ON THE LYAPUNOV TRANSFORMATION
FOR STABLE MATRICES

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ABSTRACT

The matrices studied here are positive stable (or briefly stable). These are matrices, real or complex, whose eigenvalues have positive real parts. A theorem of Lyapunov states that A is stable if and only if there exists $H > 0$ such that $AH + HA^* = I$. Let A be a stable matrix. Three aspects of the Lyapunov transformation $L_A : H \to AH + HA^*$ are discussed.

1. Let $C_1(A) = \{AH + HA^* : H \geq 0\}$ and $C_2(A) = \{H : AH + HA^* \geq 0\}$. The problems of determining the cones $C_1(A)$ and $C_2(A)$ are still unsolved. Using solvability theory for linear equations over cones it is proved that $C_1(A)$ is the polar of $C_2(A^*)$, and it is also shown that $C_1(A) = C_1(A^{-1})$. The inertia assumed by matrices in $C_1(A)$ is characterized.

2. The index of dissipation of A was defined to be the maximum number of equal eigenvalues of H, where H runs through all matrices in the interior of $C_2(A)$. Upper and lower bounds, as well as some properties of this index, are given.

3. We consider the minimal eigenvalue of the Lyapunov transform $AH + HA^*$, where H varies over the set of all positive semidefinite matrices whose largest eigenvalue is less than or equal to one. Denote it by $\psi(A)$. It is proved that if A is Hermitian and has eigenvalues $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n > 0$, then $\psi(A) = -(\mu_1 - \mu_n)^2/(4(\mu_1 + \mu_n))$. The value of $\psi(A)$ is also determined in case A is a normal, stable matrix. Then $\psi(A)$ can be expressed in terms of at most three of the eigenvalues of A. If A is an arbitrary stable matrix, then upper and lower bounds for $\psi(A)$ are obtained.

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NOTATIONS

A	$n \times n$ matrix with complex elements
A^*	complex conjugate and transpose of A
$R(A)$	$A + A^*$
$tr(A)$	trace of A
$null(A)$	null space of A
L_A	Lyapunov operator corresponding to A, see definition on page 6
$s(A)$	Stein index of A, see definition on page 13
$In(A)$	inertia of A, see definition on page 12
$Ix(A)$	index of dissipation of A, see definition on page 24
I	identity matrix
$A_1 \oplus A_2$	direct sum of the matrices A_1 and A_2
$D = diag(a_1, a_2, \ldots, a_n)$	diagonal matrix of order n with $a_1, a_2, \ldots, a_n$ on the main diagonal
V	n^2-dimensional linear space of $n \times n$ Hermitian matrices over the real numbers
H, K	Hermitian matrices
PD	set of all $n \times n$ positive definite matrices
PSD	set of all $n \times n$ positive semidefinite matrices
E^n	n-dimensional Euclidean space over the complex numbers
R^n	n-dimensional Euclidean space of the real numbers.

INTRODUCTION

This thesis is concerned with positive stable (or briefly stable) matrices. These are matrices, real or complex, whose eigenvalues have positive real parts. A most important result concerning stable matrices is Lyapunov's theorem.

Theorem (Lyapunov [13]). The $n \times n$ matrix A is stable if and only if there exists an $n \times n$ positive definite Hermitian matrix H such that $AH + HA^* = I$.

Here A denotes an $n \times n$ matrix with complex elements, and A^* is the complex conjugate and transpose of A. Throughout this work H and K denote Hermitian matrices. We write $H > 0$ if H is positive definite and $H \geq 0$ if H is positive semidefinite. The identity matrix is denoted by I.

Lyapunov's theorem is a special case of some theorems proved by Lyapunov, establishing conditions for the stability of solutions of differential equations. Because of its importance we give a brief account on some of its proofs and generalizations. Gantmacher [8] and Bellman [2] give proofs which use differential equations. Bellman even proves that if A is stable, then the unique solution of the matrix equation $AH + HA^* = K$ is given by the explicit form

$$H = \int_0^\infty e^{-At} K e^{-A^* t} \, dt \quad .$$

Other proofs are given by Hahn [10] and Taussky [21]. Givens [9] proved it via the generalized field of values $F_H(A)$ with respect to the metric $||x||_H^2 = x^*Hx$, given by the positive definite matrix H. Here $F_H(A) = \{x^*AGx: ||x||_H = 1\}$.

For a fixed matrix A, the transformation $L_A : H \to AH + HA^*$, which is a linear transformation from the real n^2-dimensional space of Hermitian matrices into itself, is called the Lyapunov transformation. The eigenvalues of this transformation are $\lambda_i + \overline{\lambda}_j$, $i, j = 1, \ldots, n$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, e.g., [9], and [22]. The elementary divisors of the Lyapunov transformation were found by Givens [9]. The fact that L_A is a linear operator, and that the positive semidefinite matrices form a closed, convex, self-polar cone enables one to use here the solvability theory of linear equations over cones. Indeed, Berman and Ben-Israel [4] proved the Lyapunov theorem in this way. This approach will be used here to obtain some further properties of the Lyapunov transformation.

To describe some generalizations of the Lyapunov theorem, we need the concept of the inertia of a matrix. For an $n \times n$ matrix A which has $\pi(A)$ eigenvalues with positive real parts, $\nu(A)$ eigenvalues with negative parts, and $\delta(A)$ purely imaginary eigenvalues, we call the ordered triple $(\pi(A), \nu(A), \delta(A))$ the inertia of A, written $\text{In}(A) = (\pi(A), \nu(A), \delta(A))$. Let A have eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Taussky [21] showed: if $\lambda_i + \overline{\lambda}_j \neq 0$, $i, j = 1, 2, \ldots, n$, then there exists H such that $AH + HA^* = I$, and this H satisfies $\text{In}(H) = \text{In}(A)$. Ostrowski and Schneider [15] proved that there exists H such that $AH + HA^* > 0$

if and only if $\delta(A) = 0$. Furthermore, $AH + HA^* > 0$ implies that $In(A) = In(H)$. Carlson [6], and Carlson and Schneider [5], investigated the matrix inequality $AH + HA^* \geq 0$. They obtained bounds for $In(H)$ in terms of $In(A)$, or as a function of rank $(AH + HA^*)$. However, the relation of $In(H)$ to $In(A)$ in this case is much more complicated and is not fully understood.

Taussky [20, 21] showed that every complex (real) stable matrix is unitarily (real orthogonally) similar to a matrix of the form $(I+S)D$, where S is skew-Hermitian (skew-symmetric) and D is a positive diagonal matrix.

There is a close connection between stable matrices and convergent matrices, namely matrices all of whose eigenvalues are of modulus less than one. The connection originates from the Cayley transformation. Thus, if A is stable then $C = (A + I)^{-1}(A - I)$ is convergent. Conversely, if C is convergent then $A = (I - C)^{-1}(I + C)$ is stable. Stein [17] proved that a matrix C is convergent if and only if there exists $H > 0$ such that $H - CHC^* > 0$. Taussky [23] showed that Lyapunov's theorem is equivalent to Stein's theorem.

Many questions concerning stable matrices and the Lyapunov transformation remain unsolved, despite the extensive research described above. It is the purpose of this thesis to consider some of them. In Chapter I the following two problems are discussed: Determine $C_1(A) = \{AH + HA^* : H \geq 0\}$ and $C_2(A) = \{H : AH + HA^* \geq 0\}$. These are the image and the inverse image of the cone of positive semi-definite matrices under the Lyapunov transformation, respectively.

These problems slightly modify problems of Taussky [24, 25], who asked what are the interiors of $C_1(A)$ and $C_2(A)$. Using the solvability theory for linear equations over cones it is proved that $C_1(A)$ is the polar of the cone $C_2(A^*)$. Thus, the two problems are not unrelated, but in fact equivalent. It is also shown that $C_1(A) = C_1(A^{-1})$.

Stein and Pfeffer [19] found the range of $BH+HB^*$, where H runs through all positive definite matrices and B varies over all matrices similar to the fixed matrix A. Restating their result in terms of A itself, we characterize the inertia vectors which are assumed by matrices in the interior of $C_1(A)$. This result leads to the characterization of inertia vectors assumed by matrices in $C_1(A)$.

The index of dissipation of a stable matrix is defined by Taussky [25] to be the maximal number of equal eigenvalues of H, where H runs through all matrices in the interior of $C_2(A)$. Upper and lower bounds, as well as some properties of this index, are given in Chapter II.

In Chapter III we consider the minimal eigenvalue of the Lyapunov transform $AH+HA^*$, where H varies over the set of all positive semidefinite matrices whose largest eigenvalue is less than or equal to one. Denote it by $\psi(A)$. The value of $\psi(A)$ is determined in case A is normal and stable, while upper and lower bounds for $\psi(A)$ are obtained for a general stable A.

CHAPTER I

ON THE RANGE OF THE LYAPUNOV TRANSFORMATION

In this chapter the range of the Lyapunov transformation is considered. More precisely, for a fixed stable matrix A, what is the image of the cone of positive semidefinite matrices under the transformation $H \to AH + HA^*$? This problem is unsolved to date, and here some observations about it are made.

Let A be a matrix of order $n \times n$ with complex elements (unless otherwise specified), and A^* be its complex conjugate and transpose. The trace of A is denoted by tr(A), and the null space of A by null(A). The trace function satisfies $tr(AB) = tr(BA)$ for every pair A, B of $n \times n$ matrices.

Let H and K denote Hermitian matrices. The space of all $n \times n$ Hermitian matrices is denoted by V. This is clearly an n^2-dimensional space over the real numbers. Moreover, an inner product can be put on this space, by defining $(H, K) = tr(HK)$. This is the ordinary inner product if we look on matrices as n^2-dimensional vectors. The set of $n \times n$ positive definite matrices is denoted by PD, and we write $H > 0$ if $H \in PD$. The set of $n \times n$ positive semidefinite matrices is denoted by PSD, and we write $H \geq 0$ if $H \in PSD$ and $H_1 \geq H_2$ if $H_1 - H_2 \geq 0$. The identity matrix is denoted by I, its order should be clear from the text.

We let $E^n(R^n)$ be the n-dimensional Euclidean space over the complex (real) numbers. The inner product in $E^n(R^n)$ is denoted by (x, y), where $x, y \in E^n(R^n)$.

The following definitions are the starting point.

Definition 1. The matrix A is called positive stable (or briefly stable) if all its eigenvalues have positive real parts.

Definition 2. For any matrix A let $R(A) = A + A^*$. Note that henceforth both $R(A)$ and $A + A^*$ are used.

Definition 3. The transformation $L_A : V \to V$ defined by

$$L_A(H) = R(AH) = AH + HA^* \tag{1}$$

is called the Lyapunov transformation. L_A is called the Lyapunov operator. Note that we use the term Lyapunov transformation, rather than the longer term Lyapunov transformation corresponding to A, since we usually consider a fixed matrix A, and no confusion should arise.

Stable matrices are characterized by the Lyapunov theorem.

Theorem 1 (Lyapunov [13]). The matrix A is stable if and only if there exists $H > 0$ such that $R(AH) = I$.

A brief survey of the various proofs of this theorem is given in the Introduction. Lyapunov's theorem characterizes stable matrices, but leaves open many questions concerning these matrices. The following two problems, due to Taussky, are discussed in this chapter.

Problem 1 [24]. Let A be a stable matrix. What is the range of the Lyapunov transformation if H runs through the set PD?

Problem 2 [25]. Let A be a stable matrix. What is the set of Hermitian matrices H such that $R(AH) > 0$?

It is our purpose to show that these seemingly unrelated problems are equivalent. The proof makes use of solvability theory of linear equations over cones. To establish the proof we need some additional theorems and definitions. The following theorem determines the eigenvalues of the Lyapunov operator L_A.

Theorem 2 [9], [22]. Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then the eigenvalues of the Lyapunov operator L_A are $\lambda_i + \overline{\lambda}_j$, $i, j = 1, 2, \ldots, n$.

The proof of Givens [9] makes use of Kronecker products of matrices. Taussky and Wielandt [22] find the required eigenvalues by choosing an appropriate basis.

Henceforth it is assumed that A is a stable matrix, unless otherwise specified.

Corollary 1. Let A be a stable matrix. Then the Lyapunov operator L_A is one-to-one. In particular, its null space consists of the 0 matrix only.

Proof. This follows immediately from Theorem 2.

Definition 4. (I) Let $S \subset E^n$, S nonempty. The set S is said to be a convex cone if

(a) $\lambda S \subset S$ for every $\lambda \geq 0$.

(b) $S + S \subset S$.

(II) Let $S \subset E^n$, S nonempty. The polar of S, written S^p, is defined by

$$S^p = \{y \in E^n : \text{Re}(y, S) \geq 0\} .$$

The polar set is defined similarly over R^n, with the Re obviously omitted. It is known that S^p is always a closed convex cone, see, e.g., [3].

The next lemma is an immediate consequence of the well-known fact that a closed convex set S of E^n and a point of E^n which does not belong to S can be separated by a hyperplane.

Lemma 1 [3]. Let $S \subset E^n$, S nonempty. Then $S = S^{pp}$ if and only if S is a closed convex cone (here $S^{pp} = (S^p)^p$).

The lemma leads to the following solvability theorem of linear equations over cones. We need this theorem to establish the equivalence of problems 1 and 2.

Theorem 3 [3]. Let T be an $m \times n$ matrix with complex elements, and let $b \in E^m$. Let $S \subset E^n$ be a closed convex cone and assume that $\text{null}(T) + S$ is closed. Then the following are equivalent.

