VARIETIES GENERATED BY MODULAR LATTICES OF WIDTH FOUR

Thesis by

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ABSTRACT

A variety (equational class) of lattices is said to be finitely based if there exists a finite set of identities defining the variety. Let $\mathcal{M}_n^\infty$ denote the lattice variety generated by all modular lattices of width not exceeding n. $\mathcal{M}_1^\infty$ and $\mathcal{M}_2^\infty$ are both the class of all distributive lattices and consequently finitely based. B. Jónsson has shown that $\mathcal{M}_3^\infty$ is also finitely based. On the other hand, K. Baker has shown that $\mathcal{M}_n^\infty$ is not finitely based for $5 \leq n < \omega$. This thesis settles the finite basis problem for $\mathcal{M}_4^\infty$. $\mathcal{M}_4^\infty$ is shown to be finitely based by proving the stronger result that there exist ten varieties which properly contain $\mathcal{M}_4^\infty$ and such that any variety which properly contains $\mathcal{M}_4^\infty$ contains one of these ten varieties.

The methods developed also yield a characterization of subdirectly irreducible width four modular lattices. From this characterization further results are derived. It is shown that the free $\mathcal{M}_4^\infty$ lattice with n generators is finite. A variety with exactly k covers is exhibited for all $k \geq 15$. It is further shown that there are $2^{\aleph_0}$ subvarieties of $\mathcal{M}_4^\infty$.

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INTRODUCTION

A variety of lattices is a class of lattices which is closed with respect to the formation of sublattices, homomorphic images and direct products. A fundamental theorem of Birkhoff [4] states that varieties of lattices are exactly those lattices defined by their identities. That is, a class $\mathcal{C}$ of lattices is a variety if the class of lattices which satisfy all the identities satisfied by all the members of $\mathcal{C}$ is the class $\mathcal{C}$. If $\mathcal{C}$ is any class of lattice then the class of all subdirect products of homomorphic images of sublattices of ultraproducts of members of $\mathcal{C}$ is the smallest variety containing $\mathcal{C}$ and is called the variety generated by $\mathcal{C}$. This theorem, which is due to Bjarni Jónsson [15], has made possible many advances in the theory of lattice varieties. Let $\mathcal{M}_n^m$ be the variety generated by all modular lattices whose width does not exceed n and whose length does not exceed m, where n and m are cardinals. It follows from the finite nature of identities that the variety generated by the finitely generated members of a class $\mathcal{C}$ is the same as the variety generated by $\mathcal{C}$. It follows from this that if n_1 and n_2 are infinite cardinals and m is any cardinal then $\mathcal{M}_{n_1}^m = \mathcal{M}_{n_2}^m$ and $\mathcal{M}_m^{n_1} = \mathcal{M}_m^{n_2}$. The symbol ∞ is used to replace any infinite cardinal. For example, the variety generated by all modular lattices of width not exceeding n, $1 \leq n < \omega$, is denoted by $\mathcal{M}_n^\infty$. This thesis makes a careful study of $\mathcal{M}_4^\infty$.

A variety is <u>finitely based</u> if it is defined by a finite set of identities. A basic problem in the theory of modular varieties is to determine the values of m and n for which $\mathcal{M}_n^m$ is finitely based (Wille [22]). R. McKenzie has shown that the variety generated by a finite lattice is finitely based [18]. From this it follows that $\mathcal{M}_n^m$ is finitely based if both m and n are finite. K. Baker has shown that $\mathcal{M}_\infty^n$ is finitely based for all n [2, 3]. $\mathcal{M}_1^\infty$ and $\mathcal{M}_2^\infty$ are both equal to the variety of all distributive lattices and thus are finitely based. B. Jónsson has shown that $\mathcal{M}_3^\infty$ is finitely based [16]. On the other hand K. Baker [2] has shown that $\mathcal{M}_n^\infty$ is not finitely based for $5 \leq n < \infty$. $\mathcal{M}_4^\infty$ is the only variety for which the above problem is not solved. This thesis completes the solution by showing that $\mathcal{M}_4^\infty$ is finitely based. This is done by showing that $\mathcal{M}_4^\infty$ is covered by ten varieties and that any variety properly containing $\mathcal{M}_4^\infty$ contains one of these ten varieties. It follows from this result that an independent set of identities which defines $\mathcal{M}_4^\infty$ has ten or less elements and there exist sets of independent identities defining $\mathcal{M}_4^\infty$ with n elements, $n = 1, 2, \ldots, 10$.

A problem closely related to Wille's problem but which appears to be more difficult is to determine which of the varieties $\mathcal{M}_n^m$ have the property that $\mathcal{M}_n^m$ is covered by a finite set of varieties such that any variety properly containing $\mathcal{M}_n^m$ contains one of these covering varieties. It is a classical theorem that the variety of all distributive lattices, which is equal to $\mathcal{M}_1^\infty$, $\mathcal{M}_2^\infty$ and $\mathcal{M}_\infty^1$, has this property. As mentioned above this thesis shows that $\mathcal{M}_4^\infty$ has this property. $\mathcal{M}_3^\infty$ and $\mathcal{M}_\infty^2$ have this property as was shown by B. Jónsson [16]. D. X. Hong has shown that $\mathcal{M}_\infty^3$ has this property [14]. Of course, $\mathcal{M}_\infty^\infty$, the variety of all modular lattices, has this property, and $\mathcal{M}_n^\infty$, $5 \leq n < \infty$

must fail to have this property. At the present time the question for $\mathcal{M}_n^m$, $5 \le n \le \infty$ and $4 \le m < \infty$, remains unsettled.

The techniques used to show that $\mathcal{M}_4^\infty$ is finitely based are also used to characterize the subdirectly irreducible members of $\mathcal{M}_4^\infty$. Two results of interest follow from this characterization. First, there are $2^{\aleph_0}$ subvarieties of $\mathcal{M}_4^\infty$. Since there are countably many finite sets of identities the above implies that there exists a subvariety of $\mathcal{M}_4^\infty$ which is not finitely based. Secondly, it is shown that all members of $\mathcal{M}_4^\infty$ are locally finite. This fact has the corollary that the free $\mathcal{M}_4^\infty$ lattice on a finite number of generators is finite (compare with Birkhoff's Problem 46 [4]). This local finiteness also has the corollary that $\mathcal{M}_4^\infty$ is generated by its finite members. This fact is known to be true for the variety of all lattices (R. Dean [7]), false for the variety of Desargian projective planes (K. Baker [1]), and unsolved for the variety of all modular lattices.

The proofs of the above results depend heavily on the development of a detailed structure theory for modular lattices. Two basic techniques are employed. First, the classical result that a modular lattice which is generated by three elements is finite is applied several times in order to obtain some of the local structure of modular lattices. In order to piece these bits of local structure together to obtain an overall picture of the lattice a second technique, the theory of projectivities, is employed.

In [8] and [9] Dilworth showed that there is a strong connection between the structure of a lattice and the notion of projectivity.

R. Thrall [21] showed that two projective quotients in a modular lattice could be connected by a sequence of transposes of a standard form (for definitions see Chapter I). G. Grätzer called such a sequence normal and applied it to the study of lattice varieties [13]. B. Jónsson defined the concept of a strongly normal sequence and showed that in most cases projective quotients in a modular lattice have subquotients connected by a strongly normal sequence. He employs this concept to solve the finite basis problem for $\mathcal{M}_3^\infty$ and $\mathcal{M}_\infty^2$.

The lattice generated by the six endpoints of three consecutive quotients in a sequence of transposes is in fact generated by three elements and thus a homomorphic image of the free modular lattice on three generators which has 28 elements. For a normal sequence the lattice generated by the endpoints of three consecutive quotients is a homomorphic image of a lattice with 15 elements. For a strongly normal sequence this number is reduced to 10. D. X. Hong further develops the theory of projectivity by showing how these various lattices generated by three consecutive quotients can fit together.

Chapter I of this thesis proves a slight extension of Hong's theorem. Chapter II studies the structure of a modular lattice generated by four elements satisfying certain relations. It is shown that any such lattice contains as a sublattice one of three specific lattices. Chapter III applies the result of Chapters I and II to prove that a modular subdirectly irreducible lattice is weakly atomic if it does not have any of the lattices $A_2, A_3, \ldots, A_{10}$ diagramed in Chapter III as a homomorphic image of a sublattice. Chapter IV applies the first three chapters to prove that a modular subdirectly irreducible lattice, which does not have any of $A_2, \ldots, A_{10}$ as a homomorphic image of a sublattice, has width not exceeding four. Chapter V applies this result to derive the applications mentioned above.

General references to lattice theory are [2] and [6], to universal algebra [5], [12], and [19], and to the theory of varieties [20].

CHAPTER I

HONG'S THEOREM

We begin with several definitions. Let L be a modular lattice. An ordered pair (a, b) in $L \times L$ with $b \geq a$ will be called a quotient of L. Instead of (a, b) we shall write b/a for this quotient. We shall use the term quotient and the symbol b/a to denote the sublattice of L consisting of the elements in the set $\{x \in L | a \leq x \leq b\}$. This will sometimes be referred to as a quotient sublattice. The quotient b/a is called a nontrivial quotient if $b > a$. f/e is a subquotient of b/a if $a \leq e \leq f \leq b$. If b/a and d/c are quotients in L we write $b/a \nearrow d/c$ and we say that b/a transposes up to d/c if $a = b \wedge c$ and $d = b \vee c$. In this situation we also say that d/c transposes down to b/a, written $d/c \searrow b/a$. We also say that b/a and d/c are transposes.

The quotient b/a is said to be projective to d/c in n steps if there exists a sequence of quotients $b/a = b_0/a_0, b_1/a_1, \ldots, b_n/a_n = d/c$ such that b_k/a_k and b_{k+1}/a_{k+1} are transposes, $k = 0, 1, \ldots, n-1$.

Much of the following notation is taken from [14] and [16]. The projective distance between b/a and d/c, written p. d. (b/a, d/c), is the smallest integer n such that there are nontrivial subquotients b_1/a_1 of b/a and d_1/c_1 of d/c which are projective in n steps. If no such integer exists then we write p. d. $(b/a, d/c) = \infty$.

A sequence of transposes $b_0/a_0, b_1/a_1, \ldots, b_n/a_n$ is called normal if the transposes alternate up and down and

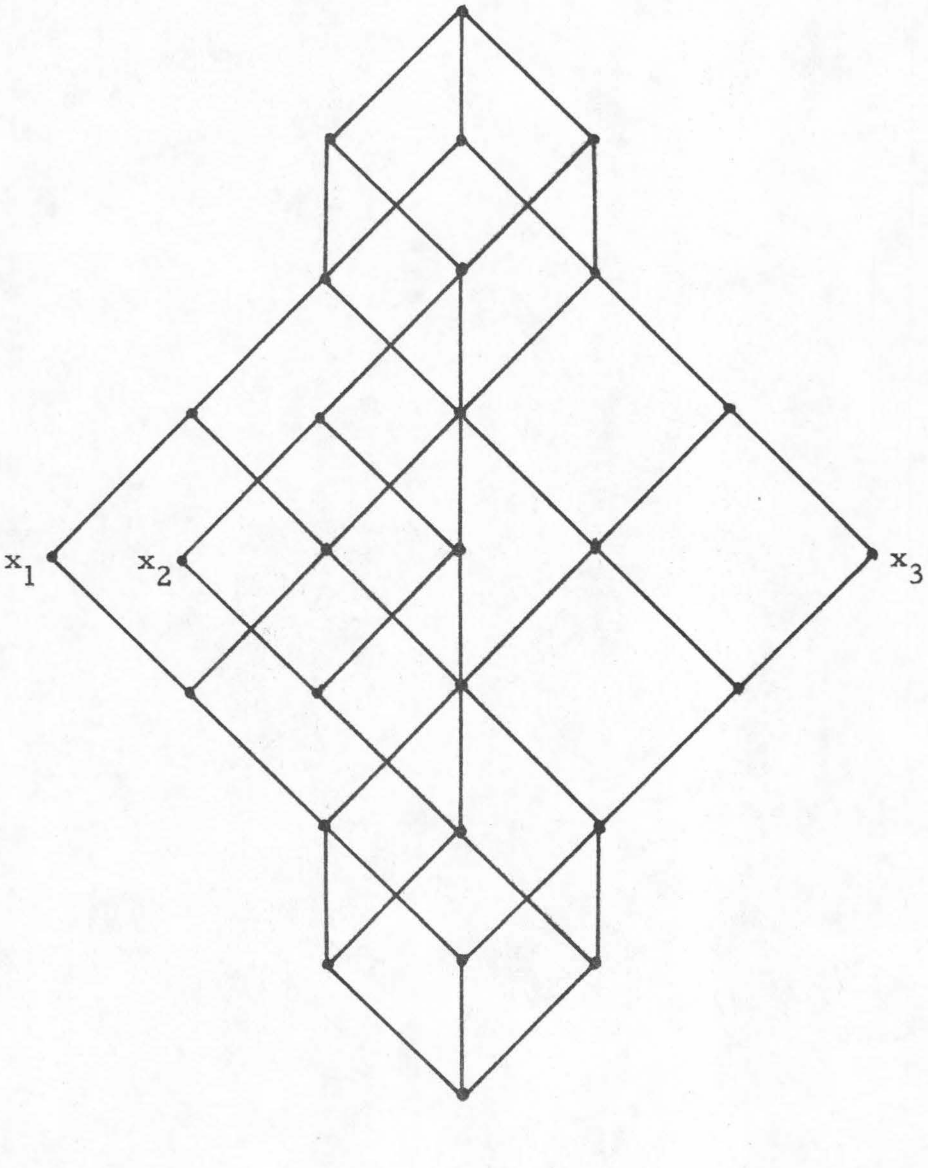
$b_{k-1}/a_{k-1} \nearrow b_k/a_k \searrow b_{k+1}/a_{k+1}$ implies $b_k = b_{k-1} \vee b_{k+1}$ and $b_{k-1}/a_{k-1} \searrow b_k/a_k \nearrow b_{k+1}/a_{k+1}$ implies $a_k = a_{k-1} \wedge a_{k+1}$. The sequence is called strongly normal if it is normal and $b_{k-1}/a_{k-1} \nearrow b_k/a_k \searrow b_{k+1}/a_{k+1}$ implies $b_{k-1} \wedge b_{k+1} \leq a_k$ and $b_{k-1}/a_{k-1} \searrow b_k/a_k \nearrow b_{k+1}/a_{k+1}$ implies $a_{k-1} \vee a_{k+1} \geq b_k$.

Suppose we have a sequence of transposes

$$(1) \qquad b_0/a_0 \nearrow b_1/a_1 \searrow b_2/a_2 \nearrow \ldots \; b_n/a_n$$

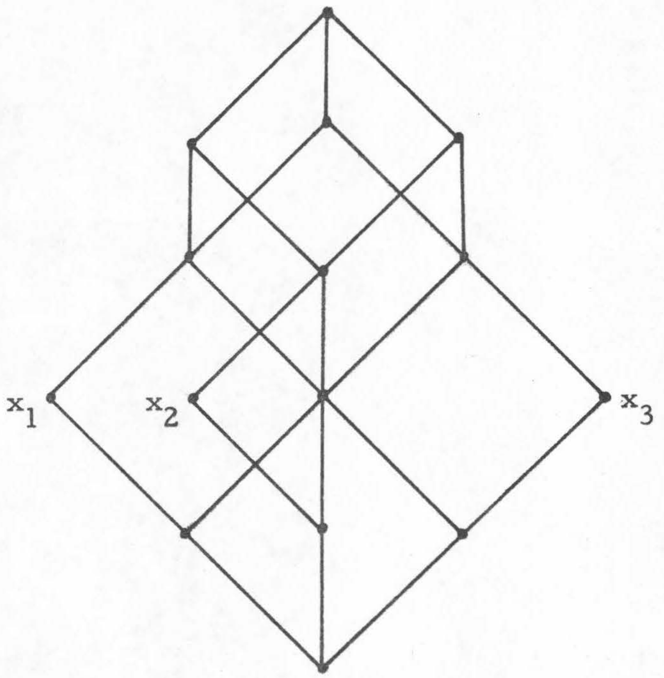
in a modular lattice. Since $a_0 = b_0 \wedge a_1$, $b_1 = b_0 \wedge a_1$, $a_2 = b_2 \wedge a_1$, $b_3 = b_2 \wedge a_3, \ldots$, the lattice L_1 generated by $a_0, b_0, a_1, b_1, \ldots$ is generated by $b_0, a_1, b_2, a_3, \ldots$. Thus L_1 is a homomorphic image of the free modular lattice on n generators, FM(n). This fact furnishes little information concerning the structure of L_1 when $n > 3$, since in this case, FM(n) is infinite. However, FM(3) is finite and has only a few homomorphic images. Hence useful information on the structure of L_1 can be obtained by considering consecutive sets of three quotients and determining the various ways in which the corresponding images of FM(3) can fit together. FM(3) and some of its homomorphic images are exhibited below.

It follows immediately from the definition of normal sequence that the endpoints of consecutive quotients generate a lattice which is a homomorphic image of G_2 (Fig. 1.2) or its dual. If the sequence is strongly normal then the endpoints of three consecutive quotients generate a homomorphic image of G_3 or its dual. More specifically



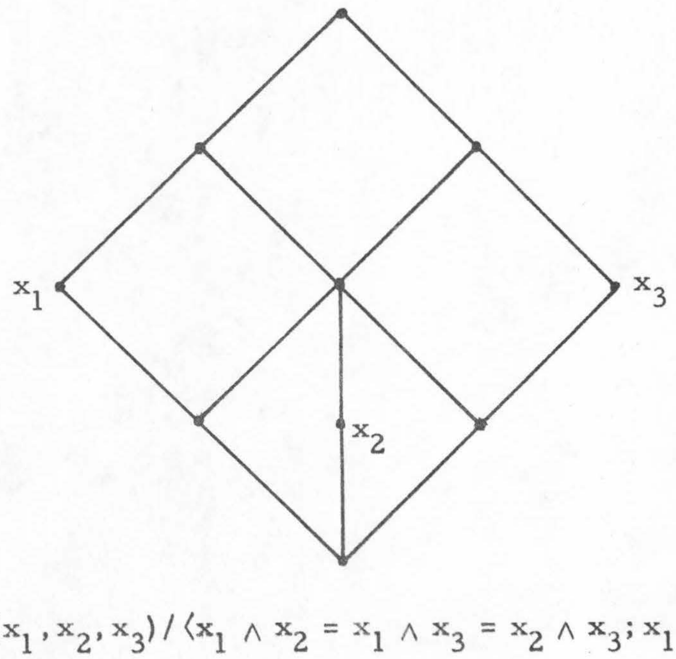
$G_1 = FM(x_1, x_2, x_3)$

Figure 1.1



$G_2 = FM(x_1, x_2, x_3)/\langle x_1 \wedge x_2 = x_1 \wedge x_3 = x_2 \wedge x_3 \rangle$

Figure 1.2



$G_3 = FM(x_1, x_2, x_3)/\langle x_1 \wedge x_2 = x_1 \wedge x_3 = x_2 \wedge x_3; x_1 \vee x_3 \geq x_2 \rangle$

Figure 1.3

if the sequence $b_{k-1}/a_{k-1} \searrow b_k/a_k \nearrow b_{k+1}/a_{k+1}$ is strongly normal then the lattice it generates is a homomorphic image of

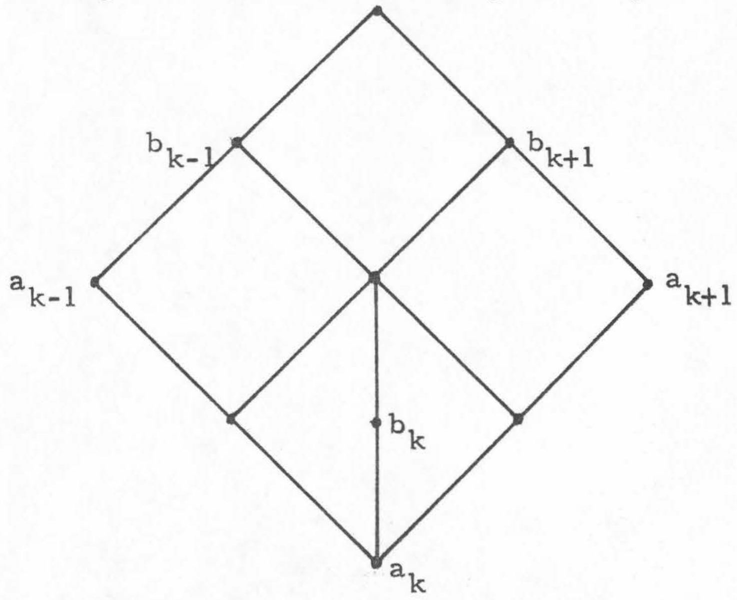


Figure 1.4

We denote the five element modular non-distributive lattice by M_3; M_3 with an addition atom is called M_4, etc.

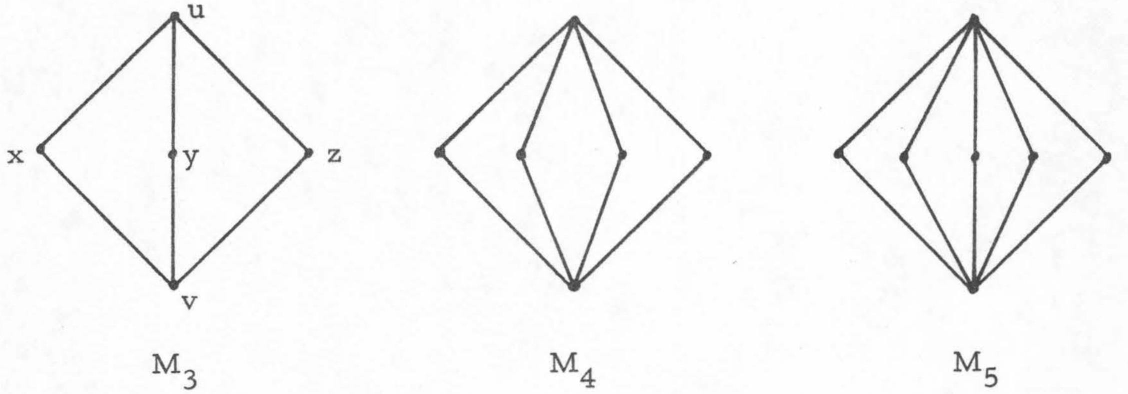


Figure 1.5

We call an ordered five-tuple (v, x, y, z, u) of elements from a modular lattice a diamond if these elements form a copy of M_3 with v and u as the bottom and top elements, respectively. Any nonidentity permutation of x, y and z yields a diamond, which by definition is distinct from the original diamond, even though they represent the same sublattice of L.

We see from Fig. 1.4 that if b_{k-1}/a_{k-1} is a nontrivial quotient then the figure contains a nontrivial diamond. More specifically, if $b_{k-1}/a_{k-1} \searrow b_k/a_k \nearrow b_{k+1}/a_{k+1}$ is part of a strongly normal sequence we let $D_k = (v_k, x_k, y_k, z_k, u_k) = (a_k, a_{k-1} \wedge b_{k+1}, b_k, a_{k+1} \wedge b_{k-1}, b_{k-1} \wedge b_{k+1})$ and if $b_{k-1}/a_{k-1} \nearrow b_k/a_k \searrow b_{k+1}/a_{k+1}$, $D_k = (v_k, x_k, y_k, z_k, u_k) = (a_{k-1} \vee a_{k+1}, b_{k-1} \vee a_{k+1}, a_k, b_{k+1} \vee a_{k-1}, b_k)$. In this way a strongly normal sequence $b_0/a_0, b_1/a_1, \ldots, b_n/a_n$ of n+1 quotients generates a sequence of n - 1 diamonds $D_1, D_2, \ldots, D_{n-1}$ which is called the <u>associated sequence of diamonds</u>.

The remainder of this chapter will be devoted to the proof of a theorem which extends slightly a result of D. X. Hong on the structure of the lattice generated by two consecutive diamonds in an associated sequence. In order to state the theorem concisely the following notation will be used. The diamond $D_1 = (v_1, x_1, y_1, z_1, u_1)$ is said to <u>translate up</u> to the diamond $D_2 = (v_2, x_2, y_2, z_2, u_2)$ if one of the quotients u_1/x_1, u_1/y_1, u_1/z_1 transposes up to one of the quotients x_2/v_2, y_2/v_2, z_2/v_2. The notation

$$D_1 \underset{(2)}{\nearrow} D_2$$

is used when u_1/z_1 transposes up to x_2/v_2 and

$$D_1 \underset{(2)}{\searrow} D_2$$

is used when z_1/v_1 transposes down to u_2/x_2. D_1 is said to <u>transpose down</u> to D_2 if $u_1/v_1 \searrow u_2/v_2$ and if $x_2 = u_2 \wedge x_1$, $y_2 = u_2 \wedge y_1$ and $z_2 = u_2 \wedge z_1$. The notation

$$D_1 \underset{(1)}{\searrow} D_2$$

means that D_1 transposes down to D_2.

If $D = (v, x, y, z, u)$ is a diamond then D^* is defined to be the diamond (v, z, x, y, u). So $D_1 \underset{(1)}{\searrow} D_2^*$ means $u_1/v_1 \searrow u_2/v_2$ and $x_1 \wedge u_2 = z_2$, $y_1 \wedge u_2 = x_2$ and $z_1 \wedge u_2 = y_2$. The theorem mentioned above can then be formulated as follows.

