MATRICES WHOSE HERMITIAN PART IS POSITIVE DEFINITE

Thesis by

Charles Royal Johnson

In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1972

(Submitted March 31, 1972)

ACKNOWLEDGMENTS

I am most thankful to my adviser Professor Olga Taussky Todd for the inspiration she gave me during my graduate career as well as the painstaking time and effort she lent to this thesis. I am also particularly grateful to Professor Charles De Prima and Professor Ky Fan for the helpful discussions of my work which I had with them at various times.

For their financial support of my graduate tenure I wish to thank the National Science Foundation and Ford Foundation as well as the California Institute of Technology.

It has been important to me that Caltech has been a most pleasant place to work. I have enjoyed living with the men of Fleming House for two years, and in the Department of Mathematics the faculty members have always been generous with their time and the secretaries pleasant to work around.

ABSTRACT

We are concerned with the class Π_n of n×n complex matrices A for which the Hermitian part $H(A) = \frac{A+A^*}{2}$ is positive definite.

Various connections are established with other classes such as the stable, D-stable and dominant diagonal matrices. For instance it is proved that if there exist positive diagonal matrices D, E such that DAE is either row dominant or column dominant and has positive diagonal entries, then there is a positive diagonal F such that $FA \in \Pi_n$.

Powers are investigated and it is found that the only matrices A for which $A^m \in \Pi_n$ for all integers m are the Hermitian elements of Π_n. Products and sums are considered and criteria are developed for AB to be in Π_n.

Since Π_n is closed under inversion, relations between $H(A)^{-1}$ and $H(A^{-1})$ are studied and a dichotomy observed between the real and complex cases. In the real case more can be said and the initial result is that for $A \in \Pi_n$, the difference $H(\text{adj}A) - \text{adj}H(A) \geqslant 0$ always and is > 0 if and only if $S(A) = \frac{A-A^*}{2}$ has more than one pair of conjugate non-zero characteristic roots. This is refined to characterize real c for which $cH(A^{-1}) - H(A)^{-1}$ is positive definite.

The cramped (characteristic roots on an arc of less than 180°) unitary matrices are linked to Π_n and characterized in several ways via products of the form $A^{-1}A^*$.

Classical inequalities for Hermitian positive definite matrices are studied in Π_n and for Hadamard's inequality two types of

generalizations are given. In the first a large subclass of Π_n in which the precise statement of Hadamard's inequality holds is isolated while in another large subclass its reverse is shown to hold. In the second Hadamard's inequality is weakened in such a way that it holds throughout Π_n. Both approaches contain the original Hadamard inequality as a special case.

TABLE OF CONTENTS

INTRODUCTION

The primary object of this thesis is the consideration of the class Π_n of $n \times n$ matrices for which $A + A^*$ is positive definite. They are a natural generalization of the positivity of the well-studied Hermitian positive definite matrices. The class Π_n has been studied in various settings. It arises naturally in operator theory where the term "dissipative" has been used (see, for instance, Phillips [36]). The finite case of Π_n, usually with entries from the real field, also arises naturally in the study of dynamic systems in economics (see Arrow [2] and Quirk [37], [38]). Carlson, Fan, Fiedler, Taussky and others have considered the class from the viewpoint of finite matrix theory. This interest flows in part from the investigation of stability via the Lyapunov Theorem (see (0.13)).

The approach of this author is to study Π_n as a natural generalization of the Hermitian positive definite matrices and thus as a further weaker generalization of positivity. In this regard certain related classes such as the positive stable matrices are peripherally considered. One goal is to delineate respects in which Π_n is like or unlike the Hermitian case, and, though there are manifest similarities, Π_n is much less well behaved. Many results which are obtained highlight the difference between Π_n and the Hermitian case by degenerating trivially in the Hermitian case.

Our methodology is largely that of finite matrices. However, many proofs are sufficiently formal or dependent upon the field of values that they extend to the operator theoretic setting. Most specific methods of proof are new.

A systematic development of several basic facts used throughout is provided in Chapter 1. Many of these facts are known or are refinements of known results (often with new proofs), but their provision here aids in indicating several different ways of looking at Π_n as well as in making the development of this thesis largely self-contained.

Chapters 2 through 6 each contain independent ideas in the theory of Π_n.

In Chapter 2 the taking of powers of elements of Π_n is studied. The main result is that the only members which remain in Π_n under all integral powers are the Hermitian ones. Here the analogy between Π_n and the complex numbers with positive real parts is quite strong.

Chapter 3 considers products and sums of matrices and develops criteria for their membership in Π_n via the characteristic roots of related matrices.

Next we compare $H(A)^{-1}$ and $H(A^{-1})$, where $H(A) = (A + A^*)/2$, in Chapter 4. A dichotomy is observed between the real and complex cases, and comparisons are drawn in each by separate methods. The real case is apparently the deeper of the two and Theorem (4.1) inspires much of the chapter including refinements and several intriguing consequences. It states that $H(\mathrm{adj}A) - \mathrm{adj}H(A)$ is positive semidefinite for all $A \in \Pi_n$ and is positive definite if and only if $(A - A^*)/2$ has more than one pair of conjugate nonzero roots. In addition the Ostrowski-Taussky inequality is strengthened.

In Chapter 5 a natural relation between Π_n and the cramped unitary matrices via products of the form $A^{-1}A^*$ is presented and two characterizations are given.

Chapter 6 asks to what extent classical inequalities involving the entries of Hermitian positive definite matrices can be extended to Π_n. In the case of Hadamard's inequality (see page 54) two different types of answers are given. The first extends the precise statement of Hadamard's inequality to a large subclass of Π_n which contains the Hermitian elements. In the process an alternate proof of Hadamard's inequality is provided as well as a second subclass of Π_n in which the reverse of Hadamard's inequality holds. The second answer economically weakens the inequality so that it holds throughout Π_n and again the original result is a special case.

CHAPTER 0

PRELIMINARIES

i) Notation, Definitions and Immediate Observations.

ii) Known Theorems of Relevance.

<u>i</u>

(0.1) <u>Definition</u>. $M_n(R)$ and $M_n(C)$ will denote the classes of $n \times n$ matrices over the real and complex fields, respectively.

(0.2) <u>Definition</u>. For $A \in M_n(C)$, denote by $F(A)$ "the field of values of A", $\{(x, Ax) \mid \|x\| = 1, x \in C^n\}$.

(0.3) <u>Definition</u>. Let $H(A) = (A + A^*)/2$ and $S(A) = (A - A^*)/2$ if $A \in M_n(C)$.

(0.4) <u>Observation</u> (Linearity). $H(A)$ is Hermitian, $S(A)$ is skew-Hermitian and $A = H(A) + S(A)$. Also, if $a, b \in R$ and $A, B \in M_n(C)$, then $H(aA + bB) = aH(A) + bH(B)$ and $S(aA + bB) = aS(A) + bS(B)$. $H(A)$ will be called the Hermitian part and $S(A)$ the skew-Hermitian part of A.

(0.5) <u>Definition</u>. Let D_n denote the set of all positive diagonal matrices in $M_n(R)$.

(0.6) <u>Definition</u>. Σ_n will denote the class of all positive definite Hermitian elements of $M_n(C)$.

Discussion of Σ_n including several useful characterizations is given in [26], and this knowledge will be assumed.

(0.7) Definition. Π_n will denote $\{A \in M_n(C) \mid H(A) \in \Sigma_n\}$, the class whose Hermitian parts are positive definite.

The study of Π_n is the primary goal of this thesis. Most frequently the general case of complex entried members of Π_n is considered, but in certain instances it is of use to either specialize or emphasize the real or complex field and in these instances $\Pi_n(R)$ or $\Pi_n(C)$ will be used.

(0.8) Definition. If $A \in M_n(C)$, $\lambda(A)$ will denote an arbitrary characteristic root of A and $\sigma(A)$ will denote the set of all characteristic roots of A.

(0.9) Definition. $SK_n \equiv \{S \in M_n(C) \mid S^* = -S\}$, the "skew-Hermitian" elements of $M_n(C)$.

(0.10) Definition. $L_n \equiv \{A \in M_n(C) \mid \lambda \in \sigma(A) \text{ implies } Re(\lambda) > 0\}$.

(0.11) Definition. $DL_n \equiv \{A \in L_n \mid D \in D_n \text{ implies } DA \in L_n\}$.

Though not the original definition, (0.11) is taken to be the definition of "D-stability" by some authors.

(0.12) Remarks. It is of note that the following sequence of containments is valid.

$$D_n \underset{\neq}{\subset} \Sigma_n \underset{\neq}{\subset} \Pi_n \underset{\neq}{\subset} D_n\Pi_n \underset{\neq}{\subset} DL_n \underset{\neq}{\subset} L_n = \Sigma_n\Pi_n \quad .$$

Thus these sets may be regarded as a sequence of successively weaker generalizations of positivity within $M_n(C)$. Note also that Π_n is just $\Sigma_n \oplus SK_n$ and that H orthogonally projects Π_n onto Σ_n.

ii

We shall assume the following three well-known theorems which are relevant to the study of Π_n.

(0.13) Theorem (Lyapunov). $A \in L_n$ if and only if for each $Q \in \Sigma_n$ there is a $P \in \Sigma_n$ such that

$$H(PA) = Q \quad .$$

Further, if $A \in L_n$, the solution P is unique.

Thus Π_n may be thought of as the set of all $A \in L_n$ such that for some $Q \in \Sigma_n$ the Lyapunov solution is $P = I$.

(0.14) Theorem (Gersgorin). If $A = (a_{ij}) \in M_n(C)$, let

$$r_i(A) = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \qquad \text{and} \qquad c_j(A) = \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}| \quad .$$

Then $\sigma(A)$ is contained in the union of the closed discs whose centers are a_{ii} and radii are $r_i(A)$ and $\sigma(A)$ is contained in the union of the closed discs with centers a_{ii} and radii $c_i(A)$, $i = 1, \ldots, n$.

(0.15) Theorem. If $A \in M_n(C)$, then F(A) is a convex set; $F(U^*AU) = F(A)$ if U is unitary; $\sigma(A) \subset F(A)$ and F(A) is the closed convex hull of $\sigma(A)$ if A is normal. If $A \in \Sigma_n$, then F(A) is a line segment on the positive real axis.

CHAPTER 1

ELEMENTARY FACTS CONCERNING Π_n

i) Preliminary Algebra and Analysis of Π_n.

ii) Geometry of Π_n.

iii) Conditions for Membership in Π_n.

We develop here a number of facts both known and previously unknown which we need to study Π_n.

<u>i</u>

(1.1) <u>Lemma</u>. Suppose $A, B \in M_n(C)$, then

(a) $F(A+B) \subset F(A) + F(B)$;

(b) if $\hat{A}$ is a principal submatrix of A, then $F(\hat{A}) \subset F(A)$; and

(c) $\{Re(c) \mid c \in F(A)\} = F(H(A))$.

<u>Proof</u>: (a) If $c \in F(A+B)$, then $c = (x, (A+B)x)$ for some $||x|| = 1$. Thus $c = (x, Ax) + (x, Bx)$ and since $||x|| = 1$, $c = d + e$ where $d \in F(A)$ and $e \in F(B)$. Therefore $c \in F(A) + F(B)$.

(b) Suppose $c \in F(\hat{A})$ and that $\hat{A}$ is determined by the set of distinct indices $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$. Then $c = (\hat{x}, \hat{A}\hat{x})$ for some $\hat{x} \in c^k$, $||\hat{x}|| = 1$. Construct $x \in c^n$ from $\hat{x}$ by making the jth component of x equal to the ℓth component of $\hat{x}$ if $j = i_\ell$ for some ℓ and making the jth component of x equal to 0 otherwise. Then $||x|| = 1$, and $(x, Ax) = (\hat{x}, \hat{A}\hat{x}) = c$, and $c \in F(A)$.

(c) We show that $\mathrm{Re}(x, Ax) = (x, H(A)x)$ for all $x \in C^n$. We have $(x, H(A)x) = (x, \frac{A+A^*}{2} x) = \frac{1}{2}(x, Ax) + \frac{1}{2}(x, A^*x) = \frac{1}{2}((x, Ax) + (Ax, x))$ $= \frac{1}{2}((x, Ax) + (\overline{x, Ax})) = \mathrm{Re}(x, Ax)$.

(1.2) Field of Values Interpretation of Π_n. $A \in \Pi_n$ if and only if $c \in F(A)$ implies $\mathrm{Re}(c) > 0$.[1]

Proof: The matrix $A \in \Pi_n \Leftrightarrow H(A) \in \Sigma_n \Leftrightarrow F(H(A)) > 0$ since $H(A)$ is Hermitian. We have $F(H(A)) > 0$ if and only if $\mathrm{Re}F(A) > 0$ by (1.1)c. Thus $A \in \Pi_n$ if and only if $\mathrm{Re}F(A) > 0$.

(1.3) Closure of Π_n under Congruences. Suppose $A \in \Pi_n$ and $B \in M_n(C)$ is invertible. Then $B^*AB \in \Pi_n$.

Proof: Since $y = Bx$ is not zero unless x is, we have that $\mathrm{Re}(x, B^*ABx)$ $= \mathrm{Re}(y, Ay) > 0$ if $x \neq 0$ by (1.2). This means $\mathrm{Re}F(B^*AB) > 0$ and yields the conclusion.

The following theorem suggests a class which generalizes Π_n [49] and provides a vehicle for extending some results proven for Π_n.

(1.4) We have $0 \notin F(A)$ if and only if $\exists \theta \in [0, 2\pi)$ such that $e^{i\theta}A \in \Pi_n$.

Proof: If $e^{i\theta}A \in \Pi_n$, then $0 \notin F(A)$ since $0 \notin F(e^{i\theta}A)$ by (1.2). By the convexity of the field of values, $0 \notin F(A)$ means that we may separate 0

---

1. The situation of this theorem is exactly that the convex set $F(A)$ lies strictly to the right of the imaginary axis. It is equivalent to say that $\mathrm{Re}(x, Ax) > 0$ for all $0 \neq x \in C^n$. This occurrence will be abbreviated by "$\mathrm{Re}F(A) > 0$", and if A is Hermitian, by "$F(A) > 0$."

and F(A) by a line. Thus we may rotate F(A) into the positive half-plane which implies the converse.

We now prove some facts to be used repeatedly in later chapters.

(1.5) Observations.

