strategic withholding.

3.5 One Leader and Two Followers

To develop the intuition for our results, we start by considering only $M = 1$ leader, $N = 2$ followers, and equal marginal costs $C = c$. Throughout this section, we denote by $\bar{\alpha}$ the normalized demand:

$$\bar{\alpha} := \frac{1}{\beta}(\alpha - C).$$

Recall that α is the maximum price that demand is willing to pay and C is the minimum price that producers need to receive for them to supply to the market. Hence, we restrict our analyses to the case where $\bar{\alpha} \geq 0$.

First, in Sections 3.5.1 and 3.5.2, we study the reactions of the followers to the leader and vice versa. In particular, we focus on the impact of followers' capacity constraints and leader's commitment power on their responses to the other producers' actions. Then, in Section 3.5.3, we study how they impact the equilibria of the market. Finally, in Sections 3.5.4, we study how followers' forward contracting impact market outcomes.

3.5.1 Follower reaction

We begin by studying how followers respond when the leader produces a fixed quantity $x \in \mathbb{R}_+$. We focus on symmetric responses, that is, those where followers take equal forward positions. Let $F : \mathbb{R}_+ \to \mathbb{P}(\mathbb{R})$ denote the symmetric reaction correspondence of the followers, i.e., for each $f \in F(x)$,

$$\begin{aligned} &\phi_1(f; f, x) \geq \phi_1(\bar{f}; f, x), \quad \forall \bar{f} \in \mathbb{R}; \\ \text{and} \quad &\phi_2(f; f, x) \geq \phi_2(\bar{f}; f, x), \quad \forall \bar{f} \in \mathbb{R}. \end{aligned}$$

Proposition 3.2 in the Appendix implies that the followers produce equal quantities $y_1(f; f, x) = y_2(f; f, x)$. Let $Y : \mathbb{R}_+ \to \mathbb{P}(\mathbb{R}_+)$ denote the production correspondence of the followers, i.e., for each $y \in Y(x)$, there exists $f \in F(x)$ such that $y_1(f; f, x) = y_2(f; f, x) = y$. Applying Propositions 3.2 and 3.3 in the Appendix, the reaction and production correspondences are given by:

$$\begin{aligned}
&F(x) = [-\bar{\alpha} + x + 3k, \infty), && Y(x) = \{k\}, && \text{if } x \leq \bar{\alpha} - 3k, \\
&F(x) = \varnothing, && Y(x) = \varnothing, && \text{if } \bar{\alpha} - 3k < x < \bar{\alpha} - \tfrac{5}{5-2\sqrt{2}}k, \\
&F(x) = \left\{\tfrac{1}{5}(\bar{\alpha} - x)\right\}, && Y(x) = \left\{\tfrac{2}{5}(\bar{\alpha} - x)\right\}, && \text{if } \bar{\alpha} - \tfrac{5}{5-2\sqrt{2}}k \leq x \leq \bar{\alpha}, \\
&F(x) = (-\infty, -\bar{\alpha} + x], && Y(x) = \{0\}, && \text{if } \bar{\alpha} \leq x.
\end{aligned}$$

Figure 3.1 shows the characteristic shapes of F and Y. There are four major segments labelled (i) – (iv). Note that the follower productions are always k in segment (i) and 0 in segment (iv). In general, one expects followers' reactions to decrease as x increases because a higher leader production

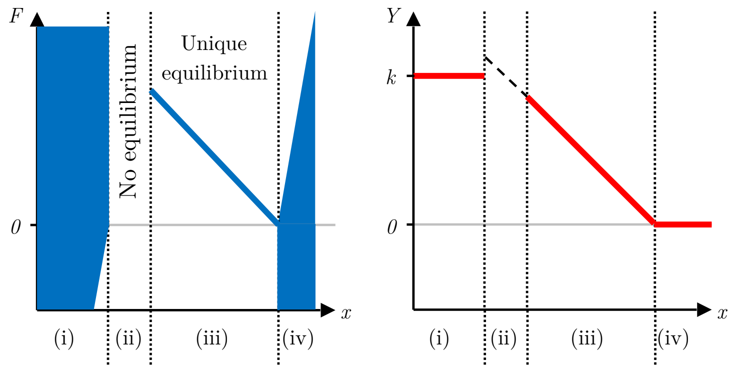


Figure 3.1: Follower reaction correspondence F and production correspondence Y.

decreases the demand in the spot market. This behavior indeed holds in segment (ii), which is also the behavior in a conventional forward market in the absence of capacity constraints. However, the capacity constraints lead to complex reactions, as seen in segments (i), (iii), and (iv).

Segments (i) and (iv): $x \leq \bar{\alpha} - 3k$ or $\bar{\alpha} \leq x$. Multiple equilibria. These are degenerate scenarios where followers have binding productions, and hence are neutral to a range of different forward positions, as they all lead to the same production outcomes. The structure of the reaction set is also intuitive. Consider segment (i), where followers produce zero quantities. If f' is a symmetric reaction, then any $f'' < f'$ is also a symmetric reaction, since decreasing forward positions create incentives to decrease productions, and productions cannot drop below zero. Hence, the reaction sets are left half-lines. A similar argument applies to segment (iv), but in this case, the reaction sets are right half-lines.

Segment (iii): $\bar{\alpha} - \frac{5}{5-2\sqrt{2}}k \leq x \leq \bar{\alpha}$. No equilibrium. This is the scenario where followers' capacity constraints create incentives for market manipulation which causes symmetric reactions to disappear. The type of symmetric reactions in segment (ii) are unsustainable here because each follower has incentive to reduce its forward position. For example, when follower 1 reduces its forward position, it induces follower 2 to increase its production. However, since follower 2 can only increase its production up to k, the total production decreases, the market price increases, and follower 1's profit increases. By symmetry, follower 2 has incentive to manipulate the market in a similar manner. Yet, should both followers reduce their forward positions, there will be excess demand in the market. Hence, there is no symmetric equilibrium between the followers.

3.5.2 Leader reaction

Next, we study how the leader responds when both followers take a fixed forward position $f \in \mathbb{R}$. Let $X : \mathbb{R} \to \mathbb{P}(\mathbb{R}_+)$ denote the leader's reaction correspondence, i.e., for each $x \in X(f)$,

$$\psi_1 (x; f, f) \geq \psi_1 (\bar{x}; f, f), \quad \forall \bar{x} \in \mathbb{R}_+.$$

Let $Y : \mathbb{R} \to \mathbb{P}(\mathbb{R}_+)$ denote the production correspondence of the followers, i.e., for each $y \in Y(f)$, there exists $x \in X(f)$ such that $y_1(f; f, x) = y_2(f; f, x) = y$. The expressions for X and Y can be obtained from Propositions 3.4 and 3.2 in the Appendix. X and Y takes three distinctive shapes depending on the value of $\bar{\alpha}$.

Low demand: $0 \leq \bar{\alpha} \leq 2k$. In this case, the reaction and production correspondences are given by

$$\begin{array}{lll}
X(f) = \{\frac{1}{2}\bar{\alpha}\}, & Y(f) = \{0\}, & \text{if } f \leq -\frac{1}{2}\bar{\alpha}, \\
X(f) = \{\bar{\alpha} + f\}, & Y(f) = \{0\}, & \text{if } -\frac{1}{2}\bar{\alpha} \leq f \leq -\frac{1}{4}\bar{\alpha}, \\
X(f) = \{\frac{1}{2}\bar{\alpha} - f\}, & Y(f) = \{\frac{1}{6}\bar{\alpha} + \frac{2}{3}f\}, & \text{if } -\frac{1}{4}\bar{\alpha} \leq f \leq \frac{1}{2}\bar{\alpha}, \\
X(f) = \{0\}, & Y(f) = \{\frac{1}{3}(\bar{\alpha} + f), & \text{if } -\frac{1}{2}\bar{\alpha} \leq f \leq 3k - \bar{\alpha}, \\
X(f) = \{0\}, & Y(f) = \{k\}, & \text{if } 3k - \bar{\alpha} \leq f.
\end{array}$$

Figure 3.2a shows the characteristic shapes of X and Y. There are four major segments labelled (i) – (iv). The follower supplies 0 in segments (i) and (ii) and supplies k for a subset of segment (iv). In general, one expects the leader's production to decrease as f increases, because larger forward positions lead to larger follower supplies, which decreases the market price. This behavior indeed holds in segment (iii). However, the capacity constraints and leader's commitment power lead to complex reactions in segments (i), (ii), and (iv).

Segment (i) and (iv): $f \leq -\frac{1}{2}\bar{\alpha}$ *or* $3k - \bar{\alpha} \leq f$. *Constant production.* These are degenerate scenarios where the leader is insensitive to the followers' forward positions. When $f \leq -\frac{1}{2}\bar{\alpha}$, it is because followers' always supply zero regardless of their forward positions. When $3k - \bar{\alpha} \leq f$, it is because followers supply large quantities, and drive prices down below the level at which it is profitable for leaders to produce.

Segment (ii): $-\frac{1}{2}\bar{\alpha} \leq f \leq -\frac{1}{4}\bar{\alpha}$. *Increasing production.* In this scenario, the leader uses its commitment power to drive the followers out of the market. As followers increase their forward positions, the leader, instead of decreasing its production (as one would typically expect), actually increases its production, as doing so allows it to depress demand below the level at which followers are willing to supply.

Medium demand: $2k < \bar{\alpha} < \frac{4}{2-\sqrt{3}}k$. In this scenario, the reaction and production correspondences are given by

$$
\begin{array}{lll}
X(f) = \{\frac{1}{2}\bar{\alpha}\}, & Y(f) = \{0\}, & \text{if } f \leq -\frac{1}{2}\bar{\alpha}, \\
X(f) = \{\bar{\alpha} + f\}, & Y(f) = \{0\}, & \text{if } -\frac{1}{2}\bar{\alpha} \leq f \leq -\frac{1}{4}\bar{\alpha}, \\
X(f) = \{-\frac{1}{2}\bar{\alpha}\}, & Y(f) = \{\frac{1}{6}\bar{\alpha} + \frac{2}{3}f\}, & \text{if } -\frac{1}{4}\bar{\alpha} \leq f \leq -\frac{\sqrt{3}-1}{2}\bar{\alpha} + \sqrt{3}k, \\
X(f) = \{\frac{1}{2}\bar{\alpha} - f, \frac{1}{2}\bar{\alpha} - k\}, & Y(f) = \{\frac{1}{6}\bar{\alpha} + \frac{2}{3}f, k\}, & \text{if } f = -\frac{\sqrt{3}-1}{2}\bar{\alpha} + \sqrt{3}k, \\
X(f) = \{\frac{1}{2}\bar{\alpha} - k\}, & Y(f) = \{k\}, & \text{if } -\frac{\sqrt{3}-1}{2}\bar{\alpha} + \sqrt{3}k < f.
\end{array}
$$

Figure 3.2b shows the characteristic shapes of X and Y. There are, again, four major segments labelled (i) – (iv). Segments (i), (ii), and (iii), are similar to that in the low demand case when $0 \leq \bar{\alpha} \leq 2k$. Segment (iv), however, is different in that, while the leader produces zero in this segment when $0 \leq \bar{\alpha} \leq 2k$, the leader now produces a strictly positive quantity in this segment. This is due to the fact that the leader's profit on each unit is given by

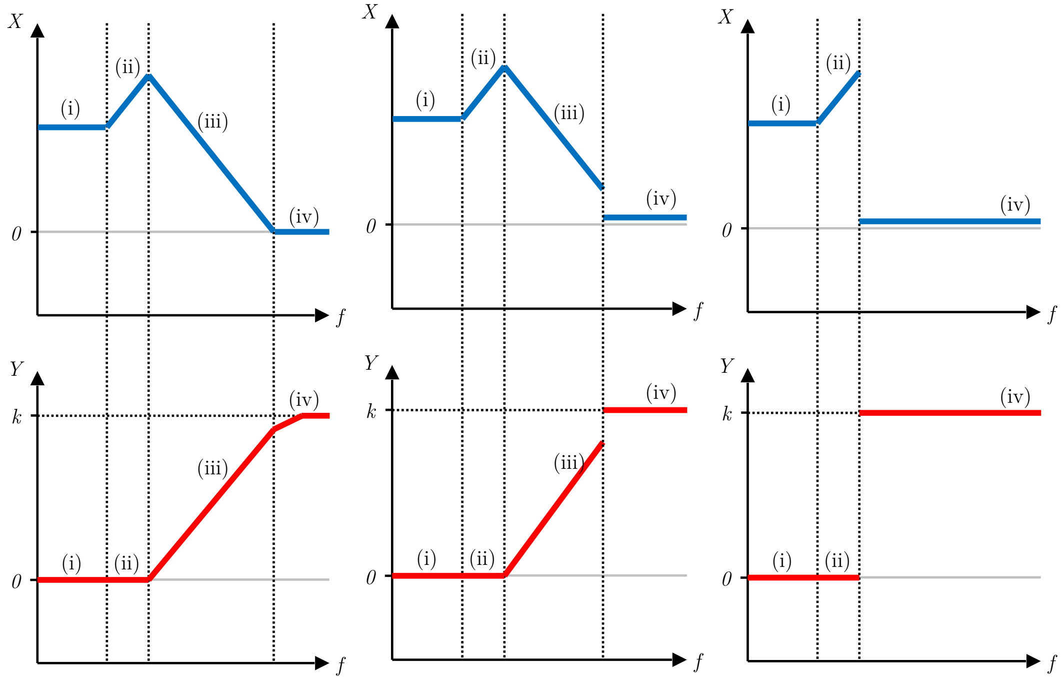
$$\alpha - \beta(y_1 + y_2 + x) - C = \beta(\bar{\alpha} - y_1 - y_2 - x),$$

and hence, when $\bar{\alpha} > 2k$, the leader is still able to profit from producing when both followers produce k. Due to this, the leader also has an incentive to exploit followers' capacity constraints, unlike previously when the leader was producing zero. The leader does so by sharply reducing its production at the end of segment (iii). This induces the followers to increase their supply, but since followers can only increase their supply up to k, the total market production decreases, the market price increases, and the leader's profit increases. Hence, there is a discontinuity in the leader's reaction curve between segments (iii) and (iv).

High demand: $\frac{4}{2-\sqrt{3}}k \leq \bar{\alpha}$. In this scenario, the reaction and production correspondences are given by

$$
\begin{array}{llll}
 & X(f) = \{\frac{1}{2}\bar{\alpha}\}, & Y(f) = \{0\}, & \text{if } f \leq -\frac{1}{2}\bar{\alpha}, \\
\text{or} & X(f) = \{\bar{\alpha} + f\}, & Y(f) = \{0\}, & \text{if } -\frac{1}{2}\bar{\alpha} \leq f < -\frac{1}{2}\bar{\alpha} + \sqrt{(\bar{\alpha} - k)k}, \\
\text{or} & X(f) = \{\bar{\alpha} + f, \frac{1}{2}\bar{\alpha} - k\}, & Y(f) = \{0, k\}, & \text{if } f = -\frac{1}{2}\bar{\alpha} + \sqrt{(\bar{\alpha} - k)k}, \\
\text{or} & X(f) = \{\frac{1}{2}\bar{\alpha} - k\}, & Y(f) = \{k\}, & \text{if } -\frac{1}{2}\bar{\alpha} + \sqrt{(\bar{\alpha} - k)k} < f.
\end{array}
$$

Figure 3.2c shows the characteristic shapes of X and Y. There are now only three segments labelled (i), (ii), and (iv). These segments are similar to segments (i), (ii), and (iv), respectively, in the medium demand case where $2k < \bar{\alpha} < \frac{4}{2-\sqrt{3}}k$. The difference is that, now, the leader decreases its production sharply once followers begin to supply to the market. As a consequence, the followers' supply jumps from 0 to k.



(a) $0 \leq \bar{\alpha} \leq 2k$ (b) $2k < \bar{\alpha} < \frac{4}{2-\sqrt{3}}k$ (c) $\frac{4}{2-\sqrt{3}}k \leq \bar{\alpha}$

Figure 3.2: Leader reaction correspondence X and follower production correspondence Y.

3.5.3 Forward market equilibrium

We now study the equilibria of the forward market. Let $Q \subseteq \mathbb{R} \times \mathbb{R}_+$ denote the set of all symmetric equilibria and $Y \subseteq \mathbb{R}_+$ denote the set of all follower productions, i.e., $(f, x) \in Q$ if (f, f, x) is a Nash equilibrium of the forward market, and $y \in Y$ if there exists $(f, x) \in Q$ such that $y_1(f; f, x) = y_2(f; f, x) = y$. From Proposition 3.5 and 3.2, the symmetric equilibria and follower productions are given by

$$
\begin{aligned}
&Q = \{(f, x) : \ f = \tfrac{1}{8}\bar{\alpha}, \ x = \tfrac{3}{8}\bar{\alpha}\}, && Y = \{\tfrac{1}{4}\bar{\alpha}\}, && \text{if } 0 \leq \bar{\alpha} \leq \tfrac{8}{4-\sqrt{3}}k, \\
&Q = \varnothing, && Y = \varnothing, && \text{if } \tfrac{8}{4-\sqrt{3}}k < \bar{\alpha} < 4k, \\
&Q = \{(f, x) : \ f \in [-\tfrac{1}{2}\bar{\alpha} + 2k, \infty), \ x = \tfrac{1}{2}\bar{\alpha} - k\}, && Y = \{k\}, && \text{if } 4k \leq \bar{\alpha}.
\end{aligned}
$$

Observe that there are three operating regimes.

Low demand: $0 \leq \bar{\alpha} \leq \frac{8}{4-\sqrt{3}}k$. There is a one symmetric equilibrium. Productions increase as $\bar{\alpha}$ increases. This regime is identical to that in the absence of capacity constraints (to see this, substitute $k = \infty$).

Medium demand: $\frac{8}{4-\sqrt{3}}k < \bar{\alpha} < 4k$. There is no symmetric equilibrium. This phenomena is due to leaders and followers withholding productions and forward contracts respectively. As observed in the separate reaction curves, each individual follower or leader has incentive to exploit the capacity constraints of the followers by reducing its position in the forward market. But should all producers do so, there will be excess demand in the market, and hence no symmetric equilibria are sustainable.

High demand: $4k \leq \bar{\alpha}$. There is a unique equilibrium leader production $\frac{1}{2}\bar{\alpha} - k$ and infinitely many equilibrium follower forward positions $[-\frac{1}{2}\bar{\alpha} + 2k, \infty)$. The latter is a right half-line because followers are supplying all their capacity and so are indifferent once forward positions exceed a certain value. There leader production increases with demand; although the rate of increase of $\frac{1}{2}$ is slower than in the case when demand is low, where it increased at the rate $\frac{3}{8}$. This distinction is due to the leader facing less competition than before when followers were not capacity-constrained.

Note that, unlike with the leader's reaction curve, there is no apparent phenomenon where leader's increase their productions to drive followers out of the market. This can be attributed to the fact that the leader and followers have equal marginal costs. In Section 3.6, we will see that the leader's commitment power does cause its equilibrium production to increase with demand, when we relax the assumption of equal marginal costs.

3.5.4 Inefficiency of the forward market

To study the efficiency of the forward market, we compare the outcome in our market against that in a Stackelberg competition, where followers do not sell forward contracts. Hence, the leader continues to commit to its production ahead of the followers.

Note that the symmetric Stackelberg equilibria are simply the symmetric reactions of the leader when followers take neutral forward positions. Hence, using the notation in Section 3.5.2, we let $X(0)$ denote the set of all symmetric Stackelberg equilibria, i.e., for each $x \in X(0)$,

$$\psi_1(x; 0, 0) \geq \psi_1(\bar{x}; 0, 0), \quad \forall \bar{x} \in \mathbb{R}_+,$$

and we let $Y \subseteq \mathbb{R}_+$ denote the set of all follower productions, i.e., $y \in Y$ if there exists $x \in X(0)$ such that $y_1(0; 0, x) = y_2(0; 0, x) = y$. From Proposition 3.6, the Stackelberg equilibria are given by

$$\begin{array}{lll} X(0) = \left\{\frac{1}{2}\bar{\alpha}\right\}, & Y = \left\{\frac{1}{6}\bar{\alpha}\right\}, & \text{if } 0 \leq \bar{\alpha} < \frac{2\sqrt{3}}{\sqrt{3}-1}k, \\ X(0) = \left\{\frac{1}{2}\bar{\alpha}, \frac{1}{2}\bar{\alpha} - k\right\}, & Y = \left\{\frac{1}{3-\sqrt{3}}k, k\right\}, & \text{if } \bar{\alpha} = \frac{2\sqrt{3}}{\sqrt{3}-1}k, \\ X(0) = \left\{\frac{1}{2}\bar{\alpha} - k\right\}, & Y = \{k\}, & \text{if } \frac{2\sqrt{3}}{\sqrt{3}-1}k < \bar{\alpha}. \end{array}$$

Observe that there are two operating regimes. The regime $0 \leq \bar{\alpha} < \frac{2\sqrt{3}}{\sqrt{3}-1}k$ is the regime of low demand. In this regime, the market has a unique equilibrium and both leader and follower productions increase with demand. The regime $\frac{2\sqrt{3}}{\sqrt{3}-1}k < \bar{\alpha}$ is the regime of high demand. In this regime, followers produce all their capacity. The leader produces $\frac{1}{2}\bar{\alpha} - k$, which is less than its production $\frac{1}{2}\bar{\alpha}$ in the low demand regime, because it faces less competition now since followers have no capacity left to supply.

By comparing the Stackelberg equilibria to the forward market equilibria, we can see that introducing a forward market does not always increase the total market production. In particular, when $4k \leq \bar{\alpha} < \frac{2\sqrt{3}}{\sqrt{3}-1}k$, the total production $\frac{1}{2}\bar{\alpha} + k$ with the forward market is less than the total production $\frac{5}{6}\bar{\alpha}$ in the Stackelberg market. In this scenario, demand is high and followers produce almost all their capacity in the Stackelberg market. Having followers trade forward contracts give them more incentive to produce and they increase their productions to k. However, this has the side effect of reducing the competition faced by the leader, and giving it an incentive to withhold its production. The net effect is a decrease in total market production. Since all producers have equal marginal costs, a decrease in total market production implies a decrease in social welfare.

3.6 Structural Insights

We now extend the insights obtained from studying the case of 1 leader, 2 followers, and equal marginal costs, to general numbers of leaders and followers with marginal costs C and c that are possibly different. In addition, we characterize the asymptotic behavior as the number of producers increase. Unless otherwise stated, the proofs for all the results in this section are provided in Appendix 3.E.

Throughout this section, we denote by α_x and α_y the normalized leader and follower demands respectively and by $\triangle C$ the normalized marginal cost difference between the leaders and followers:

$$\alpha_x = \frac{1}{\beta}(\alpha - C),$$
$$\alpha_y = \frac{1}{\beta}(\alpha - c),$$
$$\triangle C = \frac{1}{\beta}(c - C).$$

Note that $\alpha_y = \alpha_x - \triangle C$. Since $c \geq C$, it suffices to restrict our analyses to the case where $\alpha_x \geq 0$. We focus on symmetric equilibria, by which we mean equilibria where leaders have symmetric productions and followers have symmetric forward positions (which, by Proposition 3.2, implies that the latter have symmetric productions).

3.6.1 Follower reaction

Suppose all leaders produce a quantity $x \in \mathbb{R}_+$ and let $F(x) \subseteq \mathbb{R}$ denote the set of all symmetric follower reactions, i.e., for each $f \in F(x)$ and $j \in N$,

$$\phi_j(f; f\mathbf{1}, x\mathbf{1}) \geq \phi_j(\bar{f}; f\mathbf{1}, x\mathbf{1}), \quad \forall \bar{f} \in \mathbb{R}.$$

Proposition 3.3 gives the solution for $F(x)$. Observe that $F(x)$ has a similar shape to the graph in Figure 3.1. We focus on the segment where $F(x) = \varnothing$ and highlight key properties that attribute this segment to market manipulation when followers are operating just below capacity.

Lemma 3.1. *The following holds.*

1. *There exists a unique $\bar{y} < k$, such that there exists $f \in F(x)$ such that $y_j(f\mathbf{1}, x\mathbf{1}) = y$ if and only if $0 \leq y \leq \bar{y}$ or $y = k$. Moreover,*

$$\bar{y} = \left(1 - O\left(\frac{1}{N}\right)\right) k.$$

2. *There exists a unique $\underline{\xi} \in \mathbb{R}_+$, such that $x \leq \frac{1}{M}(\alpha_y - \underline{\xi})$ if and only if there exists $f \in F(x)$ such that $y_j(f\mathbf{1}, x\mathbf{1}) = k$, and a unique $\bar{\xi} < \underline{\xi}$, such that $x \geq \frac{1}{M}(\alpha_y - \bar{\xi})$ if and only if there exists $f \in F(x)$ such that $y_j(f\mathbf{1}, x\mathbf{1}) \leq \bar{y}$. Moreover, $F(x) = \varnothing$ for all $x \in \left(\frac{1}{M}\left(\alpha_y - \underline{\xi}\right), \frac{1}{M}\left(\alpha_y - \bar{\xi}\right)\right)$ and $\underline{\xi} - \bar{\xi} = \underline{\xi} \cdot O\left(\frac{1}{N}\right)$.*

Hence, there exists an open interval of symmetric leader productions inside which there is no symmetric follower reaction. Due to this interval, there is a set of symmetric follower productions just below k that are never equilibria. As N increases, this set shrinks at the rate $\frac{1}{N}$. In the limit, all symmetric follower productions could be equilibria. These asymptotic behavior are consistent with the intuition that followers have less ability to manipulate the market as their numbers increase.

3.6.2 Leader reaction

Suppose all followers take a forward position $f \in \mathbb{R}$ and let $X(f) \subseteq \mathbb{R}_+$ denote the set of all symmetric leader reactions, i.e., for each $x \in X(f)$ and $i \in M$,

$$\psi_i(x; x\mathbf{1}, f\mathbf{1}) \geq \psi_i(\bar{x}; x\mathbf{1}, f\mathbf{1}), \quad \forall \bar{x} \in \mathbb{R}_+.$$

Proposition 3.4 gives the solution for $X(f)$. When $\alpha_x \leq Nk$, we have $\eta_3 \leq k - (\alpha_x - Nk)$, and one can check that $X(f)$ has a similar shape to the graph in Figure 3.2a. When $\alpha_x > Nk$, then $X(f)$ differs from the graphs in Figures 3.2b and 3.2c in that segment (iv) may overlap with segments (iii) and (ii), i.e., there might be up to two reactions. Here, we focus on segment (ii) where the leader reaction is strictly increasing, as well as the discontinuous transition between segment (iv) and segments (ii) or (iii). The following result highlights key properties of segment (ii).

Lemma 3.2. *There exists unique $\underline{f}, \bar{f} \in \mathbb{R}$, such that $f \in \left[\underline{f}, \bar{f}\right]$ if and only if $\frac{1}{M}\left(\alpha_x - \triangle C + f\right) \in X(f)$. Moreover, the following holds:*

1. $y_j\left(f\mathbf{1}, \frac{1}{M}\left(\alpha_x - \triangle C + f\right)\mathbf{1}\right) = 0$ *for all* $f \in \left[\underline{f}, \bar{f}\right]$.

2. $\bar{f} - \underline{f} = O\left(\frac{\alpha_x}{M}\right)$.

Hence, there exists a closed interval $\left[\underline{f}, \bar{f}\right]$ of symmetric follower productions inside which there is a graph of strictly increasing leader reactions. Moreover, the followers' productions are zero. This is due to leaders using their commitment power to drive the followers out of the market. As M increases, this interval shrinks at the rate $\frac{\alpha_x}{M}$.

The next result highlights key properties of the transition between segment (iii) and (iv).

Lemma 3.3. *Suppose $\alpha_x > Nk$.*

1. *There exists a unique $\bar{y} < k$, such that there exists $f \in \mathbb{R}$ and $x \in X(f)$ such that $y_j(f\mathbf{1}, x\mathbf{1}) = y$ if and only if $0 \leq y \leq \bar{y}$ or $y = k$. Moreover,*

$$\bar{y} = \begin{cases} \left(1 - O\left(\frac{\alpha_x - Nk}{MN}\right)\right)k, & \text{if } \alpha_x \leq Nk\left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2} + \frac{(M-1)\sqrt{N+1}}{\sqrt{N+1}-1}\right), \\ 0, & \text{otherwise}. \end{cases}$$

2. *There exists a unique $\bar{f} \in \mathbb{R}$, such that $f \leq \bar{f}$ if and only if there exists $x \in X(f)$ such that $y_j(f\mathbf{1}, x\mathbf{1}) \leq \bar{y}$, and a unique $\underline{f} \leq \bar{f}$, such that $f \geq \underline{f}$ if and only if there exists $x \in X(f)$ such that $y_j(f\mathbf{1}, x\mathbf{1}) = k$. Moreover, $|X(f)| = 2$ for all $f \in \left[\underline{f}, \bar{f}\right]$. Furthermore, if $\alpha_x \leq Nk\left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right)$, then*

$$\bar{f} - \underline{f} = O\left(\frac{\alpha_x - Nk}{M\sqrt{N}}\right).$$

Hence, there exists an open interval of follower productions $(\bar{y}, k)$ that are never supported by any leader reaction. This interval is due to leaders manipulating the market when followers are operating just below capacity. As M, N, α_x increases, This interval shrinks at the rate $\frac{\alpha_x - Nk}{MN}$. In the limit, all follower productions can be sustained. Moreover, there is also an interval of follower forward positions $\left[\underline{f}, \bar{f}\right]$ inside which there are two leader reactions that have different follower productions (one equal to k and one less than $\bar{y}$). This interval shrinks at the rate $\frac{\alpha_x - Nk}{M\sqrt{N}}$.

Note that, since the follower production is continuous in f and x, the second claim in Lemma 3.2 implies that the leader reaction is discontinuous. This was also observed in the case of one leader and two followers. Also, note that followers' capacity constraints have different impacts on the reactions of the followers and that of the leaders. In the case of followers, it led to non-existence of symmetric reactions. In the case of leaders, there always exists a symmetric reaction but there is a discontinuity in the reaction correspondence.

3.6.3 Forward market equilibrium

We now present structural results for the symmetric equilibria of the forward market. Let $Q \subseteq \mathbb{R} \times \mathbb{R}_+$ denote the set of all symmetric equilibria, i.e., for each $(f, x) \in Q$, $(f\mathbf{1}, x\mathbf{1})$ is a Nash equilibrium. Proposition 3.5 gives the solution for Q.

First, we focus on the case where $\triangle C = 0$. The structure of the equilibria is almost identical to that in Section 3.5.3; the key difference is that, while there is either one or no equilibria in Section 3.5.3, there could be up to two equilibria now. This is highlighted in the following result.