<u>Theorem 1.1</u> Let b/a and d/c be nontrivial quotients in a modular lattice L such that p. d. $(b/a, d/c) = n$, $2 < n < \infty$. Then some nontrivial subquotients $\overline{b}/\overline{a}$ and $\overline{d}/\overline{c}$ of b/a and d/c can be connected by a strongly normal sequence of transposes $\overline{b}/\overline{a} = b_0/a_0, b_1/a_1, \ldots, b_n/a_n = \overline{d}/\overline{c}$ such that the associated diamonds $D_1, \ldots, D_{n-1}$ satisfy:

(i) $D_k \underset{(1)}{\nearrow} D_{k+1}^*$ or $D_k \underset{(2)}{\nearrow} D_{k+1}$ if $b_k/a_k \nearrow b_{k+1}/a_{k+1}$ and
$D_k \underset{(1)}{\searrow} D_{k+1}^*$ or $D_k \underset{(2)}{\searrow} D_{k+1}$ if $b_k/a_k \searrow b_{k+1}/a_{k+1}$
$k = 1, 2, \ldots, n-2$

(ii) If $D_k \underset{(1)}{\nearrow} D_{k+1}^*$ or $D_k \underset{(1)}{\searrow} D_{k+1}^*$ then
$D_k = D_{k+1}^*$ $k = 2, \ldots, n-2$.

The proof of this theorem is a slight modification of Hong's proof. First we need

<u>Lemma 1.2</u> (B. Jónsson [16]). Let b/a and d/c be nontrivial quotients of a modular lattice L such that p. d. $(b/a, d/c) = n$, $2 < n < \infty$.

Then

(i) Any normal sequence of n transposes from b/a to d/c is also strongly normal.

(ii) There exist nontrivial subquotients $\overline{b}/\overline{a}$ and $\overline{d}/\overline{c}$ of b/a and d/c which can be connected by a strongly normal sequence of transposes.

We give a sketch of the proof. A detailed proof appears in [16].

Suppose $b/a = b_0/a_0, b_1/a_1, \ldots, b_n/a_n = d/c$ is a normal sequence. Then, as mentioned above, the lattice generated by a_{k-1}, b_{k-1}, a_k, b_k, a_{k+1}, b_{k+1} is a homomorphic image of G_2 or its dual (Fig. 1.2). Since $n > 2$ either $k - 1 > 1$ or $k+1 < n$. Assume the former. Then L contains the configuration pictured below:

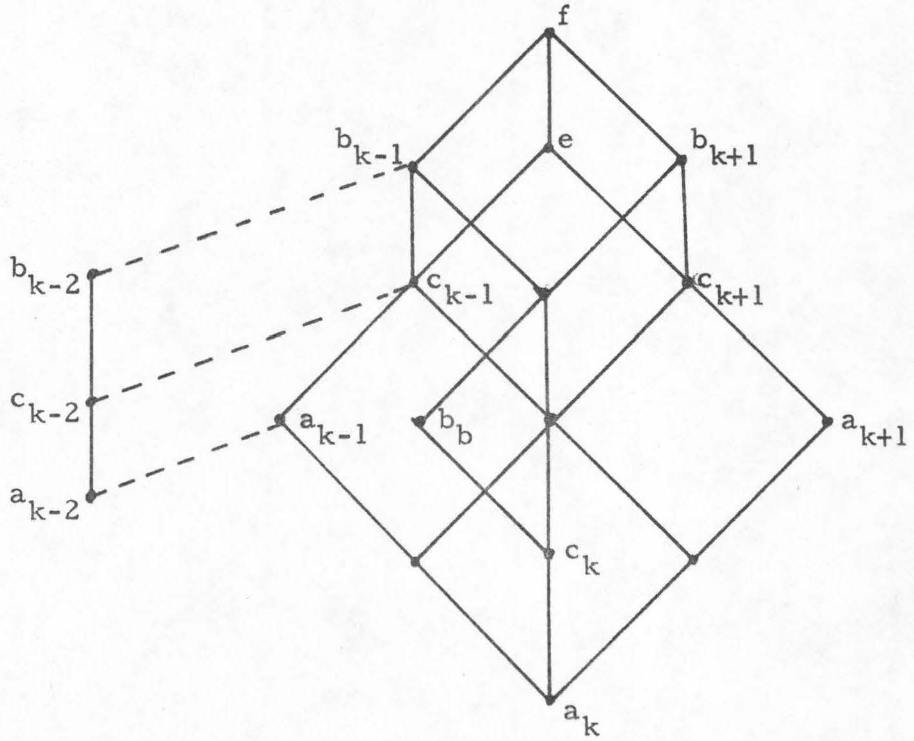


Figure 1.6

It is easily checked that $b_{k-1}/a_{k-1} \searrow b_k/a_k \nearrow b_{k+1}/a_{k+1}$ is strongly normal if and only if $c_{k-1} = b_{k-1}$. But if $c_{k-1} \neq b_{k-1}$, let c_i be the image of c_{k-1} in b_i/a_i. Then since $b_{k-2}/c_{k-2} \nearrow f/e \searrow b_{k+1}/c_{k+1}$, we have p. d. $(b/a, d/c) \leq n-1$, contrary to assumption.

To prove (ii) we take a sequence of n transposes connecting subquotients of b/a and d/c, which we know exists by definition of projective distance. It is an easy matter to replace this sequence by a normal sequence (see [13] or [21]), which, by (i), must be strongly normal.

The following lemma characterizing direct product sublattices will be needed in the proof of Theorem 1.1.

The Direct Product Lemma. If L_1 and L_2 are sublattices of a modular lattice L with greatest elements u_1 and u_2 and a common least element v such that $u_1 \wedge u_2 = v$, then the lattice generated by L_1 and L_2 is isomorphic to the direct product of L_1 and L_2.

Proof. First we show that if $a_i, b_i \in L_i$, $i = 1, 2$,

$$(1) \qquad (a_1 \vee a_2) \wedge (b_1 \vee b_2) = (a_1 \wedge b_1) \vee (a_2 \wedge b_2)$$

For

$$\begin{aligned}(a_1 \vee a_2) \wedge (b_1 \vee b_2) &= (a_1 \vee a_2) \wedge (b_1 \vee b_2) \wedge (b_1 \vee u_2) \\ &= \big([a_1 \wedge (b_1 \vee u_2)] \vee a_2\big) \wedge (b_1 \vee b_2) \\ &= \big([a_1 \wedge u_1 \wedge (b_1 \vee u_2)] \vee a_2\big) \wedge (b_1 \vee b_2) \\ &= \big([a_1 \wedge (b_1 \vee (u_1 \wedge u_2))] \vee a_2\big) \wedge (b_1 \vee b_2) \\ &= \big((a_1 \wedge b_1) \vee a_2\big) \wedge (b_1 \vee b_2)\end{aligned}$$

$$= (a_1 \wedge b_1) \vee \big(a_2 \wedge (b_1 \vee b_2)\big)$$
$$= (a_1 \wedge b_1) \vee \big(a_2 \wedge u_2 \wedge (b_1 \vee b_2)\big)$$
$$= (a_1 \wedge b_1) \vee (a_2 \wedge b_2)$$

With the aid of this it is easy to show that $(x_1, x_2) \to x_1 \vee x_2$ is an isomorphism of $L_1 \times L_2$ onto the sublattice generated by L_1 and L_2. For example, to show the map is one-to-one, let $a_1 \vee a_2 = b_1 \vee b_2$. Then $u_1 \wedge (a_1 \vee a_2) = u_1 \wedge (b_1 \vee b_2)$ which by (1) gives $a_1 = b_1$. Similarly $a_2 = b_2$.

The proof of Theorem 1.1 will be preceded by some lemmas which are more easily stated with the following notation.

Let $D = (v, x, y, z, u)$ be a diamond. We call u/x, u/y and u/z upper quotients of D and x/v, y/v and z/v lower quotients of D.

Suppose b/a is a subquotient of an upper or lower quotient of D, say $z \leq a \leq b \leq u$. If we assume that $z < a < b < u$ then the lattice generated by a, b and D is isomorphic to the lattice diagramed below (see [16]).

This lattice has three new diamonds as sublattices. We denote the upper-most diamond by $D_{u/b}$, the middle one by $D_{b/a}$ and the lowest diamond by $D_{a/z}$. More formally we have

$$D_{u/b} = \big((x \wedge b) \vee (y \wedge b),\ x \vee (y \wedge b),\ y \vee (x \wedge b),\ b,\ u\big)$$

(1) $$D_{b/a} = \big((x \wedge a) \vee (y \wedge a),\ (x \wedge b) \vee (y \wedge a),\ (x \wedge a) \vee (y \wedge b),\ a \wedge [(x \wedge b) \vee (y \wedge b)],\ (x \wedge b) \vee (y \wedge b)\big)$$

$$D_{a/z} = \big(v,\ x \wedge a,\ y \wedge a,\ z \wedge (x \vee (y \wedge a),\ (x \wedge a) \vee (y \wedge a)\big)$$

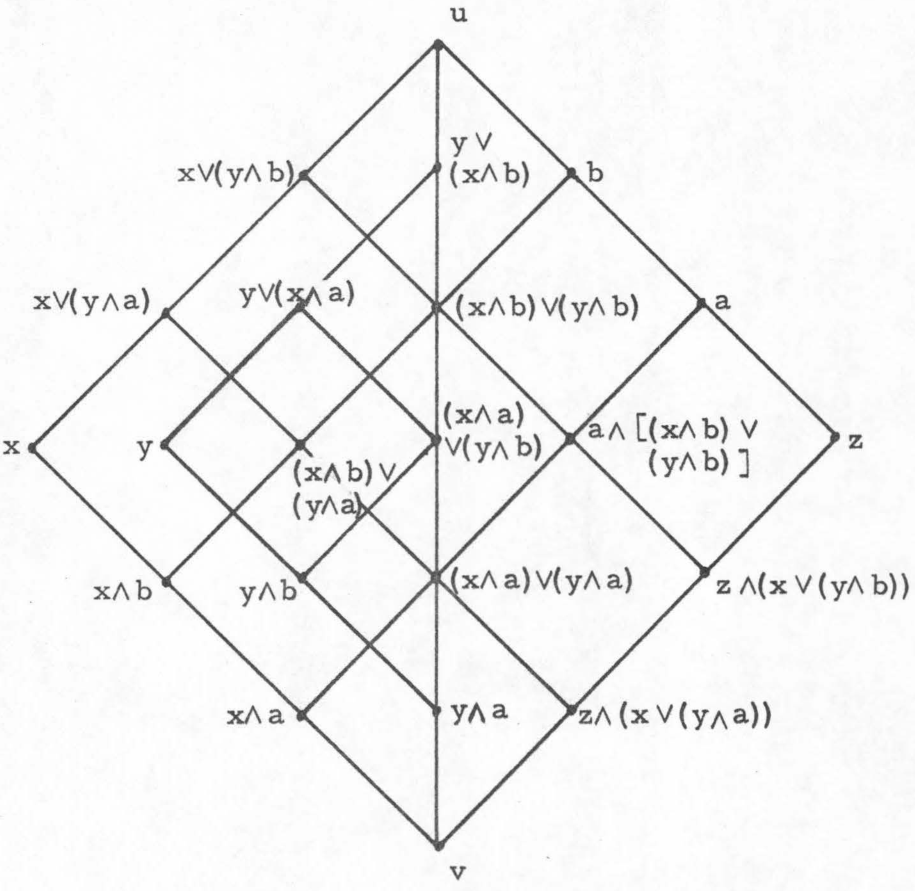


Figure 1.7

With these equations the definitions of $D_{u/b}$, $D_{b/a}$ and $D_{a/z}$ can be extended to include the possibilities u = b, b = a, or a = z. If u = b then the elements of $D_{u/b}$ are all the same; that is, $D_{u/b}$ is a single element. In this case $D_{a/b}$ is called a degenerate diamond. It should also be noted that this is the only way in which $D_{u/b}$ can be degenerate; that is, if u ≠ b the five elements of $D_{u/b}$ are distinct. Similar remarks apply for $D_{b/a}$ and $D_{a/z}$.

Similarly three diamonds (some of which may be possibly degenerate) are obtained if b/a is a subquotient of any upper or lower quotient of D. As an illustration note that if b/a is a subquotient of u/z then $x \wedge b/x \wedge a$ is a subquotient of x/v and $z \wedge (x \vee (y \wedge b))/z \wedge (x \vee (y \wedge a))$ is a subquotient z/v. It is easily checked that the diamonds $D_{b/a}$, $D_{x \wedge b/x \wedge a}$, and $D_{z \wedge (x \vee (y \wedge b))/z \wedge (x \vee (y \wedge a))}$ are the same.

The next few lemmas are due to Hong [14].

Lemma 1.3. If $D = (v, x, y, z, u)$ and $D' = (v', x', y', z', u')$ are diamonds in L with $u = u'$, $x \leq x'$, $y \leq y'$, $z \leq z'$ then $D' = D_{u/x'} = D_{u/y'} = D_{u/z'}$.

Proof: Taking $b = u$ and $z = z'$ in (1) gives

$$D_{u/z'} = \Big((x \wedge z') \vee (y \wedge z'),\ x \vee (y \wedge z'),\ y \vee (x \wedge z'),\ z',\ u\Big)$$

Now

$$\begin{aligned}(x \wedge z') \vee (y \wedge z') &= \Big((x \wedge z') \vee y\Big) \wedge z' \\ &= \Big((x \wedge x' \wedge z') \vee y\Big) \wedge z' \\ &= \Big((x \wedge v') \vee y\Big) \wedge z' \\ &= \Big((x \wedge x' \wedge y') \vee y\Big) \wedge z' \\ &= \Big((x \wedge y') \vee y\Big) \wedge z' \\ &= (x \vee y) \wedge y' \wedge z' = u \wedge v' = v'\end{aligned}$$

$$\begin{aligned}x \vee (y \wedge z') &= x \vee (x \wedge z') \vee (y \wedge z') \\ &= x \vee v' \\ &= x \vee (x' \wedge y')\end{aligned}$$

$$= x' \wedge (x \vee y')$$

$$= x' \wedge u = x'$$

Similarly $y \vee (x \wedge z') = y'$. So $D' = D_{u/z'}$. The other statements in the lemma follow by symmetry.

Corollary 1.4. Let

(1) $$b_{k-1}/a_{k-1} \nearrow b_k/a_k \searrow b_{k+1}/a_{k+1}$$

be a strongly normal sequence with associated diamond D. Let $c_i \in b_i/a_i$, $i = k-1, k, k+1$, be images of one another under the given transpositions. Let b'_{k-1}/c'_{k-1} and b'_{k+1}/c'_{k+1} be quotients such that

(2) $$b_{k-1}/c_{k-1} \nearrow b'_{k-1}/c'_{k-1} \nearrow b_k/c_k \searrow b'_{k+1}/c'_{k+1} \searrow b_{k+1}/c_{k+1}$$

and

(3) $$b'_{k-1}/c'_{k-1} \nearrow b_k/c_k \searrow b'_{k+1}/c'_{k+1}$$

is strongly normal. Then the diamond associated with (3) is $D_{b_k/c_k} = D_{u/c_k}$.

Proof: It is easily checked that the diamonds associated with (1) and (3) satisfy the hypothesis of Lemma 1.3. The corollary readily follows.

Corollary 1.5. Let

(1) $$b_{k-1}/a_{k-1} \searrow b_k/a_k \nearrow b_{k+1}/a_{k+1}$$

be a strongly normal sequence in L with associated diamond D. Let $c_i \in b_i/a_i$, $i = k-1, k, k+1$, be images of one another under the given transpositions. Then

$$(2) \qquad b_{k-1}/c_{k-1} \searrow b_k \vee (_{k-1} \wedge c_{k+1})/c_{k-1} \wedge c_{k+1} \nearrow b_{k+1}/c_{k+1}$$

is strongly normal with associated diamond $D_{b_k/c_k} = D_{y/c_k}$.

Proof: The strong normality of (1) easily implies the strong normality of (2). The diamonds associated with (1) and (2) are

$$D = (a_k, a_{k-1} \wedge b_{k+1}, b_k, b_{k-1} \wedge a_{k+1}, b_{k-1} \wedge b_{k+1})$$

$$D' = \left(c_{k-1} \wedge c_{k+1}, c_{k-1} \wedge b_{k+1}, b_k \vee (c_{k-1} \wedge c_{k+1}),\right.$$
$$\left. b_{k-1} \wedge c_{k+1}, b_{k-1} \wedge b_{k+1}\right)$$

These satisfy the hypothesis of Lemma 1.3, and thus $D' = D_{b_{k-1} \wedge b_{k+1}/b_{k-1} \wedge c_{k+1}}$. But $b_k \wedge (b_{k-1} \wedge c_{k+1}) = b_k \wedge c_{k+1} = c_k$. Thus by the remark preceding Lemma 1.3, $D' = D_{b_k/c_k}$.

The following lemma of Hong is the key to the proof of Theorem 1.1.

Lemma 1.6. Suppose

$$b_0/a_0 \nearrow b_1/a_1 \searrow b_2/a_2 \nearrow b_3/a_3$$

is a strongly normal sequence such that

$$\text{p.d.}\,(b_0/a_0, b_3/a_3) = 3$$

Then the associated diamonds,

$$D_1 = (v_1, x_1, y_1, z_1, u_1) = (a_0 \vee a_2,\ b_0 \vee a_2,\ a_1,\ a_0 \vee b_2,\ b_1)$$

$$D_2 = (v_2, x_2, y_2, z_2, u_2) = (a_2,\ a_1 \wedge b_3,\ b_2,\ b_1 \wedge a_3,\ b_1 \wedge b_3)$$

satisfy

$$D_1 \underset{(1)}{\searrow} D_2^*$$

or else one of the following holds:

(i) There exists c_0, $a_0 \leq c_0 < b_0$ such that if $c_i \in b_i/a_i$ is the image of c_0 under the given transpositions, $i = 1, 2, 3$ then

$$b_0/c_0 \nearrow b_1/c_1 \searrow b_2 \vee (c_1 \wedge c_3)/c_1 \wedge c_3 \nearrow b_3/c_3$$

is a strongly normal sequence with associated diamonds $(D_1)_{u_1/c_1} = (D_1)_{b_1/c_1}$ and $(D_2)_{y_2/c_2} = (D_2)_{b_2/c_2}$ with $(D_1)_{b_1/c_1} \underset{(2)}{\searrow} (D_2)_{b_2/c_2}$.

(ii) There exists c_0, $a_0 < c_0 \leq b_0$, c_i $i = 1, 2, 3$, the images of c_0 in b_i/c_i under the given transpositions such that

$$c_0/a_0 \nearrow c_0 \vee c_2/a_1 \wedge (c_0 \vee c_2) \searrow c_2/a_2 \nearrow c_3/a_3$$

is a strongly normal sequence with associated diamonds $(D_1)_{c_1/a_1} = (D_1)_{c_1/y_1}$ and $(D_2)_{c_2/a_2} = (D_2)_{c_2/v_2}$ with $(D_1)_{c_1/a_1} \underset{(2)}{\searrow} (D_2)_{c_2/a_2}$.

<u>Proof:</u> Note the following relations hold.

$$(1) \qquad \begin{array}{ll} v_1 \vee y_2 = z_1 & u_2 \wedge y_1 = x_2 \\ z_1 \leq v_1 \vee u_2 \leq u_1 & v_2 \leq u_2 \wedge v_1 \leq x_2 \end{array}$$

Hence either $v_1 \vee u_2 < u_1$ or $v_2 < u_2 \wedge v_1$ or else $v_1 \vee u_2 = u_1$ and $v_2 = u_2 \wedge v_1$. So we have three cases.

Case 1: $v_1 \vee u_2 = u_1$ and $v_1 \wedge u_2 = v_2$, so that

(2) $$u_2/v_2 \nearrow u_1/v_1$$

From (1) we see that this transposition maps y_2 onto z_1 and x_2 onto y_1. If this transposition sends z_2 onto x_1, i.e., if $z_2 \vee v_1 = x_1$ then $D_1 \searrow_{(1)} D_2^*$, as asserted. So let $x_1' = z_2 \vee v_1$ and suppose $x_1' \neq x_1$. Note that y_1 is a relative complement to both x_1 and x_1' in u_1/v_1. Thus x_1 and x_1' are incomparable.

Note that

(3) $$u_1/x_1' \searrow u_2/z_2 \nearrow b_3/a_3$$

and $u_1 \wedge b_3 = u_2$. It follows easily from the Direct Product Lemma that the lattice generated by u_1, x_1', u_2, z_2, b_3, a_3 is an eight-element Boolean algebra. Consequently

(4) $$u_1/x_1' \nearrow u_1 \vee b_3/x_1' \vee a_3 \searrow b_3/a_3$$

Now it is easily checked that

(5) $$b_0/b_0 \wedge x_1' \nearrow x_1/x_1 \wedge x_1' \nearrow x_1 \vee x_1'/x_1' \nearrow$$
$$x_1 \vee x_1' \vee a_3/x_1' \vee a_3 \searrow b_3 \wedge (x_1 \vee x_1' \vee a_3)/a_3$$

Since x_1 and x_1' are incomparable these quotients must be nontrivial. Thus we have p.d. $(b_0/a_0, b_3/a_3) \leq 2$, a contradiction.

Case 2: $v_1 \vee u_2 < u_1$. Let $w = v_1 \vee u_2$ and $c_1 = y_1 \vee (x_1 \wedge w)$ and let $c_i \in b_i/a_i$, $i = 0, 2, 3$ be the images of c_1 under the given transpositions. Consider

$$b_0/c_0 \nearrow b_1/c_1 \searrow b_2 \vee (c_1 \wedge c_3)/c_1 \wedge c_3 \nearrow b_3/c_3$$

This is clearly a normal sequence and so by Lemma 1.2 it is a strongly normal sequence. By Corollary 1.5 the associated diamonds are

$$(D_1)_{u_1/c_1} = (D_1)_{u_1/w} = \Big((x_1 \wedge w) \vee (y_1 \wedge w),\ x_1 \vee (y_1 \wedge w),\ c_1,\ w,\ u_1\Big)$$

$$(D_2)_{y_2/c_2} = \Big(c_1 \wedge c_3,\ c_2 \vee x_2,\ b_2 \vee (c_1 \wedge c_3),\ c_2 \vee z_2,\ u_2\Big)$$

Now $u_2 \leq w \leq (x_1 \vee u_2) \wedge (y_1 \vee u_2)$ and $x_2 \leq y_1$ so that

$$(x_1 \wedge w) \vee (y_1 \wedge w) \vee u_2 = \Big((x_1 \vee u_2) \wedge w\Big) \vee \Big((y_1 \vee u_2) \wedge w\Big) = w$$

$$\begin{aligned}\Big[(x_1 \wedge w) \vee (y_1 \wedge w)\Big] \wedge u_2 &= \Big[\Big((x_1 \wedge w) \vee y_1\Big) \wedge w\Big] \wedge u_2 \\ &= \Big((x_1 \wedge w) \vee y_1\Big) \wedge u_2 \\ &= \Big((x_1 \wedge w) \vee y_1\Big) \wedge (x_2 \vee y_2) \\ &= x_2 \vee \Big((x_1 \wedge w) \vee y_1\Big) \wedge y_2 \\ &= x_2 \vee (c_1 \wedge b_2) \\ &= x_2 \vee c_2\end{aligned}$$

Thus $w/(x_1 \wedge w) \vee (y_1 \wedge w) \searrow u_2/c_2 \vee x_2$ and thus (i) holds.

Case 3: $v_2 < v_1 \wedge u_2$. If we reverse the order of the reference of b_i/a_i and apply the dual of Case 2, we get the third alternative of the lemma.

Lemma 1.7. Suppose $D_i = (v_i, x_i, y_i, z_i, u_i)$, $i = 1, 2$ are two diamonds in L such that either

(1) $$D_1 \nearrow_{(1)} D_2^*$$

or

(2) $$D_1 \nearrow_{(2)} D_2 \ .$$

Let $c_1 \in y_1/v_1$ and let $c_2 = c_1 \vee y_2$ be its image in u_2/y_2. Then

(i) $(D_1)_{y_1/c_1} \nearrow_{(1)} (D_2^*)_{u_2/c_2}$ if (1) holds

(ii) $(D_1)_{y_1/c_1} \nearrow_{(2)} (D_2)_{u_2/c_2}$ if (2) holds.

Furthermore, if $D_1 = D_2^*$ then $(D_1)_{y_1/c_1} = (D_2^*)_{u_2/c_2}$.

Proof: Let us suppose that (2) holds.

$$x_2 \vee (c_2 \wedge z_2)/(x_2 \wedge c_2) \vee (c_2 \wedge z_2) \searrow x_2/x_2 \wedge c_2 \searrow u_1/u_1 \wedge c_2$$

Hence

$$(D_1)_{u_1/u_1 \wedge c_2} \nearrow_{(2)} (D_2)_{u_2/c_2}$$

Now $y_1 \wedge (u_1 \wedge c_2) = c_2 \wedge y_1 = c_1$ and thus by the remark preceding Lemma 1.3 $(D_1)_{u_1/u_1 \wedge c_2} = (D_1)_{y_1/c_1}$. This gives conclusion (i).

