DYNAMIC RESPONSE OF HYSTERETIC SYSTEMS WITH APPLICATION TO A SYSTEM CONTAINING LIMITED SLIP

Thesis by

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In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California

1972

(Submitted September 17, 1971)

ACKNOWLEDGMENTS

The author expresses his sincere appreciation to Professor W. D. Iwan for his invaluable guidance, suggestions, and patience throughout this investigation. The assistance of the faculty of Applied Mechanics is also gratefully appreciated.

Part of this research was sponsored by the National Science Foundation. The author is very fortunate to have received financial assistance from the Fannie and John Hertz Foundation.

The author thanks the secretarial staff, especially Mrs. Lynne Lacy, for the competent typing of the manuscript. Further thanks go to Miss Cecilia Lin for her assistance.

The author also expresses his gratitude to his wife, Florence, for her endless patience, warm encouragement, and assistance.

ABSTRACT

A general class of single degree of freedom systems possessing rate-independent hysteresis is defined. The hysteretic behavior in a system belonging to this class is depicted as a sequence of single-valued functions; at any given time, the current function is determined by some set of mathematical rules concerning the entire previous response of the system. Existence and uniqueness of solutions are established and boundedness of solutions is examined.

An asymptotic solution procedure is used to derive an approximation to the response of viscously damped systems with a small hysteretic nonlinearity and trigonometric excitation. Two properties of the hysteresis loops associated with any given system completely determine this approximation to the response: the area enclosed by each loop, and the average of the ascending and descending branches of each loop.

The approximation, supplemented by numerical calculations, is applied to investigate the steady-state response of a system with limited slip. Such features as disconnected response curves and jumps in response exist for a certain range of system parameters for any finite amount of slip.

To further understand the response of this system, solutions of the initial-value problem are examined. The boundedness of solutions is investigated first. Then the relationship between initial conditions and resulting steady-state solution is examined when multiple steady-state solutions exist. Using the approximate analysis and numerical

calculations, it is found that significant regions of initial conditions in the initial condition plane lead to the different asymptotically stable steady-state solutions.

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I. INTRODUCTION

Hysteresis is derived from a Greek word meaning "to lag". It is commonly used to describe the phenomenon exhibited by physical systems in which changes in one or more dependent system variables lag behind those of their independent variables. Hysteretic systems include both systems which are history dependent (hereditary) and those which are not.

Hysteresis is present in a wide range of physical systems. For example, the force-deflection relation of virtually all structural systems behave hysteretically as a result of the yielding of one or more elements or of interface effects between elements. In many situations it is necessary to include hysteretic behavior in formulating a mathematical model to adequately describe the dynamical nature of a physical system.

The hysteretic systems considered in this investigation are those whose motion can be described by a single variable. The mathematical models representing hysteresis in these systems are assumed to belong to a class which depends upon the previous trajectory of this variable, but not upon the rate at which it is traversed. This type of hereditary dependence is called rate-independent hysteresis. The terms hysteresis and rate-independent hysteresis are used interchangeably in the text.

The following definitions present the distinction made in this thesis between a "hereditary function" and a "slip-function" which both appear in the dynamical equation studied:

<u>hereditary function</u>: A mathematical representation for the history dependent behavior of a system. This function is formulated as a sequence of continuous, single-valued functions of x(t) where x(t) is the system response. At any given time, the current function is determined from the entire previous history of x(t). During a transition from one function to the next, the value of both functions are equal.

<u>slip-function</u>: A mathematical representation for slip-friction in a physical system. It is defined as a positive (negative) constant when $\dot{x}(t) > 0$ ($\dot{x}(t) < 0$) where x(t) is the system response.

Even though a slip-function is not a hereditary function by these definitions, the presence of slip elements, whose properties are described by slip-functions, in a physical model may result in behavior which can be described by a hereditary function. (e. g., a system with bilinear hysteresis. Figures 3 and 4 in Chapter II show the restoring force and the physical model, respectively.) Some hysteretic systems consisting of a configuration of elements must be described by an equation of motion in which both a slip-function and a hereditary function appear explicitly.

Hysteretic systems possessing a single degree of freedom have been the object of considerable study. The analysis of mathematical models describing such systems is complicated when a hereditary function of the type considered here is present since this function is formulated as a sequence of functions, each of which is determined by a set of mathematical rules concerning the entire previous response of the system. Possibly for this reason, there has been practically no effort to determine fundamental properties of solutions of this type of hysteretic system using exact analytical techniques. However, there have been analytical investigations dealing with dynamical equations in which a slip-function appears explicitly. References 1 and 2 examine the problems of existence, uniqueness, boundedness, and existence and stability of periodic solutions for certain systems of this general type.

Reference 1 considers a system containing a slip-function defined as the signum function, which vanishes when $\dot{x}(t) = 0$ where $x(t)$ is the system response. The use of this function presents some difficulties in physical interpretation and leads to certain complications in treating such problems as existence and uniqueness of solutions.

In Reference 2, a more physically meaningful representation for slip-friction is used. At $\dot{x}(t) = 0$, the slip-function is defined to have a value which leaves $\ddot{x}(t) = 0$ provided the net force on the system excluding friction is less than the current slip level. When the net force excluding friction is greater than the slip level, the slip-function is defined to have a value equal to the slip level and slipping occurs. No mathematical complications arise when this particular representation is used for the slip-function.

Neither Reference 1 nor Reference 2 considers a system with history dependence. The state of the systems examined is completely determined by specifying the current values of time, the response, and the time derivative of the response. It may be noted, however, that if the particular representation of Reference 2 for the slip-function is used to describe the properties of a slip element in a model to generate a hereditary function, no mathematical difficulties arise.

Due to the difficulty in using exact analytical techniques to determine properties of the solutions to hysteretic systems, most investigations have been made using approximate analytical techniques[3-11], numerical methods[12-17], and electric analog techniques[18-21]. The existence of solutions has been verified by the numerical and electric analog solutions. Both bounded and constantly growing solutions have been observed[3, 6-10, 13, 15, 19, 20]. The steady-state frequency response curves for most hysteretic systems are single-valued and possess a softening character[3, 6, 8, 9, 13]. Unusual features of the harmonic steady-state response have been found for models of systems in which the hysteretic behavior is limited to only an interval of values of the response[7, 10, 20]. The most striking of these is the presence of a disconnected portion of the response curve for certain values of system parameters. This feature inherently leads to the existence of multiple steady-state solutions for a given excitation frequency and to jump phenomenon. Steady-state ultraharmonic response to trigonometric excitation has also been observed in hysteretic systems. Recently, the nature of the third-order ultraharmonic oscillation in a hysteretic system has been discussed in detail[11].

The objectives of the present investigation are:

1. To define a general class of dynamical systems possessing

hysteresis and to determine fundamental properties of the exact solutions of these systems.

2. To develop, using a consistent mathematical procedure, a first order approximation to the harmonic response of a class of hysteretic systems to trigonometric excitation.
3. To apply the approximation to gain further insight into the general behavior of the harmonic response of a system with limited slip.

Chapter II begins with the mathematical description of a class of dynamical systems with rate-independent hysteresis. It is shown that there exists a unique solution to the corresponding initial-value problem. The remainder of the chapter is devoted to examining the boundedness of solutions to the initial-value problem for general excitation.

The response of a viscously damped system with a small history dependent nonlinearity and trigonometric excitation is examined in Chapter III. Since it is rarely possible to find explicit solutions to nonlinear nonautonomous systems, an approximate method of analysis is used. In Section 3.1, an asymptotic solution procedure is applied to derive an approximation (Approximation I) to the response near a periodic solution of a viscously damped system with trigonometric excitation. Only the nonlinearity is $O(\epsilon)$ as $\epsilon \to 0$. Asymptotic solution procedures similar to the one presented have been used to determine harmonic resonant response in single degree of freedom nonautonomous systems [22, 23, 24], but they are not usually applied to the case where the level of viscous damping is independent of the nonlinearity parameter.

With the additional assumptions that the levels of excitation and viscous damping and the frequency detuning are $O(\epsilon)$ as $\epsilon \to 0$, the procedure followed above is used to obtain an approximation (Approximation II) to the general harmonic response of the system. The procedure then parallels one given in Reference 22. The steady-state response predicted by Approximations I and II are identical. Thus it is shown that the steady-state response predictions of Approximation II are valid for a larger range of system parameters than are required for the validity of the general response predictions.

Section 3.2 is devoted to the examination of the response predicted by Approximation II for history dependent nonlinearities.

In Chapters IV and V, the general formulation derived in Chapter III is applied to investigate the response of a limited slip system. In addition to providing an example of the application of the general formulation, the system itself is important in several respects:

1. The system can be formulated from a physical model. Thus no arbitrary mathematical assumptions are required to determine its transient behavior.
2. The physical model is an element common in many systems. Most hysteretic models do not allow for "limiting" behavior. Thus this investigation provides insight into understanding the dynamic behavior of similar systems or more complicated systems which possess elements of this type.

3. The system exhibits unusual steady-state response behavior not found in standard hysteretic models.

In Chapter IV, the harmonic steady-state response of the limited slip system to trigonometric excitation is examined in detail. The system possesses finite viscous damping in all the examples illustrated since some viscous damping is probably present in most physical systems. One of the features of the response is the possibility of triple-valued steady-state solutions for a given frequency of excitation.

To gain further insight into the response of the system, solutions to the initial-value problem are studied in Chapter V. In Section 5.1, boundedness of solutions to the initial-value problem with general excitation is examined. In Section 5.2, the relationship between initial conditions and resulting steady-state solution is examined when multiple steady-state solutions are possible. The study of relationships of this type have been made for nonlinear systems with no hysteresis[25] where multiple steady-state solution behavior is a more common occurrence. However, this aspect of the harmonic response has not been previously investigated for rate-independent hysteretic systems.

II. DYNAMICAL SYSTEMS WITH HYSTERESIS

A class of dynamical systems with rate-independent hysteresis is formulated and discussed in this chapter. The first section deals with the description of the system. The remaining sections are concerned with properties of solutions to the corresponding initial-value problem. Existence and uniqueness of solutions are shown. With assumptions concerning the physical nature of the hysteresis, bounded solution behavior is concluded for the unforced problem. It is then shown that all solutions are bounded for a class of forced hysteretic systems with viscous dissipation. When there is no viscous dissipation, either bounded or unbounded solution behavior can occur in forced hysteretic systems. Two examples with one of the most widely used hysteretic models illustrate this behavior.

2.1 Description of the Dynamic System

Consider the system

$$\ddot{x} + \mathcal{H}\{x(t)\} + R(x, \dot{x}) + \mathcal{S}[\dot{x}, G(t) - \mathcal{H}\{x(t)\} - R(x, \dot{x}), c] = G(t) \qquad (2.1)$$

with the following assumptions:

Assumption 1 (A1): G(t) is piecewise continuous and bounded.

Assumption 2 (A2): $\mathcal{H}\{x(t)\}$ represents a hereditary function and is defined on the space of all functions (called "admissible paths") $x(\tau) \in C^1[0, t]$ with $x(0) = 0$ and $t \in [0, \infty)$. $\mathcal{H}\{x(t)\}$ satisfies the following:

1. There exists a virgin state from which time and path begin such that $t = 0$, $x(0) = 0$, and $\mathcal{H}\{x(t)\} = 0$.

2. During any admissible path, $x(\tau)$ for $\tau \in [0, t]$,

   a. $\mathcal{N}\{x(t)\}$ is defined as a sequence of continuous, single-valued functions of x between path reversals. Denote the sequence, beginning with the initial function at $\tau = 0$, by $h^{(0)}(x)$, $h^{(1)}(x)$, $\cdots$.

   b. At $\tau = 0$, $h^{(0)}(x)$ is known for $-\infty < x < \infty$.

   c. During each reversal $\dot{x} = 0$ at $x = x^{(q)}$ into $\dot{x} > 0$ ($\dot{x} < 0$), the next member of the sequence is known for $x^{(q)} \leq x < \infty$ ($-\infty < x \leq x^{(q)}$).

   d. The transition between any two successive functions $h^{(i)}(x)$ and $h^{(i+1)}(x)$ occurs only at a reversal (e. g., $x = x^{(q)}$), and $h^{(i)}(x^{(q)}) = h^{(i+1)}(x^{(q)})$.

   e. At time τ, the current $h^{(i)}(x)$ is determined uniquely by the path history from the virgin state.

3. The sequence of functions comprising $\mathcal{N}\{x(t)\}$ for a given path is independent of the rate at which the path is traversed.

<u>Assumption 3 (A3)</u>: $R(x, \dot{x})$ is a single-valued function continuous in x and $\dot{x}$.

<u>Assumption 4 (A4)</u>: $\mathcal{S}[\dot{x}, \mathcal{F}(x, \dot{x}, t, \{x(t)\}), c]$ represents a slip-function where $\mathcal{F}(x, \dot{x}, t, \{x(t)\}) \equiv G(t) - \mathcal{N}\{x(t)\} - R(x, \dot{x})$ and c is a positive constant. When $\dot{x} \neq 0$,

$$\mathcal{S}[\dot{x}, \mathcal{F}(x, \dot{x}, t, \{x(t)\}), c] = c \operatorname{sgn}(\dot{x}) \tag{2. 2}$$

where
$$\operatorname{sgn}(z) \equiv \begin{cases} 1 & \text{for } z > 0 \\ 0 & \text{for } z = 0 \\ -1 & \text{for } z < 0 \end{cases} \tag{2. 3}$$

When $\dot{x} = 0$,

$$\mathcal{S}[0, \mathcal{F}(x, 0, t, \{x(t)\}), c] = \begin{cases} c & \text{for } \mathcal{F}(x, 0, t, \{x(t)\}) > c \\ \mathcal{F}(x, 0, t, \{x(t)\}) & \text{for } |\mathcal{F}(x, 0, t, \{x(t)\})| \leq c \\ -c & \text{for } \mathcal{F}(x, 0, t, \{x(t)\}) < -c \end{cases} \tag{2. 4}$$

Remarks

The value of $\mathcal{N}\{x(t)\}$ at time t is dependent upon the previous path history. Define the state of $\mathcal{N}\{x(t)\}$ at $t = t_0$ as: A configuration of $\mathcal{N}\{x(t)\}$ at $t = t_0$ sufficient to determine the $h^{(i)}(x)$ uniquely for any $x(t) \in C^1[t_0, \infty)$. In view of Assumptions 1 and 2 of (A2), the state of $\mathcal{N}\{x(t)\}$ is known at $t_0 = 0$. In view of Assumption 3 of (A2), the state of $\mathcal{N}\{x(t)\}$ can be determined from the sequence of turning points during $t \in (0, t_0]$.

$\mathcal{N}\{x(t)\}$ is unique to within a single-valued function of x, e. g., $\mathcal{N}\{x(t)\} + R(x, \dot{x}) = \widetilde{\mathcal{N}}\{x(t)\} + \widetilde{R}(x, \dot{x})$ where $\widetilde{R}(x, \dot{x}) = R(x, \dot{x}) - g(x)$ and $\widetilde{h}^{(i)}(x) = h^{(i)}(x) + g(x)$ for $i = 0, 1, \cdots$.

$\mathcal{S}[\dot{x}, \mathcal{F}(x, \dot{x}, t, \{x(t)\}), c]$ describes the properties of slip-friction where $\mathcal{F}(x, \dot{x}, t, \{x(t)\}) \equiv G(t) - \mathcal{N}\{x(t)\} - R(x, \dot{x})$. Some investigators have used $c \operatorname{sgn}(\dot{x})$ where $\operatorname{sgn}(\dot{x})$ is defined in (2. 3). However, since $c \operatorname{sgn}(0) = 0$, a difficulty arises in some situations. Consider the case where the response of (2. 1) comes to rest from $\dot{x} > 0$ at $x = x^{(s)}$ when $t = t_0$. Also let $(G(t) - \mathcal{N}\{x(t)\} - R(x^{(s)}, 0)) \in (0, c)$ for $t \in [t_0, t_1]$ where $t_1 > t_0$. If $c \operatorname{sgn}(\dot{x})$ is used for the slip-function, then $\ddot{x}(t_0) > 0$. Thus the solution can not remain at $x = x^{(s)}$. In addition, there exists no solution of (2.1) leaving $x(t_0) = x^{(s)}$ with either $\dot{x} > 0$ or $\dot{x} < 0$. Consequently, the solution of (2.1) ceases to exist after $t = t_0$. On the other hand, if $\mathcal{S}[\dot{x}, \mathcal{F}(x, \dot{x}, t, \{x(t)\}), c]$ is used for the slip-function, then $\mathcal{S}[0, \mathcal{F}(x^{(s)}, 0, t, \{x(t)\}), c] = G(t) - \mathcal{N}\{x(t)\} - R(x^{(s)}, 0)$ for $t \in [t_0, t_1]$ and the solution remains at rest. $\mathcal{S}[\dot{x}, \mathcal{F}(x, \dot{x}, t, \{x(t)\}), c]$ is generally discontinuous.

Figure 1 shows a slip-function $\widetilde{\mathcal{S}}[\dot{x}, \mathcal{F}, c]$ which is formed by adding a part of $R(x, \dot{x})$ to $\mathcal{S}[\dot{x}, \mathcal{F}, c]$: $R(x, \dot{x}) + \mathcal{S}[\dot{x}, \mathcal{F}, c] = \widetilde{R}(x, \dot{x}) + \widetilde{\mathcal{S}}[\dot{x}, \mathcal{F}, c]$ where $\widetilde{R}(x, \dot{x}) = R(x, \dot{x}) - s(\dot{x})$, $\widetilde{\mathcal{S}}[\dot{x}, \mathcal{F}, c] = \mathcal{S}[\dot{x}, \mathcal{F}, c] + s(\dot{x})$ and $s(\dot{x}) \in C(-\infty, \infty)$. Therefore, even though (A4) describes a slip-function with only constant slip level for $\dot{x} \neq 0$, the system (2.1) can possess the more general slip properties of $\widetilde{\mathcal{S}}[x, \mathcal{F}, c]$.

2.2 Existence and Uniqueness of Solutions to the Initial-Value Problem

Consider the initial-value problem of (2.1)

$$\begin{aligned} \ddot{x} + \mathcal{N}\{x(t)\} + R(x, \dot{x}) + \mathcal{S}[\dot{x}, G(t) - \mathcal{N}\{x(t)\} - R(x, \dot{x}), c] &= G(t) \\ x(0) &= 0 \\ \dot{x}(0) &= 0 \end{aligned} \tag{2.5}$$

The properties of $\mathcal{N}\{x(t)\}$ and $\mathcal{S}[\dot{x}, \mathcal{F}(x, \dot{x}, t, \{x(t)\}), c]$ exclude any direct application of classical techniques to determine the existence and uniqueness of solutions to (2.5). For example, the Lipschitz condition cannot be applied to $\mathcal{N}\{x(t)\}$ which changes form when $\dot{x}$ changes sign, or to $\mathcal{S}[\dot{x}, \mathcal{F}, c]$ which is a discontinuous and multi-valued function of $\dot{x}$. However, $\mathcal{N}\{x(t)\}$ consists of a sequence of single-valued functions which change only at $\dot{x} = 0$, and $\mathcal{S}[\dot{x}, \mathcal{F}, c]$ becomes multi-valued only at

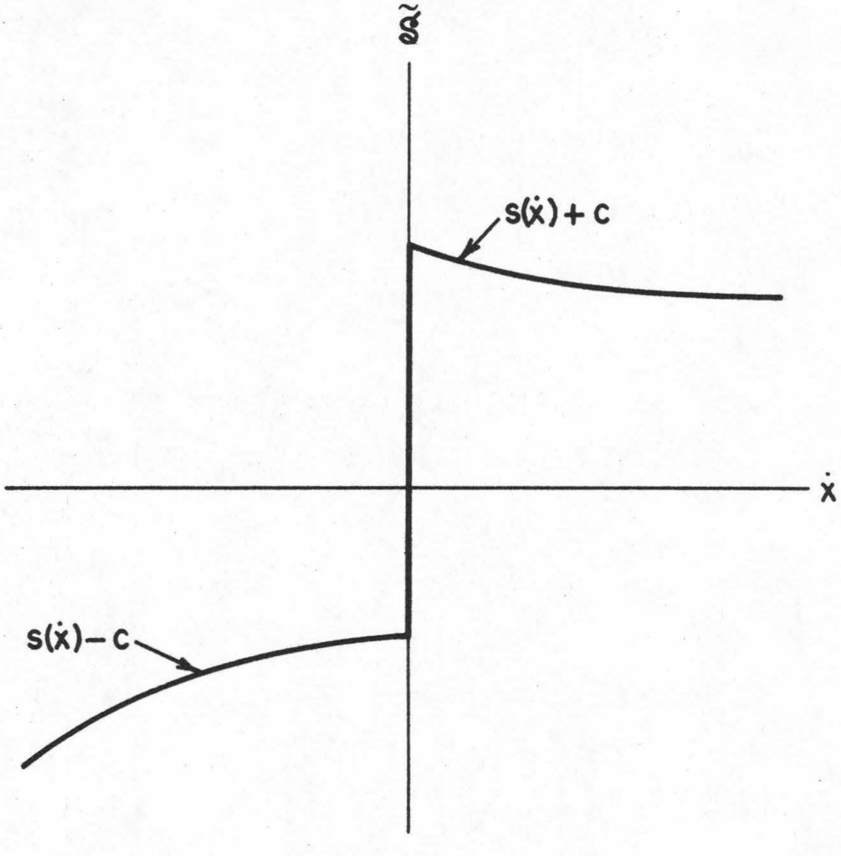


Figure 1: Slip-Function

$\dot{x} = 0$. Thus by considering the response during time intervals in which $\dot{x} \neq 0$ and by continuing the response through intervals when $\dot{x} = 0$, it may be shown that there exists a unique solution to (2. 5).

Since $\mathcal{S}[\dot{x}, \mathcal{F}, c]$ and G(t) are discontinuous, the solutions x(t) of (2. 5) are no more than continuously differentiable.

The following additional assumptions upon $R(x, \dot{x})$ and $\mathcal{N}\{x(t)\}$ will be needed. (The vector notation $\underline{z} = \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix}$ is used below.)

Assumption 5 (A5):

$$|R(x_1, x_2) - R(y_1, y_2)| \leq Q \|\underline{x} - \underline{y}\| \qquad \text{for } \|\underline{x}\|, \|\underline{y}\| < \infty \tag{2. 6}$$

where $\| \ \|$ is any vector norm and Q is a finite constant.

Assumption 6 (A6): Each $h^{(i)}(x)$ satisfies a Lipschitz condition over its respective region of definition. For the $i^{\underline{th}}$ function defined over $x \in I^{(i)}$

$$|h^{(i)}(x) - h^{(i)}(y)| \leq M^{(i)} |x - y| \quad \text{for } x, y \in I^{(i)} \tag{2. 7}$$

where $M^{(i)}$ is a finite constant.

Theorem 1

If Assumptions (A1)-(A6) are satisfied, then there exists a unique, continuously differentiable solution to (2. 5).

Preliminaries: The vector form of (2. 5) with $x_1 = x$ and $x_2 = \dot{x}$ is

$$\frac{d\underline{x}}{dt} = \begin{Bmatrix} x_2 \\ -\mathcal{N}\{x_1(t)\} - R(x_1, x_2) - \mathcal{S}[x_2, G(t) - \mathcal{N}\{x_1(t)\} - R(x_1, x_2), c] + G(t) \end{Bmatrix} \tag{2. 8}$$

$$\underline{x}(0) = \underline{0}$$

If the system changes from $x_1 = x^{(q)}$ and $x_2 = 0$ into $x_2 > 0$ or $x_2 < 0$ (choose $x_2 > 0$ for illustration), (2. 8) becomes

$$\frac{d\underline{x}}{dt} = \begin{Bmatrix} x_2 \\ -h^{(i)}(x_1) - R(x_1, x_2) - c + G(t) \end{Bmatrix} \equiv \underline{f}_i(\underline{x}, t) \quad \text{for } x_1 \geq x^{(q)} \text{ and } x_2 \geq 0 \quad (2.9)$$

where $\mathscr{H}\{x_1(t)\}$ has the form $h^{(i)}(x_1)$. Using the taxicab norm for illustration,(2. 9) implies

$$\|f_i(\underline{x}, t) - f_i(\underline{y}, t)\| = |x_2 - y_2| + |h^{(i)}(x_1) - h^{(i)}(y_1)| + |R(x_1, x_2) - R(y_1, y_2)| \quad (2.10)$$

Substituting (2. 6) and (2. 7) into (2. 10) gives

$$\|f_i(\underline{x}, t) - f_i(\underline{y}, t)\| \leq K^{(i)}\|\underline{x} - \underline{y}\| \quad \text{for } x_1, y_1 \geq x^{(q)} \text{ and } x_2, y_2 \geq 0 \quad (2.11)$$

where $K^{(i)} = \text{Max}(1, M^{(i)}, Q)$. A similar statement may be made if the system changes from $x_1 = x^{(q)}$ and $x_2 = 0$ into $x_2 < 0$. Therefore, the right member of (2. 8) satisfies a vector Lipschitz condition for time intervals between reversals.

The concept of a "stagnation strip" is helpful in understanding the behavior of (2. 8). This concept was used by Železcov[26] to examine the response of a linear oscillator with slip-friction similar to that shown in Figure 1. Consider a trajectory of (2. 8) in the space x_1, x_2, t over the time interval $t \in [t_i, t_{i+1}]$ where t_i and t_{i+1} are the times of two consecutive reversals. The stagnation strips (there may be none or several) lie in the $x_2 = 0$ plane and include all x_1 such that

$$G(t) - c \leq h^{(i)}(x_1) + R(x_1, 0) \leq G(t) + c \quad \text{for } t \in [t_i, t_{i+1}] \quad (2.12)$$

An example is shown in Figure 2. Equations (2. 4) and (2. 12) imply that a trajectory beginning in or entering the $x_2 = 0$ plane in a strip remains there with constant x_1 until it passes through the boundary of the

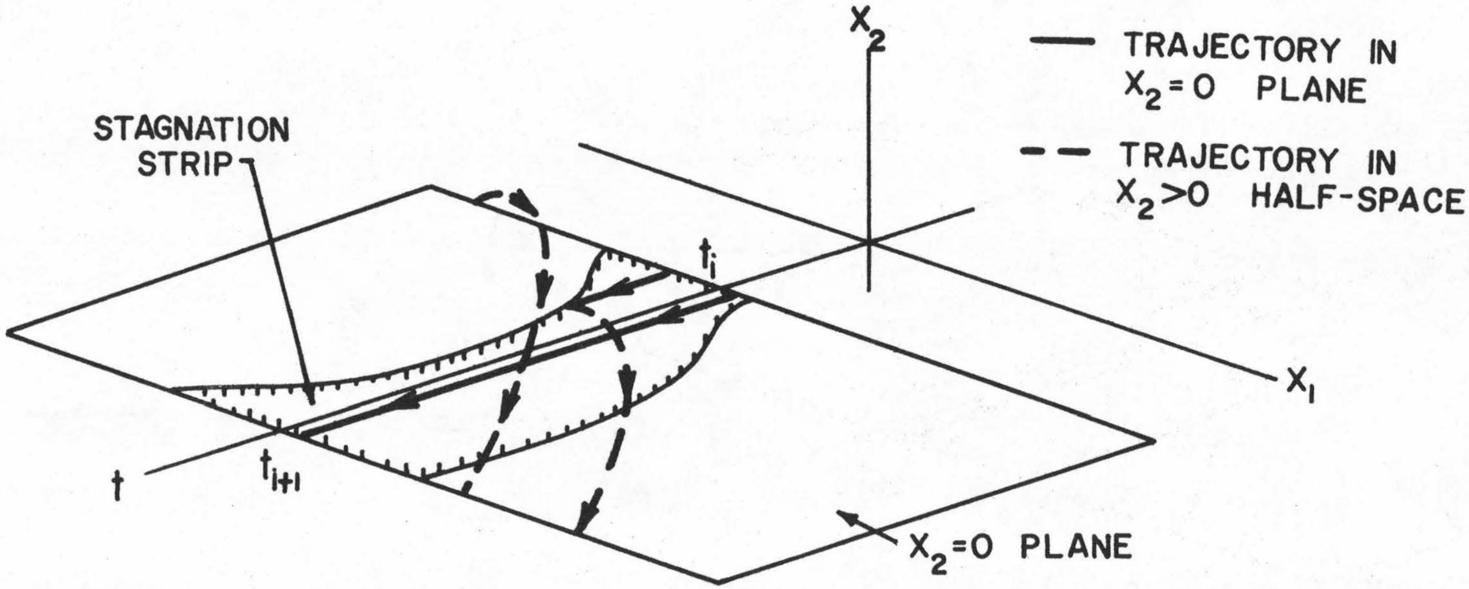


Figure 2: Stagnation Strip

strip. Then it must go into the $x_2 \operatorname{sgn}(G(t) - h^{(i)}(x_1) - R(x_1, 0)) > 0$ half-space. A trajectory beginning in or entering the $x_2 = 0$ plane outside of the strips passes immediately into the $x_2 \operatorname{sgn}(G(t) - h^{(i)}(x_1) - R(x_1, 0)) > 0$ half-space. During a reversal from $h^{(i)}(x_1)$ to $h^{(i+1)}(x_1)$, the corresponding stagnation strips do not necessarily coincide. Let the $i^{\underline{th}}$ strips denote those associated with $h^{(i)}(x_1)$. By (2. 12) and Assumption 2d of (A2) it is seen that during a reversal, if the trajectory does not lie in an $i^{\underline{th}}$ strip, it does not lie in an $i+1^{\underline{th}}$ strip; if it lies in an $i^{\underline{th}}$ strip, it lies in an $i+1^{\underline{th}}$ strip, and both $i^{\underline{th}}$ and $i+1^{\underline{th}}$ boundaries must pass through the trajectory at the same time in the same direction. Therefore, the motion is still uniquely determined independent of the stagnation strips used during the reversal.

If a trajectory is at $x_1 = x^{(q)}$ and $x_2 = 0$ when $t = \tau$, one of three types of motion can occur:

1. $x_2 = 0$ only an instant.
2. $x_2 = 0$ over a finite interval.
3. $x_2 = 0$ for $t \in [\tau, \infty)$. A necessary and sufficient condition for this to occur is

$$\max_{t\in[\tau,\infty)} G(t) - c \le h(x^{(q)}) + R(x^{(q)}, 0) \le \min_{t\in[\tau,\infty)} G(t) + c \tag{2. 13}$$

where $h(x^{(q)})$ is the value of $\mathscr{H}\{x(t)\}$ at $t = \tau$.

In any event the motion is determined uniquely.

<u>Proof</u> <u>of</u> <u>Theorem</u> <u>1</u>: At $t = 0$ (2. 8) becomes

$$\frac{d\underline{x}}{dt} = \begin{Bmatrix} x_2 \\ -h^{(0)}(x_1) - R(x_1, x_2) - \mathcal{S}[x_2, G(t) - h^{(0)}(x_1) - R(x_1, x_2), c] + G(t) \end{Bmatrix} \tag{2. 14}$$

$$\underline{x}(0) = \underline{0}$$

Three cases are possible:

1. $x_1 = 0$ is in a stagnation strip for $t \in [0, \infty)$.
2. $x_1 = 0$ is in a stagnation strip at $t = 0$ and passes through the boundary at $t = t_1$.
3. $x_1 = 0$ is not in a stagnation strip at $t = 0$.

Case 1: The system (2.14) becomes

$$\frac{d\underline{x}}{dt} = \begin{Bmatrix} x_2 \\ 0 \end{Bmatrix}$$

$$\underline{x}(0) = \underline{0} \qquad (2.15)$$

The unique solution of (2.8) is $\underline{x}(t) = \underline{0}$ for $t \in [0, \infty)$.

Cases 2 and 3: The unique solution for $t \in [0, t_1]$ (for Case 3, $t_1 = 0$) is $\underline{x}(t) = \underline{0}$. When $t > t_1$, the solution must go into either the $x_2 > 0$ or $x_2 < 0$ half-space. (Choose $x_2 > 0$ for illustration.) Then, (2.8) is

$$\frac{d\underline{x}}{dt} = \begin{Bmatrix} x_2 \\ -h^{(0)}(x_1) - R(x_1, x_2) - c + G(t) \end{Bmatrix} \qquad (2.16)$$

Consider the initial-value problem for (2.16) with

$$\underline{x}(t_1) = \underline{0} \qquad (2.17)$$

The right member of (2.16) satisfies a vector Lipschitz condition for $x_1 \geq 0$, $x_2 \geq 0$, and $t \geq t_1$. Under the hypotheses in the statement of Theorem 1, a proof[27] for the existence and uniqueness of solutions to nonautonomous initial-value problems can be applied. It follows that there exists a unique solution to (2.16) and (2.17), $\underline{x}^{(0)}(t)$, which can be continued until either $\|\underline{x}^{(0)}(t)\| \to \infty$ as $t \to t_2$, or $x_2^{(0)}(t_2) = 0$. Define the continuous function

$$\underline{x}(t) = \begin{cases} \underline{0} & \text{for } t \in [0, t_1) \\ \underline{x}^{(0)}(t) & \text{for } t \in [t_1, t_2) \end{cases} \tag{2.18}$$

If t_2 is unbounded, then (2.18) gives the unique solution of (2.8). If $\|\underline{x}^{(0)}(t)\| \to \infty$ as $t \to t_2 < \infty$, then (2.18) gives the unique solution of (2.8) which is mapped out completely for $t \in [0, t_2)$. Similar statements may be made for the existence of a unique solution for $t \in [0, t_2)$ if the solution goes into $x_2 < 0$ half-space for $t > t_1$. Finally, if $x_2^{(0)}(t_2) = 0$ for $t_2 < \infty$, then repeated application of this procedure with the appropriate $h^{(i)}(x)$ leads to the conclusion that there exists a unique, continuous solution, $\underline{x}(t)$, to (2.8). This proves Theorem 1.

Other Initial Conditions

The reasoning in the above proof may be applied to show the existence of a unique, continuously differentiable solution to the initial-value problem (2.5) with $x(0) = 0$ and $\dot{x}(0) = \dot{x}^{(0)} \neq 0$.

The initial-value problem (2.5) gives initial data at $t = 0$ to insure a unique determination of the sequence of functions $h^{(0)}(x)$, $h^{(1)}(x), \cdots$. From the Remarks in Section 2.1, it is also meaningful to consider the initial-value problem at $t = t_0 > 0$ with initial data $x(t_0) = x^{(0)}$, $\dot{x}(t_0) = \dot{x}^{(0)}$, and the state of $\mathcal{H}\{x(t)\}$ at $t = t_0$. The solution of this initial-value problem is the continuation of some unique, continuously differentiable solution of (2.5) which began at $t = 0$.

Remarks

If $\mathcal{S}[\dot{x}, \mathfrak{F}, c] \equiv 0$ and $G(t) \in C[0, \infty)$, there exists a unique, twice continuously differentiable solution to (2.5).

The proof can be generalized to include a slip-function with slip levels which are either unequal for different signs of velocity, or discontinuous functions of displacement.

The backward problem, where the solution is desired for decreasing time, has no meaning for this class of hysteresis since Assumptions 1 and 2 of (A2) define $\mathscr{H}\{x(t)\}$ only for increasing time.

2.3 Boundedness of Solutions to the Initial-Value Problem

Even including the additional mathematical restrictions to prove Theorem 1 in Section 2.2, the Assumptions (A1)-(A6) describe a quite general class of dynamical systems.

The assumptions on $\mathscr{H}\{x(t)\}$ are sufficiently broad to encompass most models presented to represent history dependent behavior in physical systems.

This section deals with viscously damped systems in which only a hereditary function appears explicitly. Systems possessing slip-friction are included as long as the effect of the friction is incorporated into the formulation of $\mathscr{H}\{x(t)\}$ (as in the bilinear hysteretic model).