(a) $\Pi_n \subset L_n$

(b) $\Sigma_n \Pi_n \subset L_n$

(c) $A \in \Sigma_n \Sigma_n$ implies A has positive real roots and linear elementary divisors

(d) $A \in \Sigma_n SK_n$ implies $\sigma(A)$ is pure imaginary.

Proof: (a) $A \in \Pi_n$ implies $\mathrm{Re}(\sigma(A)) > 0$ since $\sigma(A) \subset F(A)$. Thus $\mathrm{Re}(\sigma(A)) > 0$ means $A \in L_n$.

(b) Suppose $J \in \Sigma_n$ and $A \in \Pi_n$. Then $J^{\frac{1}{2}}$ exists in Σ_n and $J^{-\frac{1}{2}} JAJ^{\frac{1}{2}} = J^{\frac{1}{2}} AJ^{\frac{1}{2}} \in \Pi_n$ by (1.3). This means JA is similar to an element of Π_n and thus is in L_n by part (a).

(c) Let $A = HJ$ with $H, J \in \Sigma_n$, and it follows that $J^{\frac{1}{2}} AJ^{-\frac{1}{2}} = J^{\frac{1}{2}} HJ^{\frac{1}{2}}$ is Hermitian and in Σ_n by (1.3). Thus $J^{\frac{1}{2}} HJ^{\frac{1}{2}}$ (and, therefore, A) is similar to a real diagonal matrix with positive characteristic roots. This means that A has positive real roots[2] and linear elementary divisors.

(d) We now assume $A = HS$, $H \in \Sigma_n$, $S \in SK_n$, and then $H^{-\frac{1}{2}} AH^{\frac{1}{2}} = H^{\frac{1}{2}} SH^{\frac{1}{2}} \in SK_n$. Therefore by similarity $\sigma(A) = \sigma(H^{\frac{1}{2}} SH^{\frac{1}{2}})$ and is pure imaginary.

---

2. The converse is also known to be valid. See [42] or [53].

(1.6) Closure of Π_n under Inversion. $A \in \Pi_n$ if and only if A^{-1} exists and $A^{-1} \in \Pi_n$.

Proof: By (1.5), $0 \notin \sigma(A)$ and thus A^{-1} exists. To show that $A^{-1} \in \Pi_n$ it suffices to show that $\operatorname{Re}(y, A^{-1}y) > 0$ for any $0 \neq y \in C^n$. But, $\operatorname{Re}(y, A^{-1}y) = \operatorname{Re}(Ax, x)$ if $x = A^{-1}y$. Since $A \in \Pi_n$, $\operatorname{Re}(Ax, x) > 0$ or, equivalently, $\operatorname{Re}(y, A^{-1}y) > 0$ and thus $A^{-1} \in \Pi_n$. Similarly, $A^{-1} \in \Pi_n$ implies $A = (A^{-1})^{-1} \in \Pi_n$ to complete the proof.

That $A \in \Pi_n$ implies $A^{-1} \in \Pi_n$ also follows from the fact that $H(A^{-1}) = (A^{-1})^* H(A) A^{-1}$ and applying (1.3) or via the Lyapunov characterization mentioned in (0.13).

(1.7) $A \in \Pi_n$ if and only if $A^* \in \Pi_n$.

Proof: $H(A) = H(A^*)$.

(1.8) If $A, B \in \Pi_n$ and $c > 0$, then (a) $cA \in \Pi_n$ and (b) $A + B \in \Pi_n$.

Proof: The proof is immediate from (1.2), the field of values interpretation of Π_n.

(1.9) If $A \in \Pi_n$ and $\hat{A}$ is any k×k principal submatrix of A, then $\hat{A} \in \Pi_k$.[3] In particular, $\operatorname{Re} a_{ii} > 0$ for $i = 1, \ldots, n$ if $A = (a_{ij})$.

3. It is well to note that these facts about extraction of submatrices might also be proven using orthogonal projections. The union of the Π_n is closed under orthogonal projections and extraction of submatrices is a special case of an orthogonal projection.

<u>Proof</u>: By (1.1)b, $F(\hat{A}) \subset F(A)$ which means $\operatorname{Re}F(\hat{A}) > 0$. By (1.2) this implies $\hat{A} \in \Pi_k$. The special case of $k = 1$ yields that the diagonal entries have positive real parts.

Fiedler and Pták [20] have noted the following fact using different methods.

(1.10) $A \in \Pi_n(R)$ implies $\det(A) > 0$. Thus all principal minors of A are positive.

<u>Proof</u>: Since $A \in M_n(R)$, $\det(A)$ is real and could be negative only if A had a negative real characteristic root. This possibility is denied by (1.5)a and since $\det(A) \neq 0$ by (1.6), we conclude $\det(A) > 0$. By (1.9) it follows that each principal submatrix of A also has positive determinant.

(1.10.1) <u>Example</u>. Of course (1.10) does not necessarily hold even for Re det(A) if $A \in \Pi_n(C)$ as the following 2×2 example shows. If $A = \begin{pmatrix} 1+i & 0 \\ 0 & 1+2i \end{pmatrix}$, then $A \in \Pi_2$, but $\det A = -1+3i$.

Statement (1.10) is one of the more elementary of several properties which hold for $A \in \Pi_n(R)$ but, interestingly enough, have no clear analog when $A \in \Pi_n(C)$.

In this preliminary section we have found that Π_n is a class of matrices which naturally generalizes the positivity of Σ_n. The classes Π_n are closed under inversion, the taking of the Hermitian adjoint, addition, positive scalar multiplication, congruences, and the extraction of principal submatrices and may be thought of in terms of the position of the field of values. Each of these generalizes a property of Σ_n.

$\underline{\text{ii}}$

Elementary Geometry of Π_n

(1.11) Π_n is an open convex cone in $M_n(C)$ [considered as an n^2-dim vector space with Euclidean norm].

Proof: The convex cone property follows from (1.8) and the openness from the greater than sign of the definition of Π_n.

It is apparent that H defined in (0.3) may be considered as a linear transformation from $M_n(C)$ [considered as a vector space over R] onto the subspace $Q_n(C)$ of $n \times n$ Hermitian complex matrices [considered as a vector space over R]. In this context, the inverse image of an element J in $Q_n(C)$ is just $H^{-1}(J) = \{J + S \mid S \in SK_n\}$. The line $\{J + tS \mid t \in R, S \text{ fixed in } SK_n\}$ in $M_n(C)$ is contained in $H^{-1}(J)$ and thus $H^{-1}(J)$ may be thought of as a union of a class of lines passing through J in $M_n(C)$.

Since Σ_n is a convex cone in $M_n(C)$ just as Π_n is, we may view Π_n as an extension of Σ_n, $\Pi_n = H^{-1}(\Sigma_n)$. The extension is given by unioning all lines passing through points in the cone Σ_n and going in the direction of skew-Hermitian matrices.

$$\Pi_n = \bigcup_{J \in \Sigma_n} \bigcup_{S \in SK_n} \{J + tS \mid t \text{ real}\} \quad .$$

The result is the cone Π_n with Σ_n as a "core".

The map H has a number of important properties and interactions with other maps on the cone Π_n. For instance, Chapter 4 studies its relationship with inversion, and the Ostrowski-Taussky inequality (4.4) says H cannot increase determinants on Π_n and, in fact, decreases them on $\Pi_n - \Sigma_n$.

iii

Conditions for Membership in Π_n

Theorem (1.4) tells us that $A \in \Pi_n$ implies $A \in L_n$, that is that the characteristic roots of A have positive real parts. The converse is, however, hopelessly invalid.

(1.12) Example. Let $A = \begin{pmatrix} -7 & -44 \\ 22 & 92 \end{pmatrix}$. The roots of A are $\{81, 4\}$ which means $A \in L_2$. But since -7 occurs on the diagonal, $A \notin \Pi_2$ by (1.9).

The example is even stronger since the roots of A not only have positive real parts but are also real. O. Taussky [42] (see also Wigner [53]) has shown that this means that such an A can be decomposed into a product of two matrices from Σ_n if A has a complete system of eigenvectors (which, of course, our A does). (The converse (1.5)c has already been shown.) Thus this example also shows that a product of Σ_n matrices need not be in Π_n.

Example (1.12) raises the question, "under what added conditions is the converse to (1.5)a valid?"; that is, when does $A \in L_n$

imply $A \in \Pi_n$? In the case of Σ_n no such dichotomy is necessary. The non-Hermitian case is, of course, quite different, but (1.13) exhibits yet another situation where "normal" is a natural extension of "Hermitian".

Let $M_1 = \min \{\mathrm{Re}(\lambda) \mid \lambda \in \sigma(A)\}$ and $m_1 = \max_{c \in F(A)} d(c, \mathrm{Co}(\sigma(A)))$ where "d(,)" denotes Euclidean distance and "Co" the closed convex hull. From (1.2) it follows that $m_1 < M_1$ and $A \in L_n$ imply $A \in \Pi_n$. That is, if F(A) does not deviate "too much" from the convex hull of the characteristic roots of A, then $A \in L_n$ implies $A \in \Pi_n$. This answer is, unfortunately, not too helpful since no simple, effective method for determining m_1 is available. The answer does, however, serve to illustrate the magnitude of the problem of determining whether $A \in \Pi_n$ without directly consulting H(A). Some partial conditions follow.

(1.13) If A is normal, then $A \in L_n$ implies $A \in \Pi_n$.

<u>Proof</u>: Since $A \in L_n$, the convex hull of its characteristic roots lies to the right of the imaginary axis in the complex plane. Since A is normal, F(A) is the convex hull of the characteristic roots of A by (0.14), and, therefore, $\mathrm{Re}(F(A)) > 0$ which means $A \in \Pi_n$ by (1.2). Alternatively, $m_1 = 0$ and M_1 is positive.

(1.14) <u>Definition</u>. We shall define the sets GR_n and GC_n[4] as follows. $A = (a_{ij}) \in GR_n$ if and only if

4. These are the classes for which the real part of the diagonal dominates the rows and columns, respectively. Thus they might be called "real part diagonally dominant." Note that the condition implies $\mathrm{Re}\ a_{ii} > 0$.

$$\mathrm{Re}(a_{ii}) > r_i(A) = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \qquad \text{for } i = 1, \ldots, n \quad .$$

$A \in GC_n$ if and only if $A^* \in GR_n$.

(1.15) Theorem. If $A \in GR_n \cap GC_n$, then $A \in \Pi_n$.

Proof: By the triangle inequality $H(A) \in GR_n \cap GC_n$ also. Thus by Gersgorin's Theorem (0.14), the necessarily real roots of $H(A)$ are positive since they are contained in discs which must lie to the right of the imaginary axis.

In the real case $GR_n \cup GC_n$ is just the set of diagonally dominant matrices with positive diagonal elements.

Though it is a rather weak sufficiency criterion, (1.15) is quite helpful in constructing examples of members of Π_n for large n. It will also be theoretically useful in several of the theorems to follow.

Arguing as in [40] it is easy to show that $GR_n \cup GC_n \subset L_n$. Further it is easily seen that $GR_n \cup GC_n \subset DL_n$, but $GR_n \cup GC_n$ is not necessarily contained in Π_n. However the following result indicates a link between DL_n and Π_n in this case.

(1.16) Theorem. $GR_n \cup GC_n \subset D_n \Pi_n$.[5]

---

5. We may weaken the condition of the theorem to yield:
(1.16.1) If $D, E \in D_n$ are such that $DAE \in GR_n \cup GC_n$ then $A \in D_n \Pi_n$. Since DL_n is closed under multiplication from D_n, this follows from (1.3) and (1.16) by writing DAE as $E(DE^{-1}A)E$.

Proof: Consider the statement:

(α) "$A \in GR_n$ implies that there is a $D \in D_n$ such that $DA \in GR_n \cap GC_n$."

We begin by showing that the theorem follows from (α). Certainly the GR_n case of the theorem follows from (α) in virtue of (1.15). But the GC_n case also follows from (α) by the following reasoning:

$$A \in GC_n \overset{(1.14)}{\Longrightarrow} A^* \in GR_n \overset{(\alpha)}{\Longrightarrow} D \in D_n \text{ such that } DA^* \in GR_n \cap GC_n$$

$$\overset{(1.14)}{\Longrightarrow} AD \in GR_n \cap GC_n \subset \Pi_n \overset{(1.3)}{\Longrightarrow} D^{-1}ADD^{-1} = D^{-1}A \in \Pi_n$$

$$\Longrightarrow A \in D_n\Pi_n.$$

In order to prove the statement (α), we may assume without loss of generality that $A \in GR_n$ has real entries and that the off-diagonal entries are nonpositive. The general complex case then follows since the imaginary parts of the diagonal entries are irrelevant for consideration of Π_n and for the off-diagonal entries only the absolute values are relevant.

To complete the proof we shall rely on a property of the so-called "M-matrices" and employ a theorem of [19]. A matrix is of class M if (1) it has real positive diagonal entries and real nonpositive off-diagonal entries and (2) all of its principal minors are positive. Since $\mathrm{Re}(a_{ii}) > r_i(A) \geq 0$ and because of our comments above, we may assume our matrix A satisfies (1). That A also satisfies (2) follows from the fact that it is real and that each real characteristic root of each of its principal submatrices is positive by Gersgorin's Theorem

(0.14). Thus our A is an M-matrix. Theorem (4, 3) of [19] says that A is an M-matrix implies there is a $D \in D_n$ such that $DA \in GC_n$. Since left multiplication from D_n does not disturb membership in GR_n and our A is already in GR_n we have $DA \in GR_n \cap GC_n$ and the proof is complete.

As a special case of (1.3) we have

(1.17) Π_n is closed under unitary similarities.

The well-behaved nature of unitary similarities on Π_n raises the question of general similarities of matrices in Π_n. We develop here by a different approach an already known link [45] between Π_n and L_n given by similarity.

(1.18) Theorem. $A \in L_n$ if and only if A is similar to a matrix in Π_n. Further, the similarity may be provided by a matrix Q of the form $Q = DU$ where U is unitary and $D \in D_n$.

Proof: If A is similar to a matrix in Π_n, then certainly $A \in L_n$ since $\Pi_n \subset L_n$ and similarity preserves the characteristic roots.

Assume $A \in L_n$ and pick U, unitary, to triangularize A.

$T = (t_{ij}) = U^*AU$, upper triangular. Since $T \in L_n$ its diagonal entries have positive real parts. The completion of the proof now follows from two facts:

(1) that there is a $D \in D_n$ such that $DT \in GC_n \subset D_n\Pi_n$, and

(2) that the set R_n of all diagonal similarities of matrices in Π_n is exactly $D_n\Pi_n$.