Let us suppose (1) holds. Then $y_1 \vee v_2 = x_2$, $z_1 \vee v_2 = y_2$ and $x_1 \vee v_2 = z_2$. Hence

(3) $$\begin{aligned} z_2 \vee (x_2 \wedge c_2) &= z_2 \vee [(y_1 \vee v_2) \wedge c_2] \\ &= z_2 \vee v_2 \vee (y_1 \wedge c_2) \\ &= z_2 \vee (y_1 \wedge c_2) \end{aligned}$$

Recall that

$$(D_2)_{u_2/c_2} = \Big((x_2 \wedge c_2) \vee (z_2 \wedge c_2),\ x_2 \vee (z_2 \wedge c_2),\ c_2,\ z_2 \vee (x_2 \wedge c_2),\ u_2\Big)$$

Now by (3)

$$\begin{aligned} u_1 \wedge [(x_2 \wedge c_2) \vee (z_2 \wedge c_2)] &= u_1 \wedge c_2 \wedge \left(z_2 \vee (x_2 \wedge c_2)\right) \\ &= u_1 \wedge c_2 \wedge \left(z_2 \vee (y_1 \wedge c_2)\right) \\ &= u_1 \wedge c_2 \wedge (x_1 \vee y_1) \wedge \left(z_2 \vee (y_1 \wedge c_2)\right) \\ &= u_1 \wedge c_2 \wedge \left(x_1 \vee y_1 (z_2 \vee (y_1 \wedge c_2))\right) \\ &= u_1 \wedge c_2 \wedge \left(x_1 \vee (y_1 \wedge c_2)\right) \end{aligned}$$

Also

$$\begin{aligned} u_1 \wedge [z_2 \vee (x_2 \wedge c_2)] &= (x_1 \vee y_1) \wedge \left(z_2 \vee (y_1 \wedge c_2)\right) \\ &= x_1 \vee \left(y_1 \wedge (z_2 \vee y_1 \wedge c_2)\right) \\ &= x_1 \vee (y_1 \wedge c_2) \end{aligned}$$

Similarly $u_1 \wedge \left(x_2 \vee (z_2 \wedge c_2)\right) = y_1 \vee (x_1 \wedge c_2)$.

These calculations show that

$$(D_2^*)_{u_2/c_2} \searrow (D_1)_{u_1/u_1 \wedge c_2}$$

But we have already seen that $(D_1)_{u_1/u_1 \wedge c_2} = (D_1)_{y_1/c_1}$. Hence (ii) holds. The last statement of the Lemma is obvious.

One more lemma is needed.

Lemma 1.8. Let $b_0/a_0 \nearrow b_1/a_1 \searrow b_2/a_2 \nearrow b_3/a_3 \searrow b_4/a_4$ be a strongly normal sequence with associated diamonds D_1, D_2, and D_3 and let p. d. $(b_0/a_0, b_4/a_4) = 4$. Then at least one of the relations

(1) $$D_1 \underset{(1)}{\searrow} D_2^*$$
$$D_2 \underset{(1)}{\nearrow} D_3^*$$

fails to hold.

Proof: Suppose both relations hold. Since $u_1 \wedge u_3 = b_1 \wedge b_3 = u_2$, it follows from the Direct Product Lemma that $z_1, u_1, y_2, u_2, x_3, u_3$ generates an eight element Boolean algebra. Whence

$$b_0/a_0 \nearrow x_1/v_1 \nearrow u_1/z_1 \nearrow u_1 \vee u_3/z_1 \vee x_3 \searrow u_3/x_3 \searrow z_3/v_3 \searrow b_4/a_4$$

But this clearly contradicts p. d. $(b_0/a_0, b_4/a_4) = 4$.

Proof of Theorem 1. 1: It will be convenient to make an induction on n. If $n = 3$ property (ii) holds automatically and (i) follows from Lemma 1. 6. Thus we may suppose that $3 < n =$ p. d. $(b/a, d/c)$ and that the theorem holds for pairs of quotients of projective distance less than four. Since p. d. $(b/a, d/c) = n$, subquotients b_0'/a_0' of b/a and b_n'/a_n' of d/c exist which can be connected by a sequence of n transposes. Thus b_0'/a_0' transposes to a quotient b_1'/a_1' which can be connected by a sequence of $n - 1$ transposes ($n - 1$ arrows) to b_n'/a_n'. By duality it will suffice to consider the case where $b_0'/a_0' \nearrow b_1'/a_1'$. By the induction hypothesis there exist subquotients b_1''/a_1'' of b_1'/a_1' and b_n/a_n of b_n'/a_n' which can be connected by a strongly normal sequence

(1) $$b_1''/a_1'' \searrow b_2/a_2 \nearrow b_3/a_3 \quad \ldots \; b_n/a_n$$

which satisfies all the conditions of Theorem 1. 1. (Note that

$b_1''/a_1'' \nearrow b_2/a_2$ would imply p. d. $(b/a, d/c) \leq n - 1$, a contradiction.)

Let $b_0 = b_0' \wedge b_1''$ and $a_0 = b_0' \wedge a_1''$ and $b_1 = b_0 \vee b_2$ and $a_1 = a_1'' \wedge b_1$. Then

(2) $$b_0/a_0 \nearrow b_1/a_1 \searrow b_2/a_2 \nearrow \ldots\ b_n/a_n$$

is normal and hence by Lemma 1.2 strongly normal. Let $D_i = (v_i, x_i, y_i, z_i, u_i)$, $i = 1, 2, \ldots, n-1$, be the diamonds associated with (2). Then by Corollary 1.4 $D_2, D_3, \ldots, D_{n-1}$ are the diamonds associated with (1).

Now we can apply Lemma 1.6 to

$$b_0/a_0 \nearrow b_1/a_1 \searrow b_2/a_2 \nearrow b_3/a_3 \quad .$$

If $D_1 \underset{(1)}{\searrow} D_2^*$ then property (i) of Theorem 1.1 holds. By Lemma 1.8 $D_2 \underset{(1)}{\nearrow} D_3^*$ cannot hold. So by our induction hypothesis $D_2 \underset{(2)}{\nearrow} D_3$ and if $D_k \underset{(1)}{\nearrow} D_{k+1}^*$ or $D_k \searrow D_{k+1}^*$ then $D_k = D_{k+1}^*$, $k = 3, \ldots, n-1$. Thus (ii) holds in this case.

So we may now assume that either condition (i) or (ii) of Lemma 1.6 applies. If condition (i) holds, then we get a sequence

(3) $$b_0/c_0 \nearrow b_1/c_1 \searrow b_2/c_2 \nearrow \ldots\ b_n/c_n$$

which can be normalized to

(4) $$b_0'/c_0' \nearrow b_1'/c_1' \searrow b_2'/c_2' \nearrow \ldots\ b_n'/c_n'$$

by letting $c_k' = c_{k-1} \wedge c_{k+1}$ and $b_k' = b_k \vee c_k'$ for the even and $0 < k < n$ and $c_k' = c_k$ and $b_k' = b_k$ otherwise. By Lemma 1.2 the sequence (4) is strongly normal. By Corollary 1.4 and Corollary 1.5 the diamonds associated with (4) are $(D_1)_{b_1/c_1}, (D_2)_{b_2/c_2}, \ldots, (D_{n-1})_{b_{n-1}/c_{n-1}}$. By Lemma 1.6, (i) $(D_1)_{b_1/c_1} \underset{(2)}{\searrow} (D_2)_{b_2/c_2}$. Applying Lemma 1.7 to $D_2, D_3, \ldots, D_{n-1}$ we see that the rest of the diamonds associated with (4) satisfy (i) of Theorem 1.1. We may suppose that $D_2 \underset{(1)}{\nearrow} D_3^*$ since otherwise (ii) holds by the induction assumptions. Thus the situation may be described as follows: there is a strongly normal sequence $f_0/e_0 \nearrow f_1/e_1 \searrow f_2/e_2 \nearrow \ldots f_n/e_n$, where $f_i = b_i'$ and $e_i = c_i'$, $i = 0, 1, \ldots, n$, and p. d. $(f_0/e_0, f_n/e_n) = n$. Furthermore the associated diamonds, which we again denote $D_i = (v_i, x_i, y_i, z_i, u_i)$, $i = 1, \ldots, n-1$, satisfy property (i) of the theorem, $D_1 \underset{(2)}{\searrow} D_2$, $D_2 \underset{(1)}{\nearrow} D_3^*$ and property (ii) holds for $i \geq 3$.

Since $f_1 = u_1$, $u_3 = f_3 \geq v_3$ and $u_2 = f_1 \wedge f_3$, we have

$$u_1 \wedge v_3 = f_1 \wedge f_3 \wedge v_3 = u_2 \wedge v_3 = v_2$$

Thus by the Direct Product Lemma the lattice generated by the sublattices u_1/v_2 and v_3/v_2 is isomorphic to their direct product. Hence we obtain two new diamonds $D_1' = (v_1', x_1', y_1', z_1', u_1') = (v_1 \vee v_3, x_1 \vee v_3, y_1 \vee v_3, z_1 \vee v_3, u_1 \vee v_3)$ and $D_2' = (v_2', x_2', y_2', z_2', u_2') = (v_2 \vee v_3, x_2 \vee v_3, y_2 \vee v_3, z_2 \vee v_3, u_2 \vee v_3) = (v_3, z_3, x_3, y_3, u_3) = D_3^*$. See Fig. 1.8.

Consider the sequence

$$(5) \quad f_0/e_0 \nearrow u_1'/y_1' \searrow x_3/v_3 \nearrow f_3/e_3 \searrow f_4/e_4 \nearrow f_5/e_5 \searrow \ldots f_n/e_n$$

The following calculations show that (5) is a normal sequence,

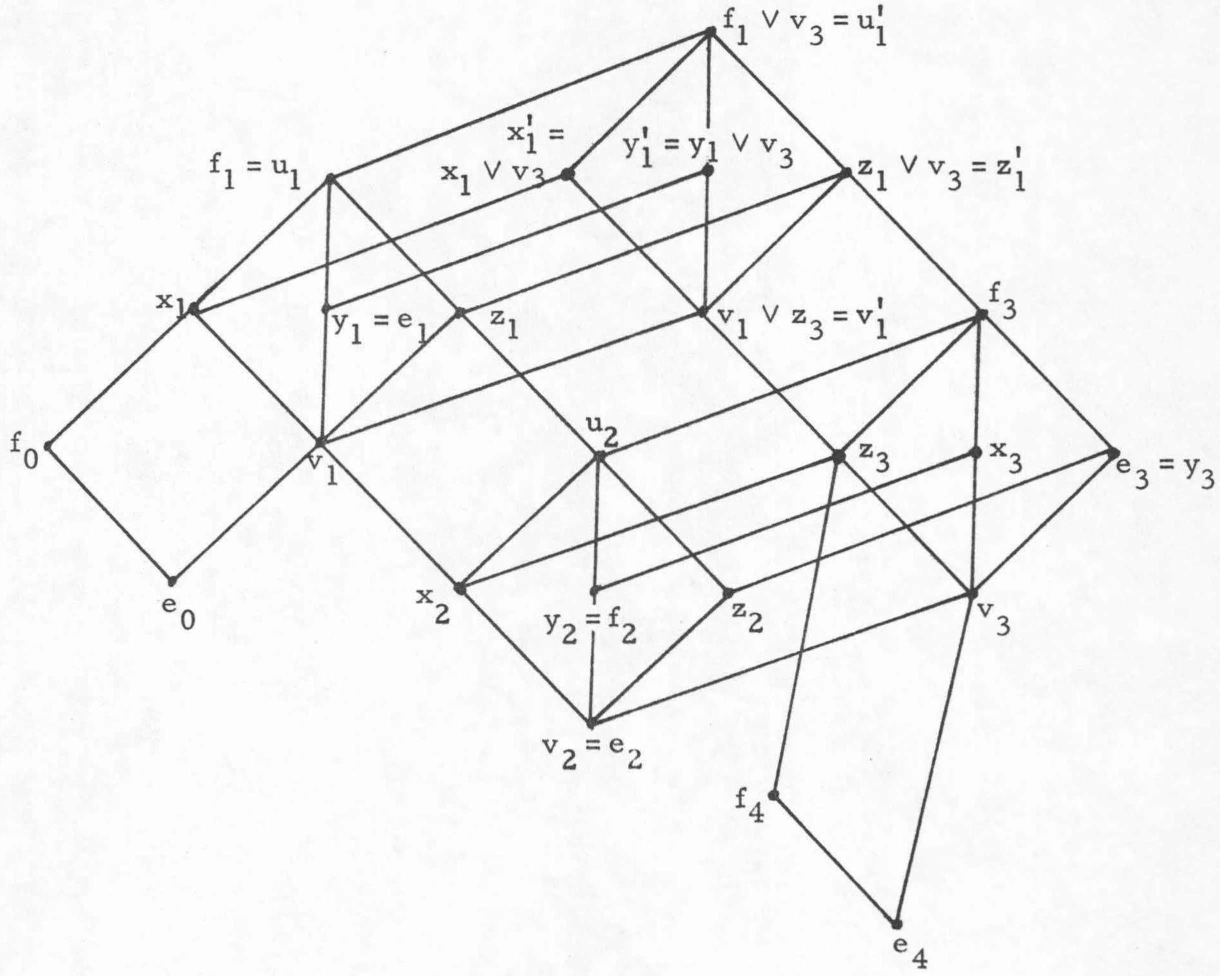


Figure 1.8

and hence, by Lemma 1.2, a strongly normal sequence. Since $f_2 = y_2 \le x_3$ and $f_1 = u_1$

(6) $$\begin{aligned} f_0 \vee x_3 &= f_0 \vee f_2 \vee x_3 \\ &= f_1 \vee x_3 \\ &= u_1' \end{aligned}$$

$$(7) \qquad y'_1 \quad e_3 = (y_1 \vee v_3) \quad e_3$$
$$= v_3 \vee (e_3 \quad y_1)$$
$$= v_3 \vee v_2$$
$$= v_3$$

Clearly $x_3 \vee f_4 = f_3$. The rest of the sequence is normal because the sequence (4) is normal.

It is easily checked that the diamonds associated with (5) are $D'_1, D'_2, D_3, D_4, \ldots, D_{n-1}$. Furthermore, the relations $D'_1 \searrow_{(2)} D'_2$ and $D'_2 = D^*_3$ are satisfied. Thus the sequence (5) satisfies properties (i) and (ii) of the theorem.

A similar argument applies if (ii) of Lemma 1.6 holds. Thus the proof of the theorem is complete.

CHAPTER II

SOME USEFUL MODULAR LATTICES WITH FOUR GENERATORS

In this chapter a theorem on modular lattices with four generators satisfying certain specific relations between the generators is proved. In addition, several corollaries are observed, which will be useful in Chapter III.

Let M_4 and A_4 be the lattices diagramed in Figure 2.1.

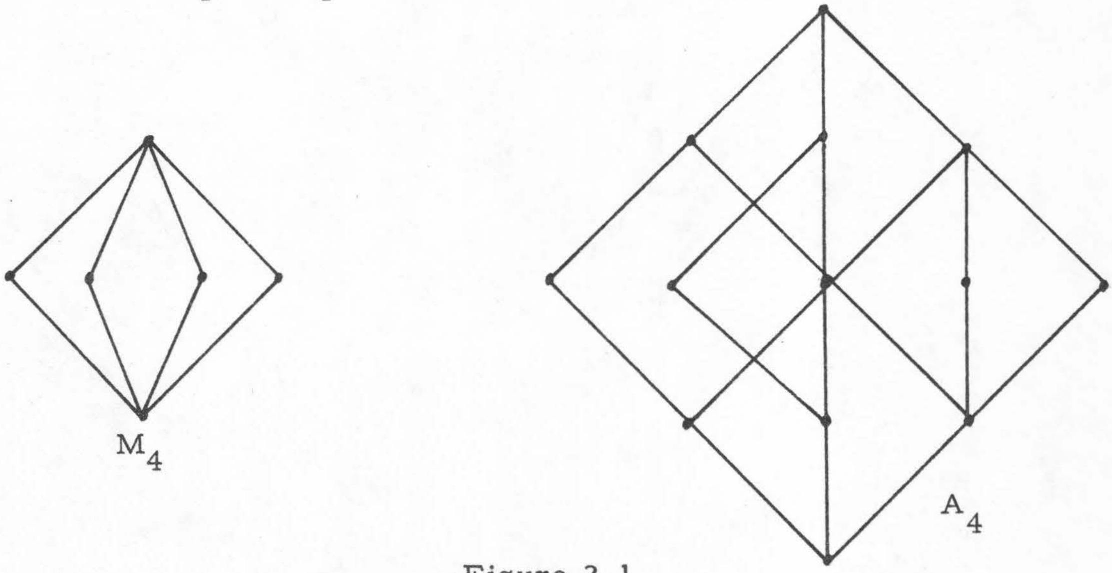


Figure 2.1

Theorem 2.1. Let L be a modular lattice with four distinct generators a, b, c, d which satisfy

(1) $a \vee b = a \vee c = a \vee d = b \vee d = c \vee d = a \vee b \vee c \vee d$

(2) $a \wedge b = a \wedge c = a \wedge d = b \wedge d = c \wedge d = a \wedge b \wedge c \wedge d$

Then either A_4 is isomorphic to a homomorphic image of a sublattice of L or L has a sublattice L' which is isomorphic to M_4 and if u is the greatest element of L' then, for one of the atoms x of L', u/x transposes up to a subquotient of $a \vee d/d$.

The hypotheses of the theorem just says that any pair of generators except possibly b and c join to the top element of L and any two except possibly b and c intersect to the bottom element of L.

Proof: We say that an ordered four-tuple (x, y, z, w) satisfies property (P) if x, y, z and w satisfy (1) with $x = a$, $y = b$, $z = c$ and $w = d$. The dual property, which is given by (2), is denoted (P^d).

Let $a_0 = a$, $b_0 = b$, $c_0 = c$, $d_0 = d$, $a_1 = a_0 \wedge (b_0 \vee c_0)$ and $d_1 = d_0 \wedge (b_0 \vee c_0)$. Then (b_0, a_1, d_1, c_0) satisfies (P). For example, $b_0 \vee a_1 = b_0 \vee (a_0 \wedge (b_0 \vee c_0)) = (b_0 \vee a_0) \wedge (b_0 \vee c_0) = b_0 \vee c_0 = b_0 \vee a_1 \vee d_1 \vee c_0$. Now if we set $b_1 = b_0 \wedge (a_1 \vee d_1)$ and $c_1 = c_0 \wedge (a_1 \vee d_1)$ then as above (a_1, b_1, c_1, d_1) satisfies (P). Inductively we define

$$a_{i+1} = a_i \wedge (b_i \vee c_i) \qquad d_{i+1} = d_i \wedge (b_i \vee c_i)$$

(3)

$$b_{i+1} = b_i \wedge (a_{i+1} \vee d_{i+1}) \qquad c_{i+1} = c_i \wedge (a_{i+1} \vee d_{i+1})$$

Thus we obtain four descending chains $a_0 \geq a_1 \geq a_2 \geq \ldots$, $b_0 \geq b_1 \geq \ldots$, $c_0 \geq c_1 \geq \ldots$, $d_0 \geq d_1 \geq \ldots$, such that (a_i, b_i, c_i, d_i) and $(b_i, a_{i+1}, d_{i+1}, c_i)$ satisfy (P).

Let $e_i = b_i \vee c_i$ and $f_i = a_i \vee d_i$. Then the lattice generated by e_i, $d_i \vee a_{i+1}$ and $d_{i+1} \vee a_i$ is a (possibly degenerate) diamond with greatest element f_i and least element f_{i+1}. Indeed, since (a_i, b_i, c_i, d_i)

has (P) we have $a_i \vee d_i = c_i \vee d_i = f_i$. Hence

$$e_i \vee d_i \vee a_{i+1} = b_i \vee c_i \vee d_i \vee a_{i+1} = f_i$$

and

$$(d_i \vee a_{i+1}) \vee (d_{i+1} \vee a_i) = a_i \vee d_i = f_i$$

From (3) we have $e_i = b_i \vee c_i \geq d_{i+1}$. Hence

$$e_i \wedge (a_i \vee d_{i+1}) = d_{i+1} \vee (e_i \wedge a_i) = d_{i+1} \vee a_{i+1} = f_{i+1}$$

and

$$\begin{aligned}(a_{i+1} \vee d_i) \wedge (a_i \vee d_{i+1}) \\ &= a_{i+1} \vee \left(d_i \wedge (a_i \vee d_{i+1})\right) \\ &= a_{i+1} \vee d_{i+1} \vee (d_i \wedge a_i) \\ &= f_{i+1} \vee (d_i \wedge a_i)\end{aligned}$$

But $a_i \wedge d_i \leq a_0 \wedge d_0$ which is the least element of L by hypothesis. The remaining two calculations are similar.

The lattice generated by f_{i+1}, $b_{i+1} \vee c_i$, $b_i \vee c_{i+1}$ is a homomorphic image of the lattice diagramed in Fig. 2.2. The proof is exactly the same as in the previous case except that $b_i \wedge c_i$ is not necessarily the least element of L.

Let us suppose that $f_2 < e_1 < f_1 < f_1 \vee (b_0 \wedge c_0) < e_0 < f_0$. Then the above agruments show that $(f_1, a_0 \vee d_1, a_1 \vee d_0, e_0, f_0) = D_0$ is a nondegenerate diamond. As was seen in Chapter I the fact that

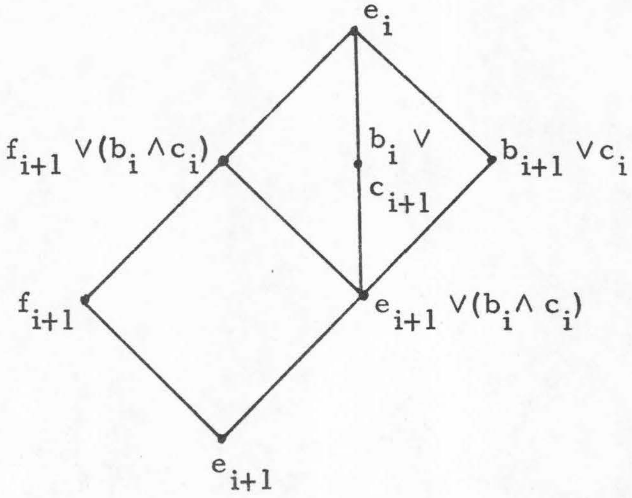


Figure 2.2

$f_1 < f_1 \vee (b_0 \wedge c_0) < e_0$ implies that D_0 and $f_1 \vee (b_0 \wedge c_0)$ generate the lattice diagramed in Fig. 2.3.

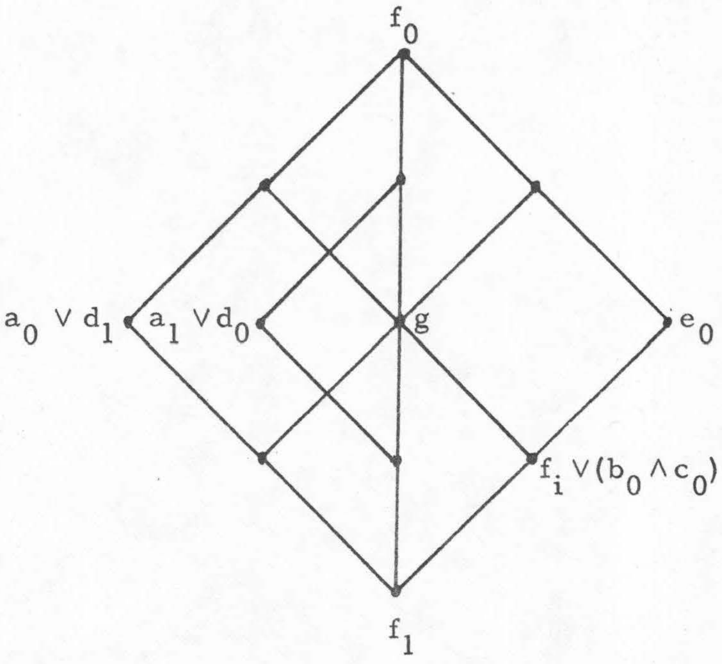


Figure 2.3

As remarked above, the elements f_1, $b_0 \vee c_1$, and $b_1 \vee c_0$ generate a sublattice which is a homomorphic image of the one diagramed in Fig. 2.2. Furthermore, since $e_1 < f_1 < f_1 \vee (b_0 \wedge c_0)$ this homomorphism must be an isomorphism. Hence the sublattice generated by f_1, $b_0 \vee c_1$, and $b_1 \vee c_0$ is isomorphic to the lattice diagramed in Fig. 2.4.

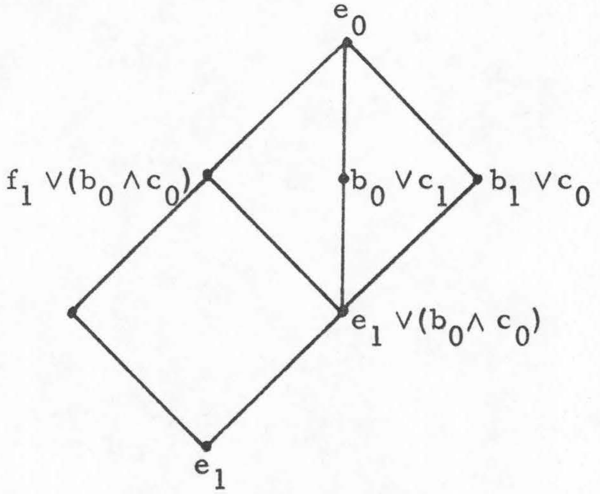


Figure 2.4

As above, the sublattice generated by e_1, $a_1 \vee d_2$, and $a_2 \vee d_1$ is diagramed in Fig. 2.5.