The system considered is

$$\begin{gathered} \ddot{x} + 2z\dot{x} + \mathscr{H}\{x(t)\} = G(t) \\ x(t_0) = x^{(0)} \\ \dot{x}(t_0) = \dot{x}^{(0)} \\ \text{State of } \mathscr{H}\{x(t)\} \text{ at } t = t_0 \end{gathered} \tag{2.19}$$

where $z \geq 0$.

Boundedness for $G(t) \equiv 0$

Thus far there have been no assumptions upon the nature of the $h^{(i)}(x)$ or the dissipative properties of $\mathcal{N}\{x(t)\}$. The following assumptions are now needed.

Assumption 7 (A7): If the path changes from $x = x^{(q)}$ and $\dot{x} = 0$ into $\dot{x} > 0$ ($\dot{x} < 0$), then the corresponding $h^{(i)}(x)$ must satisfy

$$\frac{dh^{(i)}(x)}{dx} \geq 0 \qquad \text{for } x \geq x^{(q)} (x \leq x^{(q)}) \tag{2.20}$$

and

$$\lim_{x \to \infty} h^{(i)}(x) = U \left(\lim_{x \to -\infty} h^{(i)}(x) = V \right) \tag{2.21}$$

where $0 < U \leq \infty$ ($-\infty \leq V < 0$).

Assumption 8 (A8): Define $\overline{I}^{(i)}$ to be the closed x interval between two path reversals while $\mathcal{N}\{x(t)\} = h^{(i)}(x)$. At each reversal into $\dot{x} > 0$ ($\dot{x} < 0$) from $h^{(i)}(x)$ to $h^{(i+1)}(x)$,

$$h^{(i+1)}(x) - h^{(i)}(x) \geq 0 (\leq 0) \quad \text{for } x \in \overline{I}^{(i)} \tag{2.22}$$

Assumption (A7) restricts the class of hereditary functions to those which do not decrease (increase) with increasing (decreasing) x. $\mathcal{N}\{x(t)\}$ must also resist motion as $|x|$ becomes large. If all $|h^{(i)}(x)|$ are bounded, the system is a fully yielding, saturating, or slipping type. Assumption (A8) represents a requirement upon the dissipative character of $\mathcal{N}\{x(t)\}$. Upon reversal from increasing (decreasing) x, the new function $h^{(i+1)}(x)$ is always less (greater) than or equal to $h^{(i)}(x)$ over the x interval between the previous two path reversals. Thus, if the work done by $\mathscr{H}\{x(t)\}$ is calculated along a path beginning with any given state at $x = x^{(1)}$ to $x = x^{(2)} \neq x^{(1)}$ and back to $x = x^{(1)}$, then it must be either 0 or positive. If it is 0, no energy has been dissipated; if it is positive, energy has been dissipated through some physical mechanism. Assumption (A8) is by no means a necessary condition for $\mathscr{H}\{x(t)\}$ to be dissipative. Most hysteretic models still satisfy the additional assumptions; however, (A8) excludes some. (e. g., a degrading stiffness model presented to describe the behavior of reinforced concrete[16])

<u>Theorem 2</u>

If $G(t) \equiv 0$, $z = 0$, and $\mathscr{H}\{x(t)\}$ satisfies Assumptions (A2) and (A6)-(A8), then all solutions of (2. 19) are bounded.

<u>Proof:</u> The solution of (2. 19) will be discussed in the $x, \dot{x}$ plane. Let $h^{(i)}(x)$ be the current form of $\mathscr{H}\{x(t)\}$. Then (2. 19) becomes

$$\begin{aligned} &\ddot{x} + h^{(i)}(x) = 0 \\ &x(t_0) = x^{(0)} \\ &\dot{x}(t_0) = \dot{x}^{(0)} \end{aligned} \tag{2. 23}$$

For $\dot{x}^{(0)} = 0$: Either $h^{(i)}(x^{(0)}) = 0$ or $h^{(i)}(x^{(0)}) \neq 0$. If the former is true, then $x(t) = x^{(0)}$ for $t \in [t_0, \infty)$. If the latter is true, then motion occurs. Choose $h^{(i)}(x^{(0)}) > 0$ for illustration. Then the trajectory in the $\dot{x} \leq 0$ half-plane passing through $(x^{(0)}, 0)$ is described by

$$\dot{x}^2 = -2\int_{x^{(0)}}^{x} h^{(i)}(\eta)d\eta \tag{2. 24}$$

By (A7) as $x \to -\infty$ from $x^{(0)}$, the right member of (2. 24) monotonically increases from 0, then monotonically decreases to $-\infty$. Therefore, a unique $x^{(1)} < x^{(0)}$ exists where

$$\int_{x^{(0)}}^{x^{(1)}} h^{(i)}(\eta)d\eta = 0 \tag{2.25}$$

and the motion reverses. Then $\mathscr{N}\{x(t)\}$ takes the form $h^{(i+1)}(x)$. By a similar argument, a unique $x^{(2)} > x^{(1)}$ exists where

$$\int_{x^{(1)}}^{x^{(2)}} h^{(i+1)}(\eta)d\eta = 0 \tag{2.26}$$

and the next reversal occurs. Either $x^{(2)} > x^{(0)}$ or $x^{(2)} \leq x^{(0)}$. Assume $x^{(2)} > x^{(0)}$. From (2.25) and (A8)

$$0 = \int_{x^{(0)}}^{x^{(1)}} h^{(i)}(\eta)d\eta \geq \int_{x^{(0)}}^{x^{(1)}} h^{(i+1)}(\eta)d\eta \tag{2.27}$$

which implies

$$\int_{x^{(1)}}^{x^{(0)}} h^{(i+1)}(\eta)d\eta \geq 0 \tag{2.28}$$

As $x \to \infty$ from $x^{(1)}$, $\int_{x^{(1)}}^{x} h^{(i+1)}(\eta)d\eta$ monotonically decreases from 0, then monotonically increases to ∞. Thus (2.28) contradicts (2.26). Consequently, $x^{(2)} \leq x^{(0)}$. The equality holds only if $h^{(i)}(x) = h^{(i+1)}(x)$ for $x \in [x^{(1)}, x^{(2)}]$. Similar reasoning leads to the conclusion that $x^{(3)} \geq x^{(1)}$ with equality only if $h^{(i+1)}(x) = h^{(i+2)}(x)$ for $x \in [x^{(1)}, x^{(2)}]$. This second result also applies to the initial case $\dot{x}^{(0)} = 0$ and $h^{(i)}(x^{(0)}) < 0$ if $x^{(1)} \to x^{(0)}$, $x^{(3)} \to x^{(2)}$, $h^{(i+1)}(x) \to h^{(i)}(x)$, and $h^{(i+2)}(x) \to h^{(i+1)}(x)$.

Continuing this procedure, it is seen that the solution remains bounded in the interval $x \in [x^{(1)}, x^{(0)}]$. If $\mathcal{N}\{x(t)\}$ loses its hysteretic property on some path interval in which two consecutive reversals occur, e. g., $x = x^{(i)}$ and $x = x^{(i+1)}$, then the motion is just the conservative oscillation of an autonomous system with reversals at $x = x^{(i)}$ and $x = x^{(i+1)}$. Otherwise, the distance between any two consecutive reversals must decrease as $t \to \infty$.

For $\dot{x}^{(0)} \neq 0$: The trajectory of (2. 23) in the $\dot{x}\,\mathrm{sgn}\,(\dot{x}^{(0)}) \geq 0$ half-plane passing through $(x^{(0)}, \dot{x}^{(0)})$ is

$$\dot{x}^2 = (\dot{x}^{(0)})^2 - 2\int_{x^{(0)}}^{x} h^{(i)}(\eta)d\eta \tag{2. 29}$$

By (A7) as $x\,\mathrm{sgn}\,(\dot{x}^{(0)}) \to \infty$ from $x^{(0)}$, $\int_{x^{(0)}}^{x} h^{(i)}(\eta)d\eta$ either monotonically decreases from 0 and then monotonically increases to ∞, or monotonically increases to ∞ from 0. A unique $x^{(1)}$ exists such that $x^{(1)}\,\mathrm{sgn}\,(\dot{x}^{(0)}) > x^{(0)}\,\mathrm{sgn}\,(\dot{x}^{(0)})$,

$$(\dot{x}^{(0)})^2 - 2\int_{x^{(0)}}^{x^{(1)}} h^{(i)}(\eta)d\eta = 0 \tag{2. 30}$$

and $\dot{x} = 0$. The problem has thus been reduced to that handled in the previous case. This completes the proof.

Remarks

On each segment $h^{(i)}(x)$ the system (2. 19) appears to be conservative. The dissipative property is detected only by considering a reversal and (A8).

Boundedness of solutions for $G(t) \equiv 0$ can be shown using a similar proof for a more general system. In addition to $\mathcal{H}\{x(t)\}$ satisfying (A2) and (A6)-(A8), (2.19) may also include 1) $R(x, \dot{x}) = r(\dot{x})$ where $r(\dot{x})\dot{x} \geq 0$ for all $\dot{x}$, 2) slip-friction $\mathcal{S}[\dot{x}, \mathcal{F}, c]$, or 3) both 1) and 2). For any of these cases, the distance between reversals decreases as $t \to \infty$ even if $\mathcal{H}\{x(t)\}$ loses its hysteretic property.

Boundedness for a Class of Forced Hysteretic Systems with Viscous Dissipation

The class of hereditary functions $\mathcal{H}\{x(t)\}$ considered here satisfy the following additional assumption.

Assumption 9 (A9): If the path changes from $x = x^{(q)}$ and $\dot{x} = 0$ into $\dot{x} > 0$ ($\dot{x} < 0$), then the corresponding $h^{(i)}(x)$ satisfies

$$x - s \leq h^{(i)}(x) \leq x + s \quad \text{for } x \geq x^{(q)} (x \leq x^{(q)}) \tag{2.31}$$

where s is some positive number.

Assumption (A9) requires each $h^{(i)}(x)$ to be within the band enclosed by $x - s$ and $x + s$. With the appropriate coordinate transformation, several hysteretic models, including the well known bilinear hysteretic model, satisfy (A9).

Theorem 3

If $z \in (0, 1)$, $\underset{t \in [t_0, \infty)}{\text{Max}} |G(t)| < P < \infty$, and Assumptions (A1), (A2), (A6), and (A9) are satisfied, then all solutions of (2.19) are bounded. In addition, let $\overline{\mathcal{R}}$ be the closed region in the $x, \dot{x}$ plane whose boundary is the curve described by the parametric equations

$$\begin{cases} x(\varphi) = \dfrac{-(P+s)\left(1+\coth\left(\frac{\pi z}{2\omega}\right)\right)}{\omega} e^{-z\varphi}(z \sin \omega\varphi + \omega \cos \omega\varphi) + P + s \\ \dot{x}(\varphi) = \dfrac{(P+s)\left(1+\coth\left(\frac{\pi z}{2\omega}\right)\right)}{\omega} e^{-z\varphi} \sin \omega\varphi \end{cases} \tag{2.32}$$

and

$$\begin{cases} x(\varphi) = \dfrac{(P+s)\left(1+\coth\left(\frac{\pi z}{2\omega}\right)\right)}{\omega} e^{-z\varphi}(z \sin \omega\varphi + \omega \cos \omega\varphi) - P - s \\ \dot{x}(\varphi) = \dfrac{-(P+s)\left(1+\coth\left(\frac{\pi z}{2\omega}\right)\right)}{\omega} e^{-z\varphi} \sin \omega\varphi \end{cases} \tag{2.33}$$

where $\omega = \sqrt{1-z^2}$ and the range of the parameter φ is from 0 to $\frac{\pi}{\omega}$. If $(x^{(0)}, \dot{x}^{(0)}) \in \overline{\mathbb{R}}$, then the solution trajectory $(x(t), \dot{x}(t))$ in the $x, \dot{x}$ plane remains in $\overline{\mathbb{R}}$ for $t \in [t_0, \infty)$. If $(x^{(0)}, \dot{x}^{(0)}) \notin \overline{\mathbb{R}}$, then $(x(t), \dot{x}(t))$ intersects the $\dot{x} = 0$ axis with decreasing amplitudes until it enters $\overline{\mathbb{R}}$.

Proof: The basic procedure is similar to that used to prove Theorems 1 and 2. The solution of (2.19) is considered for a time interval in which the velocity does not vanish, all possible cases which may occur are discussed, and the proof is concluded by repeated application of the results. The proof consists of four parts.

At $t = t_0$, (2.19) becomes

$$\begin{aligned} &\ddot{x} + 2z\dot{x} + h^{(i)}(x) = G(t) \\ &x(t_0) = x^{(0)} \\ &\dot{x}(t_0) = \dot{x}^{(0)} \end{aligned} \tag{2.34}$$

Part 1: Let $\dot{x}^{(0)} = 0$. Then either the solution remains at $x^{(0)}$ for $t \in [t_0, \infty)$, or it leaves $x^{(0)}$ the instant after $t = t_1 \geq t_0$.

If the solution remains at $x^{(0)}$ for $t \in [t_0, \infty)$, then (A9), (2. 34), and the hypothesis $|G(t)| < P$ imply that $|x^{(0)}| < P + s$. Thus $(x(t), 0) = (x^{(0)}, 0) \in \overline{\mathcal{R}}$ for $t \in [t_0, \infty)$.

If the solution leaves $x^{(0)}$ the instant after $t = t_1 \geq t_0$, then assume $\dot{x}(t) > 0$ for $t \in (t_1, t_2)$ where $\dot{x}(t_2) = 0$. (Thus it is necessary that $x^{(0)} < P + s$.) When $t = t_1$, (2. 19) is

$$\begin{aligned} &\ddot{x} + 2z\dot{x} + h^{(i)}(x) = G(t) \\ &x(t_1) = x^{(0)} \\ &\dot{x}(t_1) = 0 \end{aligned} \tag{2. 35}$$

The following parametric curve is considered in the $x, \dot{x}$ plane:

$$\begin{cases} x(\varphi) = \dfrac{(X^{(0)} - P - s)}{\omega} e^{-z\varphi}(z \sin \omega\varphi + \omega \cos \omega\varphi) + P + s \\ \dot{x}(\varphi) = \dfrac{(P + s - X^{(0)})}{\omega} e^{-z\varphi} \sin \omega\varphi \end{cases} \tag{2. 36}$$

where $\omega = \sqrt{1 - z^2}$, $X^{(0)} \leq -(P+s) \coth(\frac{\pi z}{2\omega})$, and the range of the parameter φ is from 0 to $\frac{\pi}{\omega}$. The expressions in (2. 36) describe a curve in the $\dot{x} \geq 0$ half-plane which intersects $\dot{x} = 0$ only at its two end-points, $x(0) = X^{(0)}$ and $x(\frac{\pi}{\omega}) \equiv X^{(1)}$. If $X^{(0)} < -(P+s) \coth(\frac{\pi z}{2\omega})$, then it can be shown that $(P+s) \coth(\frac{\pi z}{2\omega}) < X^{(1)} < -X^{(0)}$. If $X^{(0)} = -(P+s) \coth(\frac{\pi z}{2\omega})$, then $X^{(1)} = (P+s) \coth(\frac{\pi z}{2\omega})$, and (2. 36) coincides with $\partial\mathcal{R}$ for $\dot{x} \geq 0$.

The slope of (2. 36) at any point $(x(\tilde{\varphi}), \dot{x}(\tilde{\varphi})) \equiv (\tilde{x}, \tilde{\dot{x}})$ is given by

$$\frac{\left.\dfrac{d\dot{x}}{d\varphi}\right|_{\varphi=\tilde{\varphi}}}{\left.\dfrac{dx}{d\varphi}\right|_{\varphi=\tilde{\varphi}}} = \left.\frac{d\dot{x}}{dx}\right|_{\varphi=\tilde{\varphi}} = \frac{P + s - \tilde{x} - 2z\tilde{\dot{x}}}{\tilde{\dot{x}}} \tag{2. 37}$$

The slope of a trajectory of (2. 35) at the point $(x,\dot{x})$ in the $\dot{x} \geq 0$ half-plane is

$$\frac{\frac{d\dot{x}}{dt}}{\frac{dx}{dt}} = \frac{d\dot{x}}{dx} = \frac{G(t) - h^{(i)}(x) - 2z\dot{x}}{\dot{x}} \tag{2. 38}$$

Let $\mathcal{D}\left(X^{(0)}\right)$ be defined as the open region in the $\dot{x} \geq 0$ half-plane whose boundary is composed of the curve defined by (2. 36) and the $\dot{x} = 0$ axis. Then Equations (2. 37) and (2. 38), Assumption (A9), and the hypothesis $|G(t)| < P$ imply that at any point on the curve (2. 36), except at the end-points, the trajectory of (2. 35) must be directed into $\mathcal{D}\left(X^{(0)}\right)$. At $(X^{(0)}, 0)$ and $(X^{(1)}, 0)$, the slope of the curve (2. 36) and the slope of the trajectory of (2. 35) are infinite.

The case for $x^{(0)} < -(P+s)\coth\left(\frac{\pi z}{2\omega}\right)$ is examined first. Let $X^{(0)} = x^{(0)}$ in (2. 36). From (2. 37) and (2. 38) it can be seen that as soon as the trajectory of (2. 35) leaves $(x^{(0)}, 0)$, its slope in the $x, \dot{x}$ plane is less than the slope of (2. 36). Thus the trajectory immediately enters the open region $\mathcal{D}\left(x^{(0)}\right)$. From the discussion in the preceding paragraph, it can be concluded that the trajectory leaves $\mathcal{D}\left(x^{(0)}\right)$ only through its boundary at $\dot{x} = 0$. Therefore $x(t_2) \leq X^{(1)}$. Since $X^{(1)} < -X^{(0)}$, $|x(t_2)| < |x^{(0)}|$. If t_2 is unbounded, then it can be shown that $\lim_{t\to\infty} (x(t),\dot{x}(t)) = (x^{(L)}, 0) \in \overline{\mathcal{R}}$.

For $x^{(0)} \in [-(P+s)\coth\left(\frac{\pi z}{2\omega}\right), P+s)$, let $X^{(0)} = -(P+s)\coth\left(\frac{\pi z}{2\omega}\right)$. Then the trajectory of (2. 35) immediately enters $\mathcal{D}\left(-(P+s)\coth\left(\frac{\pi z}{2\omega}\right)\right)$ and leaves only through the boundary at $\dot{x} = 0$. For this case,

$(x^{(0)}, 0) \in \overline{\mathcal{R}}$. Since $\overline{\mathcal{D}}\left(-(P+s)\coth\left(\frac{\pi z}{2\omega}\right)\right) \subset \overline{\mathcal{R}}$, the trajectory remains in $\overline{\mathcal{R}}$ for $t \in [t_0, t_2]$.

Part 2: Let $\dot{x}^{(0)} > 0$. Only the time interval $t \in [t_0, t_1]$ is considered, where $t_1 > t_0$ is the first instant $\dot{x}(t_1) = 0$.

If $(x^{(0)}, \dot{x}^{(0)}) \in \overline{\mathcal{D}}\left(-(P+s)\coth\left(\frac{\pi z}{2\omega}\right)\right)$, then the trajectory of (2. 34) may leave $\overline{\mathcal{D}}\left(-(P+s)\coth\left(\frac{\pi z}{2\omega}\right)\right)$ only through the boundary at $\dot{x} = 0$. For $\dot{x}^{(0)} > 0$, $(x^{(0)}, \dot{x}^{(0)}) \in \overline{\mathcal{D}}\left(-(P+s)\coth\left(\frac{\pi z}{2\omega}\right)\right)$ if and only if $(x^{(0)}, \dot{x}^{(0)}) \in \overline{\mathcal{R}}$. Thus if $(x^{(0)}, \dot{x}^{(0)}) \in \overline{\mathcal{R}}$, then the trajectory remains in $\overline{\mathcal{R}}$ for $t \in [t_0, t_1]$.

Let $(x^{(0)}, \dot{x}^{(0)}) \notin \overline{\mathcal{D}}\left(-(P+s)\coth\left(\frac{\pi z}{2\omega}\right)\right)$. Multiplying the first of Equations (2. 34) by $2\dot{x}$ and integrating from t_0 to t gives

$$(\dot{x}(t))^2 = (\dot{x}^{(0)})^2 + 2\int_{t_0}^{t} G(\tau)\dot{x}(\tau)d\tau - 4z\int_{t_0}^{t}(\dot{x}(\tau))^2 d\tau - 2\int_{x^{(0)}}^{x(t)} h^{(i)}(\eta)d\eta \qquad (2.\ 39)$$

Using (A9) and the hypotheses $|G(t)| < P$ and $z > 0$, (2. 39) implies

$$(\dot{x}(t))^2 \le (\dot{x}^{(0)})^2 + 2P(x(t) - x^{(0)}) - (x(t))^2 + 2sx(t) + (x^{(0)})^2 - 2sx^{(0)} \qquad (2.40)$$

This gives a bound upon the velocity of the solution of (2. 34) as a function of x(t), and implies that the velocity must vanish when x(t) is finite. For $\dot{x}^{(0)} > 0$, $(x^{(0)}, \dot{x}^{(0)}) \notin \overline{\mathcal{D}}\left(-(P+s)\coth\left(\frac{\pi z}{2\omega}\right)\right)$ if and only if $(x^{(0)}, \dot{x}^{(0)}) \notin \overline{\mathcal{R}}$. Thus if $(x^{(0)}, \dot{x}^{(0)}) \notin \overline{\mathcal{R}}$, $\dot{x}(t_1) = 0$ with $x(t_1) < \infty$. If t_1 is unbounded, then it can be shown that $\lim_{t\to\infty}(x(t), \dot{x}(t)) = (x^{(L)}, 0) \in \overline{\mathcal{R}}$.

Part 3: Parts 1 and 2 discuss motion only into $\dot{x} > 0$. For motion into $\dot{x} < 0$, the substitution $x(t) = -\psi(t)$ transforms (2. 34) into an initial-value problem of the type already considered. The results in Parts 1 and 2 applied to the ψ system may readily be transformed into the x system.

The conclusions are as follows:

$\dot{x}^{(0)}=0$: $x(t)=x^{(0)}$ for $t\in[t_0, t_1]$ and $\dot{x}(t)<0$ for $t\in(t_1, t_2)$ with $\dot{x}(t_2)=0$. If t_2 is unbounded, then $\lim_{t\to\infty}(x(t), \dot{x}(t)) = (x^{(L)}, 0)\in\overline{\mathcal{R}}$. If t_2 is bounded, then either $(x^{(0)}, 0)\notin\overline{\mathcal{R}}$ and $|x(t_2)|<|x^{(0)}|$, or $(x^{(0)}, 0)\in\overline{\mathcal{R}}$ and $(x(t), \dot{x}(t))\in\overline{\mathcal{R}}$ for $t\in[t_0, t_2]$.

$\dot{x}^{(0)}<0$: $\dot{x}(t)<0$ for $t\in[t_0, t_1)$ with $\dot{x}(t_1)=0$. If t_1 is unbounded, then $\lim_{t\to\infty}(x(t), \dot{x}(t)) = (x^{(L)}, 0)\in\overline{\mathcal{R}}$. If t_1 is bounded, then either $(x^{(0)}, \dot{x}^{(0)})\in\overline{\mathcal{R}}$ and $(x(t), \dot{x}(t))\in\overline{\mathcal{R}}$ for $t\in[t_0, t_1]$, or $(x^{(0)}, \dot{x}^{(0)})\notin\overline{\mathcal{R}}$ and $|x(t_1)|<\infty$.

Part 4: For $\dot{x}^{(0)}=0$, repeated application of Parts 1 and 3 show that either $(x^{(0)}, \dot{x}^{(0)})\in\overline{\mathcal{R}}$ and $(x(t), \dot{x}(t))\in\overline{\mathcal{R}}$ for $t\in[t_0, \infty)$, or $(x^{(0)}, \dot{x}^{(0)})\notin\overline{\mathcal{R}}$ and the solution trajectory must intersect $\dot{x}=0$ with decreasing amplitudes until it enters $\overline{\mathcal{R}}$. For $\dot{x}^{(0)}\neq 0$, Parts 2 and 3 and repeated application of Parts 1 and 3 yield the same result. This concludes the proof.

Other Values of z

Corresponding theorems can be proved if $z=1$ or if $z>1$. When $z=1$, the region $\overline{\mathcal{R}}$ in the statement of Theorem 3 becomes the closed region whose boundary is the curve described by the parametric equations

$$\begin{cases} x(\varphi) = -2(P+s)(1+\varphi)e^{-\varphi}+P+s \\ \dot{x}(\varphi) = 2(P+s)\varphi e^{-\varphi} \end{cases} \tag{2.41}$$

and

$$\begin{cases} x(\varphi) = 2(P+s)(1+\varphi)e^{-\varphi}-P-s \\ \dot{x}(\varphi) = -2(P+s)\varphi e^{-\varphi} \end{cases} \tag{2.42}$$

where $\varphi \in [0,\infty)$. When $z > 1$, the boundary of the region $\bar{R}$ is the curve described by

$$\begin{cases} x(\varphi) = -\dfrac{2(P+s)}{\rho} e^{-z\varphi}(z \sinh \rho\varphi + \rho \cosh \rho\varphi) + P + s \\ \dot{x}(\varphi) = \dfrac{2(P+s)}{\rho} e^{-z\varphi} \sinh \rho\varphi \end{cases} \tag{2.43}$$

and

$$\begin{cases} x(\varphi) = \dfrac{2(P+s)}{\rho} e^{-z\varphi}(z \sinh \rho\varphi + \rho \cosh \rho\varphi) - P - s \\ \dot{x}(\varphi) = -\dfrac{2(P+s)}{\rho} e^{-z\varphi} \sinh \rho\varphi \end{cases} \tag{2.44}$$

where $\rho = \sqrt{z^2 - 1}$ and $\varphi \in [0,\infty)$. The conclusions of Theorem 3 are valid for $z = 1$ and for $z > 1$ with the respective $\bar{R}$ described above. It may be added in each case that if $(x^{(0)}, \dot{x}^{(0)}) \notin \bar{R}$, then there is at most one reversal of the solution before it enters $\bar{R}$.

Thus it may be concluded that all solutions of (2. 19) are bounded for $z > 0$. The quantitative bound upon the solutions becomes unbounded as $z \to 0$ and nothing can be concluded for the limiting value $z = 0$. This is to be expected since the simple example with $z = 0$, $\mathcal{K}\{x(t)\} = x$, and $G(t) = \cos t$ illustrates a case in which all solutions of (2. 19) are unbounded as $t \to \infty$.

<u>Boundedness for Forced Hysteretic Systems with No Viscous Dissipation</u>

When there is no viscous dissipation, it is not possible to show that all solutions to (2. 19) are bounded for general excitation. Two examples will be used to demonstrate that (2. 19) can have both bounded and unbounded solution behavior. In the first example the bilinear hysteretic model with appropriate parameters will be shown to have all solutions bounded for sufficiently small excitation. In the second example the elastoplastic model, a limiting case of the bilinear model, will be shown to have an unbounded solution for vanishingly small excitation.

Bilinear Hysteretic Model

The bilinear hysteretic model $\mathcal{B}_\alpha\{x(t)\}$ in Figure 3 can be described as follows. The virgin curve is

$$b_\alpha^{(0)}(x) = \begin{cases} \alpha x + (1-\alpha) & \text{for } x \geq 1 \\ x & \text{for } |x| < 1 \\ \alpha x - (1-\alpha) & \text{for } x \leq -1 \end{cases} \tag{2.45}$$

Define $p = 1 (p = -1)$ if a reversal occurs on the upper (lower) segment of slope α. After a reversal from either upper or lower segment of slope α at $x = \tilde{x}$, $\mathcal{B}_\alpha\{x(t)\}$ becomes

$$b_\alpha^{(k)}(x) = \begin{cases} \alpha x + p(1-\alpha) & \text{for } px \geq p\tilde{x} \\ x - (\tilde{x} - p)(1-\alpha) & \text{for } p\tilde{x} - 2 < px < p\tilde{x} \\ \alpha x - p(1-\alpha) & \text{for } px \leq p\tilde{x} - 2 \end{cases} \tag{2.46}$$

For $\alpha \in (0, 1)$ and the limiting values $\alpha = 0$ and $\alpha = 1$, $\mathcal{B}_\alpha\{x(t)\}$ satisfies (A2) and (A6)-(A8). $b_\alpha^{(k)}(x)$ changes only if a reversal occurs on either segment of slope α. The value of $x(t)$ and $\mathcal{B}_\alpha\{x(t)\}$ at $t = t_0$ specify the state of $\mathcal{B}_\alpha\{x(t)\}$ at $t = t_0$.

The motion of the bilinear hysteretic oscillator shown in Figure 4 will be considered. The system (2.19) is now

$$\left.\begin{aligned} \ddot{x} + \mathcal{B}_\alpha\{x(t)\} &= G(t) \\ x(t_0) &= x^{(0)} \end{aligned}\right\} \tag{2.47}$$

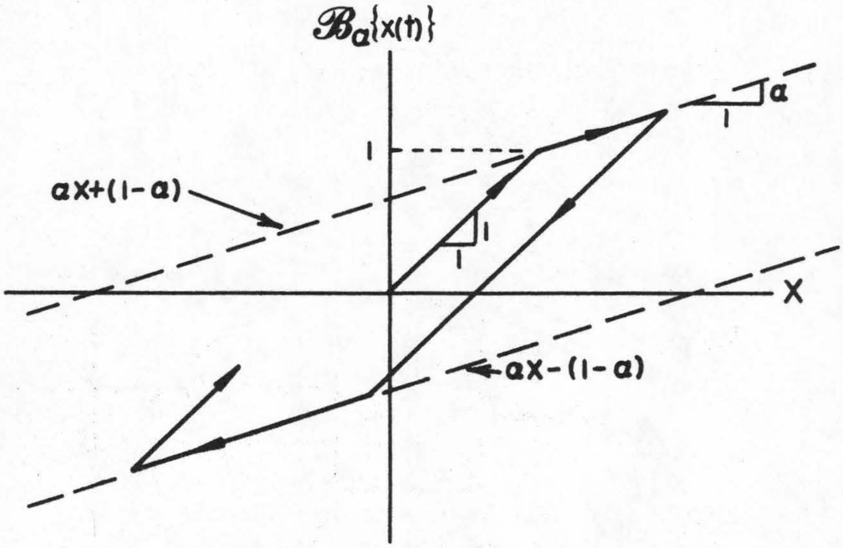
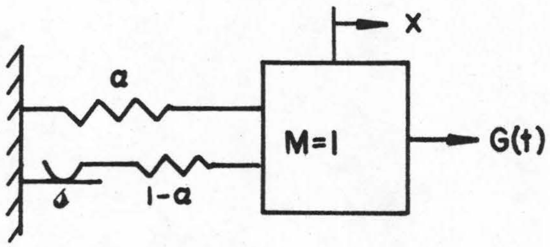


Figure 3: Bilinear Hysteretic Model



δ: Slip Element with Slip Levels = 1 - α

Figure 4: Bilinear Hysteretic Oscillator

$$\left.\begin{aligned} \dot{x}(t_0) &= \dot{x}^{(0)} \\ \mathcal{B}_\alpha\{x(t)\} &= b_\alpha^{(i)}(x) \text{ at } t = t_0 \end{aligned}\right\} \quad (2.47) \text{ cont.}$$

Example 1: Bounded Solution Behavior of the Forced Bilinear Hysteretic Oscillator

Theorem 4

If $\alpha \in [\frac{1}{2}, 1)$ and $\underset{t \in [t_0, \infty)}{\mathrm{Max}} |G(t)| \leq P < 1 - \alpha$, then all solutions $x(t)$ of (2.47) are bounded. Furthermore, if $(x^{(0)}, \dot{x}^{(0)})$ is contained in the closed region $\overline{\mathcal{R}}$ in the $x, \dot{x}$ plane whose boundary is described by

$$\dot{x} = \begin{cases} \sqrt{-\alpha(x + \frac{1-\alpha-P}{\alpha})^2 + \alpha(\hat{x} + \frac{1-\alpha-P}{\alpha})^2 - 4(1-\alpha)} & \text{for } x \in [-\hat{x}, \hat{x} - 2] \\ \sqrt{x^2(1-2\alpha) + 2x(P + (1-\hat{x})(1-\alpha)) + \hat{x}^2 - 2\hat{x}(P + 1 - \alpha)} & \text{for } x \in [\hat{x} - 2, \hat{x}] \\ -\sqrt{x^2(1-2\alpha) - 2x(P + (1-\hat{x})(1-\alpha)) + \hat{x}^2 - 2\hat{x}(P + 1 - \alpha)} & \text{for } x \in [-\hat{x}, -\hat{x} + 2] \\ -\sqrt{-\alpha(x + \frac{P+\alpha-1}{\alpha})^2 + \alpha(\hat{x} + \frac{1-\alpha-P}{\alpha})^2 - 4(1-\alpha)} & \text{for } x \in [-\hat{x} + 2, \hat{x}] \end{cases} \quad (2.48)$$

where $\hat{x} = \frac{1-\alpha}{1-\alpha-P}$, then $|x(t)| \leq \frac{1-\alpha}{1-\alpha-P}$ for $t \in [t_0, \infty)$. If $(x^{(0)}, \dot{x}^{(0)}) \notin \overline{\mathcal{R}}$, then the solution trajectory $(x(t), \dot{x}(t))$ in the $x, \dot{x}$ plane intersects the $\dot{x} = 0$ axis with decreasing amplitudes until it enters $\overline{\mathcal{R}}$.

Proof: At $t = t_0$, (2.47) becomes

$$\begin{aligned} \ddot{x} + b_\alpha^{(i)}(x) &= G(t) \\ x(t_0) &= x^{(0)} \\ \dot{x}(t_0) &= \dot{x}^{(0)} \end{aligned} \quad (2.49)$$

Part 1: Let $\dot{x}^{(0)} = 0$. Then either the solution remains at $x^{(0)}$ for $t \in [t_0, \infty)$, or it leaves $x^{(0)}$ the instant after $t = t_1 \geq t_0$.

If the solution remains at $x^{(0)}$ for $t \in [t_0, \infty)$, (2.49) implies $G(t) = b_\alpha^{(i)}(x^{(0)})$ for $t \in [t_0, \infty)$. Consequently, from the formulation of $\mathcal{B}_\alpha\{x(t)\}$ and the hypothesis $|G(t)| \le P$, $|x^{(0)}| \le \frac{P+1-\alpha}{\alpha}$. For $\alpha \in [\frac{1}{2}, 1)$ and $P < 1-\alpha$, $\frac{P+1-\alpha}{\alpha} \le \frac{1-\alpha}{1-\alpha-P}$ which implies $|x^{(0)}| \le \frac{1-\alpha}{1-\alpha-P}$. It then follows that $(x^{(0)}, 0) \in \overline{\mathcal{R}}$ and $|x(t)| \le \frac{1-\alpha}{1-\alpha-P}$ for $t \in [t_0, \infty)$.

