AN ANALYSIS OF A COMPOUND PENDULUM ROCKET SUSPENSION

Thesis by

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SUMMARY

This is an investigation of the equations of motion and physical parameters involved in stabilizing the initial flight of a vertically launched rocket by means of a booster rocket pin-connected below the main rocket. The system is designed to stabilize the flight in its early stage before the aerodynamic control surfaces become effective. Stability of the system is dependent on the pendulum action of the booster rocket.

The equations of motion were derived from Lagrange's generalized momentum equation. The differential equations thus obtained were not solved but were tested for stability by means of Routh's stability criteria. The ratio of the mass of the main rocket, M_1, to the mass of the booster rocket, M_2, was investigated for the two values $\frac{M_1}{M_2} = 1.5$ and $\frac{M_1}{M_2} = 7.75$.

The system involving a mass ratio $\frac{M_1}{M_2} = 7.75$ was found to be unstable under all conditions. However, the system involving a mass ratio $\frac{M_1}{M_2} = 1.5$ was determined to be stable in the range $1.62 < \beta < 4.54 \times 10^{10}$, where β is defined as the ratio of the distance ℓ_2 from the center of gravity of the booster M_2 to the pin connecting the strut to the main rocket M_1, divided by the radius of gyration, k_2, of the booster M_2. In this range, for any given value of β, stability was uniquely determined by

one value of the ratio $a = \frac{\ell_1}{\ell_2}$, where ℓ_1 is the length of the strut from the main rocket M_1 to the booster rocket M_2. Thus, for a given booster, stability is primarily a function of the ratio $\frac{\ell_1}{\ell_2}$, and for any given ℓ_2, ℓ_1, is uniquely determined.

Although the system was found to be theoretically stable for the mass ratio $\frac{M_1}{M_2} = 1.5$, the ratio $\frac{\ell_1}{\ell_2}$ turned out to be of such great magnitude as to make the system entirely impractical for this particular mass ratio.

TABLE OF CONTENTS

I. INTRODUCTION

When a rocket or guided missile is initially launched there is a short period when its flight is unstable due to the fact that at low velocities the aerodynamic control surfaces are ineffective.

Various procedures have been adopted to stabilize this initial part of the flight. The most obvious and most successful of these has been the use of guide rails, or launching towers which hold the missile on its course until it has attained a velocity at which the control surfaces become effective. But this method requires a bulky launcher, and special care is necessary to maintain the rails in proper alignment.

A modification of the guide rail system is the "short-length" launcher which is really a guide rail whose length has been reduced to a minimum by greatly increasing the initial acceleration and thus the velocity of the missile. This increased acceleration presents a great many problems in component design to resist the tremendous accelerative forces.

During World War II the Germans were fairly successful with a unique approach to the problem in their well-known "V-2" rocket. (Reference 1). The "V-2" was launched from a near-vertical position with no external restrictions. Instead it employed four carbon vanes mounted in the jet stream and activated by the auto-pilot to maintain the missile on its proper flight path.

The great difficulty here was that the vanes steadily burned out with a consequent diminishing of control.

The subject of this thesis is a third means of initial flight stabilization based on the concept of the action of a pendulum and independent of both external guides and internal control mechanisms.

If the missile booster is constructed as a separate unit and attached to the main rocket by a pin connection a certain distance below its own center of gravity it can be seen from Figure 1 that any motion of the main rocket will impart a proportionate motion to the booster. Specifically, if the main rocket rotates about its center of gravity in a clockwise direction, the connecting strut will cause the booster to rotate also about its own center of gravity in a clockwise direction. Thus, if the main rocket turns off course in a clockwise direction the booster immediately turns in a clockwise direction also, that is, the booster tends to re-align its thrust with the axis of the main rocket. If, by a proper choice of moment arms, the booster can be made to overshoot this position of thrust alignment slightly the thrust will have a horizontal component which acts on the strut to the main rocket and causes the main rocket to turn about its center of gravity in a counter-clockwise direction. Thus, the booster responds to an error and its response tends to counteract the error. Such a system should produce a sinusoidal flight path if the proper moment arms and damping forces were applied.

In this analysis frictional and aerodynamic forces were neglected to simplify the equations. The only damping force considered (other than the inertia forces of the system) was the jet damping force or the resistance of a jet stream to rotation. The other external forces considered were the weights of the main rocket and the booster, and the thrust of the booster. (The motor of the main rocket is considered not to be operating in this analysis). Taking into account only these forces the writer derived the equations of motion of the system in two dimensions by means of Lagrange's generalized momentum equation. Because of the complicated nature of these equations no solution of them was attempted. Instead the conditions for stability were investigated by means of Routh's stability criteria.

A hypothetical system consisting of a five-second booster pin-connected below a "V-2" rocket was first investigated. For this system the ratio of the mass of the main rocket M_1, to the mass of the booster M_2, is $\frac{M_1}{M_2} = 7.75$. To reduce the work of computation the lateral motion of the center of gravity of the system was set equal to zero and only the vertical motion of the center of gravity and the rotation of the main rocket and the booster were considered.

Since no stable solution was found, a system with mass ratio $\frac{M_1}{M_2} = 1.5$ was next investigated, again neglecting the lateral motion of the center of gravity of the system. In this case a range of stability was determined but it was of such a sensitive nature as to make impractical further investigation for this mass ratio in the general case with lateral motion taken into account.

II. DERIVATION OF THE EQUATIONS OF MOTION

In deriving the equations of motion, only four external forces are considered, namely, the weight, M_1g, of the main rocket, the weight, M_2g, of the booster, the thrust, F, of the booster, and the jet damping force, D, of the booster. These forces are indicated in Figure 1.

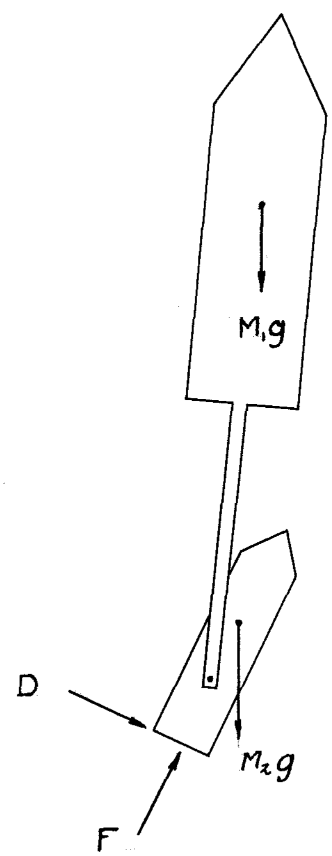
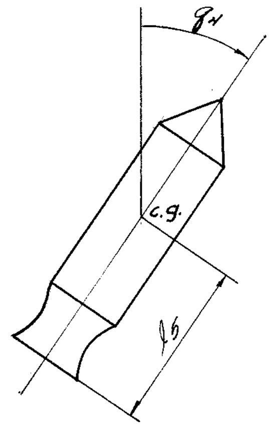


Figure 1. External Forces on Main Rocket and Booster.

The jet damping force, D, is a force which resists the rotation of the booster about its own center of gravity. If the booster is of mass M_2, then the rate of flow through the nozzle may be written as $\frac{dM_2}{dt}$, or $\dot{M}_2$. As the booster rotates about its center of gravity the tangential velocity at the nozzle exit will be $l_5 \frac{dq_4}{dt}$ or $l_5 \dot{q}_4$. Therefore the damping force, D, will be



equal to the rate of change of momentum, or $\dot{M}_2 \cdot (l_5 \dot{q}_4)$. This expression has the dimensions $\frac{M}{T} \cdot L \cdot \frac{1}{T}$, or $\frac{ML}{T}$.

Frictional forces and aerodynamic forces are neglected in this analysis.

Both the main rocket, M_1, and the booster, M_2, are assigned three degrees of freedom, namely, lateral, vertical, and rotational motion. Note, also, that only the two dimensional case is being treated. The coordinates of masses M_1 and M_2 are indicated in Figure 2. The connecting strut between M_1 and M_2 is assumed to be of zero mass and of infinite stiffness for purposes of this analysis, and consequently the coordinates of the center of gravity of the system are indicated on the straight line joining M_1 and M_2 in Figure 2.

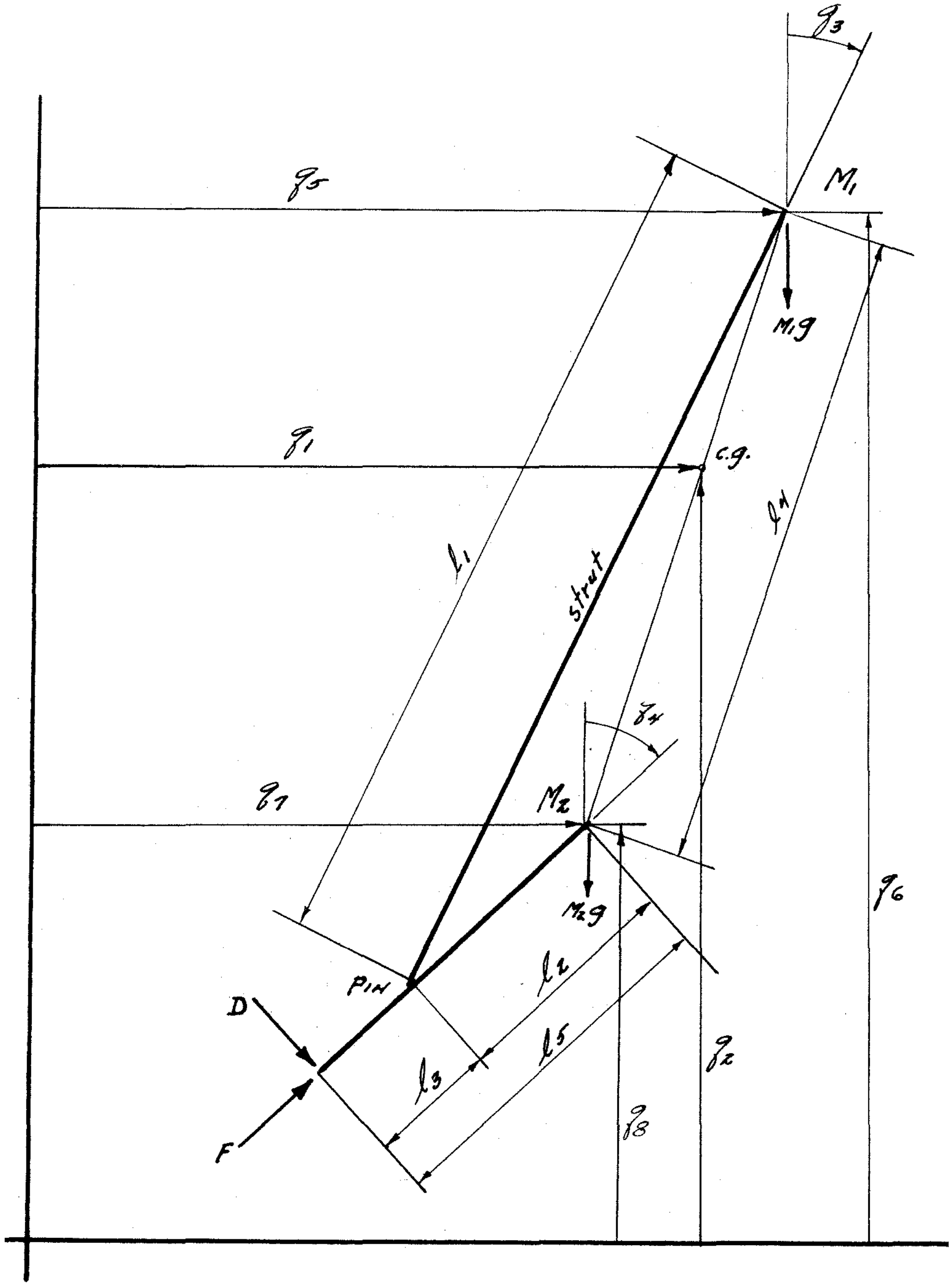


Figure 2. Schematic Diagram of Coordinate System and Dimensions.

It is assumed that the propulsion system of the main rocket, M_1, will not be operating during the initial stages of launching and consequently M_1 is a constant for this analysis. The booster, M_2, will of course be burning with a consequent change of mass during launching, and the rate of change of mass M_2, say $\dot{M}_2$, will be considered, as will be the change of position relative to M_1 and M_2 of the center of gravity of the system.

The first step of the analysis is to eliminate the extraneous coordinates q_5, q_6, q_7, and q_8 in order to write the equations of motion in terms of the rectilinear coordinates of the center of gravity of the system, q_1, and q_2, and the angular coordinates q_3 and q_4 of M_1 and M_2 about vertical axes through their respective centers of gravity. (Since only the onset of instability is of interest, the analysis will be restricted to consideration of small variations of the angles q_3 and q_4.) Once this step has been completed, Lagrange's generalized momentum equation will be used to determine the equations of motion.

ELIMINATION OF EXTRANEOUS COORDINATES

The coordinates q_5, q_6, q_7, and q_8 are always redundant because of the rigid links in the system and the known position of the center of gravity of the two masses. Therefore these extraneous coordinates may be eliminated by considering the constraints elaborated below.

The first relation between the coordinates of the individual centers of gravity and the center of gravity of the system may be found by taking moments about the center of gravity of the system indicated in Figure 3.

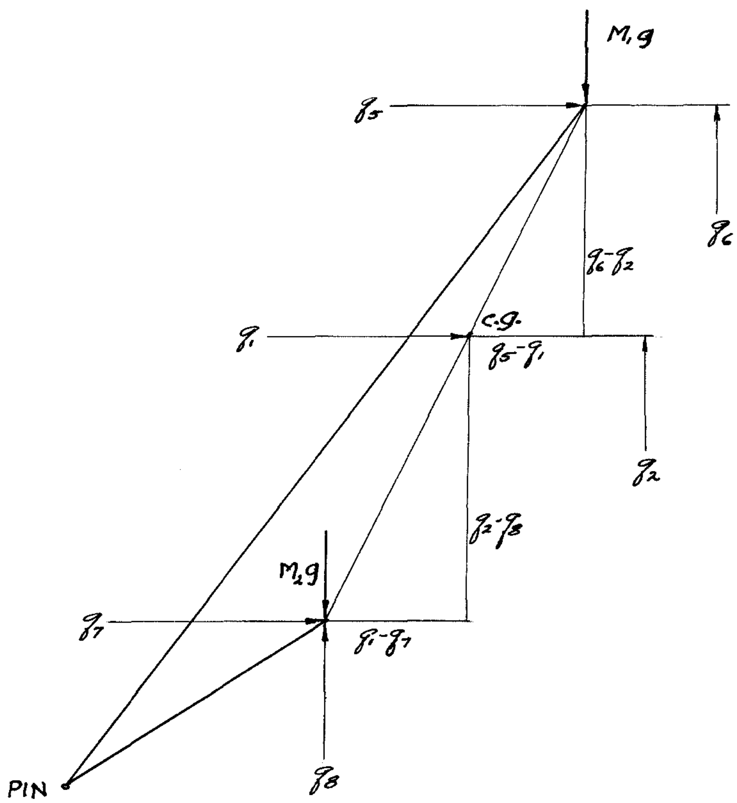


Figure 3. Relationship of M_1 and M_2 to the center of gravity of the system.

Thus

$$M_1 g (q_5 - q_1) - M_2 g (q_1 - q_7) = 0$$

or

$$M_1 (q_5 - q_1) = M_2 (q_1 - q_7) \qquad (1)$$

Furthermore the horizontal and vertical separations of the centers of gravity of M_1 and M_2 in Figure 4 are fixed by the length of the line, ℓ_4, joining the two, and by the angle, $(q_3 - \delta)$ which ℓ_4 makes with the vertical.

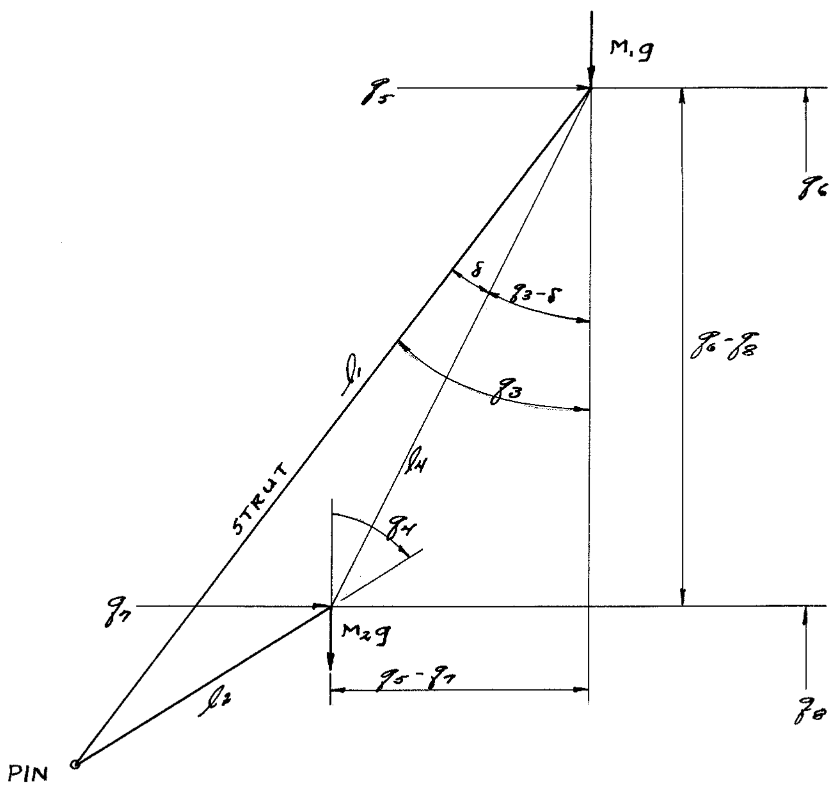


Figure 4. Relationship of M_1 and M_2 to the connecting strut.

In Figure 4 the vertical separation is

$$q_6 - q_8 = \ell_4 \cos(q_3 - \delta)$$

or $$\ell_4 = \frac{q_6 - q_8}{\cos(q_3 - \delta)} \tag{2}$$

and the horizontal separation is

$$q_5 - q_7 = \ell_4 \sin(q_3 - \delta)$$

or $$\ell_4 = \frac{q_5 - q_7}{\sin(q_3 - \delta)}. \tag{3}$$

A fourth equation follows from a consideration of the similar triangles indicated in Figure 3.

$$\frac{q_6 - q_2}{q_5 - q_1} = \frac{q_2 - q_8}{q_1 - q_7}$$

$$\frac{M_1(q_6 - q_2)}{M_1(q_5 - q_1)} = \frac{M_2(q_2 - q_8)}{M_2(q_1 - q_7)}$$

and, since the denominators of both sides of this equation are equal from Equation (1), it may be written

$$M_1(q_6 - q_2) = M_2(q_2 - q_8) \tag{4}$$

Equations (1), (2), (3), and (4) provide four linear relations between the coordinates q_5, q_6, q_7, and q_8, and q_1, q_2, q_3, and q_4. Consequently the former may be expressed in terms of the latter and thereby eliminated from subsequent equations. Solving these equations the following expressions are obtained:

From Equation (2) $$q_6 = l_4 \cos(q_3 - \delta) + q_8 \tag{5}$$

From Equation (3) $$q_5 = l_4 \sin(q_3 - \delta) + q_7$$

or $$q_7 = q_5 - l_4 \sin(q_3 - \delta) \tag{6}$$

Substituting Equation (6) into Equation (1)

$$M_1(q_5 - q_1) = M_2[q_1 - q_5 + l_4 \sin(q_3 - \delta)]$$

$$q_5(M_1 + M_2) = M_1 q_1 + M_2 q_1 + M_2 l_4 \sin(q_3 - \delta)$$

$$q_5 = q_1 + \frac{M_2}{M_1 + M_2} l_4 \sin(q_3 - \delta) \qquad (7)$$

Substituting Equation (5) into Equation (4)

$$M_1[l_4 \cos(q_3 - \delta) + q_8 - q_2] = M_2(q_2 - q_8)$$

$$q_8(M_1 + M_2) = M_1 q_2 + M_2 q_2 - M_1 l_4 \cos(q_3 - \delta)$$

$$q_8 = q_2 - \frac{M_1}{M_1 + M_2} l_4 \cos(q_3 - \delta) \qquad (8)$$

Substituting Equation (7) into Equation (6)

$$q_7 = q_1 + \frac{M_2}{M_1 + M_2} l_4 \sin(q_3 - \delta) - l_4 \sin(q_3 - \delta)$$

$$q_7 = q_1 + l_4[\sin(q_3 - \delta)]\left(\frac{M_2}{M_1 + M_2} - 1\right)$$

$$q_7 = q_1 - \frac{M_1}{M_1 + M_2} l_4 \sin(q_3 - \delta) \qquad (9)$$

From Equation (2)

$$q_6 = l_4 \cos(q_3 - \delta) + q_8$$

Substituting Equation (8) into this equation

$$q_6 = l_4 \cos(q_3 - \delta) + q_2 - \frac{M_1}{M_1 + M_2} l_4 \cos(q_3 - \delta)$$

$$q_6 = q_2 + l_4 [\cos(q_3 - \delta)] (1 - \frac{M_1}{M_1 + M_2})$$

$$q_6 = q_2 + \frac{M_2}{M_1 + M_2} l_4 \cos(q_3 - \delta) \qquad (10)$$

These expressions for the extraneous coordinates in Equations (7), (8), (9), and (10) involve, in addition to the principal coordinates, the variables l_4, the distance between the centers of gravity of the two components, and δ, the angle between the strut l_1, and the line l_4, joining the centers of gravity of the two components. These two variables will now also be expressed in terms of the principal coordinates.