Together (1) and (2) mean $T \in R_n$ as was to be shown.

Proof of (1). We shall construct a $D \in D_n$ with diagonal entries $d_1, \ldots, d_n$ so that $DT \in GC_n$. Let $d_1 = 1$ and d_2 be such that

$$d_2 \operatorname{Re}(t_{22}) > |t_{12}|$$

and, in general, let d_j, $j = 3, \ldots, n$, be such that

$$d_j \operatorname{Re}(t_{jj}) > \sum_{i=1}^{j-1} d_i |t_{ij}| \quad ,$$

assuming $d_1, \ldots, d_{j-1}$ have already been chosen. This sequential choice process may be carried out in general (thus yielding the desired result) since T is triangular and $\operatorname{Re}(t_{jj}) > 0$, $j = 1, \ldots, n$.

Proof of (2). Suppose $D \in D_n$ and $D^{\frac{1}{2}}$ is the matrix in D_n whose square is D. Then $A = DB$, where $B \in \Pi_n$, implies $(D^{\frac{1}{2}})^{-1}AD^{\frac{1}{2}} = D^{\frac{1}{2}}BD^{\frac{1}{2}} \in \Pi_n$ by (1.3), and we have $D_n\Pi_n \subset R_n$. But if $DAD^{-1} = B$ where $B \in \Pi_n$, $D \in D_n$, then $D^2A = D(DAD^{-1})D = DBD \in \Pi_n$ by (1.3) and $A \in D_n\Pi_n$ so that $R_n \subset D_n\Pi_n$.[6]

With the aid of (1.18) we may now strengthen (1.5)b to yield:

(1.19) Theorem (Taussky [43]). $\Sigma_n\Pi_n = L_n$.

---

6. The first sentence of (1.18) can also be proved via the Jordan canonical form of A as remarked in [45]. Then utilizing (1.17) and the polar decomposition, the specific form of Q mentioned in the second sentence can be realized.

Proof: By (1.5)b, $\Sigma_n \Pi_n \subset L_n$ so that it suffices to show $L_n \subset \Sigma_n \Pi_n$. Suppose $B \in L_n$, then $DUBU^*D^{-1} = C \in \Pi_n$ for a suitable D and U by (1.18). By (1.3) Π_n is closed under congruences and $U^*D^2UB = U^*DCDU = A \in \Pi_n$. Since $D \in D_n$ and U is unitary, $U^*D^2U = K \in \Sigma_n$ (thus $K^{-1} \in \Sigma_n$) and $B = K^{-1}A$, so that $B \in \Sigma_n \Pi_n$ and the proof is complete.

In [43] this theorem is deduced from Lyapunov's Theorem (0.13). Our proof is independent of Lyapunov's Theorem and, in a sense, (1.19) is a weak form of (0.13). Lyapunov's Theorem guarantees an infinite array of $\Sigma_n \Pi_n$ representations of each member of L_n while (1.19) guarantees at least one. [See also 8.]

(1.20) Remarks. It is clear that if $T \in L_n$ is triangular, then $T \in DL_n$. Thus the proof of (1.18) also shows that the triangular matrices in DL_n are in $D_n \Pi_n$ and that matrices in L_n are always unitarily similar to matrices in DL_n.

It is conceptually useful to look at the preceeding theorems from another point of view. The class Π_n is preserved when subjected to all possible unitary similarities; but under all similarities Π_n expands to exactly L_n. The interesting intermediate class of all diagonal similarities of Π_n is $D_n \Pi_n$ and is related to DL_n. The class L_n is also just the unitary similarities of DL_n.

Capitalizing again on (1.17) we may observe a simple "normal form" for Π_n matrices.

(1.21) Observation. $A \in \Pi_n$ if and only if there is a unitary $U \in M_n(C)$ such that

$$U^*AU = D + S$$

where $D \in D_n$ and $S \in SK_n$.

Proof: Pick U to diagonalize H(A) and then $S = U^*S(A)U$ to prove necessity. Sufficiency follows from the fact that $H(A) = UDU^*$.

By the construction the D of the representation (1.21) is unique up to permutation of the diagonal entries. This means that in a problem in Π_n which is unchanged under unitary equivalence we may as well assume $H(A) \in D_n$.

We close this section with an exact criterion for the membership of a square in Π_n. The proof is left to Chapter 3. (Recall that if H is Hermitian and invertible then $H^2 \in \Sigma_n$.)

(1.22) Theorem. Suppose $A \in M_n(C)$ is invertible. Then $A^2 \in \Pi_n$ if and only if the positive semi-definite Hermitian matrix

$$H(A)^{-1}S(A)S(A)^*H(A)^{-1}$$

has all its characteristic roots less than 1.

Intuitively this criterion may be thought of as saying H(A) is large in relation to S(A).

CHAPTER 2

POWERS IN Π_n: A GENERAL THEOREM

In Chapter 2, 3, and 4 we shall highlight the investigation of three different algebraic aspects of Π_n. The results will be deeper and more special but will utilize the development of Chapter 1.

In Chapter 1, Π_n was found to have the additive structure of a positive convex cone. Any investigation of multiplicative structure must consider multiplicative closure which immediately raises a question concerning powers of matrices in Π_n. Since powers of Hermitian matrices are Hermitian and for general matrices the characteristic roots of the n-th power are the n-th powers of the characteristics roots, it is apparent that Σ_n is closed under the taking of integral powers. In fact any complete set of commuting elements in Σ_n forms a multiplicative group.

In Π_n outside of Σ_n the situation is, however, systematically different, and the difference begins with the taking of integral powers. There is essentially one important result in this chapter which we shall look at in a number of different ways.

If we are to accept an analogy between $M_n(C)$ and the complex numbers, then we might reasonably identify Σ_n with the positive real numbers and Π_n with the complex numbers whose real part is positive. In this context the following result is reminiscent of the theorem of De Moivre for complex numbers.

(2.1) <u>Theorem</u>. Suppose $A \in \Pi_n$ and $A \notin \Sigma_n$ $(S(A) \neq 0)$. Then there is a positive integer m such that $A^m \notin \Pi_n$.

<u>Proof</u>: The demonstration is constructive and proceeds in six parts.

(1) <u>Reduction of problem</u>. By unitary triangularization we may reduce A to upper triangular form T by unitary equivalence.

$$U^*AU = T \quad .$$

Then $U^*A^mU = T^m$ and $A^m \in \Pi_n$ if and only if $T^m \in \Pi_n$ by (1.17). Since A is not Hermitian, either (case I) some diagonal entry of T has a nonzero imaginary part <u>or</u> (case II) all diagonal entries of T are positive real numbers and at least one entry of T above the diagonal is nonzero. In case I the main result follows immediately by De Moivre's Theorem for complex numbers and (1.5)a. Thus it remains and suffices to consider case II. In this eventuality without loss of generality by virtue of (1.8) we may assume that T is normalized so that any single diagonal entry which we specify is one. Our strategy will be to show that the Hermitian part of some power of such a matrix has negative determinant and the theorem then follows.

(2) <u>Lemma 2.1a</u>. We shall say $B = (b_{ij})$ is of the form (1) if

$$B = \begin{bmatrix} b_{11} & 0 & \cdots & 0 & b_{1n} \\ & b_{22} & & 0 & 0 \\ & & \ddots & & \vdots \\ & 0 & & \ddots & 0 \\ & & & & b_{nn} \end{bmatrix}$$

and the b_{ii} are real and positive $i = 1, \dots, n, n > 1$, and $b_{1n} \neq 0$. If $m \geqslant 1$ is an integer, B is of the form (1) and $b_{11} = 1$, then

$$B^m = \begin{bmatrix} 1 & 0 \cdots 0 & b_{1n} \sum_{j=0}^{m-1} b_{nn}^j \\ 0 & b_{22}^m & 0 \\ \vdots & \ddots & \vdots \\ \vdots & & 0 \\ 0 \cdots\cdots 0 & & b_{nn}^m \end{bmatrix},$$

and B^m is of form (1).

<u>Proof</u> (by induction on m): The assertion of the lemma is evident for $m = 1$. Assume it is valid for $k \leqslant m - 1$ and carry out the multiplication $B^m = BB^{m-1}$. Collecting terms in each entry yields the desired form. The only nonzero off-diagonal entry is the $(1, n)$-entry and it is

$$1 \cdot b_{1n} \sum_{j=0}^{m-2} b_{nn}^j + b_{1n} \cdot b_{nn}^{m-1} = b_{1n} \sum_{j=0}^{m-1} b_{nn}^j \quad .$$

(3) <u>Lemma 2.1b</u>. If B is of the form (1) and $m \geqslant 1$ is an integer and $b_{11} = 1$, then

$$\det H(B^m) = b_{22}^m \cdots b_{n-1,n-1}^m \left[b_{nn}^m - \left(\frac{|b_{1n}| \sum_{j=0}^{m-1} b_{nn}^j}{2} \right)^2 \right] .$$

Proof: Lemma 2.1a allows us to construct $H(B^m)$. Interchange the 1^{st} and $n-1^{st}$ rows of $H(B^m)$ and then the 1^{st} and $n-1^{st}$ columns. The result has the same determinant as $H(B^m)$ and is the direct sum of two matrices. One is diagonal with $b_{22}^m, \ldots, b_{n-1,n-1}^m$ down the diagonal and the other is

$$\frac{1}{2}\begin{bmatrix} 2 & b_{1n} \sum_{j=0}^{m-1} b_{nn}^j \\ b_{1n} \sum_{j=0}^{m-1} b_{nn}^j & 2\, b_{nn}^m \end{bmatrix}$$

which has determinant equal to

$$\left[b_{nn}^m - \left(\frac{|b_{1n}|}{2} \sum_{j=0}^{m-1} b_{nn}^j \right)^2 \right] .$$

Thus $\det H(B^m)$ is just the product of the determinants of these two direct summands which is the form the assertion of the lemma takes.

(4) Lemma 2.1c. Suppose $(b_{ij}) = B \in \Pi_n$ is of the form (1) and $b_{11} = 1$. Then there is an integer $m > 0$ such that $\det H(B^m) < 0$.

Proof: Let

$$b_m = b_{1n} \sum_{j=0}^{m-1} b_{nn}^j \Big/ 2 .$$

By Lemma (2.1b) it suffices to show that $b_{nn}^m - |b_m|^2 < 0$ for some such m.

We consider two cases ($b_{nn} = 1$ and $b_{nn} \neq 1$). First suppose $b_{nn} = 1$. Then $b_m = \frac{m}{2} b_{1n}$ and we must pick m so that $1 < \left(\frac{m}{2} |b_{1n}|\right)^2$ or, equivalently $1 < \frac{m}{2} |b_{1n}|$. In this case, it is clear since $b_{1n} \neq 0$ that we may take m to be the first integer greater than the positive number $2/|b_{1n}|$.

Suppose $b_{nn} \neq 1$. Then we may write

$$b_m = b_{1n} \left(\frac{b_{nn}^m - 1}{b_{nn} - 1} \right) .$$

Since

$$\left(\frac{|b_{1n}|}{2\,|b_{nn} - 1|} \right)^2$$

is positive and does not depend on m, it suffices to show that $(b_{nn}^m - 1)^2 / b_{nn}^m$ gets arbitrarily large in order to show that $b_{nn}^m - |b_m|^2$ can be made negative by choice of m in this case. But, taking square roots of positive numbers to be positive,

$$(b_{nn}^m - 1)^2 / b_{nn}^m = \left((b_{nn}^m - 1) / b_{nn}^{m/2} \right)^2 = \left(b_{nn}^{m/2} - 1 / b_{nn}^{m/2} \right)^2$$

$$= (t^m - (1/t)^m)^2$$

where $t = b_{nn}^{\frac{1}{2}}$. Since $1 \neq t > 0$, either t or $1/t$ is larger than 1,

$(t^m - (1/t)^m)^2$ grows arbitrarily large and the proof of the lemma is complete.

(5) Two observations:

(2.1d) Observation. Each nondiagonal upper triangular matrix (with positive real diagonal elements) in $M_n(C)$ has a principal submatrix which is determined by a set of consecutive indices and which is of form (1). For instance, if $T = (t_{ij})$ is our matrix and t_{ij}, $i < j$, is the first nonzero entry in the list $t_{12}, t_{23}, \ldots, t_{n-1,n}, t_{13}, t_{24}, \ldots, t_{n-2,n}, t_{14}, \ldots, \ldots, t_{1n}$, then the principal submatrix determined by the indices $i, i+1, \ldots, j$ will do.

(2.1e) Observation. Suppose $\hat{T}$ is a principal submatrix of T, triangular, in $M_n(C)$ and $\hat{T}$ is determined by the set of consecutive indices $\{i, i+1, \ldots, i+k\}$. If $m \geq 0$ is an integer, then $\hat{T}^m$ is the principal submatrix of T^m determined by the same set of consecutive indices. In other words, raising triangular matrices to powers keeps these principal submatrices intact as a simple consequence of matrix multiplication. (We might say that $\hat{T}^m = \widehat{T^m}$ or that "hatting" and "raising to powers" commute.)

(6) Completion of proof. The matrix $T \in \Pi_n$ is upper triangular, has positive real diagonal elements, and is not diagonal.

Recall by (1.9) that if T^m has a $k \times k$ principal submatrix which is not in Π_k (for any $k = 1, 2, \ldots, n$), then $T^m \notin \Pi_n$. By (2.1d, e) we may choose a principal submatrix $\hat{T}$ of T which is of form (1) and which we may focus on since the raising of T to powers leaves it intact. Without loss of generality we may normalize T via multiplication by a positive real scalar so that the first diagonal element of $\hat{T}$ is one. By (2.1c) we may choose an integer $m > 0$ so that $\hat{T}^m \notin \Pi_k$ since $\det H(\hat{T}^m) < 0$. Since $\hat{T}^m$ is a principal submatrix of T^m, we have that $T^m \notin \Pi_n$ and the proof of the theorem is complete.

Since powers of elements of Σ_n remain in Σ_n, Theorem (2.1) may be restated in the following manner.

(2.2) <u>Theorem</u>. Suppose $A \in \Pi_n$. Then $A^m \in \Pi_n$ for all integers m if and only if $A \in \Sigma_n$ ($S(A) = 0$).