If the solution leaves $x^{(0)}$ the instant after $t = t_1 \ge t_0$, then assume $\dot{x}(t) > 0$ for $t \in (t_1, t_2)$ where $\dot{x}(t_2) = 0$. (Thus it is necessary that $x^{(0)} < \frac{P+1-\alpha}{\alpha}$.) When $t = t_1$, (2.47) is

$$\begin{aligned} &\ddot{x} + b_\alpha^{(i)}(x) = G(t) \\ &x(t_1) = x^{(0)} \\ &\dot{x}(t_1) = 0 \end{aligned} \tag{2.50}$$

Multiplying the first of Equations (2.50) by $2\dot{x}$ and integrating from t_1 to t gives

$$(\dot{x}(t))^2 = 2\int_{t_1}^{t} G(\tau)\dot{x}(\tau)d\tau - 2\int_{x^{(0)}}^{x(t)} b_\alpha^{(i)}(\eta)d\eta \tag{2.51}$$

Since $|G(t)| \le P$, (2.51) implies

$$(\dot{x}(t))^2 \le 2P(x(t) - x^{(0)}) - 2\int_{x^{(0)}}^{x(t)} b_\alpha^{(i)}(\eta)d\eta \tag{2.52}$$

Define

$$B_\alpha(x; x^{(0)}) = \begin{cases} \alpha x + (1-\alpha) & \text{for } x \ge x^{(0)} + 2 \\ x - (x^{(0)}+1)(1-\alpha) & \text{for } x^{(0)} < x < x^{(0)} + 2 \\ \alpha x - (1-\alpha) & \text{for } x \le x^{(0)} \end{cases} \tag{2.53}$$

From the formulation of $\mathcal{B}_\alpha\{x(t)\}$ it can be seen that along any path increasing from $x = x^{(0)}$, $B_\alpha(x; x^{(0)}) \le b_\alpha^{(i)}(x)$ for any possible $b_\alpha^{(i)}(x)$. $B_\alpha(x; x^{(0)})$ is illustrated in Figure 5. Thus (2.52) can be written

$$(\dot{x}(t))^2 \le 2\,P(x(t) - x^{(0)}) - 2\int_{x^{(0)}}^{x(t)} B_\alpha(\eta; x^{(0)})d\eta \tag{2.54}$$

where

$$\int_{x^{(0)}}^{x} B_\alpha(\eta; x^{(0)})d\eta = \begin{cases} \frac{x^2}{2} - x(x^{(0)}+1)(1-\alpha) - \frac{(x^{(0)})^2}{2} + x^{(0)}(x^{(0)}+1)(1-\alpha) & \text{for } x \in [x^{(0)}, x^{(0)}+2] \\ \frac{\alpha x^2}{2} + x(1-\alpha) - \alpha\frac{(x^{(0)})^2}{2} - (x^{(0)}+2)(1-\alpha) & \text{for } x \ge x^{(0)}+2 \end{cases} \tag{2.55}$$

Equations (2.54) and (2.55) give a bound upon the velocity of the solution of (2.50) as a function of x(t). This bound is independent of the specific form of $b_\alpha^{(i)}(x)$ and does not depend upon time explicitly. If $x \to \infty$ from $x^{(0)}$, then $\int_{x^{(0)}}^{x} B_\alpha(\eta; x^{(0)})d\eta$ either increases to ∞ quadratically in x, or monotonically decreases and then increases to ∞ quadratically in x. Recalling that $\dot{x}(t) > 0$ for $t \in (t_1, t_2)$, it follows that a unique $X^{(1)}(x^{(0)}, 0) > x^{(0)}$ exists such that

$$P(x - x^{(0)}) - \int_{x^{(0)}}^{x} B_\alpha(\eta; x^{(0)})d\eta > 0 \quad \text{for } x \in (x^{(0)}, X^{(1)}(x^{(0)}, 0)) \tag{2.56}$$

and

$$P(X^{(1)}(x^{(0)}, 0) - x^{(0)}) - \int_{x^{(0)}}^{X^{(1)}(x^{(0)}, 0)} B_\alpha(\eta; x^{(0)})d\eta = 0 \tag{2.57}$$

Equations (2.54) to (2.57) imply that the velocity of the solution of (2.50) must vanish before $x(t) > X^{(1)}(x^{(0)}, 0)$. It is assumed that

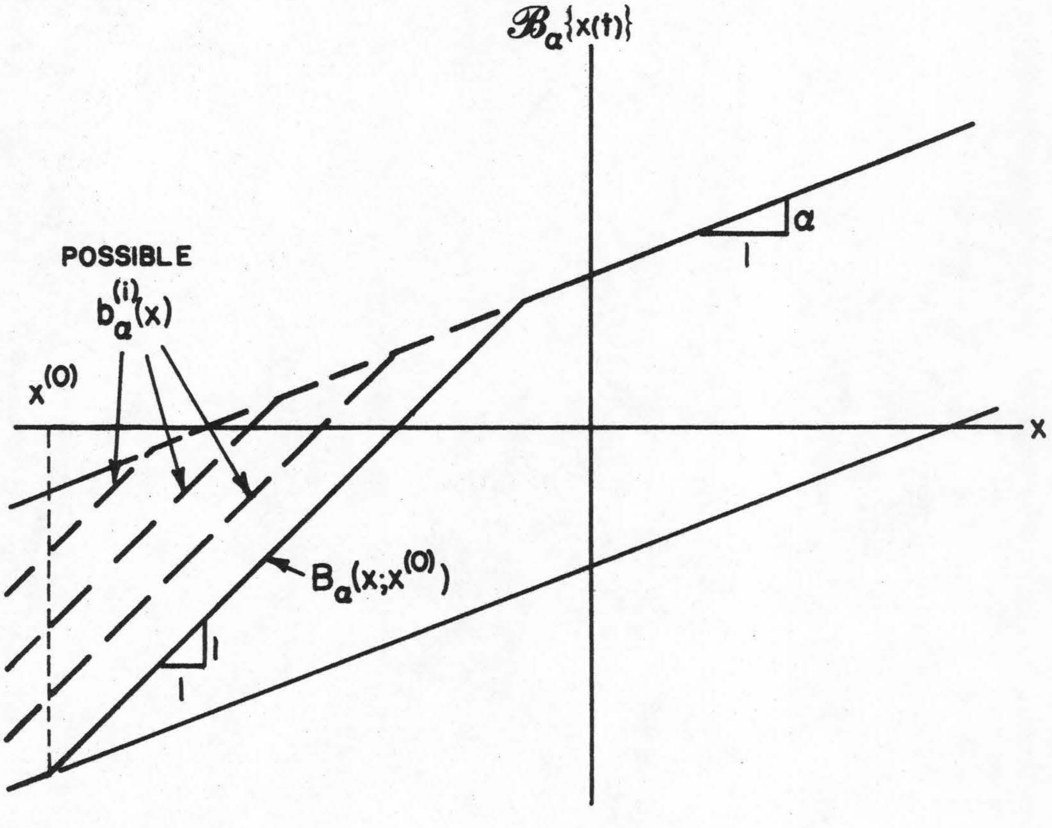


Figure 5: Illustration of

$B_\alpha(x; x^{(0)})$

$\dot{x}(t_2) = 0$. t_2 is either unbounded or bounded.

If t_2 is unbounded, then $\dot{x}(t)>0$ and $x(t) \le X^{(1)}(x^{(0)}, 0)$ for $t \in (t_1, \infty)$. Consequently, $\lim_{t\to\infty} \dot{x}(t) = 0$ and $\lim_{t\to\infty} x(t) = x^{(L)} \le X^{(1)}(x^{(0)}, 0)$. The formulation of $\mathcal{B}_\alpha\{x(t)\}$, the hypotheses $\alpha \in [\frac{1}{2}, 1)$ and $|G(t)| \le P < 1 - \alpha$, and (2.49) imply that if $|x^{(L)}| > \frac{1-\alpha}{1-\alpha-P}$, then $\ddot{x}(t)$ is bounded away from 0 for $t \in [T, \infty)$ for a sufficiently large T. This gives $\lim_{t\to\infty} \dot{x}(t) = \infty$ which is a contradiction. Therefore $|x^{(L)}| \le \frac{1-\alpha}{1-\alpha-P}$ so that $\lim_{t\to\infty} (x(t), \dot{x}(t)) = (x^{(L)}, 0) \in \overline{\mathcal{R}}$.

If t_2 is bounded, $\dot{x}(t_2) = 0$ and $x(t_2) \le X^{(1)}(x^{(0)}, 0)$. For $\alpha \in [\frac{1}{2}, 1)$ and $P < 1 - \alpha$, it can be shown that if $\frac{\alpha - 1}{1-\alpha-P} \le x^{(0)} < \frac{P+1-\alpha}{\alpha}$, then $X^{(1)}(x^{(0)}, 0) \le \frac{1-\alpha}{1-\alpha-P}$. If $x^{(0)} < \frac{\alpha-1}{1-\alpha-P}$, then $X^{(1)}(x^{(0)}, 0) < -x^{(0)}$. Therefore either $(x^{(0)}, 0) \in \overline{\mathcal{R}}$ and $|x(t)| \le \frac{1-\alpha}{1-\alpha-P}$ for $t \in [t_0, t_2]$, or $(x^{(0)}, 0) \notin \overline{\mathcal{R}}$ and $|x(t_2)| < |x^{(0)}|$.

In summary: If $x(t) = x^{(0)}$ for $t \in [t_0, \infty)$, then $(x^{(0)}, 0) \in \overline{\mathcal{R}}$ and $|x^{(0)}| \le \frac{1-\alpha}{1-\alpha-P}$. If $x(t) = x^{(0)}$ for $t \in [t_0, t_1]$ and $\dot{x}(t) > 0$ for $t \in (t_1, t_2)$ with $\dot{x}(t_2) = 0$, two cases may occur:

1. t_2 is unbounded. Then $\lim_{t\to\infty} (x(t), \dot{x}(t)) = (x^{(L)}, 0) \in \overline{\mathcal{R}}$.
2. t_2 is bounded. Then either $(x^{(0)}, 0) \in \overline{\mathcal{R}}$ and $|x(t)| \le \frac{1-\alpha}{1-\alpha-P}$ for $t \in [t_0, t_2]$ with $(x(t_2), 0) \in \overline{\mathcal{R}}$, or $(x^{(0)}, 0) \notin \overline{\mathcal{R}}$ and $|x(t_2)| < |x^{(0)}|$.

Part 2: Let $\dot{x}^{(0)} > 0$. Only the time interval $t \in [t_0, t_1]$ is considered, where $t_1 > t_0$ is the first instant that $\dot{x}(t_1) = 0$. The same procedure used in Part 1 yields

$$(\dot{x}(t))^2 \le (\dot{x}^{(0)})^2 + 2P(x(t) - x^{(0)}) - 2 \int_{x^{(0)}}^{x(t)} B_\alpha(\eta; x^{(0)}) d\eta \qquad (2.58)$$

where $\int_{x^{(0)}}^{x} B_\alpha(\eta; x^{(0)})d\eta$ is given by (2.55). There exists a unique $X^{(1)}(x^{(0)}, \dot{x}^{(0)}) > x^{(0)}$ such that

$$(\dot{x}^{(0)})^2 + 2P(x - x^{(0)}) - 2\int_{x^{(0)}}^{x} B_\alpha(\eta; x^{(0)})d\eta > 0 \text{ for } x \in (x^{(0)}, X^{(1)}(x^{(0)}, \dot{x}^{(0)})) \tag{2.59}$$

and

$$(\dot{x}^{(0)})^2 + 2P(X^{(1)}(x^{(0)}, \dot{x}^{(0)}) - x^{(0)}) - 2\int_{x^{(0)}}^{X^{(1)}(x^{(0)}, \dot{x}^{(0)})} B_\alpha(\eta; x^{(0)})d\eta = 0 \tag{2.60}$$

Equations (2.58) to (2.60) imply that the velocity of the solution of (2.49) must vanish before $x(t) > X^{(1)}(x^{(0)}, \dot{x}^{(0)})$. It is assumed that $\dot{x}(t_1) = 0$. If t_1 is unbounded, $\lim_{t\to\infty} (x(t), \dot{x}(t)) = (x^{(L)}, 0) \in \overline{\mathbb{R}}$. If t_1 is bounded, then $\dot{x}(t_1) = 0$ and $x(t_1) \leq X^{(1)}(x^{(0)}, \dot{x}^{(0)})$. For $\alpha \in [\frac{1}{2}, 1)$ and $P < 1 - \alpha$, it can be shown that if $(x^{(0)}, \dot{x}^{(0)}) \in \overline{\mathbb{R}}$, then $X^{(1)}(x^{(0)}, \dot{x}^{(0)}) \leq \frac{1-\alpha}{1-\alpha-P}$. If $(x^{(0)}, \dot{x}^{(0)}) \notin \overline{\mathbb{R}}$, then $X^{(1)}(x^{(0)}, \dot{x}^{(0)}) < \infty$. Therefore either $(x^{(0)}, \dot{x}^{(0)}) \in \overline{\mathbb{R}}$ and $|x(t)| \leq \frac{1-\alpha}{1-\alpha-P}$ for $t \in [t_0, t_1]$ with $(x(t_1), 0) \in \overline{\mathbb{R}}$, or $(x^{(0)}, \dot{x}^{(0)}) \notin \overline{\mathbb{R}}$ and $x(t_1) < \infty$.

Part 3: Parts 1 and 2 discuss motion only into $\dot{x} > 0$. For motion into $\dot{x} < 0$, the substitution of $x(t) = -\psi(t)$ transforms (2.49) into an initial-value problem of the type already considered. The results in Parts 1 and 2 applied to the ψ system may readily be transformed into the x system. The conclusions are as follows:

$\dot{x}^{(0)} = 0$: $x(t) = x^{(0)}$ for $t \in [t_0, t_1]$ and $\dot{x}(t) < 0$ for $t \in (t_1, t_2)$ with $\dot{x}(t_2) = 0$. If t_2 is unbounded, then $\lim_{t\to\infty} (x(t), \dot{x}(t)) = (x^{(L)}, 0) \in \overline{\mathbb{R}}$. If t_2 is bounded, then either $(x^{(0)}, 0) \in \overline{\mathbb{R}}$ and $|x(t)| \leq \frac{1-\alpha}{1-\alpha-P}$ for $t \in [t_0, t_2]$ with $(x(t_2), 0) \in \overline{\mathbb{R}}$, or $(x^{(0)}, 0) \notin \overline{R}$ and $|x(t_2)| < |x^{(0)}|$.

$\dot{x}^{(0)} < 0$: $\dot{x}(t) < 0$ for $t \in [t_0, t_1)$ with $\dot{x}(t_1) = 0$. If t_1 is unbounded, then $\lim_{t \to \infty} (x(t), \dot{x}(t)) = (x^{(L)}, 0) \in \overline{R}$. If t_1 is bounded, then either $(x^{(0)}, \dot{x}^{(0)}) \in \overline{R}$ and $|x(t)| \leq \frac{1 - \alpha}{1 - \alpha - P}$ for $t \in [t_0, t_1]$ with $(x(t_1), 0) \in \overline{R}$, or $(x^{(0)}, \dot{x}^{(0)}) \notin \overline{R}$ and $|x(t_1)| < \infty$.

Part 4: For $\dot{x}^{(0)} = 0$, repeated application of Parts 1 and 3 show that either $(x^{(0)}, \dot{x}^{(0)}) \in \overline{R}$ and $|x(t)| \leq \frac{1 - \alpha}{1 - \alpha - P}$ for $t \in [t_0, \infty)$, or $(x^{(0)}, \dot{x}^{(0)}) \notin \overline{R}$ and the solution trajectory must intersect $\dot{x} = 0$ with decreasing amplitudes until it enters $\overline{R}$. For $\dot{x}^{(0)} \neq 0$, Parts 2 and 3 and repeated application of Parts 1 and 3 yield the same result. This concludes the proof.

Remarks

For the limiting case of $P = 0$, Theorem 4 states that all solutions have an asymptotic bound of $|x(t)| \leq 1$. This agrees with the results of Iwan[13]. For this case Theorem 2 also implies that all solutions are bounded. In the proof of Theorem 2, it is shown that the distance between consecutive reversals must decrease until $\mathcal{B}_\alpha\{x(t)\}$ loses its hysteretic property. Thus since $\mathcal{B}_\alpha\{x(t)\}$ can lose its hysteretic property only in the interval $|x| \leq 1$ for $\alpha \in [\frac{1}{2}, 1)$, it may be concluded even from this approach that the asymptotic bound is $|x(t)| \leq 1$.

For the limiting case of $P = 1 - \alpha$, it is concluded that $\frac{1 - \alpha}{1 - \alpha - P} = \infty$. Therefore, no conclusion may be reached concerning the boundedness of solutions to (2.47).

Theorem 4 cannot be applied to the limiting case of $\alpha = 1$ since $P \geq 0$.

When the hypotheses of Theorem 4 are satisfied, it is concluded that when the solution trajectory $(x(t), \dot{x}(t))$ enters $\overline{\mathbb{R}}$ in the x, $\dot{x}$ plane, $|x(t)| \le \frac{1-\alpha}{1-\alpha-P}$ for all later time. It can also be shown that when $(x(t), \dot{x}(t))$ enters $\overline{\mathbb{R}}$, it then remains in the closed region $\overline{\mathbb{R}}^{(L)}$ whose boundary is described by

$$\dot{x}=\begin{cases} \sqrt{-\left(x-P+(\hat{x}-1)(1-\alpha)\right)^2+\left(\hat{x}+P-(\hat{x}-1)(1-\alpha)\right)^2} & \text{for } x\in[-\hat{x}, -\hat{x}+2] \\ \sqrt{-\alpha(x+\frac{1-\alpha-P}{\alpha})^2+\alpha(\hat{x}-\frac{1-\alpha-P}{\alpha})^2+4(1-\alpha)} & \text{for } x\in[-\hat{x}+2, \hat{x}] \\ -\sqrt{-\alpha(x-\frac{1-\alpha-P}{\alpha})^2+\alpha(\hat{x}-\frac{1-\alpha-P}{\alpha})^2+4(1-\alpha)} & \text{for } x\in[-\hat{x}, \hat{x}-2] \\ -\sqrt{-\left(x+P-(\hat{x}-1)(1-\alpha)\right)^2+\left(\hat{x}+P-(\hat{x}-1)(1-\alpha)\right)^2} & \text{for } x\in[\hat{x}-2, \hat{x}] \end{cases} \tag{2.61}$$

where $\hat{x}=\frac{1-\alpha}{1-\alpha-P}$.

A simple example with $P=0$ and $\alpha=\frac{1}{2}$ will be used to illustrate both the application of Theorem 4 and the differences between $\overline{\mathbb{R}}$ and $\overline{\mathbb{R}}^{(L)}$. Thus (2.47) becomes

$$\begin{aligned} &\ddot{x}+\mathbb{B}_{\frac{1}{2}}\{x(t)\}=0 \\ &x(t_0)=x^{(0)} \\ &\dot{x}(t_0)=\dot{x}^{(0)} \\ &\mathbb{B}_{\frac{1}{2}}\{x(t)\}=b_{\frac{1}{2}}^{(i)}(x) \text{ at } t=t_0 \end{aligned} \tag{2.62}$$

$\overline{\mathbb{R}}$ is now the line segment $\overline{\mathbb{R}}$: $\{(x,\dot{x})\,|\,-1\le x\le 1, \dot{x}=0\}$, and $\overline{\mathbb{R}}^{(L)}$ is the closed region $\overline{\mathbb{R}}^{(L)}$: $\{(x,\dot{x})\,|\,x^2+\dot{x}^2\le 1\}$. Figure 6 shows $\overline{\mathbb{R}}$ and $\overline{\mathbb{R}}^{(L)}$. Figure 7 shows some resulting configurations of $\mathbb{B}_{\frac{1}{2}}\{x(t)\}$ for $(x^{(0)}, \dot{x}^{(0)})\in\overline{\mathbb{R}}$. The following conclusions can be made:

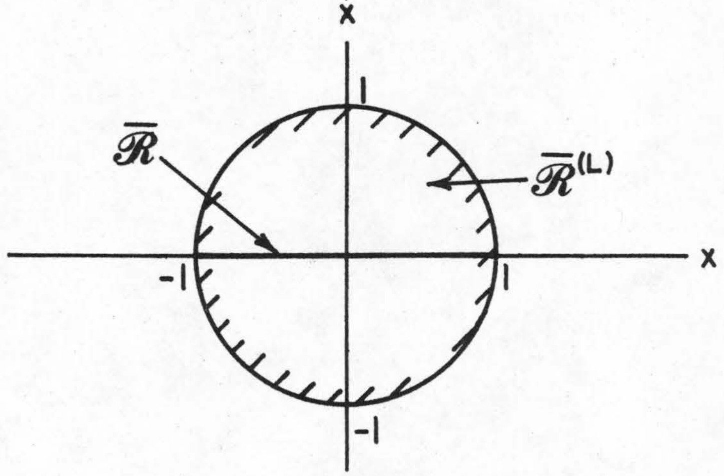


Figure 6: $\overline{\mathbb{R}}$ and $\overline{\mathbb{R}}^{(L)}$ for P=0 and $\alpha = \frac{1}{2}$

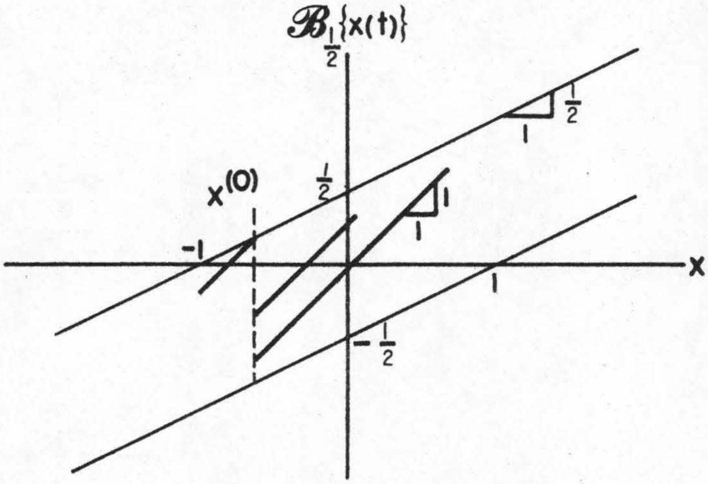


Figure 7: Some Resulting Configurations of

$\mathbb{B}_{\frac{1}{2}}\{x(t)\}$ for $(x^{(0)}, \dot{x}^{(0)}) \in \overline{\mathbb{R}}$

1. For $(x^{(0)}, \dot{x}^{(0)}) \notin \overline{\mathbb{R}}$ and any state of $\mathcal{B}_{\frac{1}{2}}\{x(t)\}$ at $t = t_0$, the solution trajectory $(x(t), \dot{x}(t))$ in the $x, \dot{x}$ plane must cross $\dot{x} = 0$ with decreasing amplitudes until it intersects $\overline{\mathbb{R}}$. Once $\overline{\mathbb{R}}$ has been intersected, the resulting motion is merely a conservative oscillation on a single-valued linear segment of slope 1.
2. For $(x^{(0)}, \dot{x}^{(0)}) \in \overline{\mathbb{R}}$ and any state of $\mathcal{B}_{\frac{1}{2}}\{x(t)\}$ at $t = t_0$, $(x(t), \dot{x}(t))$ need not remain in $\overline{\mathbb{R}}$, but must remain in $\overline{\mathbb{R}}^{(L)}$. For $(x^{(0)}, \dot{x}^{(0)}) \in \overline{\mathbb{R}}^{(L)}$ and any state of $\mathcal{B}_{\frac{1}{2}}\{x(t)\}$ at $t = t_0$, $(x(t), \dot{x}(t))$ does not necessarily remain in $\overline{\mathbb{R}}^{(L)}$. (e. g., If $x^{(0)} = 0$, $\dot{x}^{(0)} = \sqrt{\frac{3}{2}}$, and $\mathcal{B}_{\frac{1}{2}}\{x(t)\} = -\frac{1}{2}$ at $t = t_0$, then $(x^{(0)}, \dot{x}^{(0)}) \in \overline{\mathbb{R}}^{(L)}$ and the first reversal occurs at $(\frac{3}{2}, 0) \notin \overline{\mathbb{R}}^{(L)}$.)

Example 2: An Unbounded Solution for the Forced Elastoplastic Oscillator

If $\alpha = 0$, there exists unbounded solutions to (2. 47) for sufficiently large excitation since $|b_0^{(i)}(x)| \leq 1$ for every i. Clearly any hereditary function for which $|\mathcal{N}\{x(t)\}| \leq H < \infty$ exhibits unbounded solutions for sufficiently large excitation. A more interesting result is shown below.

Assertion

If $\alpha = 0$ and if $\max\limits_{t \geq 0} |G(t)| \leq \delta$, then there exists an unbounded solution to (2. 47) for any $0 < \delta < 1$.

Proof: Consider

$$\left.\begin{aligned} \ddot{x} + \mathcal{B}_0\{x(t)\} &= G(t) \\ x(T) &= -1 \end{aligned}\right\} \quad (2.63)$$

$$\left.\begin{array}{l} \dot{x}(T) = 0 \\ \mathbb{B}_0\{x(t)\} = -1 \text{ at } t = T \end{array}\right\} \quad (2.63) \text{ cont.}$$

where

1. $T = \beta + 2\sqrt{\delta} + \pi$ for $0 < \delta < 1$ and $\beta = \text{Cos}^{-1}\left(\frac{\delta - 1}{\delta + 1}\right)$

2. $G(t) = \begin{cases} \delta & \text{for } t \in [T, T + \beta) \\ 0 & \text{for } t \in [T + \beta, 2T) \end{cases}$

3. $G(t + T) = G(t)$ for $t \geq T$

During the $N^{\underline{th}}$ period of the excitation from $t = T$ where N is a positive integer, the solution is

$$x(t) \equiv \hat{x}(\tau + NT) = \begin{cases} -(1 + \delta)\cos(\tau - NT) + (2N - 1)\delta & \text{for } \tau \in [NT, NT + \beta) \\ -\frac{(\tau - NT - \beta)^2}{2} + 2(\tau - NT - \beta)\sqrt{\delta} + 1 + 2(N - 1)\delta & \\ & \text{for } \tau \in [NT + \beta, NT + \beta + 2\sqrt{\delta}) \\ \cos(\tau - NT - \beta - 2\sqrt{\delta}) + 2N\delta & \text{for } \tau \in [NT + \beta + 2\sqrt{\delta}, (N + 1)T) \end{cases} \quad (2.64)$$

where $\tau = t - NT$.

It can be seen that $x(t) \in [-1, 1 + 2\delta]$ for $t \in [T, 2T)$, $x(t) \in [-1 + 2\delta, 1 + 4\delta]$ for $t \in [2T, 3T), \cdots$, $x(t) \in [-1 + 2(N - 1)\delta, 1 + 2N\delta]$ for $t \in [NT, (N + 1)T), \cdots$. Thus as $t \to \infty$, $x(t)$ is unbounded.

The form of the periodic excitation G(t) has been chosen for mathematical convenience. During each cycle of excitation a finite amount of slipping occurs in the positive direction. The distance slipped becomes unbounded as $t \to \infty$.

2.4 Summary

A general class of dynamical systems with rate-independent hysteresis is described in this chapter. The conceptual decomposition of the system (2. 1) into three parts, each possessing distinctive characteristics, facilitates both its description and treatment.

Theorem 1 guarantees the desirable property that the related initial-value problem (2. 5) possesses a unique solution. The proof deals with the system during time intervals in which $\dot{x}(t) \neq 0$, and time intervals in which $\dot{x}(t) = 0$. This is only natural since during these time intervals (2. 5) reduces to a differential equation in which the distinctive properties of $\mathcal{V}\{x(t)\}$ and $\mathcal{S}[\dot{x}, \mathfrak{F}, c]$ are not detected. The proofs of Theorems 2, 3, and 4 follow in the same spirit.

Theorem 2 gives the intuitively obvious result of bounded solution behavior for systems with dissipative rate-independent hysteresis, provided there is no excitation. Theorem 3 shows that all solutions are bounded for a class of forced hysteretic systems with viscous dissipation. When there is no viscous dissipation, either bounded or unbounded solution behavior can occur in forced hysteretic systems as shown by Theorem 4 and the assertion in Example 2.

III. HARMONIC RESPONSE OF SLIGHTLY NONLINEAR HYSTERETIC SYSTEMS TO TRIGONOMETRIC EXCITATION

In this chapter the response of hysteretic systems to trigonometric excitation is studied. Since it is rarely possible to find analytical solutions to nonlinear dynamical systems, an approximation procedure is used. Section 3.1 is concerned with the development of an approximate solution for a system with a small nonlinearity. Section 3.2 is devoted to the examination of this solution for hysteretic nonlinearities.

3.1 An Asymptotic Solution Procedure for Harmonic Response

The system considered is

$$\ddot{x} + 2z\dot{x} + x + \epsilon f(x, \dot{x}) = r \cos \nu t$$
$$x(t_0) = x^{(0)} \qquad (3.1)$$
$$\dot{x}(t_0) = \dot{x}^{(0)}$$

where ϵ is a small positive parameter, $r>0$, $z\geq 0$, $\nu>0$, and $f(x, \dot{x})$ is a single-valued nonlinear function of x and $\dot{x}$. To assure the existence and uniqueness of solutions to (3.1), $f(x, \dot{x})$ also satisfies

$$|f(x_1, x_2) - f(y_1, y_2)| \leq F\|\underline{x}-\underline{y}\| \quad \text{for } \|\underline{x}\|, \|\underline{y}\| < \infty \qquad (3.2)$$

with F a finite constant.

Asymptotic solution procedures similar to the one presented here have been applied by investigators to study resonances in nonautonomous single degree of freedom systems possessing a small parameter[22,23,24]. Usually the systems asymptotically approach a linear, undamped oscillator (possibly with trigonometric excitation) as the small parameter approaches zero. However, (3.1) possesses viscous damping which is independent of ϵ. Approximation I derived below describes the response of (3.1) near a periodic solution. Another approximation (Approximation II) is derived to describe the response of (3.1) when this response is not necessarily near a periodic solution. For this approximation it is necessary to make the additional assumptions that r, z, and $\nu-1$ are all $O(\epsilon)$ as $\epsilon \to 0$. It is found that the steady-state response predicted by Approximation II is identical to that predicted by Approximation I. Thus when Approximation II is used, its steady-state response prediction is actually valid for values of r, z, and $|\nu-1|$ which are not necessarily small.

<u>Response Near a Periodic Solution</u> (Approximation I)

Assume (3.1) possesses a periodic solution p(t) with period $\frac{2\pi}{\nu}$. If (3.1) begins sufficiently close to p(t), then the response remains close to p(t) for some time interval with length greater than $\frac{2\pi}{\nu}$. Therefore during this interval, the response is approximately a periodic function with period $\frac{2\pi}{\nu}$. Solutions beginning sufficiently close to p(t) are assumed to have the form

$$x(t) = a\cos(\nu t-\theta) + \epsilon u_1(a,\theta,t) + \epsilon^2 u_2(a,\theta,t) + \cdots \tag{3.3}$$

$$\begin{aligned} \dot{a} &= \epsilon A_1(a,\theta) + \epsilon^2 A_2(a,\theta) + \cdots \\ \dot{\theta} &= \epsilon B_1(a,\theta) + \epsilon^2 B_2(a,\theta) + \cdots \end{aligned} \tag{3.4}$$

where $u_i(a, \theta, t)$ is periodic in θ with period 2π and in t with period $\frac{2\pi}{\nu}$, and $A_i(a, \theta)$ and $B_i(a, \theta)$ are periodic in θ with period 2π. It is also required that each $u_i(a, \theta, t)$ contains no fundamental harmonic in t with period $\frac{2\pi}{\nu}$.

When a and θ are at steady-state ($\dot{a}=\dot{\theta}=0$), (3.3) is periodic in t with period $\frac{2\pi}{\nu}$. When not at steady-state, a and θ vary slowly according to (3.4). Consequently, over any time interval of length $\frac{2\pi}{\nu}$, (3.3) is approximately a periodic function with period $\frac{2\pi}{\nu}$. Since (3.3) and (3.4) asymptotically approach the form of the periodic solution of the linear ($\epsilon=0$) system as $\epsilon \to 0$, this approximation can be expected to describe only the response of (3.1) near a periodic solution.

The explicit dependence of the solution form upon the variables a and θ can be motivated physically. For any periodic response p(t) to (3.1) with period $\frac{2\pi}{\nu}$, the energy imparted to the system by the excitation over any time interval $\frac{2\pi}{\nu}$ in length depends upon the level of excitation r, the amplitude of the fundamental harmonic of p(t), and the phase difference between the fundamental harmonic of p(t) and the excitation. Even when the response of (3.1) is not periodic with period $\frac{2\pi}{\nu}$, the energy which the excitation imparts to the system over any interval $\frac{2\pi}{\nu}$ in length still depends upon r and the amplitude and phase of the harmonic of the response with period $\frac{2\pi}{\nu}$ taken over this same interval. Thus a and θ, which are either constant or approximately constant over any interval of length $\frac{2\pi}{\nu}$, are related to the energy imparted to the system by the excitation.

The asymptotic property of the assumed solution form (3.3) and (3.4) will be used to develop a procedure to determine an approximation which satisfies (3.1) to $O(\epsilon^N)$ as $\epsilon \to 0$ for any prescribed positive integer N. It is then hoped that for a given N this approximation describes the response of (3.1) with reasonable accuracy. Due to the increasing amount of computation to determine each additional term in the expansions, N is usually taken as a small number in applications.

After substituting (3.3) into (3.1), Equations (3.4) may be used to eliminate $\dot{a}$, $\dot{\theta}$, $\ddot{a}$, and $\ddot{\theta}$. $\epsilon f(x, \dot{x})$ may then be written in the form

$$\epsilon f(x,\dot{x}) = \epsilon f(a\cos(\nu t-\theta), -a\nu\sin(\nu t-\theta)) + \epsilon^2\Big[u_1 f_x(a\cos(\nu t-\theta), -a\nu\sin(\nu t-\theta))$$
$$+ \left(A_1\cos(\nu t-\theta) + aB_1\sin(\nu t-\theta) + \frac{\partial u_1}{\partial t}\right) f_{\dot{x}}(a\cos(\nu t-\theta), -a\nu\sin(\nu t-\theta))\Big]$$
$$+ \cdots \tag{3.5}$$

Equating coefficients of like powers of ϵ, the following equations are obtained

ϵ^0:

$$(1-\nu^2)a\cos(\nu t-\theta) - 2za\nu\sin(\nu t-\theta) = r\cos\nu t \tag{3.6}$$

ϵ^1:

$$\frac{\partial^2 u_1}{\partial t^2} + 2z\frac{\partial u_1}{\partial t} + u_1 = -2(zA_1 + a\nu B_1)\cos(\nu t-\theta)$$
$$+2(\nu A_1 - azB_1)\sin(\nu t-\theta)$$
$$-f(a\cos(\nu t-\theta), -a\nu\sin(\nu t-\theta)) \tag{3.7}$$

ϵ^2:

$$\frac{\partial^2 u_2}{\partial t^2} + 2z\frac{\partial u_2}{\partial t} + u_2 =$$

$$\left(-A_1\frac{\partial A_1}{\partial a} - B_1\frac{\partial A_1}{\partial \theta} + aB_1^2 - 2zA_2 - 2a\nu B_2\right)\cos(\nu_t - \theta)$$

$$+\left(-2A_1B_1 - aA_1\frac{\partial B_1}{\partial a} - aB_1\frac{\partial B_1}{\partial \theta} + 2\nu A_2 - 2azB_2\right)\sin(\nu t - \theta)$$

$$-2A_1\left(\frac{\partial^2 u_1}{\partial t\partial a} + z\frac{\partial u_1}{\partial a}\right) - 2B_1\left(\frac{\partial^2 u_1}{\partial t\partial \theta} + z\frac{\partial u_1}{\partial \theta}\right)$$

$$-u_1 f_x(a\cos(\nu t-\theta), -a\nu\sin(\nu t-\theta))$$

$$-\left(A_1\cos(\nu t-\theta) + aB_1\sin(\nu t-\theta) + \frac{\partial u_1}{\partial t}\right)f_{\dot{x}}(a\cos(\nu t-\theta), -a\nu\sin(\nu t-\theta)) \qquad (3.8)$$

. . .

The approximation which satisfies (3. 1) to O(ϵ) is found by considering (3. 6) and neglecting terms of O(ϵ) in (3. 3) and (3. 4). Thus a and θ are constants, and (3. 3) and (3. 6) imply

$$x(t) = \frac{r\cos(\nu t - \tilde{\theta})}{\sqrt{(1-\nu^2)^2 + (2z\nu)^2}}$$

$$\tilde{\theta} = \tan^{-1}\left(\frac{2z\nu}{1-\nu^2}\right) \qquad (3.9)$$

This is the exact periodic solution of (3. 1) when $\epsilon f(x, \dot{x})$ is neglected.