Lemma 3.4. *Suppose $\triangle C = 0$.*

1. *There exists a unique $\bar{y} < k$, such that there exists $(f, x) \in Q$ such that $y_j(f\mathbf{1}, x\mathbf{1}) = y$ if and only if $0 \leq y \leq \bar{y}$ or $y = k$. Moreover,*

$$\bar{y} = \left(1 - O\left(\frac{1}{N}\right)\right) k.$$

2. *There exists a unique $\bar{\alpha}_x \in \mathbb{R}_+$, such that $\alpha_x \leq \bar{\alpha}_x$ if and only if there exists $(f, x) \in Q$ such that $y_j(f\mathbf{1}, x\mathbf{1}) \leq \bar{y}$, and a unique $\underline{\alpha}_x \in \mathbb{R}_+$, such that $\alpha_x \geq \underline{\alpha}_x$ if and only if there exists $(f, x) \in Q$ such that $y_j(f\mathbf{1}, x\mathbf{1}) = k$. Moreover, if*

$$M < N\sqrt{N+1} - 1,$$

then $\bar{\alpha}_x < \underline{\alpha}_x$, $Q = \varnothing$ for all $\alpha_x \in (\bar{\alpha}_x, \underline{\alpha}_x)$, and $\underline{\alpha}_x - \bar{\alpha}_x = \underline{\alpha}_x \cdot O\left(\frac{1}{N}\right)$. Otherwise, then $\bar{\alpha}_x \geq \underline{\alpha}_x$, $|Q| = 2$ for all $\alpha_x \in [\underline{\alpha}_x, \bar{\alpha}_x]$, and $\bar{\alpha}_x - \underline{\alpha}_x = \underline{\alpha}_x \cdot O\left(\frac{1}{N\sqrt{N}}\right)$.

Hence, there exists an open interval of follower productions $(\bar{y}, k)$ that are never symmetric equilibria. As N increases, this interval shrinks to the empty set at the rate $\frac{1}{N}$. The latter is independent of the number of leaders M or demand α_x. However, M has an impact on whether there might be no symmetric equilibria or multiple symmetric equilibria. In particular, when $M < N\sqrt{N+1} - 1$, there are no symmetric equilibria when $\underline{\alpha}_x < \alpha_x < \bar{\alpha}_x$. Otherwise, when $M \geq N\sqrt{N+1} - 1$, there are two symmetric equilibria when $\bar{\alpha}_x \leq \alpha_x \leq \underline{\alpha}_x$.

Next, we consider the case where $\triangle C > 0$. In this case, the structure of the equilibria has an additional feature that was not present when $\triangle C = 0$. In particular, when demand is low, followers might not supply to the market. The next lemma highlights the structure of the transition to strictly positive follower productions.

Lemma 3.5. *Suppose $\triangle C > 0$. Let $\zeta_1 = (M+1)\triangle C + \min\left(MN\triangle C, MNk + 2M\sqrt{Nk\triangle C}\right)$. Then there exists $(f, x) \in Q$, such that $y_j(f\mathbf{1}, x\mathbf{1}) = 0$ if and only if $\alpha_x \leq \zeta_1$. Furthermore, if*

$\alpha_x > (M+1)\triangle C$, then $y_j(f\mathbf{1}, x\mathbf{1}) = 0$ if and only if

$$(f,x) \in \left\{ (f,x) \in \mathbb{R} \times \mathbb{R}_+ \;\middle|\; x = \frac{1}{M}\left(\alpha_x - (\triangle C - f)\right) \text{ and } 0 \leq f \leq \bar{f} \right\} \subseteq Q,$$

where $\bar{f} > 0$ if $\alpha_x < \zeta_1$.

The proof is omitted as it is a straightforward observation from Proposition 3.5. As the market transitions from zero to strictly positive follower productions, there is a regime of demand where there are multiple equilibria, characterized by leaders increasing supply when followers take larger forward positions. This phenomenon is due to leaders using their commitment power to drive followers out of the market (recall Lemma 3.2). Hence, although followers are not supplying to the market, their forward positions have an impact on the efficiency of the equilibrium. Moreover, note that the size of the interval of demand values where this phenomenon occurs is $\min\left(MN\triangle C, MNk + 2M\sqrt{Nk\triangle C}\right) = \Theta(MN)$.

When demand is high, the structure of the equilibria is similar to that when $\triangle C = 0$, in that there could be two or zero equilibria, except that the threshold for M now depends on $\triangle C$. Furthermore, even in the limit as N tends to infinity, certain follower productions are never equilibria..

Lemma 3.6. *Suppose $\triangle C > 0$.*

1. *There exists a unique $\bar{y} < k$, such that there exists $(f,x) \in Q$ such that $y_j(f\mathbf{1}, x\mathbf{1}) = y$ if and only if $y \leq \bar{y}$ or $y = k$. Moreover,*

$$\bar{y} = \begin{cases} \left(1 - O\left(\frac{1}{N}\right)\right)\left(k - \frac{(\sqrt{N+1}-1)^2}{N}\triangle C\right), & \text{if } \triangle C < \frac{Nk}{(\sqrt{N+1}-1)^2}, \\ 0, & \text{otherwise.} \end{cases}$$

2. *There exists a unique $\bar{\alpha}_x \in \mathbb{R}_+$, such that $\alpha_x \leq \bar{\alpha}_x$ if and only if there exists $(f,x) \in Q$ such that $y_j(f\mathbf{1}, x\mathbf{1}) \leq \bar{y}$, and a unique $\underline{\alpha}_x \in \mathbb{R}_+$, such that $\alpha_x \geq \underline{\alpha}_x$ if and only if there exists $(f,x) \in Q$ such that $y_j(f\mathbf{1}, x\mathbf{1}) = k$. Moreover, if*

$$M < \begin{cases} \dfrac{(N+1)k - \frac{N^2+1}{N^2+(\sqrt{N+1}-1)^2}\left(Nk - \left(\sqrt{N+1}-1\right)^2\triangle C\right)}{N\triangle C - k + \frac{N+1}{N^2+(\sqrt{N+1}-1)^2}\left(Nk - \left(\sqrt{N+1}-1\right)^2\triangle C\right)}, & \text{if } \triangle C < \frac{Nk}{(\sqrt{N+1}-1)^2}, \\ \dfrac{(N+1)k - \triangle C}{Nk + \triangle C - k + 2\sqrt{Nk\triangle C}}, & \text{otherwise,} \end{cases} \tag{3.3}$$

then $\bar{\alpha}_x < \underline{\alpha}_x$ and $Q = \varnothing$ for all $\alpha_x \in (\bar{\alpha}_x, \underline{\alpha}_x)$. Otherwise, $\bar{\alpha}_x \geq \underline{\alpha}_x$ and $|Q| = 2$ for all $\alpha_x \in [\underline{\alpha}_x, \bar{\alpha}_x]$.

3.6.4 Inefficiency of the forward market

We compare the outcome against a Stackelberg competition where followers do not sell forward contracts. Note that the symmetric equilibria of a Stackelberg competition are given by the symmetric reactions of the leaders when followers take neutral forward positions, i.e., $X(0)$, where X is defined in Section 3.6.2. Proposition 3.6 gives the solution for $X(0)$. The structure is similar to the equilibria of the forward market. We highlight the key features in the following three lemmas.

Lemma 3.7. *Suppose $\triangle C = 0$.*

1. *There exists a unique $\bar{y} < k$, such that there exists $x \in X(0)$ such that $y_j(\mathbf{0}, x\mathbf{1}) = y$ if and only if $y \leq \bar{y}$ or $y = k$. Moreover,*

$$\bar{y} = \left(1 + O\left(\frac{1}{\sqrt{N}}\right)\right)\frac{k}{2}.$$

2. *There exists a unique $\bar{\alpha}_x \in \mathbb{R}_+$, such that $\alpha_x \leq \bar{\alpha}_x$ if and only if there exists $x \in X(0)$ such that $y_j(\mathbf{0}, x\mathbf{1}) \leq \bar{y}$, and a unique $\underline{\alpha}_x \leq \bar{\alpha}_x$, such that $\alpha_x \geq \underline{\alpha}_x$ if and only if there exists $x \in X(0)$ such that $y_j(\mathbf{0}, x\mathbf{1}) = k$. Moreover, $|X(0)| = 2$ for all $\alpha_x \in [\underline{\alpha}_x, \bar{\alpha}_x]$.*

Again, we see that there is an open interval of follower productions $(\bar{y}, k)$ that are never symmetric equilibria. However, as N increases, this interval, instead of shrinking as in the case of the forward market, expands at the rate $\frac{1}{\sqrt{N}}$ to a size of $\frac{k}{2}$. That is, as the followers become more competitive, the leaders are better able to exploit the capacity constraints of the followers.

When $\triangle C > 0$, followers might not supply to the market. The next lemma highlights the structure of this regime.

Lemma 3.8. *Suppose $\triangle C > 0$. Let $\zeta_1 = (M+1)\triangle C + \min\left(MN\triangle C, MNk + 2M\sqrt{Nk\triangle C}\right)$. Then there exists $x \in X(0)$ satisfying $y_j(\mathbf{0}, x\mathbf{1}) = 0$ if and only if $\alpha_x \leq \zeta_1$. Furthermore,*

$$x = \begin{cases} \frac{1}{M+1}\alpha_x & \text{if } 0 \leq \alpha_x < (M+1)\triangle C, \\ \frac{1}{M}(\alpha_x - \triangle C) & \text{if } (M+1)\triangle C \leq \alpha_x \leq \zeta_1, \end{cases}$$

The proof is omitted as it is a straightforward observation from Proposition 3.6. The key insight is that this regime exhibits different behavior depending on whether α_x is less than or greater than $(M+1)\triangle C$. The leader productions increase at a faster rate when $\alpha_x > (M+1)\triangle C$ because leaders use their commitment power to drive followers out of the market.

When demand is high, the structure of the equilibria is similar to the case when $\triangle C = 0$, except that the range of follower productions that could be equilibria is now smaller. The larger the value of $\triangle C$, the smaller the range of supportable follower productions.

Lemma 3.9. *Suppose $\triangle C > 0$.*

1. *There exists a unique $\bar{y} < k$, such that there exist $x \in X(0)$ such that $y_j(\mathbf{0}, x\mathbf{1}) = y$ if and only $y \leq \bar{y}$ or $y = k$. Moreover,*

$$\bar{y} = \begin{cases} \left(1 + O\left(\frac{1}{\sqrt{N}}\right)\right) \frac{1}{2}\left(k - \frac{(\sqrt{N+1}-1)^2}{N} \triangle C\right), & \text{if } \triangle C \leq \frac{Nk}{(\sqrt{N+1}-1)^2}, \\ 0, & \text{otherwise}. \end{cases}$$

2. *There exists a unique $\bar{\alpha}_x \in \mathbb{R}_+$, such that $\alpha_x \leq \bar{\alpha}_x$ if and only if there exists $x \in X(0)$ such that $y_j(\mathbf{0}, x\mathbf{1}) \leq \bar{y}$, and a unique $\underline{\alpha}_x \leq \bar{\alpha}_x$, such that $\alpha_x \geq \underline{\alpha}_x$ if and only if there exists $x \in X(0)$ such that $y_j(\mathbf{0}, x\mathbf{1}) = k$. Moreover, $|X(0)| = 2$ for all $\alpha_x \in [\underline{\alpha}_x, \bar{\alpha}_x]$.*

We now contrast the efficiency of the equilibria in the forward and Stackelberg markets. Given follower and leader productions y and x respectively, let $\mathsf{SW}(y, x)$ denote the social welfare:

$$\mathsf{SW}(y, x) := \int_0^{Mx+Ny} P(w)\, dw - (MCx + Ncy).$$

The next lemma highlights that adding a forward market to a Stackelberg market could be inefficient.

Lemma 3.10. *Suppose $\triangle C = 0$. Let $\underline{\alpha}_x := (M + N + 1)k$ and $\bar{\alpha}_x := \frac{(M+1)\sqrt{N+1}}{2(\sqrt{N+1}-1)} Nk$. Then, for all $\alpha_x \in [\underline{\alpha}_x, \bar{\alpha}_x]$, there exists $(f, x) \in Q$ and $x_S \in X(0)$ such that*

$$Mx + Ny_j(f\mathbf{1}, x\mathbf{1}) < Mx_S + Ny_j(\mathbf{0}, x_S\mathbf{1}),$$
$$\mathsf{SW}(y_j(f\mathbf{1}, x\mathbf{1}), x) < \mathsf{SW}(y_j(\mathbf{0}, x_S\mathbf{1}), x_S).$$

Moreover,

$$\frac{Mx_S + Ny_j(\mathbf{0}, x_S\mathbf{1})}{Mx + Ny_j(f\mathbf{1}, x\mathbf{1})} \leq \frac{(MN + M + N)(M + 1)}{M(M+1)(N+1) + 2(N + 1 - \sqrt{N+1})},$$
$$\frac{\mathsf{SW}(y_j(\mathbf{0}, x_S\mathbf{1}), x_S)}{\mathsf{SW}(y_j(f\mathbf{1}, x\mathbf{1}), x)} \leq \frac{(M+1)^2(MN + M + N)(MN + M + N + 2)}{(N+1)\left((M^2 + M + 2)\sqrt{N+1} - 2\right)\left((M^2 + 3M)\sqrt{N+1} + 2\right)},$$

where the inequalities are tight.

This inefficiency is attributed to equilibria in the forward market where followers produce k while there are equilibria in the Stackelberg market where followers produce strictly less than k. Hence, this inefficiency is due to leaders exploiting the capacity constraints of the followers in the forward market. To see this, note that this inefficiency does not disappear even with a large number of followers:

$$\lim_{N\to\infty} \frac{Mx_S + Ny_j(\mathbf{0}, x_S\mathbf{1})}{Mx + Ny_j(f\mathbf{1}, x\mathbf{1})} \leq \frac{(M+1)^2}{M^2+M+2},$$
$$\lim_{N\to\infty} \frac{\mathsf{SW}(y_j(\mathbf{0}, x_S\mathbf{1}), x_S)}{\mathsf{SW}(y_j(f\mathbf{1}, x\mathbf{1}), x)} \leq \frac{(M+1)^4}{(M^2+M+2)(M^2+3M)}.$$

On the other hand, this inefficiency disappears with a large number of leaders:

$$\lim_{M\to\infty} \frac{Mx_S + Ny_j(\mathbf{0}, x_S\mathbf{1})}{Mx + Ny_j(f\mathbf{1}, x\mathbf{1})} \leq 1,$$
$$\lim_{M\to\infty} \frac{\mathsf{SW}(y_j(\mathbf{0}, x_S\mathbf{1}), x_S)}{\mathsf{SW}(y_j(f\mathbf{1}, x\mathbf{1}), x)} \leq 1.$$

The statement of Lemma 3.10 does not specify whether there exists forward equilibria that are equally or more efficient than Stackelberg equilibria. However, it is possible to impose further conditions on the system and demand such that the Stackelberg equilibria are always strictly more efficient.

The same approach in the proof of Lemma 3.10 can be used to obtain bounds on the production and efficiency losses when $\triangle C > 0$. However, the bounds are more complicated and depend on $\triangle C$ and k.

Appendices

In the following Appendices, we derive closed-form expressions for the symmetric follower reactions, symmetric leader reactions, the symmetric forward market equilibria, and the symmetric Stackelberg equilibria. Then, we derive the structural results in Section 3.6. We denote by α_x and α_y the normalized leader and follower demands respectively, and by $\triangle C$ the normalized marginal cost gap:

$$\begin{aligned}\alpha_x &= \frac{1}{\beta}(\alpha - C),\\ \alpha_y &= \frac{1}{\beta}(\alpha - c),\\ \triangle C &= \frac{1}{\beta}(c - C).\end{aligned}$$

We use the following notation. For scalars $z, a, b \in \mathbb{R}$ such that $a \leq b$, let

$$[z]_a^b := \begin{cases} a, & \text{if } z \leq a,\\ b, & \text{if } z \geq b,\\ z, & \text{otherwise.}\end{cases}$$

We will use the following properties:

(i) For any $c \in \mathbb{R}$, $c + [z]_a^b = [z + c]_{a+c}^{b+c}$.

(ii) If $c > 0$, then $c\,[z]_a^b = [cz]_{ca}^{cb}$.

(iii) If $c < 0$, then $c\,[z]_a^b = [cz]_{cb}^{ca}$.

3.A Spot Market Analyses

Proposition 3.1. *Fix a follower $l \in N$ and suppose $f_j = f$ for every $j \neq l$. There is a unique Nash equilibrium $\mathbf{y}$ in the spot market such that, for each $j \neq l$,*

$$y_j = \left[\frac{1}{N}\left(\alpha_y + f - \sum_{i=1}^{M} x_i - y_l\right)\right]_0^k. \tag{3.4}$$

Proof. The uniqueness of the Nash equilibrium follows from Theorem 5 of [67]. Each follower $j \in N$ has a strategy set $[0, k]$ which is compact. Its payoff function in the spot market $\phi_j^{(s)}$ is continuous in all arguments and is strictly concave in y_j. Hence, from the Karush-Kuhn-Tucker (KKT) conditions, we infer that $\mathbf{y} \in [0, k]^N$ is a Nash equilibrium of the spot market, if and only if there exists $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbb{R}_+^N$ such that, for each $j \in N$:

$$\nabla_{y_j} \left[\phi_j^{(s)}(y_j; \mathbf{y}_{-j}) + \lambda_j y_j + \mu_j (k - y_j)\right] = 0, \tag{3.5}$$

$$\lambda_j y_j = \mu_j (k - y_j) = 0. \tag{3.6}$$

Take any $j \neq l$. Expanding the LHS of (3.5) gives:

$$\begin{aligned}\nabla_{y_j} \left[\phi_j^{(s)}(y_j; \mathbf{y}_{-j}) + \lambda_j y_j + \mu_j (k - y_j)\right] &= \beta \left(\alpha_y + f - \sum_{i=1}^{M} x_i - y_j - \sum_{j'=1}^{N} y_{j'}\right) + \lambda_j - \mu_j \\ &= \beta \left(\alpha_y + f - \sum_{i=1}^{M} x_i - y_l - N y_j\right) + \lambda_j - \mu_j.\end{aligned}$$

Suppose $0 < y_j < k$. Then (3.6) imply that $\lambda_j = \mu_j = 0$. From (3.5), we obtain

$$y_j = \frac{1}{N}\left(\alpha_y + f - \sum_{i=1}^{M} x_i - y_l\right). \tag{3.7}$$

Suppose $y_j = 0$. Then (3.6) imply that $\mu_j = 0$. From (3.5), we obtain

$$-\left(\alpha_y + f - \sum_{i=1}^{M} x_i - y_l\right) = \lambda_j \geq 0. \tag{3.8}$$

Suppose $y_j = k$. Then (3.6) imply that $\lambda_j = 0$. From (3.5), we obtain

$$\left(\alpha_y + f - \sum_{i=1}^{M} x_i - y_l - Nk\right) = \mu_j \geq 0. \tag{3.9}$$

Since $0 \leq y_j \leq k$, (3.7) – (3.9) together imply that

$$y_j = \begin{cases} 0, & \text{if } \frac{1}{N}\left(\alpha_y + f - \sum_{i=1}^{M} x_i - y_l\right) \leq 0, \\ k, & \text{if } \frac{1}{N}\left(\alpha_y + f - \sum_{i=1}^{M} x_i - y_l\right) \geq k, \\ \frac{1}{N}\left(\alpha_y + f - \sum_{i=1}^{M} x_i - y_l\right), & \text{otherwise,} \end{cases}$$

which is equivalent to (3.4). ∎

Proposition 3.2. *Suppose $f_j = f$ for every $j \in N$. There is a unique Nash equilibrium in the spot* *market, given by*

$$y_j = \left[\frac{1}{N+1}\left(\alpha_y + f - \sum_{i=1}^{M} x_i\right)\right]_0^k. \tag{3.10}$$

Proof. The uniqueness of the Nash equilibrium follows from Theorem 5 of [67]. Hence, it suffices to show that the given productions form a Nash equilibrium. From the optimality conditions in (3.5) – (3.6), we infer that $y \in [0, k]$ is a symmetric Nash equilibrium in the spot market, if and only if there exists scalars $\lambda, \mu \in \mathbb{R}_+$ such that,

$$\begin{aligned} \beta\left(\alpha_y + f - \sum_{i=1}^{M} x_i - (N+1)y\right) + \lambda - \mu = 0, \\ \lambda y = \mu\,(k - y) = 0. \end{aligned}$$

Let

$$\begin{aligned} \lambda &= \left[-\beta\left(\alpha_y + f - \sum_{i=1}^{M} x_i - (N+1)y\right)\right]_0^\infty, \\ \mu &= \left[\beta\left(\alpha_y + f - \sum_{i=1}^{M} x_i - (N+1)y\right)\right]_0^\infty. \end{aligned}$$

It is straightforward to show that y defined in (3.10), and λ, μ defined above, together satisfy the optimality conditions. □

3.B Follower Reaction Analyses

Proposition 3.3. *Fix the leaders' productions $\mathbf{x} \in \mathbb{R}_+^M$. Let $F \subseteq \mathbb{R}$ denote the set of symmetric follower reactions, i.e., for each $f \in F$ and $j \in N$,*

$$\phi_j\left(f; f\mathbf{1}, \mathbf{x}\right) \geq \phi_j\left(\bar{f}; f\mathbf{1}, \mathbf{x}\right), \quad \forall \bar{f} \in \mathbb{R}. \tag{3.11}$$

Let $\xi := \alpha_y - \sum_{i=1}^{M} x_i$. Then,

$$F = \begin{cases} \left(-\infty, -\xi\right], & \text{if } \xi < 0, \\ \left\{\frac{N-1}{N^2+1}\xi\right\}, & \text{if } 0 \leq \xi \leq \frac{(N^2+1)(N-1)}{N^2-2\sqrt{N}+1}k, \\ \varnothing, & \text{if } \frac{(N^2+1)(N-1)}{N^2-2\sqrt{N}+1}k < \xi < (N+1)\,k, \\ \left[-\xi + (N+1)k, \infty\right), & \text{if } (N+1)\,k \leq \xi. \end{cases} \tag{3.12}$$

Moreover, for each $f \in F$,

$$
\begin{aligned}
y_j(f\mathbf{1}, \mathbf{x}) = 0 &\iff \xi \le 0, \\
0 < y_j(f\mathbf{1}, \mathbf{x}) < k &\iff 0 < \xi \le \frac{(N^2+1)(N-1)}{N^2 - 2\sqrt{N} + 1} k, \\
y_j(f\mathbf{1}, \mathbf{x}) = k &\iff (N+1)k \le \xi.
\end{aligned}
$$

Proof. The proof proceeds in three steps. In step 1, we reformulate a follower's payoff maximization problem into a problem involving its production quantity only. In step 2, we compute its payoff maximizing production quantity. In step 3, we compute the symmetric follower forward positions that satisfy the condition that every follower is producing at its payoff maximizing quantity. The latter gives the set of symmetric follower reactions.

Step 1: Fix a follower $l \in N$ and suppose $f_j = f$ for every $j \neq l$. Using Proposition 3.1 to substitute for $y_j(f_j; \mathbf{f}_{-j}, \mathbf{x})$ for every $j \neq l$, we infer that the total production in the spot market is given by

$$
\begin{aligned}
\sum_{j=1}^{N} y_j(f_j; \mathbf{f}_{-j}, \mathbf{x}) &= y_l(f_l; f\mathbf{1}, \mathbf{x}) + (N-1)\left[\frac{1}{N}\left(\alpha_y + f - \sum_{i=1}^{M} x_i - y_l(f_l; f\mathbf{1}, \mathbf{x})\right)\right]_0^k \\
&= y_l(f_l; f\mathbf{1}, \mathbf{x}) + \left[\frac{N-1}{N}\left(\alpha_y + f - \sum_{i=1}^{M} x_i - y_l(f_l; f\mathbf{1}, \mathbf{x})\right)\right]_0^{(N-1)k} \\
&= \left[\frac{N-1}{N}\left(\alpha_y + f - \sum_{i=1}^{M} x_i\right) + \frac{1}{N} y_l(f_l; f\mathbf{1}, \mathbf{x})\right]_{y_l(f_l; f\mathbf{1}, \mathbf{x})}^{y_l(f_l; f\mathbf{1}, \mathbf{x}) + (N-1)k},
\end{aligned}
$$

By substituting the above into follower l's payoff, and using the fact that $y_l(\mathbb{R}; f\mathbf{1}, \mathbf{x}) = [0, k]$, we obtain

$$
\begin{aligned}
\sup_{f_l \in \mathbb{R}} \phi_l(f_l; f\mathbf{1}, \mathbf{x}) &= \sup_{f_l \in \mathbb{R}} \left(P\left(\left[\frac{N-1}{N}\left(\alpha_y + f - \sum_{i=1}^{M} x_i\right) + \frac{1}{N} y_l(f_l; f\mathbf{1}, \mathbf{x})\right]_{y_l(f_l; f\mathbf{1}, \mathbf{x})}^{y_l(f_l; f\mathbf{1}, \mathbf{x}) + (N-1)k} \right.\right. \\
&\qquad \left.\left. + \sum_{i=1}^{M} x_i \right) - c \right) \cdot y_l(f_l; f\mathbf{1}, \mathbf{x}) \qquad (3.13) \\
&= \sup_{y \in [0,k]} \hat{\phi}_l(y; f, \mathbf{x}), \qquad (3.14)
\end{aligned}
$$

where

$$
\hat{\phi}_l(y; f, \mathbf{x}) := \left(P\left(\left[\frac{N-1}{N}\left(\alpha_y + f - \sum_{i=1}^{M} x_i\right) + \frac{1}{N} y\right]_y^{y+(N-1)k} + \sum_{i=1}^{M} x_i \right) - c \right) \cdot y.
$$

Step 2: We solve for the solution to (3.14). Substituting for the demand function yields

$$\hat{\phi}_l(y; f, \mathbf{x}) = \beta \left(\xi - \left[\frac{N-1}{N}(\xi + f) + \frac{1}{N} y \right]_y^{y+(N-1)k} \right) y$$

$$= \begin{cases} \beta(\xi - y)\, y, & \text{if (3.15a) holds,} \\ \begin{cases} \beta\left(\frac{1}{N}\xi - \frac{N-1}{N} f - \frac{1}{N} y\right) y, & \text{if } 0 \le y < \xi + f, \\ \beta(\xi - y)\, y, & \text{if } k \ge y \ge \xi + f, \end{cases} & \text{if (3.15b) holds,} \\ \beta\left(\frac{1}{N}\xi - \frac{N-1}{N} f - \frac{1}{N} y\right) y, & \text{if (3.15c) holds,} \\ \begin{cases} \beta(\xi - y - (N-1)k)\, y, & \text{if } 0 \le y \le \xi + f - Nk, \\ \beta\left(\frac{1}{N}\xi - \frac{N-1}{N} f - \frac{1}{N} y\right) y, & \text{if } k \ge y > \xi + f - Nk, \end{cases} & \text{if (3.15d) holds,} \\ \beta(\xi - y - (N-1)k)\, y, & \text{if (3.15e) holds,} \end{cases}$$

where the second equality follows from the fact that $y \in [0, k]$ and the five cases (3.15a) – (3.15e) are defined by

$$\xi + f \le 0, \tag{3.15a}$$

$$0 < \xi + f < k, \tag{3.15b}$$

$$k \le \xi + f \le Nk, \tag{3.15c}$$

$$Nk < \xi + f < (N+1)k, \tag{3.15d}$$

$$(N+1)k \le \xi + f. \tag{3.15e}$$

We analyze each case separately.