(a) $Tx = b,\ x \in S$ has a solution.

(b) $T\overset{*}{y} \in S^p$ implies $\mathrm{Re}(b, y) \geq 0$.

To prove the equivalence of Problems 1 and 2, we define the following closed convex cones.

$$C_1(A) = \{AH + HA^* : H \geq 0\} \tag{2}$$

$$C_2(A) = \{H : AH + HA^* \geq 0\} . \tag{3}$$

It follows from Corollary 1 (since A is assumed to be stable) that the sets defined in Problems 1 and 2 are the interiors of $C_1(A)$ and $C_2(A)$, respectively. Thus, in order to prove the equivalence of Problems 1 and 2, it suffices to show that the determination of $C_1(A)$ and $C_2(A)$ are equivalent problems. This is shown in the next theorem.

Theorem 4. Let A be a stable matrix. Then,

$$C_1(A) = C_2(A^*)^p$$

and

$$C_2(A) = C_1(A^*)^p .$$

Proof. This theorem follows from Theorem 3. We replace the matrix T of Theorem 3 by L_A. Correspondingly, E^m and E^n are replaced by V, the linear space of $n \times n$ Hermitian matrices over the real numbers. We let S be PSD, the set of $n \times n$ positive semidefinite matrices. It is well-known that PSD is a closed convex cone. Moreover, PSD is a self-polar cone, namely $PSD = PSD^p$, e.g., [4]. It follows from Corollary 1 that $\mathrm{null}(L_A) = \{0\}$. Hence $\mathrm{null}(L_A) + PSD = PSD$, and the assumptions of Theorem 3 are satisfied. It remains to find L_A^*.

We show $L_A^* = L_{A^*}$. To see it, let $H, K \in V$. Then,

$$(L_A(H), K) = (AH + HA^*, K) = tr(AHK + HA^*K) = tr(HKA + HA^*K)$$

$$= (H, A^*K + KA) = (H, L_{A^*}(K)) \quad .$$

To complete the proof, let $K \in V$. Then $K \in C_1(A)$ if and only if $K = L_A(H_0)$ for some $H_0 \in PSD$. By Theorem 3 this is equivalent to:

$$L_{A^*}(H) \geq 0 \quad \text{implies} \quad (K, H) \geq 0 \quad .$$

But, $L_{A^*}(H) \geq 0$ if and only if $H \in C_2(A^*)$. Hence, $K \in C_1(A)$ if and only if $K \in C_2(A^*)^p$. This proves the first part of the theorem.

Replacing A by A^*, we get $C_1(A^*) = C_2(A)^p$. Since $C_2(A)$ is a closed convex cone, it follows from Lemma 1 that $C_2(A) = C_2(A)^{pp}$. Hence,

$$C_1(A^*)^p = C_2(A)^{pp} = C_2(A)$$

completing the proof.

Having proved the equivalence of Problems 1 and 2, we can now concentrate on Problem 1, or, equivalently, the determination of $C_1(A)$. The next theorem, which follows immediately from Lyapunov's theorem, gives us some information about $C_1(A)$.

Theorem 5 [23]. Let A be a stable matrix, and let K be a given positive definite matrix. Then the unique solution of $AH + HA^* = K$ is positive definite.

Proof. The uniqueness follows from the fact that L_A is one-to-one. By Sylvester's law of inertia, there exists nonsingular matrix T such that $TKT^* = I$. Hence,

$$(TAT^{-1})(THT^*) + (THT^*)(T^{*-1}A^*T^*) = TKT^* = I \ .$$

The matrix TAT^{-1} is stable. Thus, by Theorem 1 (Lyapunov's theorem) $THT^* > 0$, and consequently $H > 0$.

Another proof of Theorem 5 can be given, using Bellman's integral representation to the solution of $AH + HA^* = K$. For further details, see the Introduction.

Corollary 2. $$C_1(A) \supset PSD \ .$$

Proof. This follows immediately from Theorem 5 and the closedness of $C_1(A)$.

Using Theorem 4 we can prove the following theorem.

Theorem 6. Let A be a stable matrix. Then $C_1(A) = C_1(A^{-1})$.

Proof. We first show that $C_2(A) = C_2(A^{-1})$.

Let $H \in C_2(A)$, so $R(AH) \geq 0$. We want to prove that $R(A^{-1}H) \geq 0$. Since $R(AH) \geq 0$, we have

$$A^{-1}AHA^{-1^*} + A^{-1}HA^*A^{-1^*} \geq 0$$

so

$$A^{-1}H + HA^{-1^*} \geq 0 \ .$$

This proves that $R(AH) \geq 0$ implies $R(A^{-1}H) \geq 0$, or $H \in C_2(A)$ implies $H \in C_2(A^{-1})$.

Hence $C_2(A) \subset C_2(A^{-1})$. Similarly, as A^{-1} is also stable, $C_2(A^{-1}) \subset C_2(A)$. Hence $C_2(A) = C_2(A^{-1})$. Replacing A by A^*, which is also stable, we get $C_2(A^*) = C_2(A^{-1^*})$.

To finish the proof, we notice that by Theorem 4

$$C_1(A) = C_2(A^*)^p = C_2(A^{-1^*})^p = C_1(A^{-1}) \ .$$

Theorem 6 gives rise to a new problem. What matrices B satisfy $C_1(A) = C_1(B)$? This question is not discussed here.

We proceed to generalize theorems due to Stein, and Stein and Pfeffer, concerning $C_1(A)$. We notice that so far only Corollary 2 gives us some information on the structure of $C_1(A)$. In fact, almost nothing is known about $C_1(A)$, for a fixed stable A. The approach of Stein and Pfeffer is to allow the matrix A to vary. To describe their results, we introduce some more concepts.

<u>Definition 5.</u> Let A be an arbitrary matrix having $\pi(A)$ eigenvalues with positive real parts, $\nu(A)$ eigenvalues with negative real parts and $\delta(A)$ purely imaginary eigenvalues. The ordered triple $In(A) = (\pi(A), \nu(A), \delta(A))$ is called the <u>inertia</u> of the matrix A. Obviously, $\pi(A) + \nu(A) + \delta(A) = n$.

<u>Definition 6.</u> The vector $\omega = (\omega_1, \omega_2, \omega_3)$, whose coordinates are non-negative integers and satisfy $\omega_1 + \omega_2 + \omega_3 = n$, is called an <u>inertia vector</u>.

Definition 7. Let A be an arbitrary matrix, having distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$. The index of λ_i, $1 \le i \le r$, is the maximum number of linearly independent eigenvectors corresponding to it, i.e., the dimension of its eigenspace. The Stein index of A, written s(A), is defined here to be the maximum of the indices of the λ_i.

We assume now that $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the distinct eigenvalues of A. We write

$$m = s(A) \tag{4}$$

and we may assume without loss of generality that the index of λ_1 is equal to m. We can now describe the degrees of the elementary divisors corresponding to the λ_i as follows:

$$\begin{array}{ll} \lambda_1: & n_{11} \ge n_{12} \ge \ldots \ge n_{1m} > 0 \\ \lambda_2: & n_{21} \ge n_{22} \ge \ldots \ge n_{2m} \ge 0 \\ \cdot & \\ \cdot & \\ \cdot & \\ \lambda_r: & n_{r1} \ge n_{r2} \ge \ldots \ge n_{rm} \ge 0 \ . \end{array} \tag{5}$$

We agree that if $n_{ij} = 0$ for some i and j, the corresponding elementary divisor does not exist. It is obvious that the arrangement described above can be done. Also,

$$\sum_{k=1}^{r} \sum_{j=1}^{m} n_{kj} = n \ . \tag{6}$$

We are ready to state the results of Stein [18], Stein and Pfeffer [19], and their generalizations.

Theorem 7 [18]. Let $K \in V$. The following are equivalent:

(a) K has at least one positive eigenvalue, i.e., $\pi(K) \geq 1$.

(b) There exists a stable matrix A and $H > 0$ such that $R(AH) = K$.

This theorem determines what matrices can be written in the form $AH + HA^*$, where not only H runs through the set PD, but A is allowed to vary over the set of stable matrices. The proof of Stein is a constructive one, but is too complicated to be described here. Ballantine [1] gives an inductive proof.

Theorem 8 [19]. Let A be a stable matrix, and let $K \in V$ be a given matrix. Let $m = s(A)$ be the Stein index of A. Then the following are equivalent:

(a) K has at least m positive eigenvalues, i.e., $\pi(K) \geq m$.

(b) There exists a nonsingular matrix T and $H > 0$ such that

$$(TAT^{-1})H + H(TAT^{-1})^* = K \ .$$

This theorem, due to Stein and Pfeffer, gives the range of $BH + HB^*$, where H runs through the set PD, and B varies over the set of all matrices similar to the given matrix A. We want to link the Stein-Pfeffer theorem to the problem of finding $C_1(A)$. To do it, we need the following lemma.

Lemma 2. Let A be a stable matrix, let $K \in V$, and let T be a nonsingular matrix. Then there exists $H > 0$ ($H \geq 0$) such that $R(AH) = K$ if and only if there exists $H_0 > 0$ ($H_0 \geq 0$) such that $R(TAT^{-1}H_0) = TKT^*$.

Proof. Assume that $AH + HA^* = K$. Then,

$$(TAT^{-1})(THT^*) + (THT^*)(TAT^{-1})^* = TKT^* .$$

We choose $H_0 = THT^*$. Now, $H > 0$ ($H \geq 0$) implies $H_0 > 0$ ($H_0 \geq 0$), and the proof of the first part is completed. The second part follows similarly.

We can now restate Theorem 8 (Stein-Pfeffer) in terms of A alone.

Theorem 9. Let A be a stable matrix, and let $\omega = (\omega_1, \omega_2, \omega_3)$ be an inertia vector. Let $m = s(A)$ be the Stein index of A. Then the following are equivalent:

(a) $\omega_1 \geq m$.

(b) There exists $H > 0$ such that $\mathrm{In}(AH + HA^*) = \omega$.

Proof. (a)$\Longrightarrow$(b). We choose $K_0 \in V$, such that $\mathrm{In}(K_0) = \omega$. By Theorem 8, there exist T nonsingular and $H_0 > 0$ such that $R(TAT^{-1}H_0) = K_0$. We let $TKT^* = K_0$. Sylvester's law of inertia asserts that $\mathrm{In}(K) = \mathrm{In}(K_0) = \omega$. It follows now from Lemma 2 that there exists $H > 0$ such that $R(AH) = K$. This completes the first part of the proof.

(b)$\Rightarrow$(a). This follows immediately from the corresponding part in Theorem 8.

Note: Theorem 9 characterized the inertia vectors which are assumed by matrices in the interior of $C_1(A)$.

Theorems 10 and 12 below generalize Theorems 7 and 9, by allowing H to be nonnegative definite. The more interesting theorem is 12, which determined the inertia vectors assumed by matrices in $C_1(A)$.

Theorem 10. Let $K \in V$. The following are equivalent:

(a) K has at least one positive eigenvalue, i.e., $\pi(K) \geq 1$, or $K = 0$.

(b) There exists a stable matrix A and $H \geq 0$ such that $R(AH) = K$.

Proof. (a)$\Rightarrow$(b). For $K = 0$ choose any stable matrix and $H = 0$. For $K \neq 0$ with $\pi(K) \geq 1$ apply Theorem 7.

(b)$\Rightarrow$(a). Let $R(AH) = AH + HA^* = K$, where A is a stable matrix and $H \geq 0$. Assume that $K \neq 0$ and $\pi(K) = 0$. Then $-K \in PSD$. It follows from Corollary 2 that there exists $H_0 \geq 0$ such that $R(AH_0) = -K$. But the Lyapunov operator L_A is one-to-one, implying that $H = -H_0$. On the other hand, $H \geq 0$ and $H_0 \geq 0$ imply $H = H_0 = 0$. Thus $K = 0$, contradicting our assumptions. This completes the proof.

To generalize Theorem 9, we need the following theorem of Carlson, which considers the elementary divisors of a block triangular matrix (over the complex numbers). For proof, see [7].

Theorem 11 [7]. Let B be a block triangular matrix of the form

$$B = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}$$

where B_{11} and B_{22} are square matrices. Let λ be an eigenvalue of B. Let the degrees of the elementary divisors associated with λ be $a_1 \geq a_2 \geq \ldots \geq a_q$ in B, $b_1 \geq b_2 \geq \ldots \geq b_t$ in B_{11} and $c_1 \geq c_2 \geq \ldots \geq c_w$ in B_{22}. Then for all i we have

$$\begin{aligned} a_{w+i} &\leq b_i \leq a_i \\ a_{t+i} &\leq c_i \leq a_i \end{aligned} \tag{7}$$

where by definition $a_i = 0$ if $i > q$.

Theorem 12. Let A be a stable matrix, and let $m = s(A)$ be the Stein index of A. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the distinct eigenvalues of A, and that the degrees of the corresponding elementary divisors are given by (5). Let $\omega = (\omega_1, \omega_2, \omega_3)$ be an inertia vector. Denote

$$\ell = \min(\omega_1, m) \quad . \tag{8}$$

Then the following are equivalent:

(a)
$$\omega_1 + \omega_2 \leq \sum_{k=1}^{r} \sum_{j=1}^{\ell} n_{kj} \tag{9}$$

where the right-hand side is defined to be 0 if $\ell = 0$.

(b) There exists $H \geq 0$ such that $\mathrm{In}(AH + HA^*) = \omega$.