In Figure 4 by the law of cosines

$$l_4 = \sqrt{l_1^2 + l_2^2 - 2 l_1 l_2 \cos(q_4 - q_3)} \qquad (11)$$

and by the law of sines

$$\frac{l_2}{\sin\delta} = \frac{l_4}{\sin(q_4 - q_3)}$$

$$\sin\delta = \frac{l_2}{l_4} \sin(q_4 - q_3)$$

$$\delta = \sin^{-1}[\frac{l_2}{l_4} \sin(q_4 - q_3)]$$

whence $$(q_3 - \delta) = q_3 - \sin^{-1}[\frac{l_2}{l_4} \sin(q_4 - q_3)] \qquad (12)$$

These relations are unfortunately complicated. Therefore, to simplify them for later use, the fact that the variations in q_3 and q_4 are small will be taken into account. Assuming these angles sufficiently small such that terms of order q^2 may be neglected, the following approximations can be made

$$\sin q_i \approx q_i \qquad \text{and} \qquad \cos q_i \approx 1$$

From Equation (11)

$$l_4 \approx \sqrt{l_1^2 + l_2^2 - 2l_1 l_2} \qquad = l_1 - l_2$$

From Equation (12)

$$(q_3 - \delta) \approx q_3 - \frac{l_2}{l_4}(q_4 - q_3)$$

$$= q_3\left(1 + \frac{l_2}{l_4}\right) - \frac{l_2}{l_4} q_4$$

$$(q_3 - \delta) \approx q_3 \frac{l_4 + l_2}{l_4} - \frac{l_2}{l_4} q_4$$

and since

$$l_4 \approx l_1 - l_2$$

$$(q_3 - \delta) \approx \frac{l_1}{l_1 - l_2} q_3 - \frac{l_2}{l_1 - l_2} q_4 \approx \sin(q_3 - \delta)$$

$$\cos(q_3 - \delta) \approx 1$$

Thus the appropriate approximations are

$$\left.\begin{aligned} &\cos(q_3 - \delta) \approx 1 \\ &\sin(q_3 - \delta) \approx \frac{l_1}{l_1 - l_2} q_3 - \frac{l_2}{l_1 - l_2} q_4 \\ &l_4 \approx l_1 - l_2 \end{aligned}\right\} \qquad (13)$$

Substitution of the results of Equation (13) into Equations (7), (8), (9), and (10) yields the following approximate relations between the extraneous and the principal coordinates

$$q_5 \approx q_1 + \frac{M_2}{M_1+M_2}\left(l_1 q_3 - l_2 q_4\right) \tag{14}$$

$$q_6 \approx q_2 + \frac{M_2}{M_1+M_2}\left(l_1 - l_2\right) \tag{15}$$

$$q_7 \approx q_1 - \frac{M_1}{M_1+M_2}\left(l_1 q_3 - l_2 q_4\right) \tag{16}$$

$$q_8 \approx q_2 - \frac{M_1}{M_1+M_2}\left(l_1 - l_2\right) \tag{17}$$

To derive the equations of motion by the method of Lagrange the expression for the kinetic energy, T, must first be found for use in the Lagrangian equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i \tag{18}$$

where Q_i is the "generalized force" and $\dot{q}_i$ is $\frac{dq_i}{dt}$. (Reference 2). The generalized force Q_i is employed rather than the potential energy because this is a non-conservative system.

If I_i denotes the moment of inertia of the mass M_i about its own center of gravity, the kinetic energy of translation and rotation for the system may be written

$$T = \frac{1}{2} I_1 \dot{q}_3^2 + \frac{1}{2} I_2 \dot{q}_4^2 + \frac{1}{2} M_1 (\dot{q}_5^2 + \dot{q}_6^2) + \frac{1}{2} M_2 (\dot{q}_7^2 + \dot{q}_8^2) \qquad (19)$$

DERIVATION OF EQUATION OF MOTION FOR COORDINATE q_1

In applying Equation (18) to the direction of the coordinate q_1, the value of $\frac{\partial T}{\partial \dot{q}_1}$ is first computed using the kinetic energy, T, of Equation (19).

$$\frac{\partial T}{\partial \dot{q}_1} = M_1 \dot{q}_5 \frac{\partial \dot{q}_5}{\partial \dot{q}_1} + M_2 \dot{q}_7 \frac{\partial \dot{q}_7}{\partial \dot{q}_1}$$

The extraneous coordinates q_5 and q_7 may be eliminated from this expression by differentiating Equations (14) and (16) respectively. From Equation (14)

$$\dot{q}_5 \approx \dot{q}_1 + \frac{M_2}{M_1+M_2}(l_1 \dot{q}_3 - l_2 \dot{q}_4) + \frac{M_1 \dot{M}_2}{(M_1+M_2)^2}(l_1 q_3 - l_2 q_4)$$

$$\frac{\partial \dot{q}_5}{\partial \dot{q}_1} = 1$$

and from Equation (16)

$$\dot{q}_7 \approx \dot{q}_1 - \frac{M_1}{M_1+M_2}(l_1 \dot{q}_3 - l_2 \dot{q}_4) + \frac{M_1 \dot{M}_2}{(M_1+M_2)^2}(l_1 q_3 - l_2 q_4)$$

$$\frac{\partial \dot{q}_7}{\partial \dot{q}_1} = 1$$

Therefore

$$\frac{\partial T}{\partial \dot{q}_1} = M_1\left[\dot{q}_1 + \frac{M_2}{M_1+M_2}(l_1 \dot{q}_3 - l_2 \dot{q}_4) + \frac{M_1 \dot{M}_2}{(M_1+M_2)^2}(l_1 q_3 - l_2 q_4)\right]$$

$$+ M_2\left[\dot{q}_1 - \frac{M_1}{M_1+M_2}(l_1 \dot{q}_3 - l_2 \dot{q}_4) + \frac{M_1 \dot{M}_2}{(M_1+M_2)^2}(l_1 q_3 - l_2 q_4)\right]$$

or

$$\frac{\partial T}{\partial \dot{q}_1} = M_1\left[\dot{q}_1 + \frac{M_1 \dot{M}_2}{(M_1+M_2)^2}\left(l_1 q_3 - l_2 q_4\right)\right] + M_2\left[\dot{q}_1 + \frac{M_1 \dot{M}_2}{(M_1+M_2)^2}\left(l_1 q_3 - l_2 q_4\right)\right]$$

The first term in the Lagrangian expression is then

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = M_1\left[\ddot{q}_1 + \frac{M_1 \dot{M}_2}{(M_1+M_2)^2}\left(l_1 \dot{q}_3 - l_2 \dot{q}_4\right) - \frac{2 M_1 \dot{M}_2^{\,2}}{(M_1+M_2)^3}\left(l_1 q_3 - l_2 q_4\right)\right]$$

$$+ M_2\left[\ddot{q}_1 + \frac{M_1 \dot{M}_2}{(M_1+M_2)^2}\left(l_1 \dot{q}_3 - l_2 \dot{q}_4\right) - \frac{2 M_1 \dot{M}_2^{\,2}}{(M_1+M_2)^3}\left(l_1 q_3 - l_2 q_4\right)\right]$$

$$+ \dot{M}_2\left[\dot{q}_1 + \frac{M_1 \dot{M}_2}{(M_1+M_2)^2}\left(l_1 q_3 - l_2 q_4\right)\right]$$

or

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = (M_1+M_2)\ddot{q}_1 + \frac{M_1 \dot{M}_2}{M_1+M_2}\left(l_1 \dot{q}_3 - l_2 \dot{q}_4\right) - \frac{M_1 \dot{M}_2^{\,2}}{(M_1+M_2)^2}\left(l_1 q_3 - l_2 q_4\right) + \dot{M}_2 \dot{q}_1 \qquad (20)$$

By inspection of Equation (19), the second term of the Lagrangian expression is

$$\frac{\partial T}{\partial q_1} = 0 \qquad (21)$$

The generalized force $\mathcal{Q}_i$ is most easily computed by considering the work done by the external forces as the coordinate q_i is varied, all other coordinates remaining fixed.

In general, $\text{Work}_i = \mathcal{Q}_i \cdot \Delta q_i$, or $\mathcal{Q}_i = \dfrac{\text{Work}_i}{\Delta q_i}$

In particular, the force $\mathcal{Q}_1$ becomes, referring to Figure 2,

$$\mathcal{Q}_1 = \frac{\text{Work}_1}{\Delta q_1} = \frac{(F \sin q_4 + D \cos q_4)\,\Delta q_1}{\Delta q_1}$$

But within the present approximations

$$\sin q_4 \approx q_4 \qquad \text{and} \qquad \cos q_4 \approx 1$$

hence $\quad \mathcal{L}_1 \approx F q_4 + D \qquad (22)$

Substituting the results of Equations (20), (21), and (22) into Equation (18), the equation of motion in the q_1 direction becomes

$$(M_1+M_2)\ddot{q}_1 + \frac{M_1 \dot{M}_2}{M_1+M_2}(l_1\dot{q}_3 - l_2\dot{q}_4) - \frac{M_1\dot{M}_2^2}{(M_1+M_2)^2}(l_1 q_3 - l_2 q_4)$$

$$+ \dot{M}_2\dot{q}_1 = F q_4 + D \qquad (23)$$

DERIVATION OF EQUATION OF MOTION FOR COORDINATE q_2

The equation of motion in the q_2 direction is obtained in an identical manner.

From Equation (19)

$$\frac{\partial T}{\partial \dot{q}_2} = M_1 \dot{q}_6 \frac{\partial \dot{q}_6}{\partial \dot{q}_2} + M_2 \dot{q}_8 \frac{\partial \dot{q}_8}{\partial \dot{q}_2}$$

and, differentiating Equation (15)

$$\dot{q}_6 = \dot{q}_2 + \frac{M_1 \dot{M}_2}{(M_1+M_2)^2}(l_1 - l_2)$$

Consequently $\quad \dfrac{\partial \dot{q}_6}{\partial \dot{q}_2} = 1$

Similarly, differentiating Equation (17)

$$\dot{q}_8 = \dot{q}_2 + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2}(l_1 - l_2)$$

so that

$$\frac{\partial \dot{q}_8}{\partial \dot{q}_2} = 1$$

Therefore $$\frac{\partial T}{\partial \dot{q}_2} = (M_1 + M_2)\left[\dot{q}_2 + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2}(l_1 - l_2)\right]$$

and the first term of the Lagrangian expression becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) = (M_1 + M_2)\left[\ddot{q}_2 - \frac{2M_1 \dot{M}_2^2}{(M_1 + M_2)^3}(l_1 - l_2)\right] + \dot{M}_2\left[\dot{q}_2 + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2}(l_1 - l_2)\right]$$

or

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) = (M_1 + M_2)\ddot{q}_2 - \frac{M_1 \dot{M}_2^2}{(M_1 + M_2)^2}(l_1 - l_2) + \dot{M}_2 \dot{q}_2 \tag{24}$$

Furthermore $$\frac{\partial T}{\partial q_2} = 0 \tag{25}$$

The generalized force $\mathcal{Q}_2$ is, referring to Figure 2,

$$\mathcal{Q}_2 = \frac{\text{Work}_2}{\Delta q_2} = \frac{\left[-(M_1 + M_2)g + F\cos q_4 - D\sin q_4\right]\Delta q_2}{\Delta q_2}$$

and since

$$\sin q_4 \approx q_4 \qquad \text{and} \qquad \cos q_4 \approx 1$$

$$\mathcal{Q}_2 = -(M_1 + M_2)g + F - Dq_4 \tag{26}$$

Substituting the results of Equations (24), (25), and (26) into Equation (18), the equation of motion in the q_2 direction becomes

$$(M_1+M_2)\ddot{q}_2 - \frac{M_1\dot{M}_2^2}{(M_1+M_2)^2}(l_1-l_2) + \dot{M}_2\dot{q}_2 = F - g(M_1+M_2) - Dq_4 \qquad (27)$$

DERIVATION OF EQUATION OF MOTION

FOR COORDINATE q_3

The first term of the Lagrangian expression is obtained in an identical manner to that employed for coordinates q_1 and q_2. From Equation (19)

$$\frac{\partial T}{\partial \dot{q}_3} = I_1\dot{q}_3 + M_1\dot{q}_5\frac{\partial \dot{q}_5}{\partial \dot{q}_3} + M_2\dot{q}_7\frac{\partial \dot{q}_7}{\partial \dot{q}_3}$$

and, differentiating Equation (14)

$$\dot{q}_5 = \dot{q}_1 + \frac{M_2}{M_1+M_2}(l_1\dot{q}_3 - l_2\dot{q}_4) + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}(l_1q_3 - l_2q_4)$$

so that

$$\frac{\partial \dot{q}_5}{\partial \dot{q}_3} = \frac{M_2}{M_1+M_2}l_1$$

Similarly, differentiating Equation (16)

$$\dot{q}_7 = \dot{q}_1 - \frac{M_1}{M_1+M_2}(l_1\dot{q}_3 - l_2\dot{q}_4) + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}(l_1q_3 - l_2q_4)$$

so that

$$\frac{\partial \dot{q}_7}{\partial \dot{q}_3} = -\frac{M_1}{M_1+M_2}l_1$$

Therefore

$$\frac{\partial T}{\partial \dot{q}_3} = I_1 \dot{q}_3 + M_1\left[\dot{q}_1 + \frac{M_2}{M_1+M_2}(l_1\dot{q}_3 - l_2\dot{q}_4) + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}(l_1 q_3 - l_2 q_4)\right]\left(\frac{M_2}{M_1+M_2} l_1\right)$$

$$+ M_2\left[\dot{q}_1 - \frac{M_1}{M_1+M_2}(l_1\dot{q}_3 - l_2\dot{q}_4) + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}(l_1 q_3 - l_2 q_4)\right]\left(-\frac{M_1}{M_1+M_2} l_1\right)$$

or

$$\frac{\partial T}{\partial \dot{q}_3} = I_1\dot{q}_3 + \frac{M_1 M_2}{M_1+M_2} l_1(l_1\dot{q}_3 - l_2\dot{q}_4)$$

and the first term of the Lagrangian expression becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_3}\right) = I_1\ddot{q}_3 + \frac{M_1^2\dot{M}_2}{(M_1+M_2)^2} l_1(l_1\dot{q}_3 - l_2\dot{q}_4) + \frac{M_1M_2}{M_1+M_2} l_1(l_1\ddot{q}_3 - l_2\ddot{q}_4) \qquad (28)$$

From Equation (19)

$$\frac{\partial T}{\partial q_3} = M_1\left(\dot{q}_5\frac{\partial \dot{q}_5}{\partial q_3} + \dot{q}_6\frac{\partial \dot{q}_6}{\partial q_3}\right) + M_2\left(\dot{q}_7\frac{\partial \dot{q}_7}{\partial q_3} + \dot{q}_8\frac{\partial \dot{q}_8}{\partial q_3}\right) \qquad (29)$$

and, differentiating Equation (14)

$$\dot{q}_5 = \dot{q}_1 + \frac{M_2}{M_1+M_2}(l_1\dot{q}_3 - l_2\dot{q}_4) + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}(l_1q_3 - l_2q_4)$$

so that (30)

$$\frac{\partial \dot{q}_5}{\partial q_3} = \frac{M_1\dot{M}_2}{(M_1+M_2)^2} l_1$$

Similarly, differentiating Equation (16)

$$\dot{q}_7 = \dot{q}_1 - \frac{M_1}{M_1+M_2}(l_1\dot{q}_3 - l_2\dot{q}_4) + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}(l_1q_3 - l_2q_4)$$

so that (31)

$$\frac{\partial \dot{q}_7}{\partial q_3} = \frac{M_1\dot{M}_2}{(M_1+M_2)^2} l_1$$

But now the values of $\frac{\partial \dot{q}_6}{\partial q_3}$ and $\frac{\partial \dot{q}_8}{\partial q_3}$ cannot be evaluated to the first order from the linearized expressions for q_6 and q_8 in Equations (15) and (17) respectively because the q_3 term has vanished in the process of linearizing these equations. Therefore the differentiation must be carried out upon the exact expressions, Equations (10) and (8), and the results then linearized.

From Equation (10)

$$q_6 = q_2 + \frac{M_2}{M_1 + M_2} l_4 \cos\theta \qquad \text{where} \quad \theta = q_3 - \delta$$

and the time derivative is

$$\dot{q}_6 = \dot{q}_2 - \frac{M_2}{M_1+M_2} l_4 (\sin\theta)\dot{\theta} + \frac{M_2}{M_1+M_2} \dot{l}_4 \cos\theta + \frac{M_1 \dot{M}_2}{(M_1+M_2)^2} l_4 \cos\theta \tag{32}$$

Therefore

$$\frac{\partial \dot{q}_6}{\partial q_3} = -\frac{M_2}{M_1+M_2} \frac{\partial l_4}{\partial q_3} (\sin\theta)\dot{\theta} - \frac{M_2}{M_1+M_2} l_4 \frac{\partial \sin\theta}{\partial q_3} \dot{\theta}$$
$$- \frac{M_2}{M_1+M_2} l_4 \sin\theta \frac{\partial \dot{\theta}}{\partial q_3} + \frac{M_2}{M_1+M_2} \frac{\partial \dot{l}_4}{\partial q_3} \cos\theta$$
$$+ \frac{M_2}{M_1+M_2} \dot{l}_4 \frac{\partial \cos\theta}{\partial q_3} + \frac{M_1 \dot{M}_2}{(M_1+M_2)^2} \frac{\partial l_4}{\partial q_3} \cos\theta$$
$$+ \frac{M_1 \dot{M}_2}{(M_1+M_2)^2} l_4 \frac{\partial \cos\theta}{\partial q_3} \tag{33}$$

The value of $\frac{\partial l_4}{\partial q_3}$ follows from Equation (11)

$$l_4 = \sqrt{l_1^2 + l_2^2 - 2 l_1 l_2 \cos(q_4 - q_3)}$$

and upon differentiation

$$\frac{\partial l_4}{\partial q_3} = -\frac{l_1 l_2 \sin(q_4 - q_3)}{l_4}$$

which, by retaining only first order terms may be written

$$\frac{\partial l_4}{\partial q_3} \approx -\frac{l_1 l_2 (q_4 - q_3)}{l_4}$$

Similarly (34)

$$\frac{\partial l_4}{\partial t} = \dot{l}_4 = \frac{l_1 l_2 [\sin(q_4 - q_3)](\dot{q}_4 - \dot{q}_3)}{l_4}$$

The various derivatives of θ must be computed from Equation (12)

$$\theta = q_3 - \delta = q_3 - \sin^{-1}\left[\frac{l_2}{l_4} \sin(q_4 - q_3)\right]$$

Thus

$$\frac{\partial \theta}{\partial q_3} = 1 - \frac{l_2}{\sqrt{1 - \left[\frac{l_2 \sin(q_4 - q_3)}{l_4}\right]^2}} \; \frac{l_4 \cos(q_4 - q_3) - [\sin(q_4 - q_3)]\left(-\frac{l_1 l_2 \sin(q_4 - q_3)}{l_4}\right)}{l_4^2}$$

$$\frac{\partial \theta}{\partial q_3} = 1 + \frac{l_2}{\sqrt{l_4^2 - l_2^2 \sin^2(q_4 - q_3)}} \; \frac{l_4^2 \cos(q_4 - q_3) - l_1 l_2 \sin^2(q_4 - q_3)}{l_4^2}$$

and approximately

$$\frac{\partial \theta}{\partial q_3} \approx 1 + \frac{l_2}{l_4} \frac{l_4^2}{l_4^2} \approx \frac{l_1}{l_4} \qquad (35)$$

Now

$$\frac{\partial \sin\theta}{\partial q_3} = \cos\theta \, \frac{\partial \theta}{\partial q_3}$$

and substituting the results of Equation (35) and linearizing

$$\frac{\partial \sin\theta}{\partial q_3} \approx \frac{l_1}{l_4} \cos\theta \approx \frac{l_1}{l_4} \tag{36}$$

Similarly, from the results of Equations (35) and (13)

$$\frac{\partial \cos\theta}{\partial q_3} = -\sin\theta \frac{\partial \theta}{\partial q_3} \approx -\frac{l_1 q_3 - l_2 q_4}{l_4} \frac{l_1}{l_4} = -\frac{l_1(l_1 q_3 - l_2 q_4)}{l_4^2} \tag{37}$$

Substituting Equations (34), (35), (36), and (37) into Equation (33) and retaining only first order terms

$$\frac{\partial \dot q_6}{\partial q_3} = -\frac{M_2}{M_1+M_2} \frac{l_1(l_1 \dot q_3 - l_2 \dot q_4)}{l_4} - \frac{M_2}{M_1+M_2} \frac{l_1 l_2(\dot q_4 - \dot q_3)}{l_4}$$

$$- \frac{M_1 \dot M_2}{(M_1+M_2)^2} \frac{l_1 l_2 (q_4 - q_3)}{l_4} - \frac{M_1 \dot M_2}{(M_1+M_2)^2} \frac{l_1 (l_1 q_3 - l_2 q_4)}{l_4}$$

or

$$\frac{\partial \dot q_6}{\partial q_3} = -\frac{M_2}{M_1+M_2} \frac{l_1}{l_4} \dot q_3 (l_1 - l_2) - \frac{M_1 \dot M_2}{(M_1+M_2)^2} \frac{l_1}{l_4} q_3 (l_1 - l_2) \tag{38}$$

To find $\frac{\partial \dot q_8}{\partial q_3}$ the same procedure is followed. Thus, from Equation (8)

$$q_8 = q_2 - \frac{M_1}{M_1+M_2} l_4 \cos\theta \qquad \text{where} \quad \theta = q_3 - \delta$$

and the time derivative is

$$\dot q_8 = \dot q_2 + \frac{M_1}{M_1+M_2} l_4 (\sin\theta)\dot\theta - \frac{M_1}{M_1+M_2} \dot l_4 \cos\theta + \frac{M_1 \dot M_2}{(M_1+M_2)^2} l_4 \cos\theta \tag{39}$$

Therefore

$$\frac{\partial \dot{q}_8}{\partial q_3} = \frac{M_1}{M_1+M_2}\frac{\partial l_4}{\partial q_3}(\sin\theta)\dot{\theta} + \frac{M_1}{M_1+M_2} l_4 \frac{\partial \sin\theta}{\partial q_3}\dot{\theta}$$

$$+ \frac{M_1}{M_1+M_2} l_4 \sin\theta \frac{\partial \dot{\theta}}{\partial q_3} - \frac{M_1}{M_1+M_2}\frac{\partial \dot{l}_4}{\partial q_3}\cos\theta$$

$$- \frac{M_1}{M_1+M_2}\dot{l}_4 \frac{\partial \cos\theta}{\partial q_3} + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}\frac{\partial l_4}{\partial q_3}\cos\theta$$

$$+ \frac{M_1\dot{M}_2}{(M_1+M_2)^2}\cdot l_4 \frac{\partial \cos\theta}{\partial q_3} \tag{40}$$

Substituting Equations (34), (35), (36), and (37) into Equation (39) and retaining only first order terms

$$\frac{\partial \dot{q}_8}{\partial q_3} = \frac{M_1}{M_1+M_2}\frac{l_1}{l_4}(l_1\dot{q}_3 - l_2\dot{q}_4) + \frac{M_1}{M_1+M_2}\frac{l_1 l_2}{l_4}(\dot{q}_4 - \dot{q}_3)$$

$$- \frac{M_1\dot{M}_2}{(M_1+M_2)^2}\frac{l_1 l_2}{l_4}(q_4 - q_3) - \frac{M_1\dot{M}_2}{(M_1+M_2)^2}\frac{l_1}{l_4}(l_1 q_3 - l_2 q_4)$$

$$\frac{\partial \dot{q}_8}{\partial q_3} = \frac{M_1}{M_1+M_2}\frac{l_1}{l_4}\dot{q}_3(l_1 - l_2) - \frac{M_1\dot{M}_2}{(M_1+M_2)^2}\frac{l_1}{l_4}q_3(l_1 - l_2) \tag{41}$$