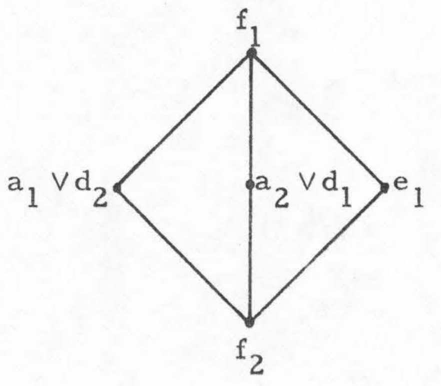


Figure 2.5

With these facts it is easy to see that the sublattice L_1 generated by $a_0 \vee d_1$, $a_1 \vee d_0$, $b_0 \vee c_1$, $b_1 \vee c_0$, $a_1 \vee d_2$, and $a_2 \vee d_1$ is isomorphic to the lattice diagramed in Fig. 2. 6.

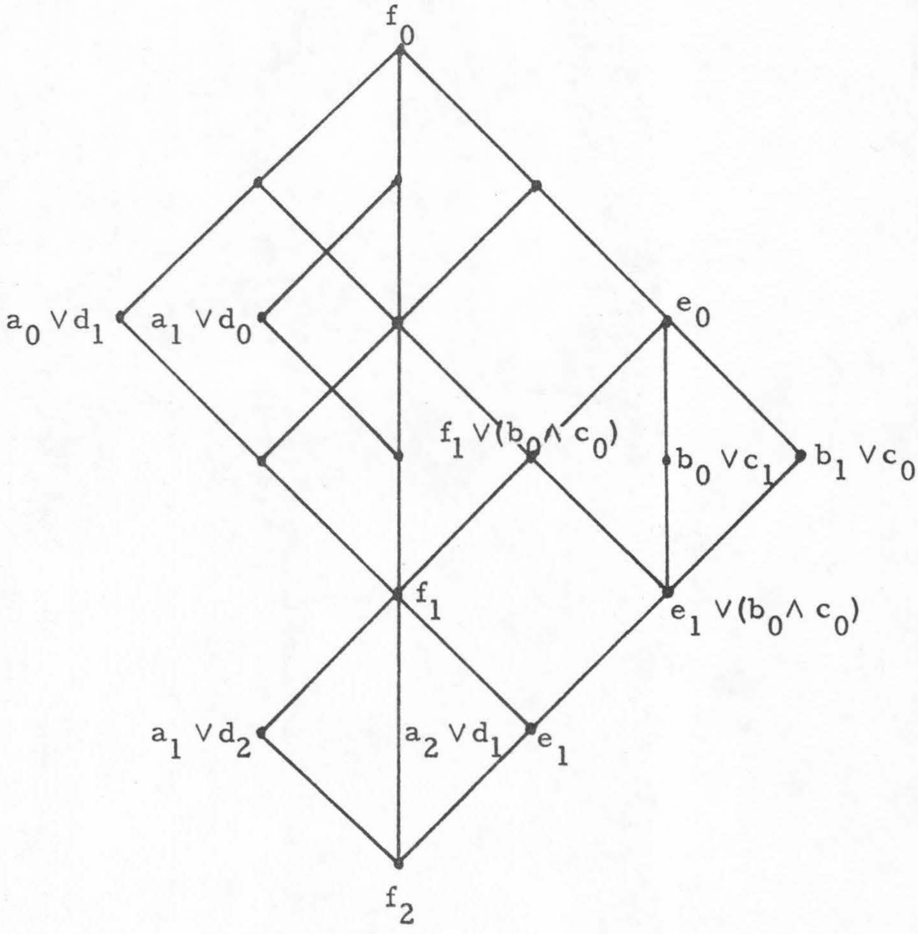


Figure 2. 6

Now $a_0 \vee d_2 \vee f_1 = a_0 \vee d_2 \vee a_1 \vee d_1 = a_0 \vee d_1$, and $f_1 \wedge (a_0 \wedge d_2)$ $= d_2 \vee (a_0 \wedge f_1) = d_2 \vee (a_0 \wedge (a_1 \vee d_1)) = d_2 \vee (a_1 \vee (a_0 \wedge d_1)) = d_2 \vee a_1$, since $a_0 \wedge d_1$ is the least element of L. Hence $a_0 \vee d_2 / a_1 \vee d_2 \nearrow a_0 \vee d_1 / f_1$. Similarly $a_2 \vee d_0 / a_2 \vee d_1 \nearrow a_1 \vee d_0 / f_1$. With these facts it is easy to show that the lattice L_2 generated by L_1, $a_0 \vee d_2$ and

$a_2 \vee d_0$ is isomorphic to the lattice diagramed in Fig. 2.7.

$L_2 =$

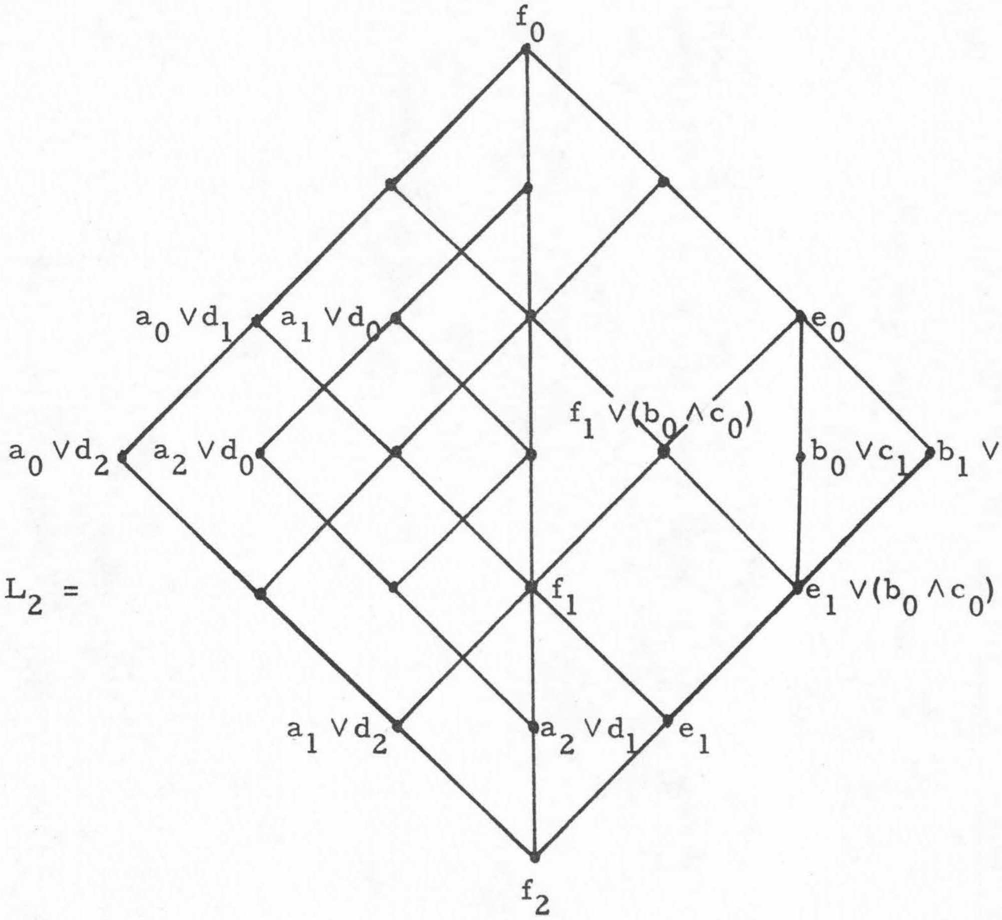


Figure 2.7

Now if $f_0 > e_0 > f_1 > e_1 > f_2$ but $e_1 = e_1 \vee (b_0 \wedge c_0)$ then Fig. 2.7 suggests, and arguments similar to those above, prove that the sub-lattice L_3 generated by L_1, $a_0 \vee d_2$ and $a_2 \vee d_0$ is isomorphic to the lattice diagramed by Fig. 2.8.

In Fig. 2.8 note that L_3 is a homomorphic image of L_2 and L_3 is isomorphic to A_4. Hence A_4 is a homomorphic image of a sub-lattice of L in these cases.

L_3 =

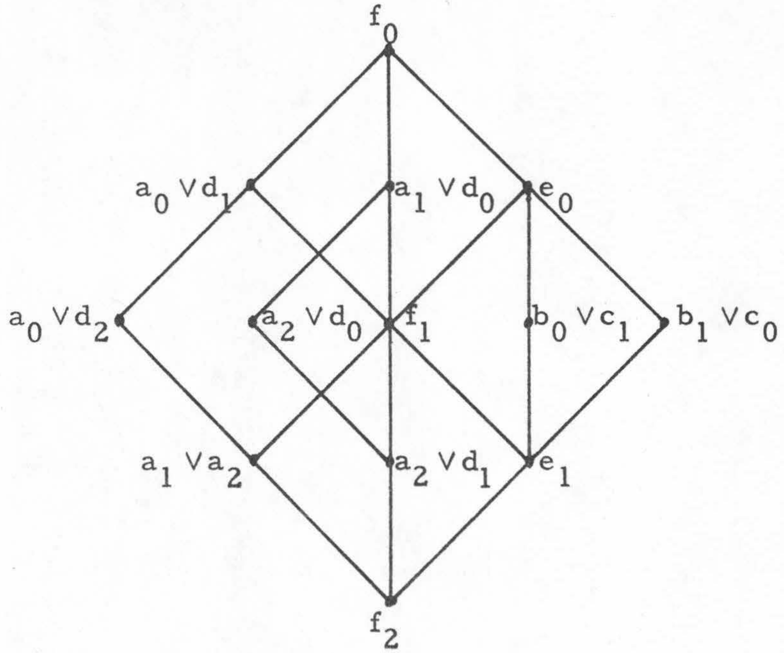


Figure 2.8

For the remaining cases we have $f_0 \geq e_0 \geq f_1 \geq e_1 \geq f_2$ and we know that there is at least one equality. It follows immediately from the definitions of these elements that any equality implies $e_1 = f_2$.

It has already been shown that $(f_2,\ a_1 \vee d_2,\ a_2 \vee d_1,\ e_1,\ f_1)$ forms a diamond, and since $e_1 = f_2$, it follows that $f_1 = e_1$. But then $a_1 \vee d_1 = b_1 \vee c_1$. This, together with the fact that (a_1, b_1, c_1, d_1) satisfies (P), shows that any two elements of $\{a_1, b_1, c_1, d_1\}$ join to f_1.

We must show that a_1, b_1, e_1, d_1 are distinct. If 0 is the bottom element of L, we note that

(4) $\quad f_1/b_1 \searrow d_1/0 \nearrow f_1/a_1 \searrow b_1/0 \nearrow f_1/d_1 \searrow a_1/0 \nearrow f_1/c_1$

Now if any two of $\{a_1, b_1, c_1, d_1\}$ are even comparable, say $a_1 \leq b_1$, then $b_1 = a_1 \vee b_1 = f_1$. Hence, by (4) $a_1 = b_1 = c_1 = d_1 = f_1$. Now

(5) $$f_1 \vee b_0 = a_1 \vee b_1 \vee b_0 = a_1 \vee b_0$$
$$= \left(a_0 \wedge (b_0 \vee c_0)\right) \vee b_0$$
$$= (a_0 \vee b_0) \wedge (b_0 \vee c_0) = e_0$$

It follows that $f_1/b_1 \nearrow e_0/b_0$. Since $f_1 = b_1$, $e_0 = b_0$. Similarly $e_0 = c_0$, a contradiction to a_0, b_0, c_0, d_0 being distinct. We conclude that a_1, b_1, c_1, d_1 are distinct.

As we have pointed out a_1, b_1, c_1, d_1 satisfy (P) and hence equation (1). Since $a_1 \leq a_0$, $b_1 \leq b_0$, $c_1 \leq c_0$ and $d_1 \leq d_0$, a_1, b_1, c_1, d_1 also satisfy (2). So the same procedure can be applied to the dual of the lattice generated by a_1, b_1, e_1, d_1. As above either A_4 is a homomorphic image of a sublattice of L or there exists $a_1' \geq a_1$, $b_1' \geq b_1$, $c_1' \geq c_1$ and $d_1' \geq d_1$ which pairwise intersect to $a_1' \wedge b_1' \wedge c_1' \wedge d_1'$. But since $a_1' \geq a_1$ etc., we also have that a_1', b_1', c_1', d_1' pairwise join to f_1. Hence the lattice generated by a_1', b_1', c_1' and d_1' is isomorphic to M_4. Moreover, $f_1/a_1' \subseteq f_1/a_1 \nearrow f_1 \vee d_0/d_0 \subseteq a_0 \vee d_0/d_0$ and so the last statement of the theorem is also true.

Let A_7 and A_9 be the lattices diagramed in Fig. 2.9 and Fig. 2.10. A_9 is the lattice of subspaces of projective plane of order two.

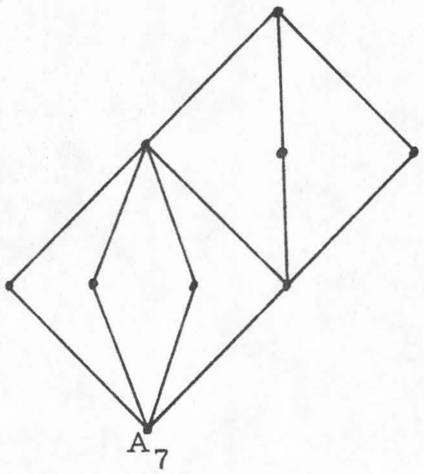


Figure 2.9

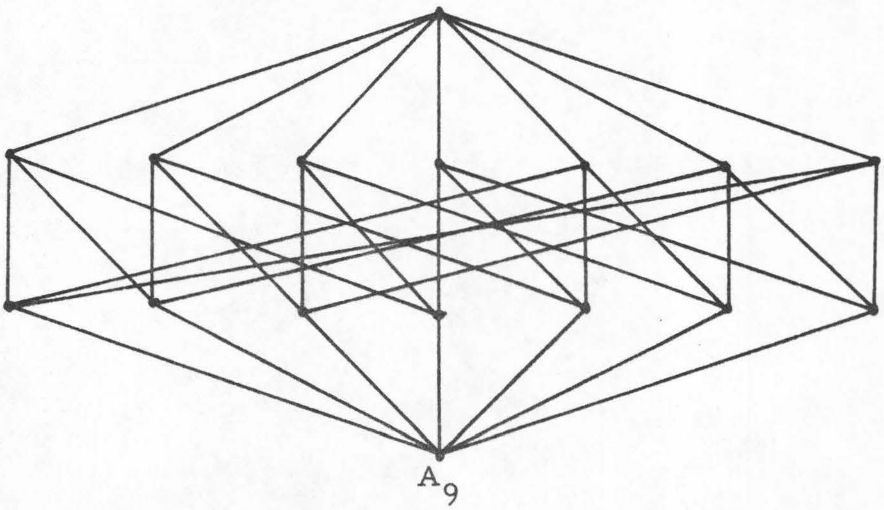


Figure 2.10

Theorem 2.2. Let the modular lattice L have diamonds $D = (v, x, y, z, u)$ and (v, z, c', v', z') such that $u \wedge z' = z$. Then either A_4, A_7 or A_9 is a homomorphic image of a sublattice of L.

The situation described in the hypotheses of the theorem is pictured in Fig. 2.11.

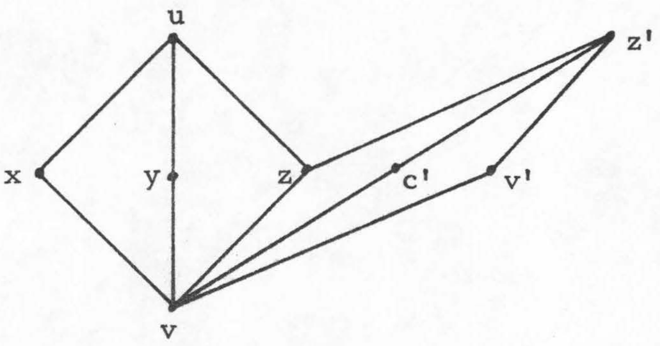


Figure 2.11

Proof: Since $u \wedge v' = u \wedge z \wedge v' = z \wedge v' = v$ the Direct Product Lemma shows that D and v' generate the lattice $M_3 \times 2$, diagramed below. In particular, there is another diamond $D' = (v \vee v', x \vee v', y \vee v', z \vee v', u \vee v') = (v', x', y', z', u')$.

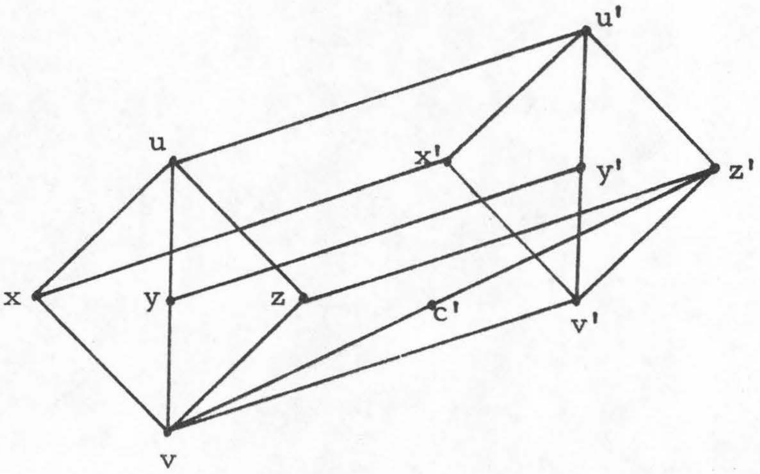


Figure 2.12

Note that

(6) $$a'/v \nearrow u'/y \searrow x'/v \nearrow u'/z \searrow y'/v \nearrow u'/x$$

Let $b \in u'/y$, $a' \in x'/v$, $c \in u'/z$, $b' \in y'/v$ and $a \in a'/x$ be the images of c' under the sequence of transposes (6). Since c' is a relative complement of both z and v' in x'/v, b is a relative complement of y' and u in u'/y. Similar statements hold for a, a', b' and c. Now let us suppose that one of the following statements fails

$$\begin{aligned} a' \vee y &= c' \vee y = b \\ a' \vee z &= b' \vee z = c \\ b' \vee x &= c' \vee x = a \\ a \wedge y' &= c \wedge y' = b' \\ a \wedge z' &= b \wedge z' = c' \\ b \wedge x' &= c \wedge x' = a' \end{aligned} \tag{7}$$

Say, for example, $c' \vee x \neq a$. Then, since $c' \vee x$ is the image of c' under the transposition $z'/v \nearrow u'/x$, we conclude that $c' \vee x$ is a relative complement of u and x' in u'/x. Since a is also a relative complement of both u and x' in u'/x, the elements u, a, $c' \vee x$, x' satisfy the hypotheses of Theorem 2.1. Since all of the quotients of (6) are isomorphic it follows that there exists elements $r, s \in z'/v$ such that z, r, s, v' satisfy the hypotheses of Theorem 2.1. Hence either A_4 is a homomorphic image of a sublattice of L or there exists a sublattice L' isomorphic to M_4 and such that if u is the greatest element of L' there is an atom of L' w such that $u/w \nearrow f/e \subseteq z'/v'$. In this case L' and $(D')_{f/e}$ together form a sublattice with A_7 as a homomorphic image (see Fig. 2.13).

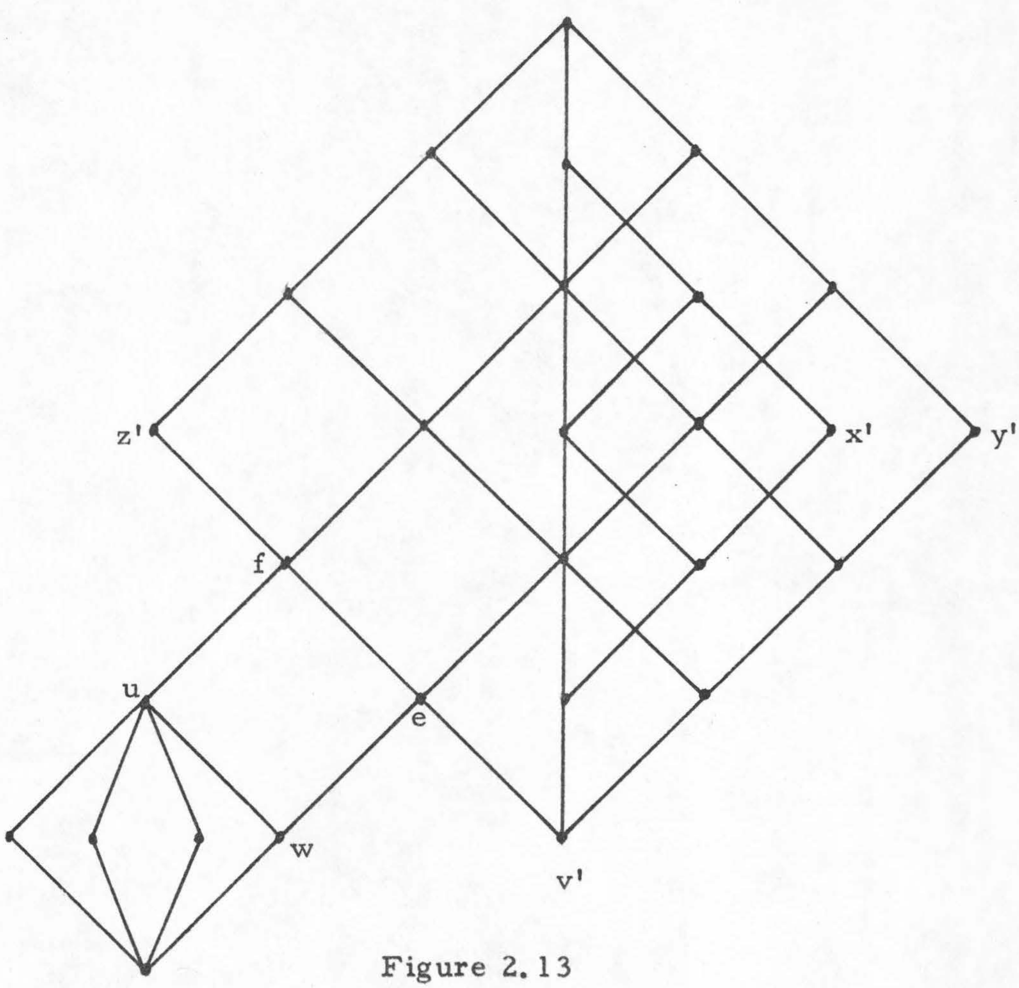


Figure 2.13

We conclude from this that the equations in (7) must all hold. In this case we claim that the sixteen element set $S = \{a, b, c, a', b', c'\} \cup D \cup D'$ form a lattice isomorphic to A_9. First we show that S is closed under joins. If $g, h \in D \cup D'$ then clearly $g \vee h \in D \cup D' \subseteq S$. Suppose $g \in \{a, b, c, a', b', c'\}$ and $h \in D \cup D'$. We wish to show that $g \vee h \in S$. The equations of (7) show that for several choices of g and h, $g \vee h \in S$. Examples of cases not covered by (7) are

$$a \vee y = a \vee x \vee y$$
$$= a \vee u = u' \in S$$

$$a \vee y' = a \vee x \vee y' = u' \in S$$

$$a \vee x' = u'$$

$$a \vee x = a$$

All other cases are similar to one of the above. Now if both g and h $\in \{a, b, c, a', b', c'\}$ then by (7) $c' = b \wedge z'$, $a' \leq b$ and hence

$$a' \vee c' = a' \vee (b \wedge z')$$
$$= b \wedge (a' \vee z')$$
$$= b \wedge u' = b \in S$$

Also $c' \vee a = a$ as $a \geq c'$ and $c' \vee c = c' \vee z \vee c = z' \vee c = u'$. The remaining cases are similar to these.

Similarly S is closed under meets. Now since we have virtually calculated all meets and joins, it can be verified directly that S is isomorphic A_9. Alternatively, it is known that a modular, simple, length three lattice, with sixteen elements whose top element is a join of its atoms is isomorphic to the projective plane of order 2, that is, A_9. It is easy to check that S has these properties.

Corollary 2.3. Let $D_1 = (v_1, x_1, y_1, z_1, u_1)$ and $D_2 = (v_2, x_2, y_2, z_2, u_2)$ be diamond sublattices of L, a modular lattice. Suppose $z_1/v_1 \searrow b/a \nearrow x_2/v_2$ and that $u_1 \wedge u_2 = b$. Then either A_4, A_7 or A_9 is a homomorphic image of a sublattice of L.

Proof: From the hypotheses we have

$$u_1 \wedge v_2 = u_1 \wedge u_2 \wedge v_2 = b \wedge v_2 = a$$

From the Direct Product Lemma we obtain a diamond $D_1' = D_1 \vee v_2 = (v_1 \vee v_2, x_1 \vee v_2, y_1 \vee v_2, z_1 \vee v_2, u_1 \vee v_2) = (v_1', x_1', y_1', z_1', u_1')$.