Proof. (a)$\Rightarrow$(b). We assume that ω satisfies (9). If $\omega_1 \geq m$, then the fact that (b) is true follows from Theorem 9, while for $\omega_1 = 0$ (b) is satisfied by $H = 0$. Hence we can assume that $0 < \omega_1 < m$, and consequently $\ell = \omega_1$.

Let j be an arbitrary positive integer such that $1 \leq j \leq m$. Let A_j be the direct sum of Jordan blocks of orders $n_{1j}, n_{2j}, \ldots, n_{rj}$, corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_r$, respectively. Thus, the order of A_j is

$$n_j = \sum_{i=1}^{r} n_{ij} .$$

The Stein index $s(A_j)$ of A_j is obviously equal to one. There exists a nonsingular matrix T such that

$$TAT^{-1} = A_1 \oplus A_2 \oplus \ldots \oplus A_m , \tag{10}$$

the right-hand side being the direct sum of the matrices $A_1, A_2, \ldots, A_m$.

We define now real diagonal matrices K_j of order n_j as follows:

$$K_j = \begin{cases} \mathrm{diag}(1, *, \ldots, *) , & j = 1, 2, \ldots, \omega_1, \\ 0 & j = \omega_1 + 1, \ldots, m . \end{cases}$$

The places denoted by * in $K_1, K_2, \ldots, K_{\omega_1}$ can be filled by any nonpositive numbers, provided that exactly ω_2 of them are negative. This is possible since (9) holds.

Consider now a fixed j, $1 \le j \le \omega_1$. Since $\pi(K_j) = s(A_j) = 1$, it follows from Theorem 9 that there exists $H_j > 0$ of order n_j such that $\mathrm{In}(A_jH_j + H_jA_j^*) = \mathrm{In}(K_j)$. We further choose $H_j = 0$ for $\omega_1 + 1 \le j \le m$, and define the following direct sums:

$$H = H_1 \oplus H_2 \oplus \ldots \oplus H_m$$

and

$$K = K_1 \oplus K_2 \oplus \ldots \oplus K_m \quad .$$

The matrix H is positive semidefinite and satisfies, by (10),

$$\mathrm{In}(R(TAT^{-1}H)) = \mathrm{In}(K) = \omega \quad .$$

Applying Lemma 2 we finish the proof that (a)$\Rightarrow$(b).

(b)$\Rightarrow$(a). The proof is by induction on n, the order of A. For $n = 1$ the proof is trivial. Assume that (b)$\Rightarrow$(a) for all stable matrices of order less than or equal to $n-1$, and consider the given matrix A. Denote $AH + HA^* = K$. By our assumption $\mathrm{In}(K) = \omega$, so in particular $\omega_1 = \pi(K)$. There are three cases.

Case 1. $\omega_1 > m$. In this case (9) holds trivially, since we have $\ell = m$. Thus, the right-hand side of (9) is equal to n, by (6).

Case 2. $\omega_1 = 0$. In this case the right-hand side of (9) is equal to 0, by definition. We also know from Theorem 10 that the left-hand side of (9) must be 0, so (9) holds in this case.

Case 3. $0 < \omega_1 < m$. Note that in this case $\ell = \omega_1$. We let $d = n - \pi(K) = n - \omega_1$.

Since λ_1 is an eigenvalue of geometric multiplicity m for A, $\bar{\lambda}_1$ is an eigenvalue of geometric multiplicity m for A^*. Denote the eigenspace of A^* corresponding to $\bar{\lambda}_1$ by S.

Let $\mu_1, \mu_2, \ldots, \mu_d$ be the d nonpositive eigenvalues of K, and let $x_1, x_2, \ldots, x_d$ be an orthonormal set of corresponding eigenvectors. Let $L[x_1, x_2, \ldots, x_d]$ be the linear space spanned by $x_1, x_2, \ldots, x_d$. We have

$$m + d = n + m - \omega_1 > n \quad ,$$

so there exists a vector y, $(y, y) = 1$, such that $y \in S \cap L[x_1, x_2, \ldots x_d]$. We write

$$y = \sum_{i=1}^{d} \alpha_i x_i \quad .$$

We have

$$A^* y = \bar{\lambda}_1 y$$

and

$$y^* K y = \sum_{i=1}^{d} |\alpha_i|^2 \mu_i \leq 0 \quad .$$

Also,

$$0 \geq y^* K y = y^*(AH + HA^*) y = (\lambda_1 + \bar{\lambda}_1) y^* H y \geq 0$$

since $H \geq 0$ and $\text{Re}(\lambda_1) > 0$. Hence,

$$y^* K y = y^* H y = 0 \quad .$$

Now, $H \geq 0$ and $y^*Hy = 0$ imply that $Hy = 0$, by the variational characterization of the eigenvalues of a Hermitian matrix [8, Vol. I, Chapter 10]. Moreover, the restriction of K to $L[x_1, x_2, \ldots, x_d]$ is negative semidefinite, so for the same reason $Ky = 0$.

We now let U be an $n \times n$ unitary matrix with the vector y in its first column. Then,

$$U^*AU = \begin{bmatrix} \lambda_1 & 0 \\ * & A_1 \end{bmatrix}$$

$$U^*HU = \begin{bmatrix} 0 & 0 \\ 0 & H_1 \end{bmatrix}$$

$$U^*KU = \begin{bmatrix} 0 & 0 \\ 0 & K_1 \end{bmatrix}$$

where H_1 and K_1 are $n\text{-}1 \times n\text{-}1$ Hermitian matrices, A_1 is an $n\text{-}1 \times n\text{-}1$ matrix, and * denotes a column vector whose coordinates are irrelevant. The equation $AH + HA^* = K$ implies

$$U^*AUU^*HU + U^*HUU^*A^*U = U^*KU \quad ,$$

so consequently,

$$A_1H_1 + H_1A_1^* = K_1 \quad .$$

The matrix K_1 satisfies $\pi(K_1) = \pi(K) = \omega_1$ and $\nu(K_1) = \nu(K) = \omega_2$, while $H_1 \geq 0$. The elementary divisors of U^*AU are equal to those of A. Let the elementary divisors of A_1 corresponding to λ_i be denoted by n'_{ij}, $j = 1, 2, \ldots, m$; $i = 1, 2, \ldots, r$. It follows from Theorem 11 that

$$n'_{ij} \leq n_{ij}, \qquad i = 1, 2, \ldots, r; \; j = 1, 2, \ldots, m \; .$$

(It also follows from Theorem 11 that $n'_{1m-1} \geq n_{1m} > 0$, so the Stein index $s(A_1) \geq m-1$, but we don't use this fact here.)

Applying the induction hypothesis for the stable matrix A_1 we find that

$$\omega_1 + \omega_2 \leq \sum_{k=1}^{r} \sum_{j=1}^{\omega_1} n'_{kj} \leq \sum_{k=1}^{r} \sum_{j=1}^{\omega_1} n_{kj} \; .$$

This completes the proof of the theorem.

Theorem 12 gives some indication on the structure of $C_1(A)$, by characterizing the inertia vectors which are assumed by matrices in $C_1(A)$. We illustrate this theorem in the next example.

Example. Let A be a stable matrix of order 45, having λ_1, λ_2, λ_3, λ_4 as its distinct eigenvalues. Let the degrees of the elementary divisors of A be given by:

λ_1:	8	6	4	2
λ_2:	5	3	3	1
λ_3:	2	2		
λ_4:	4	3	2	

The Stein index of A is 4. The following inertia vectors ω are assumed by matrices in $C_1(A)$.

All inertia vectors with $\omega_1 \geq 4$.

If $\omega_1 = 3$, all inertia vectors with $\omega_2 \leq 42-3 = 39$.

If $\omega_1 = 2$, all inertia vectors with $\omega_2 \leq 33-2 = 31$.

If $\omega_1 = 1$, all inertia vectors with $\omega_2 \leq 19-1 = 18$.

If $\omega_1 = 0$, only the vector $(0, 0, 45)$.

We finish this chapter with the following corollary.

Corollary 3. Let A be a stable matrix. Then $C_1(A) = PSD$ if and only if A is a scalar matrix, i.e., $A = \lambda I$ for some complex number λ with $Re(\lambda) > 0$.

Proof. Obviously, if A is a scalar matrix then $C_1(A) = PSD$. The converse follows immediately from Theorem 12 and the following easily verified fact: A is a scalar matrix if and only if $s(A) = n$. (Here n denotes, as usual, the order of A.)

CHAPTER II

THE INDEX OF DISSIPATION

Let A be a stable matrix. We know that there exists $H > 0$ such that $R(AH) = AH + HA^* > 0$. However, it need not be true that $R(A) = A + A^* > 0$. In fact, very little is known about the relation between the eigenvalues of A and R(A). The matrix A is said to be <u>dissipative</u> if $R(A) > 0$. It is natural to ask how "close" is a stable matrix A to a dissipative matrix. One of several possibilities to define "close" is discussed in this chapter. We need the following definition, dut to Taussky.

<u>Definition 1</u> [25]. Let A be a stable matrix. The <u>index of dissipation</u> of A, written Ix(A), is the maximum number of equal eigenvalues of H, where H runs through all positive definite matrices with $R(AH) > 0$, i.e., $H \in$ interior of $C_2(A)$. Recall that $R(AH) > 0$ implies $H > 0$, by Theorem 5, Chapter 1.

Upper and lower bounds, as well as some properties of the index of dissipation, are given in this chapter. It should be pointed out that the exact meaning of this index remains unclear.

We start with some general observations on Ix(A). <u>Throughout this chapter</u> A <u>denotes an</u> $n \times n$ <u>stable matrix, unless otherwise specified.</u>

Theorem 1. $\quad Ix(A) = Ix(A^*) = Ix(A^{-1})$.

Proof. Suppose $AH + HA^* > 0$. Then $H^{-1}(AH + HA^*)H^{-1} = A^*H^{-1} + H^{-1}A > 0$. But, if H has k equal eigenvalues so does H^{-1}, and this proves $Ix(A) \le Ix(A^*)$. Similarly, one shows $Ix(A^*) \le Ix(A)$, and the first equality follows. Also, $AH + HA^* > 0$ implies $A^{-1}(AH + HA^*)A^{-1*} = A^{-1}H + HA^{-1*} > 0$. Hence $Ix(A) \le Ix(A^{-1})$. Similarly, one shows $Ix(A^{-1}) \le Ix(A)$, completing the proof.

We let $A_1 \oplus A_2$ be the direct sum of the matrices A_1 and A_2. It is quite natural to ask:

Problem. Is

$$Ix(A_1 \oplus A_2) = Ix(A_1) + Ix(A_2) \quad ? \tag{1}$$

We would like to be able to get an affirmative answer to the question, but, unfortunately, (1) will be proved under some restrictions on A_1 or A_2. In the following we let

$$A = A_1 \oplus A_2$$

where A_1 and A_2 are stable matrices of order p and q, respectively. Here $p + q = n$.

Lemma 1. Let $A = A_1 \oplus A_2$. Then $Ix(A) \ge Ix(A_1) + Ix(A_2)$.

Proof. Let $k_i = Ix(A_i)$, $i = 1, 2$. There exists a matrix H_i, such that 1 is an eigenvalue of H_i of multiplicity k_i and $R(A_iH_i) > 0$, $i = 1, 2$.

Let $H = H_1 \oplus H_2$. Then $R(AH) = R(A_1H_1) \oplus R(A_2H_2) > 0$, and 1 is an eigenvalue of H of multiplicity $k_1 + k_2$. This completes the proof.

Before proceeding, we recall the well-known interlacing inequalities between the eigenvalues of an $n \times n$ Hermitian matrix H and an $n-1 \times n-1$ principal submatrix of H.

The Interlacing Inequalities [8, Vol. I, Chapter 10]:
Let H be an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Let K be an $n-1 \times n-1$ principal submatrix of H with eigenvalues $\eta_1 \geq \eta_2 \geq \ldots \geq \eta_{n-1}$. Then

$$\lambda_1 \geq \eta_1 \geq \lambda_2 \geq \eta_2 \geq \ldots \geq \lambda_{n-1} \geq \eta_{n-1} \geq \lambda_n \ . \qquad (2)$$

The next theorems describe situations where (1) holds.

Note: In a matrix partitioned into blocks, a * denotes a block whose entries do not matter for our purposes, while 0 denotes a block whose entries are all equal to zero.

Lemma 2. Let $A_2 = (a)$ be a 1×1 matrix, and let $A = A_1 \oplus A_2$. Then $Ix(A) = Ix(A_1) + 1$.

Proof. Let $k = Ix(A_1)$. Lemma 1 implies that $Ix(A) \geq k+1$. It remains to prove that $Ix(A) \leq k+1$. So let H be a matrix with r equal eigenvalues and satisfying $R(AH) > 0$. We can assume that 1 is an eigenvalue of H of multiplicity r. We write

$$H = \begin{bmatrix} H_1 & * \\ * & * \end{bmatrix}$$

where H_1 is an $n-1 \times n-1$ principal submatrix. Then

$$R(AH) = \begin{bmatrix} R(A_1 H_1) & * \\ * & * \end{bmatrix} > 0 \ .$$

Now $R(A_1 H_1) > 0$, since all principal submatrices of a positive definite matrix are themselves positive definite. Also, it follows from the interlacing inequalities that 1 is an eigenvalue of H_1 of multiplicity at least $r-1$. Hence $r-1 \leq k$, completing the proof.

<u>Lemma 3.</u> Let U be an $n \times n$ unitary matrix. Then $Ix(A) = Ix(U^*AU)$.

<u>Proof.</u> This follows from the identity

$$U^*(AH + HA^*)U = (U^*AU)(U^*HU) + (U^*HU)(U^*AU)^*$$

and the fact that H and U^*HU have the same eigenvalues.