The second term of the Lagrangian can now be written by substituting Equations (30), (31), (32), (38), (39) and (41) into Equation (30)

$$\frac{\partial T}{\partial q_3} = M_1\left[\dot{q}_1 + \frac{M_2}{M_1+M_2}(l_1\dot{q}_3 - l_2\dot{q}_4) + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}(l_1 q_3 - l_2 q_4)\right]\left[\frac{M_1\dot{M}_2}{(M_1+M_2)^2} l_1\right]$$

$$+ M_2\left[\dot{q}_1 - \frac{M_1}{M_1+M_2}(l_1\dot{q}_3 - l_2\dot{q}_4) + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}(l_1 q_3 - l_2 q_4)\right]\left[\frac{M_1\dot{M}_2}{(M_1+M_2)^2} l_1\right] +$$

$$+ M_1\left[\dot{q}_2 + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}(l_1-l_2)\right]\left[-\frac{M_2}{M_1+M_2}l_1\dot{q}_3 - \frac{M_1\dot{M}_2}{(M_1+M_2)^2}l_1 q_3\right]$$

$$+ M_2\left[\dot{q}_2 + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}(l_1-l_2)\right]\left[\frac{M_1}{M_1+M_2}l_1\dot{q}_3 - \frac{M_1\dot{M}_2}{(M_1+M_2)^2}l_1 q_3\right]$$

or

$$\frac{\partial T}{\partial q_3} = \frac{M_1\dot{M}_2}{M_1+M_2}l_1\dot{q}_1 + \frac{M_1^2\dot{M}_2^2}{(M_1+M_2)^3}l_1(l_1 q_3 - l_2 q_4)$$

$$-\frac{M_1\dot{M}_2}{M_1+M_2}l_1 q_3\dot{q}_2 - \frac{M_1^2\dot{M}_2^2}{(M_1+M_2)^3}l_1 q_3(l_1-l_2) \qquad (42)$$

The generalized force $\mathcal{Q}_3$ is

$$\mathcal{Q}_3 = \frac{\text{Work}_3}{\Delta q_3}$$

Therefore, referring to Figure 5,

$$\mathcal{Q}_3 = -M_1 g\frac{\partial q_6}{\partial q_3} - M_2 g\frac{\partial q_8}{\partial q_3} + F\sin q_4\frac{\partial}{\partial q_3}\left[q_7 - (l_2+l_3)\sin q_4\right]$$

$$+ F\cos q_4\frac{\partial}{\partial q_3}\left[q_8 - (l_2+l_3)\cos q_4\right] + D\cos q_4\frac{\partial}{\partial q_3}\left[q_7 - (l_2+l_3)\sin q_4\right]$$

$$- D\sin q_4\frac{\partial}{\partial q_3}\left[q_8 - (l_2+l_3)\cos q_4\right]$$

Substituting Equations (8), (9), and (10) to eliminate the extraneous coordinates q_6 , q_7 , and q_8 , and collecting terms

$$\mathcal{Q}_3 = (F\sin q_4 + D\cos q_4)\frac{\partial}{\partial q_3}\left[q_1 - \frac{M_1}{M_1+M_2} l_4\sin\theta - (l_2+l_3)\sin q_4\right]$$
$$+ (F\cos q_4 - D\sin q_4)\frac{\partial}{\partial q_3}\left[q_2 - \frac{M_1}{M_1+M_2} l_4\cos\theta - (l_2+l_3)\cos q_4\right]$$

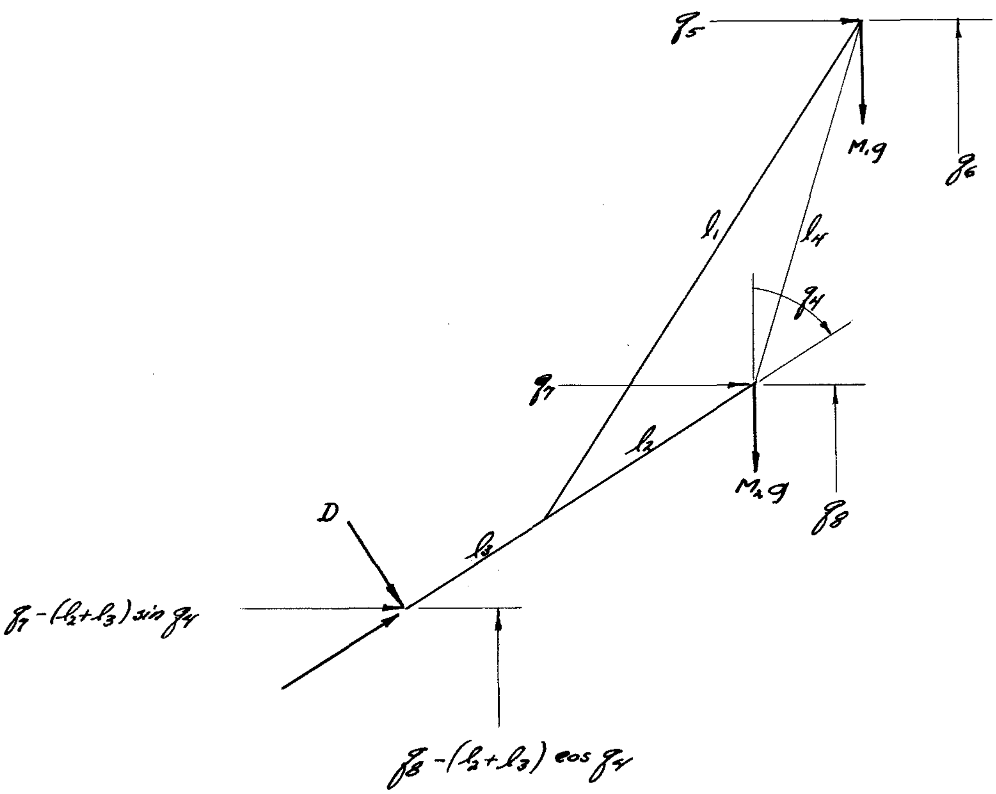


Figure 5. Schematic diagram of external forces acting on system.

Carrying out the indicated differentiation with respect to q_3

$$\mathcal{Q}_3 = (F\sin q_4 + D\cos q_4)\left[-\frac{M_1}{M_1+M_2}\left(l_4\frac{\partial\sin\theta}{\partial q_3} + \sin\theta\frac{\partial l_4}{\partial q_3}\right)\right]$$
$$+ (F\cos q_4 - D\sin q_4)\left[-\frac{M_1}{M_1+M_2}\left(l_4\frac{\partial\cos\theta}{\partial q_3} + \cos\theta\frac{\partial l_4}{\partial q_3}\right)\right]$$

Substituting Equations (34), (36), and (37) into this expression, and using the approximations $\sin q_i \approx q_i$ and $\cos q_i \approx 1$, $\mathcal{Q}_3$ becomes

$$\mathcal{Q}_3 = (Fq_4 + D)\left\{-\frac{M_1}{M_1+M_2}\left[l_1 + \left(\frac{l_1 q_3 - l_2 q_4}{l_4}\right)\left(-\frac{l_1 l_2 (q_4 - q_3)}{l_4}\right)\right]\right\}$$

$$+ (F - Dq_4)\left\{-\frac{M_1}{M_1+M_2}\left[-\frac{l_1 (l_1 q_3 - l_2 q_4)}{l_4} + \left(-\frac{l_1 l_2 (q_4 - q_3)}{l_4}\right)\right]\right\}$$

Finally, retaining only first order terms, the expression for the generalized force $\mathcal{Q}_3$ reduces to

$$\mathcal{Q}_3 = -\frac{M_1}{M_1+M_2} l_1 \left[F(q_4 - q_3) + D\right] \tag{43}$$

Therefore, by substituting the results of Equations (28), (42), and (43) into Equation (18), the equation of motion in the q_3 direction becomes

$$I_1 \ddot{q}_3 + \frac{M_1^2 \dot{M}_2}{(M_1+M_2)^2} l_1 (l_1 \dot{q}_3 - l_2 \dot{q}_4) + \frac{M_1 M_2}{M_1+M_2} l_1 (l_1 \ddot{q}_3 - l_2 \ddot{q}_4)$$

$$- \frac{M_1 \dot{M}_2}{M_1+M_2} l_1 \dot{q}_1 - \frac{M_1^2 \dot{M}_2^2}{(M_1+M_2)^3} l_1 (l_1 q_3 - l_2 q_4)$$

$$+ \frac{M_1 \dot{M}_2}{M_1+M_2} l_1 q_3 \dot{q}_2 + \frac{M_1^2 \dot{M}_2^2}{(M_1+M_2)^3} l_1 q_3 (l_1 - l_2)$$

$$= -\frac{M_1}{M_1+M_2} l_1 \left[F(q_4 - q_3) + D\right] \tag{44}$$

DERIVATION OF EQUATION OF MOTION

FOR COORDINATE q_4

The first term of the Lagrangian expression is obtained in an identical manner to that employed for coordinates q_1 , q_2 , and q_3 .

From Equation (19)

$$\frac{\partial T}{\partial \dot{q}_4} = I_2 \dot{q}_4 + M_1 \dot{q}_5 \frac{\partial \dot{q}_5}{\partial \dot{q}_4} + M_2 \dot{q}_7 \frac{\partial \dot{q}_7}{\partial \dot{q}_4}$$

and, differentiating Equation (14)

$$\dot{q}_5 = \dot{q}_1 + \frac{M_2}{M_1+M_2}(l_1\dot{q}_3 - l_2\dot{q}_4) + \frac{M_1 \dot{M}_2}{(M_1+M_2)^2}(l_1 q_3 - l_2 q_4)$$

so that

$$\frac{\partial \dot{q}_5}{\partial \dot{q}_4} = -\frac{M_2}{M_1+M_2} l_2$$

Similarly, differentiating Equation (16)

$$\dot{q}_7 = \dot{q}_1 - \frac{M_1}{M_1+M_2}(l_1\dot{q}_3 - l_2\dot{q}_4) + \frac{M_1 \dot{M}_2}{(M_1+M_2)^2}(l_1 q_3 - l_2 q_4)$$

so that

$$\frac{\partial \dot{q}_7}{\partial \dot{q}_4} = \frac{M_1}{M_1+M_2} l_2$$

Therefore,

$$\frac{\partial T}{\partial \dot{q}_4} = I_2 \dot{q}_4 + M_1\left[\dot{q}_1 + \frac{M_2}{M_1+M_2}(l_1\dot{q}_3 - l_2\dot{q}_4) + \frac{M_1 \dot{M}_2}{(M_1+M_2)^2}(l_1 q_3 - l_2 q_4)\right]\left[-\frac{M_2}{M_1+M_2} l_2\right]$$

$$+ M_2\left[\dot{q}_1 - \frac{M_1}{M_1+M_2}(l_1\dot{q}_3 - l_2\dot{q}_4) + \frac{M_1 \dot{M}_2}{(M_1+M_2)^2}(l_1 q_3 - l_2 q_4)\right]\left[\frac{M_1}{M_1+M_2} l_2\right]$$

or

$$\frac{\partial T}{\partial \dot{q}_4} = I_2 \dot{q}_4 - \frac{M_1 M_2}{M_1+M_2} l_2 (l_1\dot{q}_3 - l_2\dot{q}_4)$$

and the first term of the Lagrangian expression becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_4}\right) = I_2 \ddot{q}_4 + \dot{I}_2 \dot{q}_4 - \frac{M_1^2 \dot{M}_2}{(M_1+M_2)^2} l_2 (l_1 \dot{q}_3 - l_2 \dot{q}_4)$$

$$- \frac{M_1 M_2}{M_1+M_2} l_2 (l_1 \ddot{q}_3 - l_2 \ddot{q}_4) \tag{45}$$

From Equation (19)

$$\frac{\partial T}{\partial q_4} = M_1\left(\dot{q}_5 \frac{\partial \dot{q}_5}{\partial q_4} + \dot{q}_6 \frac{\partial \dot{q}_6}{\partial q_4}\right) + M_2\left(\dot{q}_7 \frac{\partial \dot{q}_7}{\partial q_4} + \dot{q}_8 \frac{\partial \dot{q}_8}{\partial q_4}\right) \tag{46}$$

and, differentiating Equation (14)

$$\dot{q}_5 = \dot{q}_1 + \frac{M_2}{M_1+M_2}(l_1 \dot{q}_3 - l_2 \dot{q}_4) + \frac{M_1 \dot{M}_2}{(M_1+M_2)^2}(l_1 q_3 - l_2 q_4)$$

so that (47)

$$\frac{\partial \dot{q}_5}{\partial q_4} = - \frac{M_1 \dot{M}_2}{(M_1+M_2)^2} l_2$$

Similarly, differentiating Equation (16)

$$\dot{q}_7 = \dot{q}_1 - \frac{M_1}{M_1+M_2}(l_1 \dot{q}_3 - l_2 \dot{q}_4) + \frac{M_1 \dot{M}_2}{(M_1+M_2)^2}(l_1 q_3 - l_2 q_4)$$

so that (48)

$$\frac{\partial \dot{q}_7}{\partial q_4} = - \frac{M_1 \dot{M}_2}{(M_1+M_2)^2} l_2$$

Here again the values of $\frac{\partial \dot{q}_6}{\partial q_4}$ and $\frac{\partial \dot{q}_8}{\partial q_4}$ cannot be evaluated to the first order from the linearized expressions for q_6 and q_8 in Equation (15) and (17) respectively because the q_4 term has vanished in the process of linearizing these equations. Therefore the differentiation must be carried out upon the exact expressions, Equations (10) and (8), and the results then linearized.

The time derivative of q_6, defined by Equation (10), has already been found from Equation (32)

$$\dot{q}_6 = \dot{q}_2 - \frac{M_2}{M_1+M_2} l_4 (\sin\theta)\dot{\theta} + \frac{M_2}{M_1+M_2}\dot{l}_4 \cos\theta + \frac{M_1\dot{M}_2}{(M_1+M_2)^2} l_4 \cos\theta \qquad (32)$$

Therefore

$$\begin{aligned}\frac{\partial \dot{q}_6}{\partial q_4} = &-\frac{M_2}{M_1+M_2}\frac{\partial l_4}{\partial q_4}(\sin\theta)\dot{\theta} - \frac{M_2}{M_1+M_2} l_4 \frac{\partial \sin\theta}{\partial q_4}\dot{\theta} \\ &- \frac{M_2}{M_1+M_2} l_4 \sin\theta \frac{\partial\dot{\theta}}{\partial q_4} + \frac{M_2}{M_1+M_2}\frac{\partial \dot{l}_4}{\partial q_4}\cos\theta \\ &+ \frac{M_2}{M_1+M_2}\dot{l}_4 \frac{\partial \cos\theta}{\partial q_4} + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}\frac{\partial l_4}{\partial q_4}\cos\theta \\ &+ \frac{M_1\dot{M}_2}{(M_1+M_2)^2} l_4 \frac{\partial\cos\theta}{\partial q_4}\end{aligned} \qquad (49)$$

The value of $\frac{\partial l_4}{\partial q_4}$ follows from Equation (11)

$$l_4 = \sqrt{l_1^2 + l_2^2 - 2 l_1 l_2 \cos(q_4 - q_3)}$$

and upon differentiation

$$\frac{\partial l_4}{\partial q_4} = \frac{l_1 l_2 \sin(q_4 - q_3)}{l_4}$$

which, by retaining only first order terms, may be written

$$\frac{\partial l_4}{\partial q_4} \simeq \frac{l_1 l_2 (q_4 - q_3)}{l_4}$$

Similarly

$$\frac{\partial l_4}{\partial t} = \dot{l}_4 = \frac{l_1 l_2 [\sin(q_4 - q_3)](\dot{q}_4 - \dot{q}_3)}{l_4} \qquad (50)$$

The various derivatives of θ must be computed from Equation (12)

$$\theta = q_3 - \delta = q_3 - \sin^{-1}\left[\frac{l_2}{l_4} \sin(q_4 - q_3)\right]$$

Thus

$$\frac{\partial \theta}{\partial q_4} = - \frac{l_2}{\sqrt{1 - \left[\frac{l_2 \sin(q_4 - q_3)}{l_4}\right]^2}} \; \frac{l_4 \cos(q_4 - q_3) - \sin(q_4 - q_3) \frac{l_1 l_2 \sin(q_4 - q_3)}{l_4}}{l_4^2}$$

$$\frac{\partial \theta}{\partial q_4} = - \frac{l_2}{\sqrt{l_4^2 - l_2^2 \sin^2(q_4 - q_3)}} \; \frac{l_4^2 \cos(q_4 - q_3) - l_1 l_2 \sin^2(q_4 - q_3)}{l_4^2}$$

and approximately

$$\frac{\partial \theta}{\partial q_4} \approx - \frac{l_2}{l_4} \frac{l_4^2}{l_4^2} = - \frac{l_2}{l_4} \qquad (51)$$

Now

$$\frac{\partial \sin\theta}{\partial q_4} = \cos\theta \frac{\partial \theta}{\partial q_4}$$

and, substituting the results of Equation (51) and linearizing,

$$\frac{\partial \sin\theta}{\partial q_4} \approx - \frac{l_2}{l_4} \cos\theta \approx - \frac{l_2}{l_4} \qquad (52)$$

Similarly, from the results of Equations (51) and (13),

$$\frac{\partial \cos\theta}{\partial q_4} = - \sin\theta \frac{\partial \theta}{\partial q_4} \approx \frac{l_2}{l_4} \sin\theta \approx \frac{l_2 (l_1 q_3 - l_2 q_4)}{l_4^2} \qquad (53)$$

Substituting Equations (50), (51), (52), and (53) into Equation (49) and retaining only first order terms

$$\frac{\partial \dot{q}_6}{\partial q_4} = \frac{M_2}{M_1+M_2}\frac{l_2}{l_4}(l_1\dot{q}_3 - l_2\dot{q}_4) + \frac{M_2}{M_1+M_2}\frac{l_2 l_1}{l_4}(\dot{q}_4 - \dot{q}_3)$$

$$+ \frac{M_1\dot{M}_2}{(M_1+M_2)^2}\frac{l_1 l_2(q_4-q_3)}{l_4} + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}\frac{l_2(l_1 q_3 - l_2 q_4)}{l_4}$$

or

$$\frac{\partial \dot{q}_6}{\partial q_4} = \frac{M_2}{M_1+M_2}\frac{l_2}{l_4}\dot{q}_4(l_1 - l_2) + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}\frac{l_2}{l_4}q_4(l_1 - l_2) \qquad (54)$$

To find $\frac{\partial \dot{q}_8}{\partial q_4}$ the same procedure is followed. The time derivative of q_8, defined by Equation (8), has already been found from Equation (39):

$$\dot{q}_8 = \dot{q}_2 + \frac{M_1}{M_1+M_2} l_4(\sin\theta)\dot{\theta} - \frac{M_1}{M_1+M_2}\dot{l}_4\cos\theta + \frac{M_1\dot{M}_2}{(M_1+M_2)^2} l_4\cos\theta \qquad (39)$$

Therefore

$$\frac{\partial \dot{q}_8}{\partial q_4} = \frac{M_1}{M_1+M_2}\frac{\partial l_4}{\partial q_4}(\sin\theta)\dot{\theta} + \frac{M_1}{M_1+M_2} l_4 \frac{\partial \sin\theta}{\partial q_4}\dot{\theta}$$

$$+ \frac{M_1}{M_1+M_2} l_4 \sin\theta \frac{\partial \dot{\theta}}{\partial q_4} - \frac{M_1}{M_1+M_2}\frac{\partial \dot{l}_4}{\partial q_4}\cos\theta$$

$$- \frac{M_1}{M_1+M_2}\dot{l}_4 \frac{\partial \cos\theta}{\partial q_4} + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}\frac{\partial l_4}{\partial q_4}\cos\theta$$

$$+ \frac{M_1\dot{M}_2}{(M_1+M_2)^2} l_4 \frac{\partial \cos\theta}{\partial q_4} \qquad (55)$$

Substituting Equations (50), (51), (52), and (53) into Equation (55) and retaining only first order terms

$$\frac{\partial \dot{q}_8}{\partial q_4} = -\frac{M_1}{M_1+M_2}\frac{l_2}{l_4}(l_1\dot{q}_3 - l_2\dot{q}_4) - \frac{M_1}{M_1+M_2}\frac{l_1 l_2}{l_4}(\dot{q}_4 - \dot{q}_3)$$
$$+ \frac{M_1\dot{M}_2}{(M_1+M_2)^2}\frac{l_1 l_2}{l_4}(q_4 - q_3) + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}\frac{l_2}{l_4}(l_1 q_3 - l_2 q_4)$$

or

$$\frac{\partial \dot{q}_8}{\partial q_4} = -\frac{M_1}{M_1+M_2}\frac{l_2}{l_4}\dot{q}_4(l_1 - l_2) + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}\frac{l_2}{l_4}q_4(l_1 - l_2) \qquad (56)$$

The second term of the Lagrangian can now be written by substituting Equations (32), (39), (47), (54), and (56) into Equation (46)

$$\frac{\partial T}{\partial q_4} = M_1\left[\dot{q}_1 + \frac{M_2}{M_1+M_2}(l_1\dot{q}_3 - l_2\dot{q}_4) + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}(l_1 q_3 - l_2 q_4)\right]\left[-\frac{M_1\dot{M}_2}{(M_1+M_2)^2}l_2\right]$$
$$+ M_2\left[\dot{q}_1 - \frac{M_1}{M_1+M_2}(l_1\dot{q}_3 - l_2\dot{q}_4) + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}(l_1 q_3 - l_2 q_4)\right]\left[-\frac{M_1\dot{M}_2}{(M_1+M_2)^2}l_2\right]$$
$$+ M_1\left[\dot{q}_2 + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}(l_1 - l_2)\right]\left[\frac{M_2}{M_1+M_2}l_2\dot{q}_4 + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}l_2 q_4\right]$$
$$+ M_2\left[\dot{q}_2 + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}(l_1 - l_2)\right]\left[-\frac{M_1}{M_1+M_2}l_2\dot{q}_4 + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}l_2 q_4\right]$$

or

$$\frac{\partial T}{\partial q_4} = -\frac{M_1\dot{M}_2}{M_1+M_2}l_2\dot{q}_1 - \frac{M_1^2\dot{M}_2^2}{(M_1+M_2)^3}l_2(l_1 q_3 - l_2 q_4)$$
$$+ \frac{M_1\dot{M}_2}{M_1+M_2}l_2 q_4\dot{q}_2 + \frac{M_1^2\dot{M}_2^2}{(M_1+M_2)^3}l_2 q_4(l_1 - l_2) \qquad (57)$$

The generalized force $\mathcal{Q}_4$ is

$$\mathcal{Q}_4 = \frac{\text{Work}_4}{\Delta q_4}$$

Therefore, referring to Figure 5,

$$\mathcal{Q}_4 = (F \sin q_4 + D \cos q_4) \frac{\partial}{\partial q_4}\left[q_7 - (l_2 + l_3)\sin q_4\right]$$

$$+ (F \cos q_4 - D \sin q_4) \frac{\partial}{\partial q_4}\left[q_8 - (l_2 + l_3)\cos q_4\right]$$

$$- M_2 g \frac{\partial q_8}{\partial q_4} - M_1 g \frac{\partial q_6}{\partial q_4}$$