Similarly we obtain a diamond $D_2' = D_2 \vee v_1 = (v_2 \vee v_1, x_2 \vee v_1, y_2 \vee v_1, z_2 \vee v_1, u_2 \vee v_1) = (v_2', x_2', y_2', z_2', u_2')$. Furthermore,

$$\begin{aligned} z_1' &= z_1 \vee v_2 = z_1 \vee v \vee v_2 = z_1 \vee x_2 \\ &= v_1 \vee b \vee x_2 = v_1 \vee x_2 = x_2' \end{aligned}$$

Also,

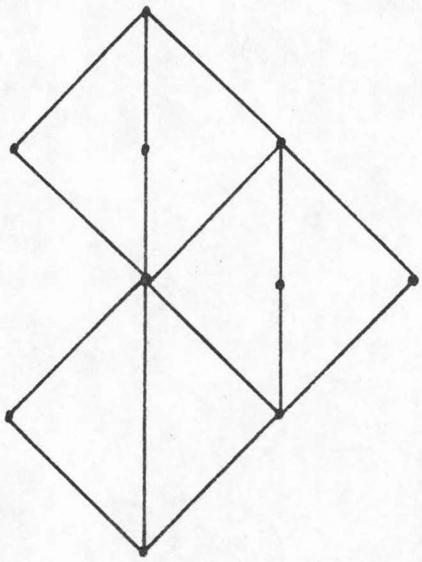
$$\begin{aligned} u_1' \wedge u_2' &= (u_1 \vee v_2) \wedge (u_2 \vee v_1) \\ &= \Big((u_1 \vee v_2) \wedge u_2\Big) \vee v_1 \\ &= v_2 \vee (u_1 \wedge u_2) \vee v_1 \\ &= v_2 \vee b \vee u_1 \\ &= v_2 \vee z_1 = z_1' = x_2' \end{aligned}$$

Thus D_1' and D_2' satisfy the hypothesis of Theorem 2.2. Since the conclusions of Theorem 2.2 are the same as Corollary 2.3, the proof is complete.

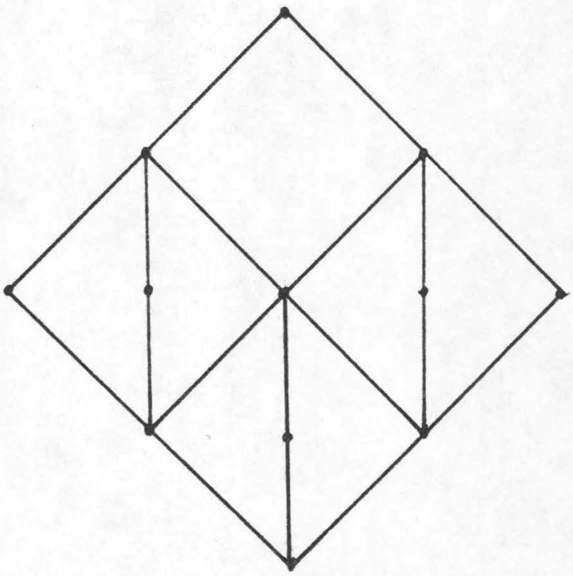
CHAPTER III

THE FUNDAMENTAL THEOREM ON WEAK ATOMICITY

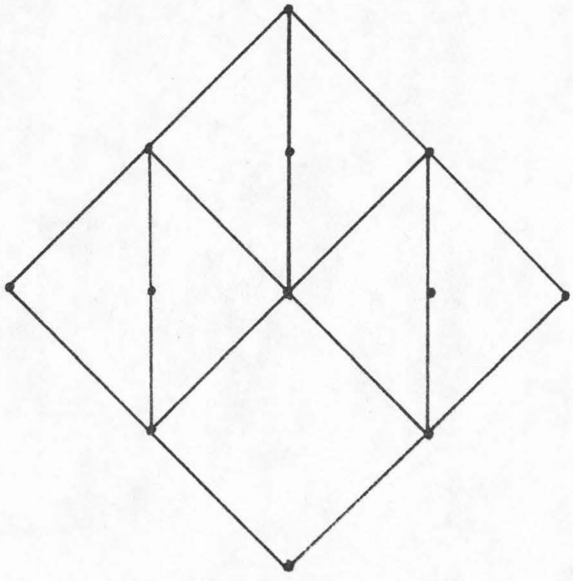
Let A_1 through A_{10} be the lattices diagramed below.



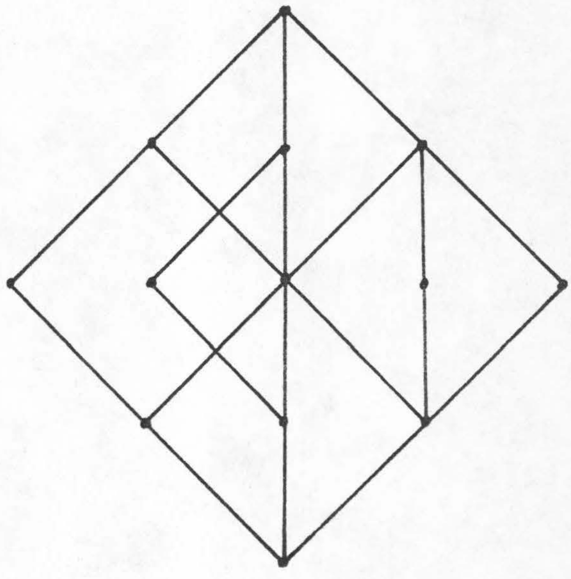
A_1



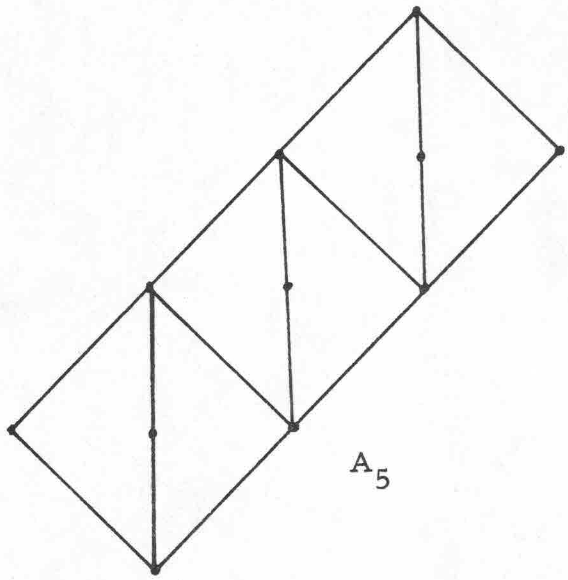
A_2



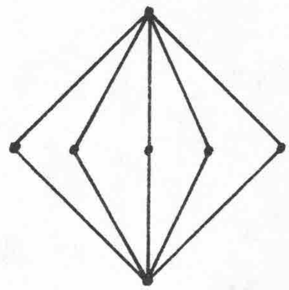
A_3



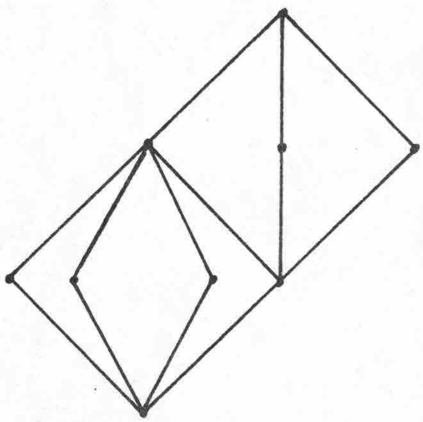
A_4



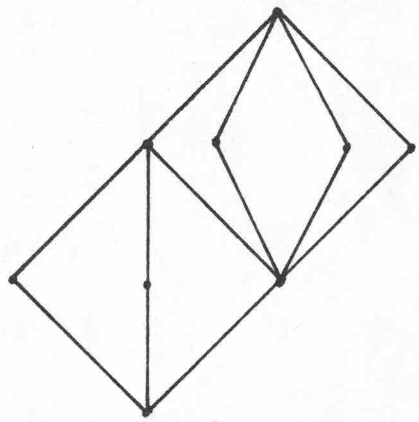
A_5



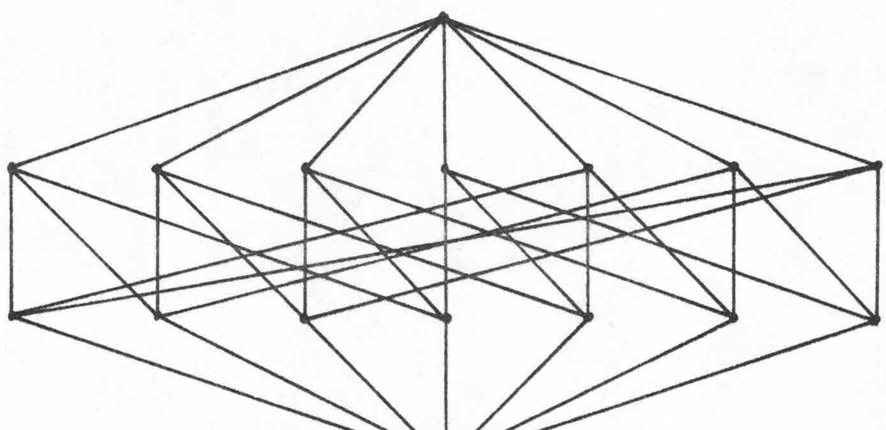
A_6



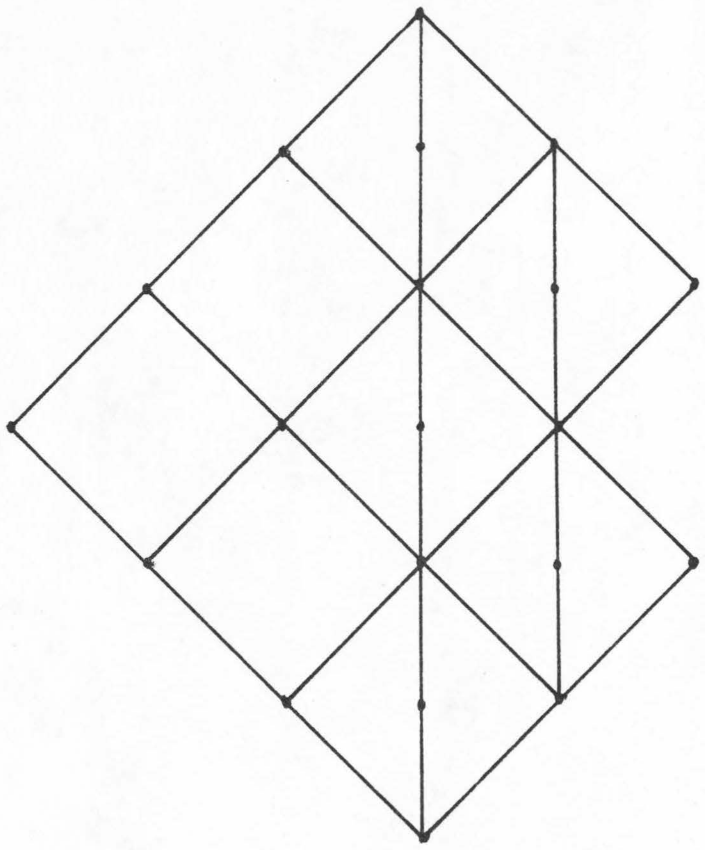
A_7



$A_8 = A_7^d$



$A_9 = PP_{(2)}$ = Fano Plane



A_{10}

Before stating the main result of this chapter we make some standard definitions. Let L be an arbitrary lattice. H(L) is the class of all lattices isomorphic to a homomorphic image of L. Within H(L) we identify isomorphic lattices. Similarly, S(L) is the class of lattices isomorphic to a sublattice of L.

If $a \leq b$ are elements of L and $a < x \leq b$ implies $x = b$, then b covers a, written $b \succ a$. The quotient b/a is called a prime quotient if $b \succ a$. L is called atomic if L has a least element 0 and if $x > 0$ there is a $y \in L$ such that $x \geq y \succ 0$. L is weakly atomic if $x > y$ implies there exists b and a such that $x \geq b \succ a \geq y$.

A sublattice L' of L is called an isometric sublattice if

$\{x \in L' | a < x \leq b\} = \{b\}$ implies $\{x \in L | a < x \leq b\} = \{b\}$ for a, b in L'. This means that a prime quotient in L' is a prime quotient in L.

We mention that in a modular, subdirectly irreducible lattice weak atomicity is equivalent to the existence of elements a and b such that $b \succ a$.

The goal of this chapter is to prove

Theorem 3.1. If L is a modular, subdirectly irreducible lattice such that none of $A_2, \ldots, A_{10}$ is a homomorphic image of a sublattice of L, then L is weakly atomic.

As we shall see in the next chapter, the weak atomicity of L is a powerful tool for analyzing the structure of L. In proving Theorem 3.1 we shall use techniques similar to those explained by Hong [14].

Lemma 3.2 (cf. [14]). Let L be a modular lattice such that $A_4 \notin S(L)$. Let $D = (v, x, y, z, u)$ be a diamond in L. Suppose that $b/a \nearrow u/x$. Then either

(i) $a \vee v = x$

or (ii) there exists x' and b', $x \leq x' < u$ and $b \leq b' < u$ such that $D_{u/x'}$ has $u = x' \vee b'$ as its greatest element and $b' \wedge x'$ as its smallest element.

Proof: It may be assumed that

$$(1) \qquad v < a \vee v < x$$

for $a \vee v \leq x$ and if $a \vee v = x$ then (i) holds. If $v = a \vee v$ then (ii) holds with x' = x and b' = b + v.

Let u_1 be the greatest element of $D_{a \vee v/v}$, which is, of course, the least element of $D_{x/a \vee v}$. That is, $u_1 = (a \vee v \vee y) \wedge (a \vee v \vee z) = (a \vee y) \wedge (a \vee z)$. By (1) both these diamonds are nondegenerate. Also, by the definition of u_1

(2) $$u_1/a \vee v \nearrow u_1 \vee x/x$$

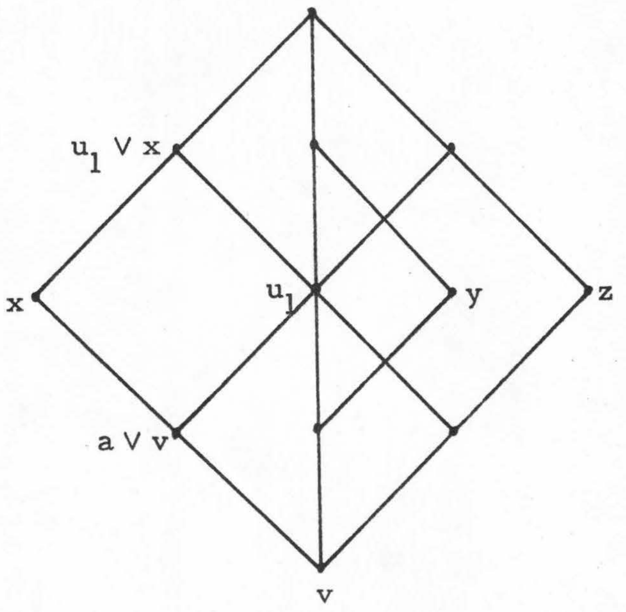


Figure 3. 1

Let $b' = b \vee v$ and $t = b' \wedge (u_1 \vee x)$. Now, since $u/x \searrow b/a$, we have

$$x \wedge t = x \wedge b' \wedge (x \vee u_1)$$
$$= x \wedge (b \vee v)$$
$$= (x \wedge b) \vee v$$

$$= a \vee v$$

$$x \vee t = x \vee [(b \vee v) \wedge (u_1 \vee x)]$$

$$= (u_1 \vee x) \wedge (x \vee b \vee v)$$

$$= (u_1 \vee x) \wedge u$$

$$= u_1 \vee x$$

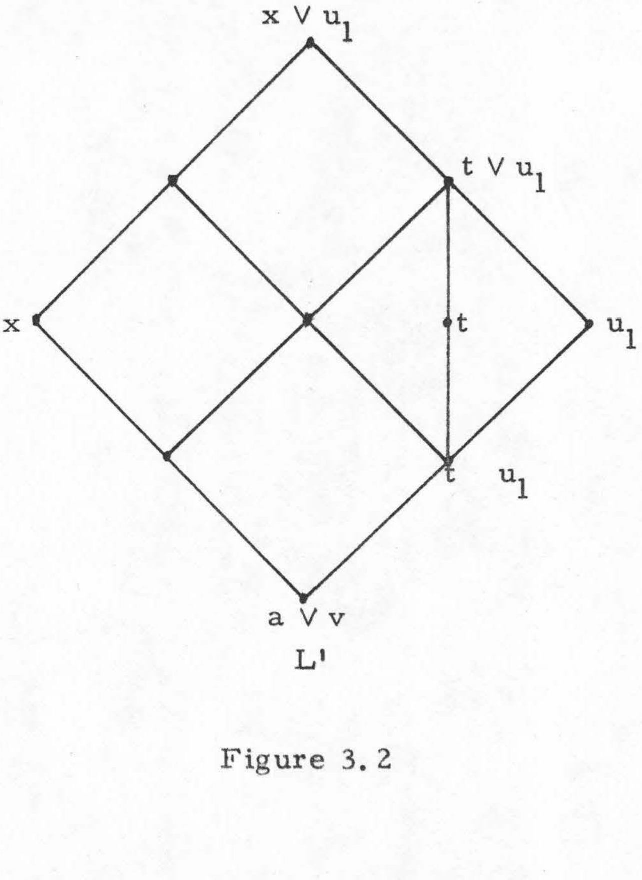
It follows that

(3) $$t/a \vee v \nearrow u_1 \vee x/x$$

Consider the sublattice generated by x, u_1 and t. By (2) and (3)

$$x \vee u_1 = x \vee t \quad \text{and} \quad x \wedge u_1 = a \vee v = x \wedge t$$

The free modular lattice with three generators subject to the above restrictions is L', which is diagramed in Fig. 3.2.



L'

Figure 3.2

That is, the sublattice generated by x, t and u_1 is a homomorphic image of L'. Notice that if the diamond in L' is collapsed then $t = u_1$. In this case (ii) holds with $x' = u_1 \vee x$, since $x' \wedge b' = (u_1 \vee x) \wedge b' = t = u_1$.

Let $D_1 = D_{a \vee v/v}$, $D_2 = D_{x/a \vee v}$, $D_3 = (D_1)_{u_1/t \wedge u_1}$, $D_4 = (D_2)_{t \vee u_1/u_1}$ and let $D_5 = (v_5, x_5, t, u_1, u_5)$ be the nondegenerate diamond of L'. Then we have

$$D_1 = \big(v,\ a \vee v,\ z \wedge (y \vee a \vee v),\ u_1,\ y \wedge (z \vee a \vee v)\big)$$

$$D_2 = (u_1,\ u_1 \vee x,\ y \vee a \vee v,\ z \vee a \vee v,\ u)$$

$$D_3 = \big(v_5 \wedge (y_1 \vee (z_1 \wedge v_5)),\ v_5,\ y_1 \vee (z_1 \wedge v_5),\ z_1 \vee (y_1 \wedge v_5),\ u_1\big)$$

$$D_4 = \big(u_1,\ u_5,\ y_2 \wedge (z_2 \vee u_5),\ z_2 \wedge (y_2 \vee u_5),\ u_5 \vee (y_2 \wedge (z_2 \vee u_5))\big)$$

Note that u_1/v_5 is an upper quotient of D_3 and a lower quotient of D_5 and u_5/u_1 is a lower quotient of D_5 and an upper quotient of D_4. Hence D_3, D_5 and D_4 together form a lattice isomorphic to A_1.

Now let $v_3 = v_5 \wedge (y_1 \vee (z_1 \wedge v_5))$ be the least element of D_3, $u_4 = u_5 \vee (y_2 \wedge (z_2 \vee u_5))$ be the greatest element of D_4 and let $y' = (y \vee v_3) \wedge u_4 = (y \wedge u_4) \vee v_3$ and let $z' = (z \vee v_3) \wedge u_4 = (z \wedge u_4) \vee v_3$.

Since

$$
\begin{aligned}
y_4 &= y_2 \wedge (z_2 \vee u_5) \\
&= (y \vee a \vee v) \wedge (z_2 \vee u_5) \\
&\leq (y \vee u_1) \wedge (z_2 \vee u_5) \\
&\leq (y \vee u_1)
\end{aligned}
$$

it follows that

$$
\begin{aligned}
y' \vee u_1 &= (y \wedge u_4) \vee v_3 \vee u_1 \\
&= (y \wedge u_4) \vee u_1 = u_4 \wedge (u_1 \vee y) \\
&= (u_5 \vee y_4) \wedge (u_1 \vee y) \\
&= y_4 \vee \left(u_5 \wedge (u_1 \wedge y) \right) \\
&= y_4 \vee \left(u_1 \vee (u_5 \wedge y) \right)
\end{aligned}
$$

Now since $u_5 \wedge y \leq x_2 \wedge y_2 = v_2 = u_1$ we have

$$y' \vee u_1 = y_4 \vee u_1 = y_4$$

Similar calculations show that $y' \wedge u_1 = y_3$, $z' \vee u_1 = z_4$ and $z' \wedge u_1 = z_3$. With these facts it follows easily that D_3, D_5, D_4, y' and z' form a lattice which is isomorphic to A_4. This contradiction proves the theorem.

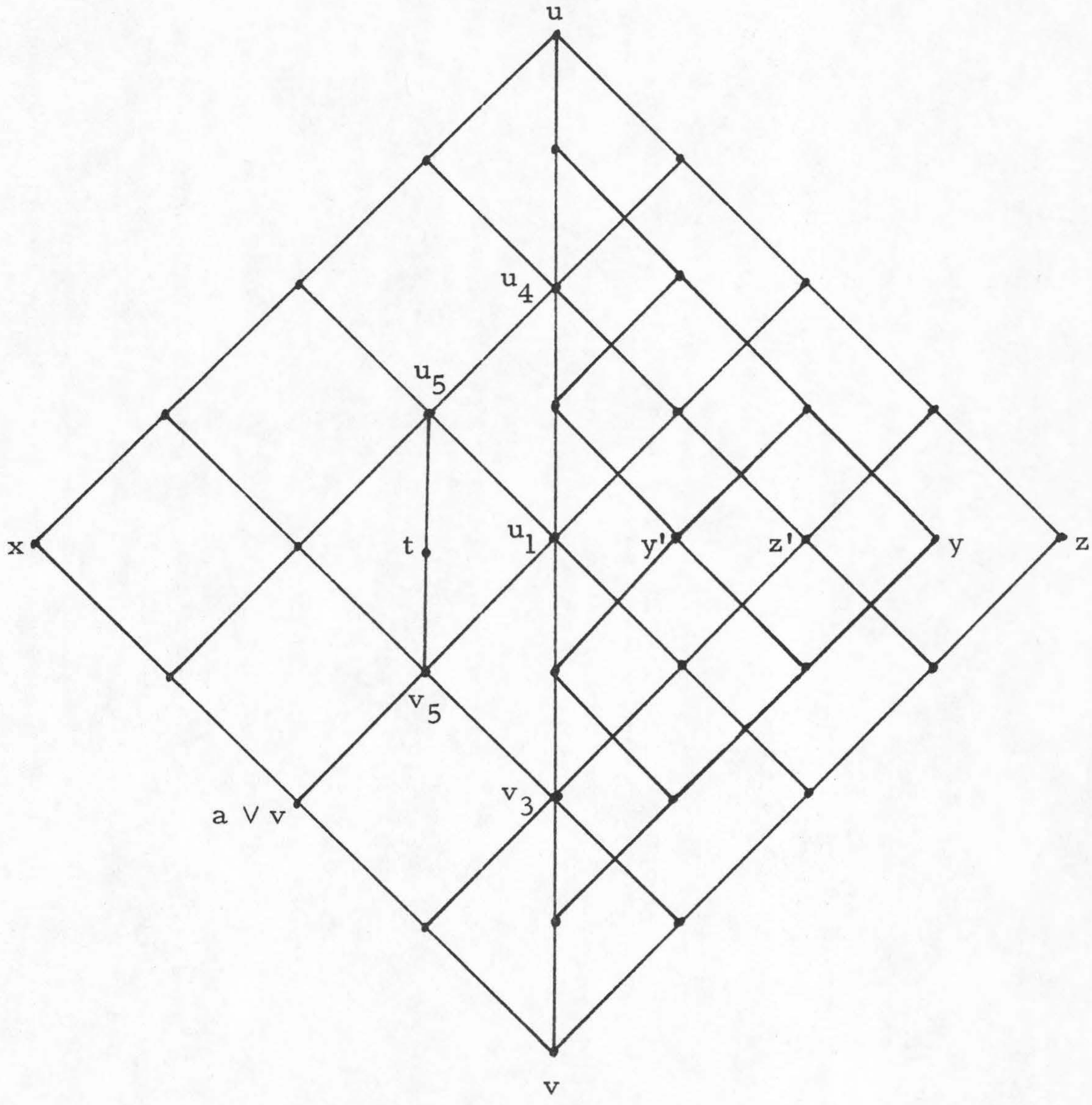


Figure 3.3

Corollary 3.3. Let L be a modular lattice such that A_4, A_7, A_8, $A_9 \notin HS(L)$. Let $D = (v, x, y, z, u)$ and $D' = (v', x', y', z', u')$ be diamond sublattices of L such that $u = u'$ and $x = x'$. Then $v = v'$.