Thus, if $A \in \Pi_n$ and $S(A) \neq 0$, then no matter how "small" S(A) is eventually A^m will pass out of Π_n. It may or may not pass out of L_n. By De Moivre's theorem it will remain in L_n if and only if all its roots are real. In this case it is evident from the proof of (2.1) that for all m greater than some M, A^m will remain in $L_n - \Pi_n$ [see Figure 1].

That A^m eventually passes out of Π_n apparently means (because of (1.2)) that as it "expands" under successive powers, $F(A^m)$ eventually crosses the imaginary axis so that $\mathrm{Re}F(A^m) > 0$ no longer holds (from the proof of (2.1) it is clear that an actual crossing takes place). Even for operators it is known that $F(A^m) \subset P^m$ where P is the positive half-plane [29] and viewing (2.1) from the point of view of the field of values may be thought of as strengthening such results.

Given any m a positive integer it is not difficult to construct an A such that $A^i \in \Pi_n - \Sigma_n$, $i = 1, \ldots, m$, so that certain subsets of

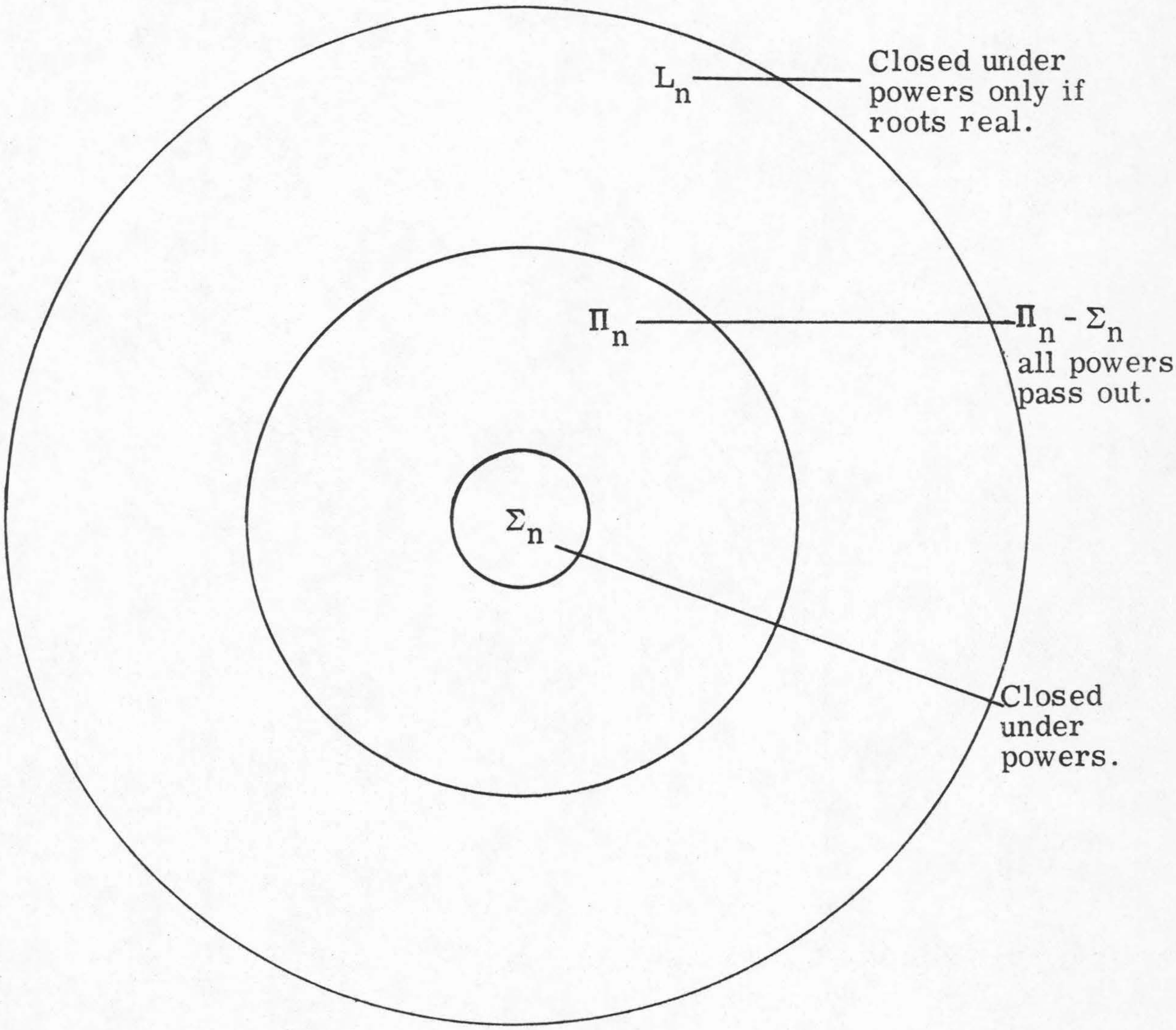
L_n
Closed under powers only if roots real.
Π_n
$\Pi_n - \Sigma_n$ all powers pass out.
Σ_n
Closed under powers.

Figure 1

$\Pi_n - \Sigma_n$ are closed under the taking of any given number of powers. Thus it is unfortunate that one bit of insight the proof of (2.1) does not provide is any close estimate of the first $m > 0$ for which $A^m \notin \Pi_n$. This could occur well before the construction in the proof indicates. If such an estimate were available it might be exploited as a measure of how strong A's membership in Π_n is or as a method of usefully classifying the elements of Π_n. The fact that

$$\Sigma_n = \bigcap_{m=1}^{\infty} \{A \in \Pi_n \mid A, A^2, \ldots, A^m \in \Pi_n\}$$

might be thought of as another way of looking at Σ_n as the "core" of Π_n (see section ii of Chapter 1).

The case of negative integral powers is entirely contained in that of positive integral powers since $A^{-m} \in \Pi_n$ if and only if $A^m \in \Pi_n$ by (1.6).

A known fact which can be shown directly to hold more generally can easily be deduced as a consequence of (2.2) and is worth mentioning here.

(2.3) Corollary. If $A \in \Pi_n$ has real roots then A is normal if and only if $A \in \Sigma_n$.

Proof: If $A \in \Pi_n$ is normal with real characteristic roots then all powers of A are normal and have positive real roots and are thus in Π_n. Therefore $A \in \Sigma_n$ by (2.2), and the converse is immediate.

CHAPTER 3

PRODUCTS AND SUMS IN Π_n

As might be expected analysis of products in Π_n is a very difficult task. Trial of only a few examples quickly convinces one that the set of products from Π_n is a large class and that Π_n is far from being multiplicatively closed. Products of elements in Σ_n even may easily pass out of Π_n (see, e.g., Ballantine [3], [4], [5], [6], on the representation of matrices as products of matrices from Σ_n). Thus we merely ask in what special cases can we guarantee that the product of two matrices in Π_n is in Π_n.

A few general lemmas are of use.

(3.1) Lemma. $A \in M_n(C)$ is normal if and only if $H(A)$ and $S(A)$ commute.

Proof: We have $AA^* = A^*A \Longleftrightarrow [H(A) + S(A)][H(A) - S(A)] = [H(A) - S(A)][H(A) + S(A)] \Longleftrightarrow H(A)^2 + S(A)H(A) - H(A)S(A) - S(A)^2 = H(A)^2 + H(A)S(A) - S(A)H(A) - S(A)^2 \Longleftrightarrow 2\,S(A)H(A) = 2\,H(A)S(A) \Longleftrightarrow H(A)$ and $S(A)$ commute.

The following lemma extends a remark of Ky Fan in [41].

(3.2) Lemma (Sums). Suppose $A \in \Pi_n$ and $B \in M_n(C)$, then $A+B \in \Pi_n$ if and only if $\lambda \in \sigma(H(A)^{-1}H(B))$ implies $\lambda > -1$.

Proof: $A+B \in \Pi_n \Longleftrightarrow H(A+B) = H(A) + H(B) \in \Sigma_n \Longleftrightarrow \lambda \in \sigma(H(A) + H(B))$ implies $\lambda > 0$. By (1.5)c the characteristic roots of $H(A) + H(B)$ are positive if and only if those of $H(A)^{-1}[H(A)+H(B)] = I + H(A)^{-1}H(B)$ are positive. The latter holds if and only if each characteristic root of $H(A)^{-1}H(B)$ is larger than -1.

(3.3) Lemma (Commutativity). Suppose $A, B \in M_n(C)$. Then the following three conditions are equivalent:

(1) A and B commute;

(2) $S(H(A)H(B)) = S(S(B)S(A))$

and $H(S(A)H(B)) = H(S(B)H(A))$;

and (3) $H(AB) = H(A)H(B) + S(A)S(B)$

and $S(AB) = S(A)H(B) + H(A)S(B)$.[1]

Proof: The proof is a computation.

(1) implies (3): $AB = [H(A)+S(A)][H(B)+S(B)] = H(A)H(B) + S(A)S(B) + S(A)H(B) + H(A)S(B) = R + T$ where $R = H(A)H(B) + S(A)S(B)$ and $T = S(A)H(B) + H(A)S(B)$. It suffices to show that

$$S(R) = 0$$

and

$$H(T) = 0$$

follow from (1). For, then, $H(AB) = H(R+T) = H(R) = R$ and $S(AB) = S(R+T) = S(T) = T$ which is what condition (3) asserts.

Let $\tilde{R} = H(B)H(A) + S(B)S(A)$ and $\tilde{T} = S(B)H(A) + H(B)S(A)$. Then $BA = \tilde{R} + \tilde{T}$, and $AB = BA$ implies $R + T = \tilde{R} + \tilde{T}$. Then $H(R+T) = H(\tilde{R}+\tilde{T})$ and $S(R+T) = S(\tilde{R}+\tilde{T})$. We shall show that

1. The two parts of (3) are equivalent statements.

$$H(R+T) = H(\tilde{R}+\tilde{T}) \quad \text{implies} \quad H(T) = 0$$

and that

$$S(R+T) = S(\tilde{R}+\tilde{T}) \quad \text{implies} \quad S(R) = 0 \quad .$$

It is easily computed that $2\,H(R) = R + \tilde{R}$ and that $2\,H(\tilde{R}) = R + \tilde{R}$. Thus $H(R) = H(\tilde{R})$, and by the linearity of the operator H $H(T)$ must equal $H(\tilde{T})$. But, $2\,H(T) = T - \tilde{T}$ and $2\,H(\tilde{T}) = \tilde{T} - T$ by direct computation. It follows that $H(T) = -H(T)$ and that $H(T) = 0$, as required. We obtain the second half of our assertion similarly. We compute that $2\,S(T) = 2\,S(\tilde{T})$ which requires that $S(R) = S(\tilde{R})$. However, $2\,S(R) = R - \tilde{R}$ and $2\,S(\tilde{R}) = \tilde{R} - R$ from which it follows that $S(R) = -S(R)$ or that $S(R) = 0$ which completes the deduction of (3) from (1).

<u>(3) implies (2)</u>: Arguing as above, (3) implies that $0 = S(R) = S(H(A)H(B)) + S(S(A)S(B)) = S(H(A)H(B)) - S(S(B)S(A))$ and this, in turn, implies $S(H(A)H(B)) = S(S(B)S(A))$, the first part of condition (2). The second part of (2) follows similarly from the second part of (3).

<u>(2) implies (1)</u>: Condition (2) implies that $S(H(A)H(B)) + S(S(A)S(B)) = 0 = H(S(A)H(B)) + H(H(A)S(B))$ which gives the conditions $S(R) = 0 = H(T)$ of above. These conditions imply (by direct computations which retrace the first portion of this proof) the commutativity of A and B. This completes the proof of the lemma.

(3.4) <u>Theorem</u>. Suppose $A, B \in M_n(C)$ commute and that $H(H(A)H(B)) \in \Sigma_n$, then $AB \in \Pi_n$ if and only if

$$\lambda \in \sigma[H(H(A)H(B))^{-1}H(S(A)S(B)^*)]$$

implies $\lambda < 1$.

Proof: The proof is an application of lemmas (3.2) and (3.3). We have that $H(H(A)H(B))^{-1}$ exists since $H(A)H(B) \in \Pi_n$ by assumption.

Since A and B commute $AB \in \Pi_n$ if and only if $H(A)H(B) + S(A)S(B) \in \Sigma_n$ by (3.3). Since $H(H(A)H(B))$ is assumed in Σ_n we may apply (3.2) and the fact that $S(B)^* = -S(B)$ to get the statement of the theorem.

When A = B the hypotheses of (3.4) are satisfied, and we have the following special case mentioned earlier.

(3.5) Theorem. If $A \in M_n(C)$ and $H(A)^{-1}$ exists, then $A^2 \in \Pi_n$ if and only if

$$\lambda \in \sigma([H(A)^{-1}S(A)][H(A)^{-1}S(A)]^*)$$

implies $\lambda < 1$.

Proof: Take A = B in (3.4). Then if $H(A)^{-1}$ exists the hypotheses are immediately satisfied. Note that $[H(A)^{-1}S(A)][H(A)^{-1}S(A)]^*$ is similar to $H(A)^{-2}S(A)S(A)^*$ and the application is proven.

Theorem (3.5) characterizes the squares in Π_n. The following lemma, included for completeness, facilitates applications of (3.5) and further results of this type as corollaries in Chapter 4.

(3.6) Lemma. If A^{-1} exists, then the following three statements are equivalent:

(1) $A^2 \in \Pi_n$

(2) $H(A)^2 + S(A)^2 > 0$

and (3) $A^{-1}A^* \in \Pi_n$.

Proof: The proof follows from two observations.

(i) $H(A^2) = H((H(A) + S(A))^2)$

$= H(H(A)^2 + S(A)^2 + H(A)S(A) + S(A)H(A))$

$= H(A)^2 + S(A)^2 + H(H(A)S(A) + S(A)H(A))$

$= H(A)^2 + S(A)^2$.

(ii) $A^{-1}\, 2\,H(A^2)(A^{-1})^* = A(A^{-1})^* + A^{-1}A^* = 2\,H(A^{-1}A^*)$.

Thus for A^2 to be in Π_n means that S(A) is "small" (in the sense of (3.5) or (3.6)) compared to H(A). Formally the conditions appear exactly like conditions for determining if the real part of the square of a complex number is positive.

If A is normal, lemma (3.1) permits the condition of theorem (3.5) to be weakened to give

(3.7) Theorem. Suppose $A \in M_n(C)$ is normal, then $A^2 \in \Pi_n$ if and only if $T = \max|\lambda(H(A)^{-1}S(A))| < 1$.