Case (i): $\xi + f \le 0$. Then $\hat{\phi}_l(y; f, \mathbf{x})$ is a smooth function in y over the interval $[0, k]$. The first and second derivatives are given by

$$\frac{\partial}{\partial y} \hat{\phi}_l(y; f, \mathbf{x}) = \beta(\xi - 2y),$$

$$\frac{\partial^2}{\partial y^2} \hat{\phi}_l(y; f, \mathbf{x}) = -2\beta < 0,$$

which implies that $\hat{\phi}_l(y; f, \mathbf{x})$ is strictly concave in y. Hence, y is a solution to (3.14) if and only if it satisfies the following first order optimality conditions:

$$\frac{\partial^+}{\partial y} \hat{\phi}_l(y; f, \mathbf{x}) \le 0, \quad \text{if } 0 \le y < k, \tag{3.16}$$

$$\frac{\partial^-}{\partial y} \hat{\phi}_l(y; f, \mathbf{x}) \ge 0, \quad \text{if } 0 < y \le k. \tag{3.17}$$

It is straightforward to show that there is a unique solution given by

$$y = \left[\frac{1}{2}\xi\right]_0^k. \tag{3.18}$$

Case (ii): $0 < \xi + f < k$. Then $\hat{\phi}_l(y; f, \mathbf{x})$ is a piecewise smooth function in y over the interval $[0, k]$. The first and second derivatives are given by

$$\begin{aligned}
\frac{\partial}{\partial y}\hat{\phi}_l(y; f, \mathbf{x}) &= \begin{cases} \beta\left(\frac{1}{N}\xi - \frac{N-1}{N}f - \frac{2}{N}y\right), & \text{if } 0 \le y < \xi + f, \\ \beta(\xi - 2y), & \text{if } k \ge y > \xi + f, \end{cases} \\
\frac{\partial^2}{\partial y^2}\hat{\phi}_l(y; f, \mathbf{x}) &= \begin{cases} -\frac{2}{N}\beta, & \text{if } 0 \le y < \xi + f, \\ -2\beta, & \text{if } k \ge y > \xi + f, \end{cases} \\
&< 0.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\left.\frac{\partial^+}{\partial y}\hat{\phi}_l(y; f, \mathbf{x})\right|_{y=\xi+f} &= \beta(-\xi - 2f) \\
&= \frac{1}{N}\beta(-N\xi - 2Nf) \\
&\le \frac{1}{N}\beta(-\xi + (N-1)f - 2Nf) \\
&= \frac{1}{N}\beta(\xi - (N-1)f - 2\xi - 2f) \\
&= \left.\frac{\partial^-}{\partial y}\hat{\phi}_l(y; f, \mathbf{x})\right|_{y=\xi+f},
\end{aligned}$$

where the inequality follows from the fact that $\xi + f > 0$. Hence, $\hat{\phi}_l(y; f, \mathbf{x})$ is concave in y over $[0, k]$. Hence, y is a solution to (3.14) if and only if it satisfies the first order optimality conditions (3.16) – (3.17). It is straightforward to show that there is a unique solution given by

$$y = \begin{cases} 0, & \text{if } \xi \le (N-1)f, \\ \frac{1}{2}(\xi - (N-1)f), & \text{if } \xi > \max((N-1)f, -(N+1)f), \\ \xi + f, & \text{if } -2f \le \xi \le -(N+1)f, \\ \frac{1}{2}\xi, & \text{if } \xi < \min(2k, -2f), \\ k, & \text{if } \xi \ge 2k. \end{cases} \tag{3.19}$$

Case (iii): $k \le \xi + f \le Nk$. Then $\hat{\phi}_l(y; f, \mathbf{x})$ is a smooth function in y over the interval $[0, k]$. The first and second derivatives are given by

$$\frac{\partial}{\partial y}\hat{\phi}_l(y; f, \mathbf{x}) = \beta\left(\frac{1}{N}\xi - \frac{N-1}{N}f - \frac{2}{N}y\right),$$
$$\frac{\partial^2}{\partial y^2}\hat{\phi}_l(y; f, \mathbf{x}) = -\frac{2}{N}\beta < 0,$$

which implies that $\hat{\phi}_l(y; f, \mathbf{x})$ is strictly concave in y. Hence, y is a solution to (3.14) if and only if it satisfies the first order optimality conditions (3.16) – (3.17). It is straightforward to show that there is a unique solution given by

$$y = \left[\frac{1}{2}\left(\xi - (N-1)f\right)\right]_0^k. \tag{3.20}$$

Case (iv): $Nk < \xi + f < (N+1)k$. Then $\hat{\phi}_l(y; f, \mathbf{x})$ is a piecewise smooth function in y over the interval $[0, k]$. The first and second derivatives are given by

$$\frac{\partial}{\partial y}\hat{\phi}_l(y; f, \mathbf{x}) = \begin{cases} \beta\left(\xi - 2y - (N-1)k\right), & \text{if } 0 \le y < \xi + f - Nk, \\ \beta\left(\frac{1}{N}\xi - \frac{N-1}{N}f - \frac{2}{N}y\right), & \text{if } k \ge y > \xi + f - Nk, \end{cases}$$
$$\frac{\partial^2}{\partial y^2}\hat{\phi}_l(y; f, \mathbf{x}) = \begin{cases} -2\beta, & \text{if } 0 \le y < \xi + f - Nk, \\ -\frac{2}{N}\beta, & \text{if } k \ge y > \xi + f - Nk, \end{cases}$$
$$< 0.$$

It is straightforward to check that $\hat{\phi}_l(y; f, \mathbf{x})$ is not concave in y over the interval $[0, k]$. However, $\hat{\phi}_l(y; f, \mathbf{x})$ is piecewise concave in y. Hence, solve the following sub-problems:

$$\sup_{y \in [0, \xi + f - (N-1)k]} \hat{\phi}_l(y; f, \mathbf{x}), \tag{3.21}$$

and

$$\sup_{y \in [\xi + f - (N-1)k, k]} \hat{\phi}_l(y; f, \mathbf{x}). \tag{3.22}$$

The solution of the sub-problem with the larger optimal value is the solution to (3.14). Using the first-order optimality conditions, the unique solution to (3.21) is given by

$$y = \left[\frac{1}{2}\left(\xi - (N-1)k\right)\right]_0^{\xi + f - Nk} =: z_1,$$

and that to (3.22) is given by

$$y = \left[\frac{1}{2}\left(\xi - (N-1)f\right)\right]_{\xi+f-Nk}^{k} =: z_2.$$

Hence, the solution(s) to (3.14) are given by:

$$y = z_1, \quad \text{if } \hat{\phi}_l(z_1; f, \mathbf{x}) > \hat{\phi}_l(z_2; f, \mathbf{x}), \tag{3.23a}$$

$$y = z_2, \quad \text{if } \hat{\phi}_l(z_1; f, \mathbf{x}) < \hat{\phi}_l(z_2; f, \mathbf{x}), \tag{3.23b}$$

$$y = z_1 \text{ or } z_2, \quad \text{if } \hat{\phi}_l(z_1; f, \mathbf{x}) = \hat{\phi}_l(z_2; f, \mathbf{x}). \tag{3.23c}$$

Case (v): $(N+1)k \le \xi + f$. Then $\hat{\phi}_l(y; f, \mathbf{x})$ is a smooth function in y over the interval $[0, k]$. The first and second derivatives are given by

$$\begin{aligned} \frac{\partial}{\partial y}\hat{\phi}_l(y; f, \mathbf{x}) &= \beta\left(\xi - (N-1)k - 2y\right), \\ \frac{\partial^2}{\partial y^2}\hat{\phi}_l(y; f, \mathbf{x}) &= -2\beta < 0, \end{aligned}$$

which implies that $\hat{\phi}_l(y; f, \mathbf{x})$ is strictly concave in y. Hence, y is a solution to (3.14) if and only if it satisfies the first order optimality conditions (3.16) – (3.17). It is straightforward to show that there is a unique solution given by

$$y = \left[\frac{1}{2}\left(\xi - (N-1)k\right)\right]_{0}^{k}. \tag{3.24}$$

Step 3: We solve for the symmetric follower forward positions that satisfy the condition that every follower is producing at its payoff maximizing quantity. The latter gives the set of symmetric follower reactions since

$$\begin{aligned} & \phi_l\left(f; f\mathbf{1}, \mathbf{x}\right) \ge \phi_l\left(\bar{f}; f\mathbf{1}, \mathbf{x}\right), \quad \forall \bar{f} \in \mathbb{R} \\ \Longleftrightarrow\ & \hat{\phi}_l\left(y_l(f; f\mathbf{1}, \mathbf{x}); f, \mathbf{x}\right) \ge \hat{\phi}_l\left(y_l(\bar{f}; f\mathbf{1}, \mathbf{x}); f, \mathbf{x}\right), \quad \forall \bar{f} \in \mathbb{R} \\ \Longleftrightarrow\ & \hat{\phi}_l\left(y; f, \mathbf{x}\right) \ge \hat{\phi}_l\left(\bar{y}; f\mathbf{1}, \mathbf{x}); f, \mathbf{x}\right), \quad \forall \bar{y} \in [0, k], \text{ and } y = \left[\frac{1}{N+1}\left(\xi + f\right)\right]_0^k. \end{aligned} \tag{3.25}$$

The first equivalence is due to (3.13). The second equivalence is due to the fact that $y_l(f; f\mathbf{1}, \mathbf{x}) = \left[\frac{1}{N+1}\left(\xi + f\right)\right]_0^k$ and $y_l\left(\mathbb{R}; f\mathbf{1}, \mathbf{x}\right) = [0, k]$. We divide the analyses into five cases depending on the value of $\xi + f$.

Case (i): $\xi + f \le 0$. Note that y is given by (3.18). Hence, the symmetric follower reactions are given by:

$$\left[\frac{1}{2}\xi\right]_0^k = \left[\frac{1}{N+1}(\xi+f)\right]_0^k \text{ and (3.15a) holds} \iff \left[\frac{1}{2}\xi\right]_0^k = 0 \text{ and (3.15a) holds}$$
$$\iff \xi \le 0 \text{ and } f \le -\xi. \tag{3.26}$$

Case (ii): $0 < \xi + f < k$. Note that y is given by (3.19). Since $0 < \xi + f < k \implies 0 < \frac{1}{N+1}(\xi+f) < k$, the symmetric follower reactions are given by:

$$\begin{aligned}
&\frac{1}{2}(\xi-(N-1)f) = \frac{1}{N+1}(\xi+f) \text{ and } \xi > \max((N-1)f, -(N+1)f) \text{ and (3.15b) holds}\\
&\text{or } \xi + f = \frac{1}{N+1}(\xi+f) \text{ and } -2f \le \xi \le -(N+1)f \text{ and (3.15b) holds}\\
&\text{or } \frac{1}{2}\xi = \frac{1}{N+1}(\xi+f) \text{ and } \xi < \min(2k, -2f) \text{ and (3.15b) holds}\\
\iff& f = \frac{N-1}{N^2+1}\xi \text{ and } \xi > \max((N-1)f, -(N+1)f) \text{ and (3.15b) holds}\\
&\text{or } f = -\xi \text{ and } -2f \le \xi \le -(N+1)f \text{ and (3.15b) holds}\\
&\text{or } f = \frac{N-1}{2}\xi \text{ and } \xi < \min(2k, -2f) \text{ and (3.15b) holds}\\
\iff& f = \frac{N-1}{N^2+1}\xi \text{ and } \xi > \max\left(\frac{(N-1)^2}{N^2+1}\xi, -\frac{N^2-1}{N^2+1}\xi\right) \text{ and } 0 < \xi < \frac{N^2+1}{N(N+1)}k\\
&\text{or } f = \frac{N-1}{2}\xi \text{ and } \xi < \min(2k, -(N-1)\xi) \text{ and } 0 < \frac{N+1}{2}\xi < k\\
\iff& f = \frac{N-1}{N^2+1}\xi \text{ and } 0 < \xi < \frac{N^2+1}{N(N+1)}k.
\end{aligned}\tag{3.27}$$

The second equivalence is due to the fact that $f = -\xi \implies \xi + f = 0$. The third equivalence is due to the fact that $\xi > 0 \implies \frac{(N-1)^2}{N^2+1}\xi \ge -\frac{N^2-1}{N^2+1}\xi$ and $\frac{N+1}{2}\xi > 0 \implies 2k > -(N-1)\xi$.

Case (iii): $k \le \xi + f \le Nk$. Note that y is given by (3.20). Hence, the symmetric follower reactions are given by:

$$\begin{aligned}
&\left[\frac{1}{2}(\xi-(N-1)f)\right]_0^k = \left[\frac{1}{N+1}(\xi+f)\right]_0^k \text{ and (3.15c) holds}\\
\iff& \frac{1}{2}(\xi-(N-1)f) = \frac{1}{N+1}(\xi+f) \text{ and (3.15c) holds and } 0 < \frac{1}{2}(\xi-(N-1)f) < k\\
\iff& f = \frac{N-1}{N^2+1}\xi \text{ and (3.15c) holds and } 0 < \frac{1}{2}(\xi-(N-1)f) < k\\
\iff& f = \frac{N-1}{N^2+1}\xi \text{ and } \frac{N^2+1}{N(N+1)}k \le \xi \le \frac{N^2+1}{N+1}k \text{ and } 0 < \xi < \frac{N^2+1}{N-\frac{1}{2}}k\\
\iff& f = \frac{N-1}{N^2+1}\xi \text{ and } \frac{N^2+1}{N(N+1)}k \le \xi \le \frac{N^2+1}{N+1}k.
\end{aligned}\tag{3.28}$$

The first equivalence is due to the fact that (3.15c) $\implies 0 < \frac{1}{N+1}(\xi + f) < k$. The last equivalence is due to the fact that $\frac{N^2+1}{N(N+1)} < \frac{N^2+1}{N+1} < \frac{N^2+1}{N-\frac{1}{2}}$.

Case (iv): $Nk < \xi + f < (N+1)k$. Note that y is described by (3.23). We show that there does not exist a symmetric follower reaction such that $y = z_1$. Suppose otherwise. By Proposition 3.1, for each $j \neq l$,

$$\begin{aligned}
y_j &= \left[\frac{1}{N}\left(\xi + f - \left[\frac{1}{2}(\xi - (N-1)k)\right]_0^{\xi+f-Nk}\right)\right]_0^k \\
&= \left[-\frac{1}{N}\left[\frac{1}{2}(-\xi - 2f - (N-1)k)\right]_{-\xi-f}^{-Nk}\right]_0^k \\
&= \left[\left[\frac{1}{2N}(\xi + 2f + (N-1)k)\right]_k^{\frac{1}{N}(\xi+f)}\right]_0^k \\
&= k
\end{aligned}$$

However, (3.15d) $\implies \frac{1}{N+1}(\xi + f) < k \implies y < k = y_j$, which contradicts with the fact that a symmetric follower reaction implies symmetric productions (by Proposition 3.2).

Hence, the symmetric follower reactions are given by:

$$\begin{aligned}
&\left[\frac{1}{2}(\xi - (N-1)f)\right]_{\xi+f-Nk}^k = \frac{1}{N+1}(\xi + f) \text{ and (3.15d) holds and } \hat{\phi}_l(z_1; f, \mathbf{x}) \leq \hat{\phi}_l(z_2; f, \mathbf{x}) \\
\iff &\frac{1}{2}(\xi - (N-1)f) = \frac{1}{N+1}(\xi + f) \text{ and (3.15d) holds and } \hat{\phi}_l(z_1; f, \mathbf{x}) \leq \hat{\phi}_l(z_2; f, \mathbf{x}) \\
\iff &f = \frac{N-1}{N^2+1}\xi \text{ and (3.15d) holds and } \hat{\phi}_l(z_1; f, \mathbf{x}) \leq \hat{\phi}_l(z_2; f, \mathbf{x}) \\
\iff &f = \frac{N-1}{N^2+1}\xi \text{ and } \frac{N^2+1}{N+1}k < \xi < \frac{N^2+1}{N}k \text{ and } \hat{\phi}_l(z_1; f, \mathbf{x}) \leq \hat{\phi}_l(z_2; f, \mathbf{x}) \\
\iff &f = \frac{N-1}{N^2+1}\xi \text{ and } \frac{N^2+1}{N+1}k < \xi \leq \frac{(N^2+1)(N-1)}{N^2 - 2\sqrt{N} + 1}k. \qquad (3.29)
\end{aligned}$$

The first equivalence follows from the fact that $\xi + f - Nk = \frac{1}{N+1}(\xi + f) \implies \xi + f = (N+1)k$ and $k = \frac{1}{N+1}(\xi + f) \implies \xi + f = (N+1)k$. The last equivalence follows from the following facts.

First, note that

$$
\begin{aligned}
z_1 &= \left[\frac{1}{2}\left(\xi - (N-1)k\right)\right]_0^{\frac{N(N+1)}{N^2+1}\xi - Nk} \\
&= \begin{cases} \frac{N(N+1)}{N^2+1}\xi - Nk, & \text{if } \frac{N^2+1}{N+1}k < \xi < \frac{N^2+1}{N}k \text{ and } \frac{1}{2}\left(\xi - (N-1)k\right) > \frac{N(N+1)}{N^2+1}\xi - Nk, \\ \frac{1}{2}\left(\xi - (N-1)k\right), & \text{if } \frac{N^2+1}{N+1}k < \xi < \frac{N^2+1}{N}k \text{ and } \frac{1}{2}\left(\xi - (N-1)k\right) \le \frac{N(N+1)}{N^2+1}\xi - Nk, \end{cases} \\
&= \begin{cases} \frac{N(N+1)}{N^2+1}\xi - Nk, & \text{if } \frac{N^2+1}{N+1}k < \xi < \frac{(N+1)(N^2+1)}{N^2+2N-1}k, \\ \frac{1}{2}\left(\xi - (N-1)k\right), & \frac{(N+1)(N^2+1)}{N^2+2N-1}k \le \xi < \frac{N^2+1}{N}k. \end{cases}
\end{aligned}
$$

where the second equality is due to $\xi > \frac{N^2+1}{N+1}k > \frac{N^2-1}{N+1}k = (N-1)k$. Hence, if $\frac{N^2+1}{N+1}k < \xi < \frac{(N+1)(N^2+1)}{N^2+2N-1}k$, then

$$
\begin{aligned}
& \hat{\phi}_l(z_1; f, \mathbf{x}) \le \hat{\phi}_l(z_2; f, \mathbf{x}) \\
\iff & \left(\xi - (N-1)k - z_1\right) z_1 \le \frac{1}{N}\left(\xi - (N-1)f - z_2\right) z_2 \\
\iff & (k - f)\left(\xi + f - Nk\right) \le \frac{1}{4N}\left(\xi - (N-1)f\right)^2 \\
\iff & \text{True.}
\end{aligned}
$$

On the other hand, if $\frac{(N+1)(N^2+1)}{N^2+2N-1}k \le \xi < \frac{N^2+1}{N}k$, then

$$
\begin{aligned}
& \hat{\phi}_l(z_1; f, \mathbf{x}) \le \hat{\phi}_l(z_2; f, \mathbf{x}) \\
\iff & \left(\xi - (N-1)k - z_1\right) z_1 \le \frac{1}{N}\left(\xi - (N-1)f - z_2\right) z_2 \\
\iff & \frac{1}{2}\left(\xi - (N-1)k\right)^2 \le \frac{1}{4N}\left(\xi - (N-1)f\right)^2 \\
\iff & \xi \le \frac{(N^2+1)(N-1)}{N^2 - 2\sqrt{N} + 1}k,
\end{aligned}
$$

where $\frac{(N+1)(N^2+1)}{N^2+2N-1}k \le \frac{(N^2+1)(N-1)}{N^2-2\sqrt{N}+1}k < \frac{N^2+1}{N}k$.

Case (v): $(N+1)k \le \xi + f$. Note that y is given by (3.24). Hence, the symmetric follower reactions are given by:

$$
\begin{aligned}
& \left[\frac{1}{2}\left(\xi - (N-1)k\right)\right]_0^k = \left[\frac{1}{N+1}\left(\xi + f\right)\right]_0^k \text{ and (3.15e) holds} \\
\iff & \left[\frac{1}{2}\left(\xi - (N-1)k\right)\right]_0^k = k \text{ and (3.15e) holds} \\
\iff & f \ge -\xi + (N+1)k \text{ and } \xi \ge (N+1)k. \qquad (3.30)
\end{aligned}
$$

Putting together the descriptions in (3.26) – (3.30) yield (3.12). □

3.C Leader Reaction Analyses

Proposition 3.4. *Fix the followers' forward positions* $\mathbf{f} = f\mathbf{1} \in \mathbb{R}^N$*. Let* $X \subseteq \mathbb{R}_+$ *denote the set of symmetric leader reactions, i.e., for each* $x \in X$ *and* $i \in M$*,*

$$\psi_i\left(x; x\mathbf{1}, f\mathbf{1}\right) \geq \psi_i\left(\bar{x}; x\mathbf{1}, f\mathbf{1}\right), \quad \forall \bar{x} \in \mathbb{R}_+. \tag{3.31}$$

Let:

$$\begin{aligned} x_1 &= \frac{1}{M+1}\left[\alpha_x\right]_0^\infty, \\ x_2 &= \frac{1}{M}\left(\alpha_x - \triangle C + f\right), \\ x_3 &= \frac{1}{M+1}\left[\alpha_x + N\left(\triangle C - f\right)\right]_0^\infty, \\ x_4 &= \frac{1}{M+1}\left[\alpha_x - Nk\right]_0^\infty. \end{aligned}$$

Then,

$$X = \left\{ x \in \mathbb{R}_+ \;\middle|\; \begin{array}{l} x = x_1 \text{ if } f - \triangle C < -\frac{\alpha_x}{M+1}, \\ \text{or } x = x_2 \text{ if } -\frac{\alpha_x}{M+1} \leq f - \triangle C \leq \min\left(-\frac{\alpha_x}{MN+M+1}, \eta_4\right), \\ \text{or } x = x_3 \text{ if } -\frac{\alpha_x}{NM+M+1} < f - \triangle C \leq \max\left(\eta_3, k - (\alpha_x - Nk)\right), \\ \text{or } x = x_4 \text{ if } f - \triangle C \geq \begin{cases} k - (\alpha_x - Nk), & \text{if } \alpha_x < Nk, \\ \eta_2, & \text{if } \alpha_x \geq \left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right) Nk, \\ \eta_1, & \text{otherwise.} \end{cases} \end{array} \right\},$$

where

$$\begin{aligned} \eta_1 &:= \; k - \frac{\alpha_x - Nk}{N}\left(\frac{2(\sqrt{N+1}-1)}{M+1}\right), \\ \eta_2 &:= \; -\frac{1}{2}\left(\frac{2(\alpha_x - Nk)}{M+1} + Nk\right)\left(1 - \sqrt{1 - \left(\frac{\frac{2(\alpha_x - Nk)}{M+1}}{\frac{2(\alpha_x - Nk)}{M+1} + Nk}\right)^2}\right), \\ \eta_3 &:= \; k - \frac{\alpha_x - Nk}{N}\left(\frac{2(\sqrt{N+1}-1)}{2+(M-1)\sqrt{N+1}}\right), \\ \eta_4 &:= \; -\frac{1}{2}\left(\frac{2(\alpha_x - MNk)}{M+1} + \left(\frac{2M}{M+1}\right)^2 Nk\right)\left(1 - \sqrt{1 - \left(\frac{\frac{2(\alpha_x - MNk)}{M+1}}{\frac{2(\alpha_x - MNk)}{M+1} + \left(\frac{2M}{M+1}\right)^2 Nk}\right)^2}\right). \end{aligned}$$

Moreover, for each $x \in X$,

$$\begin{aligned} y_j(f\mathbf{1}, x\mathbf{1}) = 0 &\iff x = x_1 \text{ or } x_2, \\ 0 < y_j(f\mathbf{1}, x\mathbf{1}) < k &\iff x = x_3, \\ y_j(f\mathbf{1}, x\mathbf{1}) = k &\iff x = x_4. \end{aligned}$$

Proof. The proof proceeds in three steps. In step 1, we solve for a leader's payoff maximizing production quantity given that all other leaders produce equal quantities. In step 2, we solve for the symmetric leader productions that satisfy the condition that every leader is producing at its payoff maximizing quantity. The latter gives the set of symmetric leader reactions. In step 3, we explain how the solutions obtained in step 2 is equivalent to X.

Step 1: Fix a leader $l \in M$ and suppose $x_i = x$ for every $i \neq l$. We solve for the solution to

$$\sup_{x_l \in \mathbb{R}_+} \psi_l \left(x_l; x\mathbf{1}, f\mathbf{1}\right). \tag{3.32}$$

Substituting for the demand function yields

$$\psi_l \left(x_l; x\mathbf{1}, f\mathbf{1}\right) = \beta \left(\alpha_x - x_l - (M-1)x - \sum_{j=1}^{N} y_j(f\mathbf{1}, x_l, x\mathbf{1}) \right) x_l$$

where the follower productions are given by

$$\begin{aligned} &\sum_{j=1}^{N} y_j(f\mathbf{1}, x_l, x\mathbf{1}) \\ &= N \left[\frac{1}{N+1} \left(\alpha_y + f - x_l - (M-1)x\right) \right]_0^k \\ &= \begin{cases} 0, & \text{if (3.33a) holds,} \\ \begin{cases} 0, & \text{if } \alpha_y + f - x_l - (M-1)x \le 0, \\ \frac{N}{N+1}\left(\alpha_y + f - x_l - (M-1)x\right), & \text{otherwise,} \end{cases} & \text{if (3.33b) holds,} \\ \begin{cases} 0, & \text{if } \alpha_y + f - x_l - (M-1)x \le 0, \\ k, & \text{if } \alpha_y + f - x_l - (M-1)x \ge (N+1)k, \\ \frac{N}{N+1}\left(\alpha_y + f - x_l - (M-1)x\right), & \text{otherwise,} \end{cases} & \text{if (3.33c) holds,} \end{cases} \end{aligned}$$

where the second equality is due to the fact that $x_l, x \ge 0$ and the three cases (3.33a) – (3.33c) are defined by

$$\alpha_y + f - (M-1)x \leq 0, \tag{3.33a}$$

$$0 < \alpha_y + f - (M-1)x \leq (N+1)k, \tag{3.33b}$$

$$(N+1)k < \alpha_y + f - (M-1)x. \tag{3.33c}$$

We analyze each case separately.

Case (i): $\alpha_y + f - (M-1)x < 0$. We obtain

$$\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right) = \beta\left(\alpha_x - x_l - (M-1)x\right)x_l.$$

Hence, $\psi_l(x_l; x\mathbf{1}, f\mathbf{1})$ is a smooth function in x_l over $\mathbb{R}_+$. The first and second derivatives are given by

$$\begin{aligned}\frac{\partial}{\partial x_l}\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right) &= \beta\left(\alpha_x - (M-1)x - 2x_l\right),\\ \frac{\partial^2}{\partial x_l^2}\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right) &= -2\beta < 0,\end{aligned}$$

which implies that $\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right)$ is strictly concave in x_l. Hence, x_l is a solution to (3.32) if and only if it satisfies the following first order optimality conditions:

$$\frac{\partial^+}{\partial x_l}\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right) \leq 0, \quad \text{if } 0 \leq x_l, \tag{3.34}$$

$$\frac{\partial^-}{\partial x_l}\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right) \geq 0, \quad \text{if } 0 < x_l. \tag{3.35}$$

It is straightforward to show that there is a unique solution is given by

$$x_l = \left[\frac{1}{2}\left(\alpha_x - (M-1)x\right)\right]_0^\infty. \tag{3.36}$$

Case (ii): $0 \leq \alpha_y + f - (M-1)x < (N+1)k$. We obtain

$$\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right) = \begin{cases}\beta\left(\alpha_x - x_l - (M-1)x\right)x_l, & \text{if } x_l \geq \alpha_y + f - (M-1)x,\\ \beta\left(\frac{1}{N+1}\alpha_x + \frac{N}{N+1}\left(\triangle C - f\right) - \frac{M-1}{N+1}x - \frac{1}{N+1}x_l\right)x_l, & \text{otherwise.}\end{cases}$$

Hence, $\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right)$ is a piecewise smooth function in x_l over $\mathbb{R}_+$. The first and second derivatives are given by

$$\frac{\partial}{\partial x_l}\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right) = \begin{cases} \beta\left(\alpha_x - (M-1)x - 2x_l\right), & \text{if } x_l > \alpha_y + f - (M-1)x, \\ \frac{\beta}{N+1}\left(\alpha_x + N\left(\triangle C - f\right) - (M-1)x - 2x_l\right), & \text{otherwise,} \end{cases}$$

$$\begin{aligned} \frac{\partial^2}{\partial x_l^2}\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right) &= \begin{cases} -2\beta, & \text{if } x_l > \alpha_y + f - (M-1)x, \\ -\frac{2}{N+1}\beta, & \text{otherwise,} \end{cases} \\ &< 0. \end{aligned}$$

Moreover, we have

$$\begin{aligned} &\left.\frac{\partial^-}{\partial x_l}\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right)\right|_{x_l = \alpha_y + f - (M-1)x} \\ &= \beta\left(\frac{1}{N+1}\alpha_x + \frac{N}{N+1}\left(\triangle C + f\right) - \frac{M-1}{N+1}x - \frac{2}{N+1}\left(\alpha_y + f - (M-1)x\right)\right) \\ &= \beta\left(-\alpha_x + 2\left(\triangle C - f\right) + \frac{N}{N+1}\left(\alpha_y + f\right) + \frac{M-1}{N+1}x\right) \\ &\geq \beta\left(-\alpha_x + 2\left(\triangle C - f\right) + \left(M-1\right)x\right) \\ &= \left.\frac{\partial^+}{\partial x_l}\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right)\right|_{x_l = \alpha_y + f - (M-1)x}, \end{aligned}$$

where the inequality follows from (3.33b). Hence, $\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right)$ is concave in x_l over $\mathbb{R}_+$. Therefore, x_l is a solution to (3.32) if and only if it satisfies the first order optimality conditions (3.34) – (3.35). It is straightforward to show that there is a unique solution given by

$$x_l = \begin{cases} 0, & \text{if (3.38a) holds,} \\ \frac{1}{2}\left(\alpha_x + N\left(\triangle C - f\right) - \left(M-1\right)x\right), & \text{if (3.38b) holds,} \\ \alpha_x - \triangle C + f - (M-1)x, & \text{if (3.38c) holds,} \\ \frac{1}{2}\left(\alpha_x - (M-1)x\right), & \text{if (3.38d) holds,} \end{cases} \tag{3.37}$$

where the cases (3.38a) – (3.38d) are defined by:

$$\alpha_x + N\left(\triangle C - f\right) \leq (M-1)x, \tag{3.38a}$$

$$(M-1)x < \min\left(\alpha_x + N(\triangle C - f), \alpha_x - (N+2)(\triangle C - f)\right), \tag{3.38b}$$

$$\alpha_x - (N+2)(\triangle C - f) \leq (M-1)x \leq \alpha_x - 2(\triangle C - f), \tag{3.38c}$$

$$\alpha_x - 2(\triangle C - f) < (M-1)x. \tag{3.38d}$$

Case (iii): $(N+1)k \le \alpha_y + f - (M-1)x$. We obtain

$$\begin{aligned}
&\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right)\\
&= \begin{cases}
\beta\left(\alpha_x - x_l - (M-1)x\right)x_l, & \text{if } x_l \ge \alpha_y + f - (M-1)x,\\
\beta\left(\alpha_x - x_l - (M-1)x - Nk\right)x_l, & \text{if } x_l \le \alpha_y + f - (M-1)x - (N+1)k,\\
\beta\left(\frac{1}{N+1}\alpha_x + \frac{N}{N+1}\left(\triangle C - f\right) - \frac{M-1}{N+1}x - \frac{1}{N+1}x_l\right)x_l, & \text{otherwise.}
\end{cases}
\end{aligned}$$

Hence, $\phi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right)$ is a piecewise smooth function in x_l over $\mathbb{R}_+$. The first and second derivatives are given by

$$\begin{aligned}
&\frac{\partial}{\partial x_l}\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right)\\
&= \begin{cases}
\beta\left(\alpha_x - (M-1)x - 2x_l\right), & \text{if } x_l > \alpha_y + f - (M-1)x,\\
\beta\left(\alpha_x - (M-1)x - Nk - 2x_l\right), & \text{if } x_l < \alpha_y + f - (M-1)x - (N+1)k,\\
\frac{\beta}{N+1}\left(\alpha_x + N\left(\triangle C - f\right) - (M-1)x - 2x_l\right), & \text{otherwise,}
\end{cases}\\
&\frac{\partial^2}{\partial x_l^2}\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right)\\
&= \begin{cases}
-2\beta, & \text{if } x_l > \alpha_y + f - (M-1)x,\\
-2\beta, & \text{if } x_l < \alpha_y + f - (M-1)x - (N+1k,\\
-\frac{2}{N+1}\beta, & \text{otherwise,}
\end{cases}\\
&< 0.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\left.\frac{\partial^-}{\partial x_l}\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right)\right|_{x_l = \alpha_y + f - (M-1)x}\\
&= \beta\left(\frac{1}{N+1}\alpha_x + \frac{N}{N+1}\left(\triangle C - f\right) - \frac{M-1}{N+1}x - \frac{2}{N+1}\left(\alpha_y + f - (M-1)x\right)\right)\\
&= \beta\left(-\alpha_x + 2(\triangle C - f) + \frac{N}{N+1}(\alpha_x - \triangle C + f) + \frac{M-1}{N+1}x\right)\\
&> \beta\left(-\alpha_x + 2(\triangle C - f) + (M-1)x\right)\\
&= \left.\frac{\partial^+}{\partial x_l}\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right)\right|_{x_l = \alpha_y + f - (M-1)x}.
\end{aligned}$$

Hence, $\phi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right)$ is concave in x_l over $[\alpha_y + f - (M-1)x - (N+1)k, \infty)$. However, it is straightforward to check that $\phi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right)$ has a non-concave kink at $x_l = \alpha_y + f - (M-1)x - (N+1)k$, and therefore $\phi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right)$ is not concave in x_l over $\mathbb{R}_+$. Hence, solve the following sub-problems:

$$\sup_{x_l \in [0, \alpha_y + f - (M-1)x - (N+1)k]} \psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right), \tag{3.39}$$

and

$$\sup_{x_l \in [\alpha_y + f - (M-1)x - (N+1)k, \infty)} \psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right). \tag{3.40}$$

The solution of the sub-problem with the larger optimal value is the solution to (3.32). Using the first order optimality conditions, the unique solution to (3.39) is given by

$$x_l = \left[\frac{1}{2}\left(\alpha_x - (M-1)x - Nk\right)\right]_0^{\alpha_y + f - (M-1)x - (N+1)k} =: z_1,$$

and that to (3.40) is given by

$$x_l = \begin{cases} \alpha_y + f - (M-1)x - (N+1)k, & \text{if (3.41a) holds,} \\ \frac{1}{2}\left(\alpha_x + N(\triangle C - f) - (M-1)x\right), & \text{if (3.41b) holds,} \\ \alpha_y + f - (M-1)x, & \text{if (3.41c) holds,} \\ \frac{1}{2}\left(\alpha_x - (M-1)x\right), & \text{if (3.41d) holds,} \end{cases} =: z_2.$$

where the cases (3.41a) – (3.41d) are defined by:

$$(M-1)x \leq \alpha_x - (N+2)(\triangle C - f) - 2(N+1)k, \tag{3.41a}$$

$$\alpha_x - (N+2)(\triangle C - f) - 2(N+1)k < (M-1)x < \alpha_x - (N+2)(\triangle C - f), \tag{3.41b}$$

$$\alpha_x - (N+2)(\triangle C - f) \leq (M-1)x \leq \alpha_x - 2(\triangle C - f), \tag{3.41c}$$

$$\alpha_x - 2(\triangle C - f) < (M-1)x. \tag{3.41d}$$

Hence, the solution(s) to (3.32) are given by:

$$x_l = z_1, \quad \text{if } \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) > \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right), \tag{3.42a}$$

$$x_l = z_2, \quad \text{if } \phi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right) > \phi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right), \tag{3.42b}$$

$$x_l = z_1 \text{ or } z_2, \quad \text{if } \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) = \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right). \tag{3.42c}$$

Step 2: We solve for the symmetric leader productions that satisfy the condition that every leader is producing at its payoff maximizing quantity. We divide the analyses into three cases depending on the value of $\alpha_y + f - (M-1)x$.