For the next higher order approximation, both (3. 6) and (3. 7) must be satisfied and terms of O(ϵ^2) may be neglected in (3. 3) and (3. 4). Since $u_1(a, \theta, t)$ can not contain harmonics in t with period $\frac{2\pi}{\nu}$, all terms in the right member of (3. 7) containing $\sin(\nu t-\theta)$ and $\cos(\nu t-\theta)$ must be eliminated. $f(a\cos(\nu t-\theta), -a\nu\sin(\nu t-\theta))$ may be expanded in the Fourier series

$$f(a\cos(\nu t-\theta), -a\nu\sin(\nu t-\theta)) =$$
$$\frac{C_0(a)}{2} + \sum_{n=1}^{\infty} [S_n(a)\sin(n(\nu t-\theta)) + C_n(a)\cos(n(\nu t-\theta))] \tag{3.10}$$

where

$$\begin{aligned} S_n(a) &= \frac{1}{\pi}\int_0^{2\pi} f(a\cos\psi, -a\nu\sin\psi)\sin(n\psi)d\psi \\ C_n(a) &= \frac{1}{\pi}\int_0^{2\pi} f(a\cos\psi, -a\nu\sin\psi)\cos(n\psi)d\psi \end{aligned} \tag{3.11}$$

The requirement upon $u_1(a, \theta, t)$ is then met by considering the following equations in place of (3. 6) and (3. 7)

$$\begin{aligned} &\left[a(1-\nu^2) + 2(z\epsilon A_1 + a\nu\epsilon B_1) + \epsilon C_1(a)\right]\cos(\nu t-\theta) \\ &+ [2(-za\nu - \nu\epsilon A_1 + az\epsilon B_1) + \epsilon S_1(a)]\sin(\nu t-\theta) = r\cos\nu t \end{aligned} \tag{3.12}$$

$$\frac{\partial^2 u_1}{\partial t^2} + 2z\frac{\partial u_1}{\partial t} + u_1 = -\frac{C_0(a)}{2} - \sum_{n=2}^{\infty} [C_n(a)\cos(n(\nu t-\theta)) + S_n(a)\sin(n(\nu t-\theta))] \tag{3.13}$$

Equation (3. 12) implies

$$\begin{aligned} \epsilon A_1(a,\theta) &= \frac{1}{2(\nu^2+z^2)}\left(-az(\nu^2+1) + r(z\cos\theta + \nu\sin\theta) - z\epsilon C_1(a) + \nu\epsilon S_1(a)\right) \\ \epsilon B_1(a,\theta) &= \frac{1}{2a(\nu^2+z^2)}\left(a\nu(\nu^2-1) + r(\nu\cos\theta - z\sin\theta) + 2\nu z^2 a - \nu\epsilon C_1(a) - z\epsilon S_1(a)\right) \end{aligned} \tag{3.14}$$

Equation (3. 13) gives

$$u_1(a, \theta, t) = -\frac{\alpha_0(a)}{2} - \sum_{n=2}^{\infty} [\alpha_n(a)\cos(n(\nu t-\theta)) + \beta_n(a)\sin(n(\nu t-\theta))] \quad (3.15)$$

with

$$\alpha_n(a) = \frac{((n\nu)^2-1)C_n(a) + 2zn\nu S_n(a)}{((n\nu)^2-1)^2 + (2zn\nu)^2}$$

$$\beta_n(a) = \frac{-2zn\nu C_n(a) + ((n\nu)^2-1)S_n(a)}{((n\nu)^2-1)^2 + (2zn\nu)^2} \quad (3.16)$$

Therefore the approximation which satisfies (3. 1) to $O(\epsilon^2)$ is

$$x(t) = a\cos(\nu t-\theta) + \epsilon u_1(a, \theta, t) \quad (3.17)$$

$$\dot{a} = \epsilon A_1(a, \theta)$$

$$\dot{\theta} = \epsilon B_1(a, \theta) \quad (3.18)$$

where $u_1(a, \theta, t)$ is given by (3. 15) and (3. 16), and $\epsilon A_1(a, \theta)$ and $\epsilon B_1(a, \theta)$ are given by (3. 14).

For the next higher order approximation, Equations (3. 6) through (3. 8) must be satisfied and terms of $O(\epsilon^3)$ may be neglected in (3. 3) and (3. 4). If the expressions (3. 14) through (3. 16) are retained for $\epsilon A_1(a, \theta)$, $\epsilon B_1(a, \theta)$, and $u_1(a, \theta, t)$, only (3. 8) needs to be satisfied. Since $u_2(a, \theta, t)$ can not possess harmonics in t with period $\frac{2\pi}{\nu}$, the terms in the right member of (3. 8) containing these harmonics must be eliminated. This is done by choosing $A_2(a, \theta)$ and $B_2(a, \theta)$ so that the coefficients of $\sin(\nu t-\theta)$ and $\cos(\nu t-\theta)$ vanish. $u_2(a, \theta, t)$ is then found by solving (3. 8).

Higher order approximations are determined in a similar way.

Qualitative features of the harmonic response are usually revealed in a first order approximation. Higher approximations require considerable calculation and usually add corrections of a small order. For this reason only the first order approximation is discussed here.

A First Order Approximation. Consider the approximation defined by (3.17) and (3.18). If $\overline{A}$ and $\overline{B}$ are average values of $A_1(a, \theta)$ and $B_1(a, \theta)$ over the interval $t \in [t_0, t_1]$, then

$$
\begin{aligned}
a(t_1) - a(t_0) &= (t_1 - t_0)\epsilon\overline{A} \\
\theta(t_1) - \theta(t_0) &= (t_1 - t_0)\epsilon\overline{B}
\end{aligned}
\tag{3.19}
$$

Hence the time interval over which a and θ change by a finite amount is $O(\frac{1}{\epsilon})$. The terms neglected in (3.4) are $O(\epsilon^2)$ and the corresponding error in a and θ over this time interval is $O(\epsilon)$. Therefore it is not necessary to carry along the term $\epsilon u_1(a, \theta, t)$ in (3.17) since the error in using (3.18) and $x(t) = a\cos(\nu t - \theta)$ is of the same order. (Similar reasoning leads to the conclusion that for higher order approximations, if terms of $O(\epsilon^M)$ are neglected in (3.3), then only terms of $O(\epsilon^{M+1})$ may be neglected in (3.4).) In light of this discussion, Approximation I is defined to be

$$x(t) = a\cos(\nu t - \theta) \tag{3.20}$$

$$
\begin{aligned}
\dot{a} &= \epsilon A_1(a,\theta) = \frac{1}{2(\nu^2+z^2)}\Big(-az(\nu^2+1) + r(z\cos\theta + \nu\sin\theta) + \nu S(a) - zC(a)\Big) \\
\dot{\theta} &= \epsilon B_1(a,\theta) = \frac{1}{2a(\nu^2+z^2)}\Big(a\nu(\nu^2-1) + r(\nu\cos\theta - z\sin\theta) + 2\nu z^2 a - zS(a) - \nu C(a)\Big)
\end{aligned}
\tag{3.21}
$$

where

$$S(a) \equiv \epsilon S_1(a) = \frac{1}{\pi}\int_0^{2\pi} \epsilon f(a\cos\psi, -a\nu\sin\psi)\sin\psi\, d\psi$$

$$C(a) \equiv \epsilon C_1(a) = \frac{1}{\pi}\int_0^{2\pi} \epsilon f(a\cos\psi, -a\nu\sin\psi)\cos\psi\, d\psi \qquad (3.22)$$

The predicted steady-state response is

$$x = \tilde{a}\cos(\nu t - \tilde{\theta}) \qquad (3.23)$$

where $(\tilde{a}, \tilde{\theta})$ is any solution pair of

$$r\sin\theta = 2za\nu - S(a)$$

$$r\cos\theta = a(1-\nu^2) + C(a) \qquad (3.24)$$

Equations (3.24) are obtained from (3.21) with $\dot{a}=\dot{\theta}=0$.

Response Not Necessarily Near A Periodic Solution (Approximation II)

Approximation I applies whenever the response of (3.1) is near a periodic solution. To determine an approximation for the response not necessarily near a periodic solution, it must in general be assumed that r, z, and ν-1 be $O(\epsilon)$. Then (3.1) is asymptotic to a linear, undamped oscillator with no excitation, and (3.3) and (3.4) are asymptotic to the form of the solution of this linear system. For $\epsilon=0$, every initial condition results in a periodic solution of (3.1). For ϵ sufficiently small, the $O(\epsilon)$ terms in (3.1) can exert only a small effect upon this response. Consequently, for any initial condition, the response is approximately a periodic function with period $\frac{2\pi}{\nu}$ over any time interval of length $\frac{2\pi}{\nu}$. This behavior is reflected in the solution form by the slow variation of a and θ. Thus (3.3) and (3.4) may be used in the derivation of this approximation also. The same procedure used above may be followed.

With the additional assumptions, the procedure reduces to that presented by Bogoliubov and Mitropolsky(22). The first order approximation to the response (not necessarily near a periodic solution) is found to be

$$x(t) = a \cos(\nu t - \theta) \tag{3.25}$$

$$\dot{a} = \frac{1}{2\nu}(-2az\nu + r \sin\theta + S(a))$$

$$\dot{\theta} = \frac{1}{2a\nu}(a(\nu^2 - 1) + r \cos\theta - C(a)) \tag{3.26}$$

where S(a) and C(a) are given by (3.22). The approximate solution given by (3.25) and (3.26) will be called <u>Approximation II</u>.

Approximation II is the same as the approximate solution obtained by the well known method of slowly varying amplitude and phase. Thus it is seen that the approximate solution obtained by the latter method is related to an asymptotic solution which is derived by a consistent mathematical procedure.

The steady-state response predicted by Approximation II is

$$x(t) = \tilde{a} \cos(\nu t - \tilde{\theta}) \tag{3.27}$$

where $(\tilde{a}, \tilde{\theta})$ is any solution pair of

$$r \sin\theta = 2za\nu - S(a)$$

$$r \cos\theta = a(1 - \nu^2) + C(a) \tag{3.28}$$

This is identical to the steady-state response (3.23) and (3.24) predicted by Approximation I. Therefore, the steady-state response prediction of Approximation II is not necessarily restricted to small r, z, and $|\nu - 1|$ as the derivation implies.

3.2 Application to Hysteretic Nonlinearities

The system considered is

$$\ddot{x} + 2z\dot{x} + x + \mathcal{N}\{x(t)\} = r \cos \nu t$$

$$x(t_0) = x^{(0)} \tag{3.29}$$

$$\dot{x}(t_0) = \dot{x}^{(0)}$$

$$\text{State of } \mathcal{N}\{x(t)\} \text{ at } t = t_0$$

where $t_0 \geq 0$, $r > 0$, $z \geq 0$, $\nu > 0$, and $\mathcal{N}\{x(t)\}$ is a hereditary function which satisfies Assumptions (A2), (A6), and (A8). It is assumed that r, z, ν-1, and $\mathcal{N}\{x(t)\}$ are $O(\epsilon)$.

The following additional assumption upon $\mathcal{N}\{x(t)\}$ is needed.

Assumption 10 (A10): When an admissible path cycles repeatedly between two turning points, $x = A$ and $x = -A$, $\mathcal{N}\{x(t)\}$ may have one of two steady-state forms.

1. $\mathcal{N}\{x(t)\}$ is a single-valued, odd function of x.
2. In the $x, \mathcal{N}$ plane, $\mathcal{N}\{x(t)\}$ traces a hysteresis loop which is symmetric about the origin.

Furthermore, for any given model $\mathcal{N}\{x(t)\}$, there exists a family of these steady-state configurations uniquely dependent upon A such that a small variation in A results in, at most, a small variation of the steady-state configuration in the $x, \mathcal{N}$ plane. Also for this family, the area enclosed by the steady-state configuration, $E_H(A)$, is a continuous function of A and possesses a piecewise continuous first derivative such that $\frac{dE_H(A)}{dA} \geq 0$ wherever it is defined.

Most hereditary functions which are used to describe rate-independent hysteretic behavior in physical systems possess a family of steady-state configurations with the properties stated in (A10).

If the steady-state configuration of $\mathcal{H}\{x(t)\}$ is single-valued, then $\mathcal{H}\{x(t)\}$ does no work over one path cycle. By (A8), if a hysteresis loop is formed, $\mathcal{H}\{x(t)\}$ does positive work over one path cycle. By Assumption 3 of (A2), each steady-state configuration of $\mathcal{H}\{x(t)\}$ is independent of the frequency of oscillation.

In the following discussion, a family of configurations of the type described in (A10) will be considered. For convenience each member will be referred to as a hysteresis loop even though it may be single-valued.

S(a) and C(a)

The nonlinearity enters into Approximation II only through S(a) and C(a). Using the transformation $\psi = \nu t$ where ν is constant, it is seen from (3.22) that S(a) and C(a) are weighted integrals of $f(y(t), \dot{y}(t))$ where $y(t) = a \cos \nu t$. y(t) passes through a trigonometric oscillation with amplitude a over the range of integration which is from 0 to $\frac{2\pi}{\nu}$. Since the solution form is oscillatory with slowly varying amplitude and phase, the hysteresis loop corresponding to amplitude a is used to evaluate S(a) and C(a).

It can be shown that

$$S(a) = -\frac{E_H(a)}{\pi a} \tag{3.30}$$

where $E_H(a)$ is the area enclosed by the hysteresis loop for amplitude a. Also,

$$C(a) = \frac{2}{\pi}\int_0^{\pi} m(a\cos\psi; a)\cos\psi\, d\psi \tag{3.31}$$

where $m(\zeta;a)$, defined over $\zeta \in [-a, a]$, is the average of the ascending and descending branches of the hysteresis loop for amplitude a. Therefore, Approximation II is completely determined by only two properties of each member of a family of hysteresis loops associated with any given $\mathscr{H}\{x(t)\}$: the area enclosed by the loop, and the average of the ascending and descending branches. These are illustrated in Fig. 8.

Steady-State Response (Simplified Approximation II)

By eliminating θ from Equations (3.28) and using (3.30), the steady-state frequency response equation is found to be

$$\nu^4 + 2\nu^2\left(2z^2 - \frac{C(a)}{a} - 1\right) + 4\nu z\,\frac{E_H(a)}{\pi a^2} + \left(\frac{E_H(a)}{\pi a^2}\right)^2 + \left(\frac{C(a)}{a} + 1\right)^2 - \left(\frac{r}{a}\right)^2 = 0 \tag{3.32}$$

It is difficult to discuss the qualitative appearance of the steady-state response curves described by (3.32). Therefore the following convenient form of (3.25) and (3.26), which is still accurate to the same order, will be examined instead.

$$x(t) = a\cos(\nu t - \theta) \tag{3.33}$$

$$\dot{a} = \frac{1}{2\nu}\left(-2az + r\sin\theta - \frac{E_H(a)}{\pi a}\right)$$
$$\dot{\theta} = \frac{1}{2a\nu}\left(a(\nu^2 - 1) + r\cos\theta - C(a)\right) \tag{3.34}$$

This will be called Simplified Approximation II.

Although Approximation II and Simplified Approximation II are accurate to the same order, their predictions will differ (unless $z = 0$

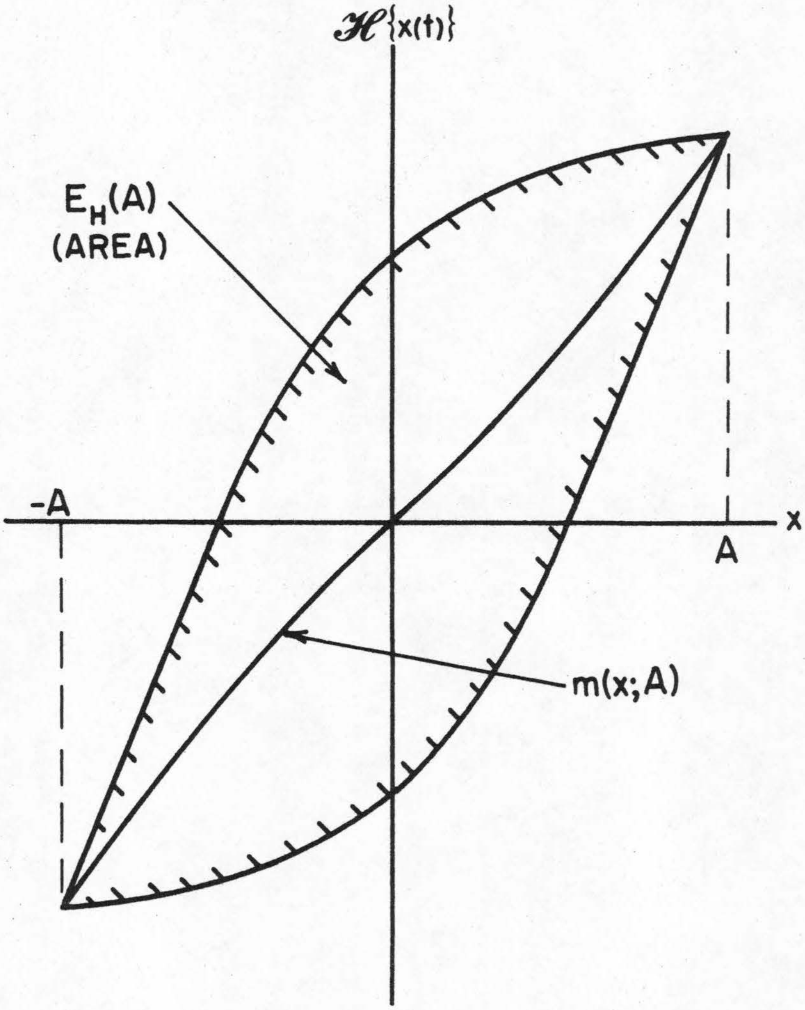


Figure 8: Hysteresis Loop

or $\nu=1$). Nevertheless, the qualitative behavior predicted will be similar for small values of z and $|\nu-1|$. For larger values of z and $|\nu-1|$, as mentioned in Section 3.1, Approximation II is expected to give more accurate steady-state predictions.

The steady-state response predicted by Simplified Approximation II is

$$x(t) = \tilde{a}\cos(\nu t-\tilde{\theta}) \tag{3.35}$$

where $(\tilde{a}, \tilde{\theta})$ is any solution pair of

$$\begin{aligned} r\sin\theta &= 2az + \frac{E_H(a)}{\pi a} \equiv e(a) \\ r\cos\theta &= a(1-\nu^2) + C(a) \end{aligned} \tag{3.36}$$

Eliminating θ from Equations (3.36) gives

$$\nu^2 = 1 + \frac{C(a)}{a} \pm \frac{1}{a}\sqrt{r^2 - e^2(a)} \tag{3.37}$$

<u>Extrema of Response.</u> If $r < e(\tilde{a})$, (3.37) implies that there can be no steady-state response with amplitude $\tilde{a}$. If $r > e(\tilde{a})$, there are at most two excitation frequencies at which steady-state response occurs with amplitude $\tilde{a}$. If $r = e(\tilde{a})$, there is at most only one frequency at which steady-state response occurs with amplitude $\tilde{a}$, and in this case $\tilde{a}$ corresponds to an extremum of response. An extremum is a relative maximum (minimum) if e(a) increases (decreases) with respect to a in some neighborhood of the amplitude at which the extremum occurs. The locus of extremum response in the ν, a plane is

$$\nu_e = \sqrt{1 + \frac{C(a)}{a}} \tag{3.38}$$

Boundedness of Response. If $z>0$, (3.37) implies that all steady-state response amplitudes are bounded by $\frac{r}{2z}$. If $z=0$ and $\lim_{a\to\infty} \frac{E_H(a)}{a}$ is unbounded, all steady-state response amplitudes are again bounded. If $z=0$ and $\frac{E_H(a)}{\pi a} \le M < \infty$ for all a, there exists unbounded resonance behavior for $r \ge M$.

Disconnected Response Curves. It is interesting to note that the response curves described by (3.37) can be disconnected. A necessary and sufficient condition for the existence of a disconnected response curve for some level of excitation is that e(a) exhibit at least one relative maximum for $a \in (0, \infty)$. An illustration which will shortly be discussed is shown in Figure 9b.

Qualitative Appearance of the Response Curves. The qualitative appearance of the steady-state frequency response curves may be determined by plotting e(a) as a function of a. Examples are shown in Figures 9a and 9b. $e_1(a)$ refers to a case with only viscous dissipation, and it is seen that for finite r, all steady-state solutions are bounded. On the other hand, unbounded amplitude resonance is predicted for sufficiently large r for the particular type of hysteretic dissipation which results in $e_2(a)$. $e_3(a)$ refers to a system with viscous dissipation and another type of hysteretic dissipation which produces isolated portions of the response curves for an interval of r. For low levels of excitation, there are single, continuous response curves similar to those corresponding to $e_1(a)$. When $r=r_4$ there is still a continuous response curve, but also an isolated point above it. As r increases, this point becomes a closed curve. When $r=r_6$, the closed curve coalesces with

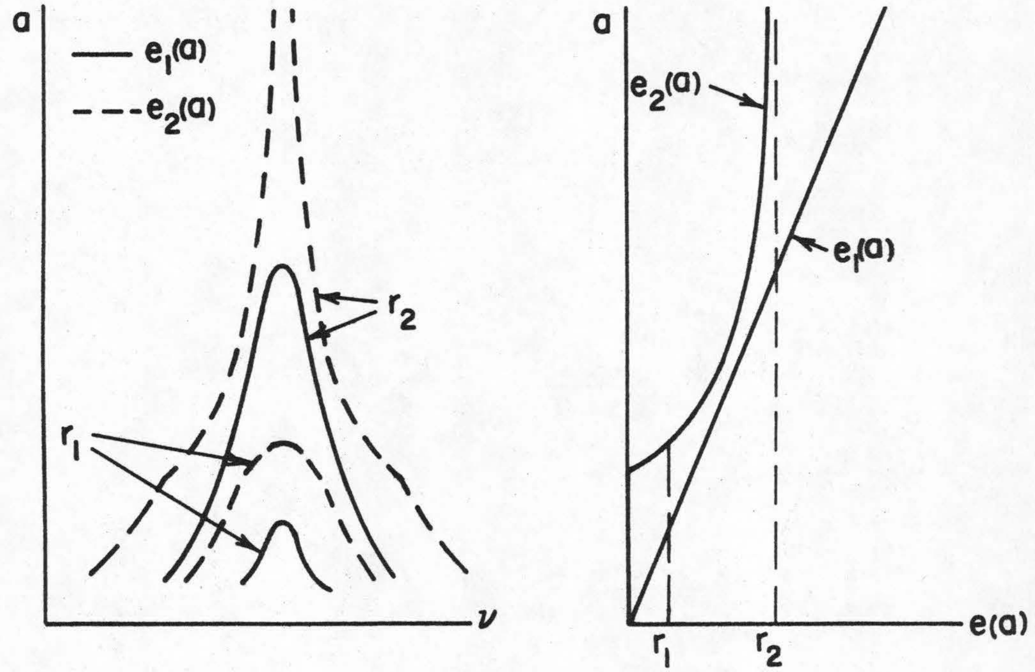


Figure 9a: Response Curves Corresponding to the Functions $e_1(a)$ and $e_2(a)$

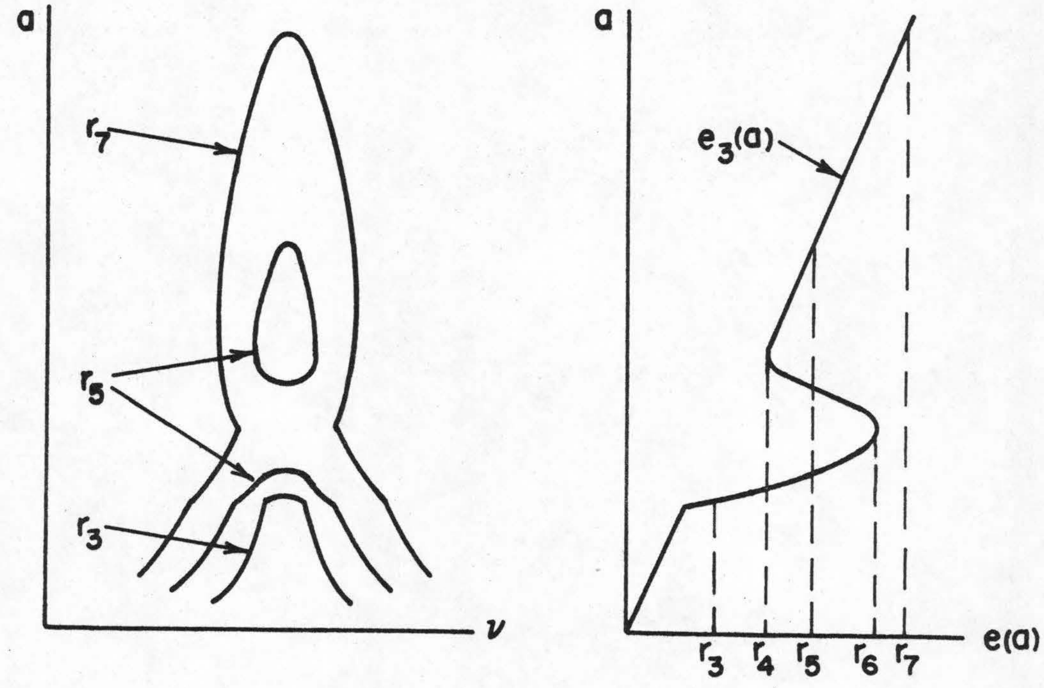


Figure 9b: Response Curves Corresponding to the Function $e_3(a)$

the lower portion of the curve to again form a single, continuous response curve. If $\frac{d}{da}\left(\frac{E_H(a)}{a}\right) \geq V > -\infty$, this type of unusual response can be eliminated by increasing the viscous damping such that $z > -\frac{V}{2\pi}$.

From (3.38) and (3.31) it can be seen that the locus of extremum response depends only upon a and m(x;a). Since the locus of extremum response is the line of symmetry of the response curves in the ν^2, a plane, m(x;a) also influences the leaning of the response curves in the ν, a plane.

Increasing the level of excitation or decreasing the viscous damping tends to raise the maxima of response, to depress the minima of response, and to increase the width of the response curves for a given amplitude. If $\mathcal{N}\{x(t)\}$ is replaced by $\overline{\mathcal{N}}\{x(t)\}$, then the same qualitative change is produced wherever $E_H(a) > \overline{E}_H(a)$. Furthermore, changing $\mathcal{N}\{x(t)\}$ generally shifts the locus of extremum response.

<u>Energy Interpretation.</u> The existence of steady-state oscillations with amplitude $\tilde{a}$ can be related to the energy balance in the system. Using the assumed steady-state solution form (3. 35), it can be shown that for each oscillation of the response with amplitude $\tilde{a}$

$E_H(\tilde{a})$ = hysteresis energy loss

$2\pi z\tilde{a}^2$ = viscous energy loss $+O(\epsilon^2)$

$\pi r\tilde{a}$ = maximum possible energy added by the excitation

If the total energy dissipated $(\pi\tilde{a}e(\tilde{a}))$ is greater than the maximum possible energy $(\pi r\tilde{a})$ which can be added to the system over an oscillation with amplitude $\tilde{a}$, then steady-state oscillations with amplitude $\tilde{a}$ cannot exist. On the other hand, if $\pi\tilde{a}e(\tilde{a}) \leq \pi r\tilde{a}$, then steady-state oscillations can exist with amplitude $\tilde{a}$, and the phase between the response and the excitation is determined to create an energy balance. The plots of e(a) and r in Figures 9a and 9b may now be interpreted as plots of energy dissipation and maximum energy input scaled by the factor $\frac{1}{\pi a}$.

Stability of Steady-State Solutions

The equations for $\dot{a}$ and $\dot{\theta}$ are of the form

$$\begin{aligned} \dot{a} &= \epsilon A_1(a, \theta) \\ \dot{\theta} &= \epsilon B_1(a, \theta) \end{aligned} \tag{3.39}$$

The steady-state values are determined from

$$\begin{aligned} \epsilon A_1(a, \theta) &= 0 \\ \epsilon B_1(a, \theta) &= 0 \end{aligned} \tag{3.40}$$

If $(\tilde{a}, \tilde{\theta})$ is a solution pair of (3.40), let

$$\begin{aligned} a &= \tilde{a} + \xi \\ \theta &= \tilde{\theta} + \eta \end{aligned} \tag{3.41}$$

Substituting (3.41) into (3.39), expanding $\epsilon A_1(a, \theta)$ and $\epsilon B_1(a, \theta)$ about $\tilde{a}$ and $\tilde{\theta}$, and using (3.40) gives

$$\begin{aligned} \dot{\xi} &= \xi \left.\frac{\partial(\epsilon A_1)}{\partial a}\right|_{\substack{a=\tilde{a} \\ \theta=\tilde{\theta}}} + \eta \left.\frac{\partial(\epsilon A_1)}{\partial \theta}\right|_{\substack{a=\tilde{a} \\ \theta=\tilde{\theta}}} + \overline{p}(\xi, \eta) \\ \dot{\eta} &= \xi \left.\frac{\partial(\epsilon B_1)}{\partial a}\right|_{\substack{a=\tilde{a} \\ \theta=\tilde{\theta}}} + \eta \left.\frac{\partial(\epsilon B_1)}{\partial \theta}\right|_{\substack{a=\tilde{a} \\ \theta=\tilde{\theta}}} + \overline{q}(\xi, \eta) \end{aligned} \tag{3.42}$$

where $\overline{p}(\xi, \eta)$ and $\overline{q}(\xi, \eta)$ contain higher order terms of ξ and η. Sufficient conditions for instability or asymptotic stability of the trivial solution of (3.42) can be obtained by examining the linear approximation

$$\dot{\bar{\xi}} = \bar{\xi}\,\frac{\partial(\epsilon A_1)}{\partial a}\bigg|_{\substack{a=\tilde{a}\\ \theta=\tilde{\theta}}} + \bar{\eta}\,\frac{\partial(\epsilon A_1)}{\partial \theta}\bigg|_{\substack{a=\tilde{a}\\ \theta=\tilde{\theta}}}$$

$$\dot{\bar{\eta}} = \bar{\xi}\,\frac{\partial(\epsilon B_1)}{\partial a}\bigg|_{\substack{a=\tilde{a}\\ \theta=\tilde{\theta}}} + \bar{\eta}\,\frac{\partial(\epsilon B_1)}{\partial \theta}\bigg|_{\substack{a=\tilde{a}\\ \theta=\tilde{\theta}}} \tag{3.43}$$

Let

$$\bar{\xi} = \hat{\xi} e^{\lambda t}$$

$$\bar{\eta} = \hat{\eta} e^{\lambda t} \tag{3.44}$$

The frequency equation is

$$\lambda^2 - Q\lambda + D = 0 \tag{3.45}$$

where

$$Q = \frac{\partial(\epsilon A_1)}{\partial a}\bigg|_{\substack{a=\tilde{a}\\ \theta=\tilde{\theta}}} + \frac{\partial(\epsilon B_1)}{\partial \theta}\bigg|_{\substack{a=\tilde{a}\\ \theta=\tilde{\theta}}}$$

$$D = \frac{\partial(\epsilon A_1)}{\partial a}\bigg|_{\substack{a=\tilde{a}\\ \theta=\tilde{\theta}}} \frac{\partial(\epsilon B_1)}{\partial \theta}\bigg|_{\substack{a=\tilde{a}\\ \theta=\tilde{\theta}}} - \frac{\partial(\epsilon A_1)}{\partial \theta}\bigg|_{\substack{a=\tilde{a}\\ \theta=\tilde{\theta}}} \frac{\partial(\epsilon B_1)}{\partial a}\bigg|_{\substack{a=\tilde{a}\\ \theta=\tilde{\theta}}} \tag{3.46}$$

The following conclusions can be made[28]. If $Q<0$ and $D>0$, the trivial solution is asymptotically stable for both systems (3.42) and (3.43). If $Q\leq 0$ and $D<0$, the trivial solution is unstable for both systems. If $Q=0$ and $D<0$, $\bar{\xi}$ and $\bar{\eta}$ are bounded and the trivial solution of (3.43) is stable, but it is possible that the trivial solution of (3.42) may be unstable, stable, or asymptotically stable. Thus if $Q\leq 0$ and $D<0$ ($Q<0$ and $D>0$), $\tilde{a}$ and $\tilde{\theta}$ are unstable (asymptotically stable).

Since it is convenient to discuss D in terms of geometric properties of the steady-state response curves in the ν, a plane, (3.40) is used to eliminate $\tilde{\theta}$ from (3.46).

For Simplified Approximation II, Equations (3.46) become

$$Q = -4z - \frac{1}{\pi\tilde{a}} \left.\frac{dE_H(a)}{da}\right|_{a=\tilde{a}}$$

$$D = \tilde{\nu}^4 - \tilde{\nu}^2\left(\frac{C(\tilde{a})}{\tilde{a}} + \left.\frac{dC(a)}{da}\right|_{a=\tilde{a}} + 2\right) + \left(\frac{C(\tilde{a})}{\tilde{a}}+1\right)\left(\left.\frac{dC(a)}{da}\right|_{a=\tilde{a}}+1\right) \tag{3.47}$$

$$+\left(2z + \frac{E_H(\tilde{a})}{\pi\tilde{a}^2}\right)\left(2z + \frac{1}{\pi}\frac{d}{da}\left(\frac{E_H(a)}{a}\right)\bigg|_{a=\tilde{a}}\right)$$

where $\tilde{a}$ is the steady-state amplitude corresponding to the excitation frequency $\tilde{\nu}$. In the following, steady-state solutions $(\tilde{\nu}, \tilde{a})$ for which $\left.\frac{dC(a)}{da}\right|_{a=\tilde{a}}$ or $\left.\frac{dE_H(a)}{da}\right|_{a=\tilde{a}}$ are not defined are excluded. Since $\frac{dE_H(a)}{da} \geq 0$ by (A10), $Q \leq 0$. It can be shown that $D = 0$ at the loci of vertical tangency of the steady-state response curves described by (3.37). Thus each locus of vertical tangency corresponds to a boundary of stability. An alternate form for D is

$$D = \left(1 + \frac{C(\tilde{a})}{\tilde{a}} - \tilde{\nu}^2\right)\frac{1}{\left.\frac{da}{d\nu}\right|_{\substack{\nu=\tilde{\nu}\\ a=\tilde{a}}}} \tag{3.48}$$

where $\left.\frac{da}{d\nu}\right|_{\substack{\nu=\tilde{\nu}\\ a=\tilde{a}}}$ is the slope of the steady-state response curve passing through $(\tilde{\nu}, \tilde{a})$. From (3.38) it can be seen that if $(\tilde{\nu}, \tilde{a})$ lies to the left of the locus of extremum response, then $D>0(<0)$ when $\left.\frac{da}{d\nu}\right|_{\substack{\nu=\tilde{\nu}\\ a=\tilde{a}}} >0(<0)$.

If $(\tilde{\nu}, \tilde{a})$ lies to the right of the locus of extremum response, then $D>0$ (<0) when $\left.\frac{da}{d\nu}\right|_{\substack{\nu=\tilde{\nu} \\ a=\tilde{a}}} < 0$ (>0). Since the loci of vertical tangency are boundaries of stability, $D>0$ (<0) at a relative maximum (minimum) of response.

Thus for $z>0$ or for steady-state amplitudes at which $\frac{dE_H(a)}{da} > 0$, the following can be concluded:

If the steady-state solution $(\tilde{\nu}, \tilde{a})$ lies to the left of the locus of extremum response, it is unstable (asymptotically stable) if $\left.\frac{da}{d\nu}\right|_{\substack{\nu=\tilde{\nu} \\ a=\tilde{a}}} < 0$ (>0). If it lies to the right of the locus of extremum response, it is unstable (asymptotically stable) if $\left.\frac{da}{d\nu}\right|_{\substack{\nu=\tilde{\nu} \\ a=\tilde{a}}} > 0$ (<0). If it is a relative minimum (maximum) of response, it is unstable (asymptotically stable).