Substituting Equations (8), (9), and (10) into this expression to eliminate the extraneous coordinates q_6 , q_7 , and q_8

$$\mathcal{Q}_4 = (F \sin q_4 + D \cos q_4) \frac{\partial}{\partial q_4}\left[q_1 - \frac{M_1}{M_1 + M_2} l_4 \sin \theta - (l_2 + l_3)\sin q_4\right]$$

$$+ (F \cos q_4 - D \sin q_4) \frac{\partial}{\partial q_4}\left[q_2 - \frac{M_1}{M_1 + M_2} l_4 \cos \theta - (l_2 + l_3)\cos q_4\right]$$

$$- M_2 g \frac{\partial}{\partial q_4}\left(q_2 - \frac{M_1}{M_1 + M_2} l_4 \cos \theta\right) - M_1 g \frac{\partial}{\partial q_4}\left(q_2 + \frac{M_2}{M_1 + M_2} l_4 \cos \theta\right)$$

and collecting terms and performing the differentiation

$$\mathcal{Q}_4 = (F \sin q_4 + D \cos q_4)\left[-\frac{M_1}{M_1 + M_2}\left(l_4 \frac{\partial \sin \theta}{\partial q_4} + \sin \theta \frac{\partial l_4}{\partial q_4}\right) - (l_2 + l_3)\cos q_4\right]$$

$$+ (F \cos q_4 - D \sin q_4)\left[-\frac{M_1}{M_1 + M_2}\left(l_4 \frac{\partial \cos \theta}{\partial q_4} + \cos \theta \frac{\partial l_4}{\partial q_4}\right) + (l_2 + l_3)\sin q_4\right]$$

Substituting Equations (13), (34), (36), and (37) into this expression, and using the approximations $\sin q_i \approx q_i$ and $\cos q_i \approx 1$, $\mathcal{Q}_4$ becomes

$$\mathcal{Q}_4 \approx (Fq_4 + \mathcal{D})\left[-\frac{M_1}{M_1+M_2}(-l_2) - (l_2 + l_3)\right]$$

$$+ (F - \mathcal{D}q_4)\left[-\frac{M_1}{M_1+M_2}\left(\frac{l_2(l_1 q_3 - l_2 q_4)}{l_4} + \frac{l_1 l_2 (q_4 - q_3)}{l_4}\right) + (l_2 + l_3) q_4\right]$$

Finally, retaining only first order terms, the expression for the generalized force $\mathcal{Q}_4$ reduces to

$$\mathcal{Q}_4 = -\mathcal{D}\left(\frac{M_2}{M_1+M_2} l_2 + l_3\right) \tag{58}$$

Therefore, by substituting the results of Equations (45), (57), and (58) into Equation (18), the equation of motion in the q_4 direction becomes

$$I_2 \ddot{q}_4 + \dot{I}_2 \dot{q}_4 - \frac{M_1^2 \dot{M}_2}{(M_1+M_2)^2} l_2 (l_1 \dot{q}_3 - l_2 \dot{q}_4) - \frac{M_1 M_2}{M_1+M_2} l_2 (l_1 \ddot{q}_3 - l_2 \ddot{q}_4)$$

$$+ \frac{M_1 \dot{M}_2}{M_1+M_2} l_2 \dot{q}_1 + \frac{M_1^2 \dot{M}_2^2}{(M_1+M_2)^3} l_2 (l_1 q_3 - l_2 q_4) - \frac{M_1 \dot{M}_2}{M_1+M_2} l_2 q_4 \dot{q}_2$$

$$- \frac{M_1^2 \dot{M}_2^2}{(M_1+M_2)^3} l_2 q_4 (l_1 - l_2) = -\mathcal{D}\left[\frac{M_2}{M_1+M_2} l_2 + l_3\right] \tag{59}$$

Thus Equations (23), (27), (44) and (59) are the four Lagrangian equations of motion corresponding to the four principal coordinates of the system.

ANALYSIS OF THE EXTERNAL FORCES AND REDUCTION OF THE EQUATIONS TO DIMENSIONLESS FORM

In carrying out the stability analysis the state of vertical motion corresponding to the q_2 direction will be considered constant during the perturbation. That is, the values of $\dot{q}_2$ and $\ddot{q}_2$ will be fixed, and their magnitudes will be determined by the time-instant at which the stability of the system is investigated. Therefore the equation of motion in the q_2 direction, Equation (27), will not be considered further in this analysis.

The thrust force, F, of the booster may be expressed in terms of the mass rate of flow, $\dot{M}_2$, and the specific impulse of the fuel, I_{sp}, as

$$F = -\dot{M}_2 \, g \, I_{sp}$$

The negative sign is necessary because $\dot{M}_2$ is numerically a negative quantity as it occurs in the Lagrangian derivation. Thus the thrust, F, will become a positive quantity when evaluated numerically in the next section.

Also, it was found in the first part of the derivation that the jet damping force, D, is proportional to the quantity $\dot{M}_2 \ell_5 \dot{q}_4$. In order to make this a numerically positive quantity, it will be written as

$$D = -\dot{M}_2 \ell_5 \dot{q}_4$$

The moment of inertia, I_i, can be written as the product of the mass, M_i, and the square of the radius of gyration, k_i, that is

$$I_i = M_i k_i^2$$

The time derivative of the moment of inertia, I_i, is

$\frac{dI_i}{dt} = \frac{dM_i}{dt} k_i^2 + M_i \frac{d(k_i^2)}{dt}$. However, the second term is small compared to the first term and will be neglected in this analysis. Therefore the following approximation can be made

$$\dot{I}_i \approx \dot{M}_i k_i^2$$

Making these four substitutions in the equations of motion in the q_1, q_3, and q_4 directions the following equations are obtained from Equations (23), (44), and (59), respectively.

From Equation (23)

$$(M_1+M_2)\ddot{q}_1 + \frac{M_1 \dot{M}_2}{M_1+M_2} l_1 \dot{q}_3 - \frac{M_1 \dot{M}_2^2}{(M_1+M_2)^2} l_1 q_3$$

$$-\left(\frac{M_1 \dot{M}_2}{M_1+M_2} l_2 - \dot{M}_2 l_5\right)\dot{q}_4 + \left(\frac{M_1 \dot{M}_2^2}{(M_1+M_2)^2} l_2 + \dot{M}_2 g I_{sp}\right) q_4 = 0 \qquad (60)$$

From Equation (44)

$$-\frac{M_1 \dot{M}_2}{M_1+M_2} l_1 \dot{q}_1 + \left(M_1 k_1^2 + \frac{M_1 M_2}{M_1+M_2} l_1^2\right)\ddot{q}_3 + \frac{M_1^2 \dot{M}_2}{(M_1+M_2)^2} l_1^2 \dot{q}_3$$

$$+\left(\frac{M_1 \dot{M}_2}{M_1+M_2} l_1 \dot{q}_2 - \frac{M_1^2 \dot{M}_2^2}{(M_1+M_2)^3} l_1 l_2 - \frac{M_1 \dot{M}_2}{M_1+M_2} l_1 g I_{sp}\right) q_3$$

$$- \frac{M_1 M_2}{M_1+M_2} l_1 l_2 \ddot{q}_4 - \left(\frac{M_1 \dot{M}_2}{M_1+M_2} l_1 l_5 + \frac{M_1^2 \dot{M}_2}{(M_1+M_2)^2} l_1 l_2 \right) \dot{q}_4$$

$$+ \left(\frac{M_1^2 \dot{M}_2^2}{(M_1+M_2)^3} l_1 l_2 - \frac{M_1 \dot{M}_2}{M_1+M_2} l_1 g I_{sp} \right) q_4 = 0 \qquad (61)$$

From Equation (59)

$$\frac{M_1 \dot{M}_2}{M_1+M_2} l_2 \dot{q}_1 - \frac{M_1 M_2}{M_1+M_2} l_1 l_2 \ddot{q}_3 - \frac{M_1^2 \dot{M}_2}{(M_1+M_2)^2} l_1 l_2 \dot{q}_3$$

$$+ \frac{M_1^2 \dot{M}_2^2}{(M_1+M_2)^3} l_1 l_2 q_3 + \left(M_2 k_2^2 + \frac{M_1 M_2}{M_1+M_2} l_2^2 \right) \ddot{q}_4$$

$$+ \left(\dot{M}_2 k_2^2 + \frac{M_1^2 \dot{M}_2}{(M_1+M_2)^2} l_2^2 - \frac{M_2 \dot{M}_2}{M_1+M_2} l_2 l_5 - \dot{M}_2 l_3 l_5 \right) \dot{q}_4$$

$$- \left(\frac{M_1 \dot{M}_2}{M_1+M_2} l_2 \dot{q}_2 + \frac{M_1^2 \dot{M}_2^2}{(M_1+M_2)^3} l_1 l_2 \right) q_4 = 0 \qquad (62)$$

To write these last three equations in dimensionless form introduce the dimensionless coordinates Q_1, Q_3, and Q_4, and the characteristic units of time $\equiv I_{sp}$, velocity $\equiv I_{sp} g$, and acceleration $\equiv g$.

Thus

$$\left.\begin{array}{lll} Q_1 = \dfrac{q_1}{I_{sp}^2 g} & \dot{Q}_1 = \dfrac{\dot{q}_1}{I_{sp} g} & \ddot{Q}_1 = \dfrac{\ddot{q}_1}{g} \\ Q_3 = q_3 & \dot{Q}_3 = I_{sp} \dot{q}_3 & \ddot{Q}_3 = I_{sp}^2 \ddot{q}_3 \\ Q_4 = q_4 & \dot{Q}_4 = I_{sp} \dot{q}_4 & \ddot{Q}_4 = I_{sp}^2 \ddot{q}_4 \end{array}\right\} \qquad (63)$$

Making the above substitutions into Equation (60) and dividing the resulting equation by $M_1 g$, the following dimensionless equation is obtained

$$\left(1+\frac{M_2}{M_1}\right)\ddot{Q}_1 + \frac{\dot{M}_2 I_{sp}}{M_1+M_2}\left(1+\frac{M_2}{M_1}\right)\dot{Q}_1 + \frac{\dot{M}_2 I_{sp}}{M_1+M_2}\,\frac{k_2}{I_{sp}^2 g}\,\frac{l_2}{k_2}\,\frac{l_1}{l_2}\,\dot{Q}_3$$

$$-\left(\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right)^2 \frac{k_2}{I_{sp}^2 g}\,\frac{l_2}{k_2}\,\frac{l_1}{l_2}\,Q_3 - \Big[\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\,\frac{k_2}{I_{sp}^2 g}\,\frac{l_2}{k_2}$$

$$-\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\left(1+\frac{M_2}{M_1}\right)\frac{l_5}{I_{sp}^2 g}\Big]\dot{Q}_4 + \Big[\left(\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right)^2 \frac{k_2}{I_{sp}^2 g}\,\frac{l_2}{k_2}$$

$$+\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\left(1+\frac{M_2}{M_1}\right)\Big]Q_4 \quad = \quad 0 \tag{64}$$

Similarly, substituting the values of Equation (63) into Equation (61) and multiplying the resulting equation by $\frac{M_1+M_2}{M_1^2 g\, l_1}$ the second dimensionless equation is obtained

$$-\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\left(1+\frac{M_2}{M_1}\right)\dot{Q}_1 + \Big[\frac{k_2}{I_{sp}^2 g}\left(\frac{k_1}{k_2}\right)^2\frac{k_2}{l_2}\,\frac{l_2}{l_1}\left(1+\frac{M_2}{M_1}\right) + \frac{M_2}{M_1}\,\frac{k_2}{I_{sp}^2 g}\,\frac{l_2}{k_2}\,\frac{l_1}{l_2}\Big]\ddot{Q}_3$$

$$+\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\,\frac{k_2}{I_{sp}^2 g}\,\frac{l_2}{k_2}\,\frac{l_1}{l_2}\,\dot{Q}_3 + \Big[\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\,\frac{\dot{g}_2}{I_{sp}^2 g}\left(1+\frac{M_2}{M_1}\right) - \left(\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right)^2\frac{k_2}{I_{sp}^2 g}\,\frac{l_2}{k_2}$$

$$+\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\left(1+\frac{M_2}{M_1}\right)\Big]Q_3 - \frac{M_2}{M_1}\,\frac{k_2}{I_{sp}^2 g}\,\frac{l_2}{k_2}\,\ddot{Q}_4 - \Big[\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\,\frac{l_5}{I_{sp}^2 g}\left(1+\frac{M_2}{M_1}\right)$$

$$+\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\,\frac{k_2}{I_{sp}^2 g}\,\frac{l_2}{k_2}\Big]\dot{Q}_4 + \Big[\left(\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right)^2\frac{k_2}{I_{sp}^2 g}\,\frac{l_3}{k_2} - \frac{\dot{M}_2 I_{sp}}{M_1+M_2}\left(1+\frac{M_2}{M_1}\right)\Big]Q_4 \quad = \quad 0 \tag{65}$$

Finally, substituting the values of Equation (63) into Equation (62) and multiplying the resulting equation by $\frac{M_1 + M_2}{M_1 M_2 g l_2}$ the third dimensionless equation is obtained

$$\frac{M_1}{M_2}\left(1+\frac{M_2}{M_1}\right)\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\dot{Q}_1 - \frac{k_2}{I_{sp}^2 g}\frac{l_2}{k_2}\frac{l_1}{l_2}\ddot{Q}_3 - \frac{M_1}{M_2}\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\frac{k_2}{I_{sp}^2 g}\frac{l_2}{k_2}\frac{l_1}{l_2}\dot{Q}_3$$

$$+\frac{M_1}{M_2}\left(\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right)^2\frac{k_2}{I_{sp}^2 g}\frac{l_2}{k_2}\frac{l_1}{l_2}Q_3 + \left[\frac{k_2}{I_{sp}^2 g}\frac{k_2}{l_2}\left(1+\frac{M_2}{M_1}\right) + \frac{k_2}{I_{sp}^2 g}\frac{l_2}{k_2}\right]\ddot{Q}_4$$

$$+\left[\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\frac{M_1}{M_2}\left(1+\frac{M_2}{M_1}\right)^2\frac{k_2}{I_{sp}^2 g}\frac{k_2}{l_2} + \frac{M_1}{M_2}\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\frac{k_2}{I_{sp}^2 g}\frac{l_2}{k_2}\right.$$

$$\left. - \frac{\dot{M}_2 I_{sp}}{M_1+M_2}\frac{l_5}{I_{sp}^2 g}\left(1+\frac{M_2}{M_1}\right) - \frac{\dot{M}_2 I_{sp}}{M_1+M_2}\left(1+\frac{M_2}{M_1}\right)^2\frac{M_1}{M_2}\frac{l_5}{I_{sp}^2 g}\left(\frac{l_5}{k_2}\frac{k_2}{l_2} - 1\right)\right]\dot{Q}_4$$

$$-\left[\frac{M_1}{M_2}\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\frac{\dot{g}_2}{I_{sp} g}\left(1+\frac{M_2}{M_1}\right) + \frac{M_1}{M_2}\left(\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right)^2\frac{k_2}{I_{sp}^2 g}\frac{l_2}{k_2}\frac{l_1}{l_2}\right]Q_4 = 0 \qquad (66)$$

Equations (64), (65), and (66) are the three dimensionless equations of motion of the system and contain the following eight dimensionless parameters

$$\frac{M_1}{M_2} \qquad \frac{k_2}{I_{sp}^2 g} \qquad \frac{\dot{M}_2 I_{sp}}{M_1+M_2} \qquad \frac{l_1}{l_2}$$

$$\frac{k_2}{k_1} \qquad \frac{l_5}{I_{sp}^2 g} \qquad \frac{\dot{g}_2}{I_{sp} g} \qquad \frac{k_2}{l_2}$$

III. NUMERICAL INVESTIGATION OF STABILITY

Rather than attempt to find a solution for Equations (64), (65), and (66), it was decided to introduce selected values of the first six dimensionless parameters and, treating $\frac{\ell_2}{k_2}$ and $\frac{\ell_1}{\ell_2}$ as variables, to use Routh's Stability Criteria to determine the range of combinations of these two variables in which the system would be stable.

To simplify the writing of the determinantal equation necessary for the use of the Routh criteria, introduce the following notation where

A_{ij} $\equiv$ coefficient of Q_i in Equation j

B_{ij} $\equiv$ coefficient of $\dot{Q}_i$ in Equation j

C_{ij} $\equiv$ coefficient of $\ddot{Q}_i$ in Equation j

and

j = 1 corresponds to Equation (64)

j = 2 corresponds to Equation (65)

j = 3 corresponds to Equation (66)

With this notation Equations (64), (65), and (66) may now be written as

$$\left.\begin{aligned}
&C_{11}\,\ddot{Q}_1 + B_{11}\,\dot{Q}_1 + B_{31}\,\dot{Q}_3 + A_{31}\,Q_3 + B_{41}\,\dot{Q}_4 + A_{41}\,Q_4 = 0\\
&B_{12}\,\dot{Q}_1 + C_{32}\,\ddot{Q}_3 + B_{32}\,\dot{Q}_3 + A_{32}\,Q_3 + C_{42}\,\ddot{Q}_4 + B_{42}\,\dot{Q}_4 + A_{42}\,Q_4 = 0\\
&B_{31}\,\dot{Q}_1 + C_{33}\,\ddot{Q}_3 + B_{33}\,\dot{Q}_3 + A_{33}\,Q_3 + C_{43}\,\ddot{Q}_4 + B_{43}\,\dot{Q}_4 + A_{43}\,Q_4 = 0
\end{aligned}\right\}(67)$$

where, letting $\frac{l_2}{k_2} \equiv \beta$ and $\frac{l_1}{l_2} \equiv \alpha$, the coefficients in Equation (67) are

$$A_{31} = -\left(\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right)^2 \frac{k_2}{I_{sp}^2 g}\,\alpha\beta$$

$$A_{32} = \frac{\dot{M}_2 I_{sp}}{M_1+M_2}\left(1+\frac{M_2}{M_1}\right)\frac{\dot{q}_2}{I_{sp} g} + \frac{\dot{M}_2 I_{sp}}{M_1+M_2}\left(1+\frac{M_2}{M_1}\right) - \left(\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right)^2 \frac{k_2}{I_{sp}^2 g}\,\beta$$

$$A_{33} = \frac{M_1}{M_2}\left(\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right)^2 \frac{k_2}{I_{sp}^2 g}\,\alpha\beta$$

$$A_{41} = \frac{\dot{M}_2 I_{sp}}{M_1+M_2}\left(1+\frac{M_2}{M_1}\right) + \left(\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right)^2 \frac{k_2}{I_{sp}^2 g}\,\beta$$

$$A_{42} = -\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\left(1+\frac{M_2}{M_1}\right) + \left(\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right)^2 \frac{k_2}{I_{sp}^2 g}\,\beta$$

$$A_{43} = -\left[\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\,\frac{M_1}{M_2}\left(1+\frac{M_2}{M_1}\right)\frac{\dot{q}_2}{I_{sp} g} + \frac{M_1}{M_2}\left(\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right)^2 \frac{k_2}{I_{sp}^2 g}\,\alpha\beta\right]$$

$$B_{11} = \frac{\dot{M}_2 I_{sp}}{M_1+M_2}\left(1+\frac{M_2}{M_1}\right)$$

$$B_{12} = -\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\left(1+\frac{M_2}{M_1}\right)$$

$$B_{13} = \frac{M_1}{M_2}\,\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\left(1+\frac{M_2}{M_1}\right)$$

$$B_{31} = \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{k_2}{I_{sp}^2 g} \alpha\beta$$

$$B_{32} = \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{k_2}{I_{sp}^2 g} \alpha\beta$$

$$B_{33} = -\frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{M_1}{M_2} \frac{k_2}{I_{sp}^2 g} \alpha\beta$$

$$B_{41} = \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \left(1 + \frac{M_2}{M_1}\right) \frac{l_5}{I_{sp}^2 g} - \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{k_2}{I_{sp}^2 g} \beta$$

$$B_{42} = -\left[\frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \left(1 + \frac{M_2}{M_1}\right) \frac{l_5}{I_{sp}^2 g} + \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{k_2}{I_{sp}^2 g} \beta\right]$$

$$B_{43} = \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{M_1}{M_2} \left(1 + \frac{M_2}{M_1}\right)^2 \frac{k_2}{I_{sp}^2 g} \frac{1}{\beta} + \frac{M_1}{M_2} \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{k_2}{I_{sp}^2 g} \beta$$

$$- \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \left(1 + \frac{M_2}{M_1}\right) \frac{l_5}{I_{sp}^2 g} - \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \left(1 + \frac{M_2}{M_1}\right)^2 \frac{M_1}{M_2} \frac{l_5}{I_{sp}^2 g} \left(\frac{l_5}{k_2} \frac{1}{\beta} - 1\right)$$

$$C_{11} = 1 + \frac{M_2}{M_1}$$

$$C_{32} = \frac{k_2}{I_{sp}^2 g} \left(\frac{k_1}{k_2}\right)^2 \left(1 + \frac{M_2}{M_1}\right) \frac{1}{\alpha\beta} + \frac{M_2}{M_1} \frac{k_2}{I_{sp}^2 g} \alpha\beta$$

$$C_{33} = -\frac{k_2}{I_{sp}^2 g} \alpha\beta$$

$$C_{42} = -\frac{M_2}{M_1} \frac{k_2}{I_{sp}^2 g} \beta$$

$$C_{43} = \frac{k_2}{I_{sp}^2 g} \left(1 + \frac{M_2}{M_1}\right) \frac{1}{\beta} + \frac{k_2}{I_{sp}^2 g} \beta$$

Inasmuch as oscillatory solutions are of main interest, assume

$$Q_i \propto e^{\lambda t}$$

Then

$$\dot{Q}_i \propto \lambda e^{\lambda t} = \lambda Q_i$$

and

$$\ddot{Q}_i \propto \lambda^2 Q_i$$

For the system of homogeneous differential equations (67) to have other than a trivial solution, the determinant of coefficients of Q_i must vanish. That is

$$\begin{vmatrix} C_{11}\lambda^2 + B_{11}\lambda & B_{31}\lambda + A_{31} & B_{41}\lambda + A_{41} \\ B_{12}\lambda & C_{32}\lambda^2 + B_{32}\lambda + A_{32} & C_{42}\lambda^2 + B_{42}\lambda + A_{42} \\ B_{13}\lambda & C_{33}\lambda^2 + B_{33}\lambda + A_{33} & C_{43}\lambda^2 + B_{43}\lambda + A_{43} \end{vmatrix} = 0 \qquad (67a)$$

Expanding the determinant of (67a)

$$C_{11} F\lambda^6 + (C_{11} H + B_{11} F)\lambda^5 + (B_{11} H + C_{11} J - B_{12} S + B_{13} W)\lambda^4$$
$$+ (C_{11} K + B_{11} J - B_{12} T + B_{13} X)\lambda^3 + (C_{11} R + B_{11} K - B_{12} U + B_{13} Y)\lambda^2$$
$$+ (B_{11} R - B_{12} V + B_{13} Z) \quad = 0$$

which can be written

$$p_0\lambda^5 + p_1\lambda^4 + p_2\lambda^3 + p_3\lambda^2 + p_4\lambda + p_5 = 0 \qquad (68)$$

where

$$p_0 = C_{11} F$$

$$p_1 = C_{11} H + B_{11} F$$

$$p_2 = B_{11} H + C_{11} J - B_{12} S + B_{13} W$$

$$p_3 = C_{11} K + B_{11} J - B_{12} T + B_{13} X$$

$$p_4 = C_{11} R + B_{11} K - B_{12} U + B_{13} Y$$

$$p_5 = B_{11} R - B_{12} V + B_{13} Z$$

and

$$F = C_{32} C_{43} - C_{33} C_{42}$$

$$H = B_{32} C_{43} + B_{43} C_{32} - B_{42} C_{33} - B_{33} C_{42}$$

$$J = A_{32} C_{43} + A_{43} C_{32} + B_{32} B_{43} - A_{42} C_{33} - A_{33} C_{42} - B_{33} B_{42}$$

$$K = A_{43} B_{32} + A_{32} B_{43} - A_{42} B_{33} - A_{33} B_{42}$$

$$R = A_{32} A_{43} - A_{33} A_{42}$$

$$S = B_{31} C_{43} - B_{41} C_{33}$$

$$T = A_{31} C_{43} + B_{31} B_{43} - A_{41} C_{33} - B_{33} B_{41}$$

$$U = A_{43} B_{31} + A_{31} B_{43} - A_{41} B_{33} - A_{33} B_{41}$$

$$V = A_{31} A_{43} - A_{33} A_{41}$$

$$W = B_{31} C_{42} - B_{41} C_{32}$$

$$X = A_{31} C_{42} + B_{31} B_{42} - A_{41} C_{32} - B_{32} B_{41}$$

$$Y = A_{31} B_{42} + A_{42} B_{31} - A_{41} B_{32} - A_{32} B_{41}$$

$$Z = A_{31} A_{42} - A_{32} A_{41}$$

The system will be stable if the coefficients of Equation (68) and the following terms of Routh's stability criteria are all of the same sign (Reference 3)

$$\left.\begin{array}{ll} 1. & p_2 - \dfrac{p_0 p_3}{p_1} \\ 2. & p_3 - \dfrac{(p_1 p_4 - p_0 p_5) p_1}{p_1 p_2 - p_0 p_3} \\ 3. & p_4 - \dfrac{p_0 p_5}{p_1} - \dfrac{(p_1 p_2 - p_0 p_3)^2 p_5}{p_1 [p_3 (p_1 p_2 - p_0 p_3) - (p_1 p_4 - p_0 p_5)]} \end{array}\right\} \quad (69)$$

NUMERICAL EXAMPLE FOR "V-2" TYPE ROCKET

To find a numerical solution to the problem the "V-2" rocket (weight approximately 28,000 lbs.) was chosen as the main rocket, M_1, and a small five-second booster providing an initial acceleration of 1-1/2 g was selected as the booster, M_2.