<u>Theorem 2.</u> Let A_2 be a $q \times q$ normal matrix, and let $A = A_1 \oplus A_2$. Then

$$Ix(A) = Ix(A_1) + I_x(A_2) = Ix(A_1) + q \ .$$

<u>Proof.</u> We first note that $Ix(A_2) = q$, because A_2 normal and stable implies $R(A_2) = A_2 + A_2^* > 0$. This proves the second equality.

A_2 is unitarily similar to a diagonal matrix. Hence, by Lemma 3, we can assume without loss of generality that A_2 is a diagonal matrix, and write $A_2 = \text{diag}(d_1, d_2, \ldots, d_q)$.

The proof is by induction on q. For $q = 1$ the situation is exactly that of Lemma 2, so the theorem is true. Let q be an integer greater than one, and assume that the theorem holds for all diagonal matrices of order less than or equal to $q - 1$. Defining $B = \text{diag}(d_1, d_2, \ldots, d_{q-1})$, we get

$$A = \begin{bmatrix} A_1 \oplus B & 0 \\ 0 & d_q \end{bmatrix} .$$

Applying again Lemma 2 we get $\text{Ix}(A) = \text{Ix}(A_1 \oplus B) + 1$, while using the induction hypothesis we find $\text{Ix}(A_1 \oplus B) = \text{Ix}(A_1) + q - 1$. Hence $\text{Ix}(A) = \text{Ix}(A_1) + q$, completing the proof.

<u>Theorem 3.</u> Let A_2 be a $q \times q$ matrix such that $R(A_2) > 0$, and let $A = A_1 \oplus A_2$. Then

$$\text{Ix}(A) = \text{Ix}(A_1) + \text{Ix}(A_2) = \text{Ix}(A_1) + q \quad .$$

<u>Proof.</u> Clearly $\text{Ix}(A_2) = q$, as $R(A_2) > 0$.

It is well-known [14, p. 67] that every matrix is unitarily similar to a lower triangular matrix. Hence, by Lemma 3, we may assume without loss of generality that A_2 is a lower triangular matrix.

The proof is by induction on q. For $q = 1$ the theorem reduces to Lemma 2. So let q be an integer greater than 1, and assume that the theorem holds for all triangular matrices of order less than or equal to q-1. We write

$$A_2 = \begin{bmatrix} A_{22} & 0 \\ A_{23} & A_{33} \end{bmatrix}$$

where A_{22} is a $q-1 \times q-1$ matrix and A_{33} is a 1×1 matrix. Accordingly, A has the following block form

$$A = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & A_{23} & A_{33} \end{bmatrix}.$$

It is enough to prove that $Ix(A) \leq Ix(A_1) + q$, since $Ix(A) \geq Ix(A_1) + q$ by Lemma 1. So let H be a Hermitian matrix having 1 as an eigenvalue of multiplicity r, and $R(AH) > 0$. Partition H conformably with A,

$$H = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{12}^* & H_{22} & H_{23} \\ H_{13}^* & H_{23}^* & H_{33} \end{bmatrix}.$$

Then

$$R(AH) = AH + HA^* = \begin{bmatrix} R(A_1H_{11}) & A_1H_{12} + H_{12}A_{22}^* & * \\ A_{22}H_{12}^* + H_{12}^*A_1^* & R(A_{22}H_{22}) & * \\ * & * & * \end{bmatrix} > 0 .$$

The positive definite principal submatrix of order n-1, sitting in the upper left corner of R(AH), is equal to

$$R\left(\begin{bmatrix} A_1 & 0 \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix}\right).$$

But 1 is an eigenvalue of $\begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix}$ of multiplicity r-1 at least, by the interlacing inequalities. Also, A_{22} satisfies $R(A_{22}) > 0$, since $R(A_2) > 0$. It follows from the induction hypothesis that $r - 1 \leq Ix(A_1) + q-1$. Hence $r \leq Ix(A_1) + q$, and the result follows.

Theorem 3 is stronger than Theorem 2, because the latter is a special case of Theorem 3. It remains to be seen whether the assumption that $R(A_2) > 0$ can be dropped in proving (1). It can be easily shown that (1) holds for matrices A_1 and A_2 with $p+q = n \leq 4$.

The remaining part of the chapter is devoted to obtaining upper and lower bounds for the index of dissipation. We recall that $\pi(H)$ denotes the number of positive eigenvalues of the Hermitian matrix H.

Theorem 4. $$Ix(A) \leq \pi(A+A^*) \quad . \qquad (3)$$

Proof. Let k = Ix(A). There exists a matrix H > 0 having 1 as an eigenvalue of multiplicity k, and R(AH) > 0. Therefore there exists a unitary matrix U such that

$$K = U^*HU = \begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix}$$

where I is the identity matrix of order k and D is a real diagonal matrix. Let $B = U^*AU$. Then

$$U^*R(AH)U = U^*(AH+HA^*)U = BK+KB^* > 0 \quad .$$

Partitioning B conformably with K,

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} ,$$

we find that

$$R(BK) = \begin{bmatrix} R(B_{11}) & * \\ * & * \end{bmatrix} > 0 \quad .$$

Hence $R(B_{11}) > 0$. The interlacing inequalities imply that $\pi(B+B^*) \geq k$. But $B+B^* = U^*(A+A^*)U$, so $\pi(A+A^*) = \pi(B+B^*) \geq k$. This completes the proof.

The next question to ask is, obviously, whether equality holds in (3). It turns out that the answer is no, and so it seems certain that the index of dissipation of a matrix has no simple meaning. An example showing that strict inequality is possible in (3) follows the next corollary.

<u>Corollary 1.</u> $\quad Ix(A) \leq \min_{\alpha \geq 0} \pi\left((A+\alpha A^{-1}) + (A+\alpha A^{-1})^*\right)$.

Proof. Let α be an arbitrary nonnegative number, and let $R(AH) > 0$. It was previously proved (see proof of Theorem 1) that $R(AH) > 0$ implies $R(A^{-1}H) > 0$, and thus $R((A+\alpha A^{-1})H) > 0$. We conclude that

$$Ix(A) \le Ix(A+\alpha A^{-1}) \le \pi\left(R(A+\alpha A^{-1})\right) ,$$

the second inequality following from Theorem 4. Since α is arbitrary, the proof is complete.

Example. We exhibit a 3×3 matrix A such that $Ix(A) = 1$, while $\pi(A+A^*) = 2$. Let

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix} .$$

Then

$$A^{-1} = \begin{bmatrix} 1 & -3 & 14/3 \\ 0 & 1/2 & -1 \\ 0 & 0 & 1/3 \end{bmatrix} ; \quad R(A) = \begin{bmatrix} 2 & 6 & 4 \\ 6 & 4 & 6 \\ 4 & 6 & 6 \end{bmatrix} ;$$

$$R(A^{-1}) = \begin{bmatrix} 2 & -3 & 14/3 \\ -3 & 1 & -1 \\ 14/3 & -1 & 2/3 \end{bmatrix} .$$

It is easy to see that $\pi(A+A^*) = 2$, because $A+A^*$ has positive trace and a negative determinant. However, the matrix

$$R(A+0.3A^{-1}) = \begin{bmatrix} 2.6 & 5.1 & 5.4 \\ 5.1 & 4.3 & 5.7 \\ 5.4 & 5.7 & 6.2 \end{bmatrix}$$

has only one positive eigenvalue, so by Corollary 1 we have $Ix(A) = 1$.

Finally, we get a lower bound for the index of dissipation. The index of dissipation is invariant under unitary similarity (Lemma 3), so without loss of generality we may assume that A is a triangular matrix. Actually, we shall assume only that A is block triangular. The following characterization of positive definite matrices, due to Haynsworth, is required for our purposes.

Theorem 5 [11]. Let

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix}$$

be a Hermitian matrix. Then $H > 0$ if and only if $H_{11} > 0$ and $H_{22} = H_{12}^* H_{11}^{-1} H_{12} > 0$. The matrix $H_{12}^* H_{11}^{-1} H_{12}$ is called the Schur complement of H_{11} in H.

Theorem 6. Let A be a block triangular matrix, partitioned into blocks as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}.$$

The matrices A_{11}, A_{22}, A_{33} are square matrices of orders p, q, r, respectively, where $p+q+r = n$ (p and/or r can be zero). If $R(A_{22}) > 0$ then $Ix(A) \geq q$.

Proof. There exist H_1 and H_3 such that $R(A_{11}H_1) > 0$ and $R(A_{33}H_3) > 0$, because A_{11} and A_{33} are stable matrices. Let β_1 and β_3 be positive real numbers, to be determined later, and let $H = \beta_1 H_1 \oplus I \oplus \beta_3 H_3$. Here I denotes the identity matrix of order q. Hence,

$$R(AH) = \begin{bmatrix} \beta_1 R(A_{11}H_1) & A_{12} & \beta_3 A_{13} H_3 \\ A_{12}^* & R(A_{22}) & \beta_3 A_{23} H_3 \\ \beta_3 H_3 A_{13}^* & \beta_3 H_3 A_{23}^* & \beta_3 R(A_{33}H_3) \end{bmatrix}.$$

We shall show that for sufficiently large β_1 and sufficiently small β_3 we have $R(AH) > 0$, and this will establish the proof.

Since $R(A_{22}) = A_{22} + A_{22}^* > 0$, also

$$A_{22} + A_{22}^* - \beta_1^{-1} A_{12}^* (A_{11}H_1 + H_1 A_{11}^*)^{-1} A_{12} > 0 \qquad (4)$$

for sufficiently large β_1, by a continuity argument. We choose β_1 which satisfies (4) and fix it. For this choice of β_1 the principal submatrix

$$\begin{bmatrix} \beta_1 R(A_{11}H_1) & A_{12} \\ A_{12}^* & R(A_{22}) \end{bmatrix}$$

of R(AH) is positive definite. It suffices to show, by Theorem 5, that its Schur complement is positive definite for sufficiently small β_3. Indeed, calculating this Schur complement we get

$$\beta_3 \left\{ R(A_{33}H_3) - \beta_3 \begin{bmatrix} H_3A_{13}^* & H_3A_{23}^* \end{bmatrix} \begin{bmatrix} \beta_1 R(A_{11}H_1) & A_{12} \\ A_{12}^* & R(A_{22}) \end{bmatrix}^{-1} \begin{bmatrix} A_{13}H_3 \\ A_{23}H_3 \end{bmatrix} \right\}. \quad (5)$$

We chose H_3 so that $R(A_{33}H_3) > 0$ and β_1 so that (4) is satisfied. If we choose β_3 sufficiently small, then the matrix given in (5) is positive definite, by a continuity argument. This completes the proof, since 1 is an eigenvalue of H of multiplicity q.

<u>Corollary 2.</u> Let A have a system of r orthonormal eigenvectors. Then $Ix(A) \geq r$.

<u>Proof.</u> Let $x_1, x_2, \ldots, x_r$ be a system of orthonormal eigenvectors of A, and let $\lambda_1 \lambda_2, \ldots, \lambda_r$ be the corresponding eigenvalues. Let U be a unitary matrix having $x_1, x_2, \ldots, x_r$ as its first r columns. Then

$$U^*AU = \begin{bmatrix} D & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_r)$. The result follows from Theorem 6, since $D + D^* > 0$.

CHAPTER III

THE MINIMAL EIGENVALUE OF THE LYAPUNOV TRANSFORM

Throughout this chapter we consider a fixed stable matrix A of order n $(n > 1)$. In Chapter I we discuss the problem of finding $C_1(A)$, the image of PSD (the cone of positive semidefinite matrices) under the Lyapunov transformation. As indicated, this problem is far from being solved. In this chapter a different aspect of the problem is discussed. Corollaries 2 and 3 of Chapter I show that $C_1(A)$ strictly contains PSD, unless A is a scalar matrix. Thus, if $A \neq \lambda I$, there exist matrices of the form $AH+HA^*$, $H \geq 0$, with negative eigenvalues. We would like to know how negative can the eigenvalues of the Lyapunov transform $AH+HA^*$ become. To make the question meaningful, we have to restrict H to a bounded subset of PSD. A precise formulation of the question to be considered will be given following some additional definitions and notation.

Let H be a Hermitian matrix. Its eigenvalues will usually be denoted by

$$\alpha_1(H) \geq \alpha_2(H) \geq \dots \geq \alpha_n(H) \ .$$

$\alpha_1(H)$ and $\alpha_n(H)$ satisfy the variational characterization

$$\alpha_1(H) = \max_{(x,x)=1} x^* H x$$
$$\alpha_n(H) = \min_{(x,x)=1} x^* H x \tag{1}$$

see, e.g., [8, Vol. I, Chapter 10]. Here x denotes a column vector.

<u>Definition 1.</u> Let S be a convex subset of a linear space over the real numbers, and let $x_1, x_2, \ldots, x_m \in S$. The linear combination $\sum_{j=1}^{m} \theta_j x_j$ is said to be a <u>convex combination</u> of $x_1, x_2, \ldots, x_m$ if $\theta_j \geq 0$, $j = 1, 2, \ldots, m$, and $\sum_{j=1}^{m} \theta_j = 1$.

<u>Definition 2.</u> Let S be a convex subset of a linear space over the real numbers, and let f be a real valued function defined on S. We say that f is a <u>concave function</u> if it satisfies for every $x, y \in S$ and $0 \leq \theta \leq 1$ the inequality

$$f(\theta x + (1-\theta)y) \geq \theta f(x) + (1-\theta)f(y) .$$

<u>Note:</u> It follows immediately from this definition that

$$f\left(\sum_{j=1}^{m} \theta_j x_j\right) \geq \sum_{j=1}^{m} \theta_j f(x_j) \tag{2}$$

for every convex combination of points $x_1, x_2, \ldots, x_m \in S$.