Case (i): $\alpha_y + f - (M-1)x < 0$. Note that x_l is given by (3.36). Hence, the symmetric leader reactions are given by:

$$\begin{aligned}
& x = 0 \text{ and } \alpha_x < (M-1)x \text{ and (3.33a) holds} \\
& \text{or } x = \frac{1}{2}\left(\alpha_x - (M-1)x\right) \text{ and } \alpha_x \geq (M-1)x \text{ and (3.33a) holds} \\
\Longleftrightarrow\ & x = 0 \text{ and } \alpha_x < 0 \text{ and (3.33a) holds} \\
& \text{or } x = \frac{1}{M+1}\alpha_x \text{ and } \alpha_x \geq 0 \text{ and (3.33a) holds.}
\end{aligned}$$

Case (ii): $0 \leq \alpha_y + f - (M-1)x < (N+1)k$. Note that x_l is given by (3.37). Hence, the symmetric leader reactions are given by:

$$\begin{aligned}
& x = 0 \text{ and (3.38a) and (3.33b) holds} \\
& \text{or } x = \frac{1}{2}\left((N+1)\alpha_x - N(\alpha_y + f) - (M-1)x\right) \text{ and (3.38b) and (3.33b) holds} \\
& \text{or } x = \alpha_y + f - (M-1)x \text{ and (3.38c) and (3.33b) holds} \\
& \text{or } x = \frac{1}{2}\left(\alpha_x - (M-1)x\right) \text{ and (3.38d) and (3.33b) holds} \\
\Longleftrightarrow\ & x = 0 \text{ and } \frac{\alpha_x}{N} \leq f - \triangle C \text{ and (3.33b) holds} \\
& \text{or } x = \frac{1}{M+1}\left(\alpha_x + N(\triangle C - f)\right) \text{ and } -\frac{\alpha_x}{NM+M+1} < f - \triangle C < \frac{\alpha_x}{N} \text{ and (3.33b) holds} \\
& \text{or } x = \frac{1}{M}\left(\alpha_x - (\triangle C - f)\right) \text{ and } -\frac{\alpha_x}{M+1} \leq f - \triangle C \leq -\frac{\alpha_x}{NM+M+1} \text{ and (3.33b) holds} \\
& \text{or } x = \frac{1}{M+1}\alpha_x \text{ and } f - \triangle C < -\frac{\alpha_x}{M+1} \text{ and (3.33b) holds.}
\end{aligned}$$

Case (iii): $(N+1)k \leq \alpha_y + f - (M-1)x$. Note that x_l is described by (3.42). Hence, the symmetric leader reactions are given by

$$x = z_1 \text{ and (3.33c) holds and } \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) > \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right) \tag{3.43}$$

$$\text{or } x = z_2 \text{ and (3.33c) holds and } \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) < \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right) \tag{3.44}$$

$$\text{or } x = z_1 \text{ or } z_2 \text{ and (3.33c) holds and } \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) = \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right). \tag{3.45}$$

We analyze the cases $x = z_1$ and $x = z_2$ separately.

Suppose $x = z_1$ is a symmetric leader reaction. Since

$$
\begin{aligned}
&\frac{\partial^-}{\partial x_l}\psi_l(x_l; x\mathbf{1}, f\mathbf{1})\bigg|_{x_l=\alpha_y+f-(M-1)x-(N+1)k} \\
&= \beta\left(-\alpha_x + 2(\triangle C - f) + (M-1)x + (N+2)k\right) \\
&\le \frac{\beta}{N+1}\left(-\alpha_x + (N+2)(\triangle C - f) + (M-1)x + 2(N+1)k\right) \\
&= \frac{\partial^+}{\partial x_l}\psi_l(x_l; x\mathbf{1}, f\mathbf{1})\bigg|_{x_l=\alpha_y+f-(M-1)x-(N+1)k},
\end{aligned}
$$

we infer that $x < \alpha_y + f - (M-1)x - (N+1)k$. Hence, we obtain

$$
\begin{aligned}
& x = z_1 \\
\iff & x = 0 \text{ and (3.33c) holds and } \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) \ge \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right) \text{ and } \alpha_x - (M-1)x - Nk \le 0 \\
& \text{or } x = \frac{1}{2}\left(\alpha_x - (M-1)x - Nk\right) \text{ and (3.33c) holds and } \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) \ge \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right) \\
& \quad \text{and } 0 < \frac{1}{2}\left(\alpha_x - (M-1)x - Nk\right) < \alpha_y + f - (M-1)x - (N+1)k \\
\iff & x = 0 \text{ and (3.33c) holds and } \alpha_x - Nk \le 0 \\
& \text{or } x = \frac{1}{M+1}\left(\alpha_x - Nk\right) \text{ and (3.33c) holds and } \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) \ge \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right) \\
& \quad \text{and } \alpha_x - Nk > 0 \text{ and } f - \triangle C > -\frac{1}{M+1}\left(\alpha_x - Nk\right) + k
\end{aligned}
$$

The second equivalence follows from solving for x in the equations, and the fact that in the case $x = 0$, the inequalities (3.33c) and $\alpha_x - Nk \le 0 \implies \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) \ge \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right)$.

Suppose $x = z_2$. Then, using the same arguments in (3.43), we infer that $x < \alpha_y + f - (M -$

$1)x - (N+1)k$. Hence, we obtain

$$
\begin{aligned}
& x = z_2 \\
\iff & x = \frac{1}{2}\left(\alpha_x + N(\triangle C - f) - (M-1)x\right) \text{ and (3.33c) and (3.41b) holds} \\
& \quad \text{and } \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) \leq \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right) \\
& \text{or } x = \alpha_x + (f - \triangle C) - (M-1)x \text{ and (3.33c) and (3.41c) holds} \\
& \quad \text{and } \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) \leq \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right) \\
& \text{or } x = \frac{1}{2}\left(\alpha_x - (M-1)x\right) \text{ and (3.33c) and (3.41d) holds} \\
& \quad \text{and } \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) \leq \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right) \\
\iff & x = \frac{1}{M+1}\left(\alpha_x + N(\triangle C - f)\right) \text{ and (3.33c) holds and } \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) \leq \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right) \\
& \quad \text{and } -\frac{1}{NM+M+1}\alpha_x < f - \triangle C < \frac{1}{NM+M+1}\left(-\alpha_x + (M+1)(N+1)k\right) \\
& \text{or } x = \frac{1}{M}\left(\alpha_x + (f - \triangle C)\right) \text{ and (3.33c) holds and } \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) \leq \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right) \\
& \quad \text{and } -\frac{1}{M+1}\alpha_x \leq f - \triangle C \leq -\frac{1}{NM+M+1}\alpha_x \\
& \text{or } x = \frac{1}{M+1}\alpha_x \text{ and (3.33c) holds and and } f - \triangle C < -\frac{1}{M+1}\alpha_x
\end{aligned}
$$

The second equivalence follows from solving for x in the equations, and the fact that in the case $x = \frac{1}{M+1}\alpha_x$, the inequalities (3.33c) and $f - \triangle C < -\frac{1}{M+1}\alpha_x \implies \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) \leq \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right)$.

Step 3: We explain how the solutions obtained in step 2 is equivalent to X. Observe that step 2 obtains five cases for x:

$$
\begin{aligned}
x &= \frac{1}{M+1}\alpha_x, \\
x &= \frac{1}{M}\left(\alpha_x - (\triangle C - f)\right), \\
x &= \frac{1}{M+1}\left(\alpha_x + N(\triangle C - f)\right), \\
x &= \frac{1}{M+1}\left(\alpha_x - Nk\right), \\
x &= 0.
\end{aligned}
$$

We analyze each case separately.

Case (i): $x = \frac{1}{M+1}\alpha_x$. This case is characterized by

$$\begin{aligned}
&\alpha_x \geq 0 \text{ and (3.33a) holds} \\
&\text{or } f - \triangle C < -\frac{\alpha_x}{M+1} \text{ and (3.33b) holds} \\
&\text{or } f - \triangle C < -\frac{\alpha_x}{M+1} \text{ and (3.33c) holds} \\
\iff &\alpha_x \geq 0 \text{ and } f - \triangle C < -\frac{\alpha_x}{M+1}.
\end{aligned}$$

The equivalence is due to the following facts. First, (3.33b) $\implies$ $\alpha_x \geq 0$ and (3.33c) $\implies$ $\alpha_x \geq 0$. Second, (3.33a) and $\alpha_x \geq 0$ $\implies$ $f - \triangle C < -\frac{1}{M+1}\alpha_x$. Third, (3.33a), (3.33b), (3.33c) $\implies$ True.

Case (ii): $x = \frac{1}{M}\left(\alpha_x - (\triangle C - f)\right)$. This case is characterized by

$$\begin{aligned}
&-\frac{\alpha_x}{M+1} \leq f - \triangle C \leq -\frac{\alpha_x}{NM+M+1} \text{ and (3.33b) holds} \\
&\text{or } -\frac{\alpha_x}{M+1} \leq f - \triangle C \leq -\frac{\alpha_x}{NM+M+1} \\
&\quad \text{and } \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) \leq \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right) \text{ and (3.33c) holds} \\
\iff &-\frac{\alpha_x}{M+1} \leq f - \triangle C \leq \min\left(-\frac{\alpha_x}{NM+M+1}, -\alpha_x + M(N+1)k\right) \\
&\text{or } -\frac{\alpha_x}{M+1} \leq f - \triangle C \leq -\frac{\alpha_x}{NM+M+1} \\
&\quad \text{and } f - \triangle C \leq \eta_4 \text{ and } f - \triangle C > -\alpha_x + M(N+1)k \\
\iff &-\frac{\alpha_x}{M+1} \leq f - \triangle C \leq \min\left(-\frac{\alpha_x}{MN+M+1}, \eta_4\right).
\end{aligned}$$

The first equivalence is due to the following facts. First, (3.33b) $\iff$ $-\alpha_x < f - \triangle C \leq -\alpha_x + M(N+1)k$. Second, $\psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) \leq \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right)$ $\iff$ $f - \triangle C \leq \eta_4$. Third, (3.33c) $\iff$ $f - \triangle C > -\alpha_x + M(N+1)k$. The second equivalence is due to the fact that $-\frac{\alpha_x}{MN+M+1} \leq -\alpha_x + M(N+1)k \implies \eta_4 > -\frac{\alpha_x}{MN+M+1}$.

Case (iii): $x = \frac{1}{M+1}\left(\alpha_x + N(\triangle C - f)\right)$. This case is characterized by

$$\begin{aligned}
& -\frac{\alpha_x}{NM+M+1} < f - \triangle C < \frac{\alpha_x}{N} \text{ and (3.33b) holds} \\
& \text{or } -\frac{\alpha_x}{NM+M+1} < f - \triangle C < \frac{1}{NM+M+1}\left(-\alpha_x + (M+1)(N+1)k\right) \\
& \quad \text{and } \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) \leq \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right) \text{ and (3.33c) holds} \\
\Longleftrightarrow\; & -\frac{\alpha_x}{NM+M+1} < f - \triangle C < \frac{\alpha_x}{N} \text{ and } f - \triangle C \leq \frac{-2\alpha_x + (M+1)(N+1)k}{M+1+N(M-1)} \\
& \text{or } -\frac{\alpha_x}{NM+M+1} < f - \triangle C < \frac{1}{NM+M+1}\left(-\alpha_x + (M+1)(N+1)k\right) \\
& \quad \text{and } f - \triangle C \leq \eta_3 \text{ and } f - \triangle C > \frac{-2\alpha_x + (M+1)(N+1)k}{M+1+N(M-1)} \\
\Longleftrightarrow\; & \alpha_x < Nk \text{ and } -\frac{\alpha_x}{NM+M+1} < f - \triangle C < \frac{\alpha_x}{N} \\
& \text{or } \alpha_x \geq Nk \text{ and } -\frac{\alpha_x}{NM+M+1} < f - \triangle C \leq \eta_3.
\end{aligned}$$

The first equivalence is due to the following facts. First, (3.33b) $\Longleftrightarrow -\frac{2\alpha_x}{M+1+N(M-1)} < f - \triangle C \leq \frac{-2\alpha_x + (M+1)(N+1)k}{M+1+N(M-1)}$. Second, (3.33c) $\Longleftrightarrow f - \triangle C > \frac{-2\alpha_x+(M+1)(N+1)k}{M+1+N(M-1)}$. Third, $\psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) \leq \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right) \Longleftrightarrow f - \triangle C \leq \eta_3$. The second equivalence is due to the following facts. First, $\frac{\alpha_x}{N} \leq \frac{-2\alpha_x+(M+1)(N+1)k}{M+1+N(M-1)} \Longleftrightarrow \alpha_x \leq Nk$. Second, $\alpha_x \geq Nk \Longrightarrow \eta_3 < \frac{1}{NM+M+1}\left(-\alpha_x + (M+1)(N+1)k\right)$.

Case (iv): $x = \frac{1}{M+1}\left(\alpha_x - Nk\right)$. This case is characterized by

$$\begin{aligned}
& \alpha_x - Nk > 0 \text{ and } f - \triangle C > -\frac{1}{M+1}\left(\alpha_x - Nk\right) + k \\
& \quad \text{and } \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) \geq \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right) \text{ and (3.33c) holds} \\
\Longleftrightarrow\; & \alpha_x - Nk > 0 \text{ and } f - \triangle C > -\frac{1}{M+1}\left(\alpha_x - Nk\right) + k \text{ and } \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) \geq \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right) \\
\Longleftrightarrow\; & Nk < \alpha_x < \left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right) Nk \text{ and } f - \triangle C \geq \eta_1 \\
& \text{or } \left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right) Nk \leq \alpha_x \text{ and } f - \triangle C \geq \eta_2.
\end{aligned}$$

The first equivalence is due to the fact that (3.33c) $\Longrightarrow f - \triangle C > -\frac{1}{M+1}(\alpha_x - Nk) + k$. The second equivalence is due to the fact that $\psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) \geq \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right) \Longleftrightarrow Nk < \alpha_x < \left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right) Nk$ and $f - \triangle C \geq \eta_1$ or $\left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right) Nk \leq \alpha_x$ and $f - \triangle C \geq \eta_2$.

Case (v): $x = 0$. This case is characterized by

$$
\begin{aligned}
& \alpha_x < 0 \text{ and (3.33a)} \\
& \text{or } \frac{\alpha_x}{N} \le f - \triangle C \text{ and (3.33b)} \\
& \text{or } \alpha_x - Nk \le 0 \text{ and (3.33c)} \\
\Longleftrightarrow\ & \alpha_x < 0 \\
& \text{or } \alpha_x \ge 0 \text{ and } \alpha_x + N(\triangle C - f) \le 0 \text{ and } 0 \le \alpha_x + (f - \triangle C) < (N+1)k \\
& \text{or } \alpha_x \ge 0 \text{ and } \alpha_x - Nk \le 0 \text{ and } (N+1)k \le \alpha_x + (f - \triangle C).
\end{aligned}
$$

The equivalence is due to the following facts. First, (3.33b) and $\alpha_x < 0 \implies \alpha_x + N(\triangle C - f) < 0$. Second, (3.33c) and $\alpha_x < 0 \implies \alpha_x - Nk < 0$. □

3.D Forward Market Equilibrium

Proposition 3.5. *Suppose $\alpha_x > 0$. Let $Q \subseteq \mathbb{R} \times \mathbb{R}_+$ denote the set of all symmetric Nash equilibria, i.e., $(f, x) \in Q$ if $(f\mathbf{1}, x\mathbf{1})$ is a Nash equilibrium of the forward market. Let:*

$$
Q_1 := \left\{ (f,x) \in \mathbb{R} \times \mathbb{R}_+ \;\middle|\; \begin{array}{l} x = \frac{1}{M+1}\alpha_x \\ f < \triangle C - \frac{1}{M+1}\alpha_x \end{array} \right\},
$$

$$
Q_2 := \left\{ (f,x) \in \mathbb{R} \times \mathbb{R}_+ \;\middle|\; \begin{array}{l} x = \frac{1}{M}\left(\alpha_x - (\triangle C - f)\right) \\ \max\left(0, \triangle C - \frac{\alpha_x}{M+1}\right) \le f \le \triangle C + \min\left(-\frac{\alpha_x}{MN+M+1}, \eta_4\right) \end{array} \right\},
$$

$$
Q_3 := \left\{ (f,x) \in \mathbb{R} \times \mathbb{R}_+ \;\middle|\; \begin{array}{l} x = \frac{N+1}{N^2+MN+M+1}\left(\alpha_x + N^2 \triangle C\right) \\ f = \frac{N-1}{N^2+MN+M+1}\left(\alpha_x - (MN+M+1)\triangle C\right) \end{array} \right\},
$$

$$
Q_4 := \left\{ (f,x) \in \mathbb{R} \times \mathbb{R}_+ \;\middle|\; \begin{array}{l} x = \frac{1}{M+1}\left(\alpha_x - Nk\right) \\ f \ge \triangle C + \begin{cases} \eta_1, & \textit{if } Nk < \alpha_x \le \left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right) Nk, \\ \eta_2, & \textit{if } \left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right) Nk < \alpha_x. \end{cases} \end{array} \right\}.
$$

where η_1, η_2, η_4 are as defined in Proposition 3.4. Then,

$$
Q = \left\{ (f,x) \in \mathbb{R} \times \mathbb{R}_+ \;\middle|\; \begin{array}{l} (f,x) \in Q_1 \textit{ if } \alpha_x \le (M+1)\triangle C, \\ \textit{or } (f,x) \in Q_2 \textit{ if } \alpha_x \le \min\left((MN+M+1)\triangle C, \zeta_1\right), \\ \textit{or } (f,x) \in Q_3 \textit{ if } (MN+M+1)\triangle C < \alpha_x \le \zeta_2, \\ \textit{or } (f,x) \in Q_4 \textit{ if } (M+1)(\triangle C + k) + Nk \le \alpha_x. \end{array} \right\},
$$

where

$$\zeta_1 = MNk + (M+1)\,\triangle C + 2M\sqrt{Nk\triangle C}, \tag{3.46}$$

$$\zeta_2 = (MN+M+1)\,\triangle C + \frac{N^2+NM+M+1}{N(N+1)+2(1-\sqrt{N+1})}\left(Nk - (\sqrt{N+1}-1)^2\triangle C\right). \tag{3.47}$$

Moreover, for each $(f,x) \in Q$*,*

$$\begin{aligned} y_j(f\mathbf{1}, x\mathbf{1}) = 0 &\iff (f,x) \in Q_1 \cup Q_2, \\ 0 < y_j(f\mathbf{1}, x\mathbf{1}) < k &\iff (f,x) \in Q_3, \\ y_j(f\mathbf{1}, x\mathbf{1}) = k &\iff (f,x) \in Q_4. \end{aligned}$$

Proof. The symmetric equilibria are given by the intersection of the follower and leader reactions obtained in Propositions 3.3 and 3.4. We divide the analyses into three separate cases depending on the value of the follower productions $y_j(f\mathbf{1}, x\mathbf{1})$.

Case (i): $y_j(f\mathbf{1}, x\mathbf{1}) = 0$. Using Propositions 3.3 and 3.4, we infer that (f,x) is a symmetric equilibrium with $y_j(f\mathbf{1}, x\mathbf{1}) = 0$ if and only if

$$f \le -\left(\alpha_x - \triangle C - Mx\right), \tag{3.48a}$$

$$0 \ge \alpha_x - \triangle C - Mx, \tag{3.48b}$$

$$x = \begin{cases} \frac{1}{M+1}\left[\alpha_x\right]_0^\infty, & \text{if } f - \triangle C < -\frac{\alpha_x}{M+1}, \\ \frac{1}{M}\left(\alpha_x - \triangle C + f\right), & \text{if } -\frac{\alpha_x}{M+1} \le f - \triangle C \le \min\left(-\frac{\alpha_x}{MN+M+1}, \eta_4\right). \end{cases} \tag{3.48c}$$

Suppose $x = \frac{1}{M+1}\left[\alpha_x\right]_0^\infty$. Since $\alpha_x > 0$, we infer that $x = \frac{1}{M+1}\alpha_x$. Substituting into (3.48a) and (3.48b) yields

$$\begin{aligned} \text{(3.48a)} &\iff f < \triangle C - \frac{\alpha_x}{M+1}, \\ \text{(3.48b)} &\iff \alpha_x \le (M+1)\triangle C. \end{aligned}$$

The above inequalities, together with (3.48c), imply that (f,x) satisfies (3.48) with $x = \frac{1}{M+1}\alpha_x$, if and only if $(f,x) \in Q_1$ and $\alpha_x \le (M+1)\triangle C$.

Suppose $x = \frac{1}{M}\left(\alpha_x - \triangle C + f\right)$. Substituting into (3.48a) and (3.48b) yields

$$\begin{aligned} \text{(3.48a)} &\iff f \le f \iff \text{True}, \\ \text{(3.48b)} &\iff f \ge 0. \end{aligned}$$

Hence, there exists (f, x) that satisfies (3.48) with $x = \frac{1}{M}(\alpha_x - \triangle C + f)$ if and only if

$$\begin{aligned}
&\left[\max\left(0, \triangle C - \frac{\alpha_x}{M+1}\right), \triangle C + \min\left(-\frac{\alpha_x}{MN+M+1}, \eta_4\right)\right] \neq \varnothing \\
\iff &0 \leq \triangle C + \min\left(-\frac{\alpha_x}{MN+M+1}, \eta_4\right) \\
\iff &\alpha_x \leq (MN+M+1)\,\triangle C \text{ and } 0 \leq \triangle C + \eta_4 \\
\iff &\alpha_x \leq (MN+M+1)\,\triangle C \text{ and } \alpha_x \leq \zeta_1.
\end{aligned}$$

Hence, (f, x) satisfies (3.48) with $x = \frac{1}{M}(\alpha_x - \triangle C + f)$, if and only if $(f, x) \in Q_2$ and $\alpha_x \leq \min\left((MN+M+1)\triangle C, \zeta_1\right)$.

Case (ii): $0 \leq y_j(f\mathbf{1}, x\mathbf{1}) \leq k$. Using Propositions 3.3 and 3.4, we infer that (f, x) is a symmetric equilibrium if and only if

$$f = \frac{N-1}{N^2+1}\left(\alpha_x - \triangle C - Mx\right), \tag{3.49a}$$

$$0 \leq \alpha_x - \triangle C - Mx \leq \xi_1, \tag{3.49b}$$

$$x = \frac{1}{M+1}\left[\alpha_x + N(\triangle C - f)\right]_0^\infty, \tag{3.49c}$$

$$-\frac{\alpha_x}{MN+M+1} < f - \triangle C \leq \max\left(k - (\alpha_x - Nk), \eta_3\right). \tag{3.49d}$$

We show that $x > 0$. Suppose otherwise. Substituting into (3.49a) implies that $f = \frac{N-1}{N^2+1}(\alpha_x - \triangle C)$. Substituting further into (3.49c) yields

$$\alpha_x + N\left(\triangle C - \frac{N-1}{N^2+1}(\alpha_x - \triangle C)\right) \leq 0 \iff \alpha_x + N\triangle C < 0,$$

which is a contradiction since $\alpha_x > 0$, $\triangle C \geq 0$, and $N \geq 2$. Henceforth, we assume that $x > 0$. Solving (3.49a) and (3.49c) gives

$$\begin{aligned}
f &= \frac{N-1}{N^2+MN+M+1}\left(\alpha_x - (MN+M+1)\triangle C\right), \\
x &= \frac{N+1}{N^2+MN+M+1}\left(\alpha_x + N^2\triangle C\right).
\end{aligned}$$

Substituting for x yields

$$\text{(3.49b)} \iff (MN+M+1)\triangle C \leq \alpha_x \leq (MN+M+1)\triangle C + \frac{(N^2+MN+M+1)(N-1)}{N^2-2\sqrt{N}+1}k.$$

Substituting for f yields

$$(3.49\text{d}) \iff (MN+M+1)\triangle C < \alpha_x \text{ and } \alpha_x \leq \begin{cases} \frac{N(M+1)}{M+N}\triangle C + \left(N + \frac{M+1}{M+N}\right)k, & \text{if } \alpha_x \leq Nk, \\ \zeta_2, & \text{if } \alpha_x > Nk, \end{cases}$$

$$\iff (MN+M+1)\triangle C < \alpha_x \text{ and } \alpha_x \leq \begin{cases} Nk, & \text{if } \alpha_x \leq Nk, \\ \zeta_2, & \text{if } \alpha_x > Nk, \end{cases}$$

$$\iff (MN+M+1)\triangle C < \alpha_x \leq \zeta_2.$$

The first equivalence is due to the fact that $k-(\alpha_x - Nk) \geq \eta_3 \iff \alpha_x \leq Nk$. The second equivalence is due to the fact that $\triangle C \geq 0$ and $k > 0$. Next, using the fact that $N \geq 2, \triangle C \geq 0, k > 0$, we obtain

$$\zeta_2 < (MN+M+1)\triangle C + \frac{(N^2+MN+M+1)(N-1)}{N^2-2\sqrt{N}+1}k,$$

from which it follows that (f,x) satisfies (3.49) if and only if $(f,x) \in Q_3$ and $(MN+M+1)\triangle C \leq \alpha_x \leq \zeta_2$.

Case (iii): $y_j(f\mathbf{1}, x\mathbf{1}) = k$. From Propositions 3.3 and 3.4, we infer that (f,x) is a symmetric equilibrium if and only if

$$f \geq -(\alpha_x - \triangle C - Mx) + (N+1)k, \tag{3.50a}$$

$$(N+1)k \leq \alpha_x - \triangle C - Mx, \tag{3.50b}$$

$$x = \frac{1}{M+1}\left[\alpha_x - Nk\right]_0^\infty, \tag{3.50c}$$

$$f - \triangle C \geq \begin{cases} k-(\alpha_x - Nk), & \text{if } \alpha_x < Nk, \\ \eta_2, & \text{if } \alpha_x \geq \left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right)Nk, \\ \eta_1, & \text{otherwise.} \end{cases} \tag{3.50d}$$

We divide the analyses into three cases depending on the value of α_x.

Suppose $0 < \alpha_x \leq Nk$. Then, (3.50c) implies $x = 0$. However, substituting into (3.50b) implies that $\alpha_x - Nk \geq k + \triangle C > 0$ which is a contradiction. Hence, there does not exist an equilibrium such that $0 < \alpha_x \leq Nk$.