For $z=0$ and $\left.\frac{dE_H(a)}{da}\right|_{a=\tilde{a}} = 0$, only the conclusions concerning unstable solutions are valid. No stability conclusions may be made about the remaining solutions by this approach.

If the nonlinearity and z are sufficiently small, the steady-state response curves of Approximation II, described by (3.32), possess only one locus of extremum response in the vicinity of $\nu=1$. It can be shown that the same conclusions concerning the stability of the steady-state response obtained for Simplified Approximation II also apply to Approximation II in the vicinity of $\nu=1$.

Transient Behavior

The preceding stability analysis provides information concerning response beginning sufficiently close to a steady-state. To investigate the general transient behavior of the approximate solution, divide $\dot{a}$ by $\dot{\theta}$ to obtain

$$\frac{da}{d\theta} = \frac{\epsilon A_1(a, \theta)}{\epsilon B_1(a, \theta)} \tag{3.49}$$

Since (3.49) is an autonomous system its solution trajectories can be discussed in the a, θ state plane. Beginning at initial values $a(t_0)$ and $\theta(t_0)$, a and θ travel along the integral curve of (3.49) passing through $(a(t_0), \theta(t_0))$ as t increases. Also, steady-state solutions $(\tilde{a}, \tilde{\theta})$ correspond to singularities of (3.49). Thus by examining the behavior of integral curves of (3.49) in the a, θ plane, the effect of initial conditions upon the behavior of a and θ may be studied. This is of special interest when multiple steady-state solutions exist for a given frequency. This method is illustrated by an example in Chapter V.

3.3 Summary

An asymptotic solution procedure has been developed to obtain an approximation (Approximation I) for the response near a periodic solution of a system with a small nonlinearity. The system is asymptotic to a linear, viscously damped oscillator with trigonometric excitation. The fact that the system possesses viscous damping which is independent of ϵ distinguishes the present solution procedure from most other standard techniques.

With the additional assumptions that the excitation and viscous damping levels and frequency detuning are $O(\epsilon)$ as $\epsilon \to 0$, Approximation II is derived to describe the transient response of the system (not necessarily near a periodic solution). The predicted steady-state response of Approximation II is identical to that of Approximation I. Thus although Approximation II is used in the application, its steady-state response prediction is actually valid for large excitation and viscous damping levels and frequency detuning.

Approximation II is applied to describe the response of a system with a small hysteretic nonlinearity, $\mathcal{N}\{x(t)\}$. $\mathcal{N}\{x(t)\}$ is assumed to possess a family of symmetric steady-state configurations depending upon the amplitude of path oscillation, a. Only two properties of each member of this family are needed to completely determine Approximation II: the area enclosed by each configuration in the $x, \mathcal{N}$ plane, and the average of the ascending and descending branches of each configuration. To facilitate discussion of the appearance of the steady-state response curves, a simplification is made in the equations of Approximation II to obtain Simplified Approximation II. In the framework of the asymptotic solution procedure, both approximations are of the same order of accuracy as $\epsilon \to 0$. Though the quantitative predictions of the different approximations will generally differ, the qualitative predictions will be similar when the level of viscous damping and the frequency detuning are small.

IV. A SYSTEM WITH LIMITED SLIP: STEADY-STATE RESPONSE

In this chapter, a system with limited slip first discussed by Iwan[10] is examined. In addition to providing an example of the application of the general formulation in Section 3.2, the present investigation complements the results found previously.

4.1 A System with Limited Slip

A physical model for a connection with limited slip is shown in Figure 10, and the corresponding restoring force $\mathcal{U}_s\{\bar{x}(t)\}$ is shown in Figure 11. It is assumed that slip-friction with maximum resisting force F_y exists between the two members. The following dimensionless form of the equation of motion of a forced oscillator with limited slip is considered:

$$\begin{gathered} \ddot{x} + 2z\dot{x} + \mathcal{U}_s\{x(t)\} = G(t) \\ x(t_0) = x^{(0)} \\ \dot{x}(t_0) = \dot{x}^{(0)} \\ \text{State of } \mathcal{U}_s\{x(t)\} \text{ at } t = t_0 \end{gathered} \tag{4.1}$$

where $z \geq 0$. $\mathcal{U}_s\{x(t)\}$ can be described as follows. The virgin curve is

$$u_s^{(0)}(x) = \begin{cases} x - s & x \geq 1 + s \\ 1 & x \in [1, 1+s] \\ x & |x| \leq 1 \end{cases} \tag{4.2}$$

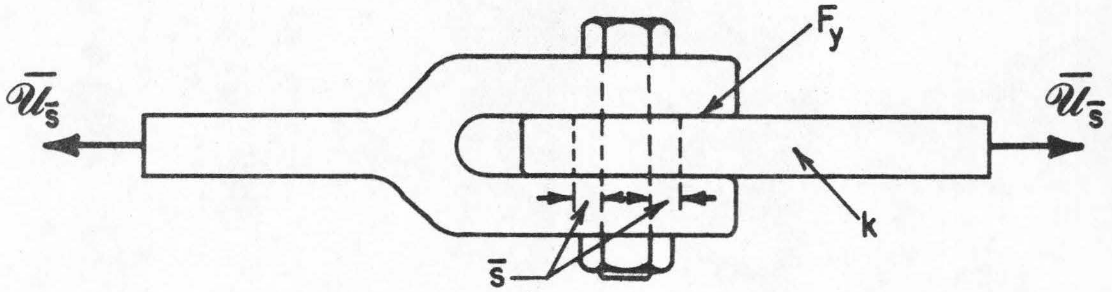


Figure 10: A Connection with Limited Slip

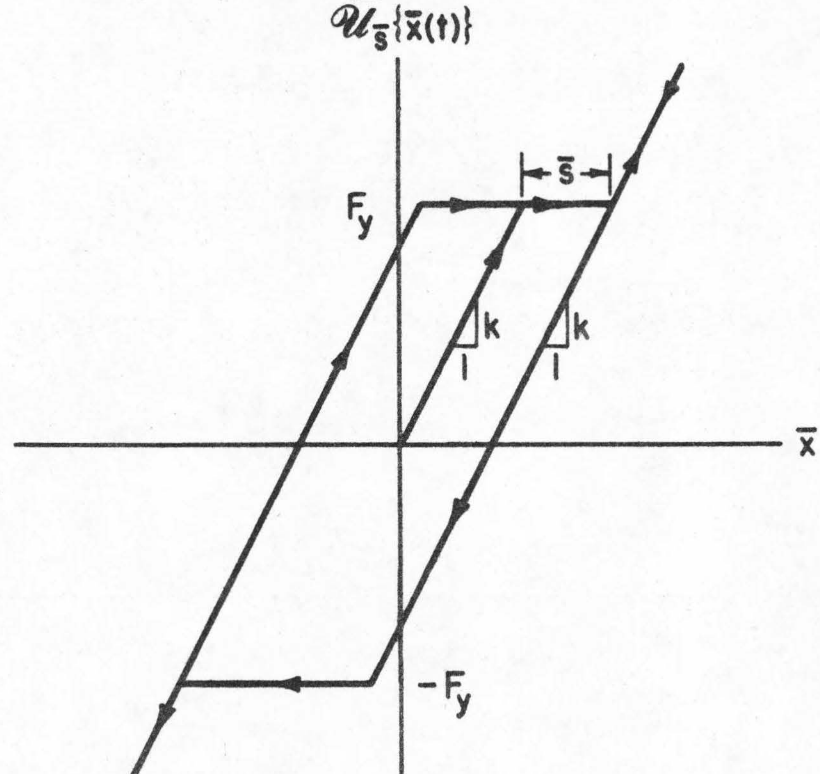


Figure 11: Restoring Force with Limited Slip

$$u_s^{(0)}(x) = \begin{cases} -1 & x \in [-1-s, -1] \\ x+s & x \le -1-s \end{cases} \qquad (4.2) \text{ cont.}$$

Define p = 1 (p = -1) if a reversal occurs on the upper (lower) segment of zero slope. After a reversal from either the upper or lower segment with zero slope at $x = \tilde{x}$, $\mathcal{U}_s\{x(t)\}$ becomes

$$u_s^{(k)}(x) = \begin{cases} x - s & \text{for} \quad x \ge 1+s \\ 1 & \text{for} \quad x \in [\tilde{x}+1-p, 1+s] \\ x - \tilde{x} + p & \text{for} \quad x \in [\tilde{x}-1-p, \tilde{x}+1-p] \\ -1 & \text{for} \quad x \in [-1-s, \tilde{x}-1-p] \\ x + s & \text{for} \quad x \le -1-s \end{cases} \qquad (4.3)$$

After a reversal at $x = \tilde{x}$ with $|\tilde{x}| \ge 1 + s$, $\mathcal{U}_s\{x(t)\}$ becomes

$$u_s^{(k)}(x) = \begin{cases} x - s & \text{for} \quad x \ge -\operatorname{sgn}(\tilde{x}) + s \\ -\operatorname{sgn}(\tilde{x}) & \text{for} \quad x \in [-\operatorname{sgn}(\tilde{x}) - s, -\operatorname{sgn}(\tilde{x}) + s] \\ x + s & \text{for} \quad x \le -\operatorname{sgn}(\tilde{x}) - s \end{cases} \qquad (4.4)$$

Thus whenever $|x(t)| < 1 + s$, the hysteretic behavior of $\mathcal{U}_s\{x(t)\}$ resembles that of an elastoplastic system. Whenever $|x(t)| \ge 1 + s$, no further slipping can occur and $\mathcal{U}_s\{x(t)\}$ is linear with an offset of either s or -s. The maximum distance slipped is 2s.

For $s \in [0, \infty)$, $\mathcal{U}_s\{x(t)\}$ satisfies (A2) and (A6) through (A10). If $|x^{(0)}| < 1 + s$, the value of x(t) and $\mathcal{U}_s\{x(t)\}$ at $t = t_0$ specify the state of $\mathcal{U}_s\{x(t)\}$ at $t = t_0$. If $|x^{(0)}| \ge 1 + s$, the state of $\mathcal{U}_s\{x(t)\}$ at $t = t_0$ can be determined from (4.4) with $\tilde{x} = x^{(0)}$.

4.2 Harmonic Steady-State Response to Trigonometric Excitation

Let $G(t) = r \cos \nu t$ and $\mathcal{N}_s\{x(t)\} = \mathcal{U}_s\{x(t)\} - x$, then (4.1) becomes

$$\ddot{x} + 2z\dot{x} + x + \mathcal{H}_s\{x(t)\} = r\cos\nu t$$
$$x(t_0) = x^{(0)} \qquad (4.5)$$
$$\dot{x}(t_0) = \dot{x}^{(0)}$$
$$\text{State of } \mathcal{H}_s\{x(t)\} \text{ at } t = t_0$$

Assume r, z, $\nu - 1$, and s are all $O(\epsilon)$, then (4.5) is a member of the class of slightly nonlinear hysteretic systems discussed in Section 3.2. There exists only one family of symmetric steady-state configurations of $\mathcal{H}_s\{x(t)\}$, and it can be described by the following equations. For $a < 1$,

$$\mathcal{H}_s\{x(t)\}\Big|_{\text{steady-state}} = 0 \qquad (4.6)$$

Define $p = 1$ ($p = -1$) for the part of the configuration for $\dot{x} \geq 0$ ($\dot{x} \leq 0$). Then for $a \in [1, 1+s]$

$$\mathcal{H}_s\{x(t)\}\Big|_{\text{steady-state}} = \begin{cases} p(a-1) & \text{for } x \in [-pa, p(2-a)] \\ p - x & \text{for } x \in [p(2-a), pa] \end{cases} \qquad (4.7)$$

For $a > 1 + s$,

$$\mathcal{H}_s\{x(t)\}\Big|_{\text{steady-state}} = \begin{cases} ps & \text{for } x \in [-pa, p(1-s)] \\ p - x & \text{for } x \in [p(1-s), p(1+s)] \\ -ps & \text{for } x \in [p(1+s), pa] \end{cases} \qquad (4.8)$$

Using (3.22) and Equations (4.6) to (4.8), $E_H(a)$ and $C(a)$ are found to be

$$E_H(a) = \begin{cases} 0 & \text{for } a \leq 1 \\ 4(a-1) & \text{for } a \in [1, 1+s] \\ 4s & \text{for } a \geq 1+s \end{cases} \qquad (4.9)$$

$$C(a) = \begin{cases} 0 & \text{for } a \leq 1 \\ \frac{a}{2\pi}(\sin 2\varphi_1 - 2\varphi_1) & \text{for } a \in [1, 1+s] \\ \frac{a}{2\pi}\left(2(\varphi_2 - \varphi_3) + \sin 2\varphi_3 - \sin 2\varphi_2\right) & \text{for } a \geq 1+s \end{cases} \quad (4.10)$$

where $\varphi_1 = \text{Cos}^{-1}\left(\frac{2}{a} - 1\right)$, $\varphi_2 = \text{Cos}^{-1}\left(\frac{1+s}{a}\right)$, and $\varphi_3 = \text{Cos}^{-1}\left(\frac{1-s}{a}\right)$.

For the family of symmetric steady-state configurations of $\mathscr{V}_s\{x(t)\}$, (4.9) is the exact expression for the hysteretic energy dissipated over each oscillation with amplitude a. When $a \leq 1$, (4.5) is linear and there is no hysteretic energy dissipation. When $a \in (1, 1+s)$, the hysteretic energy dissipated increases linearly with a. The energy dissipated by most hysteretic models usually increases as $a \to \infty$. However, since the slipping is limited in (4.5), the hysteretic energy dissipation remains constant for $a \geq 1+s$.

Qualitative Appearance of Steady-State Response Curves
(Simplified Approximation II)

Figure 12 shows e(a) plotted as a function of a for (4.5). From Section 3.2, the following can be concluded from Simplified Approximation II:

1. For $z = 0$, there exists unbounded amplitude resonance for all levels of excitation. For $z > 0$, all steady-state response amplitudes are bounded.
2. Necessary and sufficient conditions for the existence of disconnected response curves are that $z < \frac{2s}{\pi(1+s)^2}$ and $r \in \left[4\sqrt{\frac{2sz}{\pi}}, 2z(1+s) + \frac{4s}{\pi(1+s)}\right)$. Thus for any $s > 0$, there exist

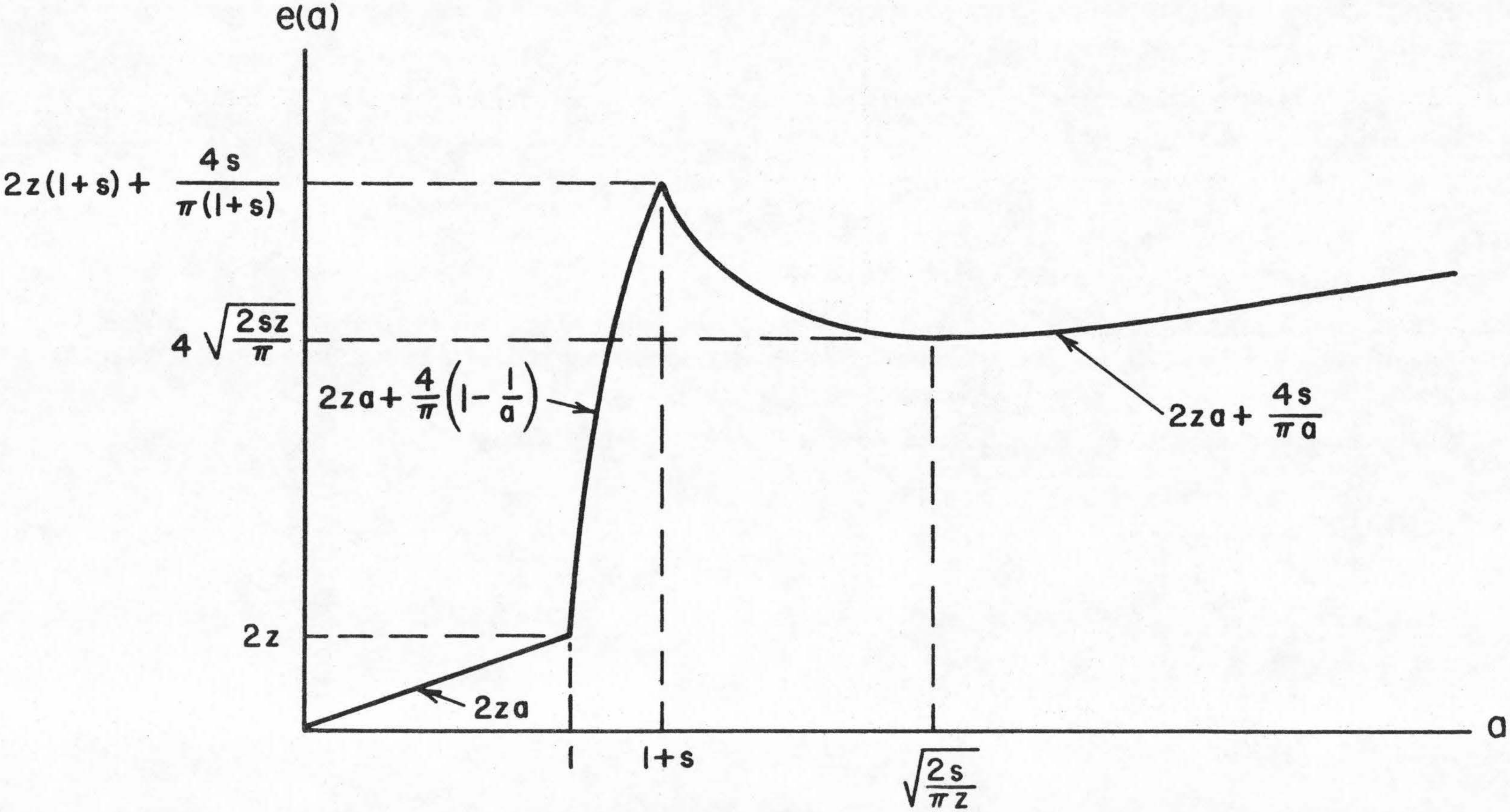


Figure 12: e(a) versus a for the Limited Slip System

some r and z for which the response curve is disconnected. Figure 13 shows, for several values of s, the relationship between r and z which results in disconnected response curves. The interval of r is largest for $z = 0$ and rapidly decreases as z is increased. It can be shown that for $s \in (0,1]$, the size of the region increases with s. It can also be shown that disconnected response curves can not occur if either $z > \frac{1}{2\pi}$ or $r > \frac{8}{\pi}$.

3. Whenever a response curve is disconnected, one portion lies entirely below $a = 1+s$, and the other, entirely above. When $r = 2z(1+s) + \frac{4s}{\pi(1+s)}$, both portions join at $a = 1+s$. If $z \neq 0$, then the upper portion degenerates to a point at $a = \sqrt{\frac{2s}{\pi z}}$ when $r = 4\sqrt{\frac{2sz}{\pi}}$. If $z = 0$, then the distance between the two portions becomes unbounded as $r \to 0$.

4. Increasing z or decreasing r decreases the width of the response curves at a given amplitude, raises the minima of response, and lowers the maxima of response. Increasing s from $\tilde{s}$ produces the same qualitative change, but only for $a > 1+\tilde{s}$. The response curves remain unchanged for $a \leq 1+\tilde{s}$.

5. The slope of e(a) is relatively large for $a \in (1, 1+s)$. Thus when $r \in \left(2z, 2z(1+s) + \frac{4s}{\pi(1+s)}\right)$, a small change in r or z results in a relatively small change in the lowest relative maximum of response. For $z \in \left(0, \frac{2s}{\pi(1+s)^2}\right)$, the slope of e(a) is relatively small in the vicinity of $a = \sqrt{\frac{2s}{\pi z}}$. Thus when r is in the vicinity of $4\sqrt{\frac{2sz}{\pi}}$, a small change in r, z, or s may result in a relatively large change in the maximum and minimum of response of the upper portion of the response curve — including

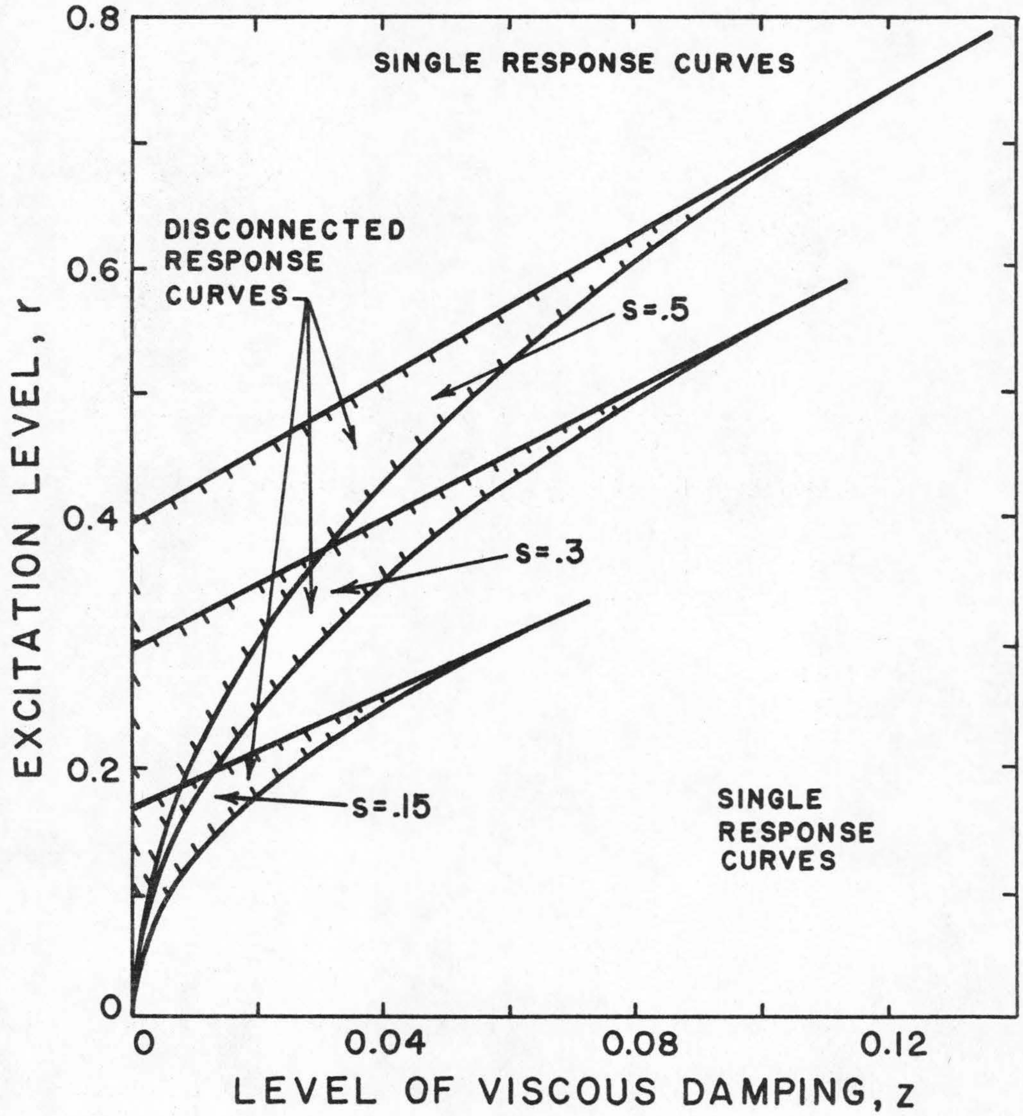


Figure 13: Response Regions

even the appearance or disappearance of this portion. If steady-state response curves are drawn with fixed s and z for values of r separated by equal small increments, then the sensitivity just discussed implies that the extrema of response are closest together for $a \in (1, 1+s)$, and farthest apart for a in the vicinity of $\sqrt{\frac{2s}{\pi z}}$.

The qualitative behavior of the locus of extremum response given by (3.38) can be determined using (4.10). For $a < 1$, the locus of extremum response is vertical at $\nu = 1$. It then decreases with amplitude, or softens, for $a \in (1, 1+s)$. For $a > 1+s$, the extremum of response begins to harden and asymptotically approaches $\nu = 1$ as $a \to \infty$. The effect of increasing s from $\tilde{s}$ is to shift the locus of extremum response to the left for $a > 1+\tilde{s}$. It remains unchanged for $a \leq 1+\tilde{s}$.

Comparison with Approximation II

Figures 14 through 17 show the steady-state response curves predicted by Simplified Approximation II and Approximation II. The loci of vertical tangents and of extremum response are shown for Approximation II only.

Figure 14 is for $s = .15$ and $z = .02$. The steady-state response curves for Simplified Approximation II have the general character previously discussed. Both approximations predict the same qualitative appearance of the steady-state response curves, but in the region where $\nu < 1$ and $a > 1$, the response curves of Simplified Approximation II are generally closer to the locus of extremum response shown than

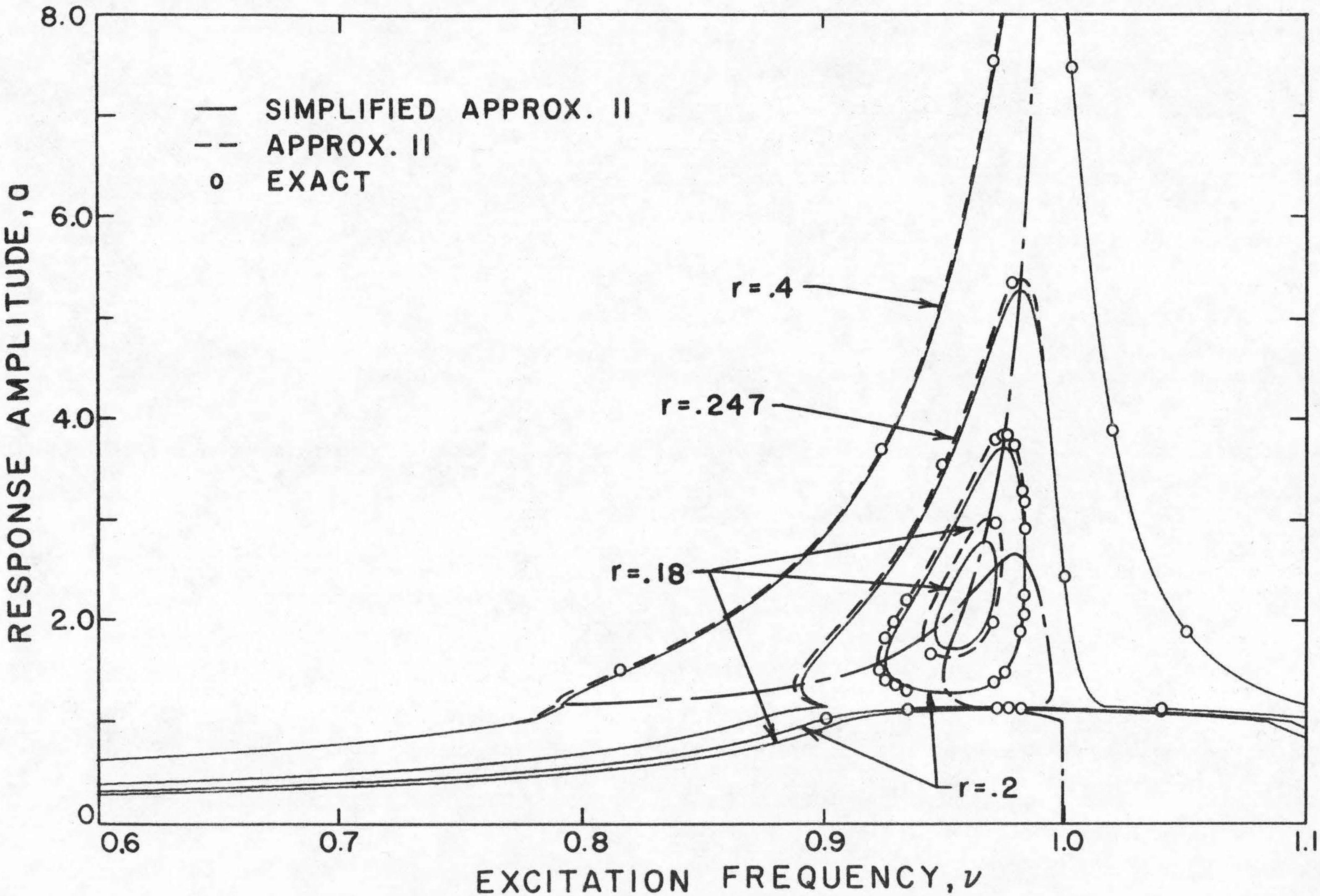


Figure 14: Steady-State Response, s = .15, z = .02

the corresponding curves of Approximation II. The quantitative agreement between the two approximations is best for $r = .4$, for the portions of the response curves either in the interval $\nu \in [1, 1.1]$ or below $a = 1$, and for those response curves which possess a maximum in the interval $a \in (1, 1+s)$. The quantitative agreement is worst near the point at which the isolated portions of the response curves vanish. This is also the region in which the maxima and minima of response for Simplified Approximation II are most sensitive to changes in r, s, or z.

Figure 15 is for $s = .15$ and $z = .04$. The most apparent effect of increasing the viscous damping from that in Figure 14 is that the response curves have been lowered. In the region where $\nu < 1$ and $a > 1$, the response curves of Simplified Approximation II are generally closer to the locus of extremum response shown than the corresponding curves of Approximation II. The quantitative agreement between the two approximations is best for the portions of the response curves either in the interval $\nu \in [1, 1.1]$ or below $a = 1$, and for the response curves which possess a maximum in the interval $a \in (1, 1+s)$. The quantitative agreement is worst for $r = .247$ since Approximation II predicts a disconnected portion of the response curve and Simplified Approximation II does not.

Figure 16 is for $s = .3$ and $z = .02$. As expected, the response curves soften much more before hardening. In the region where $\nu < 1$ and $a > 1$, the response curves of Simplified Approximation II are generally closer to the locus of extremum response shown than the corresponding curves of Approximation II. The quantitative agreement

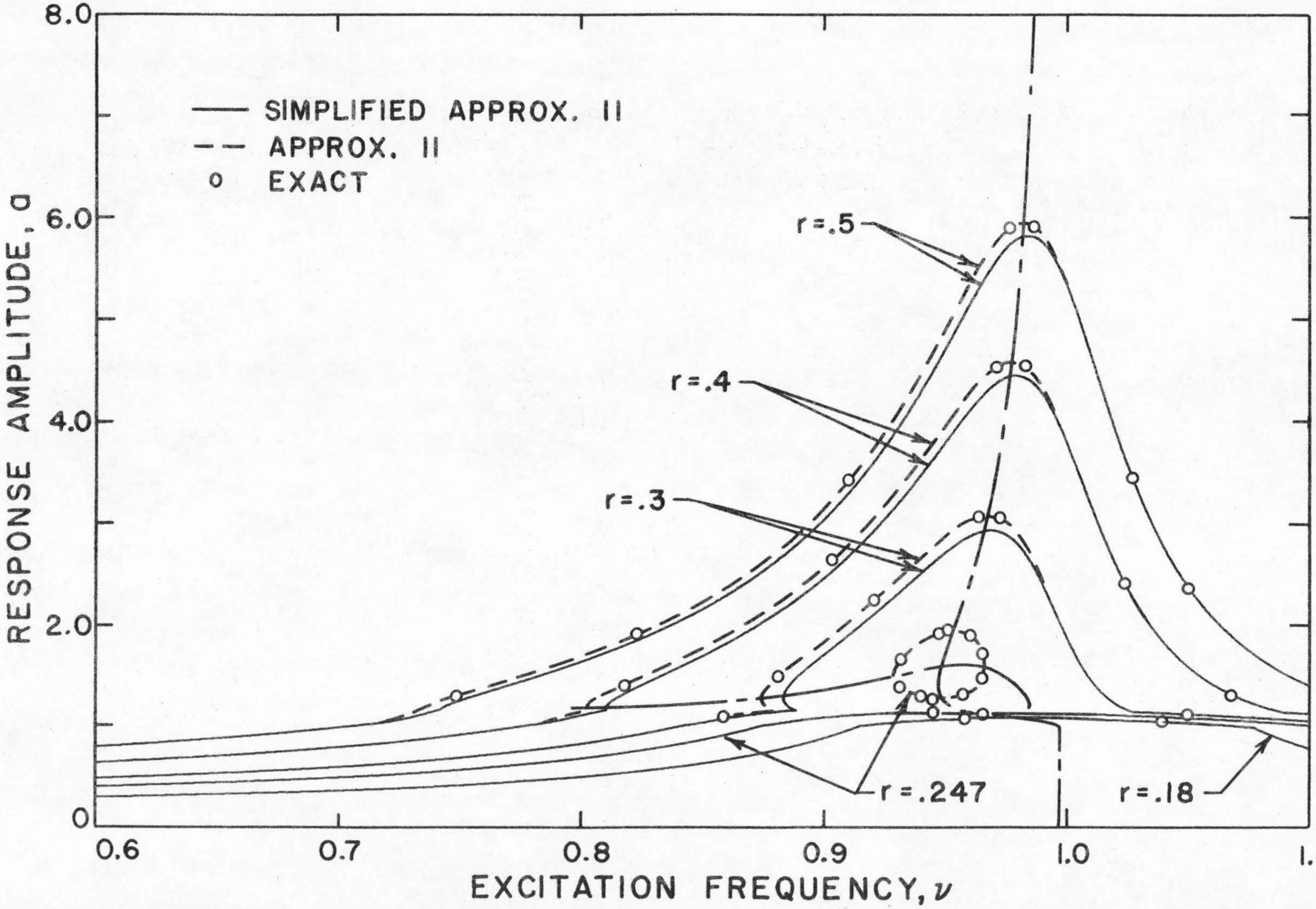


Figure 15: Steady-State Response, s = .15, z = .04

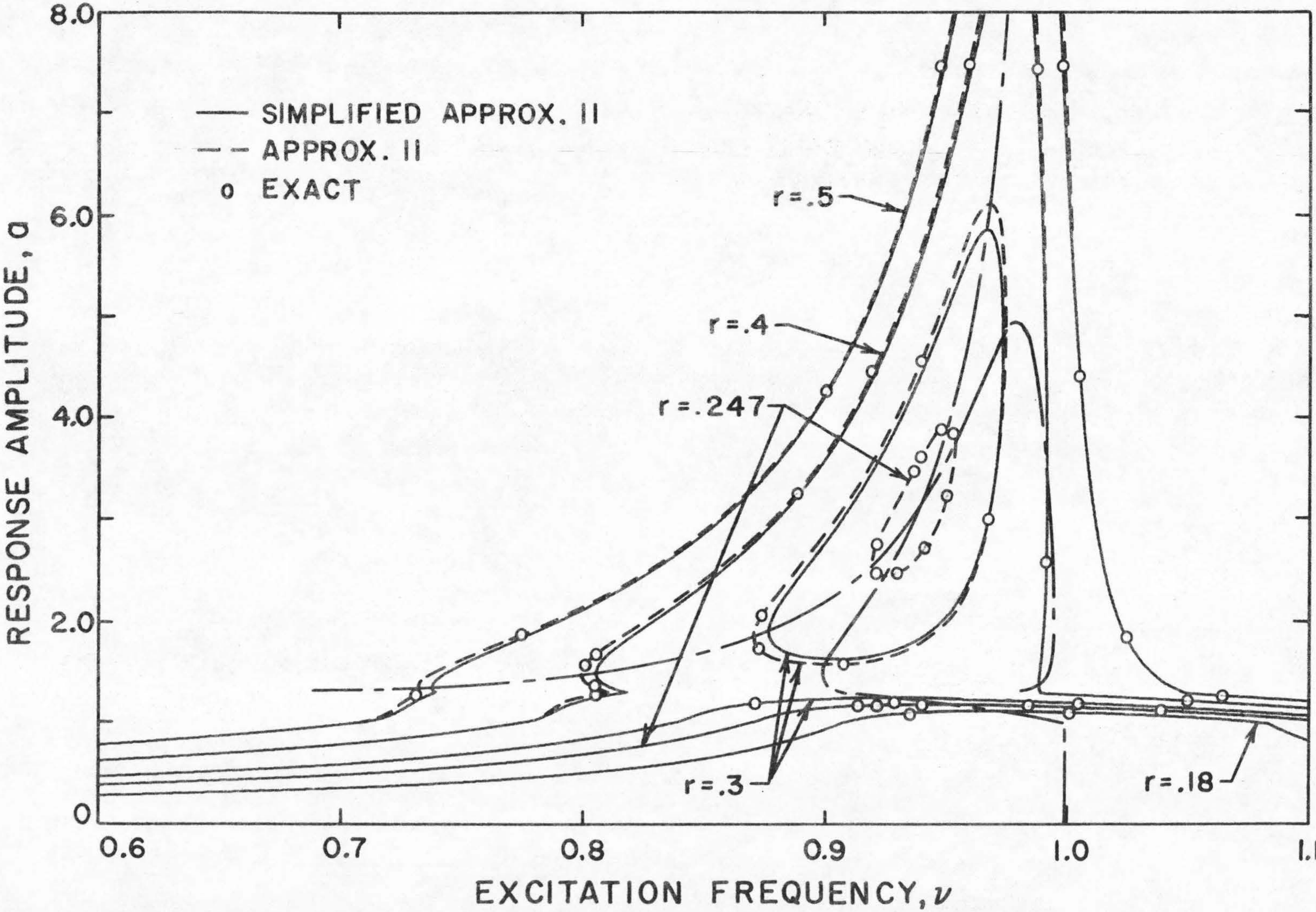


Figure 16: Steady-State Response, s = . 3, z = . 02

between the two approximations is again best for the portions of the response curves either in the interval $\nu \in [1, 1.1]$ or below $a = 1$, and for the curves which have a maximum for $a \in (1, 1+s)$. The quantitative agreement is worst for $r = .247$ since Approximation II predicts a disconnected portion of the response curve and Simplified Approximation II does not.