Proof: Let us suppose that $v \neq v'$. Then, by symmetry, we may assume that $v' \nleq v$. Apply Lemma 3.2 with $b = z'$ and $a = v'$.

The sublattice generated by D and $v \vee v'$ is denoted L' (see Fig. 3.4).

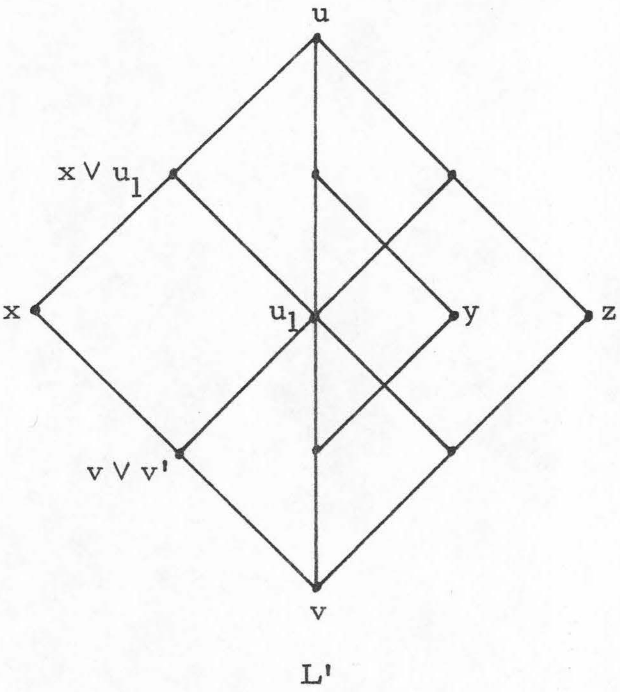


Figure 3.4

As before we let u_1 denote the top element of $D_{v \vee v'/v}$. By Lemma 3.2 there is an element b', $z' = b \leq b' < u$ such that $b' \wedge (u_1 \vee x) = u_1$ and $b' \vee u_1 \vee x = u$. Now

$$(u_1 \vee x) \wedge z' = (u_1 \vee x) \wedge b' \wedge z' = u_1 \wedge z'$$

Hence

$$\begin{aligned} x \vee (u_1 \wedge z') &= x \vee \Big((u_1 \vee x) \wedge z' \Big) \\ &= (u_1 \vee x) \wedge (x \vee z') \\ &= (u_1 \vee x) \wedge (x' \vee z') \\ &= (u_1 \vee x) \wedge u' \\ &= (u_1 \vee x) \wedge u \\ &= u_1 \vee x \end{aligned}$$

Also

$$x \wedge u_1 \wedge z' = v'$$

Hence

(1) $$u_1 \wedge z'/v' \nearrow u_1 \vee x/x$$

Since $u_1 \vee x > x$ we have $u_1 \wedge z' > v'$.

Note that, since u_1 is the top element of $D_{v \vee v'/v'}$, u_1 depends only on D and v' and not on z'. Hence, if we now let b = y' and a = v', the above argument yields that

(2) $$u_1 \wedge y'/v' \nearrow u_1 \vee x/x$$

Recall that $(x \vee u_1) \wedge b' = u_1$ so that $b' \geq u_1$. Also recall that $b' = b \vee v = z' \vee v$. Hence

(3) $$\begin{aligned}(v \vee v') \vee (u_1 \wedge z') &= v \vee (u_1 \wedge z') \\ &= u_1 \wedge (v \vee z') \\ &= u_1 \wedge b' \\ &= u_1\end{aligned}$$

Similarly

(4) $$(v \vee v') \vee (u_1 \wedge y') = u_1$$

Now consider the sublattice L'' generated by $v \vee v'$, $u_1 \wedge y'$ and $u_1 \wedge z'$. Since they are less than x, y' and z', respectively, any two of them intersect to the bottom element of L'', v'. Using this and (3) and (4) we see that L'' is a homomorphic image of the lattice diagramed in Fig. 3. 5.

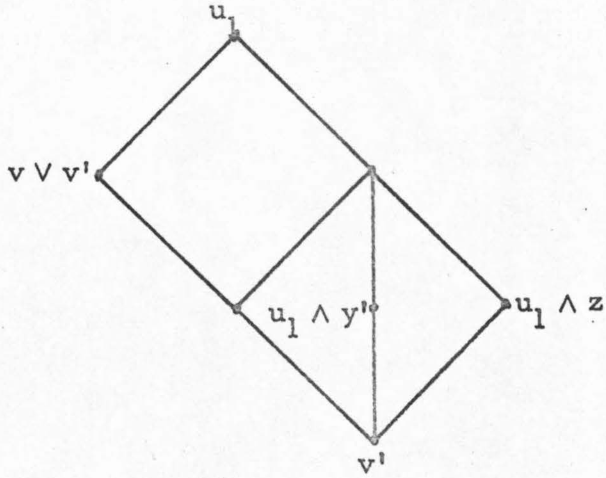


Figure 3. 5

Since $u_1 \wedge z' > u'$ we know the diamond in L'' is nondegenerate. Now the diamond $D_{v \vee v'/v}$ has u_1 as its top element and $v \vee v'$ as one of its atoms and v as its bottom element. Hence by the dual of Corollary 2. 3 either A_8, A_4 or $A_9 \in HS(L)$, a contradiction. This completes the proof.

<u>Lemma 3. 4.</u> Let L be a modular lattice such that $A_2, A_3 \notin HS(L)$. Suppose a strongly normal sequence satisfies the conditions of Theorem 1. 1. Then the associated diamonds must alternately transpose and translate. That is, the numbers below the arrows between the associated diamonds must alternate.

<u>Proof:</u> We have already seen that $D_{k-1} \underset{(1)}{\nearrow} D_k^*$, $D_k \underset{(1)}{\searrow} D_{k+1}$ is impossible. Suppose

$$D_{k-1} \underset{(2)}{\nearrow} D_k \underset{(2)}{\searrow} D_{k+1}$$

Then it is easy to verify that $D_{k-1} \cup D_k \cup D_{k+1} \cup \{u_{k-1} \wedge u_{k+1}, u_{k-1} \wedge v_{k+1}, v_{k-1} \wedge u_{k+1}, v_{k-1} \wedge v_{k+1}\}$ forms a sublattice with A_3 as a homomorphic image.

As an illustration of the last lemma suppose $b_0/a_0 \nearrow b_1/a_1$ $b_2/a_2 \nearrow \ldots \searrow b_{10}/a_{10} \nearrow b_{11}/a_{11}$ is a strongly normal sequence satisfying all the conditions of Theorem 1.1 in a modular lattice L such that $A_2, A_3 \notin HS(L)$. Let $D_1, \ldots, D_{10}$ be the associated diamonds. Suppose $D_1 \searrow_{(1)} D_2^*$. Then we must have $D_2 \nearrow_{(2)} D_3$, $D_3 = D_4^*$, $D_4 \nearrow_{(2)} D_5 = D_6^*$, $D_6 \nearrow_{(2)} D_7 = D_8^*$, $D_8 \nearrow_{(2)} D_9 = D_{10}^*$. Notice that $D_2, \ldots, D_{10}$ form a sublattice which is a homomorphic image of the sublattice pictured in Fig. 3.6.

Notice that $a_{11} \geq y_2 = b_2$.

Now we are ready to begin the proof of Theorem 3.1. Since L is subdirectly irreducible and modular we need only show that there exist elements a and b in L such that b covers a. By the results of Jónsson [16] we may assume that L has a sublattice L_1 isomorphic to the lattice diagramed in Fig. 3.7. A direct proof of this assumption will be indicated below.

If $x \succ v$ we are done. Thus let $x > x^* > v$. Now x^* and L_1 generate the sublattice diagramed in Fig. 3.8.

We conclude from these observations that L has a sublattice L_2 which is isomorphic to the lattice diagramed in Fig. 3.9.

There exist subquotients b/a of u'/x' and d/c of e/u' which are connected by a sequence of transposes satisfying the conditions of Theorem 1.1. If $b/a \nearrow b_1/a_1$ then it is clear that the sublattice

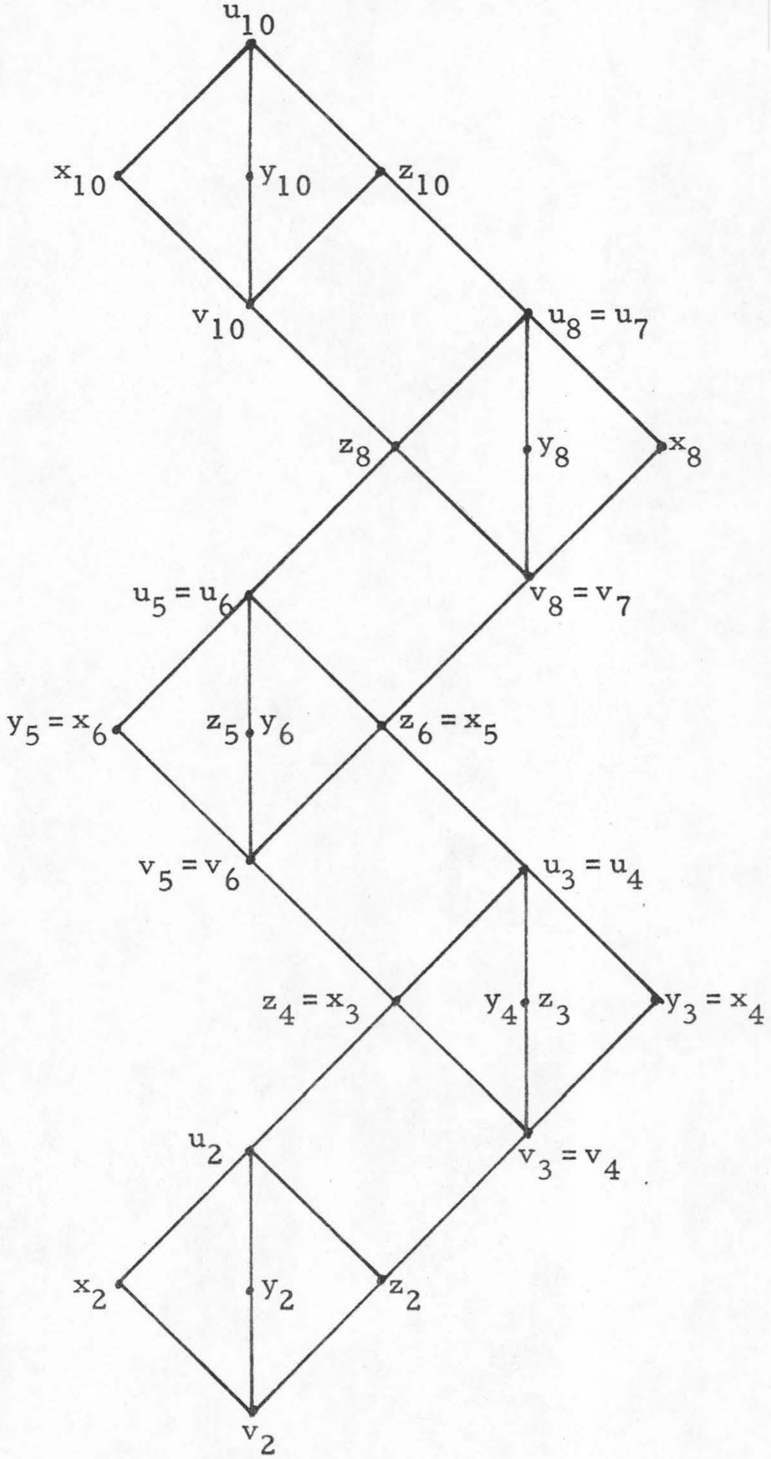

u_{10}
x_{10}
y_{10}
z_{10}
v_{10}
$u_8 = u_7$
z_8
y_8
x_8
$u_5 = u_6$
$v_8 = v_7$
$y_5 = x_6$
z_5
y_6
$z_6 = x_5$
$v_5 = v_6$
$u_3 = u_4$
$z_4 = x_3$
y_4
z_3
$y_3 = x_4$
u_2
$v_3 = v_4$
x_2
y_2
z_2
v_2

Figure 3.6

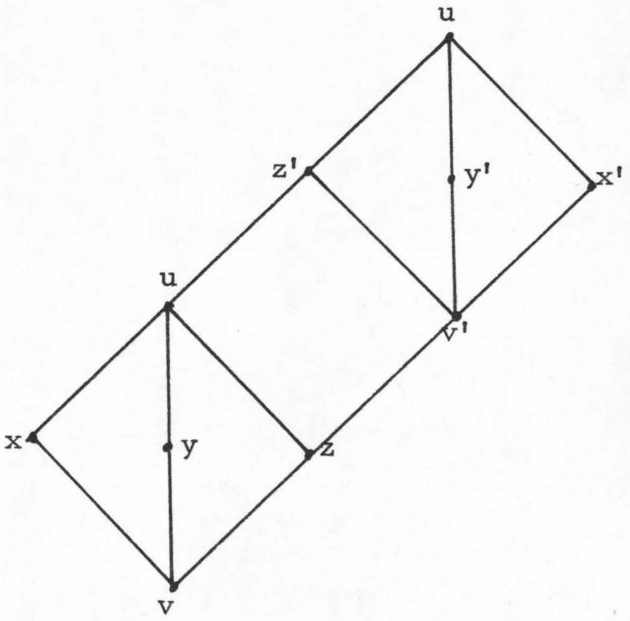


Figure 3.7

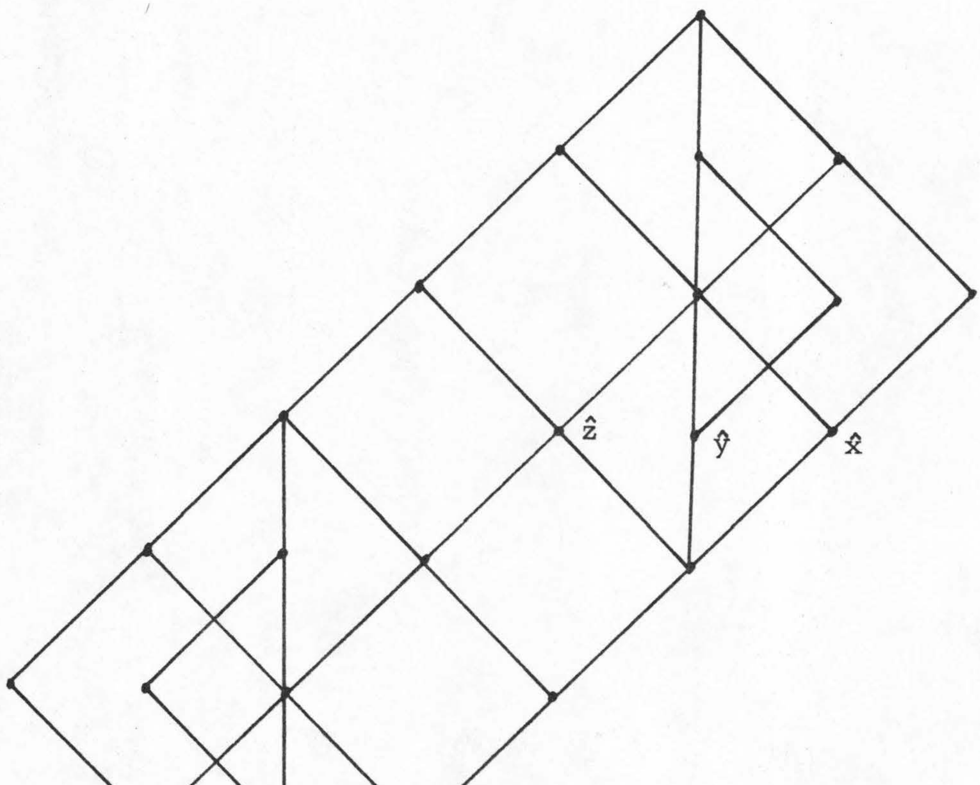


Figure 3.8

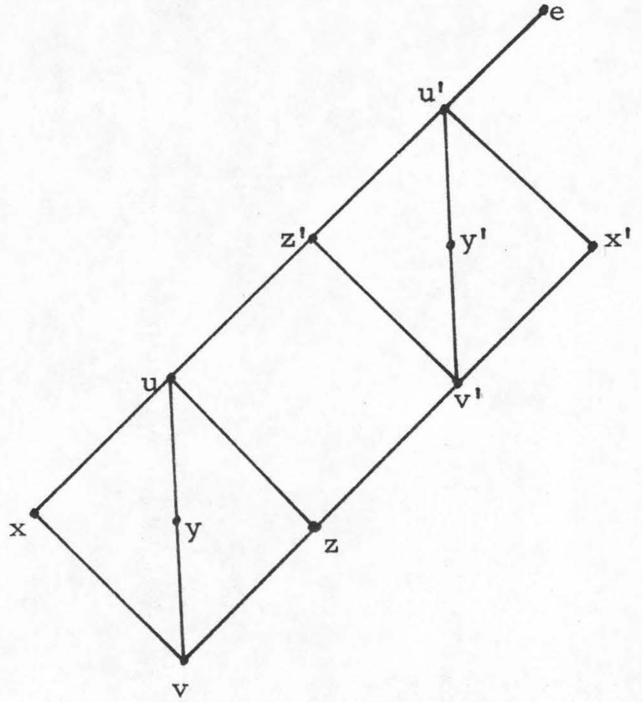


Figure 3.9

generated by D_1, $D'_{b/a}$ and $D_{u\wedge b/u\wedge a}$ has A_5 as a homomorphic image. Here D_1 is the first diamond associated with the sequence from b/a to d/c. Hence it may be assumed that

(1) $$b/a = b_0/a_0 \searrow b_1/a_1 \nearrow b_2/a_2 \searrow \dots b_n/a_n = d/c$$

Furthermore, by applying Theorem 1.1 to the sequence

(2) $$d/c = b_n/a_n \dots \nearrow b_2/a_2 \searrow b_1/a_1 \nearrow b_0/a_0 = b/a ,$$

we may assume that (1) is strongly normal satisfies condition (i) of Theorem 1.1, and

(3) $D_k \nearrow_{(1)} D^*_{k+1}$ or $D_k \searrow_{(1)} D^*_{k+1}$ imply $D_k = D^*_{k+1}$

$k = 1, 2, \ldots, n-3.$

Here $D_1, \ldots, D_{n-1}$ are the diamonds associated with (1).

Note that $b_1 \leq b_0 = b \leq c$. It is well known and easy to see that this implies p. d. $(b_1/a_1, d/c) \geq 3$. Hence $n \geq 4$ and so $n - 3 \geq 1$. Thus $D_1 \nearrow_{(1)} D_2^*$ implies $D_1 = D_2^*$. But if $D_1 = D_2^*$ we may apply Lemma 3.4 with the aid of (3) to the sequence (2) and, as the example after that lemma illustrates, $b_{n-2} \leq b_0 \leq c$. But by (1) p. d. $(b_{n-2}/a_{n-2}, d/c) = 2$. As pointed out above these two statements are contradictory. It follows that

(4) $$D_1 \nearrow_{(2)} D_2$$

The next part of the argument again uses techniques developed in [14]. Let $D_1' = (v_1', x_1', y_1', z_1', u_1') = (D')_{b/a}$ and $D_2' = (v_2', x_2', y_2', z_2', u_2') = (D)_{b \wedge u/a \wedge a}$. Let $b' = u_1 \vee u_1'$ and $a' = b' \wedge a$. Then $b'/a' \nearrow b/a$. If $b'/a' \nearrow x^*/v^*$ where x^* and v^* are elements of a diamond $D^* = (v^*, x^*, y^*, z^*, u^*)$ then D_2', D_1' and D^* form a sublattice with A_5 as a homomorphic image. From this and the fact that $b/a \searrow b'/a'$ we conclude p. d. $(b'/a', d/c) =$ p. d. $(b/a, d/c) = n$. Now it is easy to check that the sequence

(5) $$b'/a' \searrow b_1/a_1 \nearrow b_2/a_2 \searrow \ldots\ b_n/a_n = d/c$$

has $D_1, \ldots, D_{n-1}$ as its associated diamonds and satisfies all the conditions of Theorem 1.1. The situation is diagramed in Fig. 3.10.

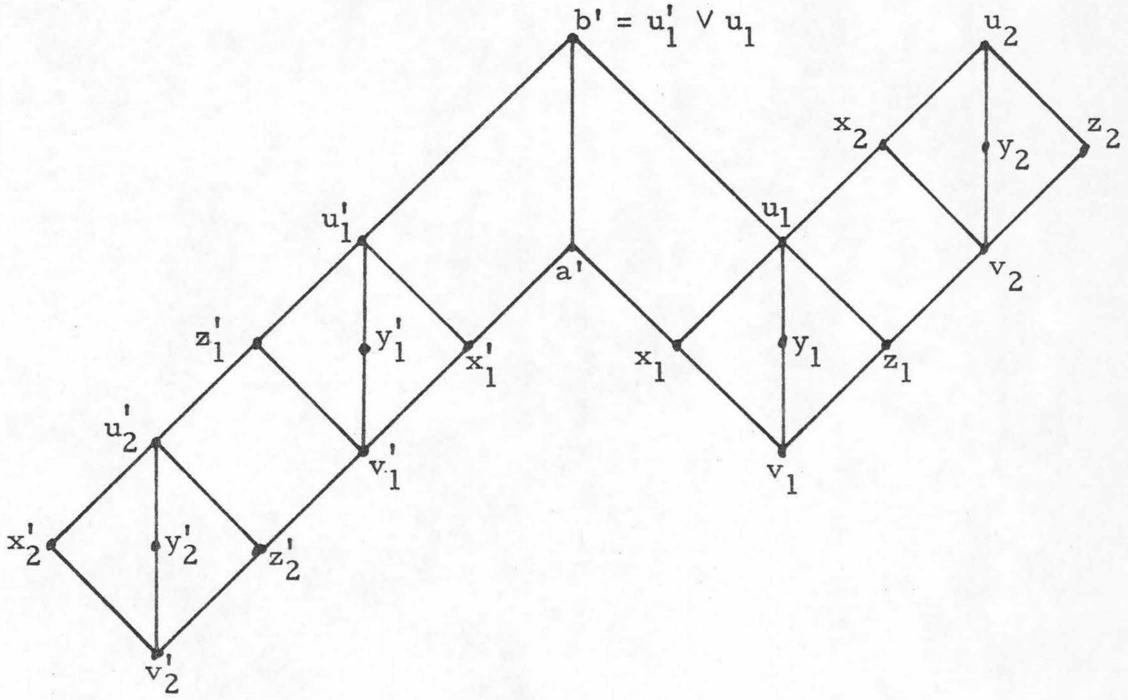


Figure 3.10

Consider the sublattice generated by y_1', a' and u_1. The fact that $u_1'/x_1' \nearrow b/a$ and the definition of b' and a' imply that $u_1'/x_1' \nearrow b'/a'$. Hence it follows that $y_1' \vee a' = b'$. Also $a' \vee u_1 = b'$. The free lattice subject to these restrictions is given in Fig. 3.11.

Suppose the diamond in Fig. 3.11, which we denote by $D_0 = (v_0, x_0, y_0, z_0, u_0)$, is nondegenerate. Then let $(D_1)_{u_1/u_1 \wedge v_0} = \overline{D}_1 = (\overline{v}_1, \overline{x}_1 = u_1 \wedge v_0, \overline{y}_1, \overline{z}_1, \overline{u}_1 = u_1)$. Let $(D_2)_{x_2/v_2 \vee \overline{z}_1} = \overline{D}_2$.

By (5) $b' \wedge u_2 = u_1$. Hence $u_0 \wedge \overline{u}_2 = u_1 = \overline{u}_1$. Also $\overline{u}_1/\overline{x}_1 \nearrow z_0/v_0$ and $\overline{u}_1/\overline{z}_1 \nearrow \overline{x}_1/\overline{v}_2$. As we noted in the proof of

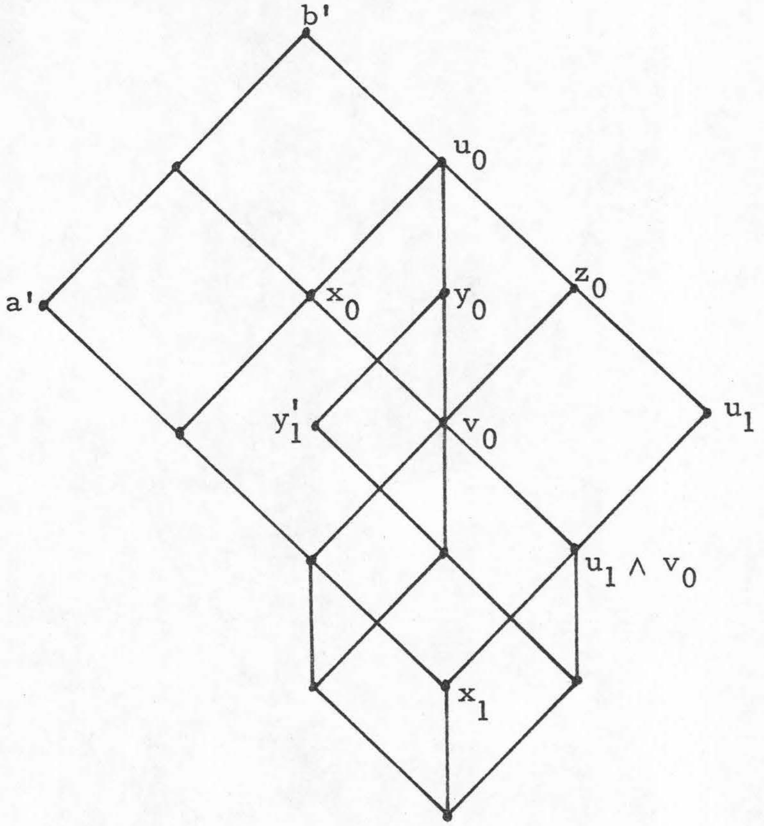


Figure 3.11

Lemma 3.4, D_0, $\overline{D}_1$, $\overline{D}_2$ generate a lattice with A_2 as a homomorphic image. We conclude from this that the diamond D_0 in Fig. 3.11 must be degenerate. That is, that sublattice generated by a', y_1' and u_1 must be distributive. Similarly the sublattice generated by a', z_1' and u_1 is distributive.