Suppose $Nk < \alpha_x \leq \left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right)Nk$. Then, (3.50c) implies $x = \frac{1}{M+1}(\alpha_x - Nk)$. Substituting for x yields

$$\begin{aligned}
(3.50\text{a}) &\iff f \geq \triangle C - \frac{1}{M+1}\left(\alpha_x - Nk\right) + k, \\
(3.50\text{b}) &\iff \triangle C - \frac{1}{M+1}\left(\alpha_x - Nk\right) + k \leq 0.
\end{aligned}$$

From (3.50d), we infer that $f \geq \triangle C + \eta_1$. Since $N \geq 2 \implies \eta_1 \geq k - \frac{\alpha_x - Nk}{M+1}$, it follows that the symmetric equilibria are characterized by

$$x = \frac{1}{M+1}(\alpha_x - Nk) \text{ and } f \geq \triangle C + \eta_1 \text{ and } \triangle C - \frac{1}{M+1}(\alpha_x - Nk) + k \leq 0. \tag{3.51}$$

Suppose $\left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right) Nk < \alpha_x$. Then, we again have

$$\begin{aligned}
(3.50\text{a}) &\iff f \geq \triangle C - \frac{1}{M+1}\left(\alpha_x - Nk\right) + k, \\
(3.50\text{b}) &\iff \triangle C - \frac{1}{M+1}\left(\alpha_x - Nk\right) + k \leq 0.
\end{aligned}$$

From (3.50d), we infer that $f \geq \triangle C + \eta_2$. Since $N \geq 2 \implies \eta_2 \geq k - \frac{\alpha_x - Nk}{M+1}$, it follows that the symmetric equilibria are characterized by

$$x = \frac{1}{M+1}(\alpha_x - Nk) \text{ and } f \geq \triangle C + \eta_2 \text{ and } \triangle C - \frac{1}{M+1}(\alpha_x - Nk) + k \leq 0. \tag{3.52}$$

By combining the characterizations in (3.51) and (3.52), we infer that (f, x) satisfies (3.50) if and only if $(f, x) \in Q_4$ and $(M+1)(\triangle C + k) + Nk \leq \alpha_x$. $\square$

Proposition 3.6. *Suppose followers' forward positions* $\mathbf{f} = \mathbf{0}$. *Let* $X \subseteq \mathbb{R}_+$ *denote the set of symmetric leader reactions, i.e., for each* $x \in X$ *and* $i \in M$*,*

$$\psi_i(x; x\mathbf{1}, \mathbf{0}) \geq \psi_i(\bar{x}; x\mathbf{1}, \mathbf{0}), \quad \forall \bar{x} \in \mathbb{R}_+.$$

Let:

$$\begin{aligned}
x_1 &= \frac{1}{M+1}\left[\alpha_x\right]_0^\infty, \\
x_2 &= \frac{1}{M}\left(\alpha_x - \triangle C\right), \\
x_3 &= \frac{1}{M+1}\left[\alpha_x + N\triangle C\right]_0^\infty, \\
x_4 &= \frac{1}{M+1}\left[\alpha_x - Nk\right]_0^\infty.
\end{aligned}$$

Then,

$$X = \left\{ x \in \mathbb{R}_+ \;\middle|\; \begin{array}{l} x = x_1 \textit{ if } \alpha_x < (M+1)\triangle C, \\ \textit{or } x = x_2 \textit{ if } (M+1)\triangle C \le \alpha_x \le \min\left((MN+M+1)\triangle C, \zeta_1\right), \\ \textit{or } x = x_3 \textit{ if } (MN+M+1)\triangle C < \alpha_x \le \zeta_2, \\ \textit{or } x = x_4 \textit{ if } \alpha_x \ge \begin{cases} Nk + \frac{N(M+1)}{2(\sqrt{N+1}-1)}(\triangle C + k), & \text{if } (\sqrt{N+1}-1)^2\triangle C < Nk, \\ Nk + (M+1)\left(\triangle C + \sqrt{Nk\triangle C}\right), & \text{otherwise.} \end{cases} \end{array} \right\}.$$

where

$$\begin{aligned} \zeta_1 &:= MNk + (M+1)\triangle C + 2M\sqrt{Nk\triangle C}, \\ \zeta_2 &:= (MN+M+1)\triangle C + \frac{(M+1)\sqrt{N+1}}{2(\sqrt{N+1}-1)}\left(Nk - \left(\sqrt{N+1}-1\right)^2 \triangle C\right). \end{aligned}$$

Moreover, for each $x \in X$,

$$\begin{aligned} y_j(\mathbf{0}, x\mathbf{1}) = 0 &\iff x = x_1 \textit{ or } x_2, \\ 0 < y_j(\mathbf{0}, x\mathbf{1}) < k &\iff x = x_3, \\ y_j(\mathbf{0}, x\mathbf{1}) = k &\iff x = x_4. \end{aligned}$$

Proof. The result is obtained by substituting $f = 0$ into Proposition 3.4 and simplifying the inequalities in X. For the case of $x = x_1$, we have

$$f - \triangle C < -\frac{\alpha_x}{M+1} \iff \alpha_x < (M+1)\triangle C.$$

For the case of $x = x_2$, we have

$$\begin{aligned} & -\frac{\alpha_x}{M+1} \le f - \triangle C \le \min\left(-\frac{\alpha_x}{MN+M+1}, \eta_4\right) \\ \iff & -\frac{\alpha_x}{M+1} \le -\triangle C \text{ and } -\triangle C \le -\frac{\alpha_x}{MN+M+1} \text{ and } -\triangle C \le \eta_4 \\ \iff & (M+1)\triangle C \le \alpha_x \text{ and } \alpha_x \le (MN+M+1)\triangle C \text{ and } \alpha_x \le \zeta_1 \\ \iff & (M+1)\triangle C \le \alpha_x \le \min\left((MN+M+1)\triangle C, \zeta_1\right), \end{aligned}$$

where the second equivalence is due to the fact that $-\triangle C \le \eta_4 \iff \alpha_x \le \zeta_1$. For the case of

$x = x_3$, we have

$$
\begin{aligned}
&-\frac{\alpha_x}{MN+M+1} < f - \triangle C \leq \max\left(\eta_3, k - (\alpha_x - Nk)\right) \\
\iff &-\frac{\alpha_x}{MN+M+1} < -\triangle C \leq \begin{cases} k - (\alpha_x - Nk), & \text{if } \alpha_x \leq Nk, \\ \eta_3, & \text{if } \alpha_x > Nk, \end{cases} \\
\iff &(MN+M+1)\triangle C < \alpha_x \leq \begin{cases} Nk, & \text{if } \alpha_x \leq Nk, \\ \zeta_2, & \text{if } \alpha_x > Nk, \end{cases} \\
\iff &(MN+M+1)\triangle C < \alpha_x \leq \zeta_2.
\end{aligned}
$$

The first equivalence is due to the fact that $k - (\alpha_x - Nk) \geq \eta_3 \iff \alpha_x \leq Nk$. The second equivalence is due to the fact that $\triangle C \geq 0$ and $k > 0$. For the case of $x = x_4$, we have

$$
f - \triangle C \geq \begin{cases} k - (\alpha_x - Nk), & \text{if } \alpha_x < Nk, \\ \eta_2, & \text{if } \alpha_x \geq \left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right) Nk, \\ \eta_1, & \text{otherwise.} \end{cases}
$$

Suppose $\alpha_x < Nk$. Then, the above inequality implies that $-\triangle C \geq k - (\alpha_x - Nk) \implies \alpha_x \geq \triangle C + (N+1)k > Nk$, which is a contradiction. Henceforth, we assume that $\alpha_x \geq Nk$, and obtain

$$
\begin{aligned}
&-\triangle C \geq \begin{cases} \eta_2, & \text{if } \alpha_x \geq \left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right) Nk, \\ \eta_1, & \text{if } Nk \leq \alpha_x < \left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right) Nk, \end{cases} \\
\iff &\alpha_x \geq \begin{cases} Nk + (M+1)\left(\triangle C + \sqrt{Nk\triangle C}\right), & \text{if } \alpha_x \geq \left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right) Nk, \\ Nk + \frac{N(M+1)}{2(\sqrt{N+1}-1)}(\triangle C + k), & \text{if } Nk \leq \alpha_x < \left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right) Nk, \end{cases} \\
\iff &\alpha_x \geq \begin{cases} Nk + \frac{N(M+1)}{2(\sqrt{N+1}-1)}(\triangle C + k), & \text{if } (\sqrt{N+1}-1)^2 \triangle C < Nk, \\ Nk + (M+1)\left(\triangle C + \sqrt{Nk\triangle C}\right), & \text{otherwise.} \end{cases}
\end{aligned}
$$

The last equivalence is due to the fact that $\frac{N(M+1)}{2(\sqrt{N+1}-1)}(\triangle C + k) < \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2} Nk \iff (\sqrt{N+1} - 1)^2 \triangle C < Nk$. $\square$

3.E Proofs of Structural Results

Proof of Lemma 3.1. From Proposition 3.3, note that $0 < y_j(f\mathbf{1}, x\mathbf{1}) < k$ if and only if $f = \frac{N-1}{N^2+1}(\alpha_x - \triangle C - Mx)$. Substituting into the follower productions from Proposition 3.2 gives

$$y_j(f\mathbf{1}, x\mathbf{1}) = \frac{N}{N^2+1}\xi,$$

which is strictly increasing in ξ. Since $\xi \leq \frac{(N^2+1)(N-1)}{N^2-2\sqrt{N}+1}k$, we obtain

$$\begin{aligned}
\bar{y} &= \left(1 - \frac{N - 2\sqrt{N} + 1}{N^2 - 2\sqrt{N} + 1}\right)k \\
&\geq \left(1 - \frac{N+1}{N^2 - 2\sqrt{N}}\right)k \\
&\geq \left(1 - \frac{1}{N}\frac{N+1}{N}\frac{N^2}{N^2 - 2\sqrt{N}}\right)k,
\end{aligned}$$

which gives the first claim.

Next, from Proposition 3.3, $\underline{\xi}$ and $\bar{\xi}$ are given by

$$\begin{aligned}
\underline{\xi} &= (N+1)k, \\
\bar{\xi} &= \frac{(N^2+1)(N-1)}{N^2 - 2\sqrt{N} + 1}k,
\end{aligned}$$

from which we obtain

$$\begin{aligned}
\frac{\underline{\xi} - \bar{\xi}}{\underline{\xi}} &= \frac{2(N^2 - N\sqrt{N} - \sqrt{N} + 1)}{(N^2 - 2\sqrt{N} + 1)(N+1)} \\
&\leq \frac{2(N^2+1)}{N(N^2 - 2\sqrt{N})} \\
&= \frac{2}{N}\frac{N^2+1}{N^2}\frac{N^2}{N^2 - 2\sqrt{N}},
\end{aligned}$$

which gives the rest of the second claim. $\square$

Proof of Lemma 3.2. From Proposition 3.4, $\underline{f}$ and $\bar{f}$ are given by

$$\begin{aligned}
\underline{f} &= \triangle C - \frac{\alpha_x}{M+1}, \\
\bar{f} &= \triangle C + \min\left(-\frac{\alpha_x}{MN + M + 1}, \eta_4\right).
\end{aligned}$$

The first claim follows from Proposition 3.4. Next,

$$\begin{aligned}\bar{f} - \underline{f} &= \frac{\alpha_x}{M+1} + \min\left(-\frac{\alpha_x}{MN+M+1}, \eta_4\right) \\ &\le \frac{\alpha_x}{M+1} - \frac{\alpha_x}{MN+M+1} \\ &= \frac{MN\alpha_x}{(M+1)(MN+M+1)} \\ &\le \frac{MN\alpha_x}{M^2(N+1)} \\ &= \frac{\alpha_x}{M}\frac{N}{N+1},\end{aligned}$$

which gives the second claim. □

Proof of Lemma 3.3. From Proposition 3.4, note that $0 < y_j(f\mathbf{1}, x\mathbf{1}) < k \implies x = x_3$. Substituting into the follower productions from Proposition 3.2 gives

$$y_j(f\mathbf{1}, x\mathbf{1}) = \left[\frac{1}{N+1}\left(\alpha_x + (f - \triangle C) - \frac{M}{M+1}(\alpha_x + N(\triangle C - f))\right)\right]_0^k,$$

which is strictly increasing in f. Note that $x = x_3$ is a reaction if and only if

$$-\frac{\alpha_x}{MN+M+1} < f - \triangle C \le \max\left(\eta_3, k - (\alpha_x - Nk)\right) \iff -\frac{\alpha_x}{MN+M+1} < f - \triangle C \le \eta_3,$$

where we used the fact that $\alpha_x > Nk \implies \eta_3 \ge k - (\alpha_x - Nk)$. Since

$$\alpha_x \le Nk\left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2} + \frac{(M-1)\sqrt{N+1}}{\sqrt{N+1}-1}\right) \implies -\frac{\alpha_x}{NM+M+1} \le \eta_3,$$

we infer the case for $\bar{y} = 0$. Otherwise, substituting for η_3 gives

$$\begin{aligned}\bar{y} &= y_j\left((\triangle C + \eta_3)\mathbf{1}, x_3\mathbf{1}\right) \\ &= k + \frac{\alpha_x - Nk}{N+1}\frac{1}{M+1}\left(\frac{2(M+1)(N+1) - (M+1)(N+2)\sqrt{N+1}}{N(2 + (M-1)\sqrt{N+1}}\right) \\ &\ge k - \frac{\alpha_x - Nk}{N}\left(\frac{N+2}{(N+1)(M-1)}\right),\end{aligned}$$

from which we obtain the first claim. From Proposition 3.4, we infer that $\bar{f} = \eta_3$ and $\underline{f} = \eta_1$ when

$\alpha_x \leq Nk\left(1+\frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right)$. Hence, we obtain

$$\begin{aligned}\bar{f}-\underline{f} &= \frac{\alpha_x - Nk}{N}\left(2(\sqrt{N+1}-1\right)\left(\frac{2+(M-1)\sqrt{N+1}-(M+1)}{(M+1)(2+(M-1)\sqrt{N+1}}\right)\\ &\leq 2\left(\frac{\alpha_x - Nk}{N}\right)\left(\frac{\sqrt{N+1}-1}{M-1}\right)\\ &\leq \frac{\alpha_x - Nk}{M\sqrt{N}}2\left(\frac{\sqrt{N+1}}{\sqrt{N}}\frac{M}{M-1}\right),\end{aligned}$$

which gives the second claim. $\square$

Proof of Lemma 3.4. From Proposition 3.5, note that $0 < y_j(f\mathbf{1}, x\mathbf{1}) < k \implies (f,x) \in Q_3$. Substituting into the follower productions from Proposition 3.2 gives

$$y_j(f\mathbf{1}, x\mathbf{1}) = \frac{N}{N^2+NM+M+1}\alpha_x,$$

which is strictly increasing in α_x. Since $\alpha_x \leq \zeta_2$, it follows that

$$\begin{aligned}\bar{y} &= \frac{N}{N^2+NM+M+1}\zeta_2\\ &= \left(1-\frac{N+2-2\sqrt{N+1}}{N^2+N+2-2\sqrt{N+1}}\right)k\\ &\geq \left(1-\frac{N}{N^2+N}\right)k\\ &= \left(1-\frac{1}{N}\frac{N}{N+1}\right)k,\end{aligned}$$

from which we obtain the first claim.

Next, from Proposition 3.5, we infer that $\bar{\alpha}_x = \zeta_2$ and $\underline{\alpha} = (M+N+1)k$. It is easy to show that $\zeta_2 < (M+N+1)k \iff M < N\sqrt{N+1}-1$. Moreover,

$$(M+N+1)k-\zeta_2 = \frac{2}{N^2+(\sqrt{N+1}-1)^2}\left((N^2+N+M+1)-(M+N+1)\sqrt{N+1}\right)k.$$

Hence, if $\underline{\alpha}_x \leq \bar{\alpha}_x$, then

$$\begin{aligned}\frac{\underline{\alpha}_x-\bar{\alpha}_x}{\underline{\alpha}_x} &= \frac{2}{N^2+(\sqrt{N+1}-1)^2}\left(1+\frac{N^2}{M+N+1}-\sqrt{N+1}\right)\\ &\leq \frac{2N}{N^2+(\sqrt{N+1}-1)^2}\\ &\leq \frac{2N}{N^2},\end{aligned}$$

from which we obtain the first part of the second claim. If $\underline{\alpha}_x \geq \bar{\alpha}_x$, then

$$\begin{aligned}
\frac{\bar{\alpha}_x - \underline{\alpha}_x}{\underline{\alpha}_x} &= \frac{2}{N^2 + (\sqrt{N+1}-1)^2}\left(\sqrt{N+1} - 1 - \frac{N^2}{M+N+1}\right) \\
&\leq \frac{2}{N^2 + (\sqrt{N+1}-1)^2}\left(\sqrt{N+1}\right) \\
&\leq \frac{2}{N^2}\sqrt{N+2} \\
&\leq \frac{2}{N\sqrt{N}}\sqrt{\frac{N+2}{N}},
\end{aligned}$$

from which we obtain the rest of the second claim. $\square$

Proof of Lemma 3.6. From Proposition 3.5, note that $0 < y_j(f\mathbf{1}, x\mathbf{1}) < k \iff (f, x) \in Q_3$. Substituting into the follower productions from Proposition 3.2 gives

$$y_j(f\mathbf{1}, x\mathbf{1}) = \frac{N}{N^2 + MN + M + 1}\left(\alpha_x - (MN + M + 1)\triangle C\right),$$

which is strictly increasing in α_x. Since $\alpha_x \leq \zeta_2$, it follows that

$$\begin{aligned}
\bar{y} &= \frac{N}{N^2 + MN + M + 1}\left(\zeta_2 - (MN + M + 1)\triangle C\right) \\
&= \frac{N^2}{N^2 + (\sqrt{N+1}-1)^2}\left(k - \frac{(\sqrt{N+1}-1)^2}{N}\triangle C\right) \\
&\geq \left(1 - \frac{N+2-2\sqrt{N+1}}{N^2}\right)\left(k - \frac{(\sqrt{N+1}-1)^2}{N}\triangle C\right) \\
&\geq \left(1 - \frac{1}{N}\right)\left(k - \frac{(\sqrt{N+1}-1)^2}{N}\triangle C\right),
\end{aligned}$$

from which we obtain the first claim.

Next, from Proposition 3.5, we infer that if $(\sqrt{N+1}-1)^2\triangle C < Nk$, then $\bar{\alpha}_x = \zeta_2$ and $\underline{\alpha}_x = (M+N+1)k$, and it is straightforward to show that $\zeta_2 < (M+N+1)k$ if and only if the first case in (3.3) holds. Otherwise, then $\bar{\alpha}_x = \zeta_1$ and $\underline{\alpha}_x = (M+N+1)k$, and it is straightforward to show that $\zeta_1 < (M+N+1)k$ if and only if the second case in (3.3) holds. $\square$

Proof of Lemma 3.7. From Proposition 3.6, note that $0 < y_j(\mathbf{0}, x\mathbf{1}) < k \iff x = x_3$. Substituting into the follower productions from Proposition 3.2 gives

$$y_j(\mathbf{0}, x\mathbf{1}) = \frac{1}{(N+1)(M+1)}\left(\alpha_x - (MN + M + 1)\triangle C\right),$$

which is strictly increasing in α_x. Since $\alpha_x \leq \zeta_2$, it follows that

$$\begin{aligned}\bar{y} &= \frac{1}{(N+1)(M+1)}\left(\zeta_2 - (MN+M+1)\triangle C\right) \\ &= \left(1 + \frac{1}{\sqrt{N+1}}\right)\frac{k}{2},\end{aligned}$$

from which we obtain the first claim.

Next, from Proposition 3.6, we infer that $\bar{\alpha}_x = \zeta_2$ and $\underline{\alpha}_x = Nk + \frac{N(M+1)}{2(\sqrt{N+1}-1)}(\triangle C + k)$. It is straightforward to show that $\bar{\alpha}_x \geq \underline{\alpha}_x$. $\square$

Proof of Lemma 3.9. From Proposition 3.6, note that $0 < y_j(\mathbf{0}, x\mathbf{1}) < k \iff x = x_3$. Substituting into the follower productions from Proposition 3.2 gives

$$y_j(\mathbf{0}, x\mathbf{1}) = \frac{1}{(N+1)(M+1)}\left(\alpha_x - (MN+M+1)\triangle C\right),$$

which is strictly increasing in α_x. Since $\alpha_x \leq \zeta_2$, it follows that

$$\begin{aligned}\bar{y} &= \frac{1}{(N+1)(M+1)}\left(\zeta_2 - (MN+M+1)\triangle C\right) \\ &= \left(1 + \frac{1}{\sqrt{N+1}}\right)\frac{1}{2}\left(k - \frac{(\sqrt{N+1}-1)^2}{N}\triangle C\right),\end{aligned}$$

from which we obtain the first claim.

Next, from Proposition 3.6, we infer that, if $(\sqrt{N+1}-1)^2\triangle C < Nk$, then $\bar{\alpha}_x = \zeta_2$ and $\underline{\alpha}_x = Nk + \frac{N(M+1)}{2(\sqrt{N+1}-1)}(\triangle C + k)$, and it is straightforward to show that $\bar{\alpha}_x \geq \underline{\alpha}_x$. Otherwise, then $\bar{\alpha}_x = \zeta_1$ and $\underline{\alpha}_x = Nk + (M+1)(\triangle C + \sqrt{Nk\triangle C})$, and it is straightforward to show that $\bar{\alpha}_x \geq \underline{\alpha}_x$. $\square$

Proof of Lemma 3.10. The proof proceeds in three steps. In step 1, we compute an equilibria with the smallest (resp. largest) market production in the forward (resp. Stackelberg) market. In step 2, we compute an equilibria with the smallest (resp. largest) social welfare in the forward (resp. Stackelberg) market. In step 3, we show that the worst case ratios of productions and efficiencies are both strictly increasing in α_x. The bounds in the lemma are obtained by evaluating those ratios at $\alpha_x = \bar{\alpha}_x$.

Step 1: We compute an equilibria with the smallest (resp. largest) market production in the forward (resp. Stackelberg) market. First, we tackle the forward market. Substituting $\triangle C = 0$ into Proposition 3.5, we infer that $(f, x) \in Q$ if and only if $(f, x) \in Q_3$ or $(f, x) \in Q_4$. By substituting into Proposition 3.2, and using the fact that $y_j(f\mathbf{1}, x\mathbf{1}) = 0$ for all $(f, x) \in Q_4$, we obtain the following market productions:

$$Mx + Ny_j(f\mathbf{1}, x\mathbf{1}) = \begin{cases} \frac{1}{N^2+MN+M+1}\left(N^2+MN+M\right)\alpha_x, & \text{if } (f,x) \in Q_3, \\ \frac{1}{M+1}\left(M\alpha_x + Nk\right), & \text{if } (f,x) \in Q_4. \end{cases}$$

Note that

$$\begin{aligned} &\frac{1}{M+1}\left[M\alpha_x + Nk\right] \\ &= \frac{1}{N^2+MN+M+1}\left[(N^2+MN+M)\alpha_x + \frac{-N^2-MN}{M+1}\alpha_x + \frac{N(N^2+MN+M+1)}{M+1}k\right] \\ &\leq \frac{1}{N^2+MN+M+1}\left[(N^2+MN+M)\alpha_x + \frac{N(M+1)(1-N-M)}{M+1}k\right] \\ &\leq \frac{1}{N^2+MN+M+1}\left(N^2+MN+M\right)\alpha_x, \end{aligned}$$

where the first inequality is due to the fact that $\alpha_x \geq \underline{\alpha}_x$ and the second inequality is due to the fact that $M \geq 1$, $N \geq 2$, and $k > 0$. Hence, we infer that the smallest equilibrium production in the forward market is given by

$$\begin{aligned} y_F &= k, \\ x_F &= \frac{1}{M+1}(\alpha_x - Nk). \end{aligned}$$

Next, we tackle the Stackelberg market. Substituting $\triangle C = 0$ into Proposition 3.6, we infer that $(0, x_s) \in X(0)$ if and only if $x_s = x_3$ or $x_s = x_4$. Suppse

$$\alpha_x < Nk + \frac{N(M+1)}{2(\sqrt{N+1}-1)}k. \tag{3.53}$$

Then, from Proposition 3.6, we conclude that $x_s = x_3$ is the only Stackelberg equilibrium, and hence it is also the equilibrium with the largest market production. Suppose, instead, that (3.53) does not hold. By substituting into Proposition 3.2, and using the fact that $y_j(\mathbf{0}, x_4\mathbf{1}) = 0$, we obtain the following market productions:

$$Mx_s + Ny_j(\mathbf{0}, x_s\mathbf{1}) = \begin{cases} \frac{MN+M+N}{(M+1)(N+1)}\alpha_x, & \text{if } x_s = x_3, \\ \frac{1}{M+1}\left(M\alpha_x + Nk\right), & \text{if } x_s = x_4. \end{cases}$$

Note that

$$
\begin{aligned}
&\frac{MN+M+N}{(M+1)(N+1)}\alpha_x \\
&= \frac{1}{M+1}\left[M\alpha_x + \frac{N}{N+1}\alpha_x\right] \\
&\geq \frac{1}{M+1}\left[M\alpha_x + \frac{N}{N+1}\left(N + \frac{N(M+1)}{2(\sqrt{N+1}-1)}\right)k\right] \\
&\geq \frac{1}{M+1}\left[M\alpha_x + \frac{N}{N+1}(N+1)\,k\right] \\
&= \frac{1}{N+1}\left[M\alpha_x + Nk\right],
\end{aligned}
$$

where the first inequality is due to the fact that (3.53) does not hold and the second inequality is due to the fact that $M \geq 1$ and $N \geq 2$. Hence, we infer that the largest equilibrium production in the Stackelberg market is given by

$$
\begin{aligned}
y_S &= \frac{1}{N+1}\left(\alpha_x - Mx_s\right), \\
x_S &= \frac{1}{M+1}\alpha_x.
\end{aligned}
$$

Step 2: We compute the equilibria with the smallest (resp. largest) social welfare in the forward (resp. Stackelberg) market. Substituting the demand function into the social welfare gives

$$
\begin{aligned}
\mathsf{SW}(y,x) &= \beta\left(\alpha_x\left(Mx+Ny\right) - \triangle CNy - \frac{1}{2}\left(Mx+Ny\right)^2\right) \\
&= \beta\left(\alpha_x\left(Mx+Ny\right) - \frac{1}{2}\left(Mx+Ny\right)^2\right),
\end{aligned}
$$

where the second equality is obtained by substituting $\triangle C = 0$. Given any two equilibrium productions (y,x) and (y',x'), we have

$$
\begin{aligned}
&\mathsf{SW}(y,x) \geq \mathsf{SW}(y',x') \\
\iff &\alpha_x(Mx+Ny) - \frac{1}{2}(Mx+Ny)^2 \geq \alpha_x(Mx'+Ny') - \frac{1}{2}(Mx'+Ny')^2 \\
\iff &\frac{1}{2}\left((Mx+Ny)-(Mx'+Ny')\right)\left(\alpha_x - (Mx+Ny) + \alpha_x - (Mx'+Ny')\right) \geq 0 \\
\iff &\frac{1}{2}\left((Mx+Ny)-(Mx'+Ny')\right)\left(\frac{1}{\beta}\left(P(Mx+Ny)-C\right) + \frac{1}{\beta}\left(P(Mx'+Ny')-C\right)\right) \geq 0 \\
\iff &Mx+Ny \geq Mx'+Ny',
\end{aligned}
$$

where the last equivalence follows from the fact that, since (y,x) and (y',x') are equilibrium productions, the profit margins $P(Mx+Ny)-C > 0$ and $P(Mx'+Ny')-C > 0$. Hence, the equilibria with the smallest (resp. largest) social welfare in the forward (resp. Stackelberg) market are those with the smallest (resp. largest) market productions, which were obtained in step 1.

Step 3: We show that the worst-case ratios of productions and social welfares are strictly increasing in α_x. From step 1, the ratio of productions is bounded from above by

$$r_P := \frac{Mx_S + Ny_S}{Mx_F + Ny_F}.$$

Taking derivatives gives

$$\begin{aligned}\frac{\partial r_P}{\partial \alpha_x} &= \frac{(Mx_F + Ny_F)\left(M\frac{\partial x_S}{\partial \alpha_x} + N\frac{\partial y_S}{\partial \alpha_x}\right) - (Mx_S + Ny_S)\left(M\frac{\partial x_F}{\partial \alpha_x} + N\frac{\partial y_F}{\partial \alpha_x}\right)}{(Mx_F + Ny_F)^2} \\ &= \frac{\frac{N(MN+M+N)k}{(M+1)^2(N+1)}}{(Mx_F + Ny_F)^2} \\ &> 0.\end{aligned}$$

Next, the ratio of social welfares is bounded from above by

$$r_W := \frac{\mathsf{SW}(y_S, x_S)}{\mathsf{SW}(y_F, x_F)}.$$

Taking derivatives gives

$$\begin{aligned}\frac{\partial r_W}{\partial \alpha_x} &= \frac{\mathsf{SW}(y_F, x_F)\frac{\partial \mathsf{SW}(y_S, x_S)}{\partial \alpha_x} - \mathsf{SW}(y_S, x_S)\frac{\partial \mathsf{SW}(y_F, x_F)}{\partial \alpha_x}}{\mathsf{SW}(y_F, x_F)^2} \\ &= \frac{\frac{\beta}{2(M+1)^4(N+1)^2}(MN+M+N)(MN+M+N+2)(\alpha_x - Nk)Nk\alpha_x}{\mathsf{SW}(y_F, x_F)^2} \\ &> 0,\end{aligned}$$

where the inequality is due to $\alpha_x \geq \underline{\alpha}_x > Nk$. Hence, r_P and r_W are both strictly increasing in α_x over $[\underline{\alpha}_x, \bar{\alpha}_x]$. By substituting $\alpha_x = \bar{\alpha}_x$ into r_P and r_W, we obtain the desired result. $\square$

Chapter 4

Optimization Decomposition for Joint Economic Dispatch and Frequency Regulation

In this chapter, we focus on the joint design of economic dispatch and frequency regulation. Recall that grid operations are divided into two timescales/layers. The timescale of 5 minutes and longer is focused on efficiency and resources are dispatched using market mechanisms and solving economic dispatch. The timescale within 5-minute intervals is focused on stability where the faster timescale resources are dispatched using engineered frequency regulation controllers. Economic dispatch and frequency regulation each have large and active literatures; however, these literatures are almost completely disparate. While there have been studies on integrating the two mechanisms more efficiently [127], we are not aware of any rigorous analysis of whether their combination solves the global system operator's goal of dispatching generation resources efficiently across both timescales.

4.1 Our Contributions

The goal of this work is to initiate a study into the joint design of economic dispatch and frequency regulation mechanisms. Our main result provides an initial answer. In the context of a DC power flow model and two classes of generators (dispatch and regulation), we show that the global system operator's problem can be decomposed into two sub-problems that correspond to the economic dispatch and frequency regulation timescales, without loss of optimality, as long as the ISO is able to estimate the difference between the average LMP in the frequency regulation periods and the LMP in the economic dispatch period (Theorem 4.1). This result can be viewed as a first-principles justification for the existing separation of power systems control into economic dispatch and frequency regulation problems. Moreover, this result provides a guide to modify the existing architecture to *optimally* control power systems across timescales. In particular, using this result, we design an optimal control policy for frequency regulation and an optimal market mechanism for economic dispatch, in a way such that the control and market mechanisms jointly solve the global system operator's problem. Our mechanims differ from existing economic dispatch and frequency regulation mechanisms in important ways.

In the case of frequency regulation (Section 4.4), our mechanism has a key advantage over the AGC mechanism in that our mechanism is efficient. The frequency regulation controller proposed in this work is built on the distributed controller in [146, 87] and controls generation based on information about generators' costs in a way such that the power system converges to an operating point that minimizes system costs. On the other hand, AGC allocates generation based on participation factors, which might not reflect actual costs, and hence the resulting allocation might not be efficient. In [84], the authors proposed a modification of the participation factors so that the AGC mechanism is cost efficient. However, unlike our mechanism, the mechanism in [84] does not respect line constraints.

In the case of economic dispatch (Section 4.5), our mechanism has a key advantage over the existing economic dispatch operations in that it coordinates efficiently with the frequency regulation timescale. This coordination does not require additional communication in the market beyond the existing mechanism used in practice. This coordination involves two main components. First, our economic dispatch mechanism communicates the supply function bids from the generators to the frequency regulation mechanism, which uses them in the distributed controllers to allocate frequency regulation resources efficiently. In contrast, the AGC mechanism allocates frequency regulation resources without regard to generation costs. Second, our economic dispatch mechanism accounts for the value that economic dispatch resources provide to frequency regulation. It does so by adjusting the resource costs in the economic dispatch objective based on the difference between the LMP in the frequency regulation periods and that in the economic dispatch period. In contrast, the existing economic dispatch objective does not perform this adjustment and hence might allocate economic dispatch resources inefficiently.

In practice, the ISO is unlikely to be able to estimate exactly the adjustment it should make to the economic dispatch objective. In Section 4.6, we investigate numerically the sensitivity of the suboptimality of our decomposition to those estimation errors on the IEEE 24-bus reliability test system.

4.2 System Model

Our aim is to understand how the combination of economic dispatch and frequency regulation can dispatch generation resources efficiently across both timescales. To this end, we formulate a model of the global objective that includes balancing supply and demand at both timescales. We use a DC power flow model and consider two generation types – dispatch and regulation – which differ in

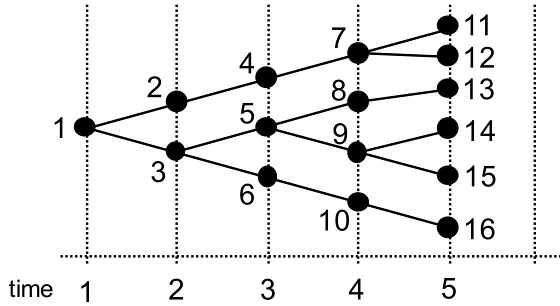


Figure 4.1: Example of a scenario tree with $S = 16$ outcomes over $K = 5$ periods. The outcomes are numbered $1, \dots, S$.

responsiveness.

Consider a connected network consisting of a set of nodes N and a set of links L. We focus on a single economic dispatch interval of the real-time market which is typically 5 minutes in existing markets. We partition this time interval into K discrete periods numbered $1, \dots, K$. In general, the length of each period may range from as little as seconds to as long as minutes. However, in this work, we focus on the case where each period is on the order tens of seconds.

4.2.1 Stochastic demand

We use a stochastic demand model motivated by the frameworks in [36, 126, 103]. Assume that there is a set of possible demand outcomes S that can be described by a scenario tree (an example is given in Figure 4.1). For each outcome $s \in S$, let $d_{s,n} \in \mathbb{R}$ denote the real power demand at node $n \in N$ and $\mathbf{d}_s := (d_{s,n}, n \in N) \in \mathbb{R}^N$ denote the vector of demands at all nodes. In addition, let $\kappa(s) \in \{1, \dots, K\}$ denote the period of this outcome and p_s denote the probability of this outcome conditioned on the information that the period is $\kappa(s)$. Hence, $\sum_{\{s|\kappa(s)=k\}} p_s = 1$ for each $k \in \{1, \dots, K\}$. Without loss of generality, we assume that $\kappa(1) = 1$ and $p_1 = 1$. That is, there exists an outcome labeled $1 \in S$ associated with period 1 and the demand in that period is deterministic.