We close this chapter with two results on products not tied so tightly to the others. One deals with the Schur product and the other is a determinantal inequality for products.

Let "o" denote the Schur (element-wise) product of matrices and recall that Schur [11] has proven that the Schur product of two positive definite matrices is positive definite, i.e., $\Sigma_n \circ \Sigma_n \subset \Sigma_n$. This may be easily shown by a field of values argument [11] and is generalized below.

(3.8) Theorem. (a) $\Pi_n \circ \Sigma_n \subset \Pi_n$ and (b) $D_n\Pi_n \circ \Sigma_n \subset D_n\Pi_n$.

Proof: (b) follows from (a) since the operations of Schur producting and diagonal multiplication commute.

To prove (a), assume $A = H(A) + S(A)$ is an arbitrary element of Π_n and J is an arbitrary element of Σ_n and consider $A \circ J = H(A) \circ J + S(A) \circ J$. Since the involution "*" is easily checked to be distributive over "o", $S(A) \circ J$ is in SK_n and $H(A) \circ J = K \in \Sigma_n$ by Schur's original result. Thus $H(A \circ J) = K \in \Sigma_n$ and $A \circ J \in \Pi_n$.

It will not be demonstrated here but can be shown that $A \in L_n - DL_n$ implies $\exists J \in \Sigma_n$ such that $A \circ J \notin L_n$. This means that between DL_n and L_n there is no intermediate class closed under the Schur product with Σ_n. Thus (3.8) by showing that $D_n\Pi_n$ is closed under "o" with Σ_n further serves to relate $D_n\Pi_n$ and DL_n (via the Schur product) and as a by-product demonstrates that $D_n\Pi_n \subset DL_n$.

General determinantal inequalities involving Π_n, products and the operator H are difficult to obtain. We close with a relatively weak result of this type which is interesting when applied to A^2.

(3.9) Theorem. Suppose $A, B, AB \in \Pi_n$, that A and B commute and that S(A) and S(B) commute. Then[2]

$$\det H(AB) \leq \det H(A)H(B) \leq |\det AB| .$$

---

2. What we shall show is that under the conditions of (3.9) $H(A)H(B) \geq H(AB)$ and that by (3.10) the inequality is preserved by applying the determinant function.

To prove (3.9) we shall use the following lemma which will be reused later.

(3.10) Lemma. If $A, B \in \Sigma_n$ and $C \in \overline{\Sigma}_n$ (closure), then $A = B + C$ implies $\det B \leq \det A$.

Proof: $\det B^{-1}A = \det B^{-1}(B+C) = \det(I + B^{-1}C) \geq 1$ since the roots of $B^{-1}C$ are nonnegative by (1.5)c.

Proof (of 3.9): $H(AB) = H(A)H(B) + S(A)S(B)$ by (3.3) and $S(A)S(B)$ is Hermitian negative semi-definite since $S(A)$ and $S(B)$ commute. Thus $H(A)H(B)$ is Hermitian positive definite and $H(A)H(B) = H(AB) - S(A)S(B)$ where $H(AB) \in \Sigma_n$ and $-S(A)S(B) \in \overline{\Sigma}_n$. Therefore $\det H(AB) \leq \det H(A)H(B)$ by (3.10). That $\det H(A)H(B) \leq |\det AB|$ follows from two applications of the Ostrowski-Taussky inequality (4.4) to be derived in Chapter 4, and the proof is complete.

Letting $A = B$ in (3.9) yields

(3.11) Theorem. If $A, A^2 \in \Pi_n$, then $\det H(A^2) \leq \det H(A)^2 \leq |\det A^2|$.

CHAPTER 4

RELATIONS BETWEEN $H(A)^{-1}$ AND $H(A^{-1})$

In Π_n we have found that we may always construct inverses (1.6) and we may always construct the Hermitian part via the linear operator H. In general these two operations do not commute and thus it is of interest to compare $H(A^{-1})$ and $H(A)^{-1}$. Here there is a dichotomy between $\Pi_n(R)$ and $\Pi_n(C)$ and the first half of this chapter studies the question in $\Pi_n(R)$ while the second half studies $\Pi_n(C)$. The main results for $\Pi_n(R)$ are (4.1) and (4.3) which we might call a "precise" inequality. Interestingly enough the systematic relation of the real case does not directly generalize to the complex case and in the second half of the chapter the main result is (4.11) which is obtained by entirely different methods.

If adjA denotes the classical adjoint of A (also called the adjugate or cofactor matrix) then the strong result of (4.3) is motivated by (4.1) which considers H(adjA) - adjH(A). For $A \in M_2(R)$ this expression is identically 0. For $A \in \Pi_3(R) - \Sigma_3(R)$ it is a rank 1 positive semi-definite matrix (i.e., $H(\mathrm{adj}A) - \mathrm{adj}H(A) = EE^*$ where $E \neq 0$ is some 3 by 1 matrix). Both these facts can be noticed computationally. For $n > 2$, $H(\mathrm{adj}A) - \mathrm{adj}H(A) = 0$ if and only if $S(A) = 0$ (A symmetric). For all n a natural relation occurs between H(adjA) and adjH(A) when $A \in \Pi_n(R)$.

(4.1) <u>Theorem</u>. If $A \in \Pi_n(R)$, then

(a) H(adjA) - adjH(A) $\geqslant$ 0 ($\in \overline{\Sigma}_n$); and

(b) H(adjA) - adjH(A) $>$ 0 ($\in \Sigma_n$) if and only if S(A) has more than one pair of conjugate nonzero eigenvalues.[1]

The proof proceeds in four parts.

Proof: Let H(A) = J and S(A) = S, then A = J + S with J $>$ 0 and S skew-symmetric. If B_1 = H(adjA) - adjH(A), then B_1 may be rewritten $B_1 = H(\text{adj}(J+S)) - \text{adj}J = \det(J+S)\cdot H((J+S)^{-1}) - \det(J)\cdot J^{-1}$.

Since det(J) $>$ 0, the definiteness of B_1 is equivalent to that of $B_2 = B_1/\det(J)$. Thus it suffices to consider

$$(1) \qquad B_2 = \frac{\det(J+S)}{\det(J)} H((J+S)^{-1}) - J^{-1} = cH((J+S)^{-1}) - J^{-1}$$

where $c = \det(I + J^{-1}S)$.

(i) The factor c.

$J^{-1}S = J^{-\frac{1}{2}}(J^{-\frac{1}{2}}SJ^{-\frac{1}{2}})J^{\frac{1}{2}}$ so that $J^{-1}S$ is similar to a skew-symmetric matrix. Therefore $J^{-1}S$ has only pure imaginary eigenvalues occurring in conjugate pairs $\pm it_j$ (and since the zero eigenspaces of $J^{-1}S$ and S have the same dimension, $J^{-1}S$ has as many nonzero conjugate pairs of eigenvalues as S has). Thus the eigenvalues of $I + J^{-1}S$ are of the form $1 \pm it_j$ and

$$(2) \qquad c = \Pi(1 \pm it_j) = \Pi(1 + t_j^2), \quad t_j \text{ real} .$$

1. By (1.6), H(adjA) - adjH(A) is the difference of two elements of Σ_n. We shall describe the situations of the theorem as H(adjA) $\geqslant$ adjH(A) and H(adjA) $>$ adjH(A), respectively.

This means c has a factor >1 for each conjugate pair of nonzero eigenvalues of S.

(ii) <u>An equivalent check for the definiteness of B_2.</u>

B_2 is symmetric and it suffices to consider

(3) (x, B_2x) for all nonzero n-vectors x to determine the definiteness of B_2.

$$(x, B_2x) = c\left(x, \frac{(J+S)^{-1} + ((J+S)^{-1})^*}{2} x\right) - (x, J^{-1}x)$$

$$= (c/2)[(x, (J+S)^{-1}x) + (x, (J-S)^{-1}x)] - (x, J^{-1}x).$$ If

$y_1 = (J+S)^{-1}x$, $y_2 = (J-S)^{-1}x$ and $y_3 = J^{-1}x$, then

$y_2 = (J-S)^{-1}(J+S)y_1$ and (x, B_2x)

$$= (c/2)[((J+S)y_1, y_1) + ((J+S)y_1, (J-S)^{-1}(J+S)y_1)] - (Jy_3, y_3)$$

$$= (c/2)[((J+S)y_1, y_1) + (y_1, (J+S)y_1)] - (Jy_3, y_3)$$

$$= (c/2)[(y_1, (J-S)y_1) + (y_1, (J+S)y_1)] - (y_3, Jy_3)$$

$$= c(y_1, Jy_1) - (y_3, Jy_3).$$

Since $y_1 = (J+S)^{-1}x = (J+S)^{-1}Jy_3 = (I+J^{-1}S)^{-1}y_3$ our expression becomes

$$c((I+J^{-1}S)^{-1}y_3, J(I+J^{-1}S)^{-1}y_3) - (y_3, Jy_3)$$

$$= (y_3, (c(I-SJ^{-1})^{-1}J(I+J^{-1}S)^{-1} - J)y_3) .$$

(iii) Thus it suffices to check the symmetric matrix

$$B_3 = c[(I - SJ^{-1})^{-1} J(I + J^{-1}S)^{-1}] - J \quad . \tag{4}$$

But $B_3 = c[(J^{-1} - J^{-1}SJ^{-1})^{-1} (I + J^{-1}S)^{-1}] - J$

$$= c[J(I - J^{-1}S)^{-1} (I + J^{-1}S)^{-1}] - J = J[c(I - (J^{-1}S)^2)^{-1} - I]$$

$$= J^{\frac{1}{2}}(J^{\frac{1}{2}}[c(I - (J^{-1}S)^2)^{-1} - I] \, J^{-\frac{1}{2}}) \, J^{\frac{1}{2}} \; .$$

This means that B_3 is congruent via an element of Σ_n to a matrix similar to

$$B_4 = c(I - (J^{-1}S)^2)^{-1} - I \quad . \tag{5}$$

Therefore B_3 is ≥ 0 or > 0 if and only if B_4 has all its eigenvalues ≥ 0 or > 0, respectively.

(iv) The eigenvalues of B_4.

The eigenvalues of B_4 are just $[c/(1+t_j^2)] - 1$ where $\pm it_j$ are the eigenvalues of $J^{-1}S$ already considered in (i). Since $c = \Pi(1 + t_j^2)$ by equation (2), the eigenvalues of B_4 become

$$\left([\Pi(1 + t_j^2)]/(1 + t_i^2)\right) - 1 = \prod_{j \neq i} (1 + t_j^2) - 1 \quad . \tag{6}$$

Thus all eigenvalues of B_4 are nonnegative and are all positive if and only if there is more than one $t_j \neq 0$, that is if and only if S(A) has more than one pair of nonzero eigenvalues.

Therefore B_2 and thus B_1 are always positive semi-definite and are positive definite if and only if S(A) has more than one conjugate pair of eigenvalues which completes the proof.

(4.2) Definition. For a given matrix $A \in M_n(C)$ let $T = \max|\lambda(H(A)^{-1}S(A))|$. Then T is a real number and a function of A; if we wish to emphasize the A, we shall write T_A.

(4.3) Theorem. If $A \in \Pi_n(R)$ and c is a real scalar, then $cH(A^{-1}) - H(A)^{-1}$ is positive definite ($\in \Sigma_n$) if and only if $c > 1 + T^2$.

Proof: Return to the proof of (4.1). Turning to parts (iii) and (iv) the matrix B_4 has all its characteristic roots positive if and only if c is greater than the largest of the $1 + t_j^2$ which is to say that $c > 1 + T^2$. By the proof of (4.1), this means that B_2 which is the same as the expression of (4.3) is positive definite if and only if $c > 1 + T^2$.

Tied up in the preceding is what may be thought of as a generalization of the spirit of an important known inequality for Π_n [51]. In equation (2) of the proof of (4.1) the fact that $c > 1$ implies the real case of the Ostrowski-Taussky inequality. The statement of (4.1) does not readily imply it, but the proof (4.1) can be adjusted to obtain more or less its original proof.

(4.4) Ostrowski-Taussky. If $A \in \Pi_n(C)$, then $\det H(A) \leqslant |\det A|$. Equality holds if and only if $S(A) = 0$ ($A \in \Sigma_n$).

Proof: By the same methods as in part (i) of the proof of (4.1), $|\det(I + H(A)^{-1}S(A))| \geqslant 1$ ($H(A)^{-1}S(A)$ has only imaginary roots) with equality only when $S(A) = 0$. This statement holds even if A is complex and means $\det H(A) \leqslant |\det[H(A) + S(A)]| = |\det A|$.

The Ostrowski-Taussky inequality says that the operator H decreases determinants on the cone Π_n--except, of course, on the "core" Σ_n. Theorem (4.4) can be generalized in several ways (see, e.g., Fan [17]) and we shall present here a new generalization which strengthens the original inequality and appears like a reverse of the triangle inequality. The real case of this generalization originally arose out of a discussion with Ky Fan by combining a result of his [18] with the result (4.1) of this author. We develop here the general complex case independently.

(4.5) Theorem. $A \in \Pi_n(C)$ implies

(1) if $n = 1$, $|\det A| \leq \det H(A) + |\det S(A)|$ with "=" if and only if $A \in \Sigma_n$; and

(2) if $n > 1$, $|\det A| \geq \det H(A) + |\det S(A)|$ with "=" if and only if either (a) $A \in \Sigma_n$ or (b) $n = 2$ and both roots of $H(A)^{-1}S(A)$ have the same absolute value.

Proof: (1) is merely the triangle inequality. For (2) it suffices to consider the validity of the inequality $|\det[I + H(A)^{-1}S(A)]| \geq 1 + |\det H(A)^{-1}S(A)|$ since $A = H(A) + S(A)$. The roots of $H(A)^{-1}S(A)$ are purely imaginary by (1.5)d and so we may call them it_j, $j = 1, \ldots, n$ where each t_j is real. Thus it suffices to show

$$\prod_j |1 + it_j| \geq 1 + |\prod_j t_j|$$

if $n > 1$.

We shall verify this by induction.

Part I: $n = 2$. It suffices to take $t_j \geqslant 0$, $j = 1,2$ since the sign of t_j, $j = 1,2$ makes no contribution to the value of either side. Thus $|1+it_1|\ |1+it_2| \geqslant 1 + t_1t_2 \Longleftrightarrow (1+t_1^2)(1+t_2^2) \geqslant (1+t_1t_2)^2 \Longleftrightarrow t_1^2 + t_2^2 \geqslant 2\,t_1t_2$ which it is since $(t_1 - t_2)^2 \geqslant 0$.