Figure 17 is for $s = .9$ and $z = .04$. Both approximations predict a distinctive "softening-hardening" character of the response curves, and the existence of disconnected response curves. The quantitative predictions agree either in the interval $\nu \in [1, 1.1]$ or below $a = 1$. For $r = .6$, Approximation II predicts an isolated portion of the response curve and Simplified Approximation II does not. For $r = .73$, Simplified Approximation II predicts an isolated portion of the response curve and Approximation II predicts that the isolated portion is still connected to the lower portion of the curve. These two cases of extreme differences in quantitative predictions are not peculiar only to $s = .9$ and $z = .04$. Obviously the excitation can be chosen even for $s = .15$ and $z = .02$ to exhibit these differences.

Discussion

The predicted response curves for Simplified Approximation II have been compared with those of Approximation II for corresponding s, z, and r. It has been found that the same qualitative features of the response curves discussed for Simplified Approximation II are predicted by Approximation II. The quantitative steady-state predictions shown in Figures 14 through 16 differ primarily in the region where

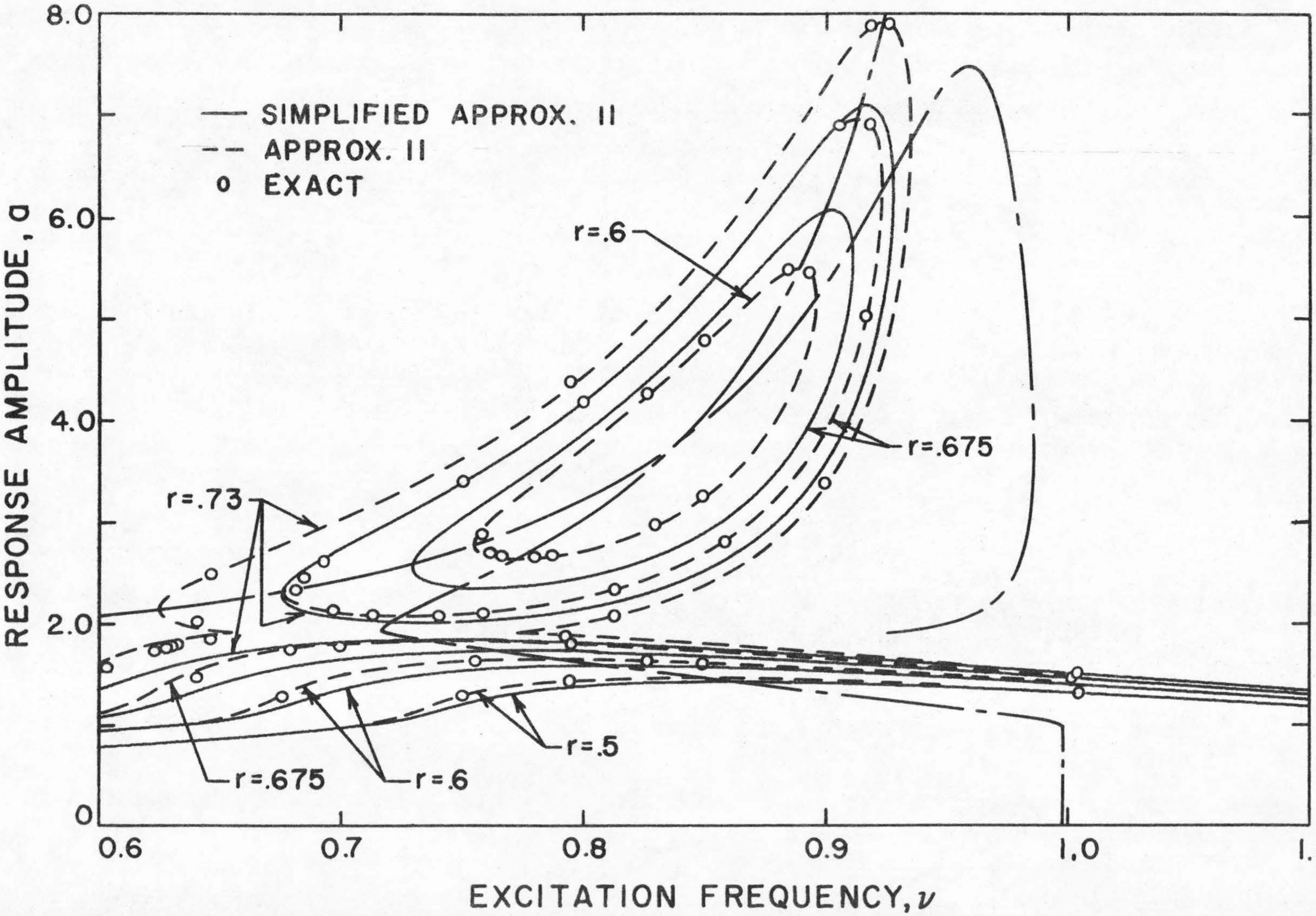


Figure 17: Steady-State Response, s = .9, z = .04

$\nu < 1$ and $a > 1$. When $s = .15$ and $z = .02$, the doubling of either s or z increases the difference between the quantitative steady-state predictions in this region. For a given level of excitation, the difference is larger when z is doubled than when s is doubled. In each of the figures, the agreement between the two approximations becomes worse in the area where the isolated portions of the response curves degenerate to a point. However, this is the same area in which Simplified Approximation II is most sensitive to changes in r, s, or z.

The existence of steady-state response at a given amplitude is dependent upon all the system parameters ν, r, s, and z. Clearly, to compare the steady-state response predictions of Simplified Approximation II and Approximation II justly, the dependence of the predictions upon the system parameters should be studied. Rather than examine the effect of variations in all the parameters, it is sufficient to consider only the dependence upon the excitation level r.

Figure 18 shows, for $s = .9$ and $z = .04$, the relationship between r and the amplitudes at which extrema of response occur for each approximation. For Simplified Approximation II, e(a) is plotted as a function of a. To obtain the curve for Approximation II, first the locus of extremum response is determined from (3.32) by setting $\frac{\partial a}{\partial \nu} = 0$. This gives

$$\nu^3 + (2z^2 - 1 - \frac{C(a)}{a})\nu + \frac{zE_H(a)}{\pi a^2} = 0 \qquad (4.11)$$

The frequency at which a given amplitude $\tilde{a}$ becomes an extremum is found numerically from (4.11). Only the frequency $\nu = \tilde{\nu}$ in the vicinity

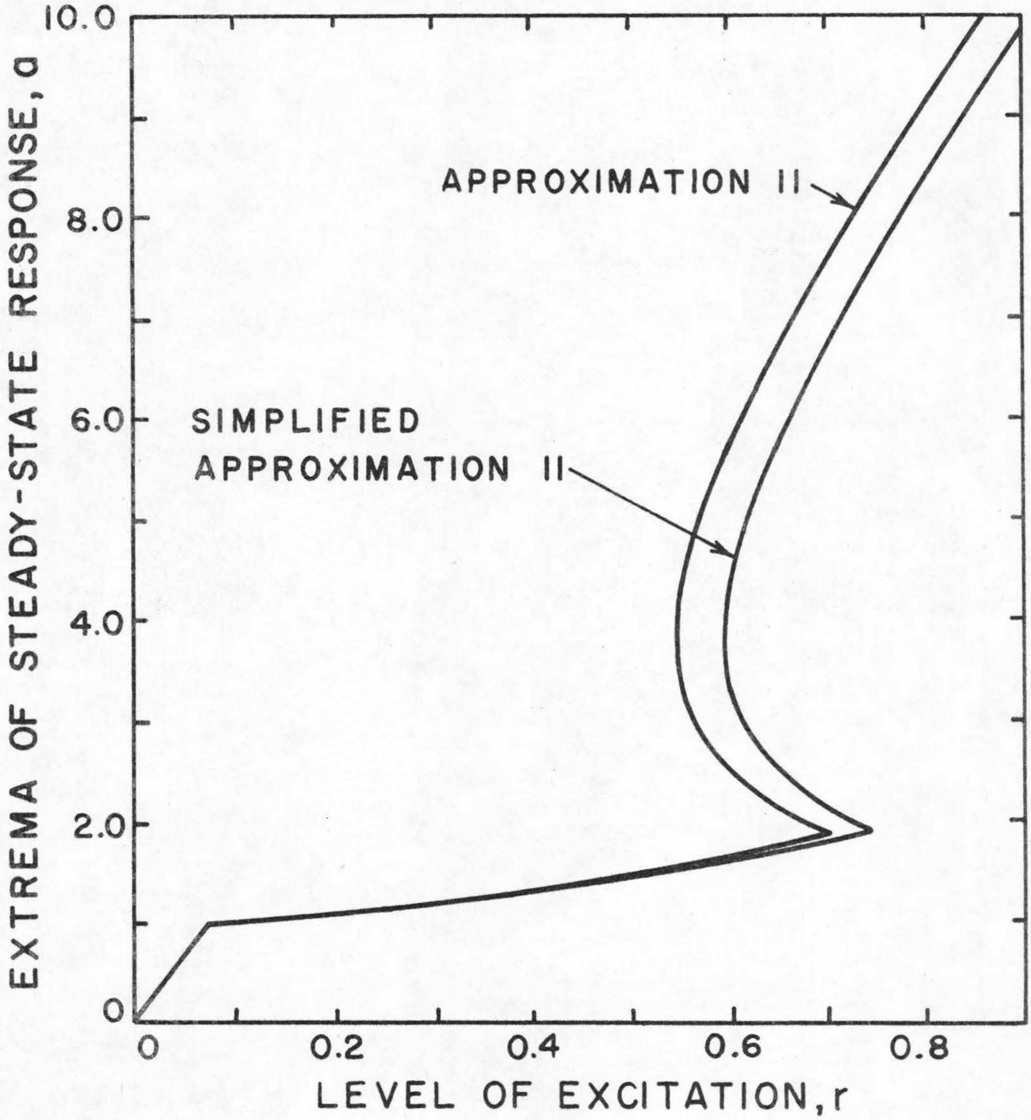


Figure 18: Extrema of Steady-State Response versus Level of Excitation, s = .9, z = .04

of $\nu = 1$ is used. Then the excitation level $\tilde{r}$ which produces that extremum is found from (3.32) with $\nu = \tilde{\nu}$ and $a = \tilde{a}$. $(\tilde{a}, \tilde{r})$ is then a point on the curve for Approximation II in Figure 18. This procedure is repeated until a curve can be drawn.

Figure 18 shows that for a given level of excitation r, each maximum (minimum) of response predicted by Simplified Approximation II is generally lower (higher) than that predicted by Approximation II. The extrema of response predicted by the two approximations can be made to agree more closely by changing the level of excitation for either approximation. Figure 19 again shows the response curves of the two approximations for $s = .9$ and $z = .04$, but in this figure the excitation levels for Simplified Approximation II have been adjusted using Figure 18. The relative adjustments are less than 8%, and the maximum absolute adjustment is .05. The agreement between the quantitative predictions of the two approximations is considerably better. Similarly, small relative adjustments in r for the response curves in Figures 14 through 16 provide better agreement for the isolated portions of the curves. Therefore the quantitative differences between the steady-state response predictions of the two approximations are reasonable if the sensitivity upon r, for example, is considered.

The accuracy of the steady-state response predictions is investigated next.

Comparison with Exact Steady-State Response

The response is calculated by numerically continuing the analytic solutions to the linear parts of (4.5). "Exact" periodic

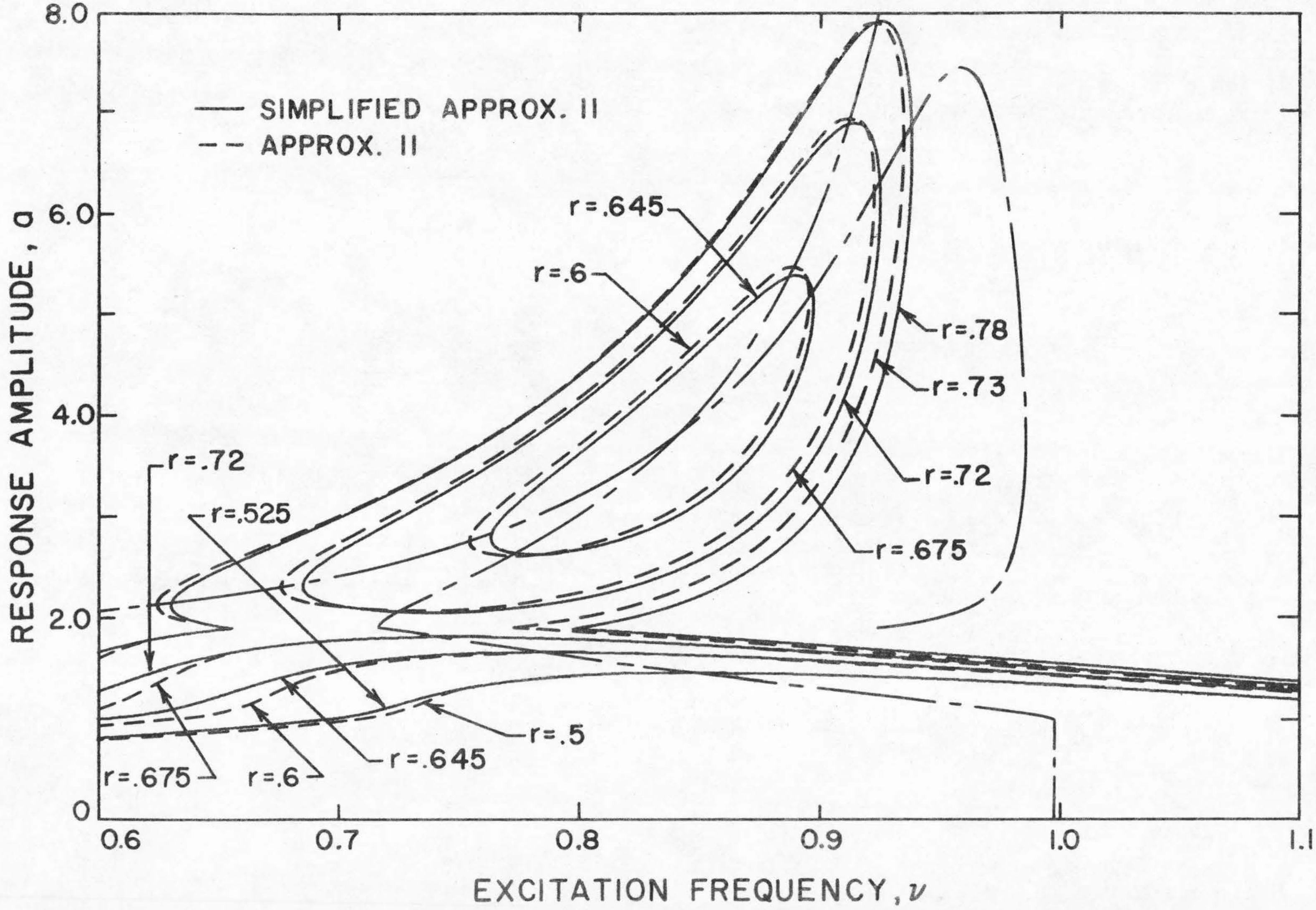


Figure 19: Steady-State Response, s = .9, z = .04
(Adjusted Excitation Levels for Simplified Approximation II)

solutions of (4.5) with the same symmetry property as the approximate solutions, $x(t) = -x(t+\frac{\pi}{\nu})$, are then obtained using an iterative scheme. The exact solution points are shown in Figures 14 through 17. No exact solution points are shown when (4.5) is linear ($a \leq 1$) since Approximation II then provides the exact response curves. Each exact point in Figure 17 corresponds to the level of excitation for the nearest response curve of Approximation II.

Approximation II generally gives much better agreement with the exact solution points than Simplified Approximation II. There is practically no difference between the predictions of Approximation II and the exact solution points in Figures 14 through 16. In Figure 17, for $s = .9$ and $z = .04$, there is a difference of less than 5% in ν and 5% in a between the exact solution points and steady-state solutions predicted by Approximation II. This is surprising since the validity of the assumptions of Approximation II are questionable for the relatively large nonlinearity.

For Figures 14 through 16, the steady-state solutions predicted by Simplified Approximation II agree very well with the exact solutions either in the interval $\nu \in [1, 1.1]$ or below $a = 1$, and for the curves which possess maxima in the interval $a \in (1, 1+s)$. For $s = .15$, $z = .02$, and $r = .18$, the difference between exact solution points and steady-state solutions predicted by Simplified Approximation II is less than 5% in ν and 5% in a. The agreement is better for $r = .247$ and $r = .4$, but worse for the isolated portion of $r = .18$. As shown in Figures 15 and 16, the agreement is made worse by either doubling z or by doubling s. In Figure 17, Simplified Approximation II agrees with the exact points for portions of the response curves either near $\nu = 1$ or below $a = 1$. The agreement with the remaining points is poor.

Discussion

It is seen that Approximation II generally provides more accurate quantitative predictions of steady-state response amplitudes than Simplified Approximation II for all s and z investigated. However, in view of the sensitivity of the predictions of both approximations to the system parameters, it may be concluded that the approximations are equivalent when some of the parameters are known only to within a few percent. In this situation, both approximations predict the same steady-state frequency-amplitude behavior and dependence of this behavior upon changes in the system parameters.

The usefulness of Simplified Approximation II is reflected in the simple form of the equation for steady-state response given by (3.37), as compared with that of Approximation II given by (3.32). As an initial investigation, Equation (3.37) is easily used to determine qualitative characteristics of the frequency response curves and the dependence of these characteristics upon variations of system parameters without even plotting the curves. (A more time consuming and complicated parameter study of (3.32) is required.) Simplified Approximation II may even be used for quantitative predictions if s and z are sufficiently small and the steady-state predictions desired are not near isolated portions of the response curves.

Stability of Steady-State Response

In both Simplified Approximation II and Approximation II, only the steady-state solutions with trigonometric symmetry are considered. When a steady-state solution is predicted with amplitude $a = \tilde{a} \in (1, 1 + s)$, it is asymptotically stable even for $z = 0$ according to the stability analysis in Section 3.2. However, if there actually exists a periodic solution with this amplitude, examination of (4.5) leads to the conclusion that there in fact exists an infinite number of periodic solutions with a peak to peak amplitude of $2\tilde{a}$. These solutions differ only by a displacement translation which is less than or equal to $(1 + s - \tilde{a})$. There is no reason for any particular one of these solutions to be the preferred steady-state for arbitrary initial conditions. (This "translation" of the origin is typical for elastoplastic systems.) Henceforth, any translation of steady-state solutions is neglected.

The following discussion refers to both Simplified Approximation II and Approximation II when $z > 0$. From the stability analysis, it may be concluded that steady-state solutions in the open region bounded by the locus of vertical tangents and $a = 1 + s$ are unstable. Steady-state solutions in the remaining region, excluding those occurring at a discontinuity in slope of a response curve, are asymptotically stable. Thus it is seen from Figures 14 through 17 that when three steady-state solutions are possible for a given frequency, the one with intermediate amplitude is unstable. The remaining solutions are asymptotically stable if they do not occur at $a = 1$ or $a = 1 + s$. Since one steady-state solution is unstable, only two, at most, may actually occur in a physical situation.

The steady-state ultimately approached by a solution of (4.5) depends upon initial conditions. This dependence is discussed in the next chapter.

Jump Phenomenon

It may be noted that there exists the possibility of two types of jumps in the response amplitude as the frequency is varied slowly. One type occurs when there exists an isolated portion of the response curve. If the response begins on the lower portion of the curve and the frequency is varied, only elastoplastic behavior is detected. If the response begins on the isolated portion and the frequency is varied, downward jumps are possible. The other type occurs when there is only one connected curve possessing vertical tangents. The most general situation is illustrated by Simplified Approximation II and Approximation II in Figure 16 for $r = .4$. Both downward and upward jumps are possible on either side of the locus of extremum response. These jumps may be detected by a slow frequency sweep in either direction. The difference in amplitudes is greater for the steeper part of the response curve near $\nu=1$.

4.3 Summary of Results

Some unusual features of the harmonic steady-state response of a system with limited slip have been indicated in this section. These features, which are present for any finite slip, can be summarized as follows:

1. A general "softening-hardening" of the response curves. The

standard rate-independent hysteretic models exhibit only a softening character.

2. Unbounded amplitude resonance for any level of excitation if $z = 0$. Usually for hysteretic models, the level of excitation must be greater than a critical value to exhibit unbounded resonance.
3. The existence of disconnected portions of the response curve for certain ranges of system parameters. This unusual behavior is not common for rate-independent hysteretic models.
4. The existence of triple-valued response curves for a certain range of system parameters even if the curve is connected. Multiple-valued response curves are not found in standard hysteretic models as pointed out by Iwan[8].
5. Jump phenomenon in steady-state response amplitudes associated with the triple-valued response curves. Two types of jumps exist in the present system. One type occurs when a response curve is triple-valued and not disconnected. Both upward and downward jumps are possible, and these can always be detected by a slow frequency sweep. The other type occurs when there is a disconnected portion of the response curve. For this type, however, the jumps can occur only if the response is on the disconnected portion, and all jumps are downward.

V. A SYSTEM WITH LIMITED SLIP: INITIAL-VALUE PROBLEM

The steady-state response of a limited slip system to trigonometric excitation is investigated in Chapter IV. However, the steady-state response is only a part of the total response possible. To gain further insight into the general response of the system, solutions to the initial-value problem are studied in this chapter. Section 5.1 is concerned with the boundedness of the response of the present system to arbitrary excitation. In Section 5.2 the excitation is again taken to be trigonometric. The relationship between initial conditions and the resulting steady-state solution is examined when multiple steady-state solutions are possible.

5.1 Boundedness of Solutions to the Initial-Value Problem

In this section it is concluded that the response of the limited slip system (4.1) is bounded either when there is no excitation or when there is excitation and viscous dissipation. When there is no viscous dissipation and the excitation is sufficiently small, it is shown that solutions of (4.1) are bounded provided that the initial conditions $(x^{(0)}, \dot{x}^{(0)})$ lie in a certain region in the $x, \dot{x}$ plane.

Boundedness for the Unforced Limited Slip System

If $G(t) \equiv 0$ and $z = 0$, then Theorem 2 in Section 2.3 implies that all solutions of (4.1) are bounded. It may also be concluded that when reversals occur outside the interval $|x| \leq 1 + s$, the distance between consecutive reversals decreases. It can be seen from the formulation of $\mathcal{U}_s\{x(t)\}$ that when a reversal occurs in the interval $|x| \leq 1+s$, $\mathcal{U}_s\{x(t)\}$

loses its hysteretic property. The solution then becomes the free oscillation of a linear, undamped system with the origin possibly translated by a maximum distance of s.

If $G(t) \equiv 0$ and $z \neq 0$, then an extension of Theorem 2 to include viscous dissipation implies that all solutions of (4.1) are bounded. Again it may be concluded that the distance between consecutive reversals decreases until a reversal occurs in the interval $|x| \leq 1+s$. The solution then becomes the free vibrations of a viscously damped linear oscillator with the origin possibly translated by a maximum distance of s.

Boundedness for the Forced Limited Slip System

Application of Theorem 3 shows that all solutions of (4.1) are bounded for $z > 0$ and bounded excitation. However, Theorem 3 can not be applied to the limiting case of $z = 0$. It is shown below that when $z \geq 0$ and the excitation is sufficiently small, all solutions of (4.1) for a certain class of initial conditions are bounded. The technique used in proving Theorem 4 in Section 2.3 is applied.

Theorem 5

Assume $z \geq 0$ and $\underset{t \in [t_0, \infty)}{\mathrm{Max}} |G(t)| \leq P < \frac{1}{4}\left(\sqrt{(1+s)^2 + 8s} - 1 - s\right)$, and let $\overline{R}_{\hat{x}}$ be the closed region in the $x,\dot{x}$ plane whose boundary is described by

$$\dot{x} = \begin{cases} \sqrt{-(x - P + s)^2 + (\hat{x} - P - s)^2 + 4s(1 - P)} & \text{for } x \in [-\hat{x}, -1 - s] \\ \sqrt{2(P - 1)x + (\hat{x} - P - s)^2 + 2s(1 - P) - P^2 - 3} & \text{for } x \in [-1 - s, -1 + s] \\ \sqrt{-(x - P - s)^2 + (\hat{x} - P - s)^2} & \text{for } x \in [-1 + s, \hat{x}] \end{cases} \quad (5.1)$$

$$\dot{x} = \begin{cases} -\sqrt{-(x+P+s)^2 + (\hat{x}-P-s)^2} & \text{for } x \in [-\hat{x}, 1-s] \\ -\sqrt{2(1-P) + (\hat{x}-P-s)^2 + 2s(1-P) - P^2 - 3} & \text{for } x \in [1-s, 1+s] \\ -\sqrt{-(x+P-s)^2 + (\hat{x}-P-s)^2 + 4s(1-P)} & \text{for } x \in [1+s, \hat{x}] \end{cases} \quad (5.1) \text{ cont.}$$

If $(x^{(0)}, \dot{x}^{(0)}) \in \overline{\mathcal{R}}_{\hat{x} = \frac{s}{P}}$, then the solution of (4. 1) is bounded with $|x(t)| \leq \frac{s}{P}$ for $t \in [t_0, \infty)$. Furthermore, if $(x^{(0)}, \dot{x}^{(0)}) \in \overline{\mathcal{R}}_{\hat{x} = 2P+s+1}$, then the solution is bounded with $|x(t)| \leq 2P+s+1$. If $(x^{(0)}, \dot{x}^{(0)}) \in (\overline{\mathcal{R}}_{\hat{x} = \frac{s}{P}} - \overline{\mathcal{R}}_{\hat{x} = 2P+s+1})$, then the solution trajectory $(x(t), \dot{x}(t))$ in the x, $\dot{x}$ plane intersects the $\dot{x} = 0$ axis with decreasing amplitudes until it enters $\overline{\mathcal{R}}_{\hat{x} = 2P+s+1}$.

<u>Proof:</u> At $t = t_0$ (4. 1) becomes

$$\begin{aligned} \ddot{x} + 2z\dot{x} + u_s^{(i)}(x) &= G(t) \\ x(t_0) &= x^{(0)} \\ \dot{x}(t_0) &= \dot{x}^{(0)} \end{aligned} \qquad (5.2)$$

Part 1: Let $\dot{x}^{(0)} = 0$. Then either the solution remains at $x^{(0)}$ for $t \in [t_0, \infty)$, or it leaves $x^{(0)}$ the instant after $t = t_1 \geq t_0$.

If the solution remains at $x^{(0)}$ for $t \in [t_0, \infty)$, the formulation of $\mathcal{U}_s\{x(t)\}$, (5. 2), and the hypothesis $|G(t)| \leq P$ imply that $|x^{(0)}| \leq P+s$. Thus $(x^{(0)}, 0) \in \overline{\mathcal{R}}_{\hat{x} = 2P+s+1}$ and $|x(t)| = |x^{(0)}| \leq 2P+s+1$ for $t \in [t_0, \infty)$.

If the solution leaves $x^{(0)}$ the instant after $t = t_1 \geq t_0$, assume $\dot{x}(t) > 0$ for $t \in (t_1, t_2)$ where $\dot{x}(t_2) = 0$. (Thus it is necessary that $x^{(0)} < P+s$.) When $t = t_1$, (4. 1) is

$$\ddot{x} + 2z\dot{x} + u_s^{(i)}(x) = G(t)$$

$$x(t_1) = x^{(0)} \tag{5.3}$$

$$\dot{x}(t_1) = 0$$

Multiplying the first of Equations (5.3) by $2\dot{x}$ and integrating from t_1 to t gives

$$(\dot{x}(t))^2 = 2\int_{t_1}^{t} G(\tau)\dot{x}(\tau)d\tau - 4z\int_{t_1}^{t}(\dot{x}(\tau))^2 d\tau - 2\int_{x^{(0)}}^{x(t)} u_s^{(i)}(\eta)d\eta \tag{5.4}$$

Since $|G(t)| \le P$ and $z \ge 0$, (5.4) implies

$$(\dot{x}(t))^2 \le 2P(x(t) - x^{(0)}) - 2\int_{x^{(0)}}^{x(t)} u_s^{(i)}(\eta)d\eta \tag{5.5}$$

For $x^{(0)} \in (-1 - s, -1 + s)$, define

$$U_s(x;x^{(0)}) = \begin{cases} x - s & \text{for } x > 1 + s \\ 1 & \text{for } x \in [x^{(0)} + 2, 1 + s] \\ x - x^{(0)} - 1 & \text{for } x \in (x^{(0)}, x^{(0)} + 2) \\ -1 & \text{for } x \in [-1 - s, x^{(0)}] \\ x + s & \text{for } x < -1 - s \end{cases} \tag{5.6}$$

For $P + s > x^{(0)} \ge -1 + s$ or $x^{(0)} \le -1 - s$, define

$$U_s(x;x^{(0)}) = \begin{cases} x - s & \text{for } x \ge -\operatorname{sgn}(x^{(0)} + 1) + s \\ -\operatorname{sgn}(x^{(0)} + 1) & \text{for } x \in (-\operatorname{sgn}(x^{(0)} + 1) - s, -\operatorname{sgn}(x^{(0)} + 1) + s) \\ x + s & \text{for } x \le -\operatorname{sgn}(x^{(0)} + 1) - s \end{cases} \tag{5.7}$$

From the formulation of $\mathcal{U}_s\{x(t)\}$ it can be seen that along any path increasing from $x = x^{(0)}$, $U_s(x;x^{(0)}) \le u_s^{(i)}(x)$ for any possible $u_s^{(i)}(x)$.

$U_s(x;x^{(0)})$ is illustrated in Figure 20. Thus (5.5) can now be written

$$(\dot{x}(t))^2 \leq 2P(x(t) - x^{(0)}) - 2\int_{x^{(0)}}^{x(t)} U_s(\eta;x^{(0)})d\eta \qquad (5.8)$$

The expression (5.8) gives a bound upon the velocity of the solution of (5.3) as a function of x(t). This bound is independent of the specific form of $u_s^{(i)}(x)$ and does not depend upon time explicitly. If $x \to \infty$ from $x^{(0)}$, then $\int_{x^{(0)}}^{x} U_s(\eta; x^{(0)})d\eta$ either increases to ∞ quadratically in x, or monotonically decreases and then monotonically increases such that when $x \geq 1 + s$ the increase is quadratic in x. Recalling that $\dot{x}(t) > 0$ for $t \in (t_1, t_2)$, it follows that a unique $X^{(1)}(x^{(0)}, 0) > x^{(0)}$ exists such that

$$P(x - x^{(0)}) - \int_{x^{(0)}}^{x} U_s(\eta; x^{(0)})d\eta > 0 \quad \text{for } x \in (x^{(0)}, X^{(1)}(x^{(0)}, 0)) \qquad (5.9)$$

and

$$P(X^{(1)}(x^{(0)}, 0) - x^{(0)}) - \int_{x^{(0)}}^{X^{(1)}(x^{(0)}, 0)} U_s(\eta; x^{(0)})d\eta = 0 \qquad (5.10)$$

Equations (5.8) to (5.10) imply that the velocity of the solution of (5.3) must vanish before $x(t) > X^{(1)}(x^{(0)}, 0)$. It is assumed that $x(t_2) = 0$. t_2 is either unbounded or bounded.

If t_2 is unbounded, then it can be shown that $\lim_{t \to \infty} (x(t), \dot{x}(t)) = (x^{(L)}, 0) \in \mathcal{R}_{\hat{x} = 2P + s + 1}$.

If t_2 is bounded, $\dot{x}(t_2) = 0$ and $x(t_2) \leq X^{(1)}(x^{(0)}, 0)$. For $P < \frac{1}{4}\left(\sqrt{(1+s)^2 + 8s} - 1 - s\right)$, it can be shown that if $-2P - s - 1 \leq x^{(0)} \leq P + s$, then $X^{(1)}(x^{(0)}, 0) \leq 2P + s + 1$. If $-\frac{s}{P} \leq x^{(0)} < -2P - s - 1$, then $X^{(1)}(x^{(0)}, 0) < -x^{(0)}$. Therefore, if $(x^{(0)}, 0) \in \mathcal{R}_{\hat{x} = 2P + s + 1}$, then $|x(t)| \leq 2P + s + 1$ for

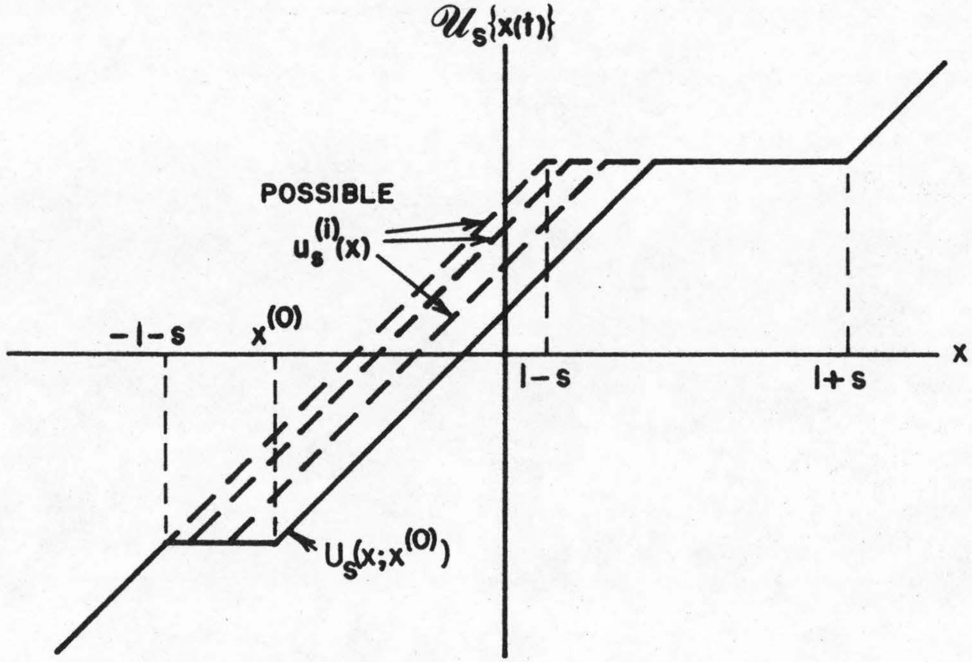


Figure 20: Illustration of $U_s(x; x^{(0)})$

$t \in [t_0, t_2]$. If $(x^{(0)}, 0) \in (\overline{\mathcal{R}}_{\hat{x}=\frac{s}{P}} - \overline{\mathcal{R}}_{\hat{x}=2P+s+1})$, then $|x(t_2)| < |x^{(0)}|$.