4.2.2 Generation

We assume that each node $n \in N$ has two generators – a dispatch generator and a regulation generator – where the regulation generator is more responsive than the dispatch generator. To model the differing responsiveness, we assume that the dispatch generator produces at a constant level over the entire economic dispatch interval while the regulation generator may change its production level every period after uncertain demand is realized [50]. Formally, we assume that the dispatch generator produces $q_n^b \in \mathbb{R}$ in all outcomes, and the regulation generator produces $q_n^p \in \mathbb{R}$ in period 1 and

$q_n^p + r_{s,n}^p \in \mathbb{R}$ in each subsequent outcome $s \in S \setminus \{1\}$. Hence, q_n^p and $r_{s,n}^p$ can be interpreted as the regulation generator's setpoint and recourse respectively. To simplify notations, we define a dummy variable $r_{1,n}^p := 0$ so that we may write the regulation generator's production in period 1 as $q_n^p + r_{1,n}^p$. We assume that the regulation and dispatch generators have capacity constraints $[\underline{q}_n^p, \bar{q}_n^p]$ and $[\underline{q}_n^b, \bar{q}_n^b]$ respectively, and incur costs $c_n^p(q_n^p + r_{s,n}^p)$ and $c_n^b(q_n^b)$ respectively in period $\kappa(s)$, where the functions $c_n^p : [\underline{q}_n^p, \bar{q}_n^p] \to \mathbb{R}_+$ and $c_n^b : [\underline{q}_n^b, \bar{q}_n^b] \to \mathbb{R}_+$ are strictly convex and continuously differentiable.

Define vectors $\mathbf{q}^p := (q_n^p, n \in N)$, $\mathbf{r}_s^p := (r_{s,n}^p, n \in N)$, $\mathbf{q}^b := (q_n^b, n \in N)$, $\underline{\mathbf{q}}^p := (\underline{q}_n^p, n \in N)$, $\underline{\mathbf{q}}^b := (\underline{q}_n^b, n \in N)$, $\bar{\mathbf{q}}^p := (\bar{q}_n^p, n \in N)$, $\bar{\mathbf{q}}^b := (\bar{q}_n^b, n \in N)$. Then the generation constraints in outcome $s \in S$ are given by:

$$\underline{\mathbf{q}}^b \leq \mathbf{q}^b \leq \bar{\mathbf{q}}^b, \tag{4.1}$$

$$\underline{\mathbf{q}}^p \leq \mathbf{q}^p + \mathbf{r}_s^p \leq \bar{\mathbf{q}}^p. \tag{4.2}$$

We also let the vector $\mathbf{r}^p := (\mathbf{r}_s^p, s \in S)$.

4.2.3 Network constraints

Note that $\mathbf{q}^b + \mathbf{q}^p + \mathbf{r}_s^p - \mathbf{d}_s$ is the vector of nodal injections for $s \in S$. Thus, the supply-demand balance constraint is:

$$\mathbf{1}^\top(\mathbf{q}^b + \mathbf{q}^p + \mathbf{r}_s^p - \mathbf{d}_s) = 0, \tag{4.3}$$

where $\mathbf{1} \in \mathbb{R}^N$ denotes the vector of all ones.

We adopt the DC power flow model for line flows. Let $\theta_{s,n}$ denote the phase angle of node n. Without loss of generality, assign each link l an arbitrary orientation and let $i(l)$ and $j(l)$ denote the tail and head of the link respectively. Let B_l denote the sensitivity of the flow with respect to changes in the phase difference $\theta_{s,i(l)} - \theta_{s,j(l)}$ and let $v_{s,l}$ denote its power flow. Define the vectors $\boldsymbol{\theta}_s := (\theta_{s,n}, n \in N)$ and $\mathbf{v}_s := (v_{s,l}, l \in L)$ and the matrix $\mathbf{B} := \mathrm{diag}(B_l, l \in L)$. Then, the line flows are given by $\mathbf{v}_s = \mathbf{B}\mathbf{C}^\top\boldsymbol{\theta}_s$, where $\mathbf{C} \in \mathbb{R}^{N \times L}$ is the incidence matrix of the directed graph. And the injections are:

$$\mathbf{q}^b + \mathbf{q}^p + \mathbf{r}_s^p - \mathbf{d}_s = \mathbf{C}\mathbf{v}_s = \mathbf{L}\boldsymbol{\theta}_s, \tag{4.4}$$

where $\mathbf{L} := \mathbf{C}\mathbf{B}\mathbf{C}^\top$.

Note that (4.3) and (4.4) are equivalent. For any set of injections that satisfy (4.3), we can always find $\boldsymbol{\theta}_s$ that satisfies (4.4). Conversely, since $\mathbf{1}^\top\mathbf{C} = 0$, any injections that satisfy (4.4) also satisfy (4.3). Hence, the line flows can be written in terms of the power injections:

$$\mathbf{v}_s = \mathbf{B}\mathbf{C}^\top \mathbf{L}^\dagger (\mathbf{q}^b + \mathbf{q}^p + \mathbf{r}_s^p - \mathbf{d}_s),$$

where $\mathbf{L}^\dagger$ denotes the pseudo-inverse of $\mathbf{L}$. Let $\mathbf{H} := \mathbf{B}\mathbf{C}^\top \mathbf{L}^\dagger$. Let f_l denote the capacity of line l and define the vector $\mathbf{f} := (f_l, l \in L)$. Then the line flow constraints are:

$$-\mathbf{f} \leq \mathbf{H}\left(\mathbf{q}^b + \mathbf{q}^p + \mathbf{r}_s^p - \mathbf{d}_s\right) \leq \mathbf{f}. \tag{4.5}$$

To simplify notations, we define the set $\Omega(\mathbf{d}_s)$ of feasible generation for a given demand vector $\mathbf{d}_s$ as:

$$\Omega(\mathbf{d}_s) := \left\{(\mathbf{q}^b, \mathbf{q}^p, \mathbf{r}_s^p) : (4.1), (4.2), (4.3), (4.5) \text{ holds}\right\}.$$

4.2.4 System operator's objective

The global system operator's objective is to allocate the dispatch and regulation generations $(\mathbf{q}^b, \mathbf{q}^p, \mathbf{r}^p)$ to minimize the expected cost of satisfying demand and operating constraints. This is formalized as follows.

$$\begin{aligned} SYSTEM: \min_{\mathbf{q}^b, \mathbf{q}^p, \mathbf{r}^p} \quad & \sum_{s \in S} p_s \sum_{n \in N} \left(c_n^b(q_n^b) + c_n^p(q_n^p + r_{s,n}^p)\right) \\ \text{s.t.} \quad & (\mathbf{q}^b, \mathbf{q}^p, \mathbf{r}_s^p) \in \Omega(\mathbf{d}_s), \quad \forall s \in S, \\ & \mathbf{r}_1^p = \mathbf{0}. \end{aligned}$$

We assume that this optimization is feasible. Note that $SYSTEM$ differs from the existing economic dispatch mechanism which minimize costs under the assumption that the demand during all the K periods in the economic dispatch interval is equal to the demand $\mathbf{d}_1$ in period 1.

Let λ_s and $(\underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s)$ be the Lagrange multipliers associated with constraints (4.3) and (4.5) respectively in $SYSTEM$. Then, the function $\boldsymbol{\pi} : \mathbb{R} \times \mathbb{R}_+^{2L} \to \mathbb{R}^N$, defined by:

$$\boldsymbol{\pi}(\lambda_s, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) := \lambda_s \mathbf{1} + \mathbf{H}^\top (\underline{\boldsymbol{\mu}}_s - \bar{\boldsymbol{\mu}}_s), \tag{4.6}$$

gives the nodal prices in outcome $s \in S$.

4.3 Architectural Decomposition

Our main result is a decomposition of $SYSTEM$ into setpoint and recourse sub-problems. Importantly, our decomposition identifies a rigorous connection between the setpoint and recourse sub-problems that ensures that the combination solves $SYSTEM$. In particular, our decomposition divides $SYSTEM$ into sub-problems ED and FR defined by:

$$
\begin{aligned}
ED(\mathbf{d}_1): \quad & \min_{\mathbf{q}^b, \mathbf{q}^p} \quad \sum_{n \in N} \left(Kc_n^b(q_n^b) + Kc_n^p(q_n^p) - \delta_n q_n^b \right) \\
& \quad \text{s.t.} \quad (\mathbf{q}^b, \mathbf{q}^p, \mathbf{0}) \in \Omega(\mathbf{d}_1), \\
FR(\mathbf{q}^b, \mathbf{q}^p, \mathbf{d}_s): \quad & \min_{\mathbf{r}_s^p} \quad \sum_{n \in N} c_n^p(q_n^p + r_{s,n}^p) \\
& \quad \text{s.t.} \quad (\mathbf{q}^b, \mathbf{q}^p, \mathbf{r}_s^p) \in \Omega(\mathbf{d}_s),
\end{aligned}
$$

where $\boldsymbol{\delta} \in \mathbb{R}^N$ is a constant. $ED(\mathbf{d}_1)$ is implemented in time period 1 and $FR(\mathbf{q}^b, \mathbf{q}^p, \mathbf{d}_s)$ is implemented in subsequent time periods $\kappa(s) > 1$.

We denote the first optimization problem by ED, since it optimizes only generation setpoints $(\mathbf{q}^b, \mathbf{q}^p)$ assuming constant demand $\mathbf{d}_1$ over the K time periods, and hence it is on the same timescale as the existing economic dispatch mechanism. We denote the second optimization problem by FR, since it optimizes regulation generators' recourse production $\mathbf{r}_s^p$ in subsequent time periods, and hence it is on the same timescale as the existing frequency regulation mechanism.

Definition 4.1. *We say that $SYSTEM$ can be optimally decomposed into ED-FR if $(\mathbf{q}^b, \mathbf{q}^p, \mathbf{r}^p)$ is an optimal solution to $SYSTEM$ if and only if $\mathbf{r}_1^p = \mathbf{0}$, $(\mathbf{q}^b, \mathbf{q}^p)$ is an optimal solution to $ED(\mathbf{d}_1)$, and $\mathbf{r}_s^p$ is an optimal solution to $FR(\mathbf{q}^b, \mathbf{q}^p, \mathbf{d}_s)$ for all $s \in S$.*

Theorem 4.1 (Decomposition)**.** *Let λ_s and $(\underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s)$ be any Lagrange multipliers associated with constraints* (4.3) *and* (4.5) *respectively in $SYSTEM$.*

(a) If $\boldsymbol{\delta}$ is the average, over all time periods, of the difference between the expected nodal prices in each period and that in period 1*, that is, for each $n \in N$,*

$$
\delta_n = \sum_{s \in S} p_s \left(\pi_n(\lambda_s, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) - \pi_n(\lambda_1, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1) \right), \tag{4.7}
$$

then $SYSTEM$ can be optimally decomposed into ED-FR.

(b) If $SYSTEM$ can be optimally decomposed into ED-FR, then for all n such that $\underline{q}_n^b < q_n^b < \bar{q}_n^b$ and $\underline{q}_n^p < q_{1,n}^p < \bar{q}_n^p$, (4.7) *holds.*

The proof of Theorem 4.1 is given in the Appendix. The result follows from analyzing the Karush-Kuhn-Tucker conditions of the system operator's problem and those of ED and FR. As mentioned, we denote the two sub-problems by ED and FR because they focus on the economic dispatch and frequency regulation timescales respectively. Hence, these sub-problems can serve as guides for the optimal design of economic dispatch and frequency regulation mechanisms. The insights are immediate in the case of economic dispatch and we show how ED leads to an improved market mechanism in Section 4.5. However, the insights may not be as clear in the case of frequency regulation. We show in Section 4.4 that FR can in fact be solved via distributed frequency control algorithms, although these algorithms deviate from current practice that do not optimize generation costs.

The most important feature of Theorem 4.1 is that one way to choose generation setpoints optimally at the economic dispatch timescale, is to include, in the optimization objective, an offset of the dispatch generators' marginal costs by the expected changes in nodal prices during the frequency regulation timescale. The latter can be interpreted as the expected changes in the marginal value of dispatch generation. Hence, if the latter is zero, then generation setpoints can be chosen optimally at the economic dispatch timescale without regard to the behavior of the system in the frequency regulation timescale [28].

An important extension of this result is to understand the efficiency loss of the decomposition when we are unable to estimate the RHS of (4.7) accurately. Note that negative estimation errors cause $ED(\mathbf{d}_1)$ to use less than optimal dispatch resources (and more than optimal regulation resources) and vice versa. We investigate the efficiency loss in Section 4.6. In such situations, the dispatch generation $\mathbf{q}^b$ might not be optimal, and therefore $FR(\mathbf{q}^b, \mathbf{q}^p, \mathbf{d}_s)$ might not be feasible. To ensure that $FR(\mathbf{q}^b, \mathbf{q}^p, \mathbf{d}^s)$ is feasible, we may modify $ED(\mathbf{d}_1)$ into a robust optimization problem by adding constraints $(\mathbf{q}^b, \mathbf{q}^p, \mathbf{r}^p_s) \in \Omega(\mathbf{d}_s)$ for all $s \in S \setminus \{1\}$. The size of such a problem is exponential in S but can be reduced using the technique in [93]. Note that this should not be viewed as a drawback of our decomposition, as the current practice based on AGC might also not be feasible. In practice, the risks of infeasibility are mitigated using reserves. Moreover, our decomposition has the advantage that it coordinates the economic dispatch and frequency regulation resources efficiently, and hence, may reduce reserve requirements.

Theorem 4.1 is close in spirit to work in communication networks that use optimization decomposition to justify and optimize protocol layering [39, 104, 31]. Hence, *Theorem 4.1 provides a rigorous way to think about architectural design of power networks.*

4.4 Distributed Frequency Regulation

This section illustrates how to implement the solution to FR using distributed frequency regulation controllers. Besides achieving optimality, a practical implementation should preserve network stability, be robust to unexpected system events, aggregate network information in a distributed manner, and satisfy constraints (4.2), (4.3) and (4.5). The distributed algorithm that we provide in this section satisfies all the above characteristics. It can be interpreted as performing distributed frequency regulation by sending different regulation signals to each bus.

4.4.1 Dynamic model

Before introducing our algorithm we add dynamics to our system model to describe the system behavior within a single time period. Let t denote the time evolution within the time period of outcome s, and assume without loss of generality that $t \in (k, k+1]$ where $k = \kappa(s)$. Let $\mathbf{r}_s^p(t) := (r_{s,n}^p(t), n \in N)$ denote the recourse quantities generated by the regulation generators at time t. We assume that dispatch generation and demand do not change within the time period.

Then, the system changes within the time period are governed by the swing equations, which we assume to be:

$$\dot{\boldsymbol{\theta}}_s(t) = \boldsymbol{\omega}_s(t); \tag{4.8a}$$

$$\mathbf{M}\dot{\boldsymbol{\omega}}_s(t) = \mathbf{q}^b + \mathbf{q}^p + \mathbf{r}_s^p(t) - \mathbf{d}_s - \mathbf{D}\boldsymbol{\omega}_s(t) - \mathbf{L}\boldsymbol{\theta}_s(t), \tag{4.8b}$$

where $\boldsymbol{\omega}_s(t) := (\omega_{s,n}(t), n \in N)$ are the frequency deviations from the nominal value at time t, $\boldsymbol{\theta}_s(t) := (\theta_{s,n}(t), n \in N)$ are the phase angles at time t, $\mathbf{M} := \mathrm{diag}(M_1, \ldots, M_N)$ where M_n is the aggregate inertia of the generators at node n, and $\mathbf{D} := \mathrm{diag}(D_1, \ldots, D_N)$ where D_n is the aggregate damping of the generators at node n. The notation $\dot{x}$ denotes the time derivative, i.e. $\dot{x} = dx/dt$. Equation (4.8) is a linearized version of the nonlinear network dynamics [19, 18], and has been widely used in the design of frequency regulation controllers. See, e.g., [45, 46].

4.4.2 Distributed frequency regulation

We now introduce a distributed, continuous-time algorithm that provably solves FR *while preserving system stability*. Our solution is based on a novel reverse and forward engineering approach for distributed control design in power systems [147, 145, 87, 84, 143, 70]. The algorithm operates as follows. Each regulation generator n updates its power generation using

$$r_{s,n}^p(t) = [c_n^{p\prime -1}(-\omega_{s,n}(t) - \pi_{s,n}^p(t))]_{\underline{q}_n^p - q_n^p}^{\bar{q}_n^p - q_n^p}, \tag{4.9}$$

where $c_n^{p\prime}(x) = \frac{\partial}{\partial x} c_n^p(x)$ and $c_n^{p\prime -1}$ denotes its inverse. The projection $[r]_{\underline{q}_n^p - q_n^p}^{\bar{q}_n^p - q_n^p}$ ensures that $\underline{q}_n^p - q_n^p \leq r \leq \bar{q}_n^p - q_n^p$ (or equivalently $\underline{q}_n^p \leq r + q_n^p \leq \bar{q}_n^p$) and $\pi_{s,n}^p(t)$ is a control signal generated using:

$$DFR: \; \dot{\boldsymbol{\pi}}_s^p(t) = \boldsymbol{\zeta}^\pi \left(\mathbf{q}^b + \mathbf{q}^p + \mathbf{r}_s^p(t) - \mathbf{d}_s - \boldsymbol{L}\boldsymbol{\phi}_s(t)\right); \tag{4.10a}$$

$$\dot{\bar{\boldsymbol{\mu}}}_s(t) = \boldsymbol{\zeta}^{\bar{\mu}} \left[\mathbf{B}\mathbf{C}^\top \boldsymbol{\phi}_s(t) - \mathbf{f}\right]_{\bar{\boldsymbol{\mu}}_s}^+; \tag{4.10b}$$

$$\dot{\underline{\boldsymbol{\mu}}}_s(t) = \boldsymbol{\zeta}^{\underline{\mu}} \left[-\mathbf{f} - \mathbf{B}\mathbf{C}^\top \boldsymbol{\phi}_s(t)\right]_{\underline{\boldsymbol{\mu}}_s}^+; \tag{4.10c}$$

$$\dot{\boldsymbol{\phi}}_s(t) = \boldsymbol{\chi}^\phi \left(\mathbf{L}\boldsymbol{\pi}_s^p(t) - \mathbf{C}\mathbf{B}(\bar{\boldsymbol{\mu}}_s(t) - \underline{\boldsymbol{\mu}}_s(t))\right), \tag{4.10d}$$

where $\boldsymbol{\zeta}^{\pi} := \mathrm{diag}(\zeta_1^{\pi}, \ldots, \zeta_N^{\pi})$, $\boldsymbol{\zeta}^{\bar{\mu}} := \mathrm{diag}(\zeta_1^{\bar{\mu}}, \ldots, \zeta_L^{\bar{\mu}})$, $\boldsymbol{\zeta}^{\underline{\mu}} := \mathrm{diag}(\zeta_1^{\underline{\mu}}, \ldots, \zeta_L^{\underline{\mu}})$, $\boldsymbol{\chi}^{\phi} := \mathrm{diag}(\chi_1^{\phi}, \ldots, \chi_N^{\phi})$ denote the respective control gains. The element-wise projection $[\mathbf{y}]_{\mathbf{x}}^+ := ([y_n]_{x_n}^+, n \in N)$ ensures that the dynamics $\dot{\mathbf{x}} = [\mathbf{y}]_{\mathbf{x}}^+$ have a solution $\mathbf{x}(t)$ that remains in the positive orthant, that is, $[y_n]_{x_n}^+ = 0$ if $x_n = 0$ and $y_n < 0$, and $[y_n]_{x_n}^+ = y_n$ otherwise.

The proposed solution (4.9) – (4.10) can be interpreted as a frequency regulation algorithm in which each regulation generator receives a different regulation signal (4.9) depending on its location in the network. The key step in the design of DFR is reformulating FR into the following equivalent optimization problem:

$$
\begin{aligned}
FR'(\boldsymbol{q}^b, \boldsymbol{q}^p, \boldsymbol{d}_s): \quad \min_{\mathbf{r}_s^p, \boldsymbol{\omega}_s, \mathbf{v}_s, \boldsymbol{\phi}_s} \quad & \sum_{n\in N} \left(c_n^p(q_n^p + r_{s,n}^p) + D_n \omega_{s,n}^2/2 \right) \\
\text{s.t.} \quad & \mathbf{q}^b + \mathbf{q}^p + \mathbf{r}_s^p - \mathbf{d}_s - \mathbf{D}\boldsymbol{\omega}_s = \mathbf{C}\mathbf{v}_s; && (4.11a) \\
& \mathbf{q}^b + \mathbf{q}^p + \mathbf{r}_s^p - \mathbf{d}_s = \mathbf{L}\boldsymbol{\phi}_s; && (4.11b) \\
& -\mathbf{f} \le \mathbf{B}\mathbf{C}^\top \boldsymbol{\phi}_s \le \mathbf{f}; && (4.11c) \\
& \underline{\mathbf{q}}^p \le \mathbf{q}^p \le \bar{\mathbf{q}}^p. && (4.11d)
\end{aligned}
$$

Recall from Section 4.2.3 that $\mathbf{v}_s$ denote line flows. Constraint (4.11a) is reformulated from the per node supply-demand balance constraint (4.4), and makes explicit the fact that, whenever supply and demand do not match, the mismatch is compensated by a change in the frequency. Constraint (4.11b) ensures that $\boldsymbol{\omega}_s = 0$ at the optimal solution so that supply and demand are balanced. Constraint (4.11c) imposes line flow limits. However, instead of using actual line flows $\mathbf{v}_s$, these limits are imposed on *virtual flows* $\mathbf{B}\mathbf{C}^\top\boldsymbol{\phi}_s$, which are identical to line flows at the optimal solution [87].

It can be shown that FR' has a primal-dual algorithm that contains the component (4.8) resembling power network dynamics and the components (4.9) – (4.10) that can be implemented via distributed communication and computation. This new problem FR' also makes explicit the role of frequency in maintaining supply-demand balance.

The next proposition formally relates the optimal solutions of FR and FR' and guarantees the optimality of (4.9) – (4.10).

Proposition 4.1 (Optimality)**.** *Let $\mathbf{r}_s^p$ and $(\mathbf{r}_s^{p\prime}, \boldsymbol{\omega}_s', \mathbf{v}_s', \boldsymbol{\phi}_s')$ be optimal solutions of FR and FR' respectively. Then, the following statements are true: (i) Frequency restoration: $\boldsymbol{\omega}_s' = 0$; (ii) Generation equivalence: $\mathbf{r}_s^p = \mathbf{r}_s^{p\prime}$; (iii) Line flow equivalence: $\mathbf{H}\left(\mathbf{q}^b + \mathbf{q}^p + \mathbf{r}_s^p - \mathbf{d}_s\right) = \mathbf{B}\mathbf{C}^\top\boldsymbol{\phi}_s'$. Moreover, there exists $\boldsymbol{\theta}_s' \in \mathbb{R}^N$ and $\mathbf{y}_s' \in \mathbb{R}^L$, satisfying $\mathbf{C}\mathbf{y}_s' = 0$, such that $\mathbf{v}_s' = \mathbf{B}\mathbf{C}^\top\boldsymbol{\theta}_s' + \mathbf{y}_s'$ and $\mathbf{B}\mathbf{C}^\top\boldsymbol{\phi}_s' = \mathbf{B}\mathbf{C}^\top\boldsymbol{\theta}_s'$. And $(\mathbf{r}_s^{p\prime}, \boldsymbol{\omega}_s', \boldsymbol{\theta}_s', \boldsymbol{\phi}_s', \boldsymbol{\pi}_s^{p\prime}, \underline{\boldsymbol{\mu}}_s', \bar{\boldsymbol{\mu}}_s')$ is an equilibrium point of* (4.8) – (4.10) *if and only if $(\mathbf{r}_s^{p\prime}, \boldsymbol{\omega}_s', \mathbf{v}_s', \boldsymbol{\phi}_s', \boldsymbol{\pi}_s^{p\prime}, \underline{\boldsymbol{\mu}}_s', \bar{\boldsymbol{\mu}}_s')$ is a primal-dual optimal solution of FR', where $\boldsymbol{\omega}_s'$, $\boldsymbol{\pi}_s^{p\prime}$, and $(\underline{\boldsymbol{\mu}}_s', \bar{\boldsymbol{\mu}}_s')$ are the Lagrange multipliers associated with constraints* (4.11a)*,* (4.11b)*, and* (4.11c)*, respectively.*

The proof of Proposition 4.1 is given in the Appendix. What remains is to guarantee the convergence of the distributed frequency regulation algorithm.

Proposition 4.2 (Convergence)**.** *If c^p_n is twice continuous differentiable with $c^{p\prime\prime}_n \geq \alpha > 0$ (i.e., α-strictly convex) and $c^p_n(q^p_n + r^p_{s,n}) \to +\infty$ as $q^p_n + r^p_{s,n} \to \{\underline{q}^p_n, \bar{q}^p_n\}$, then $\mathbf{r}^p_s(t)$ in* (4.8) – (4.10) *converge globally to an optimal solution of FR.*

The proof of Proposition 4.2 follows from [87] and uses the machinery developed in [38] to handle projections (4.10b) – (4.10c). By substituting the line flows $\mathbf{v}_s(t) = \mathbf{B}\mathbf{C}^\top \boldsymbol{\theta}_s(t)$ into (4.8) and eliminating $\boldsymbol{\theta}_s(t)$, we can show that the entire system (4.8) – (4.10) is a primal-dual algorithm of FR' (see [87, Theorem 5]). Therefore, Theorem 10 in [87] guarantees global asymptotic convergence to an equilibrium point which by Proposition 4.1 is an optimal solution of both FR' and FR. Our setup is simpler than the controllers in [87], which had additional states, but the same proof technique applies. Although Proposition 4.2 requires costs to blow up as regulation generations approach minimum and maximum capacities, this assumption is not restrictive, as it can be achieved by adding a barrier function to the actual cost before implementing in the controllers.

4.5 Market Mechanism for Economic Dispatch

This section illustrates how to implement the solution to ED through a market mechanism for economic dispatch. The mechanism works in the following manner. In the first time period, the ISO collects supply function bids from generators (both dispatch and regulation) and uses those bids to solve ED. Then, in subsequent time periods, the ISO uses the regulation generators' supply function bids to implement the controller in (4.9). This mechanism is efficient if $SYSTEM$ can be decomposed into ED-FR and does not require any more communication than the existing market mechanisms used in practice.

4.5.1 Market model

We assume that generators are price-takers. Let π^b_n denote the price paid to dispatch generator n in each period and $\pi^p_{s,n}$ denote the price paid to regulation generator n in outcome s. Then, the expected profit of the dispatch and regulation generators at node n are:

$$
\begin{aligned}
&\mathrm{PF}^b_n(q^b_n, \pi^b_n) := K\left(\pi^b_n q^b_n - c^b_n(q^b_n)\right), \\
&\mathrm{PF}^p_n((q^p_n + r^p_{s,n}, \pi^p_{s,n}), s \in S) \\
&:= \sum_{s \in S} p_s \left(\pi^p_{s,n}\left(q^p_n + r^p_{s,n}\right) - c^p_n(q^p_n + r^p_{s,n})\right).
\end{aligned}
$$

Note that the regulation generator's profit is a function of its total production $q_n^p + r_{s,n}^p$ in each outcome $s \in S$. The supply function bids indicate the quantities the generators are willing to produce at every price. We assume that these bids are chosen from a parameterized family of functions. In particular, for node n, we represent the dispatch and regulation generators' supply functions by parameters $\alpha_n^b > 0$ and $\alpha_n^p > 0$ respectively, and these bids indicate that the dispatch generator is willing to supply the quantity $q_n^b = [\alpha_n^b s_n^b(\pi_n^b)]_{\underline{q}_n^b}^{\bar{q}_n^b}$ in the first time period and the regulation generator is willing to supply the quantity $q_n^p + r_{s,n}^p = [\alpha_n^p s_n^p(\pi_{s,n}^p)]_{\underline{q}_n^p}^{\bar{q}_n^p}$ in outcome s, for some fixed functions $s_n^b : [\underline{q}_n^b, \bar{q}_n^b] \to \mathbb{R}_+$ and $s_n^p : [\underline{q}_n^p, \bar{q}_n^p] \to \mathbb{R}_+$.[1] We also assume that $s_n^b(\pi_n^b) \neq 0$ for all $\pi_n^b \in \mathbb{R}$ and $s_n^p(\pi_{s,n}^p) \neq 0$ for all $\pi_{s,n}^p \in \mathbb{R}$.[2] The generators choose their bids to maximize their profits subject to their capacity constraints. Note that the regulation generator submits only one supply function for all possible outcomes. Hence, its bid in the economic dispatch timescale is also used as its bid in the frequency regulation timescale.

The system operator interprets bids α_n^b and α_n^p as signals that the dispatch and regulation generators at node n have marginal costs π_n^b and $\pi_{s,n}^p$ respectively when supplying quantities $\alpha_n^b s_n^b(\pi_n^b)$ and $\alpha_n^p s_n^p(\pi_{s,n}^p)$ respectively. Hence, it associates with the generators the following bid cost functions:

$$\hat{c}_n^b(q_n^b) := \int_{\underline{q}_n^b}^{q_n^b} (s_n^b)^{-1}(w/\alpha_n^b)\, dw, \tag{4.12}$$

$$\hat{c}_n^p(q_n^p) := \int_{\underline{q}_n^p}^{q_n^p} (s_n^p)^{-1}(w/\alpha_n^p)\, dw. \tag{4.13}$$

Let $\boldsymbol{\alpha}^b := (\alpha_n^b, n \in N)$ and $\boldsymbol{\alpha}^p := (\alpha_n^p, n \in N)$ denote the vectors of bids. Given bids $(\boldsymbol{\alpha}^b, \boldsymbol{\alpha}^p)$, the system operator solves ED to minimize expected bid costs. The prices for the regulation generator in the first time period are the nodal prices in ED while the prices for the dispatch generator are the nodal prices offset by $\boldsymbol{\delta}$. Then, in each subsequent outcome $s \in S$, the system operator implements the controller in (4.9) using regulation generators' bid costs. The prices are the nodal prices in FR (which are computed by DFR).

4.5.2 Market equilibrium

Our focus is on understanding the efficiency of the mechanism. Formally, we consider the following notion of a competitive equilibrium.

Definition 4.2. *We say that bids $(\boldsymbol{\alpha}^b, \boldsymbol{\alpha}^p)$ are a competitive equilibrium if there exists prices $\boldsymbol{\pi}^b \in$*

[1] Numerous studies have explored different functional forms of the supply functions and their impact on market efficiency, e.g., see [111, 15, 68, 14, 76]. The focus of this work is on illustrating that ED can be implemented using a simple market mechanism. Hence, we restrict ourselves to linearly parameterized supply functions and leave the analyses of other more sophisticated supply functions to future work. We refer the reader to [68] for some appealing properties of linearly parameterized supply functions.

[2] This assumption is a technical condition to avoid the degenerate situation where a generator's supply quantity is not sensitive to its bid parameter which would occur if $s_n^b(\pi_n^b) = 0$ or $s_n^p(\pi_{s,n}^p) = 0$.