The dimensionless parameters of the system were then estimated to be

$$\frac{M_1}{M_2} = 7.75 \qquad \frac{k_2}{I_{sp}^2 g} = 1.9 \times 10^{-6} \qquad \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} = -2.21$$

$$\frac{k_1}{k_2} = 7.75 \qquad \frac{l_5}{I_{sp}^2 g} = 2.48 \times 10^{-6}$$

$$\frac{\dot{g}_2}{I_{sp} g} = 0 \quad \text{(at launching)}$$

The stability of the system was then to be investigated for various values of the two variables $\alpha = \frac{l_1}{l_2}$ and $\beta = \frac{l_2}{k_2}$.

Because of the inherent complexity of the constants of Equation (69), a vast amount of time would be required to investigate every likely combination of the variables $\alpha = \frac{l_1}{l_2}$ and $\beta = \frac{l_2}{k_2}$. However, the writer optimistically set out to evaluate these constants for various combinations of α and β . Inasmuch as no combination was found where even the constants p_1 to p_5 were of the same sign, much less where the test functions of Equation (69) were of the same sign, it was decided to reduce the complexity of the system in order to expedite the computation.

Accordingly, $\dot{Q}_1$ and $\ddot{Q}_1$ were assumed to be zero, that is, the system was restricted to vertical and rotational movement of the center of gravity. This assumption reduced the system of equations to two, namely, Equations (65) and (66), and reduced the labor involved in applying the Routh stability criteria substantially. Once a stable range of α and β had been determined, these values were then to have been used in the general case for closer investigation.

STABILITY OF SYSTEM WITH TWO DEGREES OF FREEDOM

If both $\dot{Q}_1$ and $\ddot{Q}_1$ are assumed to be zero, Equations (65) and (66) may be re-written (using the abbreviated notation introduced earlier) as

$$C_{32}\ddot{Q}_3 + B_{32}\dot{Q}_3 + A_{32}Q_3 + C_{42}\ddot{Q}_4 + B_{42}\dot{Q}_4 + A_{42}Q_4 = 0$$

$$C_{33}\ddot{Q}_3 + B_{33}\dot{Q}_3 + A_{33}Q_3 + C_{43}\ddot{Q}_4 + B_{43}\dot{Q}_4 + A_{43}Q_4 = 0$$

and assuming $Q_i \propto e^{\lambda t}$, as in the more general case, the determinant for this case becomes

$$\begin{vmatrix} C_{32}\lambda^2 + B_{32}\lambda + A_{32} & C_{42}\lambda^2 + B_{42}\lambda + A_{42} \\ C_{33}\lambda^2 + B_{33}\lambda + A_{33} & C_{43}\lambda^2 + B_{43}\lambda + A_{43} \end{vmatrix} = 0$$

The expanded determinant may then be written

$$(C_{32}C_{43} - C_{42}C_{33})\,\lambda^4 + (C_{32}B_{43} + B_{32}C_{43} - C_{42}B_{33} - B_{42}C_{33})\,\lambda^3$$
$$+ (C_{32}A_{43} + A_{32}C_{43} + B_{32}B_{43} - C_{42}A_{33} - A_{42}C_{33} - B_{42}B_{33})\,\lambda^2$$
$$+ (B_{32}A_{43} + A_{32}B_{43} - B_{42}A_{33} - A_{42}B_{33})\,\lambda + (A_{32}A_{43} - A_{42}A_{33}) = 0 \qquad (70)$$

Furthermore, since the parameter $\frac{\dot{g}_2}{I_{sp}g}$ equals zero at launching, the last term of Equation (70) becomes

$$(A_{32}A_{43} - A_{42}A_{33}) = -\left[\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\left(1+\frac{M_2}{M_1}\right)\right]\left[\frac{M_1}{M_2}\left(\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right)^2 \frac{k_2}{I_{sp}^2 g}\,\alpha\beta\right]$$
$$+\left[\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\left(1+\frac{M_2}{M_1}\right)\right]\left[\frac{M_1}{M_2}\left(\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right)^2 \frac{k_2}{I_{sp}^2 g}\,\alpha\beta\right] = 0$$

Therefore Equation (70) reduces to a third order equation of the form

$$p_0\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 = 0 \qquad (71)$$

where

$$p_0 = C_{32}C_{43} - C_{42}C_{33}$$
$$p_1 = C_{32}B_{43} + B_{32}C_{43} - C_{42}B_{33} - B_{42}C_{33}$$
$$p_2 = C_{32}A_{43} + A_{32}C_{43} + B_{32}B_{43} - C_{42}A_{33} - A_{42}C_{33} - B_{42}B_{33}$$
$$p_3 = B_{32}A_{43} + A_{32}B_{43} - B_{42}A_{33} - A_{42}B_{33}$$

and the Routh stability criteria require that p_0, p_1, p_2, p_3, and $\left(p_2 - \frac{p_0 p_3}{p_1}\right)$ all be of the same sign. (Reference 3).

The coefficients of Equation (71) may now be evaluated by substituting the parameters previously selected for the "V-2" rocket with five second booster. Collecting terms these coefficients may be expressed as follows

$$\rho_0 = \frac{1}{\alpha}\left(a_0 \alpha^2 + c_0\right) \tag{72}$$

where

$$a_0 = 5.24 \times 10^{-13}$$

$$c_0 = \frac{2.76 \times 10^{-10}}{\beta^2} + 2.45 \times 10^{-10}$$

$$\rho_1 = \frac{1}{\alpha}\left(a_1 \alpha^2 + c_1\right) \tag{73}$$

where

$$a_1 = -1.91 \times 10^{-12}$$

$$c_1 = \frac{3.71 \times 10^{-9}}{\beta^2} - \frac{6.15 \times 10^{-9}}{\beta} - 4.2 \times 10^{-9}$$

$$\rho_2 = a_2 \alpha^2 + b_2 \alpha + c_2 \tag{74}$$

where

$$a_2 = -1.76 \times 10^{-11} \beta^2$$

$$b_2 = 3.53 \times 10^{-11} \beta^2 + 4.73 \times 10^{-6} \beta - 1.21 \times 10^{-10}$$

$$c_2 = -\left(1.76 \times 10^{-11} \beta^2 + 4.73 \times 10^{-6} \beta + \frac{5.33 \times 10^{-6}}{\beta} - 9.3 \times 10^{-9}\right)$$

$$\rho_3 = a_3 \alpha^2 + b_3 \alpha + c_3 \tag{75}$$

where

$$a_3 = 3.02 \times 10^{-10} \beta^2$$

$$b_3 = -\left(6.06 \times 10^{-10} \beta^2 + 8.12 \times 10^{-5} \beta\right)$$

$$c_3 = 3.03 \times 10^{-10} \beta^2 + 8.12 \times 10^{-5} \beta + 1.19 \times 10^{-4} - \frac{7.18 \times 10^{-5}}{\beta}$$

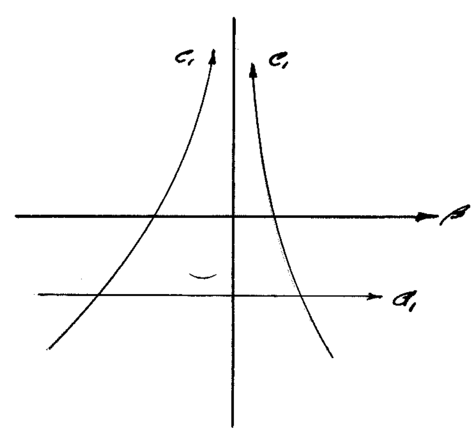
Before computing numerical values of the coefficients of Equation (71) it will be profitable to draw certain conclusions from an inspection of Equations (72) through (75).

α is defined as the quantity $\frac{\ell_1}{\ell_2}$, and β is defined as the quantity $\frac{\ell_2}{k_2}$, where ℓ_1 and k_2 are inherently positive quantities by the physical nature of the problem. Therefore, it can be concluded that for any real solution of the physical problem, both α and β must be of the same sign, that is, either both positive, or both negative.

In Equation (72) the sign of p_0 is independent of β and takes the sign of α. Therefore numerical values of p_0 need not be determined at this stage of the investigation since the sign of p_0 will be evident by inspection.

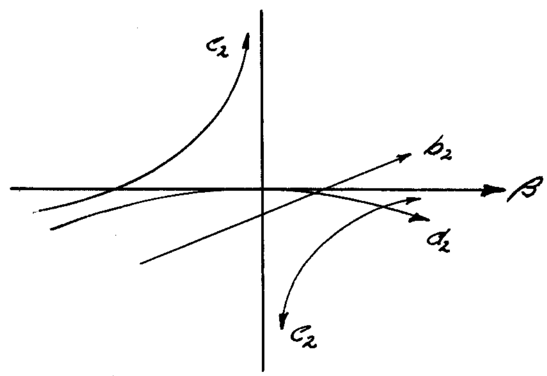
In Equation (73) a_1 is always negative, and c_1 has roots at $\beta = 0.456$ and $\beta = -1.93$.

Since both a_1 and c_1 are negative for $\beta < 1.93$ and since α is negative when β is negative, it follows that p_1 is always positive in this region.



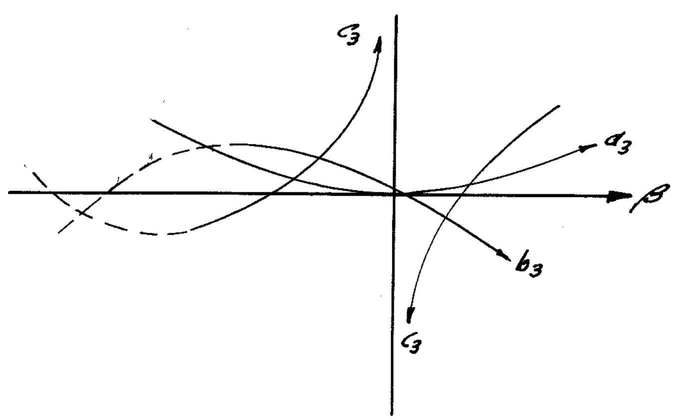
Negative α yields negative p_0 from Equation (72). Therefore p_1 and p_2 will be of opposite sign for all values of $\beta < 1.93$, and, by Routh's stability criteria, stability is impossible for this case. By analogous reasoning stability is impossible in the range $0 < \beta < 0.456$.

In Equation (74) $a_2 = 0$ when $\beta = 0$, and a_2 is negative for all other values of β. Furthermore, b_2 has a root at $\beta = 2.54 \times 10^{-4}$, and c_2 has a root at $\beta = -2.68 \times 10^{-5}$. Since a_2, b_2, and c_2 are all negative in the range $0 < \beta < 2.54 \times 10^{-4}$



it can be concluded that stability is impossible in this range by reasoning analogous to that in the case of p_1.

In Equation (75) $a_3 = 0$ when $\beta = 0$, and a_3 is positive for all other values of β. Furthermore b_3 has a root at $\beta = -1.34 \times 10^{-5}$ and at $\beta = 0$. Also c_3 has roots at $\beta = 0.456$, $\beta = -1.92$, and $\beta = -2.68 \times 10^{-5}$. No conclusions can be drawn in this case.



To summarize the conclusions of the preceding paragraphs, the possibility of stability need only be investigated in the ranges $-1.93 < \beta < 0$ and $2.54\times10^{-4} < \beta < 0.456$.

In evaluating the coefficients p_1, p_2, and p_3 the following procedure was adopted. Selected values of β within the positive range $2.54\times10^{-4} < \beta < 0.456$ were used to find the constant coefficients of Equations (73), (74), and (75). Then the positive roots of the resulting equations were computed as functions of α. Negative roots were not investigated since α must be of the same sign as β. These values are tabulated in Table I.

Similarly, values of the coefficients and corresponding negative roots of Equations (73), (74), and (75) were computed and are tabulated in Table II.

The data of Tables I and Table II are plotted qualitatively in Figure 6 and Figure 7.

In Figure 8 and Figure 9 are plotted the logarithms of maximum and minimum limits of α for which p_1, p_2, and p_3 are each positive.

It is apparent from Figure 8 that there is no single value of α for which p_1, p_2, and p_3 can all be positive since α_{min} for positive p_3 greatly exceeds α_{max} for positive p_1 for all values of $2.54\times10^{-4} < \beta < 0.456$.

Likewise, in Figure 9 there is no single value of α for which p_1, p_2, and p_3 can all be negative since $|a_{min}|$ for negative p_2 greatly exceeds $|a_{max}|$ for negative p_1 for all values of $-1.93 < \beta < 0$.

Therefore, it is clear that Equations (65) and (66) do not have a stable solution for any combination of a and β for the parameters chosen.

STABILITY OF SYSTEM WITH REDUCED MASS RATIO

In view of the negative results obtained in the previous case it was decided to investigate the effect of changing the parameter $\frac{M_1}{M_2}$. The following set of dimensionless parameters was then selected

$$\frac{M_1}{M_2} = 1.5 \qquad \frac{k_2}{I_{sp}^2 g} = 6.72 \times 10^{-6} \qquad \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} = -2.21$$

$$\frac{k_1}{k_2} = 1.5 \qquad \frac{A_5}{I_{sp}^2 g} = 8.75 \times 10^{-6}$$

$$\frac{\dot{q}_2}{I_{sp}\, g} = 0 \qquad \text{(at launching)}$$

Note that $\frac{M_1}{M_2}$ was decreased from 7.75 to 1.5, $\frac{k_2}{I_{sp}^2 g}$ was increased from 1.96×10^{-6} to 6.72×10^{-6}, and $\frac{A_5}{I_{sp}^2 g}$ was increased from 2.48×10^{-6} to 8.75×10^{-6}. The remaining parameters were unchanged. Thus, this second set of parameters describe a system essentially the same as that first investigated except that the mass ratio $\frac{M_1}{M_2}$ has been markedly reduced.

The coefficients of Equation (71) are again evaluated by substituting these new parameters. Collecting terms the coefficients may be expressed as follows

$$p_0 = \frac{1}{\alpha}(a_0\alpha^2 + c_0) \quad (76)$$

where $a_0 = 5.04 \times 10^{-11}$

$$c_0 = \frac{2.82 \times 10^{-10}}{\beta^2} + 1.70 \times 10^{-10}$$

$$p_1 = \frac{1}{\alpha}(a_1\alpha^2 + c_1) \quad (77)$$

where $a_1 = 2.49 \times 10^{-11}$

$$c_1 = \frac{1.08 \times 10^{-9}}{\beta^2} - \frac{1.21 \times 10^{-9}}{\beta} - 5.62 \times 10^{-10}$$

$$p_2 = a_2\alpha^2 + b_2\alpha + c_2 \quad (78)$$

where $a_2 = -2.22 \times 10^{-10}\beta^2$

$$b_2 = 4.43 \times 10^{-10}\beta^2 + 2.48 \times 10^{-5}\beta - 6.36 \times 10^{-10}$$

$$c_2 = -\left(2.21 \times 10^{-10}\beta^2 + 2.48 \times 10^{-5}\beta + 1.61 \times 10^{-9} + \frac{4.13 \times 10^{-5}}{\beta}\right)$$

$$p_3 = a_3\alpha^2 + b_3\alpha + c_3 \quad (79)$$

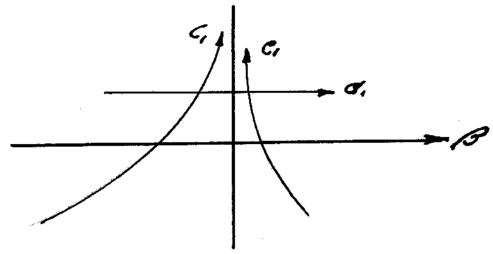
where $a_3 = 7.32 \times 10^{-10}\beta^2$

$$b_3 = -\left(1.46 \times 10^{-9}\beta^2 + 8.22 \times 10^{-5}\beta\right)$$

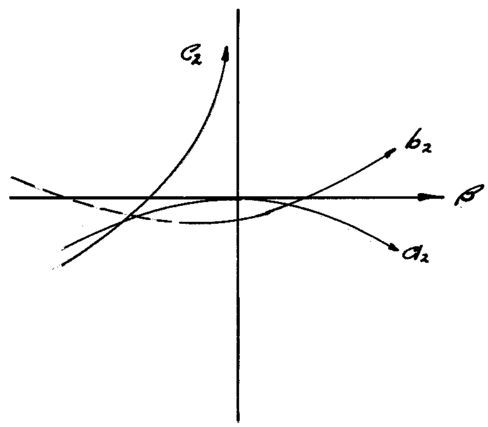
$$c_3 = 7.32 \times 10^{-10}\beta^2 + 8.22 \times 10^{-5}\beta + 1.77 \times 10^{-4} - \frac{1.57 \times 10^{-4}}{\beta}$$

As previously stated, a and β must be of the same sign. Again, the sign of p_o is independent of β and takes the sign of a.

In Equation (77) a_1 is always positive. Also c_1 has roots at $\beta = 0.68$ and at $\beta = -2.84$.

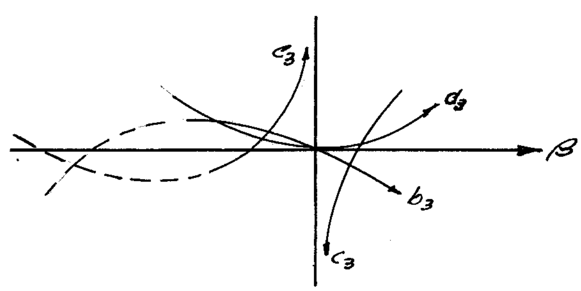


In Equation (78) $a_2 = 0$ when $\beta = 0$, and a_2 is negative for all other values of β. Furthermore b_2 has roots at $\beta = -5.6 \times 10^{5}$ and at $\beta = 2.55 \times 10^{-5}$. Also c_2 has a root at $\beta = -1.12 \times 10^{5}$.



Since a_2, b_2, and c_2 are all negative in the range $0 < \beta < 2.55 \times 10^{-5}$ stability is not possible in this range.

In Equation (79) $a_3 = 0$ when $\beta = 0$, and a_3 is positive for all other values of β. Furthermore b_3 has roots at $\beta = 0$ and at $\beta = -5.62 \times 10^{4}$. Also, c_3 has roots at $\beta \simeq 0.625$, $\beta = -2.83$ and $\beta = -1.12 \times 10^{5}$.



It may be concluded that the possibility of stability need only be investigated outside the range $0 < \beta < 2.55 \times 10^{-5}$.

As in the previous case selected values of both positive and negative β were substituted into Equations (77), (78), and (79), to compute the constant coefficients of Equation (71), and the resulting equations were then solved for maximum and minimum allowable values of a. The results are tabulated in Table III and Table IV, and are plotted qualitatively in Figure 10 and Figure 11.

Figure 12 and Figure 13 are again logarithmic plots of the maximum and minimum limits of a versus the logarithm of β .

In Figure 12 it can be seen that stability is possible in the range $10^{-5} < \beta < 4.5 \times 10^{4}$ only if a_{max} for positive p_2 is greater than a_{min} for positive p_3 . (The curves appear to coincide.) However, an analysis of Table III reveals that for all values of $10^{-5} < \beta < 0.68$ both c_2 and c_3 are negative whereas a_2 is negative while a_3 is positive in this range. Therefore $a_{2_{max}} = \left|\frac{b_2}{2a_2}\right| \left[1 + \sqrt{1 - \left|\frac{4a_2 c_2}{b_2^2}\right|}\right]$ while $a_{3_{min}} = \left|\frac{b_3}{2a_3}\right| \left[1 + \sqrt{1 + \left|\frac{4a_3 c_3}{b_3^2}\right|}\right]$ Thus it follows that $a_{3_{min}}$ is greater than $a_{2_{max}}$ in this range and stability is impossible.

Similarly, in Figure 13 stability is only possible if $|a_{max}|$ for negative p_3 is greater than $|a_{min}|$ for negative p_2 .

However, an analysis of Table IV reveals that for all values of $-2.8 < \beta < -10^{-5}$ both c_2 and c_3 are positive while a_2 is negative and a_3 is positive in this range. Therefore, $|\alpha_{2\,min}| = \left|\frac{b_2}{2a_2}\right|\left[1 + \sqrt{1 + \left|\frac{4a_2c_2}{b_2^2}\right|}\right]$ while $|\alpha_{3\,max}| = \left|\frac{b_3}{2a_3}\right|\left[1 + \sqrt{1 - \left|\frac{4a_3c_3}{b_3^2}\right|}\right]$ and thus $|\alpha_{2\,min}|$ is greater than $|\alpha_{3\,max}|$.