Part 2: Let $\dot{x}^{(0)} > 0$. Only the interval $t \in [t_0, t_1]$ is considered where $t_1 > t_0$ is the first instant that $\dot{x}(t_1) = 0$. The same procedure used in Part 1 yields

$$(\dot{x}(t))^2 \le (\dot{x}^{(0)})^2 + 2P(x(t) - x^{(0)}) - 2\int_{x^{(0)}}^{x(t)} U_s(\eta; x^{(0)})d\eta \tag{5. 11}$$

There exists a unique $X^{(1)}(x^{(0)}, \dot{x}^{(0)}) > x^{(0)}$ such that

$$(\dot{x}^{(0)})^2 + 2P(x - x^{(0)}) - 2\int_{x^{(0)}}^{x} U_s(\eta; x^{(0)})d\eta > 0 \ \text{ for } x \in (x^{(0)}, X^{(1)}(x^{(0)}, \dot{x}^{(0)})) \tag{5. 12}$$

and

$$(\dot{x}^{(0)})^2 + 2P(X^{(1)}(x^{(0)}, \dot{x}^{(0)}) - x^{(0)}) - 2\int_{x^{(0)}}^{X^{(1)}(x^{(0)}, \dot{x}^{(0)})} U_s(\eta; x^{(0)})d\eta = 0 \tag{5. 13}$$

Equations (5. 11) to (5. 13) imply that the velocity of the solution of (5. 2) must vanish before $x(t) > X^{(1)}(x^{(0)}, \dot{x}^{(0)})$. It is assumed that $\dot{x}(t_1) = 0$. If t_1 is unbounded, it can be concluded that $\lim_{t \to \infty} (x(t), \dot{x}(t)) = (x^{(L)}, 0) \in \overline{\mathcal{R}}_{\hat{x}=2P+s+1}$. If t_1 is bounded, then $\dot{x}(t_1) = 0$ and $x(t_1) \le X^{(1)}(x^{(0)}, \dot{x}^{(0)})$. For $P < \frac{1}{4}\left(\sqrt{(1+s)^2 + 8s} - 1 - s\right)$, it can be shown that if $(x^{(0)}, \dot{x}^{(0)}) \in \overline{\mathcal{R}}_{\hat{x}=2P+s+1}$, then $X^{(1)}(x^{(0)}, \dot{x}^{(0)}) \le 2P + s + 1$. If $(x^{(0)}, \dot{x}^{(0)}) \in (\overline{\mathcal{R}}_{\hat{x}=\frac{s}{P}} - \overline{\mathcal{R}}_{\hat{x}=2P+s+1})$, then $X^{(1)}(x^{(0)}, \dot{x}^{(0)}) \le \frac{s}{P}$. Therefore, if $(x^{(0)}, \dot{x}^{(0)}) \in \overline{\mathcal{R}}_{\hat{x}=2P+s+1}$, then $|x(t)| \le 2P + s + 1$ for $t \in [t_0, t_1]$. If $(x^{(0)}, \dot{x}^{(0)}) \in (\overline{\mathcal{R}}_{\hat{x}=\frac{s}{P}} - \overline{\mathcal{R}}_{\hat{x}=2P+s+1})$, then $|x(t)| \le \frac{s}{P}$ for $t \in [t_0, t_1]$.

Part 3: Parts 1 and 2 discuss motion only into $\dot{x} > 0$. For motion into $\dot{x} < 0$, the substitution of $x(t) = -\psi(t)$ transforms (5.2) into an initial-value problem of the type already considered. The results in Parts 1 and 2 applied to the ψ system may readily be transformed into the x system. The conclusions are as follows:

$\dot{x}^{(0)} = 0$: $x(t) = x^{(0)}$ for $t \in [t_0, t_1]$ and $\dot{x}(t) < 0$ for $t \in (t_1, t_2)$ with $\dot{x}(t_2) = 0$. If t_2 is unbounded, then $\lim_{t \to \infty} (x(t), \dot{x}(t)) = (x^{(L)}, 0) \in \overline{\mathcal{R}}_{\hat{x} = 2P+s+1}$. If t_2 is bounded and $(x^{(0)}, 0) \in \overline{\mathcal{R}}_{\hat{x} = 2P+s+1}$, then $|x(t)| \le 2P+s+1$ for $t \in [t_0, t_2]$. If t_2 is bounded and $(x^{(0)}, 0) \in (\overline{\mathcal{R}}_{\hat{x} = \frac{s}{P}} - \overline{\mathcal{R}}_{\hat{x} = 2P+s+1})$, then $|x(t_2)| < |x^{(0)}|$.

$\dot{x}^{(0)} < 0$: $\dot{x}(t) < 0$ for $t \in [t_0, t_1)$ with $\dot{x}(t_1) = 0$. If t_1 is unbounded, then $\lim_{t \to \infty} (x(t), \dot{x}(t)) = (x^{(L)}, 0) \in \overline{\mathcal{R}}_{\hat{x} = 2P+s+1}$. If t_1 is bounded and $(x^{(0)}, \dot{x}^{(0)}) \in \overline{\mathcal{R}}_{\hat{x} = 2P+s+1}$, then $|x(t)| \le 2P+s+1$ for $t \in [t_0, t_1]$. If t_1 is bounded and $(x^{(0)}, \dot{x}^{(0)}) \in (\overline{\mathcal{R}}_{\hat{x} = \frac{s}{P}} - \overline{\mathcal{R}}_{\hat{x} = 2P+s+1})$, then $|x(t)| \le \frac{s}{P}$ for $t \in [t_0, t_1]$.

Part 4: For $\dot{x}^{(0)} = 0$, repeated application of Parts 1 and 3 show that if $(x^{(0)}, \dot{x}^{(0)}) \in \overline{\mathcal{R}}_{\hat{x} = 2P+s+1}$, $|x(t)| \le 2P+s+1$ for $t \in [t_0, \infty)$. If $(x^{(0)}, \dot{x}^{(0)}) \in (\overline{\mathcal{R}}_{\hat{x} = \frac{s}{P}} - \overline{\mathcal{R}}_{\hat{x} = 2P+s+1})$, then the solution trajectory must intersect $\dot{x} = 0$ with decreasing amplitudes until it enters $\overline{\mathcal{R}}_{\hat{x} = 2P+s+1}$. For $\dot{x}^{(0)} \ne 0$, Parts 2 and 3 and repeated application of Parts 1 and 3 yield the same result. This concludes the proof.

Remarks

For the limiting case of $P = 0$, Theorem 5 states that all solutions are bounded and that the asymptotic bound is $|x(t)| \le 1 + s$. This agrees with the previous discussion for $G(t)\equiv 0$.

Theorem 5 cannot be applied to the limiting case of $s = 0$ since $P \geq 0$.

In addition to the bounds stated in Theorem 5, it may also be shown that if $(x^{(0)}, \dot{x}^{(0)}) \in \overline{\mathcal{R}}_{\hat{x}=\frac{s}{P}}$, then the solution trajectory $(x(t), \dot{x}(t))$ in the $x, \dot{x}$ plane remains in the closed region whose boundary is described by

$$\dot{x} = \begin{cases} \sqrt{-(x-P+s)^2+(P+s(\frac{1}{P}-1))^2} & \text{for } x \in [-\frac{s}{P}, 1-s] \\ \sqrt{2(P-1)x+s^2(1-\frac{1}{P})^2+1} & \text{for } x \in [1-s, 1+s] \\ \sqrt{-(x-P-s)^2+(P+s(1-\frac{1}{P}))^2} & \text{for } x \in [1+s, \frac{s}{P}] \\ -\sqrt{-(x+P-s)^2+(P+s(\frac{1}{P}-1))^2} & \text{for } x \in [-\frac{s}{P}, -1-s] \\ -\sqrt{2(1-P)x+s^2(1-\frac{1}{P})^2+1} & \text{for } x \in [-1-s, -1+s] \\ -\sqrt{-(x+P+s)^2+(P+s(1-\frac{1}{P}))^2} & \text{for } x \in [-1+s, \frac{s}{P}] \end{cases} \tag{5.14}$$

When $(x(t), \dot{x}(t))$ enters $\overline{\mathcal{R}}_{\hat{x}=2P+s+1}$, it then remains in the closed region whose boundary is described by

$$\dot{x} = \begin{cases} \sqrt{-(x-P+s)^2+(P+1)^2+4s(1-P)} & \text{for } x \in [-2P-s-1, 1-s] \\ \sqrt{2(P-1)x+2s(1-P)+2(P+1)} & \text{for } x \in [1-s, 1+s] \\ \sqrt{-(x-P-s)^2+(P+1)^2} & \text{for } x \in [1+s, 2P+s+1] \end{cases} \tag{5.15}$$

$$\dot{x} = \begin{cases} -\sqrt{-(x+P-s)^2+(P+1)^2+4s(1-P)} & \text{for } x\in[-2P-s-1,\,-1-s] \\ -\sqrt{2(1-P)x+2s(1-P)+2(P+1)} & \text{for } x\in[-1-s,\,-1+s] \\ -\sqrt{-(x+P+s)^2+(P+1)^2} & \text{for } x\in[-1+s,\,2P+s+1] \end{cases} \qquad (5.15)\ \text{cont.}$$

If $\underset{t\in[t_0,\infty)}{\text{Max}} |G(t)| \le P < \frac{1}{4}\left(\sqrt{(1+s)^2+8s} - 1 - s\right)$, then Theorem 5 may be used in conjunction with Theorem 3 to obtain a sharper asymptotic bound for all solutions of (4.1) for an interval of z. This interval is $z\in(z_1, z_2)$ where z_1 and z_2 can be determined from

$$(P+s)\coth\left(\frac{\pi z_1}{2\sqrt{1-z_1^2}}\right) = \frac{s}{P}$$

$$(P+s)\coth\left(\frac{\pi z_2}{2\sqrt{1-z_2^2}}\right) = 2P+s+1 \qquad (5.16)$$

Figure 21 illustrates the asymptotic bound of the solutions as a function of z. (The term asymptotic bound is used in the sense that if $(x^{(0)}, \dot{x}^{(0)})$ lies in a certain closed region in the x, $\dot{x}$ plane, $|x(t)|$ is less than or equal to the bound for $t\in[t_0,\infty)$. If $(x^{(0)}, \dot{x}^{(0)})$ is not in this region, then the motion continually decreases until $|x(t)|$ becomes less than or equal to this bound.)

5.2 Relationship Between Initial Conditions and Steady-State Solutions

This section continues the investigation of the response of the viscously damped limited slip system to trigonometric excitation. It is shown in Chapter IV that the steady-state response curves can be

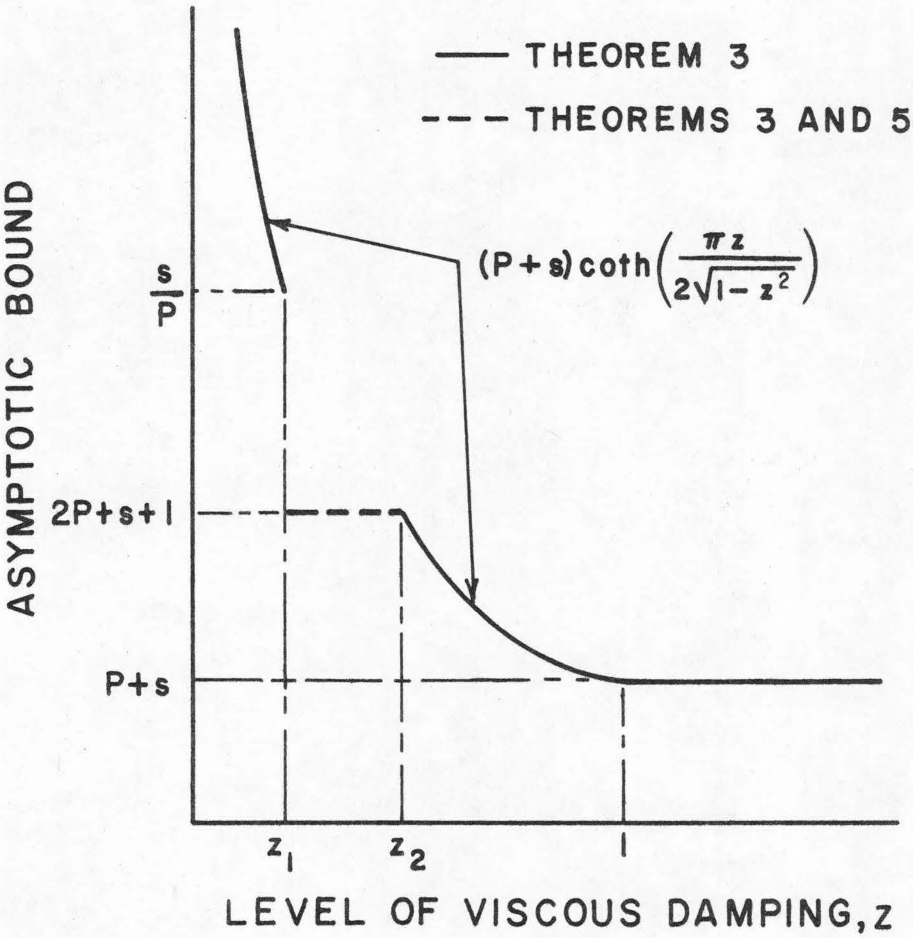


Figure 21: Asymptotic Bound for Solutions of (4. 1) with

$$|G(t)| \le P < \frac{1}{4}\left(\sqrt{(1+s)^2+8s}-1-s\right)$$

triple-valued for a given frequency of excitation. Triple steady-state solutions may occur when there is no disconnected portion of the associated response curve. For this situation the two asymptotically stable portions of the steady-state response curve can always be detected by a slow frequency sweep in both directions. Triple steady-state solutions also occur when there is a disconnected portion of the response curve. However, for this second situation, the detection of the upper asymptotically stable branch depends critically upon the frequency and initial conditions. The presence of the upper branch of the response curve can never be detected by a slow frequency sweep if the response is on the lower portion of the curve. The relationship between initial conditions and the resulting steady-state solution is studied in this section. The investigation is concerned with the situation for which the response curve is disconnected since the detection of the upper portion then depends critically upon the initial conditions.

Transient Response for a Family of Initial States (Approximation II)

The Equations for $\dot{a}$ and $\dot{\theta}$ are of the same complexity in either Simplified Approximation II or Approximation II. Since Approximation II is more accurate in its steady-state predictions, it is used in this investigation.

In the derivation of Approximation II, it has been assumed that a remains finite as $\epsilon \to 0$. Thus the fact that $\dot{\theta}$, given by the second of Equations (3.26), can become unbounded as $a \to 0$ is not surprising — Approximation II is not valid near $a = 0$. With this in mind, only $a > 0$

is considered. Using (3. 26) and (3. 30), Equation (3. 49) becomes

$$\frac{da}{d\theta} = \frac{-2\pi a^2 z\nu + \pi a r \sin\theta - E_H(a)}{\pi\left(a(\nu^2 - 1) + r\cos\theta - C(a)\right)} \tag{5. 17}$$

For convenience a and θ will be taken as polar coordinates.

Integral curves of (5. 17) are shown in Figure 22 for s = . 15, z = . 02, r = . 18, and ν = . 97. Steady-state solutions ($\dot{a} = \dot{\theta} = 0$) correspond to singularities of (5. 17). As expected, two singularities are asymptotically stable and the other is unstable. The unstable singularity is a saddle. (From the stability analysis in Section 4. 2, it may be concluded that every solution in the unstable region of the ν, a plane corresponds to a saddle singularity.) If $a(t_0) = a^{(0)}$ and $\theta(t_0) = \theta^{(0)}$, the motion of a and θ follows the integral curve passing through $(a^{(0)}, \theta^{(0)})$ and ultimately approaches a singular point. Thus it is seen that the two trajectories entering the saddle, referred to as a separatrix, separate the trajectories leading to the upper solution, and those leading to the lower solution.

Figures 23 through 25 are for other s and z investigated in the steady-state analysis. Only the trajectories entering or leaving the saddle are shown. It may be noted from the figures that the region of initial conditions for solutions which approach the isolated portion of the response curve is not negligible. For the area shown, this region may be even larger than the one for solutions which approach the lower portion of the curve. When $z > 0$, the qualitative behavior of the separatrix on a larger scale may be determined from Equations (3. 26), (3. 30),

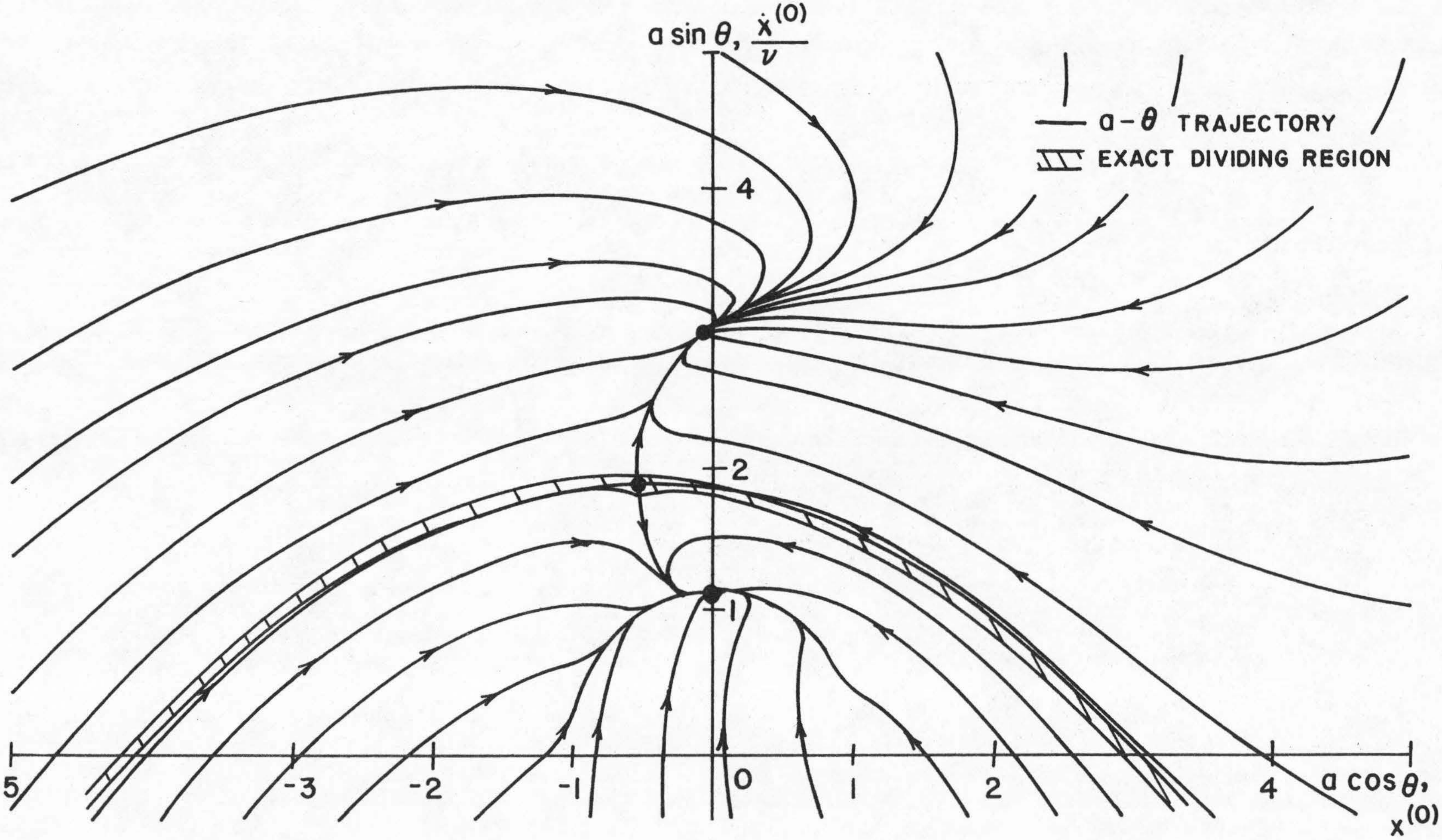


Figure 22: a-θ Trajectories and Exact Dividing Region for s = .15, z = .02, r = .18, ν = .97

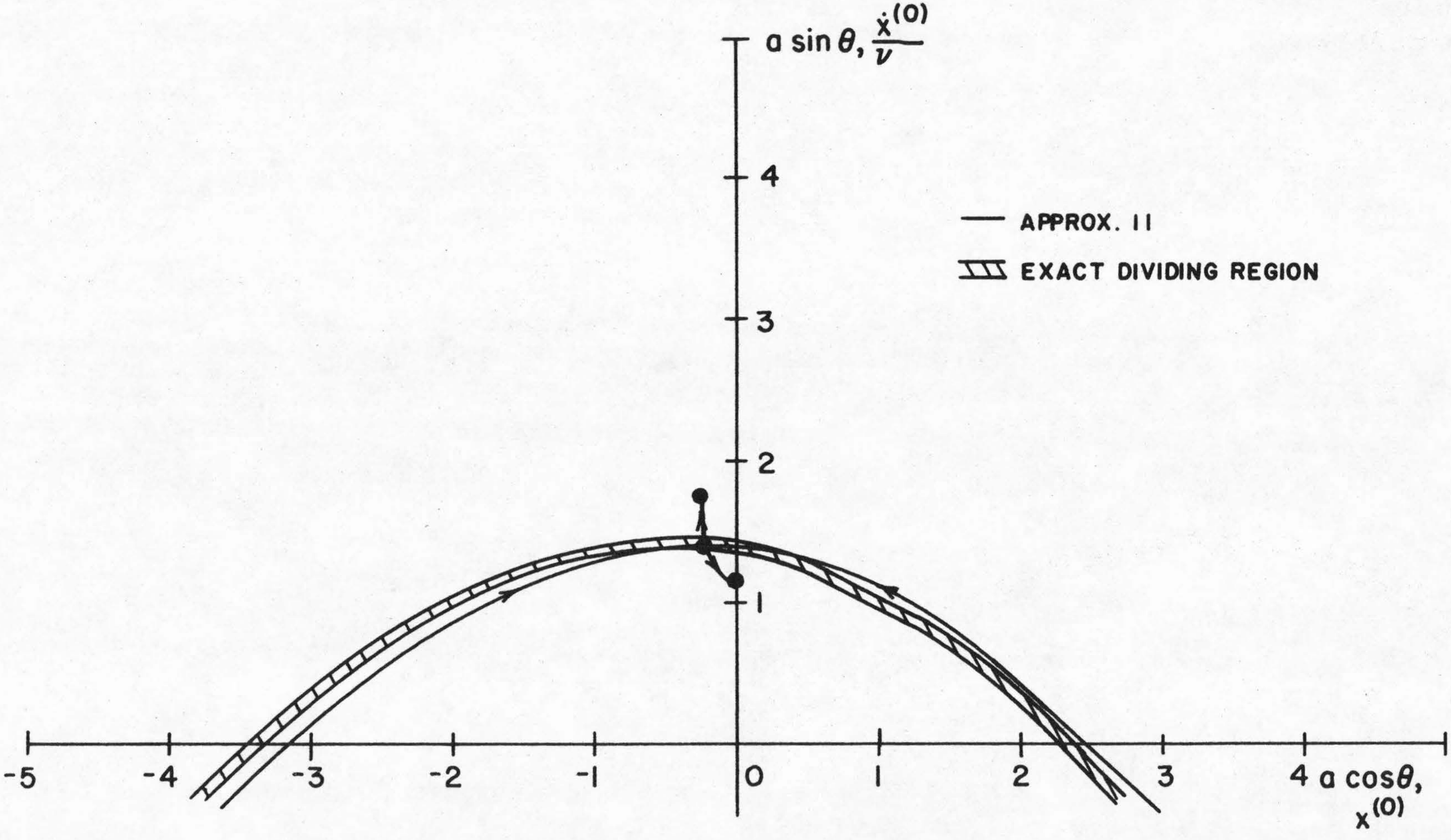


Figure 23: Separatrix and Exact Dividing
Region for s = .15, z = .04, r = .247, ν = .965

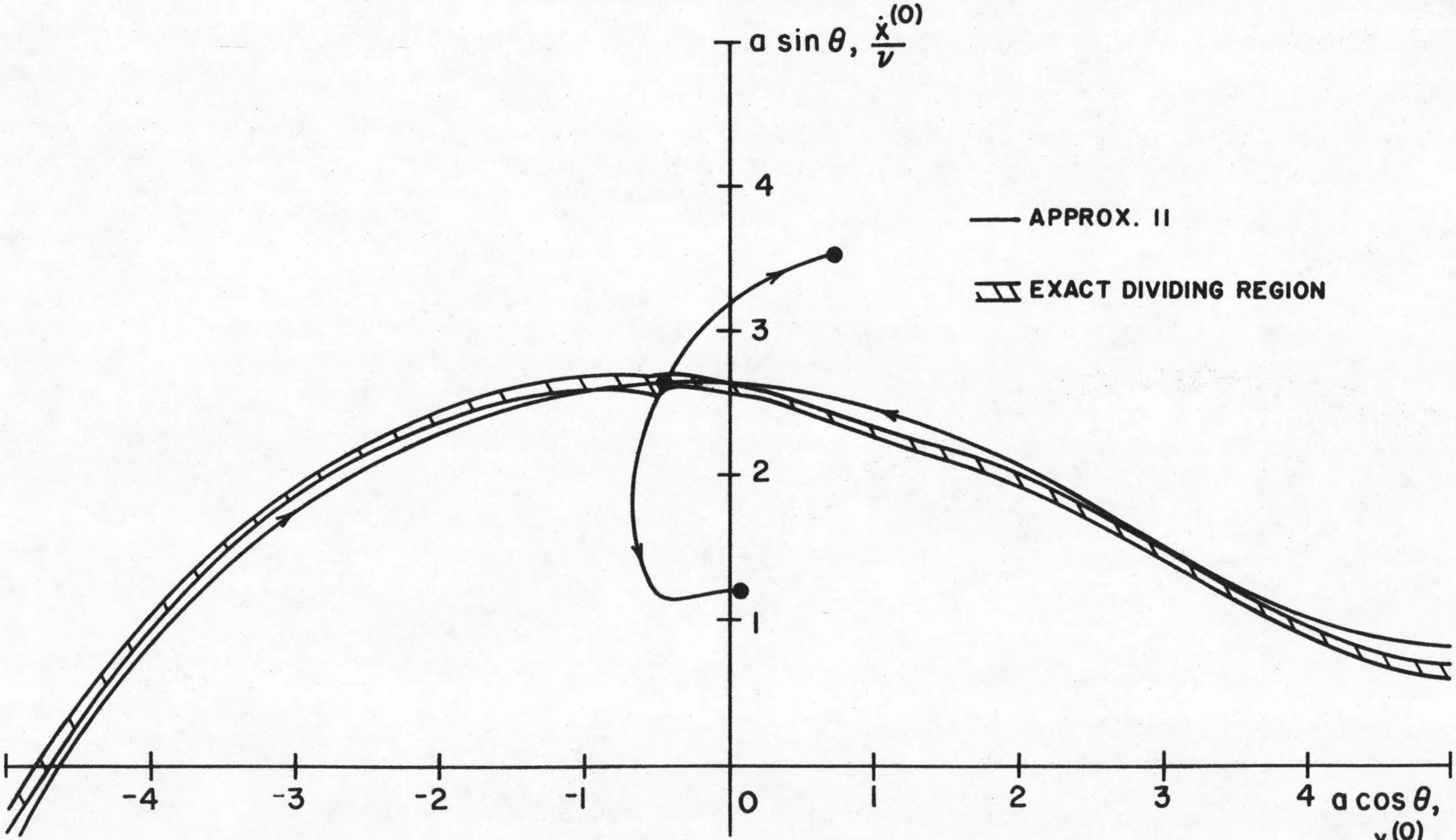


Figure 24: Separatrix and Exact Dividing Region for s = .3, z = .02, r = .247, ν = .94

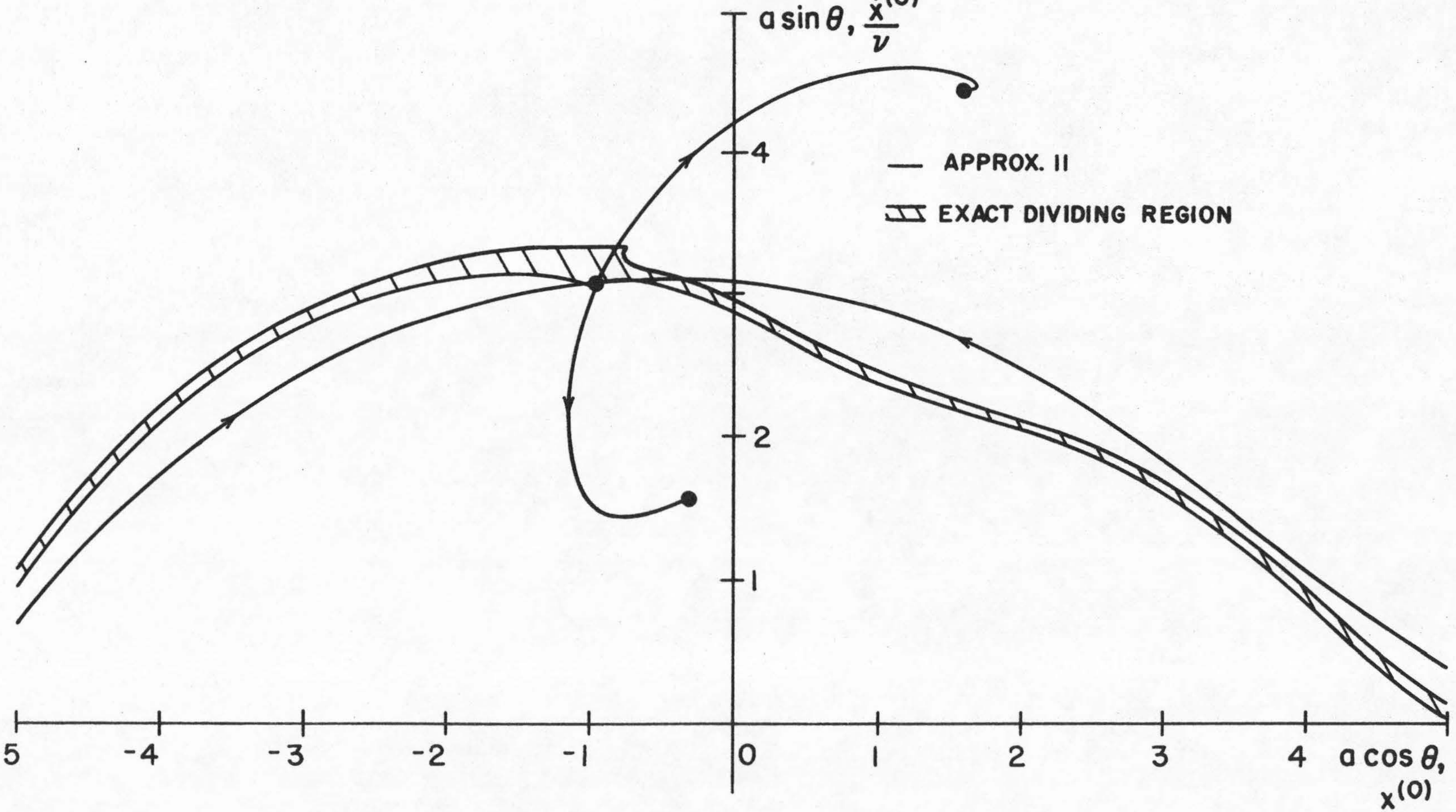


Figure 25: Separatrix and Exact Dividing Region for s = .9, z = .04, r = .6, ν = .85

(4. 9), and (4. 10). For sufficiently large a, $\dot{a} < 0$. Thus Approximation II predicts that all solutions (not only steady-state solutions) are bounded. Also for sufficiently large a, $\dot{\theta}<0(>0)$ when $\nu < 1(>1)$. Therefore, it may be concluded that for large initial amplitudes, all trajectories of a and θ begin spiraling clockwise (counterclockwise) toward the origin for $\nu < 1(>1)$. On a large scale, the separatrix resembles that shown in Figure 26. Along any line of initial conditions drawn for fixed θ, the intervals of initial amplitudes leading to a steady-state on the isolated portion of the response curve actually alternate with those leading to a steady-state on the lower portion.

Comparison with Exact Solutions

The solution assumed for Approximation II is $x(t) = a(t) \cos (\nu t - \theta(t))$. To the same order of accuracy, it may be concluded that $\dot{x}(t) = - a(t)\nu \sin (\nu t - \theta(t))$. Thus the approximate solution with $x(t_0) = x^{(0)}$ and $\dot{x}(t_0) = \dot{x}^{(0)}$ is associated with the trajectory of a and θ corresponding to the initial conditions $a(t_0) = \sqrt{(x^{(0)})^2 + \left(\frac{\dot{x}^{(0)}}{\nu}\right)^2}$ and $\theta(t_0) = \nu t_0 - \tan^{-1}\left(\frac{-\dot{x}^{(0)}}{\nu x^{(0)}}\right)$. For convenience, t_0 is taken to be some interger multiple of $\frac{2\pi}{\nu}$ denoted by $\tilde{t}$. Then $x(\tilde{t}) = a(\tilde{t}) \cos \theta(\tilde{t})$ and $\dot{x}(\tilde{t}) = a(\tilde{t})\nu \sin \theta(\tilde{t})$. Thus Figures 22 through 25 may be interpreted as the plane of initial conditions for the approximate solution at $t_0 = \tilde{t}$. The horizontal axis is $x^{(0)}$ and the vertical axis, $\frac{\dot{x}^{(0)}}{\nu}$. If the initial condition $(x^{(0)}, \dot{x}^{(0)})$ lies above the separatrix in the figures, it follows that the corresponding approximate solution will approach a steady-state on the isolated portion of the response curve. If it lies below the

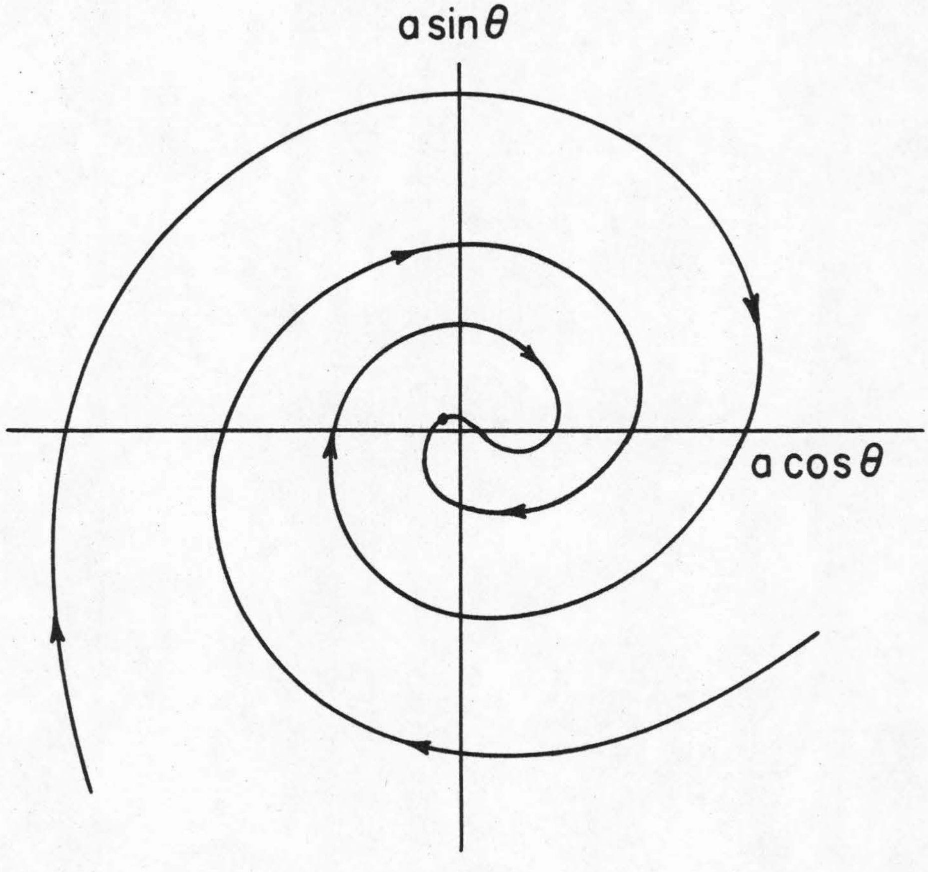


Figure 26: Separatrix (Large Scale)

separatrix, the solution will approach a steady-state on the lower portion of the response curve.