A similar argument shows that if the sublattice generated by u_1', a' and y_1 or the sublattice lattice generated by u_1', a' and z_1 is not distributive then there exist s_1/r_1 and s_2/r_2 subquotients of u_1'/x_1' and of u_2'/z_2', respectively, such that the diamond in $\langle u_1', a', y_1 \rangle$

and $(D_1')_{s_1/r_1}$ and $(D_2')_{s_2/r_2}$ form a sublattice with A_5 as a homomorphic image. We conclude that $\{u_1', a', y_1\}$ and $\{u_1', a', z_1\}$ generate distributive sublattices. Thus

$$\begin{aligned}(6) \qquad v_1' \vee u_1 &= (a' \wedge y_1') \vee u_1 \vee (a' \wedge z_1') \vee u_1 \\ &= [(a' \vee u_1) \wedge (y_1' \vee u_1)] \vee [a' \vee u_1) \wedge (z_1' \vee u_1)] \\ &= [b' \wedge (y_1' \vee u_1)] \vee [b' \wedge (z_1' \vee u_1)] \\ &= y_1' \vee u_1 \vee z_1' \vee u_1 \\ &= u_1' \vee u_1 = b'\end{aligned}$$

Similarly

$$(7) \qquad v_1 \vee u_1' = b'$$

By the Direct Lemma Product there exist diamonds $D_3' = D_1' \wedge u_1 = (v_1' \wedge u_1, x_1' \wedge u_1, y_1' \wedge u_1, z_1' \wedge u_1, u_1' \wedge u_1)$ and $D_4' = D_1 \wedge u_1' = (v_1 \wedge u_1', x_1 \wedge u_1', y_1 \wedge u_1', z_1 \wedge u_1', u_1 \wedge u_1')$.

Since

$$\begin{aligned}(8) \qquad u_1 \wedge x_1' &= u_1 \wedge a \wedge u_1' \\ &= x_1 \wedge u_1'\end{aligned}$$

we have $x_3' = x_4'$. Since $u_3' = u_4'$, Corollary 3.3 implies that $v_3' = v_4'$. By the construction of D_3' and D_4' we know that

$$(9) \qquad D_3' \nearrow_{(1)} D_1' \quad \text{and} \quad D_4' \nearrow_{(1)} D_1$$

Now $z_4' \in u_3'/v_3' = u_4'/v_4'$ and u_3'/v_3' transposes up to u_1'/v_1'. This transposition is of course an isomorphism; let $\overline{z}_4'$ be the image of z_4' in u_1'/v_1'. Then, as $\overline{z}_4' < u_1' \leq b'$, we have

(10) $$\begin{aligned} \bar{z}'_4 \wedge u_2 &= \bar{z}'_4 \wedge b' \wedge u_2 \\ &= \bar{z}'_4 \wedge u_1 \\ &= \bar{z}'_4 \wedge u'_1 \wedge u_1 \\ &= \bar{z}'_4 \wedge u'_4 \\ &= z'_4 \end{aligned}$$

Hence the Direct Product Lemma may be applied to the sublattices $\bar{z}'_4/z'_4$ and u_2/z'_4 to obtain a diamond $D'_5 = D_2 \vee \bar{z}'_4$. Since $u'_4/z'_4 \nearrow x_2/v_2$, $u'_4 \vee \bar{z}'_4/z'_4 \vee \bar{z}'_4 = u'_1/\bar{z}'_4 \nearrow x_2 \vee \bar{z}'_4/v_2 \vee \bar{z}'_4$. (See Fig. 3.12.)

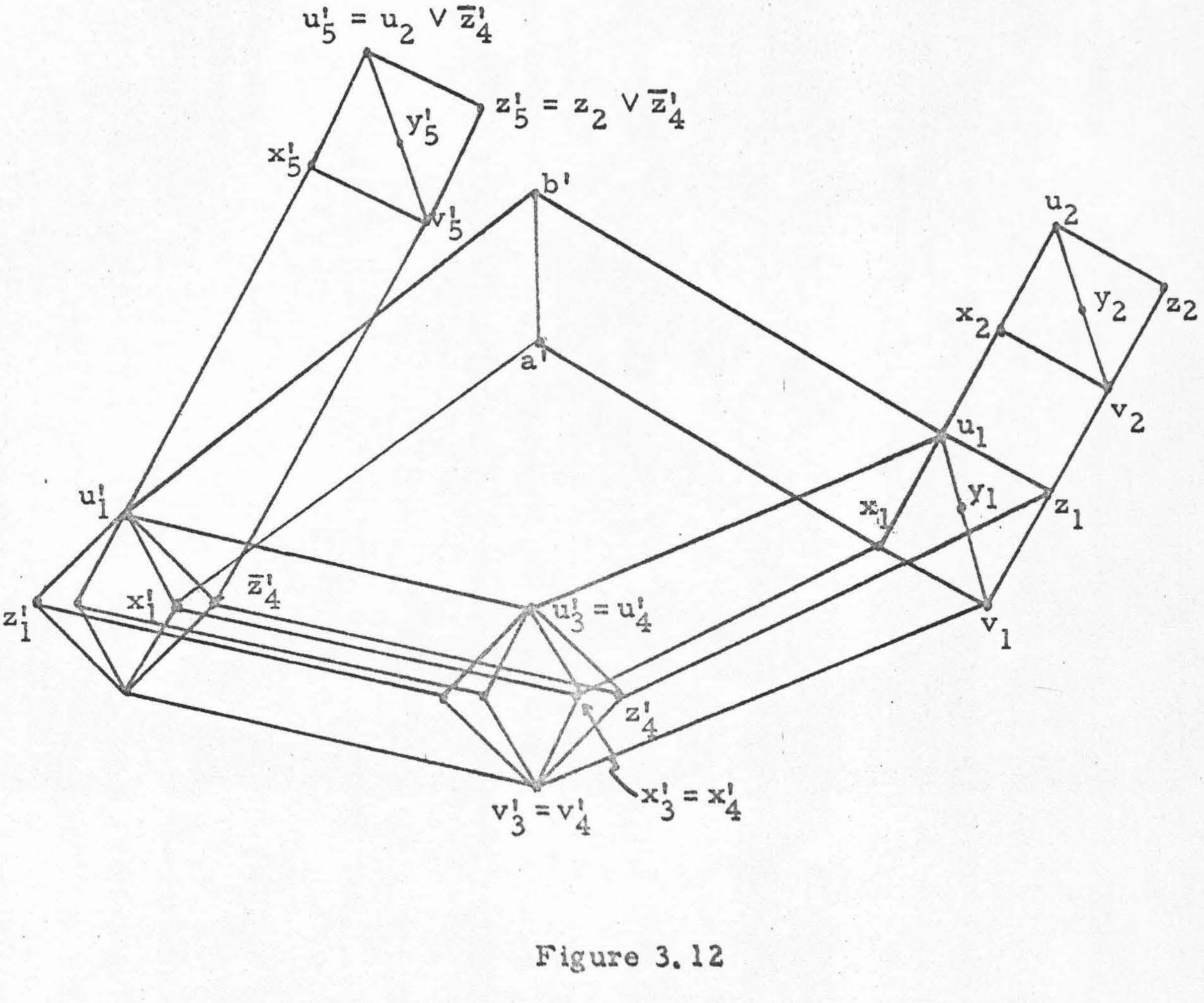


Figure 3.12

Since $x_3' = x_4'$ and since z_4' is a relative complement of x_4' in $u_4'/v_4' = u_3'/v_3'$, and $\bar{z}_4'$ is the image of z_4' under the isomorphism $u_3'/v_3' \nearrow u_1'/v_1'$, it follows that $\bar{z}_4'$ is a relative complement of x_1' in u_1'/v_1'. Hence the sublattice generated by z_1', x_1' and $\bar{z}_4'$ is a homomorphic image of the lattice diagramed in Fig. 3.13.

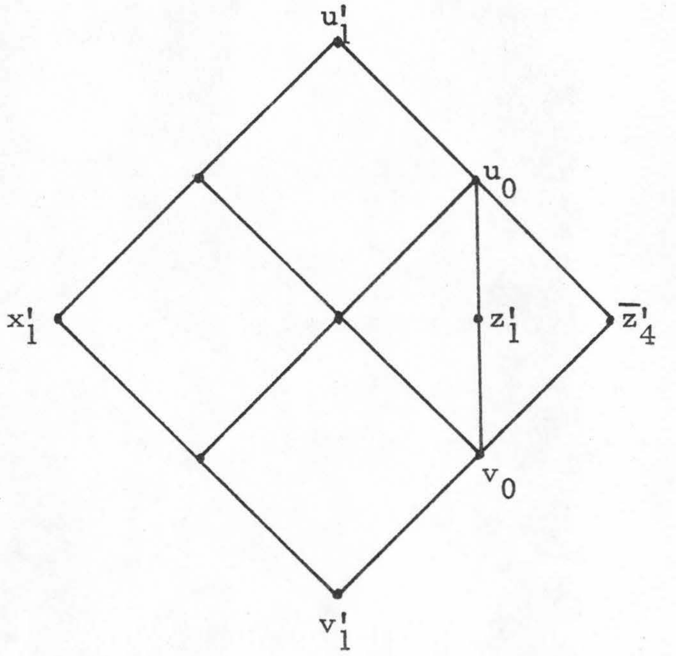


Figure 3.13

Let D_0 denote the diamond in this sublattice lattice. If this diamond is nondegenerate then D_0, $(D_2')_{u_2'/u_2' \wedge v}$ and $(D_5')_{u_0 \vee v_5'/v_5'}$ form a sublattice which has A_5 as a homomorphic image. Hence D_0 is degenerate, which implies $\bar{z}_4' = z_1'$. In this case D_2', D_1' and D_5' form one of the lattices pictured in Fig. 3.14.

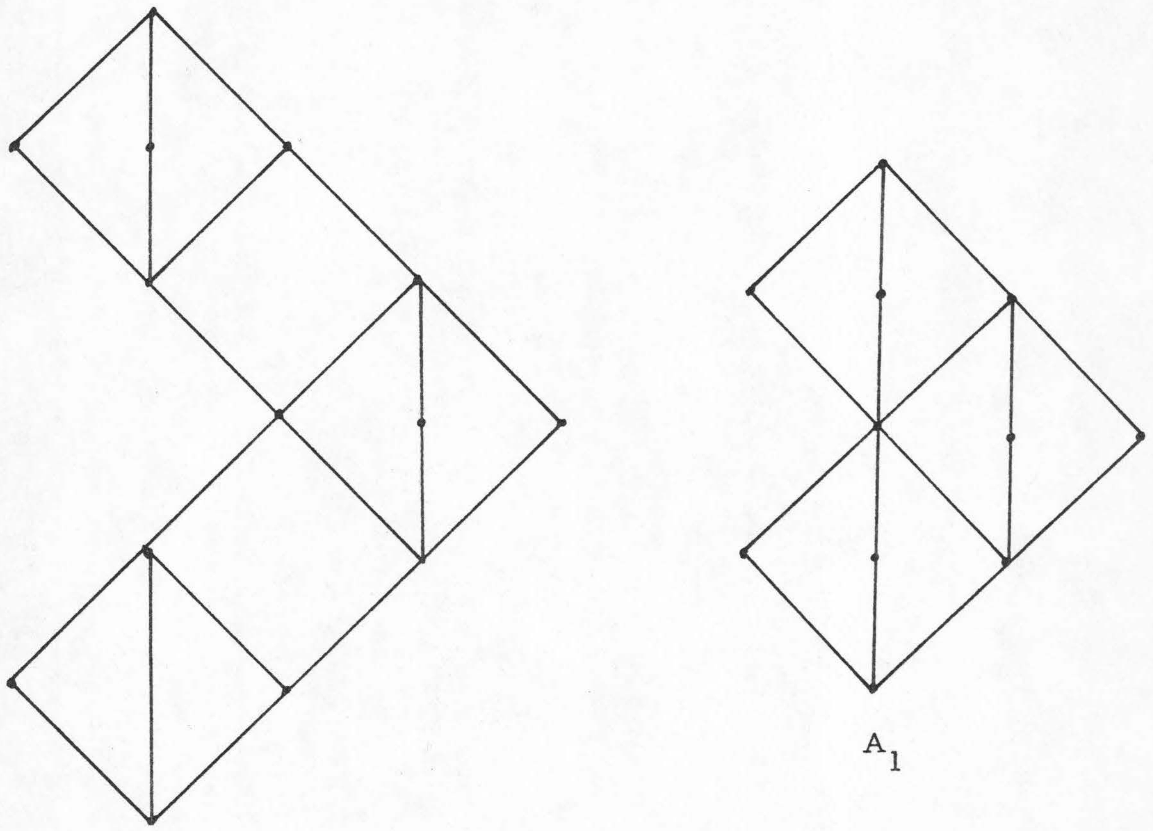


Figure 3.14

Remarks. The above arguments show that if L has two diamond sublattices D = (v, x, y, z, u) and D' = (v', x', y', z', u') such that u/z ↗ z'/v' and u' is not the greatest element of L then one of the lattices of Fig. 3.14 is a sublattice of L. Furthermore, the two lower diamonds of Fig. 3.14 are $(D)_{b/a}$ and $(D')_{b \vee v'/a \vee v'}$ for some a, b such that $z \leq a < b \leq u$.

The same arguments can also be used to show that if D = (v, x, y, z, u) is a sublattice of L such that u is not the greatest element of L then L has one of the following sublattices.

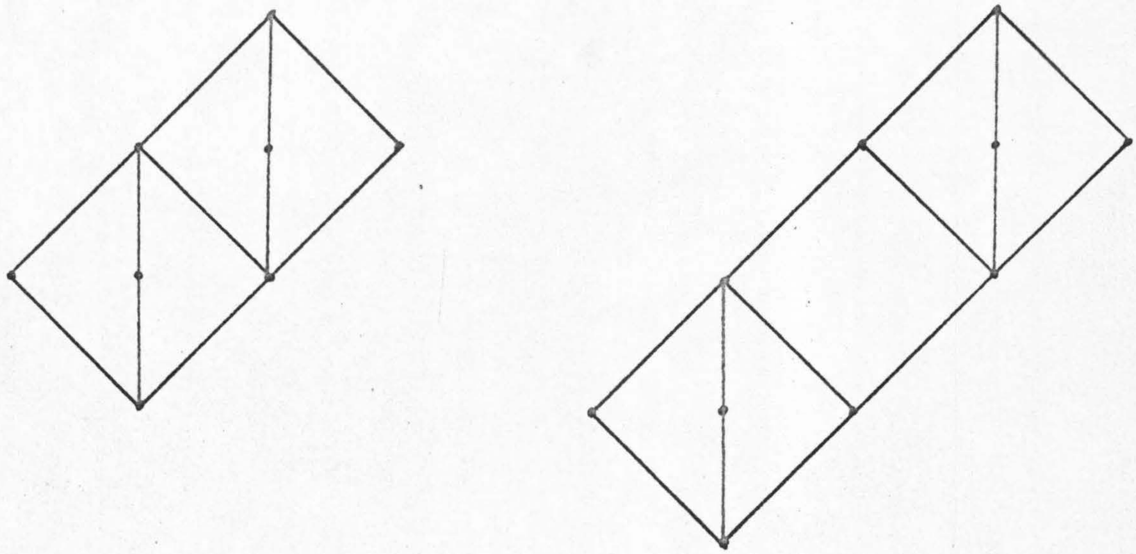


Figure 3.15

Furthermore, the lower diamond of these lattices is $D_{b/a}$ for some $z \le a < b \le u$.

Before continuing the proof of Theorem 3.1 three additional lemmas will be needed.

Lemma 3.5. Let L be a modular lattice such that $A_2, \dots, A_{10} \notin HS(L)$. Let

$$d/c = b_0/a_0 \nearrow b_1/a_1 \searrow b_2/a_2 \nearrow \cdots \searrow b_n/a_n = f/e$$

be a strongly normal sequence from d/c to f/e. Let us also assume that the associated diamonds satisfy

$$D_1 \searrow_{(1)} D_2^*,\ D_2 \nearrow_{(2)} D_3 = D_4^*,\ D_4 \nearrow_{(2)} D_5 = D_6^*, \dots, D_{n-2} \nearrow_{(2)} D_{n-1}$$

Then f $\not\leq$ c.

<u>Proof</u>: Since $D_1 \underset{(1)}{\searrow} D_2^*$ $u_2 \wedge x_1 = z_2$. We also know that $b_1 \wedge b_3 = u_1 \wedge u_3 = u_2$. Hence

$$\begin{aligned} u_3 \wedge x_1 &= u_3 \wedge u_1 \wedge x_1 \\ &= u_2 \wedge x_1 = z_2 \end{aligned}$$

Applying the Direct Product Lemma we obtain a diamond $D_3' = (v_3 \vee x_1,\ x_3 \vee x_1,\ y_3 \vee x_1,\ z_3 \vee x_1,\ u_3 \vee x_1) = (v_3', x_3', y_3', z_3', u_3')$ such that $u_1/x_1 \nearrow x_3'/v_3'$ and $D_3 \underset{(1)}{\nearrow} D_3'$ (see Fig. 3.16).

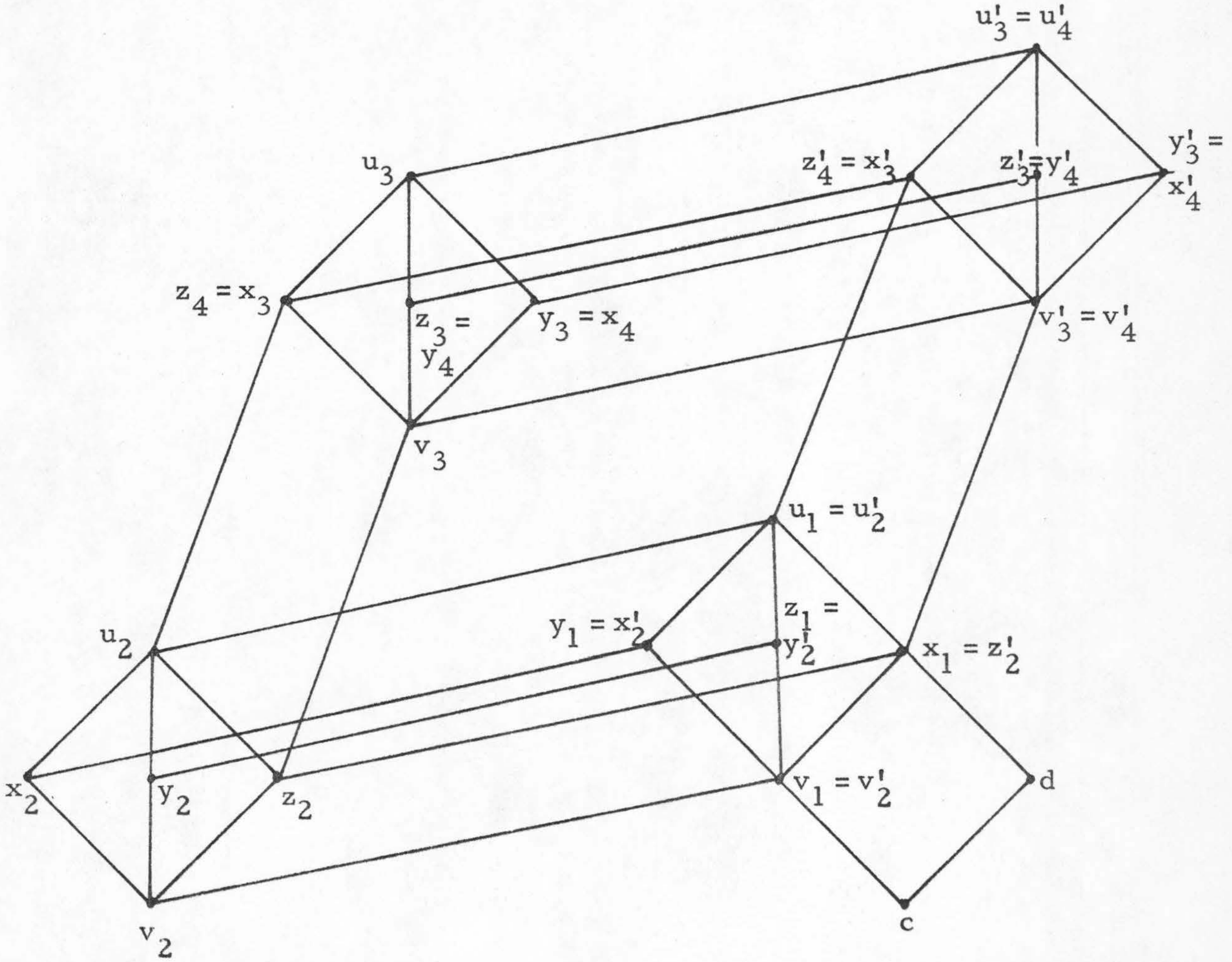


Figure 3.16

We also set $D_2' = (v_2', x_2', y_2', z_2', u_2') = (v_2 \vee v_1, x_2 \vee v_1, y_2 \vee v_1, z_2 \vee v_1, u_2 \vee v_1) = (v_1, y_1, z_1, x_1, u_1) = (D_1^*)^*$. Also set $D_4' = (D_3')^*$.

Let $r = z_4' \wedge v_5$ and $s = r \vee u_3$. Then it follows that

$$u_4'/z_4' \searrow s/r \nearrow x_5/v_5$$

is a normal sequence of transposes. From this it follows that $y_5 \wedge s = r = s \wedge z_4'$. Hence the lattice generated by y_5, s and z_4' is a homomorphic image of the lattice given in Fig. 3.17.

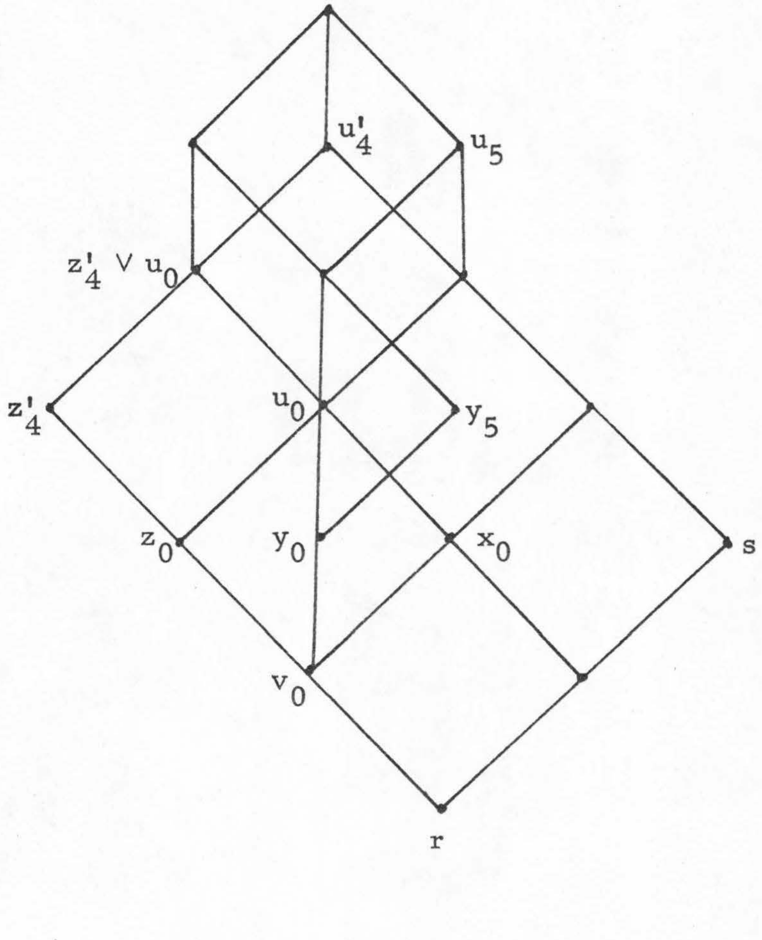


Figure 3.17

If the diamond of this lattice, which we denote D_0, is non-degenerate then, since $v_0 \vee v'_4 = v_0 \vee z_4 \vee v'_4 = z'_4$, we can invoke Corollary 2.3 on the diamonds D_0 and $(D'_4)_{z'_4 \vee u_0 / z'_4}$ to arrive at a contradiction. Hence the sublattice generated by z'_4, y_5 and s is distributive. Similarly, the sublattice generated by z'_4, z_5 and s is distributive. Hence

$$\begin{aligned} u_5 \wedge z'_4 &= (s \vee y_5) \wedge z'_4 \wedge (s \vee z_5) \wedge z'_4 \\ &= [(s \wedge z'_4) \vee (y_5 \wedge z'_4)] \wedge [(s \wedge z'_4) \vee (z_5 \wedge z'_4)] \\ &= [r \vee (y_5 \wedge z'_4)] \wedge [r \vee (z_5 \wedge z'_4)] \\ &= y_5 \wedge z'_4 \wedge z_5 \wedge z'_4 \\ &= v_5 \wedge z'_4 \\ &= r \end{aligned}$$

The Direct Product Lemma yields a diamond $D'_5 = D_5 \vee z'_4 = (v_5 \vee z'_4,\ x_5 \vee z'_4,\ y_5 \vee z'_4,\ z_5 \vee z'_4,\ u_5 \vee z'_4)$ such that $D_5 \nearrow_{(1)} D'_5$ and $u'_4/z'_4 \nearrow_{(1)} x'_5/v'_5$. Let $D'_6 = (D'_5)^*$. Continuing in this way we obtain diamonds $D'_1 = D_1, D'_2, D'_3, \ldots, D'_{n-1}$ such that $D_k \nearrow_{(1)} D'_k$ and such that $v'_k \geq c$. From the definition of the associated diamonds we know $f/e \nearrow z_{n-1}/v_{n-1}$. We also know $z_{n-1}/v_{n-1} \nearrow z'_{n-1}/v'_{n-1}$. Hence $f/e \nearrow z'_{n-1}/v'_{n-1}$. But, since $v'_{n-1} \geq c$, this clearly implies $f \nleq c$.