Therefore it is only necessary to investigate more closely the range $0.68 < \beta < 4.5\text{x}10^4$ and $-10^3 < \beta < -2.8$.

Returning to Equation (71) and multiplying the analytic expressions for the coefficients of p_2 and p_3 the following equations are derived if only first order terms are retained.

$$p_2 = a_2\alpha^2 + b_2\alpha + c_2$$

$$= -\left[\left(\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right)^2\left(\frac{k_2}{I_{sp}^2 g}\right)^2\beta^2\right]\alpha^2$$

$$+\left[2\left(\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right)^2\left(\frac{k_2}{I_{sp}^2 g}\right)^2\beta^2 - \left(\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right)\frac{k_2}{I_{sp}^2 g}\left(1+\frac{M_2}{M_1}\right)\beta\right]\alpha$$

$$+\left[\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\frac{k_2}{I_{sp}^2 g}\left(1+\frac{M_2}{M_1}\right)\beta + \frac{\dot{M}_2 I_{sp}}{M_1+M_2}\frac{k_2}{I_{sp}^2 g}\left(1+\frac{M_2}{M_1}\right)^2\frac{1}{\beta}\right.$$

$$\left. - \left(\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right)^2\left(\frac{k_2}{I_{sp}^2 g}\right)^2\left(1+\frac{M_2}{M_1}\right)\left(\frac{M_1}{M_2}\frac{k_1^2}{k_2^2} + 1\right)\right] \quad (80)$$

and

$$p_3 = a_3\alpha^2 + b_3\alpha + c_3$$

$$= -\left[\left(\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right)^3\left(\frac{k_2}{I_{sp}^2 g}\right)^2\frac{M_1}{M_2}\beta^2\right]\alpha^2 \quad +$$

$$+\left[2\left(\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right)^3\left(\frac{k_2}{I_{sp}^2 g}\right)^2 \frac{M_1}{M_2}\beta^2 - \left(\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right)^2 \frac{k_2}{I_{sp}^2 g}\left(1+\frac{M_2}{M_1}\right)\frac{M_1}{M_2}\beta\right]\alpha$$

$$+\left[\left(\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right)^2 \frac{k_2}{I_{sp}^2 g}\left(1+\frac{M_2}{M_1}\right)\frac{M_1}{M_2}\beta + \left(\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right)^2 \frac{k_2}{I_{sp}^2 g}\left(1+\frac{M_2}{M_1}\right)^3\frac{M_1}{M_2}\left(1-\frac{l_5^2}{k_2^2}\right)\frac{1}{\beta}\right.$$

$$\left.+\left(\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right)^2\left(1+\frac{M_2}{M_1}\right)^2\frac{M_1}{M_2}\frac{l_5}{I_{sp}^2 g}\right] \qquad (81)$$

Factoring $\frac{M_1}{M_2}\ \frac{\dot{M}_2 I_{sp}}{M_1+M_2}$ from Equation (81)

$$b_3 = \frac{M_1}{M_2}\ \frac{\dot{M}_2 I_{sp}}{M_1+M_2}\left\{-\left[\left(\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right)^2\left(\frac{k_2}{I_{sp}^2 g}\right)^2\beta^2\right]\alpha^2\right.$$

$$+\left[2\left(\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right)^2\left(\frac{k_2}{I_{sp}^2 g}\right)^2\beta^2 - \frac{\dot{M}_2 I_{sp}}{M_1+M_2}\ \frac{k_2}{I_{sp}^2 g}\left(1+\frac{M_2}{M_1}\right)\beta\right]\alpha$$

$$+\left[\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\ \frac{k_2}{I_{sp}^2 g}\left(1+\frac{M_2}{M_1}\right)\beta + \frac{\dot{M}_2 I_{sp}}{M_1+M_2}\ \frac{k_2}{I_{sp}^2 g}\left(1+\frac{M_2}{M_1}\right)^3\left(1-\frac{l_5^2}{k_2^2}\right)\frac{1}{\beta}\right.$$

$$\left.\left.+\ \frac{\dot{M}_2 I_{sp}}{M_1+M_2}\left(1+\frac{M_2}{M_1}\right)^2\frac{l_5}{I_{sp}^2 g}\right]\right\} \qquad (82)$$

Comparing Equation (80) with Equation (82) it can be seen that both a_3 and b_3 are multiples of a_2 and b_2 respectively by the factor $\frac{M_1}{M_2}\ \frac{\dot{M}_2 I_{sp}}{M_1+M_2}$. For positive β, stability exists only if $a_{2\,max}$ is greater than $a_{3\,min}$, that is, if $c_2 < \dfrac{c_3}{\frac{M_1}{M_2}\ \frac{\dot{M}_2 I_{sp}}{M_1+M_2}}$

. But the factor $\frac{M_1}{M_2}\,\frac{\dot{M}_2 I_{sp}}{M_1+M_2}$ is numerically negative so that the condition for stability can be written

$$c_2 > \frac{c_3}{\left|\frac{M_1}{M_2}\,\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right|}$$

Substituting the values of c_2 and c_3 from Equation (80) and Equation (82) this condition becomes

$$\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\,\frac{k_2}{I_{sp}^2 g}\left(1+\frac{M_2}{M_1}\right)^2\frac{1}{\beta} - \left(\frac{\dot{M}_2 I_{sp}}{M_1+M_2}\right)^2\left(\frac{k_2}{I_{sp}^2 g}\right)^2\left(1+\frac{M_2}{M_1}\right)\left(\frac{M_1}{M_2}\,\frac{k_1^2}{k_2^2}+1\right)$$

$$> \frac{\dot{M}_2 I_{sp}}{M_1+M_2}\,\frac{k_2}{I_{sp}^2 g}\left(1+\frac{M_2}{M_1}\right)^3\left(1-\frac{\ell_5^2}{k_2^2}\right)\frac{1}{\beta} + \frac{\dot{M}_2 I_{sp}}{M_1+M_2}\left(1+\frac{M_2}{M_1}\right)^2\frac{\ell_5}{I_{sp}^2 g}$$

which, upon collecting terms, becomes

$$\beta > \frac{\left(1+\frac{M_2}{M_1}\right)\left[1-\left(1+\frac{M_2}{M_1}\right)\left(1-\frac{\ell_5^2}{k_2^2}\right)\right]}{\left(1+\frac{M_2}{M_1}\right)\frac{\ell_5}{k_2} + \frac{k_2}{I_{sp}^2 g}\left(\frac{M_1^3}{M_2^3}+1\right)\frac{\dot{M}_2 I_{sp}}{M_1+M_2}}$$

Inserting the numerical values of the parameters, the second term of the denominator is of the order 10^{-6} since $\frac{k_2}{I_{sp}^2 g} = 6.72 \times 10^{-6}$.

Therefore this term can be neglected in comparison with the first term of the denominator and the expression simplifies to

$$\beta > \frac{1-\left(1+\frac{M_2}{M_1}\right)\left(1-\frac{\ell_5^2}{k_2^2}\right)}{\frac{\ell_5}{k_2}} \qquad (83)$$

Solving this inequality with the parameters chosen, the result is

$$\beta > 1.62$$

As an example, choose $\beta = 2$. Inserting values from Table III into Equation (78) and Equation (79), for $\beta = 2$ the following values of a are obtained

$$\alpha = -\frac{b}{2a}\left[1 \pm \sqrt{1-\frac{4ac}{b^2}}\right] \approx -\frac{b}{2a}\left[1 \pm \left(1-\frac{2ac}{b^2}\right)\right]$$

$$\alpha \approx -\frac{b}{a}\left[1 - \frac{ac}{b^2}\right]$$

$$\alpha_2 = 5.6 \times 10^4 \left(1 - 2.54 \times 10^{-5}\right)$$

$$\alpha_3 = 5.6 \times 10^4 \left(1 - 9.85 \times 10^{-5}\right)$$

Thus p_2 changes from positive to negative when $a_2 = 5.6\times10^4(1-2.54\times10^{-5})$, while p_3 changes from negative to positive when $a_3 = 5.6\times10^4(1-9.85\times10^{-5})$, and both p_2 and p_3 are positive in the range $(5.6\times10^4 - 5.52) < a < (5.6\times10^4 - 1.42)$. Results with a similar sensitivity are obtained for larger values of β but smaller values of a. (It can be seen from Figure 12 that the product of a and β is approximately constant and equal to 10^5.)

There is yet to be applied the final Routh criterion, namely that $\left(p_2 - \frac{p_0 p_3}{p_1}\right)$ be positive also. Choosing p_3 at its smallest allowable value, say 0_+, and calling the corresponding value of p_2 some small positive value, say ε, $\left(p_2 - \frac{p_0 p_3}{p_1}\right) = \varepsilon$. Thus all the stability criteria are satisfied and the system is stable. Therefore, it can be concluded that the system is stable for $\beta > 1.62$ but that, for any given value of β in this stable range, a is extremely restricted.

Finally, there remains the negative range of β to be investigated, namely $-10^3 < \beta < -2.8$. But, for negative values of β, β must be greater than -1.62 by Equation (83) in order that p_2 and p_3 both be negative. Therefore stability is not possible for negative values of β.

IV. DISCUSSION OF RESULTS

Only two possible sets of parameters for a launching system of this type have been investigated. In the first case where the mass ratio was 7.75 it was found that stability was impossible, whereas, in the second case where the mass ratio was 1.5 it was found that stability was theoretically possible for values of β between 1.62 and 4.5×10^4 if the effect of lateral motion of the center of gravity of the system was neglected. For any given value of in this range stability was possible for only one corresponding value of α.

Returning to the physical aspects of the problem consider now the meaning of the variables α and β. β is defined as $\frac{\ell_2}{k_2}$ where ℓ_2 is the distance from the center of gravity of the booster, M_2, to the pin connecting it to the strut to the main rocket, M_1, and k_2 is the radius of gyration of the booster, M_2. For any system of this type k_2 will always be of the order of one foot or greater. Therefore, $\ell_2 = \beta k_2$ will always be equal to or greater than β. That is, as a first approximation, ℓ_2 is approximately numerically equal to β. Thus, from a practical consideration ℓ_2, and therefore β, must certainly not exceed some small number of the order of 1 to 10, depending on the size of the booster, for otherwise the pin will be located too near the booster exhaust.

Now α is defined as $\frac{\ell_1}{\ell_2}$ where ℓ_1 is the length of the strut from the pin in the booster M_2 to the center of gravity of the main rocket, M_1. Here again there is a practical upper limit for ℓ_1, first, because the strut was assumed to be of zero mass in the derivation of the equations of motion, and second, because the strut was assumed to be of infinite rigidity. A long strut would present structural problems in rigidity since it is essentially a column acting under both axial and bending loads. Restricting ℓ_2 to values of the order of a few feet it is clear that α should not exceed 100 or 200 at most, since these values correspond to an ℓ_1 of 25 to 100 feet.

In view of the foregoing remarks consider the product of the two variables α and β. Thus $\alpha\beta = \frac{\ell_1}{\ell_2}\frac{\ell_2}{k_2} = \frac{\ell_1}{k_2}$. If k_2 is of the order of 1 to 10 feet, say, and ℓ_1 is restricted to 25 to 100 feet, then it can be seen that the product $\alpha\beta$ should not exceed 25. Returning to Figure 12 note that, in the range $1.62 < \beta < 4.54 \times 10^4$, $\alpha\beta$ is approximately constant and equal to about 10^5. Thus, even though theoretically stable, this system is of no practical significance for the mass ratio investigated.

V. CONCLUSIONS

1. A compound pendulum system of launching stabilization is not stable for a mass ratio 7.75.

2. Such a system is theoretically stable for a mass ratio 1.5 if the effect of lateral motion is neglected. However, even in this restricted case the system is of no practical significance due to the excessive length requirements for the strut from the main rocket to the booster.

3. Since the system is quite sensitive to the particular parameters chosen, several more investigations of specific cases are required before any general conclusions can be drawn as to the practicality of this method of stabilization.

TABLE I

Data from Equations (73), (74), and (75).

($2.54 \times 10^{-4} < \beta < .455$)

β	2.54×10^{-4}	10^{-3}	3×10^{-3}	0.01	0.04	0.05	0.1	0.2	0.3	0.4	0.455
a_1	-1.91×10^{-12}	-1.91×10^{-12}	-1.91×10^{-12}	-1.91×10^{-12}	-1.91×10^{-12}	-1.91×10^{-12}	-1.91×10^{-12}	-1.91×10^{-12}	-1.91×10^{-12}	-1.91×10^{-12}	-1.91×10^{-12}
c_1	5.76×10^{-2}	3.71×10^{-3}	4.1×10^{-4}	3.65×10^{-5}	2.17×10^{-6}	1.38×10^{-6}	3.05×10^{-7}	5.9×10^{-8}	1.68×10^{-8}	4.2×10^{-9}	2.1×10^{-10}
α_{max}	1.73×10^{5}	4.42×10^{4}	1.46×10^{4}	4.42×10^{3}	1.06×10^{3}	850	400	172	86.4	47	10
α_{min}	0	0	0	0	0	0	0	0	0	0	0
a_2	-1.13×10^{-18}	-1.76×10^{-17}	-1.58×10^{-16}	-1.76×10^{-15}	-2.82×10^{-14}	-4.41×10^{-14}	-1.76×10^{-13}	-7.05×10^{-13}	-1.59×10^{-12}	-2.82×10^{-12}	-3.67×10^{-12}
b_2	1.08×10^{-7}	4.63×10^{-9}	1.44×10^{-8}	4.74×10^{-8}	1.90×10^{-7}	2.38×10^{-7}	4.76×10^{-7}	9.52×10^{-7}	1.32×10^{-6}	1.9×10^{-6}	2.16×10^{-6}
c_2	-2.1×10^{-2}	-5.33×10^{-3}	-1.78×10^{-3}	-5.33×10^{-4}	-1.33×10^{-4}	-1.06×10^{-4}	-5.38×10^{-5}	-2.75×10^{-5}	-1.9×10^{-5}	-1.51×10^{-5}	-1.39×10^{-5}
α_{max}	9.37×10^{8}	2.7×10^{8}	8.92×10^{7}	2.69×10^{7}	6.76×10^{6}	5.4×10^{6}	2.7×10^{6}	1.35×10^{6}	8.4×10^{5}	6.74×10^{5}	5.88×10^{5}
α_{min}	2.07×10^{7}	1.12×10^{6}	1.25×10^{5}	1.12×10^{4}	7.02×10^{2}	444	113	29	14	8	6.4
a_3	1.95×10^{-17}	3.02×10^{-16}	2.72×10^{-15}	3.02×10^{-14}	4.84×10^{-13}	7.57×10^{-13}	3.02×10^{-12}	1.2×10^{-11}	2.72×10^{-11}	4.83×10^{-11}	6.27×10^{-11}
b_3	-2.06×10^{-9}	-8.12×10^{-8}	-2.43×10^{-7}	-8.12×10^{-7}	-3.25×10^{-6}	-4.06×10^{-6}	-8.12×10^{-6}	-1.62×10^{-5}	-2.43×10^{-5}	-3.24×10^{-5}	-3.7×10^{-5}
c_3	-2.83×10^{-1}	-7.17×10^{-2}	-2.37×10^{-2}	-7.06×10^{-3}	-1.67×10^{-3}	-1.31×10^{-3}	-5.91×10^{-4}	-2.24×10^{-4}	-9.3×10^{-5}	-2.81×10^{-5}	1.5×10^{-6}
α_{max}	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	.040, ∞
α_{min}	1.07×10^{9}	2.69×10^{8}	8.97×10^{7}	2.7×10^{7}	6.72×10^{6}	5.46×10^{6}	2.7×10^{6}	1.35×10^{6}	8.94×10^{5}	6.7×10^{5}	0, 5.89×10^{5}

TABLE II
Data from Equations (73), (74), and (75)
($-1.91 < \beta < 0$)

β	-1.90	-1.4	-1	-.8	-.4	-.1	-.01	-10^{-3}	-10^{-4}	-10^{-5}	-10^{-6}	-10^{-7}	-10^{-8}	-10^{-9}
a_1	-1.91×10^{-12}	-1.91×10^{-12}	-1.91×10^{-12}	-1.91×10^{-12}	-1.91×10^{-12}	-1.91×10^{-12}	-1.91×10^{-12}	-1.91×10^{-12}	-1.91×10^{-12}	-1.91×10^{-12}	-1.91×10^{-12}	-1.91×10^{-12}	-1.91×10^{-12}	-1.91×10^{-12}
c_1	6.9×10^{-11}	2.07×10^{-9}	5.66×10^{-9}	9.28×10^{-9}	3.43×10^{-8}	4.28×10^{-7}	3.77×10^{-5}	3.71×10^{-3}	3.71×10^{-1}	3.71×10^{1}	3.71×10^{3}	3.71×10^{5}	3.71×10^{7}	3.71×10^{9}
α_{max}	-6.8	-32.4	-53.4	-69	-1.34×10^{2}	-4.74×10^{2}	-4.44×10^{3}	-4.42×10^{4}	-4.42×10^{5}	-4.42×10^{6}	-4.42×10^{7}	-4.42×10^{8}	-4.42×10^{9}	-4.42×10^{10}
α_{min}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a_2	-6.35×10^{-11}	-3.45×10^{-11}	-1.76×10^{-11}	-1.12×10^{-11}	-2.82×10^{-12}	-1.76×10^{-13}	-1.76×10^{-15}	-1.76×10^{-17}	-1.76×10^{-19}	-1.76×10^{-21}	-1.76×10^{-23}	-1.76×10^{-25}	-1.76×10^{-27}	-1.76×10^{-29}
b_2	-9.04×10^{-6}	-6.65×10^{-6}	-4.76×10^{-6}	-3.80×10^{-6}	-1.90×10^{-6}	-4.76×10^{-7}	-4.79×10^{-8}	-4.88×10^{-9}	-5.97×10^{-10}	-1.68×10^{-10}	-1.26×10^{-10}	-1.21×10^{-10}	-1.21×10^{-10}	-1.21×10^{-10}
c_2	1.18×10^{-5}	1.04×10^{-5}	1.00×10^{-5}	1.04×10^{-5}	1.52×10^{-5}	5.37×10^{-5}	5.33×10^{-4}	5.33×10^{-3}	5.33×10^{-2}	5.33×10^{-1}	5.33	5.33×10^{1}	5.33×10^{2}	5.33×10^{3}
α_{max}	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
α_{min}	-1.42×10^{5}	-1.92×10^{5}	-2.7×10^{5}	-3.38×10^{5}	-6.74×10^{5}	-2.7×10^{6}	-2.7×10^{7}	-2.78×10^{8}	-3.38×10^{9}	-9.57×10^{10}	-7.15×10^{12}	-6.9×10^{14}	-6.88×10^{16}	-6.88×10^{18}
a_3	1.09×10^{-9}	5.93×10^{-10}	3.02×10^{-10}	9.68×10^{-11}	4.84×10^{-11}	3.02×10^{-12}	3.02×10^{-14}	3.02×10^{-16}	3.02×10^{-18}	3.02×10^{-20}				
b_3	1.54×10^{-4}	1.13×10^{-4}	8.12×10^{-5}	6.5×10^{-5}	3.25×10^{-5}	8.12×10^{-6}	8.12×10^{-7}	8.12×10^{-8}	8.12×10^{-9}	8.12×10^{-10}				
c_3	2.5×10^{-6}	5.66×10^{-5}	1.10×10^{-4}	2.09×10^{-4}	2.98×10^{-4}	8.37×10^{-4}	7.3×10^{-3}	7.19×10^{-2}	7.18×10^{-1}	7.18				
α_{max}	-1.42×10^{5}	-1.92×10^{5}	-2.68×10^{5}	-6.71×10^{5}	-6.72×10^{5}	-2.63×10^{6}	-2.63×10^{7}	-2.69×10^{8}	-2.6×10^{9}	Imag		Imaginary		
α_{min}	-.02	-.5	-1.36	-3.2	-9.2	-1.03×10^{2}	-8.98×10^{3}	-8.95×10^{5}	-9.02×10^{7}	Imag				

TABLE III
Data from Equations (77), (78), and (79)
($\beta > 2.55 \times 10^{-5}$)

β	10^{-5}	10^{-3}	10^{-2}	10^{-1}	.68	1	2	10	10^{2}	10^{5}
a_1	2.49×10^{-11}	2.49×10^{-11}	2.49×10^{-11}	2.49×10^{-11}	2.49×10^{-11}	2.49×10^{-11}	2.49×10^{-11}	2.49×10^{-11}	2.49×10^{-11}	2.49×10^{-11}
c_1	1.08×10^{1}	1.08×10^{-3}	1.08×10^{-5}	9.55×10^{-8}	5.60×10^{-11}	-7.00×10^{-10}	-8.00×10^{-10}	-6.72×10^{-10}	-5.74×10^{-10}	-5.62×10^{-10}
α_{max}	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
α_{min}	0	0	0	0	0	5.68	5.67	5.2	4.8	4.75
a_2	-2.22×10^{-20}	-2.22×10^{-16}	-2.22×10^{-14}	-2.22×10^{-12}	-1.02×10^{-10}	-2.22×10^{-10}	-8.86×10^{-10}	-2.22×10^{-8}	-2.22×10^{-6}	-2.22
b_2	-3.88×10^{-16}	2.42×10^{-8}	2.47×10^{-7}	2.48×10^{-6}	1.68×10^{-5}	2.48×10^{-5}	4.96×10^{-5}	2.48×10^{-4}	2.48×10^{-3}	6.91
c_2	-4.13	-4.13×10^{-2}	-4.13×10^{-3}	-4.13×10^{-4}	-7.76×10^{-5}	-6.61×10^{-5}	-7.02×10^{-5}	-2.52×10^{-4}	-2.48×10^{-3}	-4.69
α_{max}	Imag	1.08×10^{8}	1.12×10^{7}	1.12×10^{6}	1.65×10^{5}	1.12×10^{5}	5.58×10^{4}	1.12×10^{4}	1.12×10^{3}	2.11
α_{min}		1.75×10^{6}	1.67×10^{4}	1.67×10^{2}	4.64	2.68	1.41	1.01	1+	1+
a_3	7.32×10^{-20}	7.32×10^{-16}	7.32×10^{-14}	7.32×10^{-12}	3.38×10^{-10}	7.32×10^{-10}	2.93×10^{-9}	7.32×10^{-8}	7.32×10^{-6}	7.32
b_3	-8.22×10^{-10}	-8.22×10^{-8}	-8.22×10^{-7}	-8.22×10^{-6}	-5.6×10^{-5}	-8.22×10^{-5}	-1.64×10^{-4}	-8.22×10^{-4}	-8.22×10^{-3}	-2.28×10^{1}
c_3	-1.57×10^{1}	-1.57×10^{-1}	1.56×10^{-2}	-1.39×10^{-3}	2.10×10^{-6}	1.07×10^{-4}	9.15×10^{-4}	9.83×10^{-4}	8.80×10^{-3}	1.55×10^{1}
α_{max}	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
α_{min}	2.13×10^{10}	1.12×10^{8}	1.12×10^{7}	1.12×10^{6}	1.65×10^{5}	1.12×10^{5}	5.58×10^{4}	1.12×10^{4}	1.12×10^{3}	3.02
α'_{max}					3.75×10	1.3	1.3	1.2	1.02	0.01
α'_{min}					0	0	0	0	0	0