We are ready to state the problem to be considered in this chapter. Let

$$J = \{H : H \in V \quad \text{and} \quad 0 \leq H \leq I\} . \tag{3}$$

Problem. Find

$$\psi(A) = \min_{H \in J} \{\alpha_n(AH + HA^*)\} \quad . \tag{4}$$

The number $\psi(A)$ is well defined, since $\alpha_n(AH+HA^*)$ is a continuous function of H, and J is a compact set. It gives us the minimal eigenvalue of the Lyapunov transform $AH+HA^*$, where H runs through all matrices in J. In what follows the value of $\psi(A)$ is determined in case A is a normal matrix, while lower and upper bounds for $\psi(A)$ are given in the general case.

To find $\psi(A)$ we start with some observations on J and the function $\alpha_n(AH+HA^*)$. First, note that for every $n \times n$ unitary matrix U we have

$$UJU^* = J \quad . \tag{5}$$

Next, the set J is compact and convex (in the space V). Moreover, it follows from (1) that

$$\alpha_n(\theta H + (1-\theta)K) \geq \theta\alpha_n(H) + (1-\theta)\alpha_n(K)$$

for every $H, K \in J$ and $0 \leq \theta \leq 1$, implying that $\alpha_n(AH+HA^*)$ is a concave function on the convex set J. We claim now that in order to find $\psi(A)$ it is enough to consider $\alpha_n(AH+HA^*)$ only on the extreme points of J. This follows immediately from (2) and from the following lemma. This lemma is essentially known [16] even in more general spaces, but a brief matrix theory proof is given for the sake of completeness.

Lemma 1. Let

$$P_j = \text{diag}(\underbrace{1, 1, \ldots, 1}_{j \text{ times}}, 0, \ldots, 0) , \qquad j = 0, 1, \ldots, n . \qquad (6)$$

The extreme points of J are exactly the projection matrices, i.e., the matrices of the form UP_jU^*, where U is an arbitrary $n \times n$ unitary matrix and $j = 0, 1, \ldots, n$. Moreover, every $H \in J$ can be written as a convex combination of (a finite number of) extreme points of J.

Proof. From (5) it follows that H is an extreme point of J if and only if UHU^* is, where U is an arbitrary unitary matrix. Hence it suffices to find what real diagonal matrices are extreme points of J. So let $D = \text{diag}(d_1, d_2, \ldots, d_n) \in J$. Since every number in the open interval (0, 1) is a convex combination of 0 and 1, the matrix D cannot be an extreme point, unless all the main diagonal entries are equal to 0 or 1. Conversely, we show that if $d_j = 0$ or $d_j = 1$ for all j, $1 \leq j \leq n$, D is an extreme point. We can assume that $D = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$, where I is the identity matrix of order q for some $0 \leq q \leq n$. Suppose that $D = \theta H + (1-\theta)K$ for some $H, K \in J$ and $0 < \theta < 1$. Partitioning H and K conformably with D,

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix} \qquad K = \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^* & K_{22} \end{bmatrix} ,$$

we find that

$$\theta H_{11} + (1-\theta)K_{11} = 0.$$

But $H \geq 0$ and $K \geq 0$ imply $H_{11} = K_{11} = 0$, and consequently $H_{12} = K_{12} = 0$. Furthermore,

$$\theta H_{22} + (1-\theta) K_{22} = I$$

together with $0 \leq H_{22} \leq I$ and $0 \leq K_{22} \leq I$ imply $H_{22} = K_{22} = I$. This completes the first part of the proof.

In proving the second part of the lemma we can again consider only real diagonal matrices. We let $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$, where we may assume that $1 \geq d_1 \geq d_2 \geq \ldots \geq d_n \geq 0$. The decomposition

$$D = \sum_{j=0}^{n} (d_j - d_{j+1}) P_j \quad ,$$

where $d_{n+1} = 0$ and $d_0 = 1 - d_1$, describes D as a convex combination of extreme points of J, completing the proof.

We denote now

$$\mathcal{P}_j = \{ U P_j U^* : U \text{ an arbitrary unitary } n \times n \text{ matrix} \}, \qquad j = 0, 1, \ldots, n \ , \tag{7}$$

$$\psi_j(A) = \min_{H \in \mathcal{P}_j} \{ \alpha_n (AH + HA^*) \} \ , \quad j = 0, 1, \ldots, n \ . \tag{8}$$

The sets $\mathcal{P}_j$ are compact, hence the $\psi_j(A)$ are well defined. Moreover, from (4), (8) and Lemma 1 it follows that

$$\psi(A) = \min_{0 \leq j \leq n} \psi_j(A) \ . \tag{9}$$

We shall use (9) to find $\psi(A)$ by evaluating the $\psi_j(A)$. Note that $\mathcal{P}_0 = \{0\}$ and $\mathcal{P}_n = \{I\}$, so $\psi_0(A) = 0$ and $\psi_n(A) = \alpha_n(A + A^*)$.

Notation: From now on all vectors are column vectors with n components, denoted by x, y and these letters with subscripts. The complex conjugate and transpose of x is denoted by x^*, and the transpose of x by x^t.

Lemma 2. Let $1 \leq j \leq n$. The following are equivalent:

(a) $H \in \mathcal{P}_j$.

(b) There exist vectors $x_1, x_2, \ldots, x_j$ such that

$$(x_k, x_\ell) = \delta_{k\ell}, \qquad k, \ell = 1, 2, \ldots, j \;,$$

and (10)

$$H = \sum_{k=1}^{j} x_k x_k^* \;.$$

Proof. (a)$\Rightarrow$(b). Let $H \in \mathcal{P}_j$. Hence $H = UP_jU^*$ for some unitary matrix U. Denote the columns of U by $x_1, x_2, \ldots, x_n$. Also, denote the unit vector with 0 in all places except 1 in the k-th entry by e_k, $k = 1, 2, \ldots, n$. Then

$$H = UP_jU^* = \sum_{k=1}^{j} Ue_k e_k^* U^* = \sum_{k=1}^{j} x_k x_k^* \;.$$

Hence the vectors $x_1, x_2, \ldots, x_j$ satisfy (10).

(b)$\Rightarrow$(a). Suppose that $x_1, x_2, \ldots, x_j$ satisfy (10). We can find vectors $x_{j+1}, \ldots, x_n$ such that $x_1, x_2, \ldots, x_n$ form an orthonormal basis. The matrix U, whose k-th column is x_k, is a unitary matrix and satisfies $H = UP_jU^*$.

Corollary 1. $H \in \mathcal{P}_{n-1}$ if and only if there exists a vector x such that $(x, x) = 1$ and $H = I - xx^*$.

Proof. This follows immediately from Lemma 2.

Relations among the $\psi_j(A)$

The purpose of the next theorems is to prove that $\psi(A) = \psi_1(A)$. For each vector x, such that $(x, x) = 1$, we define

$$M(A, x) = Axx^* + xx^*A^* \quad . \tag{11}$$

Lemma 3. Let x be a vector such that $(x, x) = 1$. Then

$$\alpha_1(M(A, x)) \geq 0 = \alpha_2(M(A, x)) = \ldots$$
$$\ldots = \alpha_{n-1}(M(A, x)) \geq \alpha_n(M(A, x)) \quad . \tag{12}$$

Proof. There exists an $n \times n$ unitary matrix U such that $Ux = e_1$, where e_1 is the unit vector whose first component is equal to 1 and all other components are equal to 0. Let $B = UAU^* = (b_{ij})$. Then

$$UM(A, x)U^* = Be_1e_1^* + e_1e_1^*B^* \quad .$$

Let $f(\lambda)$ be the characteristic polynomial of $M(A, x)$. We have

$$f(\lambda) = \lambda^{n-2}\left[\lambda^2 - (b_{11} + \overline{b}_{11})\lambda - \sum_{i=2}^{n} |b_{i1}|^2\right] \quad .$$

This completes the proof.

Lemma 4. $\psi_1(A) = \psi_1(A^*)$.

Proof. Recall that the trace function satisfies $tr(A_1A_2) = tr(A_2A_1)$ for every pair of matrices A_1 and A_2 of orders $m\times p$ and $p\times m$, respectively. Moreover, the trace of a 1×1 matrix is equal to its single entry. Using these properties, Lemma 2 and the variational characterization of the smallest eigenvalue of a Hermitian matrix, we get

$$\psi_1(A) = \min_{(x,x)=1} \alpha_n(M(A,x)) = \min_{(x,x)=1} \min_{(y,y)=1} y^*(Axx^* + xx^*A^*)y$$

$$= \min_{(x,x)=1} \min_{(y,y)=1} [tr(y^*Axx^*y + y^*xx^*A^*y)]$$

$$= \min_{(x,x)=1} \min_{(y,y)=1} [tr(Axx^*yy^*) + tr(yy^*xx^*A^*)] \ .$$

Replacing A by A^* we get

$$\psi_1(A^*) = \min_{(x,x)=1} \min_{(y,y)=1} [tr(A^*xx^*yy^*) + tr(yy^*xx^*A)]$$

$$= \min_{(x,x)=1} \min_{(y,y)=1} [tr(xx^*yy^*A^*) + tr(Ayy^*xx^*)]$$

$$= \min_{(x,x)=1} \min_{(y,y)=1} [tr(yy^*xx^*A^*) + tr(Axx^*yy^*)] = \psi_1(A) \ .$$

Theorem 1. $\psi_n(A) \geq \psi_1(A) \geq \psi_2(A) \geq \dots \geq \psi_{n-1}(A)$.

Proof. We first prove that $\psi_n(A) \geq \psi_1(A)$. We have

$$\psi_n(A) = \psi_n(A^*) = \alpha_n(A + A^*) \ .$$

Also, it follows from (12) and the trace properties that for each vector x, such that $(x, x) = 1$,

$$\alpha_1(M(A, x)) + \alpha_n(M(A, x)) = \mathrm{tr}(M(A, x)) = \mathrm{tr}(Axx^* + xx^*A^*) = x^*(A + A^*)x \ .$$

Hence, by Lemma 3,

$$\alpha_n(M(A, x)) \leq x^*(A + A^*)x \ .$$

We conclude that

$$\psi_1(A) = \min_{(x,x)=1} \alpha_n(M(A, x)) \leq \min_{(x,x)=1} x^*(A + A^*)x = \alpha_n(A + A^*) = \psi_n(A) \ .$$

It remains to prove $\psi_1(A) \geq \psi_2(A) \geq \ldots \geq \psi_{n-1}(A)$. Let $1 \leq j \leq n-2$. We prove that $\psi_j(A) \geq \psi_{j+1}(A)$.

There exists a matrix $H_j \in \mathcal{P}_j$ such that $\alpha_n(AH_j + H_jA^*) = \psi_j(A)$. By Lemma 2 we can write

$$H_j = \sum_{k=1}^{j} x_k x_k^* \ ,$$

where $x_1, x_2, \ldots, x_j$ satisfy (10). There exists a vector y such that $(y, y) = 1$ and

$$\psi_j(A) = \alpha_n(AH_j + H_jA^*) = y^*(AH_j + H_jA^*)y \ . \qquad (13)$$

We look now on the linear subspace spanned by $x_1, x_2, \ldots, x_j, y$. Since $j \leq n-2$, there exists a vector x_{j+1} such that

$$\begin{aligned} (x_{j+1}, x_k) &= \delta_{j+1,k} \ , \qquad k = 1, 2, \ldots, j+1 \ , \\ (x_{j+1}, y) &= 0 \ . \end{aligned} \qquad (14)$$

Define now

$$H_{j+1} = \sum_{k=1}^{j+1} x_k x_k^* = H_j + x_{j+1}x_{j+1}^* \quad .$$

It follows from Lemma 2 and (14) that $H_{j+1} \in \mathcal{P}_{j+1}$. Moreover,

$$y^*(AH_{j+1} + H_{j+1}A^*)y = y^*(AH_j + H_jA^*)y$$

$$+ y^*(Ax_{j+1}x_{j+1}^* + x_{j+1}x_{j+1}^*A^*)y = \psi_j(A) \quad ,$$

the last equality following from (13) and (14). Applying again the variational characterization (1), we get

$$\psi_{j+1}(A) \leq \alpha_n(AH_{j+1} + H_{j+1}A^*) \leq y^*(AH_{j+1} + H_{j+1}A^*)y = \psi_j(A) \quad .$$

Theorem 2. $\qquad \psi_1(A) = \psi_2(A) = \ldots = \psi_{n-1}(A) \quad .$

Proof. It is enough to show that $\psi_1(A) = \psi_{n-1}(A)$, by Theorem 1. Using Corollary 1, (11), properties of the trace function and Lemmas 3 and 4, we get:

$$\psi_{n-1}(A) = \min_{H \in \mathcal{P}_{n-1}} \alpha_n(AH + HA^*) = \min_{(x,x)=1} \alpha_n[A(I-xx^*) + (I-xx^*)A^*]$$

$$= \min_{(x,x)=1} \min_{(y,y)=1} y^*[A(I-xx^*) + (I-xx^*)A^*]y$$

$$= \min_{(y,y)=1} \min_{(x,x)=1} [y^*(A+A^*)y - y^*(Axx^* + xx^*A^*)y]$$

$$= \min_{(y,y)=1} [y^*(A+A^*)y - \max_{(x,x)=1} y^*(Axx^* + xx^*A^*)y]$$

$$= \min_{(y,y)=1} [y^*(A+A^*)y - \max_{(x,x)=1} \mathrm{tr}(y^*Axx^*y + y^*xx^*A^*y)]$$

$$= \min_{(y,y)=1} [y^*(A+A^*)y - \max_{(x,x)=1} \mathrm{tr}(x^*yy^*Ax + x^*A^*yy^*x)]$$

$$= \min_{(y,y)=1} [y^*(A+A^*)y - \max_{(x,x)=1} x^*(yy^*A + A^*yy^*)x]$$

$$= \min_{(y,y)=1} [y^*(A+A^*)y - \alpha_1(M(A^*, y))]$$

$$= \min_{(y,y)=1} [y^*(A+A^*)y - \{\mathrm{tr}(M(A^*, y)) - \alpha_n(M(A^*, y))\}]$$

$$= \min_{(y,y)=1} [y^*(A+A^*)y - y^*(A+A^*)y + \alpha_n(M(A^*, y))]$$

$$= \psi_1(A^*) = \psi_1(A).$$

<u>Corollary 2.</u>

$$\psi(A) = \psi_1(A)$$

and

$$\psi(A) = \psi(A^*) \ .$$

<u>Proof.</u> The proof follows from Theorems 1 and 2, Lemma 4 and (9).