Assuming the state of $\mathcal{X}_s\{x(t)\}$ at $t = t_0$ to be the steady-state configuration for path oscillation with amplitude $a = \sqrt{(x^{(0)})^2 + \left(\frac{\dot{x}^{(0)}}{\nu}\right)^2}$, "exact" solutions are calculated by numerically continuing the analytical solutions to the linear parts of (4. 5). The numerical results imply that there is a dividing line corresponding to the separatrix referred to in the approximate analysis, but due to time limitations only a region containing the line has been determined for each case. The exact dividing region is shown as a shaded band in Figures 22 through 25. The numerical calculations also verify the predicted stability of the steady-state solutions.

The exact dividing regions agree closely with the separatrices determined from Approximation II in Figures 22 through 24. In Figure 25, for $s = .9$ and $z = .04$, there is poor agreement. This is to be expected for the relatively large nonlinearity. In all the figures, the exact dividing region is higher (lower) for initial displacements to the left (right) of the displacement at which the unstable singularity is located.

Other Initial Times

If t_0 is not a multiple of $\frac{2\pi}{\nu}$, the polar diagram of the trajectories of a and θ may still be interpreted as an initial condition plane since (5. 17) is autonomous. The diagram must be rotated clockwise through an angle νt_0. Then the horizontal and vertical axes correspond to

$a \cos(\theta - \nu t_0)$ and $a \sin(\theta - \nu t_0)$, respectively. Thus after the rotation, the polar diagram may be interpreted as the initial condition plane for the approximate solution at the initial time t_0. The horizontal axis is $x^{(0)}$ and the vertical axis, $\frac{\dot{x}^{(0)}}{\nu}$. Since (4. 5) is a nonautonomous system, the exact dividing regions are not valid even if similarly rotated, provided t_0 is not a multiple of $\frac{\pi}{\nu}$. However, from the symmetry of (4. 5) with respect to t and the dependent variable, it may be concluded that only initial times in the interval $t_0 \in \left[\frac{2\pi}{\nu}, \frac{3\pi}{\nu}\right)$ need to be considered to determine the exact relationship between initial conditions and resulting steady-state for any initial time. The initial time chosen for the examples is of no special significance to (4. 5). Thus it is reasonable to conclude that for other initial times, there will be agreement between the exact dividing region and separatrix for parameters used in Figures 22 through 24.

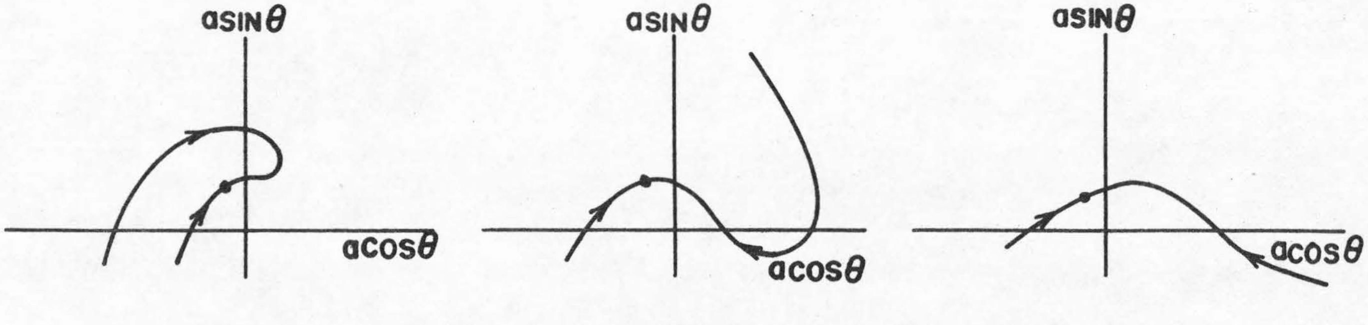
Discussion

For each initial time, Approximation II predicts the existence of a dividing line in the initial condition plane for the cases examined. This line separates the regions of initial conditions leading to the different steady-states possible. The author has seen no proof for the existence of such a dividing line of this type for nonautonomous systems. However, numerical results have verified the existence of at least one dividing region of this type for a portion of the initial condition plane.

For the examples considered, it is found that the size of the region of initial conditions leading to steady-state on the upper portion of the response curve is significant. However, it may be concluded that with sufficiently small initial displacement and velocity at any initial time, the solution approaches a steady-state on the lower portion of the curve. Thus, if the system is initially at equilibrium before the excitation is applied, it will always approach a steady-state on the lower portion of the response curve. The amplitude of oscillations at steady-state is then the smaller of the two which can occur. If the system has an appropriate initial displacement or velocity, (this can also correspond to an appropriate impulse while in motion) it approaches oscillations with the larger of the two possible amplitudes.

Frequency Dependence

In each of Figures 22 through 25, the frequency is in the vicinity of the maximum of both upper and lower portions of the associated disconnected response curve. The figures show, for initial times which are a multiple of $\frac{2\pi}{\nu}$, the relative regions of the initial condition plane which lead to different steady-state solutions. It has been mentioned that the regions predicted by Approximation II are merely rotated if the initial time is changed. No indication of the dependence of these regions upon the excitation frequency is given. It is found that the separatrices corresponding to excitation frequencies near the left edge, middle, and right edge of the isolated portion of a response curve resemble those shown in Figures 27a, b, and c, respectively. The region leading to a steady-state on the upper portion of the response curve changes considerably with frequency. Rather than investigate the frequency dependence of this region, the frequency



(a) Left (b) Middle (c) Right

Figure 27: Appearance of Separatrix for Frequencies near Left Edge (a), Middle (b), and Right Edge (c) of an Isolated Portion of a Response Curve

dependence of the corresponding interval on the $x^{(0)}$ axis is investigated. Thus the situation considered is that in which the system is initially displaced and at rest when the excitation is applied. The initial time is again taken to be a multiple of $\frac{2\pi}{\nu}$.

Figures 28 and 29 show, for various values of s, z, and r, the relationship between frequency and initial displacements which leads to different steady-state solutions. The boundaries have been determined numerically using (5.17), (4.9), and (4.10). Initial displacements in the shaded areas lead to steady-state solutions on the isolated portion of the corresponding response curve. The following is observed:

1. For sufficiently small displacements, only steady-state solutions on the lower portion of the response curve are approached.
2. There is a significant region of initial displacements which leads to steady-state solutions on the upper portion of the response curve.
3. The region of initial displacements leading to a steady-state on the upper portion of the response curve is larger for negative displacements than that for positive displacements.
4. When there exists an isolated portion of the response curve, the size of the region leading to steady-state on this portion decreases when r is decreased. When r is decreased, the new shaded region is contained in the old region, and the minimum initial displacement which leads to steady-state on the upper portion of the response curve is increased.

To verify the predicted relationship between the frequency and initial displacement which leads to different steady-state solutions,

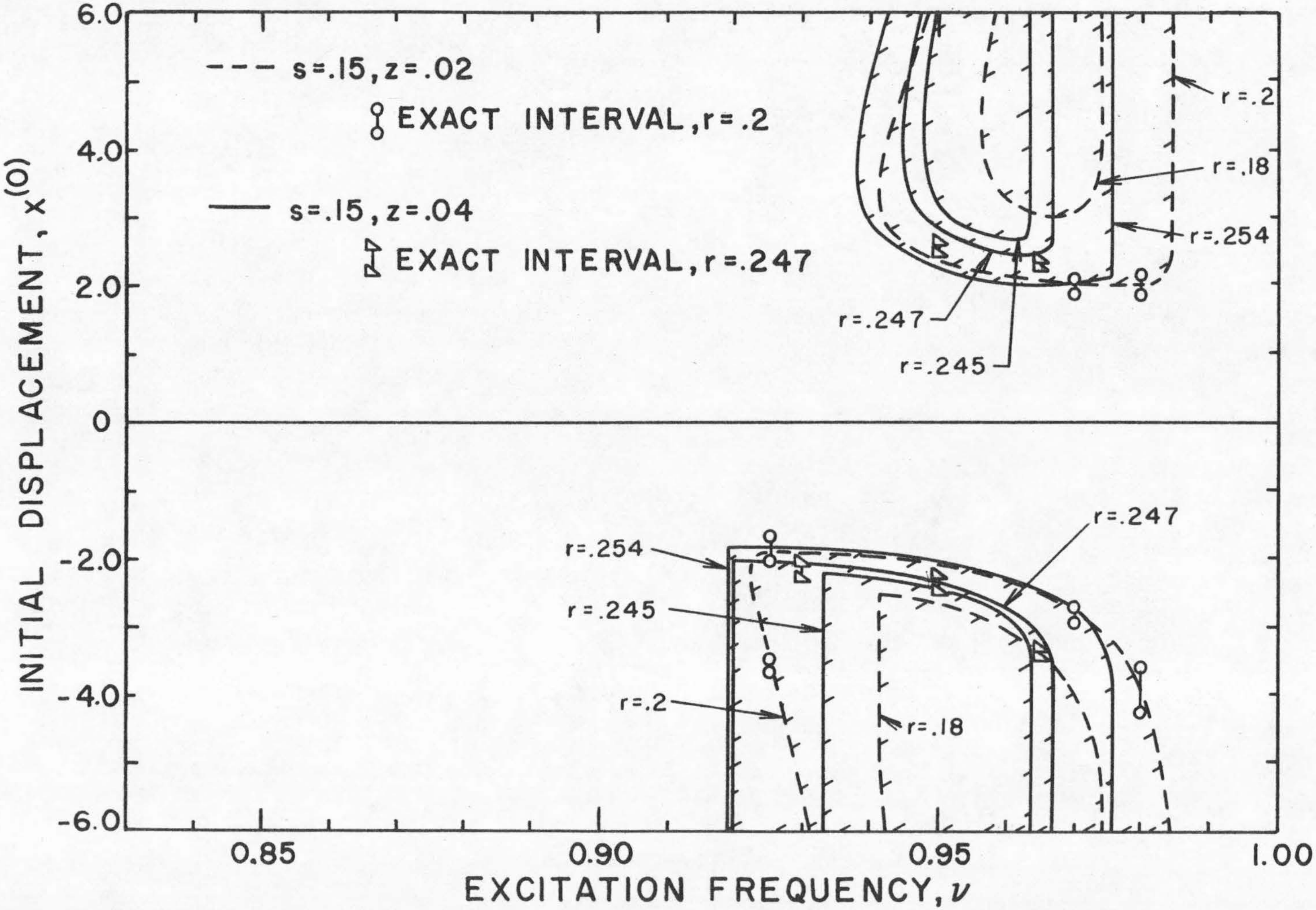


Figure 28: Frequency versus $x^{(0)}$, s = .15 (Shaded (Unshaded) Region Leads to Steady-State on Upper (Lower) Portion of Response Curve)

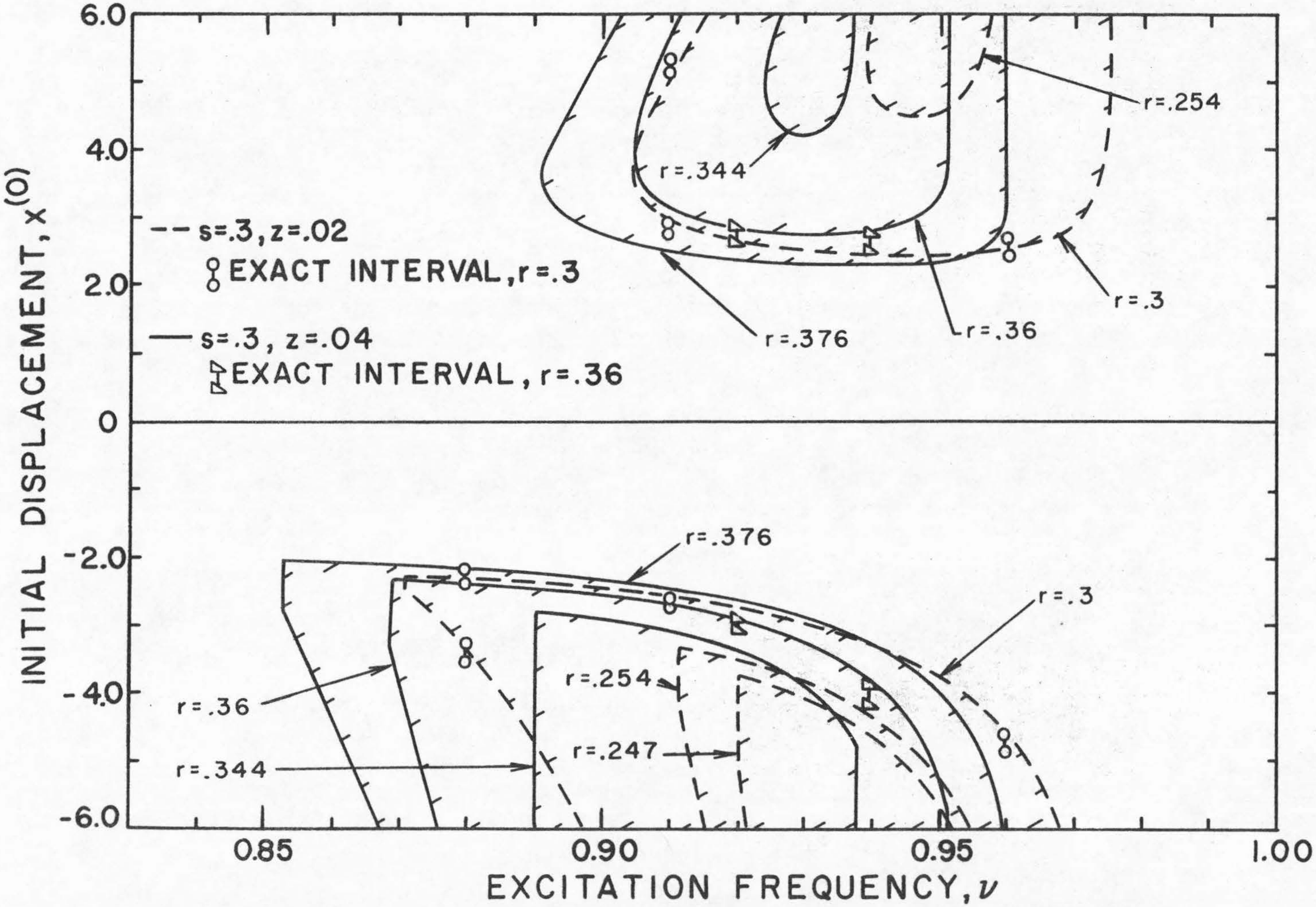


Figure 29: Frequency versus $x^{(0)}$, s = .3 (Shaded (Unshaded) Region Leads to Steady-State on Upper (Lower) Portion of Response Curve)

"exact" solutions have been computed by numerically continuing the analytic solutions to the linear parts of (4. 5). Two initial displacements connected by a line are plotted for several sets of the parameters s, z, r, and ν in Figures 28 and 29. The line segment between a pair of points at a given frequency corresponds to a dividing interval which separates initial displacements leading to the different steady-state solutions. The exact intervals agree well with the boundaries predicted by Approximation II.

Other Initial States

One of the important assumptions in the application of the approximate analysis to (4. 5) is that the solution is oscillatory. In view of this assumption, the state of $\mathcal{N}_s\{x(t)\}$ at $t = t_0$ is always taken to be a symmetric steady-state configuration in the numerical computation of the exact transient solutions. The configuration used is associated with the path oscillation with amplitude $a = \sqrt{(x^{(0)})^2 + \left(\frac{\dot{x}^{(0)}}{\nu}\right)^2}$. If $|x^{(0)}| \geq 1+s$, there is only one possible state of $\mathcal{N}_s\{x(t)\}$ at $t = t_0$. Thus the exact dividing regions shown in Figures 22 through 25 are unique for $|x^{(0)}| \geq 1+s$. It may also be concluded that the exact points are unique in Figures 28 and 29. However, there are an infinite number of possible initial configurations of $\mathcal{N}_s\{x(t)\}$ for $|x^{(0)}| < 1+s$. Thus for other states of $\mathcal{N}_s\{x(t)\}$ at $t = t_0$, the exact dividing regions in Figures 22 through 25 change where $|x^{(0)}| < 1+s$.

For an indication of the effect of other initial states upon the exact dividing region, a different configuration for the state of $\mathcal{N}_s\{x(t)\}$

at $t = t_0$ is used to compute exact solutions. The configuration chosen corresponds to the virgin curve of $\mathcal{N}_s\{x(t)\}$ and can be determined from (4.2) by recalling that $\mathcal{N}_s\{x(t)\} = \mathcal{U}_s\{x(t)\} - x$. The initial-value problem (4.5) now corresponds to the situation in which the system is displaced from $x = 0$ when $\mathcal{N}_s\{x(t)\}$ is in its virgin state, and then given an initial velocity at $t = t_0$. The region of the initial condition plane for which the virgin curve differs from the family of steady-state configurations originally used is shaded in Figure 30. For an initial time which is again a multiple of $\frac{2\pi}{\nu}$, exact solutions have been calculated for the parameters of Figures 22 and 25. Figures 31 and 32 show the dividing regions superposed upon Figures 22 and 25, respectively. The dividing region does not differ appreciably for the smaller parameters, but does for the larger parameters. Although Approximation II is not expected to describe the behavior of the solutions of (4.5) with this initial state of $\mathcal{N}_s\{x(t)\}$, the dividing region still agrees with the separatrix for the smaller parameters. However, the agreement is made worse in Figure 32 where $s = .9$ and $z = .04$.

5.3 Summary

The investigation of the response of the limited slip system is continued in this chapter. It is shown that the response of the forced system is bounded provided viscous dissipation is present. When there is no viscous dissipation, the response is bounded provided that it begins in a certain region in the $x, \dot{x}$ plane and that the excitation is sufficiently small.

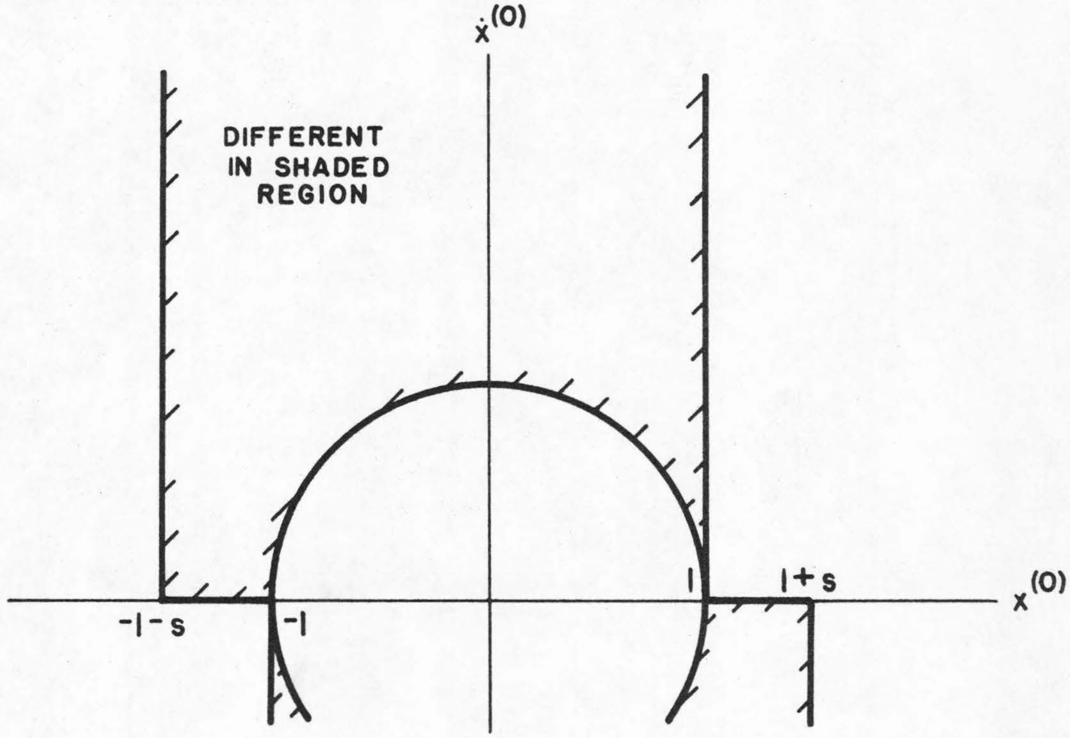


Figure 30: Region of the Initial Condition Plane in which the Virgin Curve Differs from the Steady-State Configuration

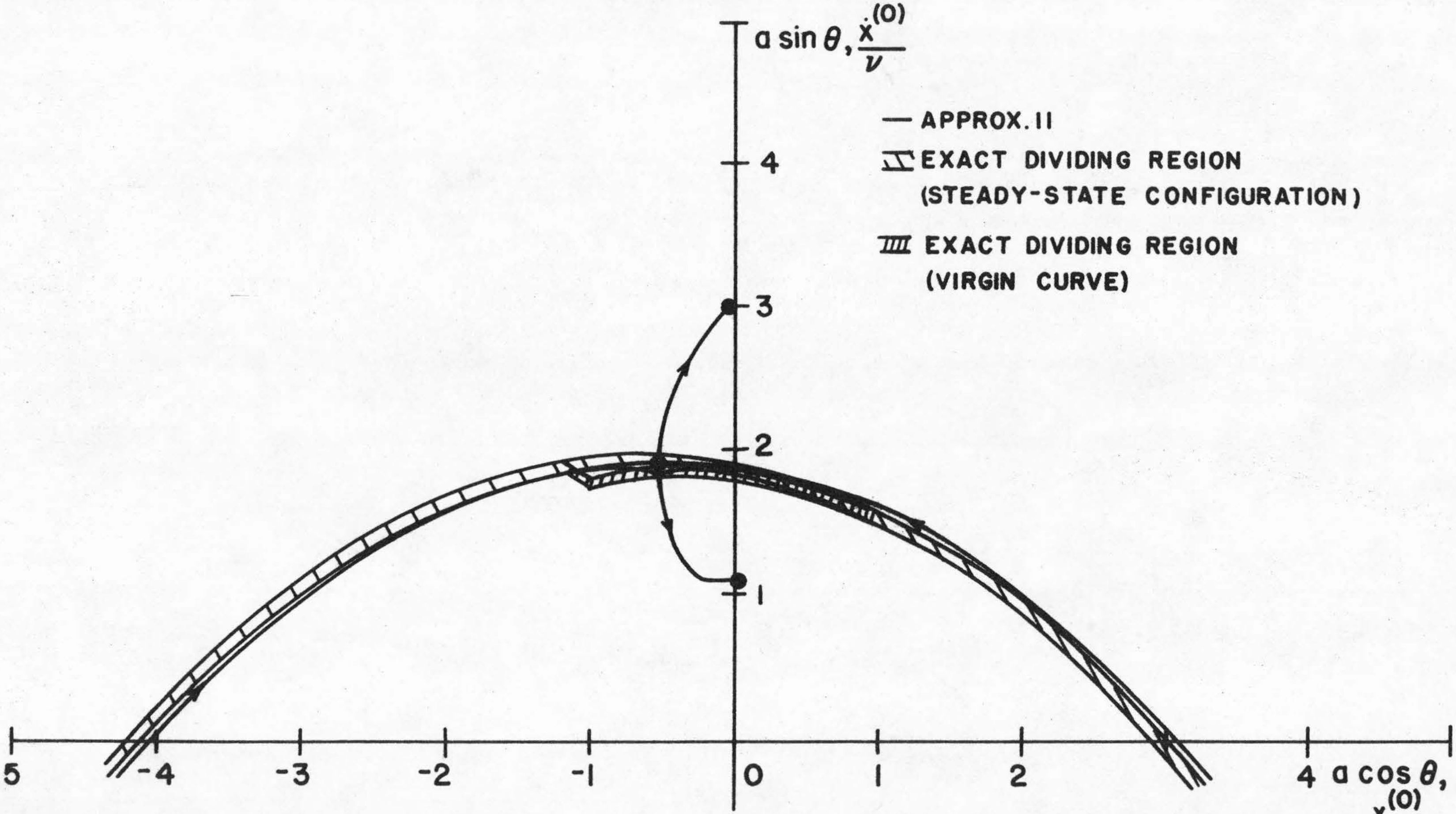


Figure 31: Exact Dividing Regions for Different Initial States, $s = .15$, $z = .02$, $r = .18$, $\nu = .97$

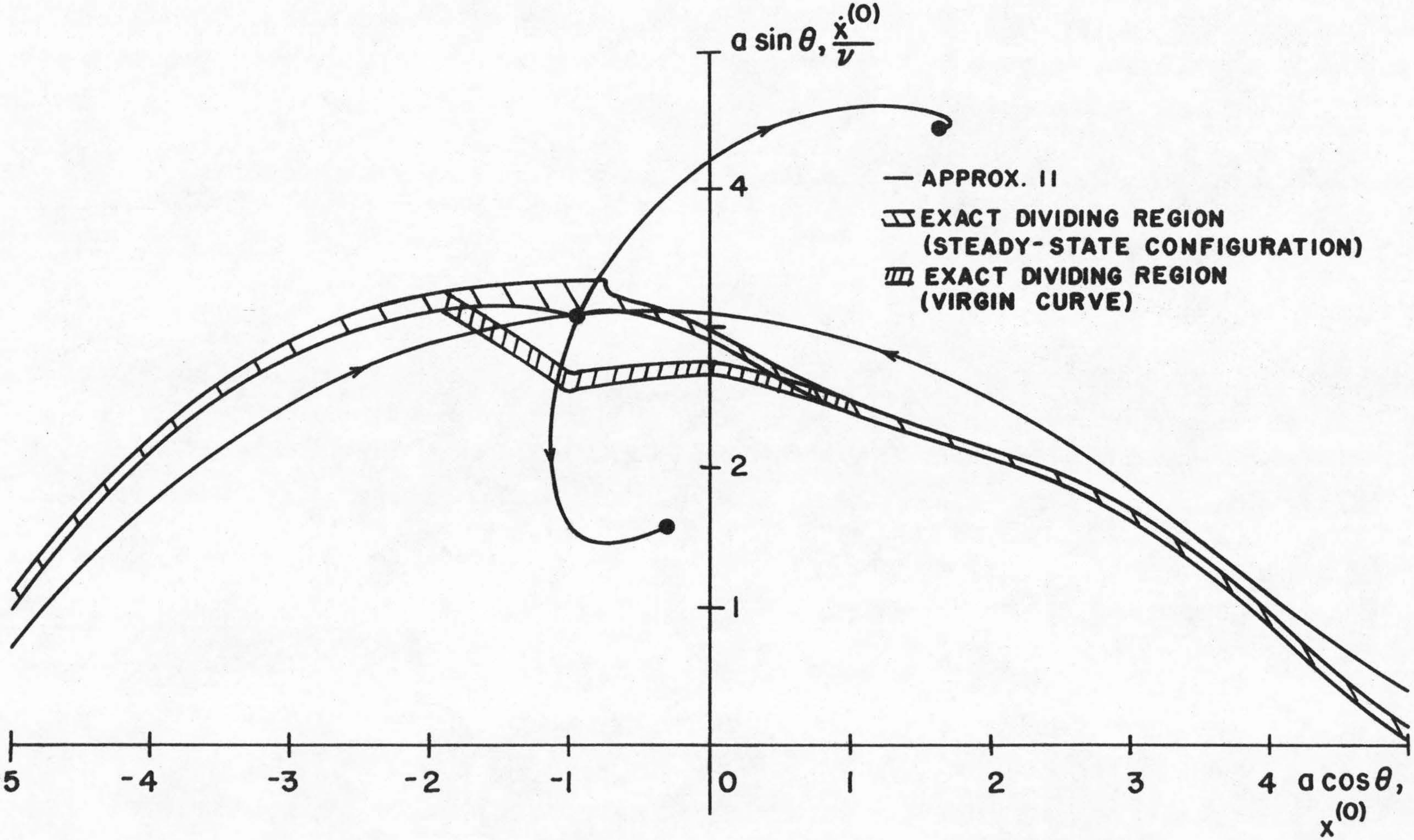


Figure 32: Exact Dividing Regions for Different Initial States, $s = .9$, $z = .04$, $r = .6$, $\nu = .85$

It is shown in Chapter III that when the excitation is trigonometric, the steady-state response curves can be triple-valued. In this chapter the relationship between initial conditions and resulting steady-state response is examined when multiple steady-states can occur. For a given initial time, regions of initial conditions leading to different steady-states are found for several sets of parameters. The dependence of the regions upon initial time, excitation frequency, and the state of $\mathcal{N}_s\{x(t)\}$ at $t = t_0$ is also discussed.

VI. SUMMARY AND CONCLUSIONS

The hysteretic behavior in the systems considered is depicted by a family of nonlinear functions depending upon one or more parameters. The current function parameters are determined from the previous response of the system.

Solution properties of a general class of dynamical systems with hysteresis are examined in Chapter II. The class is first identified mathematically. The following are concluded:

1. Solutions to the corresponding initial-value problem exist and are unique. (Theorem 1)
2. All solutions to systems with dissipative hysteresis are bounded, provided there is no excitation. (Theorem 2)
3. If the range of each member of the family of functions characterizing the hysteretic behavior, $h^{(i)}(x)$, is in the region bounded by the lines $x+s$ and $x-s$, then all solutions to the forced hysteretic system are bounded, provided there is viscous damping. (Theorem 3) A quantitative asymptotic bound upon the solution is given.
4. When there is no viscous damping, both bounded and unbounded solution behavior occur for forced hysteretic systems. Two examples are provided for illustration.

The response of a hysteretic system to trigonometric excitation is examined in Chapter III. Using an asymptotic solution procedure, an approximation (Approximation II) is derived for the general harmonic response of a viscously damped system with a small nonlinearity and trigonometric excitation. It is assumed that the nonlinearity, the levels of viscous damping and excitation, and the frequency detuning are $O(\varepsilon)$ as $\varepsilon \to 0$. However, the steady-state response prediction is still valid if only the nonlinearity is $O(\varepsilon)$.

Approximation II is applied to describe the response of the system when the nonlinearity is a hereditary function $\mathscr{H}\{x(t)\}$. $\mathscr{H}\{x(t)\}$ is assumed to possess a family of symmetric steady-state configurations depending upon the amplitude of oscillation, a. Only two properties of each member of this family are needed to completely determine Approximation II: the area enclosed by each configuration in the $x, \mathscr{H}$ plane ($E_H(a)$) and the average of the ascending and descending branches of each configuration ($m(x;a)$). To facilitate discussion of the steady-state response curves, a simplification is made in the equations of Approximation II to obtain Simplified Approximation II. In the framework of the asymptotic solution procedure, both approximations are of the same order of accuracy as $\varepsilon \to 0$. Though the quantitative predictions of the different approximations will generally differ, the qualitative predictions will be similar when the level of viscous damping and the frequency detuning are small.

Given only $E_H(a)$ and the level of viscous damping, the following information concerning the frequency-response curves can be easily determined from Simplified Approximation II: existence (non-existence) of steady-state response for a given amplitude of response and level of excitation, amplitudes of maxima and minima of response for a given level of excitation, width of the response curve at a given amplitude for a given level of excitation, existence of bounded (unbounded) resonance behavior, and existence of disconnected response curves. Also the qualitative effect of changing the level of excitation, level of viscous damping, or $E_H(a)$ upon the appearance of the response curves can be predicted.

The locus of extremum response depends upon m(x;a). Since this locus is the line of symmetry of the response curves in the ν^2, a plane, m(x;a) also influences the leaning of the response curves in the ν, a plane.

The accuracy of the approximations is examined in Chapter IV.

In Chapter IV, the harmonic steady-state response of a system with limited slip and trigonometric excitation is examined in detail. The distinguishing characteristic of the system is that the hysteretic behavior is limited to a finite interval of values of the response.

With even the slightest amount of slip present, the steady-state response curves possess distinctive characteristics: a "softening-hardening" behavior of the response curves; unbounded amplitude resonance for any level of excitation if no viscous damping is present; the existence of disconnected portions of the response curves for certain levels of viscous damping and excitation; the existence of triple-valued response curves for a certain range of system parameters even if the response curve is not disconnected; and the jump phenomenon associated with the triple-valued response curves. With the exception of unbounded amplitude resonance, none of the characteristics above are found in linear systems. The unbounded amplitude resonance is interesting in that it occurs for this system with dissipative rate-independent hysteresis even for a vanishingly small level of excitation.

To gain further insight into the general response of the system, solutions to the initial-value problem are studied in Chapter V. The boundedness of solutions is first examined and it is shown that:

1. All solutions are bounded, provided there is no excitation. (Application of Theorem 2)
2. All solutions to the forced system are bounded, provided there is viscous damping. (Application of Theorem 3)
3. When there is no viscous damping, all solutions beginning in a certain region of the $x, \dot{x}$ plane are bounded, provided the excitation is sufficiently small. A quantitative asymptotic bound upon the solution is given.

In the remainder of Chapter V, the relationship between initial conditions and resulting steady-state solution is investigated for the present system. Only the situations in which triple-valued response occurs as a result of a disconnected portion of the response curve are examined. For several sets of system parameters, a dividing region is found in the initial condition plane which separates initial conditions leading to different possible steady-state solutions. The initial condition plane is associated with an initial time and a class of initial states of the system. The region of initial conditions leading to steady-state solutions on the upper portion of the curve is significant in each case examined. For sufficiently small initial displacement and velocity for any initial time, the steady-state always occurs on the lower portion of the response curve.

The effect of excitation frequency upon a class of initial conditions leading to the different steady-states is also studied. The class consists of an initial time which is a multiple of $\frac{2\pi}{\nu}$, and initial data for which $\dot{x}^{(0)} = 0$. Frequency-displacement diagrams are used to illustrate the relationship between frequency and initial displacement for which steady-state occurs on the upper portion of the response curve. It is found for the cases studied that for sufficiently small initial displacements, only steady-state solutions on the lower portion of the curve are approached; that there is a significant region of initial displacements which lead to a steady-state on the upper portion of the curve; and that when the excitation level is decreased, the regions become smaller.

Since the possibility of triple-valued response curves is not common for rate-independent hysteretic models, their presence may not be expected in physical systems where they actually occur. When the response curve is connected, multiple-valued response can always be detected by a slow frequency sweep. However, when the response curve is disconnected, the detection of multiple-valued response depends strongly upon frequency and initial conditions. In this case, the examples investigated imply that there are some system parameters for which the resulting steady-state always lies on the lower portion of the response curve, provided the initial conditions are sufficiently small. Large initial conditions do not necessarily lead to steady-state on the upper portion of the response curve. Thus even using a systematic testing procedure, multiple-valued response may not be detected. If the system has appropriate initial conditions or receives an appropriate impulse while in motion, it may lead to steady-state on the isolated portion of the response curve. This behavior is significant from the design point of view since the larger of the two possible steady-state amplitudes may be several times the smaller.

It is hoped that this study furthers the understanding of the dynamic response of systems with hysteresis, and that the results and techniques used will be useful to other investigators in this field.

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