TABLE IV

Data from Equations (77), (78), and (79)

($\beta < 0$)

β	-10^{-5}	-10^{-3}	-10^{-1}	-2	-2.8	-10	-10^{2}	-10^{3}
a_1	2.49×10^{-11}	2.49×10^{-11}	2.49×10^{-11}	2.49×10^{-11}	2.49×10^{-11}	2.49×10^{-11}	2.49×10^{-11}	2.49×10^{-11}
c_1	1.08×10^{1}	1.08×10^{-3}	1.16×10^{-8}	3.15×10^{-10}	-8.67×10^{-10}	-4.4×10^{-10}	-5.50×10^{-10}	-5.60×10^{-10}
α_{max}	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
α_{min}	0	0	0	3.36	-5.9	-4.21	-4.70	-4.75
a_2	-2.2×10^{-20}	-2.2×10^{-16}	-2.2×10^{-12}	-8.86×10^{-10}	-1.74×10^{-9}	-2.2×10^{-8}	-2.2×10^{-6}	-2.2×10^{-4}
b_2	-8.84×10^{-10}	-2.54×10^{-8}	-2.48×10^{-6}	-4.96×10^{-5}	-6.94×10^{-5}	-2.48×10^{-4}	-2.48×10^{-3}	-2.44×10^{-2}
c_2	4.13	4.13×10^{2}	4.16×10^{-4}	7.02×10^{-5}	8.42×10^{-5}	2.44×10^{-4}	2.48×10^{-3}	2.46×10^{-2}
α_{max}	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
α_{min}	-4.38×10^{10}	-1.15×10^{8}	-1.12×10^{6}	-5.60×10^{4}	-4.00×10^{4}	-1.12×10^{4}	-1.12×10^{3}	-1.12×10^{2}
a_3	7.32×10^{-20}	7.32×10^{-16}	7.32×10^{-12}	2.93×10^{-9}	5.73×10^{-9}	7.32×10^{-8}	7.32×10^{-6}	7.32×10^{-4}
b_3	8.22×10^{-10}	8.22×10^{-8}	8.22×10^{-6}	1.64×10^{-4}	2.30×10^{-4}	8.22×10^{-4}	8.21×10^{-3}	8.08×10^{-2}
c_3	1.57×10^{1}	1.57×10^{-1}	1.75×10^{-3}	9.15×10^{-4}	3.50×10^{-6}	-6.29×10^{-4}	-8.04×10^{-3}	-8.13×10^{-2}
α_{max}	Imag	-1.10×10^{8}	-1.12×10^{6}	-5.60×10^{4}	-4.00×10^{4}	-1.12×10^{4}	-1.12×10^{3}	-1.12×10^{2}
α_{min}		-1.98×10^{6}	-2.12×10^{2}	5.57	-1.52×10^{-2}	0	0	0

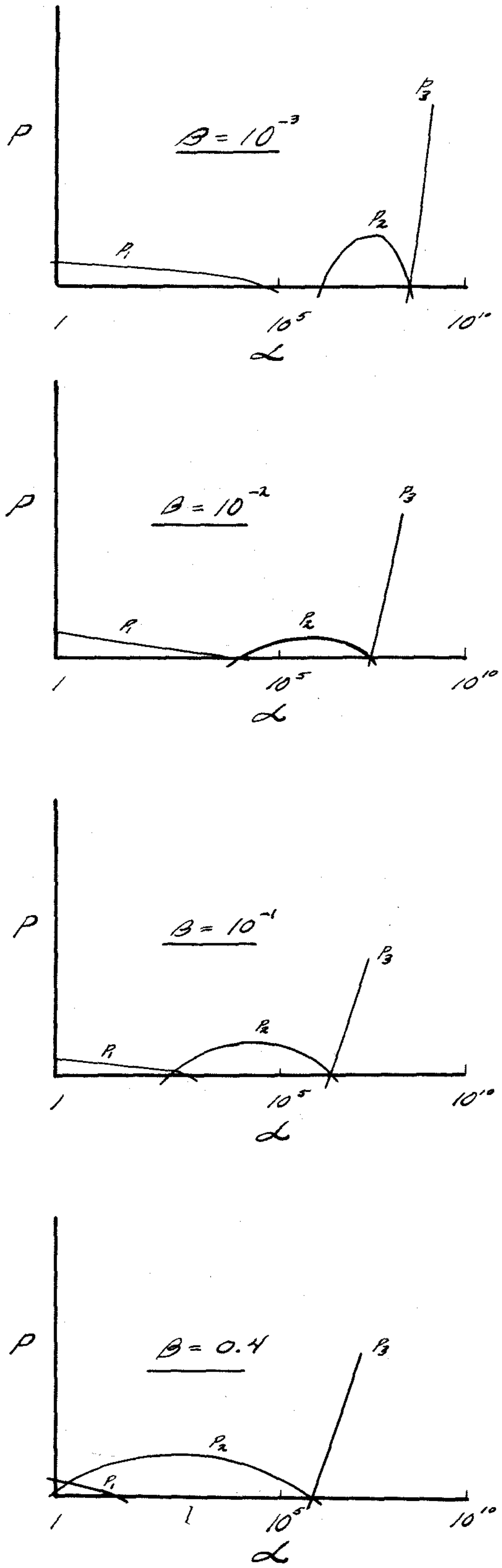


Figure 6. Variation of p_1, p_2, and p_3 with α for several values of positive β. ($\frac{M_1}{M_2} = 7.75$)

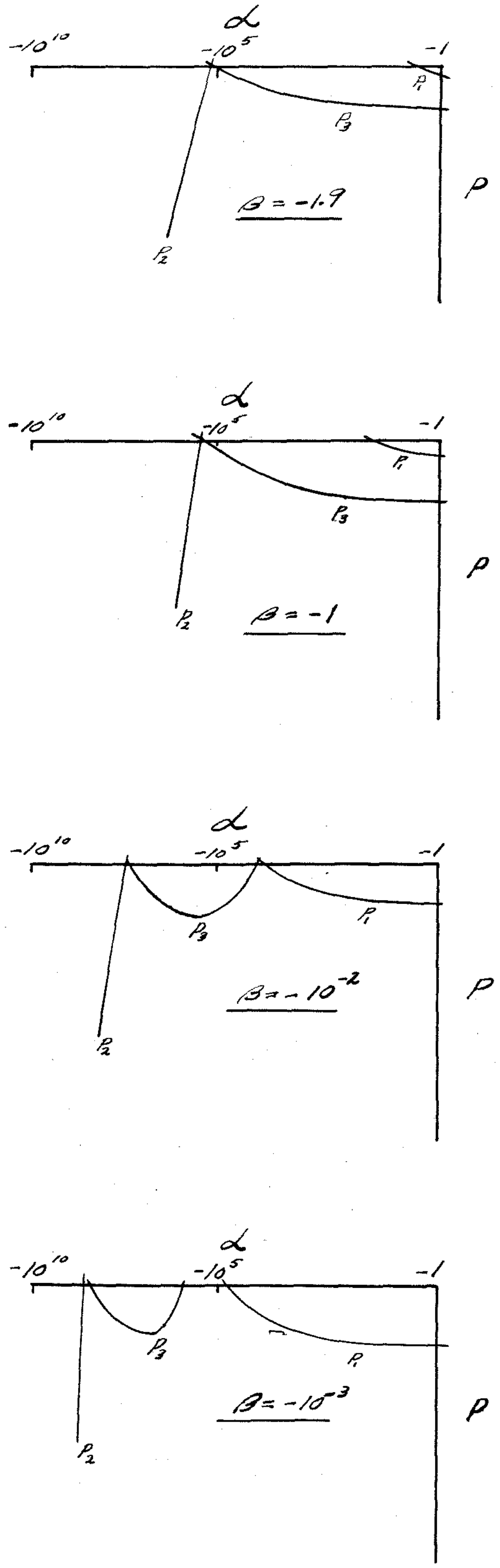


Figure 7. Variation of p_1 , p_2 , and p_3 with α for several values of negative β . ($\frac{M_1}{M_2}$ = 7.75)

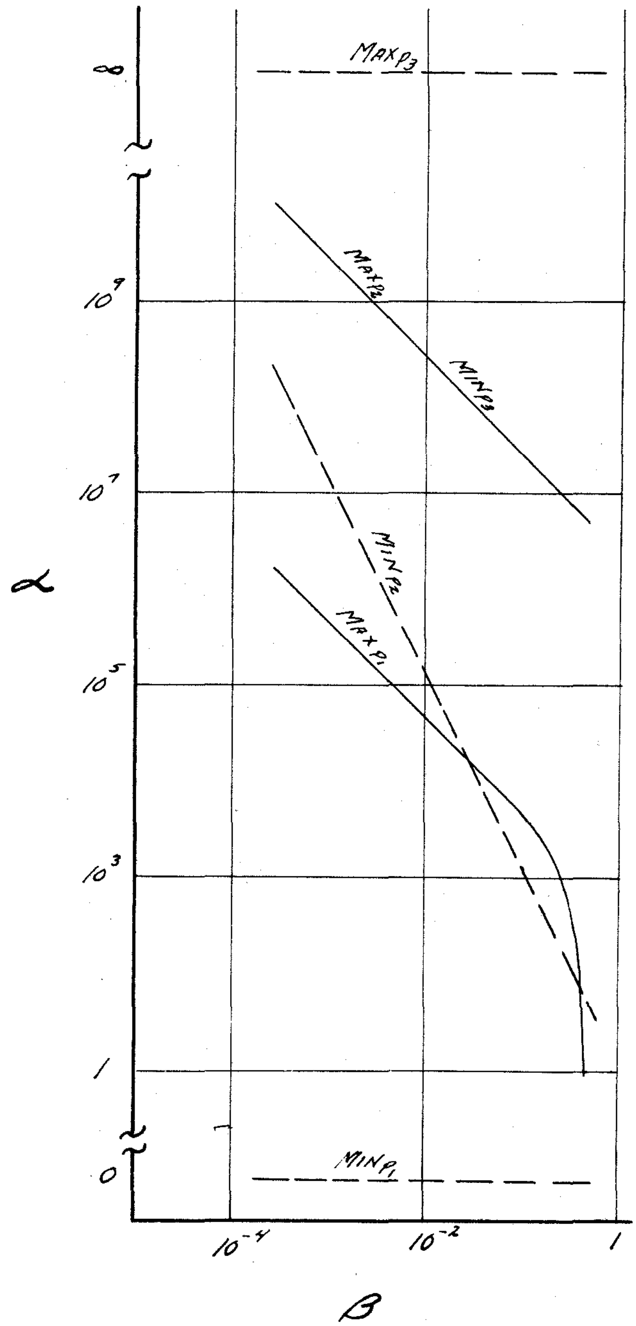


Figure 8. Minimum and maximum limits of α which yield positive p_i for positive β. ($\frac{M_1}{M_2} = 7.75$)

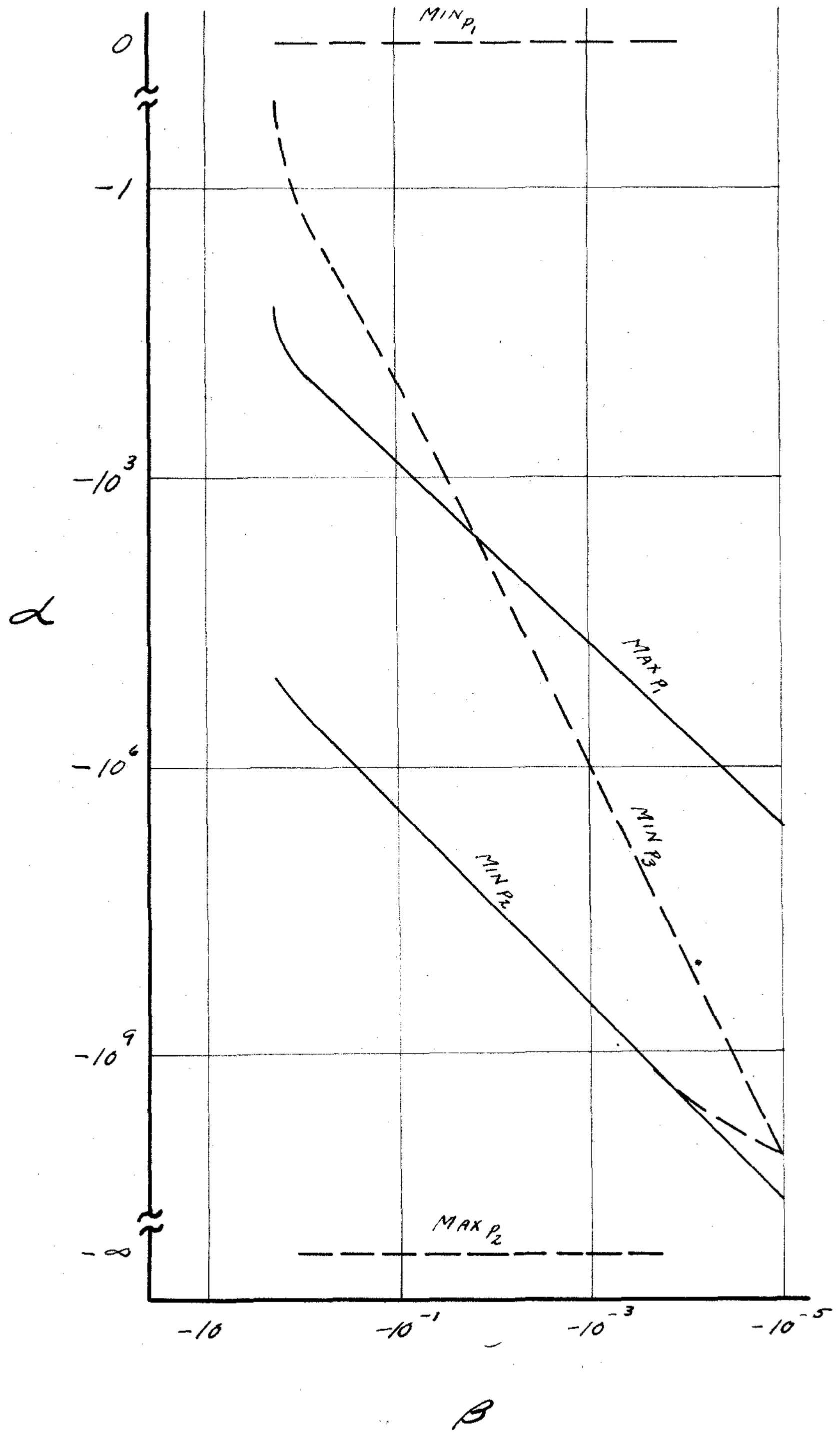


Figure 9. Minimum and maximum limits of α which yield negative p_i for negative β. ($\frac{M_1}{M_2} = 7.75$)

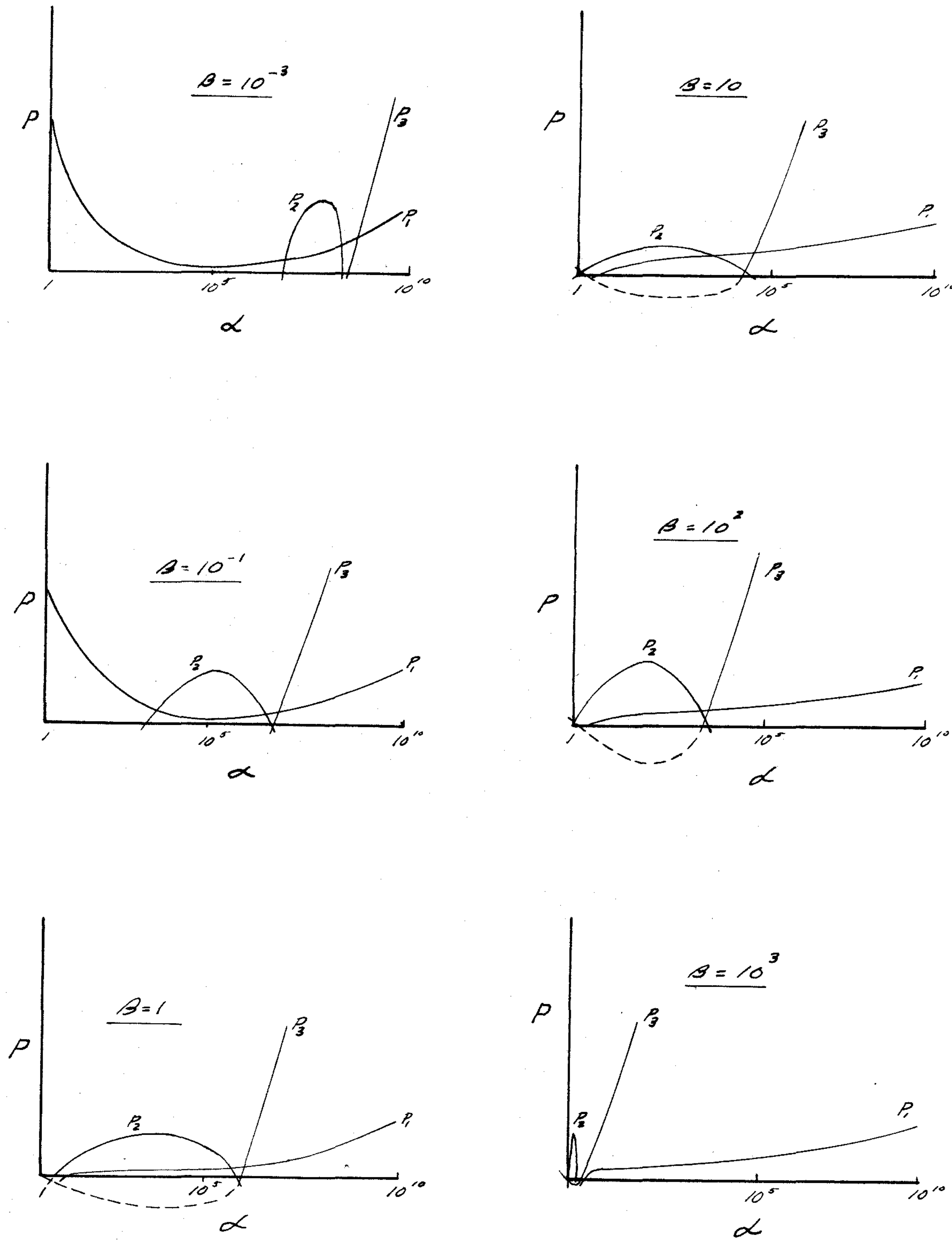


Figure 10. Variation of p_1, p_2, and p_3 with α for several values of positive β. ($\frac{M_1}{M_2} = 1.5$)

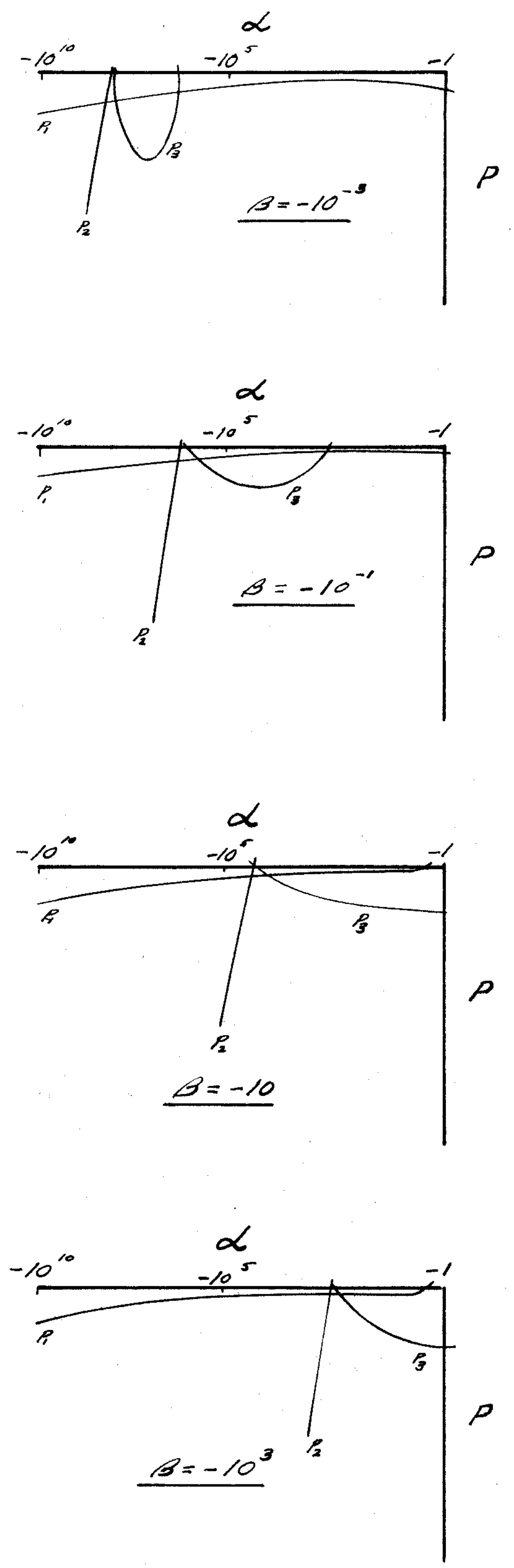


Figure 11. Variation of p_1, p_2, and p_3 with α for several values of negative β. ($\frac{M_1}{M_2} = 1.5$)