<u>Computation of $\psi(A)$ for Normal and Hermitian Matrices</u>

We use Corollary 2 to find $\psi(A)$, by computing $\psi_1(A)$. In case A is a normal and stable matrix $\psi(A)$ is determined precisely, while for a general stable matrix only upper and lower bounds are given here.

We assume now that A is a given normal and stable matrix. From (5) and the identity

$$U(AH + HA^*)U^* = (UAU^*)(UHU^*) + (UHU^*)(UAU^*)^*$$

it follows that $\psi(A) = \psi(U^*AU)$, for every $n \times n$ unitary matrix U. Hence we can assume that A is a diagonal matrix, and we write

$$A = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \ . \tag{15}$$

Here

$$\lambda_j = \mu_j + i\nu_j \ , \qquad j = 1, 2, \ldots, n \ , \tag{16}$$

where $i = \sqrt{-1}$, and $\mu_1, \mu_2, \ldots, \mu_n$ and $\nu_1, \nu_2, \ldots, \nu_n$ are real numbers satisfying

$$\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n > 0 \ .$$

We define

$$\begin{aligned} D_R &= \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_n) \\ D_I &= \mathrm{diag}(\nu_1, \nu_2, \ldots, \nu_n) \end{aligned} \tag{17}$$

so $A = D_R + iD_I$. Obviously, $A = A^*$ if and only if $D_I = 0$.

Let $x = (\xi_j)$ be a column vector with coordinates $\xi_1, \xi_2, \ldots, \xi_n$, satisfying $(x, x) = 1$. Recall that

$$\psi(A) = \psi_1(A) = \min_{(x,x)=1} \alpha_n(M(A, x))$$

where $M(A, x)$ is defined by (11). Let $f_x(\lambda)$ be the characteristic polynomial of $M(A, x)$. Since A is a diagonal matrix, it is easy to verify that

$$f_x(\lambda) = \lambda^{n-2}\left\{\lambda^2 - 2\left(\sum_{j=1}^{n} \mu_j |\xi_j|^2\right)\lambda + \right.$$

$$\left. \sum_{1 \leq j<k \leq n} |\xi_j \xi_k|^2 \left[(\lambda_j + \overline{\lambda}_j)(\lambda_k + \overline{\lambda}_k) - (\lambda_j + \overline{\lambda}_k)(\overline{\lambda}_j + \lambda_k)\right]\right\}$$

$$= \lambda^{n-2}\left\{\lambda^2 - 2\left(\sum_{j=1}^{n} \mu_j|\xi_j|^2\right)\lambda - \sum_{1\le j<k\le n} |\xi_j\xi_k|^2|\lambda_j-\lambda_k|^2\right\} .$$

Note: $f_x(\lambda)$ depends only on the $|\xi_j|$, so we can assume from now on (for the case of normal A) that x has only real entries. This and the computation of $\alpha_n(M(A,x))$ from $f_x(\lambda)$ give

$$\alpha_n(M(A, x)) = \sum_{j=1}^{n} \mu_j\xi_j^2 - \left[\left(\sum_{j=1}^{n} \mu_j\xi_j^2\right)^2 + \sum_{1\le j<k\le n} |\lambda_j-\lambda_k|^2 \xi_j^2\xi_k^2\right]^{\frac{1}{2}} . \quad (18)$$

Denoting

$$\Delta(x) = \left(\sum_{j=1}^{n} \mu_j\xi_j^2\right)^2 + \sum_{1\le j<k\le n} |\lambda_j-\lambda_k|^2 \xi_j^2\xi_k^2 \quad (19)$$

we have $\Delta(x) > 0$ and

$$\Delta(x) = \sum_{j=1}^{n} \mu_j^2\xi_j^4 + 2 \sum_{1\le j<k\le n} \mu_j\mu_k\xi_j^2\xi_k^2$$

$$+ \sum_{1\le j<k\le n} [\mu_j^2 - 2\mu_j\mu_k + \mu_k^2 + (\nu_j - \nu_k)^2]\xi_j^2\xi_k^2 .$$

Since $(x, x) = 1$, we get

$$\Delta(x) = \sum_{j=1}^{n} \mu_j^2\xi_j^2 + \sum_{1\le j<k\le n} (\nu_j - \nu_k)^2\xi_j^2\xi_k^2 . \quad (20)$$

Also,

$$\sum_{j=1}^{n} \mu_j^2\xi_j^2 + \sum_{1\le j<k\le n} (\nu_j^2 - 2\nu_j\nu_k + \nu_k^2)\xi_j^2\xi_k^2 = \sum_{j=1}^{n} \mu_j^2\xi_j^2$$

$$+\sum_{j=1}^{n} \nu_j^2\xi_j^2(1-\xi_j^2) - 2\sum_{1\le j<k\le n} \nu_j\nu_k\xi_j^2\xi_k^2 = \sum_{j=1}^{n}(\mu_j^2+\nu_j^2)\xi_j^2 - \left(\sum_{j=1}^{n}\nu_j\xi_j^2\right)^2$$

so $\Delta(x)$ can be written in the following form:

$$\Delta(x) = \sum_{j=1}^{n}(\mu_j^2+\nu_j^2)\xi_j^2 - \left(\sum_{j=1}^{n}\nu_j\xi_j^2\right)^2 = x^t(D_R^2+D_I^2)x - (x^tD_Ix)^2 . \quad (21)$$

Thus,

$$\alpha_n(M(A,x)) = \sum_{j=1}^{n}\mu_j\xi_j^2 - \Delta(x)^{\frac{1}{2}}$$

$$= x^tD_Rx - [x^tD_R^2x + x^tD_I^2x - (x^tD_Ix)^2]^{\frac{1}{2}} .$$

The last expression can be written in a homogeneous form of degree two, using the fact that $(x,x) = x^tx = 1$. Indeed,

$$\alpha_n(M(A,x)) = x^tD_Rx - [(x^tx)\{x^t(D_R^2+D_I^2)x\} - (x^tD_Ix)^2]^{\frac{1}{2}} . \quad (22)$$

We are interested in the minimum of $\alpha_n(M(A,x))$, subject to the equality constraint $(x,x) = 1$. We use the method of Lagrange multipliers and look on the function

$$\alpha_n(M(A,x)) - \beta(x,x)$$

where β is a Lagrange multiplier. The real vector $x = (\xi_j)$ minimizes $\alpha_n(M(A,x))$, subject to $(x,x) = 1$, only if the equations

$$\frac{\partial\alpha_n(M(A,x))}{\partial\xi_j} - 2\beta\xi_j = 0 , \qquad j = 1,2,\ldots,n ,$$

and

$$(x, x) = \sum_{j=1}^{n} \xi_j^2 = 1$$

are satisfied. Writing these equations in detail we get:

$$[2\mu_j - \Delta(x)^{-\frac{1}{2}} \{\mu_j^2 + \nu_j^2 + x^t(D_R^2 + D_I^2)x - 2(x^t D_I x)\nu_j\} - 2\beta]\xi_j = 0 \ , \quad j = 1, 2, \ldots, n \ ,$$

$$\sum_{j=1}^{n} \xi_j^2 = 1 \ . \tag{23}$$

Since $\alpha_n(M(A, x))$, in the form given by (22), and (x, x) are homogeneous functions of degree two, it follows from Euler's theorem on homogeneous functions that if x and β satisfy (23) then

$$\beta = \alpha_n(M(A, x)) \ . \tag{24}$$

To simplify subsequent computations we make, for the time being, the following assumptions:

Assumption (a). The matrix A is given by (15). We may assume without loss of generality that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct. Otherwise, suppose that $\lambda_1, \lambda_2, \ldots, \lambda_r$ $(r < n)$ are the distinct numbers among $\lambda_1, \lambda_2, \ldots, \lambda_n$, and define $B = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_r)$. It follows from (22) that $\psi_1(A) = \psi_1(B)$, and B is a matrix of order r with distinct eigenvalues.

Assumption (b). Consider the points (μ_j, ν_j), $j = 1, 2, \ldots, n$, in the Cartesian plane. We assume that no four of these points lie on one proper circle. This assumption will be removed later by a continuity argument.

Let the vector x minimize $\alpha_n(M(A, x))$, subject to $(x, x) = 1$. Then for all indices j such that $\xi_j \neq 0$ the equation

$$2\mu_j - \Delta(x)^{-\frac{1}{2}} \{\mu_j^2 + \nu_j^2 + x^t(D_R^2 + D_I^2)x - 2(x^t D_I x)\nu_j\} - 2\beta = 0 \qquad (25)$$

must hold. Hence all the corresponding points (μ_j, ν_j) lie on one proper circle. Because of assumption (b) it follows that x has at most three nonzero components, and in the Hermitian case at most two nonzero components. Thus, in order to find $\psi(A)$, we need only consider vectors with at most three nonzero components (two at most in the Hermitian case) and satisfying (23). We investigate three cases.

Case 1. The vector x has exactly one nonzero component. In this case it follows from (18) that $\alpha_n(M(A, x)) = 0$. As indicated in the beginning of this chapter, $\psi(A) \leq 0$ with equality holding if and only if A is a scalar matrix. But, by assumption (a), A is not a scalar matrix, and hence the required minimum is not attained at x.

Case 2. The vector x has exactly two nonzero components. So let $1 \leq k < \ell \leq n$, and assume that all the components of x are equal to zero except ξ_k and ξ_ℓ. We denote

$$\begin{aligned} \mu_{k\ell} &= \mu_k - \mu_\ell \geq 0 \\ \nu_{k\ell}^2 &= (\nu_k - \nu_\ell)^2 \end{aligned} \qquad (26)$$

and

$$\eta_j = \xi_j^2 , \qquad j = k, \ell . \tag{27}$$

Equations (23) are rewritten here as

$$2\mu_j - \Delta(x)^{-\frac{1}{2}} \{\mu_j^2 + \nu_j^2 + \eta_k(\mu_k^2 + \nu_k^2) + \eta_\ell(\mu_\ell^2 + \nu_\ell^2)$$

$$- 2\nu_j(\eta_k\nu_k + \eta_\ell\nu_\ell)\} - 2\beta = 0 , \qquad j = k, \ell , \tag{28}$$

and

$$\eta_k + \eta_\ell = 1 ; \quad \eta_k > 0 ; \quad \eta_\ell > 0 . \tag{29}$$

In order to solve these equations we assume further that $\nu_{k\ell}^2 > 0$. It is not difficult to remove this restriction after finishing the calculations.

We use here the expression (20) for $\Delta(x)$, so we have

$$\Delta(x) = \mu_k^2\eta_k + \mu_\ell^2\eta_\ell + \nu_{k\ell}^2\eta_k\eta_\ell .$$

Subtracting the equations for $j = k$ and $j = \ell$ in (28) gives

$$2\mu_{k\ell} - [\mu_k^2\eta_k + \mu_\ell^2\eta_\ell + \nu_{k\ell}^2\eta_k\eta_\ell]^{-\frac{1}{2}} \{(\mu_k^2 - \mu_\ell^2) + (\nu_k^2 - \nu_\ell^2)$$

$$- 2(\nu_k - \nu_\ell)(\eta_k\nu_k + \eta_\ell\nu_\ell)\} = 0$$

and substituting $\eta_\ell = 1 - \eta_k$ gives the equation

$$2\mu_{k\ell} = \frac{\mu_k^2 - \mu_\ell^2 + \nu_{k\ell}^2 - 2\nu_{k\ell}^2\eta_k}{[\mu_\ell^2 + (\mu_k^2 - \mu_\ell^2 + \nu_{k\ell}^2)\eta_k - \nu_{k\ell}^2\eta_k^2]^{\frac{1}{2}}} . \tag{30}$$

We show that (30) possesses a solution in $(0, \frac{1}{2}]$ and, conversely, every solution of (30) must lie in $(0, \frac{1}{2}]$. This is obvious in case $\mu_k = \mu_\ell$, for then $\eta_k = \frac{1}{2}$ is the only solution of (30). In case $\mu_k > \mu_\ell$ the right-hand side of (30) for $\eta_k = 0$ is larger than the left-hand side of (30), while the right-hand side of (30) for $\eta_k \geq \frac{1}{2}$ is smaller than the left-hand side.