Part II: Suppose

$$\prod_{j=1}^{n-1} |1 + it_j| \geqslant 1 + \left|\prod_{j=1}^{n-1} t_j\right| .$$

Multiply both sides by $|1 + it_n|$ which is positive and greater than both 1 and $|t_n|$ unless $t_n = 0$. Thus

$$\prod_{j=1}^{n} |1+it_j| = |1+it_n| \prod_{j=1}^{n-1} |1+it_j| \geqslant |1+it_n| \left[1 + \prod_{j=1}^{n-1} |t_j|\right]$$

$$= |1+it_n| + |1+it_n| \prod_{j=1}^{n-1} |t_j| \geqslant 1 + \left|\prod_{j=1}^{n} t_j\right|$$

which was to be shown and completes the induction proof.

(a) If $A \in \Sigma_n$, then $S(A) = 0$ and "=" holds as in Ostrowski-Taussky (4.4).

(b) If $n = 2$, then "=" holds only when $|t_1| = |t_2|$ by part I. Since "=" cannot hold if $n \geqslant 3$ unless $A \in \Sigma_n$, as is seen from the induction step, the proof is complete.

Some additional corollaries to (4.3) which expand on Chapter 3 are of note.

(4.6) Corollary. If $A \in \Pi_n(R)$ and is normal then $A^{-1}A^* \in \Pi_n$ if and only if $T < 1$. In this case $0 < \text{Re}\,\lambda(A^{-1}A^*) \leq 1$.

Proof: By (4.3) $c > 1 + T^2$ if and only if $c(A^{-1} + (A^{-1})^*)/2 > [(A + A^*)/2]^{-1}$ or $(c/4)\,[2I + (A^{-1})^*A + A^{-1}A^*] > I$ since the left-hand side is Hermitian by the normality of A. Since A normal implies $A^{-1}A^*$ is unitary we have $((A^{-1})^*A) = (A^{-1}A^*)^{-1} = (A^{-1}A^*)^*$ and $(c/4)\,[2I + 2H(A^{-1}A^*)] > I$ or $H(A^{-1}A^*) > \frac{2-c}{c} I$. Then $A^{-1}A^* \in \Pi_n \Leftrightarrow c < 2 \Leftrightarrow T^2 < 1$. When $A^{-1}A^* \in \Pi_n$ all real parts of its characteristic roots must be positive and since $A^{-1}A^*$ is unitary they are also less than or equal to 1.

If $A \in \Pi_n(R)$ but not necessarily normal a similar approach still yields information about $A^{-1}A^*$ via (4.3). The condition of (4.6) is still necessary, but no longer sufficient.

Since the product of two positive definite matrices has all its roots real and positive (1.5), $cH(A^{-1}) - H(A)^{-1} > 0$ if and only if $cH(A^{-1})H(A) - I$ has all roots positive. Equivalently $H(A^{-1})H(A)$ has all roots $> 1/c$. But $H(A^{-1})H(A) = I/2 + (A^{-1}A^* + (A^{-1}A^*)^{-1})/4$ and we may conclude by Theorem (4.2) that $A^{-1}A^* + (A^{-1}A^*)^{-1}$ has all its eigenvalues $> 2(2-c)/c$ if and only if $c > 1 + T^2$. Thus $A^{-1}A^* + (A^{-1}A^*)^{-1}$ has all its roots positive if and only if $1 < c < 2$, that is, $T^2 > 1$. The smallest eigenvalue of $H(B)$ is always less than or equal to the smallest of the real parts of those of B for any square matrix B and $A^{-1}A^*$ and $(A^{-1}A^*)^{-1}$ are either both in or both not in Π_n. If both are, their sum is and we may conclude as a consequence of (4.2).

(4.7) Corollary. If $A \in \Pi_n(R)$, then $A^{-1}A^* \in \Pi_n$ implies $T < 1$.

It could be that $T < 1$ and $A^{-1}A^* \notin \Pi_n$ if A is not normal. A simple example is $A = \begin{pmatrix} 6 & -3 \\ 3 & 2 \end{pmatrix}$. It, of course, depends on how much smaller the smallest root of $H(A^{-1}A^*)$ is than the smallest of the real parts of the roots of $A^{-1}A^*$, that is, on how much the field of values of $A^{-1}A^*$ deviates from the convex hull of its eigenvalues.

Lemma (3.6) allows us to translate the preceeding results to consideration of A^2.

(4.8) Corollary. Suppose $A \in \Pi_n(R)$, then

(1) $A^2 \in \Pi_n$ implies $T < 1$, and

(2) if A is normal $A^2 \in \Pi_n$ if and only if $T < 1$.

Just as with $A^{-1}A^*$, $T < 1$ does not imply $A^2 \in \Pi_n$ if A is not normal (the same example suffices).

We now turn to consideration of $\Pi_n(C)$ in order to more generally compare $H(A)^{-1}$ and $H(A^{-1})$.

First we prove a useful lemma which is interesting by itself. It is an extension of the remark by Ky Fan [16] and the proof is new. The lemma is more general than the Π_n case, but the proof depends essentially on the Π_n case. Since the arguments involving F(A) do not depend on finite matrices, the proof is valid for the operator case as well.

(4.9) Lemma. $0 \notin F(A)$ implies $A^{-1}A^*$ is similar to a unitary matrix.

Proof: By (1.4) $O \notin F(A)$ implies $e^{i\theta}A = B \in \Pi_n$ for some θ. Since $B^{-1}B^* = e^{-2i\theta}A^{-1}A^*$ and $e^{-2i\theta}$ is a unitary scalar, $A^{-1}A^*$ is similar to a unitary matrix if and only if $B^{-1}B^*$ is, and it suffices to assume $A \in \Pi_n$. [Note "$A^{-1}A^*$ similar to a unitary matrix if $A \in \Pi_n$" has been shown by Fan by alternate means.]

Assume $A \in \Pi_n$ and write $A = H + S$ where $H \in \Sigma_n$ and S is skew-Hermitian. We shall show $A^{-1}A^*$ is similar to a unitary matrix by a Hermitian positive definite matrix.

$$\begin{aligned}
A^{-1}A^* &= (H+S)^{-1}(H-S) = (H+S)^{-1}HH^{-1}(H-S)\\
&= [H^{-1}(H+S)]^{-1}[H^{-1}(H-S)] = (I+H^{-1}S)^{-1}(I-H^{-1}S)\\
&= (I+H^{-1}S)^{-1}(I+SH^{-1})^*\\
&= H^{-\frac{1}{2}}H^{\frac{1}{2}}(I+H^{-1}S)^{-1}H^{-\frac{1}{2}}H^{\frac{1}{2}}(I+SH^{-1})^*H^{-\frac{1}{2}}H^{\frac{1}{2}}\\
&= H^{-\frac{1}{2}}[H^{\frac{1}{2}}(I+H^{-1}S)H^{-\frac{1}{2}}]^{-1}[H^{-\frac{1}{2}}(I+SH^{-1})H^{\frac{1}{2}}]^*H^{\frac{1}{2}}\\
&= H^{-\frac{1}{2}}(I+H^{-\frac{1}{2}}SH^{-\frac{1}{2}})^{-1}(I+H^{-\frac{1}{2}}SH^{-\frac{1}{2}})^*H^{\frac{1}{2}}\\
&= H^{-\frac{1}{2}}(I+\tilde{S})^{-1}(I+\tilde{S})^*H^{\frac{1}{2}}
\end{aligned}$$

where $\tilde{S} = H^{-\frac{1}{2}}SH^{-\frac{1}{2}} \in SK_n$. The last expression in the sequence of equalities is similar to $(I+\tilde{S})^{-1}(I+\tilde{S})^*$ via $H^{\frac{1}{2}} \in \Sigma_n$, and $(I+\tilde{S})^{-1}(I+\tilde{S})^*$ is necessarily unitary since $(I+\tilde{S})$ is normal.

In order to facilitate the main result to follow,

(4.10) Definition. Let $M = \max \operatorname{Re} \lambda(A^{-1}A^*)$ and $m = \min \operatorname{Re} \lambda(A^{-1}A^*)$.

(4.11) Theorem. If $A \in \Pi_n(C)$, then

(1) $cH(A^{-1}) - H(A)^{-1} > 0$ if and only if $c > \frac{2}{m+1}$ (equivalently $m > \frac{2-c}{c}$), and

(2) $dH(A)^{-1} - H(A^{-1}) > 0$ if and only if $d > \frac{M+1}{2}$ (equivalently $M < 2d-1$).

In view of (4.9) and the fact that $I + A^{-1}A^* = A^{-1}(A+A^*)$ is invertible, we have necessarily that $m > -1$ and that the c and d of the theorem are positive.

Proof: (1) $cH(A^{-1}) - H(A)^{-1} > 0 \overset{(1.5)}{\Longleftrightarrow}$

$$\Longleftrightarrow \lambda([cH(A^{-1})]^{-1}(cH(A^{-1}) - H(A)^{-1})) > 0$$

$$\Longleftrightarrow \lambda(I - [cH(A)H(A^{-1})]^{-1}) > 0$$

$$\Longleftrightarrow \lambda(-[cH(A)H(A^{-1})]^{-1}) > -1$$

$$\Longleftrightarrow \lambda([cH(A)H(A^{-1})]^{-1}) < 1$$

$$\Longleftrightarrow \lambda(cH(A)H(A^{-1})) > 1$$

$$\Longleftrightarrow \lambda(H(A)H(A^{-1})) > \frac{1}{c}$$

$$\Longleftrightarrow \lambda\left(\frac{A+A^*}{2} \cdot \frac{A^{-1}+(A^{-1})^*}{2}\right) > \frac{1}{c}$$

$$\Longleftrightarrow \lambda\left(\frac{I}{2} + \frac{A^*A^{-1} + AA^{-1*}}{4}\right) > \frac{1}{c}$$

$$\Longleftrightarrow \lambda\left(\frac{A^*A^{-1} + (A^*A^{-1})^{-1}}{4}\right) > \frac{1}{c} - \frac{1}{2} = \frac{2-c}{2c}$$

$$\Leftrightarrow \lambda(A^*A^{-1} + (A^*A^{-1})^{-1}) > 2(\frac{2-c}{c})$$

$$\Leftrightarrow \operatorname{Re}\lambda(A^*A^{-1}) = \operatorname{Re}\lambda(A^{-1}A^*) > \frac{2-c}{c} \qquad \text{[See footnote 2.]}$$

$$\Leftrightarrow m > \frac{2-c}{c} \Leftrightarrow cm > 2-c \Leftrightarrow cm + c > 2$$

$$\Leftrightarrow c(m+1) > 2 \Leftrightarrow c > \frac{2}{m+1},$$

and (1) is complete.

The proof of (2) is similar.

(2) $$dH(A)^{-1} - H(A^{-1}) > 0$$

$$\Leftrightarrow \lambda([dH(A)^{-1}]^{-1}[dH(A)^{-1} - H(A^{-1})]) > 0$$

$$\Leftrightarrow \lambda(I - \frac{1}{d}H(A)H(A^{-1})) > 0$$

$$\Leftrightarrow \lambda(-\frac{1}{d}H(A)H(A^{-1})) > -1$$

$$\Leftrightarrow \lambda(-H(A)H(A^{-1})) > -d$$

$$\Leftrightarrow \lambda(H(A)H(A^{-1})) < d$$

$$\Leftrightarrow \lambda\left(\frac{I}{2} + \frac{A^*A^{-1} + (A^*A^{-1})^{-1}}{4}\right) < d$$

$$\Leftrightarrow \lambda(A^*A^{-1} + (A^*A^{-1})^{-1}) < (d - \tfrac{1}{2})4$$

$$\Leftrightarrow \operatorname{Re}\lambda(A^*A^{-1}) = \operatorname{Re}\lambda(A^{-1}A^*) < (d - \tfrac{1}{2})2$$

$$\Leftrightarrow M < 2d - 1 \Leftrightarrow M+1 < 2d$$

$$\Leftrightarrow d > \frac{M+1}{2}$$

---

2. Note at this point: each matrix mentioned has had necessarily real roots. In particular, since the roots of $A^*A^{-1} + (A^*A^{-1})^{-1}$, or equivalently $A^{-1}A^* + (A^{-1}A^*)^{-1}$, are necessarily real, we have by this calculation alone that any complex roots of $A^{-1}A^*$, $A \in \Pi_n$, must be 1 in absolute value. This pleasantly agrees with (4.9).

and the proof is complete.

(4.12) Corollary. $A \in \Pi_n$ implies $H(A)^{-1} \geq H(A^{-1})$.

Proof: $d = 1$ must satisfy $d \geq \frac{M+1}{2}$ since M is at most 1 by (4.9) and the corollary follows from part (2) of (4.11).

Recalling (3.10), (4.12) yields the following determinantal inequalities.

(4.13) Corollary. $A \in \Pi_n$ implies

$$[\det H(A)]^{-1} \geq \det H(A^{-1})$$

and

$$\det H(A)H(A^{-1}) \leq 1 \quad .$$

Applying (4.11) to $\Pi_n(R)$ and comparing it to (4.3) reveals:

(4.14) Corollary. $A \in \Pi_n(R)$ implies

$$T = \sqrt{\frac{1-m}{1+m}} \qquad \text{or} \qquad m = \frac{1-T^2}{1+T^2} \quad .$$

Finally, inspection of (4.11) produces:

(4.15) Corollary. If c and d are any real numbers which satisfy parts (1) and (2) of (4.11), respectively, then

$$\frac{2-c}{c} < \operatorname{Re} \lambda(A^{-1}A^*) < 2d - 1 \quad .$$

CHAPTER 5

UNITARY AND CRAMPED UNITARY MATRICES AND Π_n

(5.1) Definition. A unitary matrix U shall be called cramped if and only if all of its roots fall within a sector of less than 180° on the unit circle. [Notice U is cramped if and only if $0 \notin F(U)$.]

The results of this chapter are obtained in a straightforward manner but do accomplish two goals. They characterize the cramped unitary matrices and link them to Π_n in a rather pleasant way and provide an area for the theoretical application of criteria previously developed in this paper.