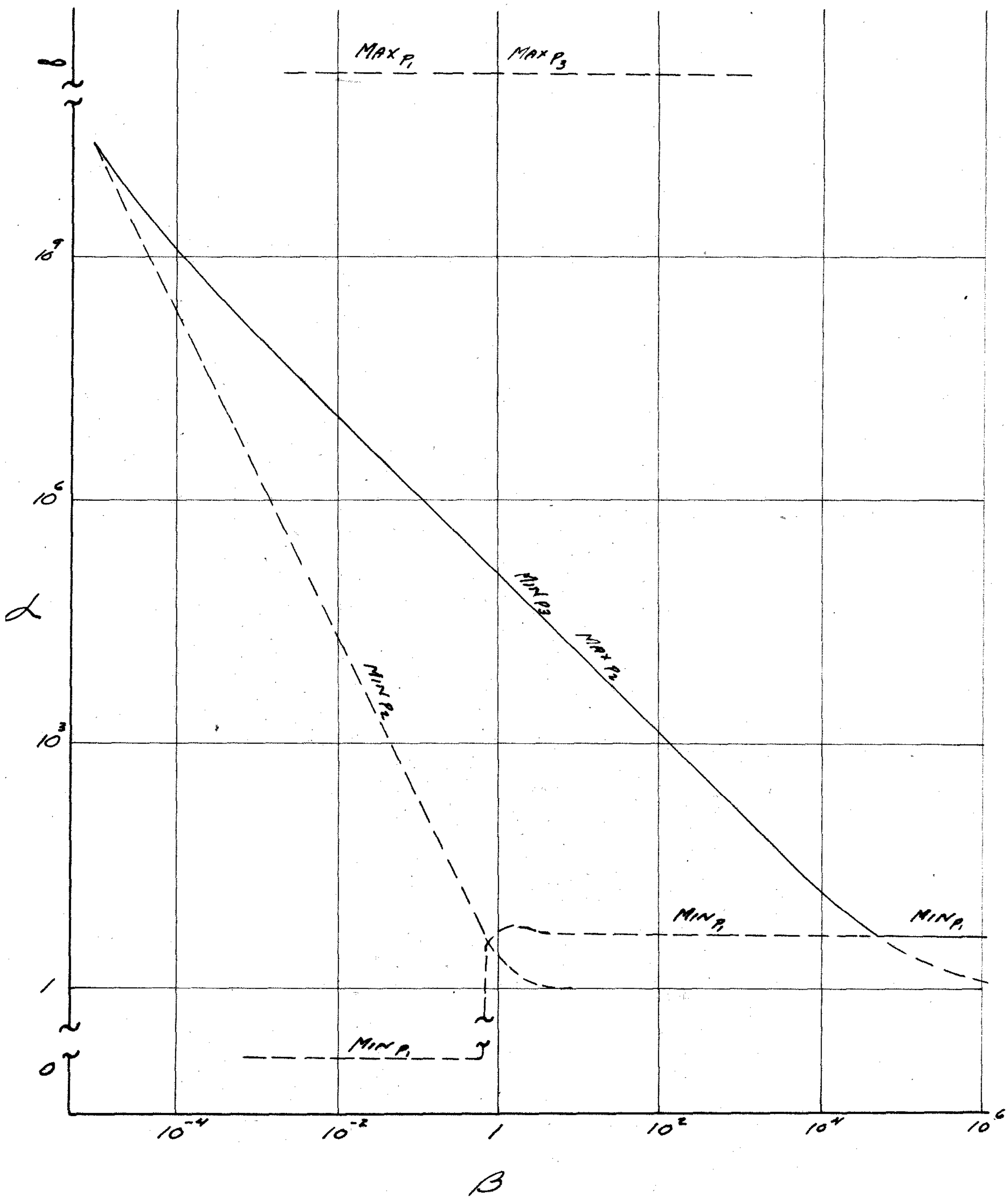


Figure 12. Minimum and maximum limits of α which yield positive p_i for positive β. ($\frac{M_1}{M_2} = 1.5$)

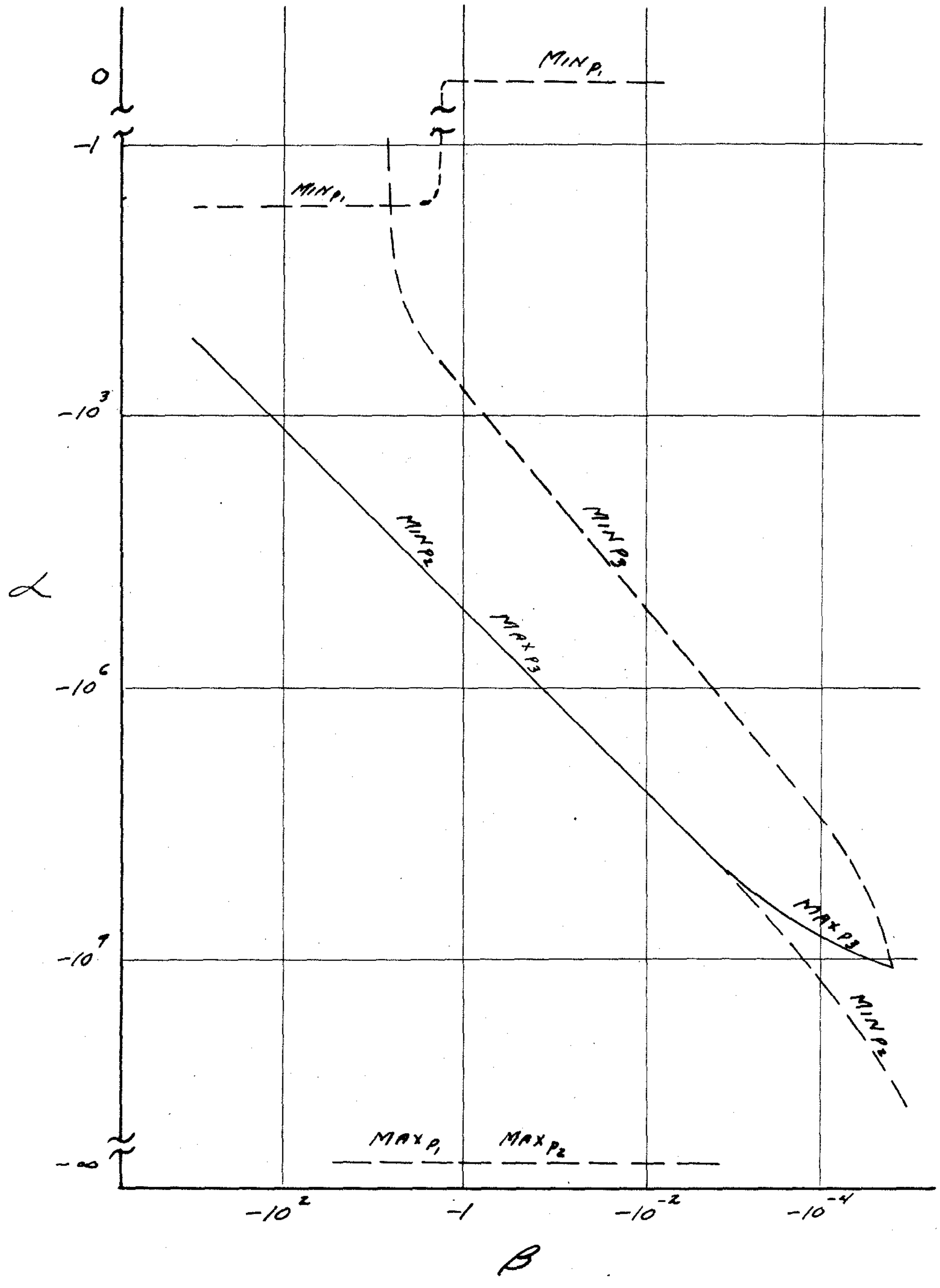


Figure 13. Minimum and maximum limits of α which yield negative p_i for negative β. ($\frac{M_1}{M_2} = 1.5$)

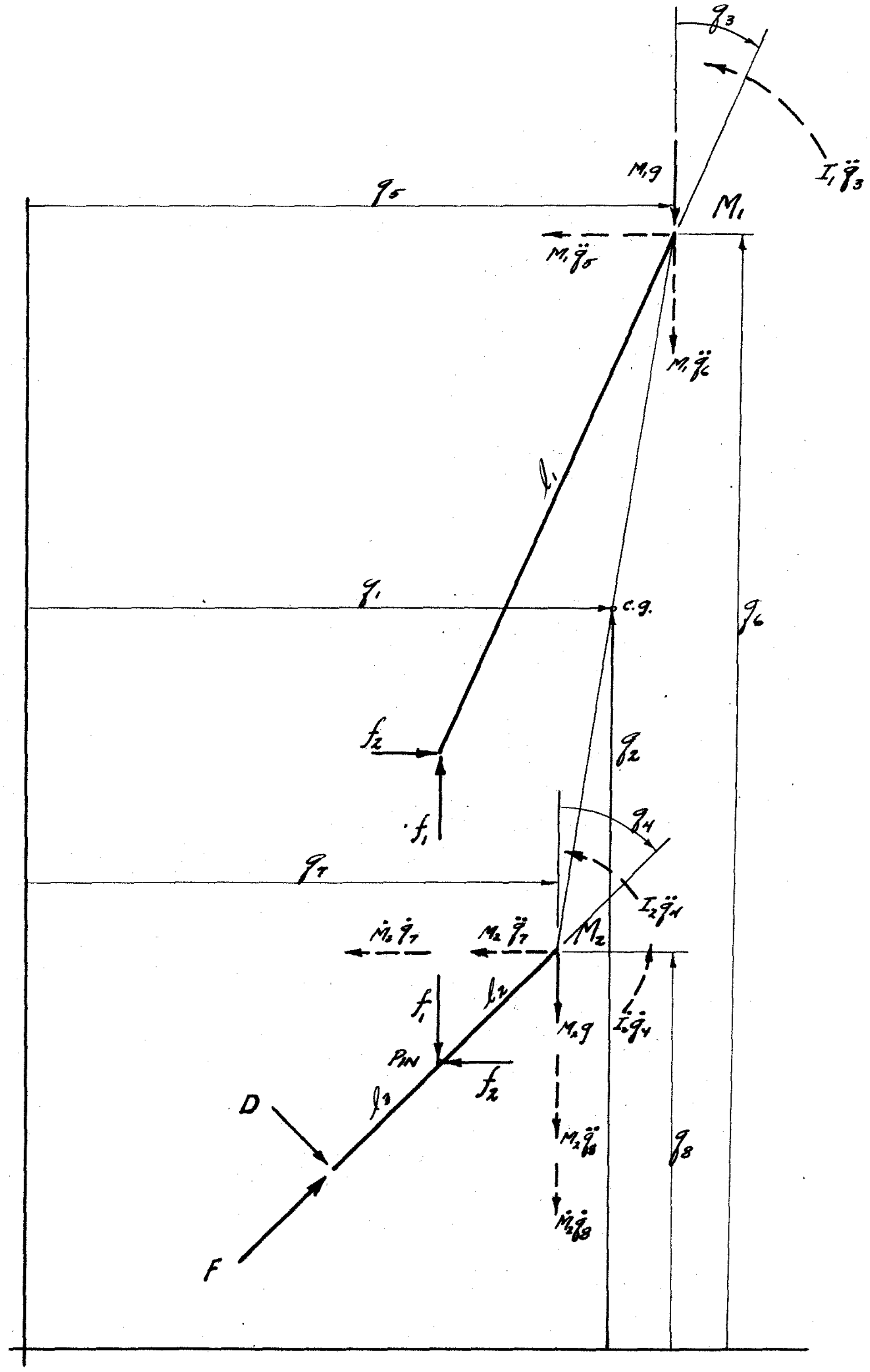


Figure 14. Schematic diagram of system showing inertia forces for use in Newtonian derivation of equations of motion.

APPENDIX

NEWTONIAN DERIVATION OF EQUATIONS OF MOTION

The equations of motion for the two dimensional system may be derived by applying Newton's Second Law of Motion provided that proper consideration is given to the inertia forces of the system. In Figure 14 these forces are indicated by broken lines. In this derivation the main rocket, M_1 , and the booster, M_2 , will be treated separately as free bodies and the reaction of the pin connection will be replaced in Figure 14 by its two components, f_1 in the vertical direction, and f_2 in the horizontal direction. The pin is assumed to be frictionless and therefore no moment is transmitted.

Treating the main rocket, M_1 , as a free body first, the sum of the vertical forces acting on it are

$$\Sigma V = f_1 - M_1 g - M_1 \ddot{q}_6$$

and therefore

$$f_1 = M_1 g + M_1 \ddot{q}_6 \qquad (1)$$

Similarly, the sum of the horizontal forces acting on M_2 are

$$\Sigma H = f_2 - M_1 \ddot{q}_5$$

and therefore

$$f_2 = M_1 \ddot{q}_5 \qquad (2)$$

Finally, the sum of the moments acting on M_1 are

$$\Sigma M = I_1\ddot{q}_3 + f_2 l_1 \cos q_3 - f_1 l_1 \sin q_3$$

or

$$I_1\ddot{q}_3 = f_1 l_1 \sin q_3 - f_2 l_1 \cos q_3 \tag{3}$$

Treating the booster, M_2, as a free body, the sum of the vertical forces acting on it are

$$\Sigma V = F\cos q_4 - D\sin q_4 - f_1 - M_2 g - M_2\ddot{q}_8 - \dot{M}_2\dot{q}_8$$

and therefore

$$f_1 = F\cos q_4 - D\sin q_4 - M_2 g - M_2\ddot{q}_8 - \dot{M}_2\dot{q}_8 \tag{4}$$

Furthermore, the sum of the horizontal forces acting on M_2 are

$$\Sigma H = F\sin q_4 + D\cos q_4 - f_2 - M_2\ddot{q}_7 - \dot{M}_2\dot{q}_7$$

and therefore

$$f_2 = F\sin q_4 + D\cos q_4 - M_2\ddot{q}_7 - \dot{M}_2\dot{q}_7 \tag{5}$$

Finally, the sum of the moments acting on M_2 are

$$\Sigma M = \dot{I}_2\dot{q}_4 + I_2\ddot{q}_4 + f_1 l_2 \sin q_4 - f_2 l_2 \cos q_4 + D(l_2 + l_3)$$

or

$$\dot{I}_2\dot{q}_4 + I_2\ddot{q}_4 = f_2 l_2 \cos q_4 - f_1 l_2 \sin q_4 - D(l_2 + l_3) \tag{6}$$

Substituting the results of Equation (1) into Equation (4) to eliminate f_1

$$M_1 g + M_1\ddot{q}_6 = F\cos q_4 - D\sin q_4 - M_2 g - M_2\ddot{q}_8 - \dot{M}_2\dot{q}_8 \tag{7}$$

Similarly, substituting the results of Equation (2) into Equation (5) to eliminate f_2

$$M_1\ddot{q}_5 = F\sin q_4 + D\cos q_4 - M_2\ddot{q}_7 - \dot{M}_2\dot{q}_7 \tag{8}$$

Substituting the results of Equations (4) and (5) into Equation (3) to eliminate f_1 and f_2

$$I_1\ddot{q}_3 = (F\cos q_4 - D\sin q_4 - M_2 g - M_2\ddot{q}_8 - \dot{M}_2\dot{q}_8)\, l_1 \sin q_3$$
$$- (F\sin q_4 + D\cos q_4 - \dot{M}_2\dot{q}_7)\, l_1 \cos q_3 \qquad (9)$$

Similarly, substituting the results of Equations (4) and (5) into Equation (6) to eliminate f_1 and f_2

$$I_2\ddot{q}_4 + \dot{I}_2\dot{q}_4 = (F\sin q_4 + D\cos q_4 - M_2\ddot{q}_7 - \dot{M}_2\dot{q}_7)\, l_2 \cos q_4$$
$$- (F\cos q_4 - D\sin q_4 - M_2 g - M_2\ddot{q}_8 - \dot{M}_2\dot{q}_8)\, l_2 \sin q_4$$
$$- D(l_2 + l_3) \qquad (10)$$

Assuming the angles q_3 and q_4 sufficiently small such that terms of order q^2 may be neglected, the following approximations can be made

$$\sin q_i \approx q_i \quad \text{and} \quad \cos q_i \approx 1$$

Therefore, Equations (7), (8), (9), and (10) may be rewritten in simpler form. Thus

From Equation (7)

$$M_1\ddot{q}_6 = F - Dq_4 - (M_1 + M_2)g - M_2\ddot{q}_8 - \dot{M}_2\dot{q}_8 \qquad (11)$$

From Equation (8)

$$M_1\ddot{q}_5 = Fq_4 + D - M_2\ddot{q}_7 - \dot{M}_2\dot{q}_7 \qquad (12)$$

From Equation (9)

$$I_1\ddot{q}_3 = (F - Dq_4 - M_2 g - M_2\ddot{q}_8 - \dot{M}_2\dot{q}_8)\, l_1 q_3$$
$$- (Fq_4 + D - M_2\ddot{q}_7 - \dot{M}_2\dot{q}_7)\, l_1 \qquad (13)$$

From Equation (10)

$$I_2\ddot{q}_4 + \dot{I}_2\dot{q}_4 = (Fq_4 + D - M_2\ddot{q}_7 - \dot{M}_2\dot{q}_7)l_2$$
$$-(F - Dq_4 - M_2 g - M_2\ddot{q}_8 - \dot{M}_2\dot{q}_8)l_2 q_4 - D(l_2 + l_3) \qquad (14)$$

The following approximate relations between the principal coordinates q_1, q_2, q_3, and q_4, and the extraneous coordinates q_5, q_6, q_7, q_8 were proved earlier in Part I of this paper

$$q_5 \approx q_1 + \frac{M_2}{M_1+M_2}(l_1 q_3 - l_2 q_4) \qquad (15)$$

$$q_6 \approx q_2 + \frac{M_2}{M_1+M_2}(l_1 - l_2) \qquad (16)$$

$$q_7 \approx q_1 - \frac{M_1}{M_1+M_2}(l_1 q_3 - l_2 q_4) \qquad (17)$$

$$q_8 \approx q_2 - \frac{M_1}{M_1+M_2}(l_1 - l_2) \qquad (18)$$

Taking the first and second time derivatives of Equation (15)

$$\dot{q}_5 = \dot{q}_1 + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}(l_1 q_3 - l_2 q_4) + \frac{M_2}{M_1+M_2}(l_1\dot{q}_3 - l_2\dot{q}_4) \qquad (19)$$

and

$$\ddot{q}_5 = \ddot{q}_1 - 2\frac{M_1\dot{M}_2^2}{(M_1+M_2)^3}(l_1 q_3 - l_2 q_4) + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}(l_1\dot{q}_3 - l_2\dot{q}_4)$$
$$+ \frac{M_1\dot{M}_2}{(M_1+M_2)^2}(l_1\dot{q}_3 - l_2\dot{q}_4) + \frac{M_2}{M_1+M_2}(l_1\ddot{q}_3 - l_2\ddot{q}_4) \qquad (20)$$

Taking the first and second time derivatives of Equation (16)

$$\dot{q}_6 = \dot{q}_2 + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}(l_1 - l_2) \qquad (21)$$

and

$$\ddot{q}_6 = \ddot{q}_2 - 2\frac{M_1\dot{M}_2^2}{(M_1+M_2)^3}(l_1 - l_2) \qquad (22)$$

Taking the first and second time derivatives of Equation (17)

$$\dot{q}_7 = \dot{q}_1 + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}(l_1 q_3 - l_2 q_4) - \frac{M_1}{M_1+M_2}(l_1\dot{q}_3 - l_2\dot{q}_4) \qquad (23)$$

and

$$\ddot{q}_7 = \ddot{q}_1 - 2\frac{M_1\dot{M}_2^2}{(M_1+M_2)^3}(l_1 q_3 - l_2 q_4) + \frac{M_1\dot{M}_2}{(M_1+M_2)^3}(l_1\dot{q}_3 - l_2\dot{q}_4)$$

$$+ \frac{M_1\dot{M}_2}{(M_1+M_2)^2}(l_1\dot{q}_3 - l_2\dot{q}_4) - \frac{M_1}{M_1+M_2}(l_1\ddot{q}_3 - l_2\ddot{q}_4) \qquad (24)$$

Taking the first and second time derivatives of Equation (18)

$$\dot{q}_8 = \dot{q}_2 + \frac{M_1\dot{M}_2}{(M_1+M_2)^2}(l_1 - l_2) \qquad (25)$$

and

$$\ddot{q}_8 = \ddot{q}_2 - 2\frac{M_1\dot{M}_2^2}{(M_1+M_2)^3}(l_1 - l_2) \qquad (26)$$

Equation (11) can now be rewritten in terms of the principal coordinates only by substituting the results of Equations (22), (25) and (26) for the extraneous coordinates

$$(M_1+M_2)\ddot{q}_2 - \frac{M_1\dot{M}_2^2}{(M_1+M_2)^2}(l_1 - l_2) + \dot{M}_2\dot{q}_2 = F - g(M_1+M_2) - Dq_4 \qquad (27)$$

Note that Equation (27) is identical with Equation (27) of Part II of this paper.

In a similar manner, Equation (12) can be rewritten in terms of the principal coordinates by substituting the results of Equations (20), (23), and (24) for the extraneous coordinates

$$(M_1+M_2)\ddot{q}_1 + \frac{M_1\dot{M}_2}{M_1+M_2}(l_1\dot{q}_3 - l_2\dot{q}_4) - \frac{M_1\dot{M}_2^2}{(M_1+M_2)^2}(l_1 q_3 - l_2 q_4) + \dot{M}_2\dot{q}_1 = Fq_4 + D \qquad (28)$$

which is identical with Equation (23) of Part II of this paper.

Substitution of Equations (23), (24), (25), and (26) in Equation (13) yields

$$\begin{aligned} & I_1\ddot{q}_3 + \frac{M_1^2\dot{M}_2}{(M_1+M_2)^2}\, l_1(l_1\dot{q}_3 - l_2\dot{q}_4) + \frac{M_1M_2}{M_1+M_2}\, l_1(l_1\ddot{q}_3 - l_2\ddot{q}_4) - \frac{M_1\dot{M}_2}{M_1+M_2}\, l_1\dot{q}_1 \\ & - \frac{M_1^2\dot{M}_2^2}{(M_1+M_2)^3}\, l_1(l_1q_3 - l_2q_4) + \frac{M_1\dot{M}_2}{M_1+M_2}\, l_1 q_3\dot{q}_2 + \frac{M_1^2\dot{M}_2^2}{(M_1+M_2)^2}\, l_1 q_3(l_1 - l_2) \\ & = -\frac{M_1}{M_1+M_2}\, l_1\left[F(q_4 - q_3) + D\right] \end{aligned} \tag{29}$$

which is identical with Equation (44) of Part II of this paper.

Finally, substitution of Equations (23), (24), (25), and (26) in Equation (14) yields

$$\begin{aligned} & I_2\ddot{q}_4 + \dot{I}_2\dot{q}_4 - \frac{M_1^2\dot{M}_2}{(M_1+M_2)^2}\, l_2(l_1\dot{q}_3 - l_2\dot{q}_4) - \frac{M_1M_2}{M_1+M_2}\, l_2(l_1\ddot{q}_3 - l_2\ddot{q}_4) \\ & + \frac{M_1\dot{M}_2}{M_1+M_2}\, l_2\dot{q}_1 + \frac{M_1^2\dot{M}_2^2}{(M_1+M_2)^3}\, l_2(l_1q_3 - l_2q_4) - \frac{M_1\dot{M}_2}{M_1+M_2}\, l_2 q_4\dot{q}_2 \\ & - \frac{M_1^2\dot{M}_2^2}{(M_1+M_2)^3}\, l_2 q_4(l_1 - l_2) = -D\left[\frac{M_2}{M_1+M_2}\, l_2 + l_3\right] \end{aligned} \tag{30}$$

which is identical with Equation (59) of Part II of this paper.

TABLE OF SYMBOLS

a - $\frac{\ell_1}{\ell_2}$

β - $\frac{\ell_2}{k_2}$

D - Jet damping force

F - Thrust force

g - Acceleration of gravity

I - Moment of inertia

I_{sp} - Specific Impulse

k - Radius of gyration

M - Mass

$\dot{M}$ - Mass rate of flow

q - Lagrangian generalized space coordinate

Q - Dimensionless Lagrangian generalized space coordinate

$\mathcal{Q}$ - Lagrangian generalized force

T - Kinetic energy

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