Rewriting (30) leads to the following quadratic equation,

$$\nu_{k\ell}^2(\mu_{k\ell}^2 + \nu_{k\ell}^2)\eta_k^2 - (\mu_{k\ell}^2 + \nu_{k\ell}^2)(\mu_k^2 - \mu_\ell^2 + \nu_{k\ell}^2)\eta_k + \frac{1}{4}(\mu_k^2 - \mu_\ell^2 + \nu_{k\ell}^2)^2 - \mu_\ell^2\mu_{k\ell}^2 = 0 .$$

Its only solution in $(0, \frac{1}{2}]$ is

$$\eta_k = \frac{(\mu_{k\ell}^2 + \nu_{k\ell}^2)(\mu_k^2 - \mu_\ell^2 + \nu_{k\ell}^2) - [\mu_{k\ell}^2(\mu_{k\ell}^2 + \nu_{k\ell}^2)\,\theta]^{\frac{1}{2}}}{2\nu_{k\ell}^2(\mu_{k\ell}^2 + \nu_{k\ell}^2)} \tag{31}$$

where we define

$$\theta = (\mu_k^2 - \mu_\ell^2 + \nu_{k\ell}^2)^2 + 4\mu_\ell^2\nu_{k\ell}^2 \quad .$$

Hence

$$\theta = \nu_{k\ell}^4 + 2\nu_{k\ell}^2(\mu_k^2 + \mu_\ell^2) + \mu_{k\ell}^2(\mu_k + \mu_\ell)^2 = (\mu_{k\ell}^2 + \nu_{k\ell}^2)\,[\nu_{k\ell}^2 + (\mu_k + \mu_\ell)^2] \; . \tag{32}$$

It remains to determine $\alpha_n(M(A, x))$. We know that $\eta_k = \xi_k^2$ is given by (31), and $\eta_\ell = \xi_\ell^2 = 1 - \eta_k$. Hence the modulii of ξ_k and ξ_ℓ are uniquely determined. Substituting these values into $\alpha_n(M(A, x))$ and using (30), we get

$$\begin{aligned}
\alpha_n(M(A, x)) &= \eta_k\mu_k + \eta_\ell\mu_\ell - [\eta_k\mu_k^2 + \eta_\ell\mu_\ell^2 + \eta_k\eta_\ell\nu_{k\ell}^2]^{\frac{1}{2}} \\
&= \mu_\ell + \eta_k\mu_{k\ell} - [\mu_\ell^2 + \eta_k(\mu_k^2 - \mu_\ell^2) + \eta_k(1 - \eta_k)\nu_{k\ell}^2]^{\frac{1}{2}} \\
&= \mu_\ell + \frac{\mu_{k\ell}\{(\mu_{k\ell}^2 + \nu_{k\ell}^2)(\mu_k^2 - \mu_\ell^2 + \nu_{k\ell}^2) - [\mu_{k\ell}^2(\mu_{k\ell}^2 + \nu_{k\ell}^2)\,\theta]^{\frac{1}{2}}\}}{2\nu_{k\ell}^2(\mu_{k\ell}^2 + \nu_{k\ell}^2)}
\end{aligned}$$

$$- \frac{\mu_k^2 - \mu_\ell^2 + \nu_{k\ell}^2}{2\mu_{k\ell}}$$

$$+ \frac{\nu_{k\ell}^2\{(\mu_{k\ell}^2 + \nu_{k\ell}^2)(\mu_k^2 - \mu_\ell^2 + \nu_{k\ell}^2) - [\mu_{k\ell}^2(\mu_{k\ell}^2 + \nu_{k\ell}^2)\theta]^{\frac{1}{2}}\}}{2\mu_{k\ell}\nu_{k\ell}^2(\mu_{k\ell}^2 + \nu_{k\ell}^2)}$$

Hence,

$$2\mu_{k\ell}\nu_{k\ell}^2(\mu_{k\ell}^2 + \nu_{k\ell}^2)\alpha_n(M(A, x)) = 2\mu_\ell\mu_{k\ell}\nu_{k\ell}^2(\mu_{k\ell}^2 + \nu_{k\ell}^2)$$

$$+ \mu_{k\ell}^2(\mu_{k\ell}^2 + \nu_{k\ell}^2)(\mu_k^2 - \mu_\ell^2 + \nu_{k\ell}^2) - \mu_{k\ell}^2[\mu_{k\ell}^2(\mu_{k\ell}^2 + \nu_{k\ell}^2)\theta]^{\frac{1}{2}}$$

$$- (\mu_k^2 - \mu_\ell^2 + \nu_{k\ell}^2)\nu_{k\ell}^2(\mu_{k\ell}^2 + \nu_{k\ell}^2) + \nu_{k\ell}^2(\mu_{k\ell}^2 + \nu_{k\ell}^2)(\mu_k^2 - \mu_\ell^2 + \nu_{k\ell}^2)$$

$$- \nu_{k\ell}^2[\mu_{k\ell}^2(\mu_{k\ell}^2 + \nu_{k\ell}^2)\theta]^{\frac{1}{2}} = (\mu_{k\ell}^2 + \nu_{k\ell}^2)\mu_{k\ell}[\mu_{k\ell}^2(\mu_k + \mu_\ell) + \mu_{k\ell}\nu_{k\ell}^2 + 2\mu_\ell\nu_{k\ell}^2]$$

$$- (\mu_{k\ell}^2 + \nu_{k\ell}^2)[\mu_{k\ell}^2(\mu_{k\ell}^2 + \nu_{k\ell}^2)\theta]^{\frac{1}{2}}$$

and

$$\alpha_n(M(A, x)) = \frac{(\mu_k + \mu_\ell)(\mu_{k\ell}^2 + \nu_{k\ell}^2) - [(\mu_{k\ell}^2 + \nu_{k\ell}^2)\theta]^{\frac{1}{2}}}{2\nu_{k\ell}^2} .$$

Substituting for θ from (32) we finally get

$$\alpha_n(M(A, x)) = \frac{(\mu_{k\ell}^2 + \nu_{k\ell}^2)(\mu_k + \mu_\ell)}{2\nu_{k\ell}^2}\left\{1 - \left[1 + \frac{\nu_{k\ell}^2}{(\mu_k + \mu_\ell)^2}\right]^{\frac{1}{2}}\right\}$$

or

$$\alpha_n(M(A, x)) = -\frac{\mu_{k\ell}^2 + \nu_{k\ell}^2}{2\{(\mu_k + \mu_\ell) + [(\mu_k + \mu_\ell)^2 + \nu_{k\ell}^2]^{\frac{1}{2}}\}} . \qquad (33)$$

The last formula was obtained under the assumption that $\nu_{k\ell}^2 > 0$. However, (33) remains meaningful if we put $\nu_{k\ell} = 0$. Indeed, if $\nu_{k\ell} = 0$ we can still show by similar calculations that if x satisfies (28) and (29), then $\alpha_n(M(A, x))$ is given by (33).

We define

$$\varphi_{k\ell} = -\frac{\mu_{k\ell}^2 + \nu_{k\ell}^2}{2\{(\mu_k + \mu_\ell) + [(\mu_k + \mu_\ell)^2 + \nu_{k\ell}^2]^{\frac{1}{2}}\}}$$
$$= -\frac{(\mu_k - \mu_\ell)^2 + (\nu_k - \nu_\ell)^2}{2\{(\mu_k + \mu_\ell) + [(\mu_k + \mu_\ell)^2 + (\nu_k - \nu_\ell)^2]^{\frac{1}{2}}\}}, \quad 1 \le k < \ell \le n . \tag{34}$$

It follows from the preceding calculations that if the vector x has exactly two nonzero components and satisfies equations (23), then

$$\alpha_n(M(A, x)) = \varphi_{k\ell} \qquad \text{for some } k, \ell;\ 1 \le k < \ell \le n . \tag{35}$$

Case 3. The vector x has exactly three nonzero components. So let $1 \le k < \ell < m \le n$, and assume that all the components of x are equal to zero except ξ_k, ξ_ℓ and ξ_m.

It follows from (25) that the points (μ_j, ν_j), $j = k, \ell, m$, must lie on a proper circle. Hence, if this is not the case, there exists no vector x which satisfies (23) and whose only nonzero components are ξ_k, ξ_ℓ, ξ_m. This explains the fact that for Hermitian matrices we have to consider only Cases 1 and 2.

If there exists a proper circle passing through (μ_j, ν_j), $j = k, \ell, m$, let $(c_{k\ell m}, d_{k\ell m})$ and $r_{k\ell m}$ denote its center and radius, respectively. Thus,

$$(\mu_j - c_{k\ell m})^2 + (\nu_j - d_{k\ell m})^2 = r_{k\ell m}^2 , \qquad j = k, \ell, m . \tag{36}$$

But, rewriting (25) we get

$$(\mu_j - \Delta(x)^{\frac{1}{2}})^2 + (\nu_j - x^t D_I x)^2 = -2\beta \Delta(x)^{\frac{1}{2}} - x^t(D_R^2 + D_I^2)x$$
$$+ \Delta(x) + (x^t D_I x)^2 , \quad j = k, \ell, m . \tag{37}$$

Comparing (36) and (37) and substituting for $\Delta(x)$ using (21), we find that the following equations must be satisfied by x:

$$\Delta(x)^{\frac{1}{2}} = [x^t(D_R^2 + D_I^2)x - (x^t D_I x)^2]^{\frac{1}{2}} = c_{k\ell m} \tag{38}$$

$$x^t D_I x = d_{k\ell m} \tag{39}$$

$$x^t x = 1 \tag{40}$$

$$\beta = -\frac{r_{k\ell m}^2}{2c_{k\ell m}} . \tag{41}$$

Let

$$\eta_j = \xi_j^2 > 0 , \qquad j = k, \ell, m .$$

Since $\Delta(x) > 0$, it follows that x satisfies equations (38), (39), and (40) if and only if

$$c_{k\ell m} > 0$$

and the following linear system has a positive solution for η_k, η_ℓ, η_m:

$$\eta_k + \eta_\ell + \eta_m = 1$$

$$\nu_k \eta_k + \nu_\ell \eta_\ell + \nu_m \eta_m = d_{k\ell m} \tag{42}$$

$$(\mu_k^2 + \nu_k^2)\eta_k + (\mu_\ell^2 + \nu_\ell^2)\eta_\ell + (\mu_m^2 + \nu_m^2)\eta_m = c_{k\ell m}^2 + d_{k\ell m}^2 .$$

We are ready to conclude the discussion of Case 3. We see that it is possible that there exists no vector x which satisfies (23) and whose only nonzero components are ξ_k, ξ_ℓ, ξ_m. Such a vector exists if and only if the following conditions are satisfied.

(i) There exists a proper circle passing through (μ_j, ν_j), $j = k, \ell, m$.

(ii) $c_{k\ell m} > 0$.

(iii) The linear system (42) has a positive solution for η_k, η_ℓ, η_m .

If these conditions are satisfied, the modulii of ξ_k, ξ_ℓ, ξ_m are uniquely determined. Moreover, we conclude from (24) and (41) that

$$\alpha_n(M(A,x)) = \beta = -\frac{r^2_{k\ell m}}{2c_{k\ell m}} , \tag{43}$$

so $\alpha_n(M(A, x))$ is uniquely determined.

Finally, we define

$$\Phi_{k\ell m} = \begin{cases} -\dfrac{r^2_{k\ell m}}{2c_{k\ell m}} & \text{if conditions (i), (ii), (iii) are satisfied} \\ & \qquad 1 \leq k < \ell < m \leq n . \\ 0 & \text{otherwise} \end{cases} \tag{44}$$

This completes the investigation of the three cases. Recall that this investigation was carried out under assumptions (a) and (b). We are ready to find $\psi(A)$ for a normal matrix A, following the next elementary lemma.

<u>Lemma 5</u>. Let $\xi > 0$, $\eta > 0$ and $\xi \geq \eta$. Let

$$h(\xi, \eta) = \frac{(\xi - \eta)^2}{\xi + \eta} \quad .$$

Then $\qquad h_\xi(\xi, \eta) \geq 0 \quad$ and $\quad h_\eta(\xi, \eta) \leq 0$.

(Here, naturally, h_ξ and h_η denote partial derivatives.)

Proof. Trivial.

Theorem 3. Let $A = A^*$ be a positive definite matrix with eigenvalues $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n > 0$. Then

$$\psi(A) = -\frac{(\mu_1 - \mu_n)^2}{4(\mu_1 + \mu_n)} \quad .$$

Proof. Again we may assume without loss of generality that $A = D_R$, where D_R is given by (17). We further assume that all the eigenvalues of A are distinct. We know that

$$\psi(A) = \psi_1(A) = \min_{(x,x)=1} \alpha_n(M(A, x)) \quad .$$

$\psi_1(A)$ is attained at x, such that $(x, x) = 1$, only if x satisfies (23). Under our assumptions, this vector x has at most two nonzero components. The discussion of Cases 1 and 2 shows that there exist k and ℓ, $1 \leq k < \ell \leq n$, such that $\alpha_n(M(A, x)) = \varphi_{k\ell}$, where

$$\varphi_{k\ell} = -\frac{(\mu_k - \mu_\ell)^2}{4(\mu_k + \mu_\ell)} \quad .$$

Here the expression for $\varphi_{k\ell}$ is a specialization of (34) for the

Hermitian case. Since there are only a finite number of $\varphi_{k\ell}$, we get

$$\psi(A) = \min_{1 \le k < \ell \le n} \left\{ -\frac{(\mu_k - \mu_\ell)^2}{4(\mu_k + \mu_\ell)} \right\} = -\frac{(\mu_1 - \mu_n)^2}{4(\mu_1 + \mu_n)} ,$$

where the second equality follows from Lemma 5.