A very natural link exists with Π_n and the discussion will be centered around products of the form $A^{-1}A^*$ (see, e.g., Fan [17] and Taussky [48], [50]). If A is normal then it is an easy computation that $A^{-1}A^*$ is unitary (compare (4.9)) and if $A^{-1}A^* = U$, unitary, then A can only be normal (as an aside this means that the normal matrices are just those matrices which differ from their * by only a unitary matrix since $A^* = UA$).

(5.2) Definition. Let $N(U) = \{A \mid U = A^{-1}A^* \text{ and } A \text{ is normal}\}$.

Taussky [50] has given a general discussion of the decompositions of a unitary U into $A^{-1}A^*$ and characterized N(U).

(5.3) Lemma. U is unitary if and only if N(U) is nonempty.

Proof: Sufficiency of the condition has already been mentioned. The necessity follows from the fact that any unitary matrix has an invertible square root $U^{-\frac{1}{2}}$ which is unitary and $(U^{-\frac{1}{2}})^{-1}(U^{-\frac{1}{2}})^{*} = U$. Other proofs involving the characteristic roots are available and straightforward.

Thus we can always pick an element from N(U) if U is unitary without worrying about the case $N(U) = \phi$.

Since U cramped unitary means $0 \notin F(U)$ we may observe the following by virtue of (1.4).

(5.4) Lemma. A unitary matrix U is cramped if and only if $\exists \theta \in [0, 2\pi)$ such that

$$e^{i\theta} U \in \Pi_n \quad .$$

Relying partly upon (1.4) and (3.6), we are now in a position to present two parallel characterizations of the cramped unitary matrices.

(5.5a) Theorem. U is cramped unitary if and only if $\exists_A \in N(U)$ and $\exists \varphi \in [0, 2\pi)$ such that $A = e^{i\varphi}\tilde{A}$ and

$$\tilde{A}, \tilde{A}^2 \in \Pi_n \quad .$$

(5.5b) Theorem. U is cramped unitary if and only if $\exists_A \in N(U)$ such that $O \notin F(A)$ and $O \notin F(A^2)$.

Proof: Statement (5.5b) follows from (5.5a) by (1.4) and we shall constructively demonstrate (5.5a).

By (5.4) U is cramped if and only if $\exists\, \theta \in [0, 2\pi)$ such that

$U_\theta = e^{i\theta}U \in \Pi_n$. Clearly U_θ is unitary. Suppose V is unitary such that $V^*U_\theta V = \mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})$. Let $\tilde{A} = V\,\mathrm{diag}(e^{-i\theta_1/2}, \ldots, e^{-i\theta_n/2})V^*$. Now $\tilde{A} \in N(U_\theta)$ and $\tilde{A} \in \Pi_n$ by (1.12) and $\tilde{A}^2 \in \Pi_n$ by (3.6) since $U_\theta \in \Pi_n$. Since $A = e^{i\theta/2}\tilde{A} \in N(U)$ and each argument is reversible, the theorem follows.

Applying Theorem (3.7) to (5.5)a we obtain

(5.6) Theorem. A unitary matrix U is cramped if and only if

$$\min_{\substack{\theta \in [0, 2\pi] \\ A \in N(U)}} \max \lambda(B_\theta) < 1$$

where $B_\theta = [H(e^{i\theta}A)^{-1}S(e^{i\theta}A)]\,[H(e^{i\theta}A)^{-1}S(e^{i\theta}A)]^*$.

CHAPTER 6

CLASSICAL INEQUALITIES IN Π_n: TWO EXTENSIONS OF HADAMARD'S INEQUALITY

The inequality of Hadamard [24] holds for a matrix in $M_n(C)$ when the absolute value of its determinant is dominated ($\leq$) by the absolute value of the product of its diagonal elements. This very beautiful relationship holds throughout Σ_n and is one of the most basic inequalities in Σ_n. In considering Π_n as a generalization of Σ_n it is therefore appropriate to ask to what extent Hadamard's inequality holds in Π_n. Relevant partial information has been provided by Gantmacher, Krein, Koteljanskii and Fan [22], [30], [14].

That Hadamard's inequality does not hold in its pure form throughout Π_n is easily demonstrated by the following examples.

(6.1) <u>Examples</u>.

(a) Let $A = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}$. Then $A \in \Pi_2$, but $\det A = 10 > 9 = 3 \cdot 3$

(b) Let $B = \begin{pmatrix} 8 & 1 & 6 \\ 7 & 10 & 1 \\ 1 & 8 & 12 \end{pmatrix}$. Then $B \in \Pi_3$, but $\det B = 1089 > 960 = 8 \cdot 10 \cdot 12$.

In this chapter we generalize Hadamard's inequality into Π_n in two directions. In one (Theorem (6.6)) the inequality is generalized in its pure form to a large subclass of Π_n which includes Σ_n. This provides an intriguing independent proof of Hadamard's original inequality, and the theorem indicates as well a class where the reverse of Hadamard's inequality necessarily holds. In the second, Theorem (6.9) and (6.10), the inequality is weakened in such a way that it holds throughout Π_n. This direction also includes the full force of the original inequality as a special case. Both directions differ essentially from the work of GKK and F as an example will illustrate.

To facilitate the discussion, certain definitions are in order.

(6.2) Definition (Triangular Part). If $H = (h_{ij})$ is Hermitian, then $T(H) \equiv (t_{ij})$ where

$$t_{ij} \equiv \begin{cases} 2\,h_{ij} & \text{if} \quad i < j \\ h_{ij} & \text{if} \quad i = j \\ 0 & \text{if} \quad j < i \end{cases} \quad .$$

If $A \in M_n(C)$, then $T(A) \equiv T(H(A))$.

(6.3) Observations. The operator T is linear over the reals as H was (in fact T and H have similar matrix representations) and H "ignores" T since $H(T(A)) = H(A)$ just as $T(H(A)) = T(A)$. Thus $T(A)$ is just that upper triangular matrix whose Hermitian part is $H(A)$. The diagonal elements of $T(A)$ are real and $\det T(A)$ is just the product of those diagonal elements. Thus for $H \in \Sigma_n$, Hadamard's inequality states that $\det(H) \leq \det T(H)$.

(6.4) Definition. Let $\alpha \in R$ and define

$$\Pi_n^{(\alpha)} = \{\alpha T(A) + (1-\alpha)T(A)^* \mid A \in \Pi_n\} \quad .$$

(6.5) Observations. If $\alpha = \frac{1}{2}$, $\Pi_n^{(\alpha)} = \Sigma_n$. If

$$A = \begin{pmatrix} 1 & 2\alpha & \cdots & 2\alpha \\ 2(1-\alpha) & 1 & \ddots & \vdots \\ \vdots & & \ddots & 2\alpha \\ 2(1-\alpha) & \cdots & 2(1-\alpha) & 1 \end{pmatrix}$$

and "o" denotes the Schur (element-wise) product of matrices, then $\Pi_n^{(\alpha)} = \{A \circ H \,|\, H \in \Sigma_n\}$. For the purpose of simplicity of description this is probably the easiest way to think of $\Pi_n^{(\alpha)}$. However, the original definition is more convenient for displaying proofs. Since $H(\alpha T(A) + (1-\alpha)T(A)^*) = H(A)$, we have $\Pi_n^{(\alpha)} \subset \Pi_n$ for all $\alpha \in R$.

We now present our most general form of the pure extension of Hadamard's inequality.

(6.6) Theorem. Suppose $A \in \Pi_n^{(\alpha)}$ and $D, E \in D_n$. Let $(b_{ij}) = B = DAE$, then

(i) if $\alpha \in [0, 1]$, B satisfies Hadamard's inequality, and

(ii) if $\alpha \notin [0, 1]$,

$$\prod_{i=1}^{n} b_{ii} \leq |\det B| ,$$

the reverse of Hadamard's inequality.

Proof: Since pre or post multiplication by a diagonal matrix effects both the determinant and the product of the diagonal elements in an identical manner, it suffices to take $B \in \Pi_n^{(\alpha)}$ to prove the general result.

Suppose $B = \alpha T + (1-\alpha)T^*$, where $T = T(C)$, $C \in \Pi_n$. We first show that $S(B) = (2\alpha - 1)S(T)$.

$$S(B) = (B-B^*)/2 = \frac{[\alpha T + (1-\alpha)T^*] - [\alpha T^* + (1-\alpha)T]}{2}$$

$$= [(2\alpha-1)T - (2\alpha-1)T^*]/2 = (2\alpha-1)S(T) \quad .$$

As noted before, $H(B) = H(T)$ and thus $T = H(T) + S(T)$ and $B = H(T) + (2\alpha-1)S(T)$, $H(T) \in \Sigma_n$. Call $H(T) = H$ and $S(T) = S$.

Now consider the quotient TB^{-1}. The theorem follows if $|\det TB^{-1}| \geq 1$ for $\alpha \in [0,1]$ and $|\det TB^{-1}| \leq 1$ for $\alpha \notin [0,1]$.

But

$$\det TB^{-1} = \frac{\det(H+S)}{\det(H+(2\alpha-1)S)} = \frac{\det(I+H^{-\frac{1}{2}}SH^{-\frac{1}{2}})}{\det(I+(2\alpha-1)H^{-\frac{1}{2}}SH^{-\frac{1}{2}})} \quad .$$

Since $H \in \Sigma_n$, $H^{-\frac{1}{2}}$ exists in Σ_n and $\hat{S} = H^{-\frac{1}{2}}SH^{-\frac{1}{2}}$ is skew-Hermitian. Thus the roots of $I+\hat{S}$ are of the form $1 + i\lambda_j$ while those of $I+(2\alpha-1)\hat{S}$ are of the form $1+i(2\alpha-1)\lambda_j$, λ_j real, $j = 1,\ldots,n$.

Now,

$$|\det TB^{-1}| = \prod_j |1+i\lambda_j| \Big/ \prod_j |1+i(2\alpha-1)\lambda_j| = \prod_j |1+i\lambda_j / 1+i(2\alpha-1)\lambda_j| \quad .$$

Thus we have

$$|2\alpha-1| \leq 1 \quad \text{if} \quad \alpha \in [0, 1] \quad \text{and} \quad |2\alpha-1| \geq 1 \quad \text{if} \quad \alpha \notin [0, 1] \quad .$$

This means

$$|1 + i\lambda_j / 1 + i(2\alpha-1)\lambda_j| \geq 1$$

if

$$\alpha \in [0, 1] \quad \text{and} \quad |1 + i\lambda_j / 1 + i(2\alpha-1)\lambda_j| \leq 1$$

if

$$\alpha \notin [0, 1] \quad .$$

And, finally,

$$\prod_j |1 + i\lambda_j / 1 + i(2\alpha-1)\lambda_j| \geq 1$$

if

$$\alpha \in [0, 1] \quad \text{and} \leq 1 \text{ if} \quad \alpha \notin [0, 1] \quad .$$

Thus we conclude:

$$\det T \geq |\det B| \quad \text{if} \quad \alpha \in [0, 1]$$

and

$$\det T \leq |\det B| \quad \text{if} \quad \alpha \notin [0, 1]$$

which is equivalent to the statement of the theorem. The special case $\alpha = \frac{1}{2}$ is Hadamard's result.

We might briefly state:

$$|\det A| \leq \det T(A) \quad \text{if} \quad A \in \Pi_n^{(\alpha)}, \ \alpha \in [0, 1] \quad .$$

It should be noted that the method of proof indicates the function $f(\alpha) = \det(\alpha T + (1-\alpha)T^*)$, $T \in \Pi_n$, attains a min for $\alpha = \frac{1}{2}$ and is decreasing everywhere to the left and increasing everywhere to the right (see Figure 2).

Finally, then, if $A \in \Pi_n$ we have verified the following sequence of inequalities:

$$(6.7) \qquad \det H(A) \leq \underbrace{|\det(\alpha T(A) + (1-\alpha)T(A)^*)|}_{\alpha \in [0,1]}$$

$$\leq \det T(A) \leq \underbrace{|\det(\beta T(A) + (1-\beta)T(A)^*)|}_{\beta \in R - [0,1]} \quad .$$

It is worth noting that the classes $\Pi_n^{(\alpha)}$, $\alpha \in [0, 1]$, for which we have verified Hadamard's inequality differ from and are not contained in the class for which GKK and F extended the Hadamard and Szasz inequalities.

(6.8) <u>Example</u>. Let

$$A = \begin{pmatrix} 3 & 3 & -3 \\ 1 & 4 & -6 \\ -1 & -2 & 5 \end{pmatrix} \quad .$$

Then $A \in \Pi_n^{(\frac{3}{4})}$ by direct verification, but A is not a GKK matrix [14] since

$$\det\begin{pmatrix} 3 & -3 \\ 4 & -6 \end{pmatrix} \det\begin{pmatrix} 1 & 4 \\ -1 & -2 \end{pmatrix} = (-6)(2) = -12 \not\geq 0 \quad .$$

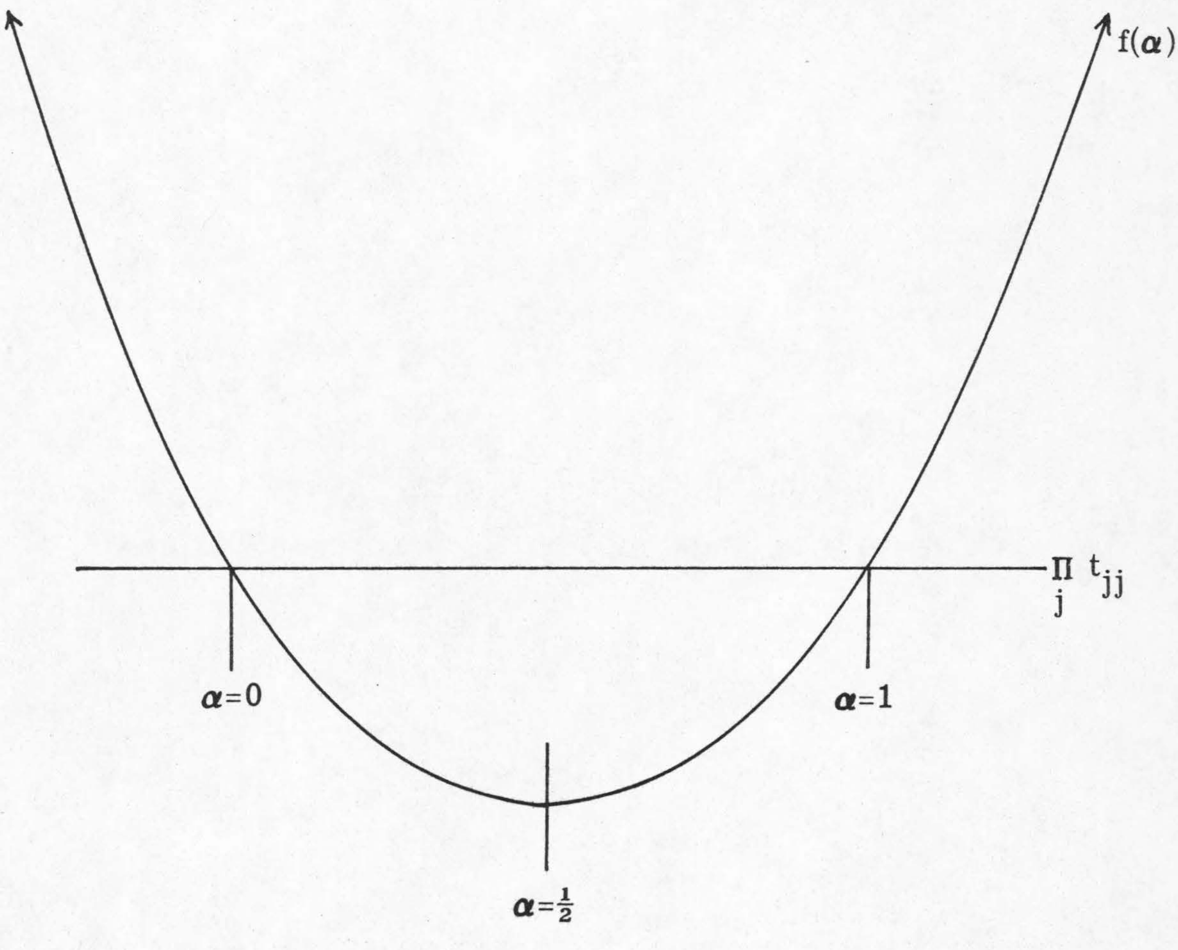

Figure 2

An elaboration on the systematic reversal of Hadamard's inequality in some subclasses of Π_n (such as in part (ii) of (6.6)) is given in the following.