$\mathbb{R}^N$ *and* $\boldsymbol{\pi}^p = (\boldsymbol{\pi}^p_s, s \in S) \in \mathbb{R}^{NS}$ *such that:*

(a) *For all* n, α^b_n *is an optimal solution to:*

$$\max_{\hat{\alpha}^b_n > 0} \quad \mathrm{PF}^b_n \left([\hat{\alpha}^b_n s^b_n(\pi^b_n)]^{\bar{q}^b_n}_{\underline{q}^b_n}, \pi^b_n \right).$$

(b) *For all* n, α^p_n *is an optimal solution to:*

$$\max_{\hat{\alpha}^p_n > 0} \quad \mathrm{PF}^p_n \left(([\hat{\alpha}^p_n s^p_n(\pi^p_{s,n})]^{\bar{q}^p_n}_{\underline{q}^p_n}, \pi^p_{s,n}), s \in S \right).$$

(c) $\boldsymbol{\pi}^b = (1/K)\left(\boldsymbol{\pi}(\lambda_1, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1) + \boldsymbol{\delta}\right)$ *and* $\boldsymbol{\pi}^p_1 = (1/K)\boldsymbol{\pi}(\lambda_1, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1)$ *where* λ_1 *and* $(\underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1)$ *are the Lagrange multipliers associated with constraints* (4.3) *and* (4.5) *respectively in:*

$$\begin{aligned} \hat{ED}(\mathbf{d}_1): \quad & \min_{\mathbf{q}^b, \mathbf{q}^p} \quad \sum_{n \in N} \left(K\hat{c}^b_n(q^b_n) + K\hat{c}^p_n(q^p_n) - \delta_n q^b_n \right) \\ & \text{s.t.} \quad (\mathbf{q}^b, \mathbf{q}^p, \mathbf{0}) \in \Omega(\mathbf{d}_1). \end{aligned}$$

(d) *For all* $s \in S$, $\boldsymbol{\pi}^p_s = \boldsymbol{\pi}(\lambda_s, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s)$ *where* λ_s *and* $(\underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s)$ *are the Lagrange multipliers associated with constraints* (4.3) *and* (4.5) *respectively in:*

$$\begin{aligned} \hat{FR}(\mathbf{q}^b, \mathbf{q}^p, \mathbf{d}_s): \quad & \min_{\mathbf{r}^p_s} \quad \sum_{n \in N} \hat{c}^p_n(q^p_n + r^p_{s,n}) \\ & \text{s.t.} \quad (\mathbf{q}^b, \mathbf{q}^p, \mathbf{r}^p_s) \in \Omega(\mathbf{d}_s), \end{aligned}$$

where $\mathbf{q}^b = \left([\alpha^b_n s^b_n(\pi^b_n)]^{\bar{q}^b_n}_{\underline{q}^b_n}, n \in N \right)$ *and* $\mathbf{q}^p = \left([\alpha^p_n s^p_n(\pi^p_{1,n})]^{\bar{q}^p_n}_{\underline{q}^p_n}, n \in N \right)$.

At each node $n \in N$, *the dispatch and regulation generators produce at setpoints* $[\alpha^b_n s^b_n(\pi^b_n)]^{\bar{q}^b_n}_{\underline{q}^b_n}$ *and* $[\alpha^p_n s^p_n(\pi^p_{1,n})]^{\bar{q}^p_n}_{\underline{q}^p_n}$ *respectively in period* 1, *and the regulation generator produces an additional quantity* $[\alpha^p_n s^p_n(\pi^p_{s,n})]^{\bar{q}^p_n}_{\underline{q}^p_n} - [\alpha^p_n s^p_n(\pi^p_{1,n})]^{\bar{q}^p_n}_{\underline{q}^p_n}$ *in outcome* $s \in S$.

The following is our main result for this section. It highlights that, as a consequence of Theorem 4.1, any competitive equilibrium is efficient.

Proposition 4.3 (Efficiency)**.** *Suppose that, for each* $n \in N$, *the functions* $s^b_n(\cdot) = c^{b\prime -1}_n(\cdot)/\gamma^b_n$ *and* $s^p_n(\cdot) = c^{p\prime -1}_n(\cdot)/\gamma^p_n$ *for some constants* $\gamma^b_n, \gamma^p_n > 0$. *Let* λ_s *and* $(\underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s)$ *be the Lagrange multipliers associated with constraints* (4.3) *and* (4.5) *respectively in* $SYSTEM$. *Suppose that* (4.7) *holds. Then:*

(a) *Any competitive equilibrium has a production schedule that solves* $SYSTEM$.

(b) *Any production schedule that solves* $SYSTEM$ *can be sustained by a competitive equilibrium.*

Table 4.1: Generators on test system.

Unit Group	Unit Type	Production Range (MW)	Marginal Cost Range ($/MWh)	Assignment
U12	Oil/Steam	$[10, 60]$	$[58.14, 64.446]$	Dispatch
U20	Oil/CT	$[64, 80]$	$[130.0, 130.0]$	Regulation
U50	Hydro	$[60, 300]$	$[0.001, 0.001]$	Regulation
U76	Coal/Steam	$[60, 304]$	$[16.511, 18.231]$	Dispatch
U100	Oil/Steam	$[75, 300]$	$[46.295, 54.196]$	Dispatch
U155	Coal/Steam	$[216, 620]$	$[13.294, 14.974]$	Dispatch
U197	Oil/Steam	$[207, 591]$	$[49.57, 51.405]$	Dispatch
U350	Coal/Steam	$[140, 350]$	$[13.22, 15.276]$	Dispatch
U400	Nuclear	$[200, 800]$	$[4.466, 4.594]$	Dispatch

Proposition 4.3 resembles classical welfare theorems, e.g., [68, 91, 69, 131]. However, it differs from typical competitive equilibria frameworks because each regulation generator is restricted to bidding a single supply function over the entire economic dispatch interval even though there are multiple fast timescale instances. The latter creates challenges in guaranteeing existence and efficiency of equilibria that do not arise in typical competitive equilibria frameworks. In particular, the space of bid functions needs to be sufficiently expressive for generators to convey their costs over multiple fast timescale instances via a single bid function. This is not an issue in market frameworks where separate bids are collected for separate market instances. Proposition 4.3 circumvented this challenge by restricting supply functions to be in the linear space of regulation generators' true cost functions. An important extension is to understand the existence and efficiency of equilibria under less restrictive bid spaces. Proposition 4.3 also highlights that nodal pricing is not always efficient and that the pricing mechanism needs to be jointly designed and analyzed with decomposition principles in order to achieve efficiency.

4.6 Case Study

The efficiency of the mechanisms in Sections 4.4 and 4.5 depends on how accurately the system operator can predict the RHS of equation (4.7). In this section, we investigate the sensitivity of the performance of the decomposition to the value of $\boldsymbol{\delta}$ using a case study on the IEEE 24-bus reliability test system [137].

Table 4.1 summarizes the properties of the generators on the system. We assume that the hydro and combustion turbine (CT) generators are regulation resources while all other generators are dispatch resources. Note that, the hydro resources, which generate between 60 to 300 MW, have the lowest marginal cost, while the CT resources, which generate between 64 to 80 MW, have the highest marginal cost.

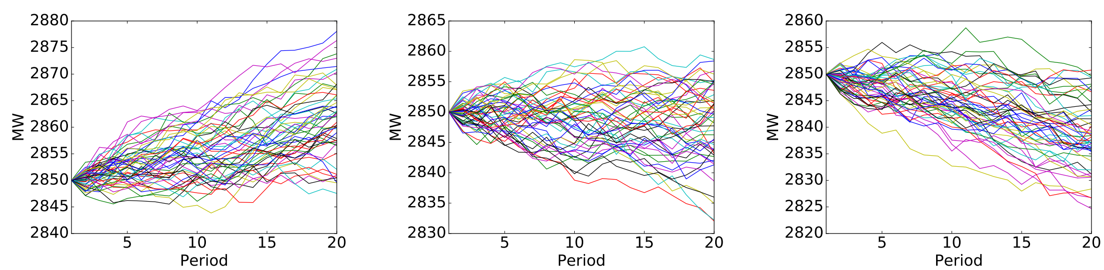
We assume that there are $K = 20$ time periods in the economic dispatch interval. Hence, each time period lasts 15 seconds. We construct the scenario tree as follows. Abusing notation, let $\mathbf{d}_k \in \mathbb{R}^N$ denote the demand at all nodes in period k. Set $\mathbf{d}_1$ to the values in the test system data and let

$$\mathbf{d}_k = \text{diag}\left(\mathbf{1} + \sum_{k'=1}^{k-1} \mathbf{w}_{k'}\right) \cdot \mathbf{d}_1,$$

where $\mathbf{w}_{k'} \sim \mathcal{N}(\mu_d \mathbf{1}, (0.002^2)\mathbf{I})$ is a Gaussian vector with mean $\mu_d \mathbf{1}$ and covariance $(0.002^2)\mathbf{I}$. We simulate $\mu_d = -0.0002, 0, +0.0002$ to model scenarios with increasing, constant, and decreasing demands, respectively. For each value of μ_d, we generate 50 length-K samples of the random process and assign equal probabilities to all the samples. Hence, the scenario tree is a tall tree, where the root node has 50 children, and all other nodes either have one child or is a leaf node. Figure 4.1 shows the sample trajectories of total system demand. The RHS of equation (4.7) have values $\delta_n^* = -80.34, 49.67, 49.72$ (in \$/MWh) corresponding to $\mu_d = -0.0002, 0, +0.0002$, respectively. Note that the optimal δ_n is non-zero even when the demand evolution has zero mean. To study the impact when $\boldsymbol{\delta}$ deviates from $\boldsymbol{\delta}^*$, we consider

$$\boldsymbol{\delta} = \boldsymbol{\delta}^* + \boldsymbol{\epsilon},$$

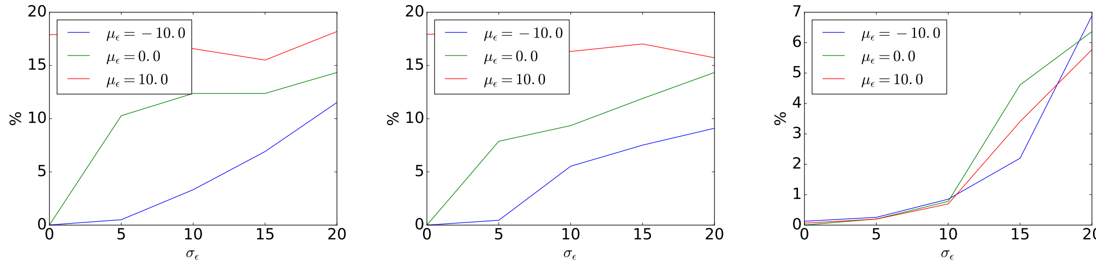
where $\boldsymbol{\epsilon} \sim \mathcal{N}(\mu_\epsilon \mathbf{1}, \sigma_\epsilon^2 \mathbf{I})$. Hence, μ_ϵ and σ_ϵ can be interpreted as the bias and standard deviation of the prediction errors. Given any specified μ_ϵ and σ_ϵ, we generate 50 samples of $\boldsymbol{\delta}$ and randomly match these samples to the 50 demand samples. Figure 4.2 shows the percentage increase in average total costs under the decomposition (compared to the optimal solution) for different values of μ_ϵ and σ_ϵ.



(a) Increasing Demand (b) Constant Demand (c) Decreasing Demand

Figure 4.1: Samples paths of total demand.

Observe the asymmetry in the plots between the different choices of μ_ϵ. In particular, when demand is constant ($\mu_d = 0$) or increasing ($\mu_d = 0.0002$), the sub-optimality is less sensitive to negative prediction errors ($\mu_\epsilon = -10$) than to positive prediction errors ($\mu_\epsilon = 10$). This phenomenon is due to high utilization of regulation resources at the optimal solution. Since majority of the



(a) Increasing Demand (b) Constant Demand (c) Decreasing Demand

Figure 4.2: Percentage change in average total costs under $ED\text{-}FR$.

regulation resources are hydro resources with low marginal costs, almost all the regulation resources are dispatched in the first time period, and there are less than 30MW of unused regulation capacity. Recall that negative prediction errors create incentive for economic dispatch to use less-than-optimal dispatch resources and more-than-optimal regulation resources. However, in the scenario with increasing demand, system demand increases by up to about 30MW. This deviation must be met by regulation resources, and since there are only 30MW of unused regulation capacity, the economic dispatch mechanism is unable to significantly increase the usage of regulation resources in the first time period, and therefore the sub-optimality is small under negative prediction errors. On the other hand, at the optimal solution, there is significant excess capacity of dispatch resources. Hence, positive prediction errors could lead to significantly more usage of dispatch resources and less usage of regulation resources, and cause a larger increase in total costs. We do not observe this asymmetry when demand is decreasing. This is due to the fact that $\delta_n^* = -80.34$ and hence a larger positive prediction error is needed for the asymmetry to manifest.

The simulations illustrate the complex interactions between $\boldsymbol{\delta}$ and the performance of $ED\text{-}FR$. In particular, both marginal costs and feasibility constraints have crucial impacts on the performance of $ED\text{-}FR$.

Appendices

4.A Proof of Theorem 4.1

Proof. The result follows from analyzing the Karush-Kuhn-Tucker (KKT) conditions of $SYSTEM$, ED, and FR. However, we first reformulate the problems as the notations are simpler with the reformulations. Define $\mathbf{q}^p_s := \mathbf{q}^p + \mathbf{r}^p_s$. Note that, due to the constraint that $\mathbf{r}^p_1 = \mathbf{0}$, there is a bijection between the set of feasible $(\mathbf{q}^b, \mathbf{q}^p, \mathbf{r}^p)$ and the set of feasible $(\mathbf{q}^b, \mathbf{q}^p_1, \ldots, \mathbf{q}^p_S)$. Hence, $SYSTEM$ can be reformulated as:

$$\begin{aligned} \min_{\mathbf{q}^b, \mathbf{q}^p_1, \ldots, \mathbf{q}^p_S} \quad & \sum_{s \in S} p_s \sum_{n \in N} \left(c^b_n(q^b_n) + c^p_n(q^p_{s,n}) \right) \\ \text{s.t.} \quad & (\mathbf{q}^b, \mathbf{q}^p_1, \mathbf{q}^p_s - \mathbf{q}^p_1) \in \Omega(\mathbf{d}_s), \quad \forall s \in S. \end{aligned} \tag{4.14}$$

Also, $ED(\mathbf{d}_1)$ can be reformulated as:

$$\begin{aligned} \min_{\mathbf{q}^b, \mathbf{q}^p_1} \quad & \sum_{n \in N} \left(K c^b_n(q^b_n) + K c^p_n(q^p_{1,n}) - \delta_n q^b_n \right) \\ \text{s.t.} \quad & (\mathbf{q}^b, \mathbf{q}^p_1, \mathbf{0}) \in \Omega(\mathbf{d}_1). \end{aligned} \tag{4.15}$$

And, $FR(\mathbf{q}^b, \mathbf{q}^p, \mathbf{d}_s)$ can be reformulated as:

$$\begin{aligned} \min_{\mathbf{q}^p_s} \quad & \sum_{n \in N} c^p_n(q^p_{s,n}) \\ \text{s.t.} \quad & (\mathbf{q}^b, \mathbf{q}^p_1, \mathbf{q}^p_s - \mathbf{q}^p_1) \in \Omega(\mathbf{d}_s). \end{aligned} \tag{4.16}$$

Hence, $SYSTEM$ can be optimally decomposed into ED-FR if $(\mathbf{q}^b, \mathbf{q}^p_1, \ldots, \mathbf{q}^p_S)$ is an optimal solution to (4.14) if and only if $(\mathbf{q}^b, \mathbf{q}^p_1)$ is an optimal solution to (4.15) and $\mathbf{q}^p_s$ is an optimal solution to (4.16) for all $s \in S$.

Next, we prove (a). It is easy to see that (4.14) has compact sub-level sets. Moreover, its objective function is strictly convex. Hence, (4.14) has a unique optimal solution. By similar arguments, we conclude that (4.15) has a unique optimal solution, and that (4.16) has a unique optimal solution if the set $\left\{ \mathbf{q}^p_s \in \mathbb{R}^N : (\mathbf{q}^b, \mathbf{q}^p_1, \mathbf{q}^p_s - \mathbf{q}^p_1) \in \Omega(\mathbf{d}_s) \right\}$ is non-empty. Hence, to prove (a), it suffices to show the forward implication, that is, if (4.7) holds, then $(\mathbf{q}^b, \mathbf{q}_1^p, \ldots, \mathbf{q}_S^p)$ is an optimal solution to (4.14) implies that $(\mathbf{q}^b, \mathbf{q}_1^p)$ is an optimal solution to (4.15) and $\mathbf{q}_s^p$ is an optimal solution to (4.16) for all $s \in S$. The reverse implication follows from the existence and uniqueness of the optimal solutions.

Let the Lagrangian of (4.14) be denoted by:

$$\begin{aligned}
&L(\mathbf{q}^b, \mathbf{q}_1^p, \ldots, \mathbf{q}_S^p, \underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\xi}}, \underline{\boldsymbol{\nu}}, \bar{\boldsymbol{\nu}}, \underline{\boldsymbol{\mu}}, \bar{\boldsymbol{\mu}}, \boldsymbol{\lambda}) \\
&:= \sum_{s \in S} p_s \sum_{n \in N} \left(c_n^b(q_n^b) + c_n^p(q_{s,n}^p) \right) + L^b(\mathbf{q}^b, \underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\xi}}) \\
&\quad + \sum_{s \in S} p_s L^p(\mathbf{q}_s^p, \underline{\boldsymbol{\nu}}_s, \bar{\boldsymbol{\nu}}_s) + \sum_{s \in S} p_s L^f(\mathbf{q}^b, \mathbf{q}_s^p, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) \\
&\quad - \sum_{s \in S} p_s \lambda_s \mathbf{1}^\top \left(\mathbf{q}^b + \mathbf{q}_s^p - \mathbf{d}_s \right),
\end{aligned}$$

where:

$$\begin{aligned}
L^b(\mathbf{q}^b, \underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\xi}}) &:= \underline{\boldsymbol{\xi}}^\top \left(\underline{\mathbf{q}}^b - \mathbf{q}^b \right) + \bar{\boldsymbol{\xi}}^\top \left(\mathbf{q}^b - \bar{\mathbf{q}}^b \right) \\
L^p(\mathbf{q}_s^p, \underline{\boldsymbol{\nu}}_s, \bar{\boldsymbol{\nu}}_s) &:= \underline{\boldsymbol{\nu}}_s^\top \left(\underline{\mathbf{q}}^p - \mathbf{q}_s^p \right) + \bar{\boldsymbol{\nu}}_s^\top \left(\mathbf{q}_s^p - \bar{\mathbf{q}}^p \right) \\
L^f(\mathbf{q}^b, \mathbf{q}_s^p, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) &:= \underline{\boldsymbol{\mu}}_s^\top \left(-\mathbf{f} - \mathbf{H} \left(\mathbf{q}^b + \mathbf{q}_s^p - \mathbf{d}_s \right) \right) \\
&\quad + \bar{\boldsymbol{\mu}}_s^\top \left(\mathbf{H} \left(\mathbf{q}^b + \mathbf{q}_s^p - \mathbf{d}_s \right) - \mathbf{f} \right).
\end{aligned}$$

Note that we scaled the constraints by their probabilities, and $\underline{\boldsymbol{\xi}} \in \mathbb{R}_+^N$, $\bar{\boldsymbol{\xi}} \in \mathbb{R}_+^N$, $\underline{\boldsymbol{\nu}} = (\underline{\boldsymbol{\nu}}_s, s \in S) \in \mathbb{R}_+^{NS}$, $\bar{\boldsymbol{\nu}} = (\bar{\boldsymbol{\nu}}_s, s \in S) \in \mathbb{R}_+^{NS}$, $\underline{\boldsymbol{\mu}} = (\underline{\boldsymbol{\mu}}_s, s \in S) \in \mathbb{R}_+^{LS}$, $\bar{\boldsymbol{\mu}} = (\bar{\boldsymbol{\mu}}_s, s \in S) \in \mathbb{R}_+^{LS}$, $\boldsymbol{\lambda} = (\lambda_s, s \in S) \in \mathbb{R}^S$ are appropriate Lagrange multipliers.

Since (4.14) has a convex objective and linear constraints, from the KKT conditions, we infer that $(\mathbf{q}^b, \mathbf{q}_1^p, \ldots, \mathbf{q}_S^p)$ is an optimal solution to (4.14) if and only if $(\mathbf{q}^b, \mathbf{q}_1^p, \mathbf{q}_s^p - \mathbf{q}_1^p) \in \Omega(\mathbf{d}_s)$ for all $s \in S$ and there exists $\underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\xi}} \in \mathbb{R}_+^N, \underline{\boldsymbol{\nu}}, \bar{\boldsymbol{\nu}} \in \mathbb{R}_+^{NS}, \underline{\boldsymbol{\mu}}, \bar{\boldsymbol{\mu}} \in \mathbb{R}_+^{LS}, \boldsymbol{\lambda} \in \mathbb{R}^S$ such that:

$$\left(K c_n^{b\prime}(q_n^b), n \in N \right) + \bar{\boldsymbol{\xi}} - \underline{\boldsymbol{\xi}} - \sum_{s \in S} p_s \boldsymbol{\pi}(\lambda_s, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) = 0; \tag{4.17a}$$

$$L^b(\mathbf{q}^b, \underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\xi}}) = 0; \tag{4.17b}$$

$$\left(c_n^{p\prime}(q_{s,n}^p), n \in N \right) + \bar{\boldsymbol{\nu}}_s - \underline{\boldsymbol{\nu}}_s - \boldsymbol{\pi}(\lambda_s, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) = 0; \tag{4.17c}$$

$$L^p(\mathbf{q}_s^p, \underline{\boldsymbol{\nu}}_s, \bar{\boldsymbol{\nu}}_s) = 0; \tag{4.17d}$$

$$L^f(\mathbf{q}^b, \mathbf{q}_s^p, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) = 0, \tag{4.17e}$$

for all $s \in S$.

Similarly, $(\mathbf{q}^b, \mathbf{q}_1^p)$ is an optimal solution to (4.15) if and only if $(\mathbf{q}^b, \mathbf{q}_1^p, \mathbf{0}) \in \Omega(\mathbf{d}_1)$ and there exists $\underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\xi}} \in \mathbb{R}_+^N, \underline{\boldsymbol{\nu}}_1, \bar{\boldsymbol{\nu}}_1 \in \mathbb{R}_+^N, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1 \in \mathbb{R}_+^L, \lambda_1 \in \mathbb{R}$ such that:

$$\big(Kc_n^{b\prime}(q_n^b), n \in N\big) + \bar{\boldsymbol{\xi}} - \underline{\boldsymbol{\xi}} - \boldsymbol{\pi}(\lambda_1, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1) - \boldsymbol{\delta} = 0; \tag{4.18a}$$

$$L^b(\mathbf{q}^b, \underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\xi}}) = 0; \tag{4.18b}$$

$$\big(Kc_n^{p\prime}(q_{1,n}^p), n \in N\big) + \bar{\boldsymbol{\nu}}_1 - \underline{\boldsymbol{\nu}}_1 - \boldsymbol{\pi}(\lambda_1, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1) = 0; \tag{4.18c}$$

$$L^p(\mathbf{q}_1^p, \underline{\boldsymbol{\nu}}_1, \bar{\boldsymbol{\nu}}_1) = 0; \tag{4.18d}$$

$$L^f(\mathbf{q}^b, \mathbf{q}_1^p, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1) = 0. \tag{4.18e}$$

And $\mathbf{q}_s^p$ is an optimal solution to (4.16) if and only if $(\mathbf{q}^b, \mathbf{q}_1^p, \mathbf{q}_s^p - \mathbf{q}_1^p) \in \Omega(\mathbf{d}_s)$ and there exists $\underline{\boldsymbol{\nu}}_s, \bar{\boldsymbol{\nu}}_s \in \mathbb{R}_+^N, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s \in \mathbb{R}_+^L, \lambda_s \in \mathbb{R}$ such that:

$$\big(c_n^{p\prime}(q_{s,n}^p), n \in N\big) + \bar{\boldsymbol{\nu}}_s - \underline{\boldsymbol{\nu}}_s - \boldsymbol{\pi}(\lambda_s, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) = 0; \tag{4.19a}$$

$$L^p(\mathbf{q}_s^p, \underline{\boldsymbol{\nu}}_s, \bar{\boldsymbol{\nu}}_s) = 0; \tag{4.19b}$$

$$L^f(\mathbf{q}^b, \mathbf{q}_s^p, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) = 0. \tag{4.19c}$$

Suppose $(\mathbf{q}^b, \mathbf{q}_1^p, \ldots, \mathbf{q}_S^p)$ is an optimal solution to (4.14) with associated Lagrange multipliers $(\underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\xi}}, \underline{\boldsymbol{\nu}}, \bar{\boldsymbol{\nu}}, \underline{\boldsymbol{\mu}}, \bar{\boldsymbol{\mu}}, \boldsymbol{\lambda})$. Note that $(\mathbf{q}^b, \mathbf{q}_1^p, \mathbf{0}) \in \Omega(\mathbf{d}_1)$. From the fact that the variables $(\mathbf{q}^b, \underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\xi}}, \underline{\boldsymbol{\mu}}, \bar{\boldsymbol{\mu}}, \lambda)$ satisfy (4.17a) and (4.7) and the fact that $\sum_{s\in S} p_s = K$, we infer that the variables $(\mathbf{q}^b, \underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\xi}}, K\underline{\boldsymbol{\mu}}_1, K\bar{\boldsymbol{\mu}}_1, K\lambda_1)$ satisfy (4.18a). From the fact that $(\mathbf{q}^b, \mathbf{q}_s^p, \bar{\boldsymbol{\xi}}, \underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\nu}}_s, \underline{\boldsymbol{\nu}}_s, \bar{\boldsymbol{\mu}}_s, \underline{\boldsymbol{\mu}}_s, \lambda_s)$ satisfy (4.17b) – (4.17e), we infer that the variables $(\mathbf{q}^b, \mathbf{q}_1^p, \bar{\boldsymbol{\xi}}, \underline{\boldsymbol{\xi}}, K\bar{\boldsymbol{\nu}}_1, K\underline{\boldsymbol{\nu}}_1, K\bar{\boldsymbol{\mu}}_1, K\underline{\boldsymbol{\mu}}_1, K\lambda_1)$ satisfy (4.18b) – (4.18e). Hence, $(\mathbf{q}^b, \mathbf{q}_1^p)$ is an optimal solution to (4.15). Note also that $(\mathbf{q}^b, \mathbf{q}_1^p, \mathbf{q}_s^p - \mathbf{q}_1^p) \in \Omega(\mathbf{d}_s)$ for all $s \in S$. From the fact that the variables $(\mathbf{q}^b, \mathbf{q}_s^p, \underline{\boldsymbol{\nu}}_s, \bar{\boldsymbol{\nu}}_s, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s, \lambda_s)$ satisfy (4.17c) – (4.17e), we infer that those variables satisfy (4.19). Hence, $\mathbf{q}_s^p$ is an optimal solution to (4.16) for all $s \in S$.

Next, we prove (b). Let $(\mathbf{q}^b, \mathbf{q}_1^p, \ldots, \mathbf{q}_S^p)$ be a solution to (4.14) such that $(\mathbf{q}^b, \mathbf{q}_1^p)$ is a solution to (4.15). If $\underline{q}_n^b < q_n^b < \bar{q}_n^b$ and $\underline{q}_n^p < q_{1,n}^p < \bar{q}_n^p$, then the complementary slackness conditions imply that $\underline{\xi}_n = \bar{\xi}_n = 0$ and $\underline{\nu}_{1,n} = \bar{\nu}_{1,n} = 0$. From the KKT conditions of (4.14), which are given by (4.17), we infer that:

$$Kc_n^{b\prime}(q_n^b) - \sum_{s\in S} p_s \pi_n(\lambda_s, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) = 0; \tag{4.20}$$

$$c_n^{p\prime}(q_{1,n}^p) - \pi_n(\lambda_1, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1) = 0, \tag{4.21}$$

where $(\underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s, \boldsymbol{\lambda})$ are the associated Lagrange multipliers. From the KKT conditions of (4.15), which are given by (4.18), we infer that:

$$Kc_n^{b\prime}(q_n^b) - \pi_n(\lambda_1', \underline{\boldsymbol{\mu}}_1', \bar{\boldsymbol{\mu}}_1') - \delta_n = 0; \tag{4.22}$$

$$Kc_n^{p\prime}(q_{1,n}^p) - \pi_n(\lambda_1', \underline{\boldsymbol{\mu}}_1', \bar{\boldsymbol{\mu}}_1') = 0, \tag{4.23}$$

where $(\underline{\boldsymbol{\mu}}_s', \bar{\boldsymbol{\mu}}_s', \boldsymbol{\lambda}')$ are the associated Lagrange multipliers. It follows that:

$$\begin{aligned}
\delta_n &= \sum_{s\in S} p_s \pi_n(\lambda_s, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) - \pi_n(\lambda_1', \underline{\boldsymbol{\mu}}_1', \bar{\boldsymbol{\mu}}_1') \\
&= \sum_{s\in S} p_s \pi_n(\lambda_s, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) - K\pi_n(\lambda_1, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1) \\
&= \sum_{s\in S} p_s \left(\pi_n(\lambda_s, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) - \pi_n(\lambda_1, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1)\right).
\end{aligned}$$

The first equality follows from comparing (4.20) and (4.22). The second equality follows from comparing (4.21) and (4.23). The last equality follows from the fact that $\sum_{s\in S} p_s = K$. □

4.B Proof of Proposition 4.1

Proof. We provide a proof sketch of this result. The skipped details can be found in [87]. (i) follows from the KKT conditions of $FR'(\mathbf{q}^b, \mathbf{q}^p, \mathbf{d}_s)$ and is shown in [87, Lemma 2]. Since $\boldsymbol{\omega}_s' = 0$, it follows from constraints (4.11a) and (4.11b) of $FR'(\mathbf{q}^b, \mathbf{q}^p, \mathbf{d}_s)$ that $\mathbf{L}\boldsymbol{\theta}_s' = \mathbf{L}\boldsymbol{\phi}_s'$, which, since the null space of $\mathbf{L}$ is span$\{\mathbf{1}\}$, implies that $\boldsymbol{\theta}_s' = \boldsymbol{\phi}_s' + \alpha\mathbf{1}$ for some $\alpha \in \mathbb{R}$. This implies that $\mathbf{BC}^\top\boldsymbol{\phi}_s' = \mathbf{BC}^\top\boldsymbol{\theta}_s'$. Therefore, without loss of generality, we can substitute constraint (4.11a) in $FR'(\mathbf{q}^b, \mathbf{q}^p, \mathbf{d}_s)$ by the constraint $\boldsymbol{\omega}_s = 0$. Then, using the definition of $\mathbf{H}$ and the equivalence between (4.3) and (4.4), we infer that the feasible sets of $FR(\mathbf{q}^b, \mathbf{q}^p, \mathbf{d}_s)$ and $FR'(\mathbf{q}^b, \mathbf{q}^p, \mathbf{d}_s)$ are equivalent. Finally, since $c_n^p(\cdot)$ is strictly convex, by uniqueness of the optimal solutions, we get (ii). Lastly, (iii) follows from the definition of $\mathbf{H}$ and $\mathbf{BC}^\top\boldsymbol{\phi}_s' = \mathbf{BC}^\top\boldsymbol{\theta}_s'$. The final statement of the proposition follows directly from [87, Theorem 8]. □

4.C Proof of Proposition 4.3

Proof. Our proof proceeds in 6 steps: (1) Characterizing regulation generators' optimal bids $\boldsymbol{\alpha}^p$ given their prices $\boldsymbol{\pi}^p$; (2) Characterizing dispatch generators' optimal bids $\boldsymbol{\alpha}^b$ given their prices $\boldsymbol{\pi}^b$; (3) Characterizing prices $(\boldsymbol{\pi}^b, \boldsymbol{\pi}^p)$ given bids $(\boldsymbol{\alpha}^b, \boldsymbol{\alpha}^p)$ using KKT conditions; (4) Showing that, at an equilibrium, the production schedule is the unique optimal solution to $\hat{ED}$-$\hat{FR}$; (5) Showing that any production schedule $(\mathbf{q}^b, \mathbf{q}^p, \mathbf{r}^p)$ that solves $SYSTEM$ can be obtained using bids $(\boldsymbol{\gamma}^b, \boldsymbol{\gamma}^p)$ and the latter satisfy the equilibrium characterizations in steps 1 to 3; and (6) Showing that any bids $(\boldsymbol{\alpha}^b, \boldsymbol{\alpha}^p)$ that satisfy the equilibrium characterizations in steps 1 to 3 give the same production schedule as that under bids $(\boldsymbol{\gamma}^b, \boldsymbol{\gamma}^p)$ (which also solves $SYSTEM$). Note that part (a) follows from step 6 and part (b) follows from step 5.