The theorem remains true if the eigenvalues of A are not distinct by the remark of assumption (a). This completes the proof.

Theorem 4. Let A be a normal stable matrix of order n $(n \ge 2)$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and suppose that $A \ne A^*$.

(1) If the eigenvalues of A are distinct, then

$$\psi(A) = \begin{cases} \min\left\{ \min\limits_{1 \le k < \ell \le n} \varphi_{k\ell} ; \min\limits_{1 \le k < \ell < m \le n} \Phi_{k\ell m} \right\} & \text{for } n \ge 3 \\ \varphi_{12} & \text{for } n = 2 \end{cases} \tag{45}$$

where the numbers $\varphi_{k\ell}$ and $\Phi_{k\ell m}$ are defined by (34) and (44), respectively.

(2) If the eigenvalues of A are not distinct, suppose that $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the distinct numbers among $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let B be any normal matrix of the order r $(r < n)$ whose eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_r$. Then $\psi(A) = \psi(B)$, where $\psi(B)$ can be calculated by part (1).

Proof. It is enough to prove part (1) of the theorem by the remark of assumption (a). Thus assume that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct.

We may again assume without loss of generality that A is a diagonal matrix, given by (15) and (16). We further assume that no four of the points (μ_j, ν_j), $j = 1, 2, \ldots, n$, lie on a proper circle.

The value $\psi(A) = \psi_1(A)$ is attained at x, such that $(x, x) = 1$, only if x satisfies (23). Under our assumptions, this vector x has at most three nonzero components. The discussion of cases 1, 2, 3 shows that there exist k and ℓ, $1 \leq k < \ell \leq n$, such that $\alpha_n(M(A, x)) = \varphi_{k\ell}$, or k, ℓ, m, $1 \leq k < \ell < m \leq n$, such that $\alpha_n(M(A, x)) = \Phi_{k\ell m}$. Since there are only a finite number of $\varphi_{k\ell}$ and $\Phi_{k\ell m}$, the result follows.

To complete the proof, we notice that every normal matrix can be arbitrarily approximated by a normal matrix with the property that no proper circle passes through four of the points (μ_j, ν_j), $j = 1, 2, \ldots, n$. Since $\psi(A)$ depends continuously on the entries of A, the proof is complete.

Remark. We conclude from Theorem 3 that if A is a Hermitian matrix, then $\psi(A)$ depends only on its largest and smallest eigenvalues. Theorem 4 provides an algorithm for the computation of $\psi(A)$ for a normal matrix A. It also proves that for a normal, but not Hermitian, matrix $\psi(A)$ can be expressed in terms of at most three eigenvalues of A. However, the precise identification of these eigenvalues depends on the location of the points $(\mu_1, \nu_1), (\mu_2, \nu_2), \ldots, (\mu_n, \nu_n)$ in the Cartesian plane.

Finally, it is clear from the proof of Theorem 4 that $\psi(A)$ can be determined in a number of steps which is proportional to n^3, if A is an $n \times n$ diagonal matrix with distinct eigenvalues and n is sufficiently large.

We illustrate Theoren 4 in the following examples.

Examples.

(1) Let A = diag(2, 2+i, 1+i, 1). Easy calculations give

$$\begin{aligned}
\varphi_{12} &= -\tfrac{1}{2}(17^{\frac{1}{2}} - 4) \\
\varphi_{13} = \varphi_{24} &= -(10^{\frac{1}{2}} - 3) \\
\varphi_{14} = \varphi_{23} &= -\frac{1}{12} \\
\varphi_{34} &= -\tfrac{1}{2}(5^{\frac{1}{2}} - 2) \\
\Phi_{123} = \Phi_{124} &= 0 \\
\Phi_{134} = \Phi_{234} &= -\frac{1}{6} \quad .
\end{aligned}$$

Here $\Phi_{123} = \Phi_{124} = 0$ by (44), since in both cases the system (42) has no positive solution. Hence,

$$\psi(A) = \Phi_{134} = \Phi_{234} = -\frac{1}{6}$$

(2) Let A = diag(2, 2+i, 2+2i, 1+4i). Hence,

$$\begin{aligned}
\varphi_{12} = \varphi_{23} &= -\tfrac{1}{2}(17^{\frac{1}{2}} - 4) \\
\varphi_{13} &= -(5^{\frac{1}{2}} - 2) \\
\varphi_{14} &= -\frac{17}{16} \\
\varphi_{24} &= -\frac{5}{3}(2^{\frac{1}{2}} - 1) \\
\varphi_{34} &= -\frac{5}{8}(13^{\frac{1}{2}} - 3) \\
\Phi_{123} = \Phi_{124} = \Phi_{134} = \Phi_{234} &= 0 \quad .
\end{aligned}$$

Here $\Phi_{123} = 0$ by (44), since the points (2, 0), (2, 1) and (2. 2) are on a line. Also, $\Phi_{124} = \Phi_{134} = \Phi_{234} = 0$ by (44), since c_{124}, c_{134} and c_{234} are negative. We conclude that

$$\psi(A) = \varphi_{14} = -\frac{17}{16} .$$

Upper and Lower Bounds for $\psi(A)$

In this section upper and lower bounds for $\psi(A)$ are given. We first consider normal matrices. Although $\psi(A)$ was precisely determined for a normal matrix A, the following inequalities seem to be useful.

Theorem 5. Let A be an $n \times n$ normal, stable matrix with eigenvalues $\mu_1 + i\nu_1, \mu_2 + i\nu_2, \ldots, \mu_n + i\nu_n$, where $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n > 0$. Let

$$\nu = \max_{1 \leq j < k \leq n} |\nu_j - \nu_k| .$$

Then

$$-\frac{(\mu_1 - \mu_n)^2}{4(\mu_1 + \mu_n)} - [\frac{n-1}{2n}]^{\frac{1}{2}} \nu \leq \psi(A) \leq -\frac{(\mu_1 - \mu_n)^2}{4(\mu_1 + \mu_n)} .$$

Proof. We use the results of the previous section. We can assume that A is a diagonal matrix. Let $x = (\xi_j)$ be a real vector (see the note preceding (18)) such that $(x, x) = 1$. We have by (18) and (19)

$$\alpha_n(M(A, x)) = \sum_{j=1}^{n} \mu_j \xi_j^2 - \Delta(x)^{\frac{1}{2}} .$$

Using (20) for $\Delta(x)$ we get

$$\left[\sum_{j=1}^{n} \mu_j^2 \xi_j^2\right]^{\frac{1}{2}} \le \Delta(x)^{\frac{1}{2}} \le \left[\sum_{j=1}^{n} \mu_j^2 \xi_j^2\right]^{\frac{1}{2}} + \nu \left[\sum_{1 \le j < k \le n} \xi_j^2 \xi_k^2\right]^{\frac{1}{2}} .$$

Hence,

$$\psi(A) \le \min_{(x,x)=1} \left\{ \sum_{j=1}^{n} \mu_j \xi_j^2 - \left[\sum_{j=1}^{n} \mu_j^2 \xi_j^2\right]^{\frac{1}{2}} \right\} \tag{46}$$

and

$$\psi(A) \ge \min_{(x,x)=1} \left\{ \sum_{j=1}^{n} \mu_j \xi_j^2 - \left[\sum_{j=1}^{n} \mu_j^2 \xi_j^2\right]^{\frac{1}{2}} \right\} - \nu \max_{(x,x)=1} \left[\sum_{1 \le j < k \le n} \xi_j^2 \xi_k^2\right]^{\frac{1}{2}}$$

where x varies only over real vectors.

Let D_R be the real diagonal matrix defined by (17). It follows from (20) that the right-hand side of (46) is exactly $\psi(D_R)$, so by Theorem 3

$$\psi(A) \le - \frac{(\mu_1 - \mu_n)^2}{4(\mu_1 + \mu_n)} .$$

Also, $(x, x) = \sum_{j=1}^{n} \xi_j^2 = 1$ implies

$$\sum_{1 \le j < k \le n} \xi_j^2 \xi_k^2 = \frac{1}{2} \sum_{\substack{j,k=1 \\ j \ne k}}^{n} \xi_j^2 \xi_k^2$$

$$= \frac{1}{2} \left[\sum_{j,k=1}^{n} \xi_j^2 \xi_k^2 - \sum_{j=1}^{n} \xi_j^4 \right] = \frac{1}{2} - \frac{1}{2} \sum_{j=1}^{n} \xi_j^4 .$$

Hence,

$$\max_{(x,x)=1} \left[\sum_{1 \le j < k \le n} \xi_j^2 \xi_k^2 \right]^{\frac{1}{2}} = \left[\tfrac{1}{2} - \tfrac{1}{2} \min_{(x,x)=1} \left\{ \sum_{j=1}^{n} \xi_j^4 \right\} \right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{2} - \frac{1}{2n}\right]^{\frac{1}{2}} = \left[\frac{n-1}{2n}\right]^{\frac{1}{2}} ,$$

completing the proof.

We turn now to a general stable matrix A. The upper and lower bounds for $\psi(A)$ are expressed in terms of Hermitian or normal matrices, for which we know how to find the ψ.

To get bounds for $\psi(A)$, we use a theorem due to Hoffman and Wielandt. Define for $A = (a_{jk})$

$$||A||^2 = \sum_{j,k=1}^{n} |a_{jk}|^2 = tr(A^* A) .$$

It is well-known that $||A+B|| \le ||A|| + ||B||$ and $||AB|| \le ||A|| \, ||B||$ for arbitrary $n \times n$ matrices.

Theorem 6 [12]. Let H and K be $n \times n$ Hermitian matrices with eigenvalues $\alpha_1(H) \ge \alpha_2(H) \ge \ldots \ge \alpha_n(H)$ and $\alpha_1(K) \ge \alpha_2(K) \ge \ldots \ge \alpha_n(K)$, respectively. Then

$$\sum_{j=1}^{n} [\alpha_j(H) - \alpha_j(K)]^2 \le ||H - K||^2 .$$

The proof of this theorem uses convexity arguments, see [12].

Recall that $\psi(A) = \psi(U^* A U)$ for any unitary matrix U, so we can assume without loss of generality that A is a triangular matrix.

Theorem 7. Let $A = D + T$, where D is a stable, diagonal matrix and T is a strictly upper triangular matrix (so $D = \mathrm{diag}(a_{11}, a_{22}, \ldots, a_{nn})$). Then

$$|\psi(A) - \psi(D)| \le 2||T||$$

where $\psi(D)$ can be determined from Theorem 3 or 4.

Proof. There exists a vector x such that $(x, x) = 1$ and

$$\psi(A) = \psi_1(A) = \alpha_n(M(A, x)) \quad .$$

Also,

$$M(A, x) = M(D, x) + M(T, x) \quad ,$$

so Theorem 6 implies

$$|\alpha_n(M(A, x)) - \alpha_n(M(D, x))| \le ||M(T, x)|| \quad .$$

But

$$||M(T, x)|| = ||Txx^* + xx^*T^*|| \le 2||T|| \; ||xx^*|| = 2||T|| \quad .$$

Hence,

$$\psi(A) = \alpha_n(M(X, x)) \ge \alpha_n(M(D, x)) - 2||T|| \ge \psi(D) - 2||T|| \quad .$$

Similarly one proves $\psi(D) \ge \psi(A) - 2||T||$, completing the proof.

To get another upper bound for $\psi(A)$, we have to drop the assumption that A is a stable matrix. We note the following:

Remark. The assumption that A is a stable matrix is not essential in the discussion of this chapter. In fact, $\psi(A), \psi_1(A), \psi_2(A), \ldots, \psi_n(A)$, and the theorems proved about these numbers, remain valid even if A is not stable. In particular, $\psi(A) = \psi_1(A)$ is still true, and we can find $\psi(A)$ by minimizing $\alpha_n(M(A, x))$. The only difference occurs in the previous section, where we compute $\psi(A)$ for a normal and stable matrix A. There are still three cases to consider even if A is not stable, but we can no longer disregard Case 1, while Case 2 becomes more complicated. We do not repeat the calculations, but state the generalization of Theorem 3 to an arbitrary Hermitian matrix. As indicated, the proof differs only slightly from the proof of Theorem 3.

Theorem 8. Let $A = A^*$ have eigenvalues $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$. Then

$$\psi(A) = \begin{cases} -\dfrac{(\mu_1 - \mu_n)^2}{4(\mu_1 + \mu_n)} & \text{if} \quad \mu_1 + 3\mu_n > 0 \\ \\ 2\mu_n & \text{if} \quad \mu_1 + 3\mu_n \leq 0 \end{cases}$$

The last theorem enables us to get an upper bound for $\psi(A)$.

Theorem 9. Let A be an arbitrary matrix. Then $\psi(A) \leq \psi(\frac{1}{2}(A + A^*))$.

Proof. There exists a matrix $H \in J$ such that

$$\psi(\tfrac{1}{2}(A + A^*)) = \alpha_n(\tfrac{1}{2}(A + A^*)H + \tfrac{1}{2}H(A + A^*)) = \tfrac{1}{2}\alpha_n(AH + HA^* + A^*H + HA)$$

$$\geq \tfrac{1}{2}\alpha_n(AH + HA^*) + \tfrac{1}{2}\alpha_n(A^*H + HA) \geq \tfrac{1}{2}\psi(A) + \tfrac{1}{2}\psi(A^*) = \psi(A) ,$$

the last equality following from Corollary 2. This completes the proof.

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