(6.8.5) Observation. If $A = (a_{ij}) \in \Pi_n$ is in the normal form given by (1.21), then

$$\prod_{i=1}^{n} \operatorname{Re}(a_{ii}) \leq |\det A| \quad ,$$

the reverse of Hadamard's inequality. Equality holds if and only if $A \in \Sigma_n$.

Proof: Merely notice that for this class of matrices our statement is equivalent to the Ostrowski-Taussky inequality (4.4) since $H(A) = D$ and $\det D \leq |\det D+S|$.

The interesting way in which the classes $\Pi_n^{(\alpha)}$ parallel Σ_n suggests that they may provide fruitful ground for generalizing other classical inequalities for Σ_n. Inequalities due to Szasz [34], Thompson [52], Marcus [31] and Bergstrom [11] each in some respect generalize Hadamard's inequality within Σ_n and thus would be natural to consider in $\Pi_n^{(\alpha)}$.

In the second direction of generalization of Hadamard's inequality we are able to consider all of Π_n and give a systematic estimate for how close the inequality is to being valid. We consider $\Pi_n(R)$ and $\Pi_n(C)$ separately.

(6.9) Theorem. Suppose $A \in \Pi_n(R)$. Then

$$\det A \leqslant K \det T(A)$$

where

$$K = \det c(I+B)^{-1}, \quad B = H(A)^{-1}S(A)$$

and

$$c = 1 + \max |\lambda(B)|^2 \quad .$$

[Note 1: $0 \leqslant \det(I+B)^{-1} \leqslant 1$, $c \geqslant 1$.]

[Note 2: $A \in \Sigma_n$ implies $K = 1$ which is the original Hadamard result.]

Proof: $\det A = \det H(A) \det(I+B)$ implies $\det A^{-1} = \det(I+B)^{-1} \det H(A)^{-1}$ $\leqslant \det(I+B)^{-1} \det c H(A^{-1})$ by (4.3). But $\det(I+B)^{-1} \det c H(A^{-1}) = \det c(I+B)^{-1}$ $\det H(A^{-1}) = K \det H(A^{-1}) \leqslant K \det T(A^{-1})$ by the original Hadamard inequality. Since $\Pi_n(R)$ is closed under inversion (1.5) we may just as well replace A by A^{-1} to yield $\det A \leqslant K \det T(A)$, the desired result. [Note: By (1.9) and (1.10) both quantities are positive real numbers.]

(6.10) Theorem. Suppose $A \in \Pi_n(C)$. Then

$$|\det A| \leqslant K \det T(A)$$

where

$$K = |\det c(I+B)^{-1}|, \quad B = H(A)^{-1}S(A)$$

and

$$c = \frac{2}{m+1} \quad \text{and} \quad m = \min \operatorname{Re} \lambda(A^{-1}A^*) \quad .$$

Proof: The proof proceeds formally just as that of (6.9) except that the alternate definition of c is justified by (4.11).

To show that the type of estimate of (6.9) or (6.10) is actually economical, consider in closing an example.

(6.11) Example. Let $A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$. $A \in \Pi_2(R)$ and we will check all the quantities required by (6.9).

$$\det A = 3 \ ; \ \det T(A) = 2 \ ;$$

$$B = \begin{pmatrix} 0 & -\frac{1}{2} \\ 1 & 0 \end{pmatrix}; \quad c = 1 + \frac{1}{2} = \frac{3}{2} \ ;$$

$$(I+B) = \begin{pmatrix} 1 & -\frac{1}{2} \\ 1 & 1 \end{pmatrix}; \quad (I+B)^{-1} = \frac{2}{3}\begin{pmatrix} 1 & \frac{1}{2} \\ -1 & 1 \end{pmatrix};$$

$$c(I+B)^{-1} = \frac{3}{2}\frac{2}{3}\begin{pmatrix} 1 & \frac{1}{2} \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ -1 & 1 \end{pmatrix};$$

and $K = \det c(I+B)^{-1} = 3/2$, and we have $\det A = 3 = (3/2)2 = K \det T(A)$, a case in which equality is realized in (6.9).

BIBLIOGRAPHY

1. A. R. Amir-Moéz and A. Horn, Singular Values of a Matrix, Am. Math. Monthly, Vol. 65, No. 10, December, 1958, pp. 742-748.

2. K. J. Arrow and M. McManus, A Note on Dynamic Stability, Econometrica, Vol. 26, 1958, pp. 448-454.

3. C. S. Ballantine, Products of Positive Definite Matrices. I, Pacific J. Math., Vol. 23, 1967, pp. 427-433.

4. C. S. Ballantine, Products of Positive Definite Matrices. II, Pacific J. Math., Vol. 24, 1968, pp. 7-17.

5. C. S. Ballantine, Products of Positive Definite Matrices. III, J. Algebra, Vol. 10, 1968, pp. 174-182.

6. C. S. Ballantine, Products of Positive Definite Matrices. IV, Linear Algebra and Appl., Vol. 3, 1970, pp. 79-114.

7. C. S. Ballantine, Stabilization by a Diagonal Matrix, Proc. Am. Math. Soc., Vol. 25, 1970, pp. 728-734.

8. S. Barnett and C. Storey, Analysis and Synthesis of Stability Matrices, J. of Differential Equations, Vol. 3, 1967, pp. 414-422.

9. S. Barnett and C. Storey, Some Applications of the Lyapunov Matrix Equation, J. Inst. Maths. Appl., Vol. 4, 1968, pp. 33-42.

10. S. Barnett and C. Storey, "Matrix Methods in Stability Theory," Nelson, London, 1970.

11. R. Bellman, "Introduction to Matrix Analysis," McGraw-Hill, New York, 1960.

12. D. Carlson, A New Criterion for H-Stability of Complex Matrices, Linear Algebra and Appl., Vol. 1, 1968, No. 1, pp. 59-64.

13. D. Carlson and H. Schneider, Inertia Theorems for Matrices: The Semidefinite Case, J. Math. Anal. Appl., Vol. 6, 1963, pp. 430-446.

14. K. Fan, Subadditive Functions on a Distributive Lattice and an Extension of Szasz's Inequality, J. of Mathematical Analysis and Appl., Vol. 18, 1967, pp. 262-268.

15. K. Fan, An Inequality for Subadditive Functions on a Distributive Lattice with Applications to Determinantal Inequalities, Linear Algebra and Appl., Vol. 1, 1968, pp. 33-38.

16. K. Fan, Lecture, Auburn Matrix Theory Conference, June 1970.

17. K. Fan, Generalized Cayley Transforms and Strictly Dissipative Matrices, Linear Algebra and Appl., Vol. 5, 1972, pp. 155-172.

18. K. Fan, On Real Matrices with Positive Definite Symmetric Component, to appear.

19. M. Fiedler and V. Pták, On Matrices with Non-Positive Off-Diagonal Elements and Positive Principal Minors, Czech. Mathematical Journal, Vol. 12, 1962, pp. 382-400.

20. M. Fiedler and V. Pták, Some Generalizations of Positive Definiteness and Monotonicity, Numerische Mathematic, Vol. 9, 1966, pp. 163-172.

21. F. R. Gantmacher, "Matrix Theory," Vols. I, II, Chelsea, New York, 1959.

22. F. R. Gantmacher and M. G. Krein, "Oszillationsmatrizen, Oszillationskerne und kleine Schwingungen mechanischer Systeme," Akademie-Verlag, Berlin, 1960.

23. J. Genin and J. S. Maybee, A Stability Theorem for a Class of Damped Dynamic Systems and Some Applications, J. Inst. Maths. Appl., Vol. 2, 1966, pp. 343-357.

24. J. Hadamard, Resolution d'une Question Relative aux Determinants, Bull Sci. Math., Vol. 2, 1893, pp. 240-248.

25. E. Hille and R. S. Phillips, "Functional Analysis and Semigroups," American Mathematical Society, Providence, 1957.

26. C. R. Johnson, Positive Definite Matrices, Am. Math. Monthly, Vol. 77, No. 3, March 1970, pp. 259-264.

27. C. R. Johnson, An Inequality for Matrices Whose Symmetric Part Is Positive Definite, Linear Algebra and Appl., to appear.

28. G. T. Joyce and S. Barnett, Remarks on the Inertia of a Matrix, Linear Algebra and Appl., Vol. 3, 1970, pp. 1-5.

29. T. Kato, Some Mapping Theorems for the Numercial Range, Proc. Japan Acad., Vol. 41, No. 8, 1965, pp. 652-655.

30. D. M. Koteljanskii, The Theory of Nonnegative and Oscillating Matrices (Russian), Ukrain, Math. Z., Vol. 2, 1950, pp. 94-101. [English translation, Am. Math. Soc. Transl., Ser. 2, Vol. 27, 1963, pp. 1-8.]

31. M. Marcus, The Hadamard Theorem for Permanents, Proc. Am. Math. Soc., Vol. 15, 1964, pp. 967-973.

32. M. Marcus and H. Minc, "A Survey of Matrix Theory and Matrix Inequalities," Allyn and Bacon, Boston, 1964.

33. M. Marcus and W. Watkins, Partitioned Hermitian Matrices, Duke Math. J., Vol. 38, 1971, pp. 237-249.

34. L. Mirsky, On a Generalization of Hadamard's Determinantal Inequality Due to Szasz, Archiv. Mathematik, Vol. 8, 1957, pp. 274-275.

35. A. Ostrowski and H. Schneider, Some Theorems on the Inertia of General Matrices, J. of Math. Analysis and Appl., Vol. 4, 1962, pp. 72-84.

36. R. S. Phillips, Dissipative Hyperbolic Systems, Transactions AMS, Vol. 86, 1957, pp. 109-173.

37. J. Quirk and R. Saposnik, "Introduction to General Equilibrium Theory and Welfare Economics," New York, McGraw-Hill, 1968.

38. J. Quirk and R. Ruppert, Qualitative Economics and the Stability of Equilibrium, Review of Economic Studies, Vol. 32, 1965, pp. 311-325.

39. J. Schur, Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen, J. für die reine und angewandte Mathematik, Vol. 140, 1911, pp. 1-28.

40. O. Taussky, A Recurring Theorem on Determinants, Am. Math. Monthly, Vol. 56, No. 10, December 1949, pp. 672-676.

41. O. Taussky, A Determinantal Inequality of H. P. Robertson, I, J. of the Washington Academy of Sciences, Vol. 47, No. 8, August 1957, pp. 263-264.

42. O. Taussky, Problem 4846, Am. Math. Monthly, Vol. 66, 1959, p. 427.

43. O. Taussky, A Remark on a Theorem of Lyapunov, J. of Math. Analysis and Appl., Vol. 2, No. 1, February 1961, pp. 105-107.

44. O. Taussky, A Generalization of a Theorem of Lyapunov, J. of the Society of Industrial and Applied Mathematics, Vol. 9, No. 4, December 1961, pp. 640-643.

45. O. Taussky, On Stable Matrices, from Colloques Internationaux Centre National De La Recherche Scientifique, No. 165, Programmation En Mathématiques Numériques, 1966.

46. O. Taussky, Positive Definite Matrices, in "Inequalities," 1967, Academic Press, New York, pp. 309-319.

47. O. Taussky, Positive Definite Matrices and Their Role in the Study of the Characteristic Roots of General Matrices, Advances in Mathematics, Vol. 2, June 1968, Academic Press, New York, pp. 175-186.

48. O. Taussky, A Remark Concerning the Similarity of a Finite Matrix A and A^*, Mathematische Zeitschrift, Vol. 117, 1970, pp. 189-190.

49. O. Taussky, The Role of Symmetric Matrices in the Study of General Matrices, Auburn Matrix Symposium, June 1970, and J. of Linear Algebra, Vol. 5, 1972, pp. 147-154.

50. O. Taussky, Hilbert's Theorem 90 in Matrix Rings, to appear.

51. A. M. Ostrowski and O. Taussky, On the Variation of the Determinant of a Positive Definite Matrix, Proc. Koninkl. Nederl. Acad. Wetensch. Amsterdam, Ser. A, Vol. 54, 1951, pp. 383-385.

52. R. C. Thompson, A Determinantal Inequality for Positive Definite Matrices, Canadian Math. Bull., Vol. 4, No. 1, January 1961, pp. 57-62.

53. E. P. Wigner, On Weakly Positive Matrices, Canadian Journal of Mathematics, Vol. 15, 1963, pp. 313-317.