Step 1: Characterizing regulation generators' optimal bids $\boldsymbol{\alpha}^p$ given their prices $\boldsymbol{\pi}^p$. Since c_n^p is strictly convex and $c_n^p(q_{s,n}^p) \to +\infty$ as $q_{s,n}^p \to \{\underline{q}_n^p, \bar{q}_n^p\}$, $c_n^{p\prime}$ is invertible. Let $\sigma : S \to S$ be any permutation function that satisfies:

$$c_n^{p\prime-1}(\pi^p_{\sigma(1),n}) \leq c_n^{p\prime-1}(\pi^p_{\sigma(2),n}) \leq \ldots \leq c_n^{p\prime-1}(\pi^p_{\sigma(S),n}),$$

and let integers $i, j \in \{0, 1, \ldots, S\}$ be such that:

$$c_n^{p\prime-1}(\pi^p_{\sigma(s),n}) \leq \underline{q}_n^p \qquad \forall s = 1, \ldots, i; \tag{4.24a}$$

$$\underline{q}_n^p < c_n^{p\prime-1}(\pi^p_{\sigma(s),n}) < \bar{q}_n^p \qquad \forall s = i+1, \ldots, j; \tag{4.24b}$$

$$\bar{q}_n^p \leq c_n^{p\prime-1}(\pi^p_{\sigma(s),n}) \qquad \forall s = j+1, \ldots, S. \tag{4.24c}$$

We now show that $\alpha_n^p \in \mathbb{R}_{++}$ maximizes PF_n^p if and only if:

$$\alpha_n^p s_n^p(\pi^p_{\sigma(s),n}) \leq \underline{q}_n^p \qquad \forall s = 1, \ldots, i; \tag{4.25a}$$

$$\alpha_n^p s_n^p(\pi^p_{\sigma(s),n}) = c_n^{p\prime-1}(\pi^p_{\sigma(k),n}) \qquad \forall s = i+1, \ldots, j; \tag{4.25b}$$

$$\alpha_n^p s_n^p(\pi^p_{\sigma(s),n}) \geq \bar{q}_n^p \qquad \forall s = j+1, \ldots, S. \tag{4.25c}$$

For notational brevity, in the rest of this step, we abuse notation and let:

$$q_{s,n}^p(\alpha_n^p) = [\alpha_n^p s_n^p(\pi^p_{\sigma(s),n})]_{\underline{q}_n^p}^{\bar{q}_n^p}.$$

To prove our characterization, it suffices to show that, given any $\alpha_n^p \in \mathbb{R}_{++}$ that satisfies (4.25), the vector of per-outcome profits

$$\begin{aligned} &\left(\pi^p_{\sigma(s),n} q_{s,n}^p(\alpha_n^p) - c_n^p\left(q_{s,n}^p(\alpha_n^p)\right), s \in S\right) \\ &\gneq \left(\pi^p_{\sigma(s),n} q_{s,n}^p(\bar{\alpha}_n^p) - c_n^p\left(q_{s,n}^p(\bar{\alpha}_n^p)\right), s \in S\right) \end{aligned} \tag{4.26}$$

for any $\bar{\alpha}_n^p$ that does not satisfy (4.25). Since $p_{\sigma(s)} > 0$ for all $s \in S$, it then follows that:

$$\begin{aligned}\mathrm{PF}_n^p|_{\alpha_n^p} &= \sum_s p_{\sigma(s)} \left(\pi_{\sigma(s),n}^p q_{s,n}^p(\alpha_n^p) - c_n^p\left(q_{s,n}^p(\alpha_n^p)\right)\right) \\ &> \sum_s p_{\sigma(s)} \left(\pi_{\sigma(s),n}^p q_{s,n}^p(\bar{\alpha}_n^p) - c_n^p\left(q_{s,n}^p(\bar{\alpha}_n^p)\right)\right) \\ &= \mathrm{PF}_n^p|_{\bar{\alpha}_n^p}.\end{aligned}$$

Suppose $s \in \{1, \dots, i\}$. From (4.24a) and the fact that c_n^p is strictly convex, we infer that $\pi_{\sigma(s),n}^p \le c_n^{p\prime}(\underline{q}_n^p)$. From (4.25a), we infer that $q_{s,n}^p(\alpha_n^p) = \underline{q}_n^p$. Then:

$$\begin{aligned}& c_n^p(q_{s,n}^p(\bar{\alpha}_n^p)) \\ &\ge c_n^p(\underline{q}_n^p) + c_n^{p\prime}(\underline{q}_n^p)\left(q_{s,n}^p(\bar{\alpha}_n^p) - \underline{q}_n^p\right) \\ &\ge c_n^p(\underline{q}_n^p) + \pi_{\sigma(s),n}^p\left(q_{s,n}^p(\bar{\alpha}_n^p) - \underline{q}_n^p\right) \\ &= c_n^p(q_{s,n}^p(\alpha_n^p)) + \pi_{\sigma(s),n}^p\left(q_{s,n}^p(\bar{\alpha}_n^p) - q_{s,n}^p(\alpha_n^p)\right),\end{aligned}$$

where the first inequality follows from the fact that c_n^p is strictly convex, the second inequality follows from $\pi_{\sigma(s),n}^p \le c_n^{p\prime}(\underline{q}_n^p)$ and $q_{s,n}^p(\bar{\alpha}_n^p) \ge \underline{q}_n^p$, and the last equality follows from $q_{s,n}^p(\alpha_n^p) = \underline{q}_n^p$. Furthermore, if $q_{s,n}^p(\bar{\alpha}_n^p) > \underline{q}_n^p$, then the first inequality is strict, and hence:

$$\begin{aligned}& c_n^p(q_{s,n}^p(\bar{\alpha}_n^p)) \\ &> c_n^p(q_{s,n}^p(\alpha_n^p)) + \pi_{\sigma(s),n}^p\left(q_{s,n}^p(\bar{\alpha}_n^p) - q_{s,n}^p(\alpha_n^p)\right).\end{aligned}$$

Suppose $s \in \{i+1, \dots, j\}$. From (4.24b) and (4.25b), we infer that $q_{s,n}^p(\alpha_n^p) = c_n^{p\prime -1}(\pi_{\sigma(s),n}^p)$ and $\underline{q}_n^p < q_{s,n}^p(\alpha_n^p) < \bar{q}_n^p$. From $\underline{q}_n^p < q_{s,n}^p(\alpha_n^p) < \bar{q}_n^p$, and the fact that $s_n^p(\pi_{\sigma(s),n}^p) \ne 0$ and $\bar{\alpha}_n^p \ne \alpha_n^p$, we infer that $q_{s,n}^p(\bar{\alpha}_n^p) \ne q_{s,n}^p(\alpha_n^p)$. Then:

$$\begin{aligned}& c_n^p(q_{s,n}^p(\bar{\alpha}_n^p)) \\ &> c_n^p(q_{s,n}^p(\alpha_n^p)) + c_n^{p\prime}(q_{s,n}^p(\alpha_n^p))\left(q_{s,n}^p(\bar{\alpha}_n^p) - q_{s,n}^p(\alpha_n^p)\right) \\ &= c_n^p(q_{s,n}^p(\alpha_n^p)) + \pi_{\sigma(s),n}^p\left(q_{s,n}^p(\bar{\alpha}_n^p) - q_{s,n}^p(\alpha_n^p)\right),\end{aligned}$$

where the first inequality follows from the fact that c_n^p is strictly convex and $q_{s,n}^p(\bar{\alpha}_n^p) \ne q_{s,n}^p(\alpha_n^p)$ and the equality follows from $q_{s,n}^p(\alpha_n^p) = c_n^{p\prime -1}(\pi_{\sigma(s),n}^p)$.

Suppose $s \in \{i+1, \dots, S\}$. From (4.24c) and the fact that c_n^p is strictly convex, we infer that

$\pi^p_{\sigma(s),n} \geq c^{p\prime}_n(\bar{q}^p_n)$. From (4.25c), we infer that $q^p_{s,n}(\alpha^p_n) = \bar{q}^p_n$. Then:

$$
\begin{aligned}
& c^p_n(q^p_{s,n}(\bar{\alpha}^p_n)) \\
& \geq c^p_n(\bar{q}^p_n) + c^{p\prime}_n(\bar{q}^p_n)\left(q^p_{s,n}(\bar{\alpha}^p_n) - \bar{q}^p_n\right) \\
& \geq c^p_n(\bar{q}^p_n) + \pi^p_{\sigma(s),n}\left(q^p_{s,n}(\bar{\alpha}^p_n) - \bar{q}^p_n\right) \\
& = c^p_n(q^p_{s,n}(\alpha^p_n)) + \pi^p_{\sigma(s),n}\left(q^p_{s,n}(\bar{\alpha}^p_n) - q^p_{s,n}(\alpha^p_n)\right),
\end{aligned}
$$

where the first inequality follows from the fact that c^p_n is strictly convex, the second inequality follows from $\pi^p_{\sigma(s),n} \geq c^{p\prime}_n(\bar{q}^p_n)$ and $q^p_{s,n}(\bar{\alpha}^p_n) \leq \bar{q}^p_n$, and the last equality follows from $q^p_{s,n}(\alpha^p_n) = \bar{q}^p_n$. Furthermore, if $q^p_{s,n}(\bar{\alpha}^p_n) < \bar{q}^p_n$, then the first inequality is strict, and hence:

$$
\begin{aligned}
& c^p_n(q^p_{s,n}(\bar{\alpha}^p_n)) \\
& > c^p_n(q^p_{s,n}(\alpha^p_n)) + \pi^p_{\sigma(s),n}\left(q^p_{s,n}(\bar{\alpha}^p_n) - q^p_{s,n}(\alpha^p_n)\right).
\end{aligned}
$$

Hence, for all $s \in S$:

$$
\begin{aligned}
& c^p_n(q^p_{s,n}(\bar{\alpha}^p_n)) \\
& \geq c^p_n(q^p_{s,n}(\alpha^p_n)) + \pi^p_{\sigma(s),n}\left(q^p_{s,n}(\bar{\alpha}^p_n - q^p_{s,n}(\alpha^p_n)\right).
\end{aligned}
\tag{4.27}
$$

Moreover, this inequality is strict for some $s \in S$. If $i < j$, the inequality is strict for $s \in \{i+1, \ldots, j\}$. If $i = j$, then, since $\bar{\alpha}^p_n$ does not satisfy (4.25), there exists some $s \in \{1, \ldots, i\}$ such that $\alpha^p_n s^p_n(\pi^p_{\sigma(s),n}) > \underline{q}^p_n$ or some $s \in \{i+1, \ldots, S\}$ such that $\alpha^p_n s^p_n(\pi^p_{\sigma(s),n}) < \bar{q}^p_n$, and hence there exists some $s \in \{1, \ldots, i\}$ such that $q^p_{s,n}(\bar{\alpha}^p_n) > \underline{q}^p_n$ or some $s \in \{i+1, \ldots, S\}$ such that $q^p_{s,n}(\bar{\alpha}^p_n) < \bar{q}^p_n$, and the inequality in (4.27) is strict for that s. Hence, we conclude that:

$$
\begin{aligned}
& \left(c^p_n(q^p_{s,n}(\bar{\alpha}^p_n)), s \in S\right) \\
& \gneq \left(c^p_n(q^p_{s,n}(\alpha^p_n)) + \pi^p_{\sigma(s),n}\left(q^p_{s,n}(\bar{\alpha}^p_n) - q^p_{s,n}(\alpha^p_n)\right), s \in S\right)
\end{aligned}
$$

for any $\bar{\alpha}^p_n$ that does not satisfy (4.25). By rearranging terms, we obtain (4.26).

Step 2: Characterizing dispatch generators' optimal bids $\boldsymbol{\alpha}^b$ given their prices $\boldsymbol{\pi}^b$. Note that the profit maximization problem for a dispatch generator is a special case of that for a regulation generator with $S = 1$. By applying the characterization in step 1, we infer that $\alpha^b_n \in \mathbb{R}_{++}$ maximizes

PF_n^b if and only if:

$$\alpha_n^b s_n^b(\pi_n^b) \le \underline{q}_n^b, \qquad \text{if} \qquad c_n^{b\prime -1}(\pi_n^b) \le \underline{q}_n^b; \tag{4.28a}$$

$$\alpha_n^b = \gamma_n^b, \qquad \text{if } \underline{q}_n^b < c_n^{b\prime -1}(\pi_n^b) < \bar{q}_n^b; \tag{4.28b}$$

$$\alpha_n^b s_n^b(\pi_n^b) \ge \bar{q}_n^b, \qquad \text{if } \bar{q}_n^b \le c_n^{b\prime -1}(\pi_n^b). \tag{4.28c}$$

Step 3: Characterizing prices $(\boldsymbol{\pi}^b, \boldsymbol{\pi}^p)$ given bids $(\boldsymbol{\alpha}^b, \boldsymbol{\alpha}^p)$ using KKT conditions. First, we take the same approach as in the proof of Theorem 4.1 and reformulate $\hat{ED}$ and $\hat{FR}$ before applying the KKT conditions. Relabeling the variable $\mathbf{q}^p$ to $\mathbf{q}_1^p$ in $\hat{ED}$ gives:

$$\begin{aligned} \min_{\mathbf{q}^b, \mathbf{q}_1^p} \quad & \sum_{n \in N} \left(K\hat{c}_n^b(q_n^b) + K\hat{c}_n^p(q_{1,n}^p) - \delta_n q_n^b \right) \\ \text{s.t.} \quad & (\mathbf{q}^b, \mathbf{q}_1^p, \mathbf{0}) \in \Omega(\mathbf{d}_1). \end{aligned} \tag{4.29}$$

And substituting $\mathbf{q}_s^p = \mathbf{q}^p + \mathbf{r}_s^p$ in $\hat{FR}$ gives:

$$\begin{aligned} \min_{\mathbf{q}_s^p} \quad & \sum_{n \in N} \hat{c}_n^p(q_{s,n}^p) \\ \text{s.t.} \quad & (\mathbf{q}^b, \mathbf{q}_1^p, \mathbf{q}_s^p - \mathbf{q}_1^p) \in \Omega(\mathbf{d}_s). \end{aligned} \tag{4.30}$$

Substituting $s_n^b = c_n^{b\prime -1}(\cdot)/\gamma_n^b$ and $s_n^p = c_n^{p\prime -1}(\cdot)/\gamma_n^p$ into the definition of $\hat{c}_n^b$ and $\hat{c}_n^p$ implies that:

$$\hat{c}_n^b(q_n^b) = \int_{\underline{q}_n^b}^{q_n^b} c_n^{b\prime}((\gamma_n^b/\alpha_n^b)w)\, dw,$$

$$\hat{c}_n^p(q_n^p) = \int_{\underline{q}_n^p}^{q_n^p} c_n^{p\prime}((\gamma_n^p/\alpha_n^p)w)\, dw.$$

Hence, (4.29) has a continuous and strictly convex objective and linear constraints. Thus, from the KKT conditions, $(\mathbf{q}^b, \mathbf{q}_1^p)$ is an optimal solution to (4.29) if and only if $(\mathbf{q}^b, \mathbf{q}_1^p, \mathbf{0}) \in \Omega(\mathbf{d}_1)$ and there exists $\underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\xi}} \in \mathbb{R}_+^N, \underline{\boldsymbol{\nu}}_1, \bar{\boldsymbol{\nu}}_1 \in \mathbb{R}_+^N, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1 \in \mathbb{R}_+^L, \lambda_1 \in \mathbb{R}$ such that:

$$\left(Kc_n^{b\prime}((\gamma_n^b/\alpha_n^b)q_n^b), n \in N\right) + \bar{\boldsymbol{\xi}} - \underline{\boldsymbol{\xi}} - K\boldsymbol{\pi}^b = 0; \tag{4.31a}$$

$$L^b(\mathbf{q}^b, \underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\xi}}) = 0; \tag{4.31b}$$

$$\left(Kc_n^{p\prime}((\gamma_n^p/\alpha_n^p)q_{1,n}^p), n \in N\right) + \bar{\boldsymbol{\nu}}_1 - \underline{\boldsymbol{\nu}}_1 - K\boldsymbol{\pi}_1^p = 0; \tag{4.31c}$$

$$L^p(\mathbf{q}_1^p, \underline{\boldsymbol{\nu}}_1, \bar{\boldsymbol{\nu}}_1) = 0; \tag{4.31d}$$

$$L^f(\mathbf{q}^b, \mathbf{q}_1^p, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1) = 0, \tag{4.31e}$$

where:

$$\boldsymbol{\pi}^b = (1/K)\left(\boldsymbol{\pi}(\lambda_1, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1) + \boldsymbol{\delta}\right); \tag{4.31f}$$

$$\boldsymbol{\pi}_1^p = (1/K)\boldsymbol{\pi}(\lambda_1, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1). \tag{4.31g}$$

Similarly, from the KKT conditions, $\mathbf{q}_s^p$ is an optimal solution to (4.30) if and only if $(\mathbf{q}^b, \mathbf{q}_1^p, \mathbf{q}_s^p - \mathbf{q}_1^p) \in \Omega(\mathbf{d}_s)$ and there exists $\underline{\boldsymbol{\nu}}_s, \bar{\boldsymbol{\nu}}_s \in \mathbb{R}_+^N, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s \in \mathbb{R}_+^L, \lambda_s \in \mathbb{R}$ such that:

$$\left(c_n^{p\prime}((\gamma_n^p/\alpha_n^p)q_{s,n}^p), n \in N\right) + \bar{\boldsymbol{\nu}}_s - \underline{\boldsymbol{\nu}}_s - \boldsymbol{\pi}_s^p = 0; \tag{4.32a}$$

$$L^p(\mathbf{q}_s^p, \underline{\boldsymbol{\nu}}_s, \bar{\boldsymbol{\nu}}_s) = 0; \tag{4.32b}$$

$$L^f(\mathbf{q}^b, \mathbf{q}_s^p, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) = 0, \tag{4.32c}$$

where:

$$\boldsymbol{\pi}_s^p = \boldsymbol{\pi}(\lambda_s, \underline{\boldsymbol{\mu}}_s, \boldsymbol{\mu}_s). \tag{4.32d}$$

Step 4: Showing that, at an equilibrium, the production schedule is the unique optimal solution to $\hat{ED}$-$\hat{FR}$. Let $(\mathbf{q}^b, \mathbf{q}^p)$ be an optimal solution to $\hat{ED}(\mathbf{d}_1)$ and $\mathbf{r}_s^p$ be an optimal solution to $\hat{FR}(\mathbf{q}^b, \mathbf{q}^p, \mathbf{d}_s)$. We will show that:

$$\begin{aligned}
\mathbf{q}^b &= \left([\alpha_n^b s_n^b(\pi_n^b)]_{\underline{q}_n^b}^{\bar{q}_n^b}, n \in N\right); \\
\mathbf{q}^p &= \left([\alpha_n^p s_n^p(\pi_{1,n}^p)]_{\underline{q}_n^p}^{\bar{q}_n^p}, n \in N\right); \\
\mathbf{r}_s^p &= \left([\alpha_n^p s_n^p(\pi_{s,n}^p)]_{\underline{q}_n^p}^{\bar{q}_n^p} - [\alpha_n^p s_n^p(\pi_{1,n}^p)]_{\underline{q}_n^p}^{\bar{q}_n^p}, n \in N\right).
\end{aligned}$$

It suffices to show that, if $(\mathbf{q}^b, \mathbf{q}_1^p)$ is an optimal solution to (4.29) and $\mathbf{q}_s^p$ is an optimal solution to (4.30), then:

$$\mathbf{q}^b = \left([\alpha_n^b s_n^b(\pi_n^b)]_{\underline{q}_n^b}^{\bar{q}_n^b}, n \in N\right); \tag{4.33}$$

$$\mathbf{q}_s^p = \left([\alpha_n^p s_n^p(\pi_{s,n}^p)]_{\underline{q}_n^p}^{\bar{q}_n^p}, n \in N\right). \tag{4.34}$$

By rewriting (4.31a) for dispatch generator n, we infer that:

$$q_n^b = \alpha_n^b s_n^b\left(\pi_n^b + \underline{\xi}_n/K - \bar{\xi}_n/K\right).$$

If $\underline{q}_n^b < q_n^b < \bar{q}_n^b$, then from (4.31b), we infer that $\bar{\xi}_n = \underline{\xi}_n = 0$, which implies that $q_n^b = \alpha_n^b s_n^b(\pi_n^b)$. If $q_n^b = \underline{q}_n^b$, then from (4.31b), we infer that $\bar{\xi}_n = 0$ and $\underline{\xi}_n \geq 0$, which implies that $\underline{q}_n^b = q_n^b =$

$\alpha_n^b s_n^b(\pi_n^b + \underline{\xi}_n/K) \geq \alpha_n^b s_n^b(\pi_n^b)$, where the last inequality follows from the fact that c_n^b is strictly convex. If $q_n^b = \bar{q}_n^b$, then from (4.31b), we infer that $\underline{\xi}_n = 0$ and $\bar{\xi}_n \geq 0$, which implies that $\bar{q}_n^b = q_n^b = \alpha_n^b s_n^b(\pi_n^b - \bar{\xi}_n/K) \leq \alpha_n^b s_n^b(\pi_n^b)$, where the last inequality follows from the fact that c_n^b is strictly convex. Hence, we conclude that $\mathbf{q}^b$ is given by (4.33). By making similar arguments, we conclude that $\mathbf{q}_s^p$ is given by (4.34).

Step 5: Showing that any production schedule $(\mathbf{q}^b, \mathbf{q}^p, \mathbf{r}^p)$ that solves $SYSTEM$ can be obtained using bids $(\boldsymbol{\gamma}^b, \boldsymbol{\gamma}^p)$ and the latter satisfy the characterizations in steps 1 to 3. By Theorem 4.1, $(\mathbf{q}^b, \mathbf{q}^p)$ is the unique solution to $ED(\mathbf{d}_1)$ and $\mathbf{r}_s^p$ is the unique solution to $FR(\mathbf{q}^b, \mathbf{q}^p, \mathbf{d}_s)$. Under bids $(\boldsymbol{\gamma}^b, \boldsymbol{\gamma}^p)$, the problems $ED(\mathbf{d}_1)$ and $\hat{ED}(\mathbf{d}_1)$ are equivalent. Hence, $(\mathbf{q}^b, \mathbf{q}^p)$ is the unique solution to $\hat{ED}$, and by step 4, the production in the first time period is $(\mathbf{q}^b, \mathbf{q}^p)$. Under bids $(\boldsymbol{\gamma}^b, \boldsymbol{\gamma}^p)$, the problems $FR(\mathbf{q}^b, \mathbf{q}^p, \mathbf{d}_s)$ and $\hat{FR}(\mathbf{q}^b, \mathbf{q}^p, \mathbf{d}_s)$ are equivalent. Hence, $\mathbf{r}_s^p$ is the unique solution to $\hat{FR}(\mathbf{q}^b, \mathbf{q}^p, \mathbf{d}_s)$, and by step 4, the recourse production is $\mathbf{r}_s^p$. Hence, the production schedule is $(\mathbf{q}^b, \mathbf{q}^p, \mathbf{r}^p)$.

It suffices to show that bids $(\boldsymbol{\gamma}^b, \boldsymbol{\gamma}^p)$ constitute an equilibrium. It is easy to check that $\boldsymbol{\alpha}^p = \boldsymbol{\gamma}^p$ and $\boldsymbol{\alpha}^b = \boldsymbol{\gamma}^b$ satisfy conditions (4.25) and (4.28) respectively for any prices $(\boldsymbol{\pi}^b, \boldsymbol{\pi}^p)$. Hence, simply choose $(\boldsymbol{\pi}^b, \boldsymbol{\pi}^p)$ based on equations (4.31) and (4.32). This proves part (a) of the proposition.

Step 6: Showing that any bids $(\boldsymbol{\alpha}^b, \boldsymbol{\alpha}^p)$ that satisfy the characterizations in steps 1 to 3 give the same dispatch as that under bids $(\boldsymbol{\gamma}^b, \boldsymbol{\gamma}^p)$. Suppose that $(\boldsymbol{\alpha}^b, \boldsymbol{\alpha}^p)$ satisfy the characterizations in step 4 with productions $(\mathbf{q}^b, \mathbf{q}_1^p, \ldots, \mathbf{q}_S^p)$, Lagrange multipliers $(\underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\xi}}, \underline{\boldsymbol{\nu}}, \bar{\boldsymbol{\nu}}, \underline{\boldsymbol{\mu}}, \bar{\boldsymbol{\mu}}, \boldsymbol{\lambda})$, and prices $(\boldsymbol{\pi}^b, \boldsymbol{\pi}^p)$. We will construct $\underline{\boldsymbol{\xi}}', \bar{\boldsymbol{\xi}}' \in \mathbb{R}_+^N$ and $\underline{\boldsymbol{\nu}}_1', \bar{\boldsymbol{\nu}}_1' \in \mathbb{R}_+^N$ such that:

$$\big(Kc_n^{b\prime}(q_n^b), n \in N\big) + \bar{\boldsymbol{\xi}}' - \underline{\boldsymbol{\xi}}' - K\boldsymbol{\pi}^b = 0; \tag{4.35a}$$

$$L^b(\mathbf{q}^b, \underline{\boldsymbol{\xi}}', \bar{\boldsymbol{\xi}}') = 0; \tag{4.35b}$$

$$\big(Kc_n^{p\prime}(q_{1,n}^p), n \in N\big) + \bar{\boldsymbol{\nu}}_1' - \underline{\boldsymbol{\nu}}_1' - K\boldsymbol{\pi}_1^p = 0; \tag{4.35c}$$

$$L^p(\mathbf{q}_1^p, \underline{\boldsymbol{\nu}}_1', \bar{\boldsymbol{\nu}}_1') = 0, \tag{4.35d}$$

and $\underline{\boldsymbol{\nu}}_s', \bar{\boldsymbol{\nu}}_s' \in \mathbb{R}_+^N$ for all $s \in S \setminus \{1\}$ such that:

$$\big(c_n^{p\prime}(q_{s,n}^p), n \in N\big) + \bar{\boldsymbol{\nu}}_s' - \underline{\boldsymbol{\nu}}_s' - \boldsymbol{\pi}_s^p = 0; \tag{4.36a}$$

$$L^p(\mathbf{q}_s^p, \underline{\boldsymbol{\nu}}_s', \bar{\boldsymbol{\nu}}_s') = 0, \tag{4.36b}$$

which are the KKT conditions for (4.29) and (4.30) under bids $(\boldsymbol{\gamma}^b, \boldsymbol{\gamma}^p)$. Then, step 5 allows us to infer that the production schedule is an optimal solution to $SYSTEM$. Our construction is given by:

$$\underline{\xi}'_n = \begin{cases} K\left(c_n^{b\prime}(\underline{q}_n^b) - \pi_n^b\right), & \text{if } q_n^b = \underline{q}_n^b; \\ 0, & \text{else,} \end{cases}$$

$$\bar{\xi}'_n = \begin{cases} K\left(\pi_n^b - c_n^{b\prime}(\bar{q}_n^b)\right), & \text{if } q_n^b = \bar{q}_n^b; \\ 0, & \text{else,} \end{cases}$$

$$\underline{\nu}'_{1,n} = \begin{cases} K\left(c_n^{p\prime}(\underline{q}_n^p) - \pi_{1,n}^p\right), & \text{if } q_{1,n}^p = \underline{q}_n^p; \\ 0, & \text{else,} \end{cases}$$

$$\bar{\nu}'_{1,n} = \begin{cases} K\left(\pi_{1,n}^p - c_n^{p\prime}(\bar{q}_n^p)\right), & \text{if } q_{1,n}^p = \bar{q}_n^p; \\ 0, & \text{else,} \end{cases}$$

and:

$$\underline{\nu}'_{s,n} = \begin{cases} c_n^{p\prime}(\underline{q}_n^p) - \pi_{s,n}^p, & \text{if } q_{s,n}^p = \underline{q}_n^p; \\ 0, & \text{else,} \end{cases}$$

$$\bar{\nu}'_{s,n} = \begin{cases} \pi_{s,n}^p - c_n^{p\prime}(\bar{q}_n^p), & \text{if } q_{s,n}^p = \bar{q}_n^p; \\ 0, & \text{else,} \end{cases}$$

for all $s \in S \setminus \{1\}$.

First, we show that $\underline{\boldsymbol{\xi}}', \bar{\boldsymbol{\xi}}', \underline{\boldsymbol{\nu}}'_s, \bar{\boldsymbol{\nu}}'_s \geq 0$. Suppose $q_n^b = \underline{q}_n^b$. Then, from (4.28a), we infer that $c_n^{b\prime-1}(\pi_n^b) \leq \underline{q}_n^b$, and since c_n^b is strictly convex, we infer that $\pi_n^b \leq c_n^{b\prime}(\underline{q}_n^b)$, and hence $\underline{\xi}'_n \geq 0$. Suppose $q_n^b = \bar{q}_n^b$. Then, from (4.28c), we infer that $c_n^{b\prime-1}(\pi_n^b) \geq \bar{q}_n^b$, and since c_n^b is strictly convex, we infer that $\pi_n^b \geq c_n^{b\prime}(\bar{q}_n^b)$, and hence $\bar{\xi}'_n \geq 0$. By similar arguments, we infer that $\underline{\nu}'_{s,n} \geq 0$ and $\bar{\nu}'_{s,n} \geq 0$.

Second, we show that this construction satisfies (4.35) and (4.36). It is easy to check that the complementary slackness conditions (4.35b), (4.35d), (4.36b) are satisfied. Suppose $\underline{q}_n^b < c_n^{b\prime-1}(\pi_n^b) < \bar{q}_n^b$. From (4.28b), we infer that $\alpha_n^b = \gamma_n^b$. From the fact that $q_n^b = \left[\alpha_n^b s_n^b(\pi_n^b)\right]_{\underline{q}_n^b}^{\bar{q}_n^b} = c_n^{b\prime-1}(\pi_n^b)$, we infer that $\underline{q}_n^b < q_n^b < \bar{q}_n^b$. From (4.31b), we infer that $\underline{\xi}_n = \bar{\xi}_n = 0$. Substituting into (4.31a), we infer that our construction satisfies (4.35a). Suppose $c_n^{b\prime-1}(\pi_n^b) \leq \underline{q}_n^b$. From (4.28a), we infer that $q_n^b = \underline{q}_n^b$. Hence, our construction satisfies (4.35a). Suppose $\bar{q}_n^b \leq c_n^{b\prime-1}(\pi_n^b)$. From (4.28c), we infer that $q_n^b = \bar{q}_n^b$. Hence, our construction satisfies (4.35a). Using similar arguments, we can infer that our construction satisfies (4.35c) and (4.36a). This proves part (b) of the proposition. $\square$

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