<u>Lemma 3.6.</u> Let $D = (v, x, y, z, u)$ be a diamond in a modular lattice. Set $w_0 = v$, $w_4 = x$ and let $w_0 \leq w_1 \leq w_2 \leq w_3 \leq w_4$. Then there exist elements $x = t_0 \leq t_1 \leq t_2 \leq t_3 \leq t_4 = u$ and diamonds $D_i = (v_i, x_i, y_i, z_i, u_i) = D_{w_i/w_{i-1}}$ such that $w_i/w_{i-1} \nearrow x_i/v_i$ and $u_i/x_i \nearrow t_i/t_{i-1}$, $i = 1, 2, 3, 4$.

Lemma 3. 7. Assume the hypothesis of the previous lemma. Suppose also that there is another diamond $D' = (v', x', y', z', u')$ such that $u/z \nearrow z'/v'$. Let $w_i' = w_i \vee v'$, $i = 0, 1, 2, 3, 4$ and let $D_i' = (v_i', x_i', y_i', z_i', u_i')$ be the diamonds obtained by applying Lemma 3. 6 to D' and w_0', w_1', w_2', w_3', w_4' (with z' playing the role of x). Then

$$w_i/w_{i-1} \nearrow x_i/v_i \nearrow u_i/z_i \nearrow w_i \vee z/w_{i-1} \vee z \nearrow w_i'/w_{i-1}' \nearrow z_i'/v_i'$$

Furthermore $w_i \vee z = z_i' \wedge u$, $i = 0, 1, 2, 3, 4$ (see Fig. 3. 18).

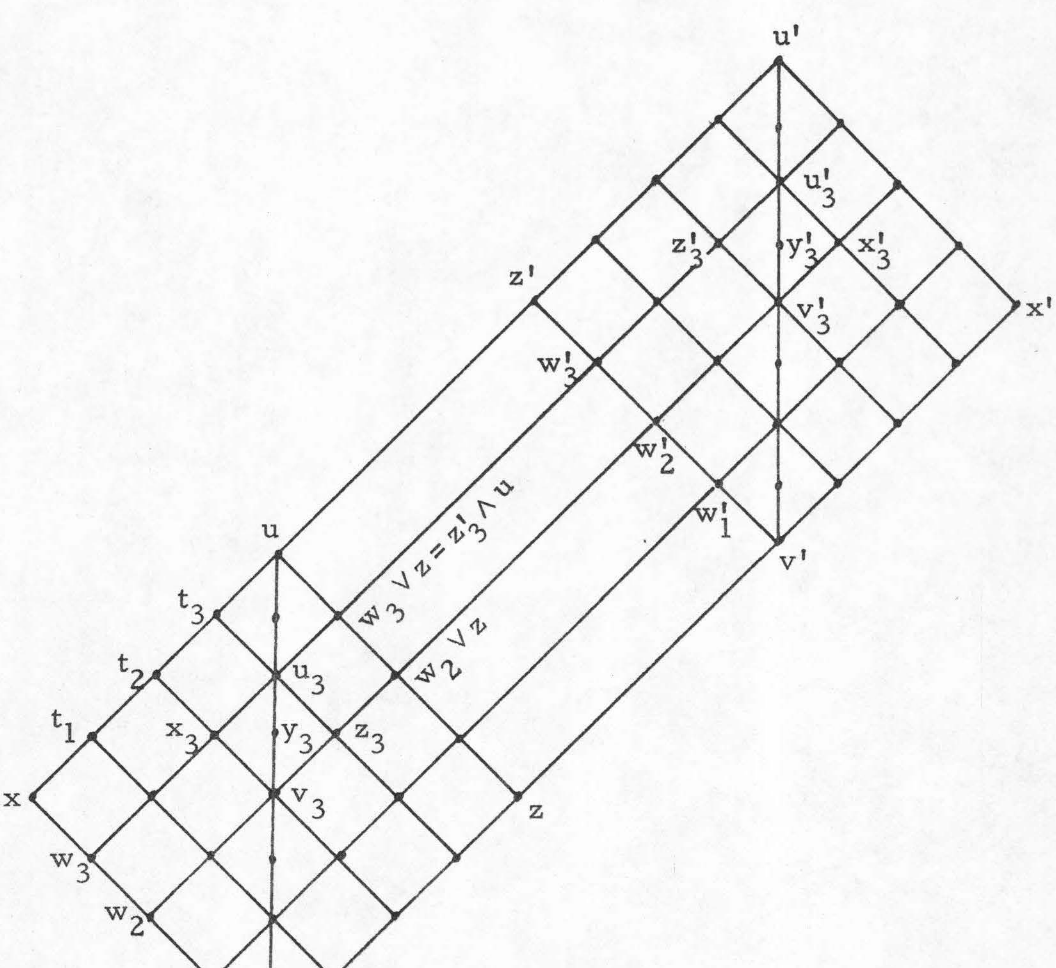


Figure 3. 18

<u>Proofs:</u> Let $u_i = (w_i \vee y) \wedge (w_i \vee z)$, $i = 1, 2, 3, 4$, $v_1 = v$, $v_i = u_{i-1}$, $i = 2, 3, 4$, $x_i = v_i \vee w_i$, $i = 1, 2, 3, 4$, $y_i = u_i \wedge (y \vee w_{i-1})$ and $z_i = u_i \wedge (z \vee w_{i-1})$. Straightforward calculations show that v_i, x_i, y_i, z_i and w_i form a diamond and that $w_i/w_{i-1} \nearrow x_i/v_i$. This is the conclusion of Lemma 3. 6.

The proof of Lemma 3. 7 will also be complete if we show $u_i/z_i \nearrow w_i \vee z/w_{i-1} \vee z \nearrow w_i'/w_{i-1}'$, and $w_i \vee z = z_i' \wedge u$.

$$\begin{aligned} u_i \vee w_{i-1} \vee z &= w_{i-1} \vee z \vee \Big((w_i \vee y) \wedge (w_i \vee z)\Big) \\ &= w_{i-1} \vee \Big((w_i \vee z) \wedge (w_i \vee y \vee z)\Big) \\ &= w_{i-1} \vee w_i \vee z = w_i \vee z \end{aligned}$$

Also $u_i \wedge (w_{i-1} \vee z) = z_i$ by definition. Hence $u_i/z_i \nearrow w_i \vee z/w_{i-1} \vee z$

Now, as $z \leq v'$

$$(w_i \vee z) \vee (w_{i-1} \vee v') = w_i \vee v'$$

and as $w_i \wedge v' \leq x \wedge v' = v \leq z$

$$\begin{aligned} (w_i \vee z) \wedge (w_{i-1} \vee v') &= w_{i-1} \vee \Big((w_i \vee z) \wedge v'\Big) \\ &= w_{i-1} \vee \Big(z \vee (w_i \wedge v')\Big) \\ &= w_{i-1} \vee z \end{aligned}$$

To see that $w_i \vee z = z_i' \wedge u$, first note $u_i' = (w_i' \vee y') \wedge (w_i' \vee x')$ $= (w_i \vee v' \vee y') \wedge (w_i \vee v' \vee x') = (w_i \vee y') \wedge (w_i \vee x')$ and $z_i' = v_i' \vee w_i'$ $= v_i' \vee w_i$ where $v_i' = u_{i-1}'$, $i = 2, 3, 4$ and $v_1' = v'$. Also, as $u \leq z'$, $u \wedge (w_i \vee y') = w_i \vee (u \wedge y') = w_i \vee (y \wedge z' \wedge y') = w_i \vee (u \wedge v') = w_i \vee z$.

Similarly $u \wedge (w_i \vee x') = w_i \vee z$. Hence

$$u \wedge u_i' = u \wedge (w_i \vee y') \wedge (w_i \vee x') \wedge u = w_i \vee z$$

Thus if i is 2, 3, or 4

$$u \wedge z_i' = u \wedge (v_i' \vee w_i) = u \wedge (u_{i-1}' \vee w_i)$$
$$= w_i \vee (u \wedge u_{i-1}') = w_i \vee w_i \vee z = w_i \vee z$$

If i = 1 then

$$u \wedge z_1' = u \wedge (v_1' \vee w_1) = u \wedge (v' \vee w_1)$$
$$= w_1 \vee (u \wedge v') = w_1 \vee z$$

This completes the proof.

Now we return to the proof of Theorem 3.1. Recall that we have shown that L has three diamond sublattices $D = (v, x, y, z, u)$, $D' = (v', x', y', z', u')$ and $D'' = (v'', x'', y'', z'', u'')$ such that

(11) $\quad u/z \nearrow z'/v' \quad$ and $\quad u'/z' \nearrow z''/v''$

The diamonds D, D', D'' form one of the sublattices of Fig. 3.14.

If these diamonds are isometric diamonds the theorem is true. Hence there exists $w_1 \in L$ such that $v < w_1 < x$. Applying the previous two lemmas to the diamonds D and D' and also D' and D'' we obtain diamonds $D_2' = (v_2', x_2', y_2', z_2', u_2') = D_{w_1/v}$, $D_1' = (v_1', x_1', y_1', z_1', u_1') = (D')_{w_1 \vee v'/v'}$, $D_4' = (v_4', x_4', y_4', z_4', u_4') = D_{x/w_1}$, $D_3' = (v_3', x_3', y_3', z_3', u_3') = (D')_{x \vee v'/w_1 \vee v'} = (D')_{z'/w_1 \vee v'}$, $D_5' = (v_5', x_5', y_5', z_5', u_5') = (D'')_{u_1' \vee v''/v_1' \vee v''}$ and $D_6' = (v_6', x_6', y_6', z_6', u_6') = (D'')_{u_3' \vee v''/v_3' \vee v''}$ such that

$$x/w_1 \nearrow x_4'/v_4' \nearrow u_4'/z_4' \nearrow z'/v' \vee w_1 \nearrow z_3'/v_3'$$

$$u_1'/z_1' \nearrow z_5'/v_5' \qquad \text{and} \qquad u_3'/z_3' \nearrow z_6'/v_6'$$

This is represented in Fig. 3.19.

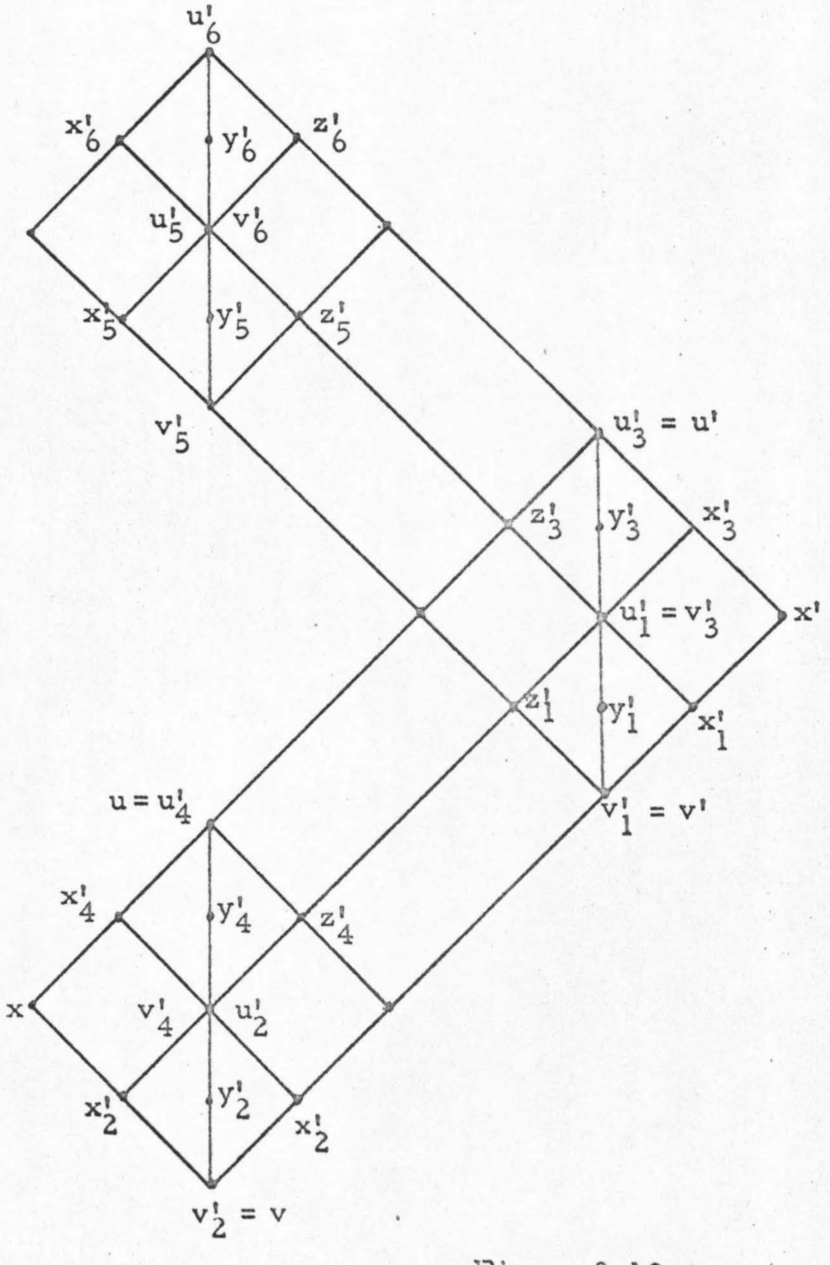


Figure 3.19

Since L is subdirectly irreducible and $x_1' \leq v_3'$ there exist subquotients d/c of x_3'/v_3' and f/e of x_1'/v_1' which are connected by a strongly normal sequence of transposes. If $d/c \searrow b_1/a_1 \nearrow \dots f/e$, then the first associated diamond, D_1, together with $(D_3')_{d/c}$ and $(D_5')_{v_6' \vee d/v_6 \vee c}$ form a sublattice which has A_5 as a homomorphic image. Similarly, if $b_{n-1}/a_{n-1} \nearrow f/e$ then D_{n-1}, $(D_1')_{f/e}$ and $(D_5')_{v_5' \vee f/v_5' \vee e}$ form a sublattice with A_5 as a homomorphic image. Hence it may be assumed that the sequence connecting d/c to f/e has the form:

(12) $$d/c \nearrow b_1/a_1 \searrow b_2/a_2 \nearrow \dots \nearrow b_{n-1}/a_{n-1} \searrow b_n/a_n = f/e$$

Furthermore, we may assume this sequence satisfies the conditions of Theorem 1.1. With the aid of Lemma 3.5, Lemma 3.4 and Theorem 1.1, we can conclude that the diamonds associated with (12) satisfy

(13) $$D_1 \underset{(2)}{\searrow} D_2 = D_3^*,\ D_3 \underset{(2)}{\searrow} D_4 = D_5^*, \dots,\ D_{n-3} \underset{(2)}{\searrow} D_{n-2} = D_{n-1}^*$$

It follows from (12) and (13) that

(14) $$v_1' \leq e \leq v_k', \qquad k = 1, \dots, n-1$$

Applying Lemma 3.6 and the dual of Lemma 3.7 to the elements $v' < x_1' \leq c \wedge x' \leq d \wedge x' \leq x'$, diamonds $D_7' = (v_7', x_7', y_7', z_7', u_7')$ and $D_{10}' = (v_{10}', x_{10}', y_{10}', z_{10}', u_{10}')$ are obtained such that

(15) $$d/c \nearrow x_7'/v_7'$$

and

(16) $u_7'/x_7' \searrow z_7'/v_7' \searrow z' \wedge z_7'/z' \wedge v_7' \searrow u \wedge z_7'/u \wedge v_7' \searrow u_{10}'/z_{10}'$

and

(17) $$v_{10}' \vee v_1' = z' \wedge v_7'$$

Since $u_{10}'/z_{10}' \nearrow z' \wedge z_7'/z' \wedge v_7'$ by (16), $z_{10}' \leq z' \wedge v_7'$. Hence by (17)

(18) $$z_{10}' \leq v_{10}' \vee v_1'$$

Now let $D_8' = (D_6')_{v_6' \vee d/v_6' \vee c'}$ and let $c' = v_1 \wedge v_7'$ and $d' = d \vee c'$. The situation is represented in Fig. 3.20.

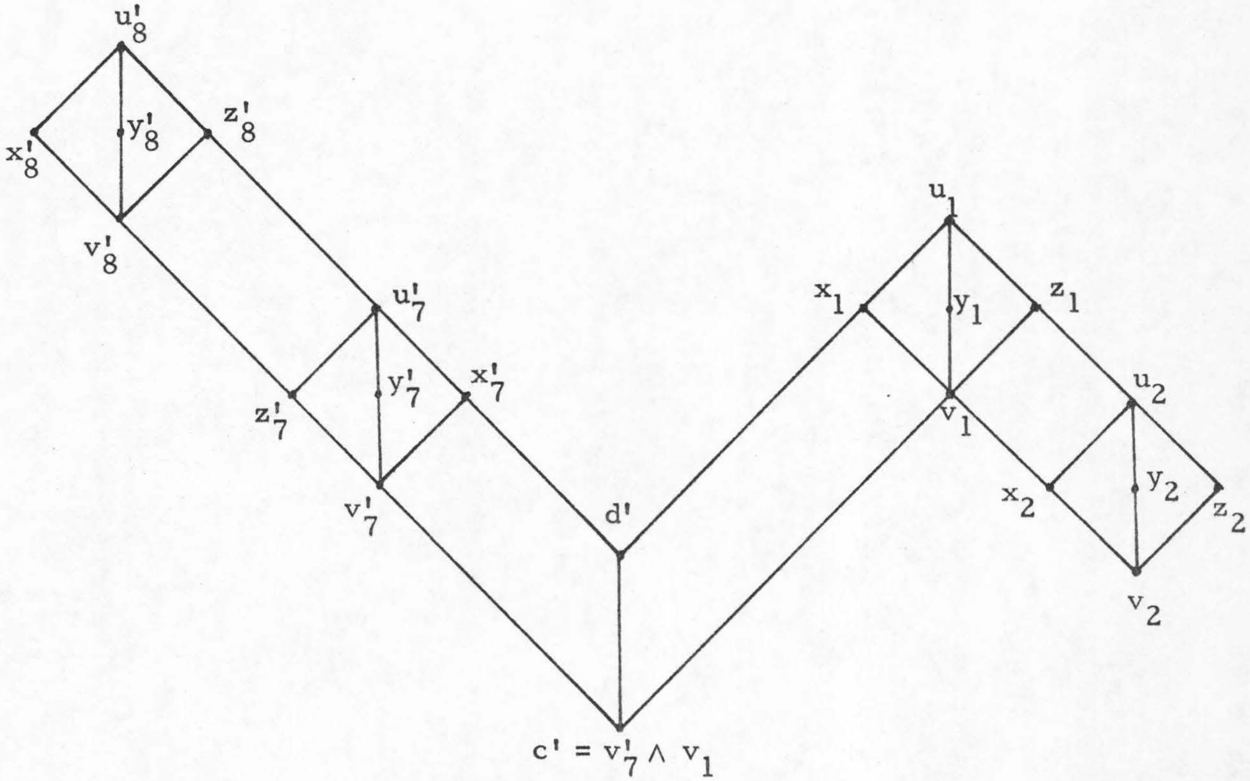


Figure 3.20

Notice that this situation is the dual of the situation represented in Fig. 3.10. By using the dual arguments used in that case, we can conclude that there exists a diamond $D_9' = D_2 \wedge z_7'$ such that

$z_7'/v_7' \searrow u_2 \wedge z_7'/x_2 \wedge z_7' = u_9'/x_9'$.

Let $s = u_9' \vee u_{10}'$ and $r = s \wedge v_7'$. This situation is represented in Fig. 3.21.

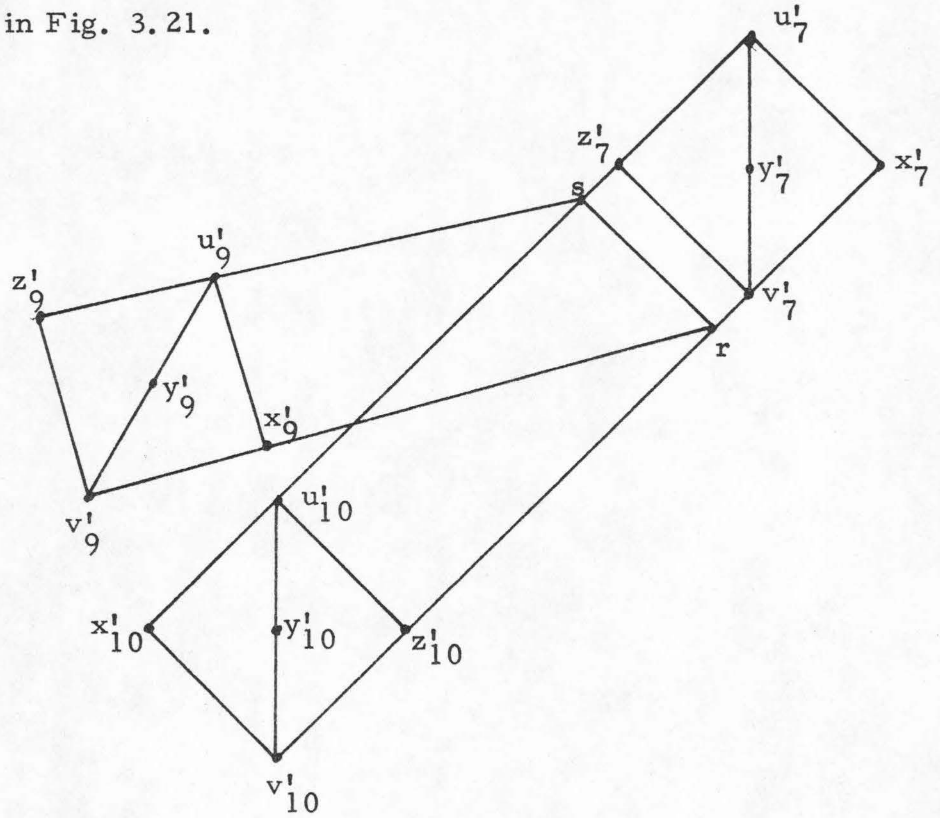


Figure 3.21

Let L' be the sublattice generated by u_9', r and x_{10}'. L' is a homomorphic image of the lattice given in Fig. 3.11 with $a' = r$, $b' = s$, $u_1 = u_9'$, $x_1 = x_9'$ and $y_1' = x_{10}'$. If the diamond D_0 in this sublattice is nondegenerate then as before D_0, $(D_7')_{v_7' \vee u_0/v_7 \vee x_0}$ and $(D_9')_{u_9' \wedge z_0/u_9' \wedge v_0}$ form a sublattice with A_5 as a homomorphic image (see Fig. 3.11). Similar arguments show that the sublattices generated by $\{u_9', r, y_{10}'\}$, $\{u_{10}', r, z_9'\}$ and $\{u_{10}', r, y_9'\}$ are distributive. As before this implies that

(19) $$v'_{10} \vee u'_9 = v'_9 \vee u'_{10} = s$$

By the Direct Product Lemma this yields two new diamonds $D'_{11} = D'_9 \wedge u'_{10}$ and $D'_{12} = D'_{10} \wedge u'_9$. Since $u'_{12} = u'_{10} \wedge u'_9 = u'_{11}$ and $x'_{11} = x'_9 \wedge u'_{10} = u'_9 \wedge r \wedge u'_{10} = u'_9 \wedge z'_{10} = z'_{12}$, we may apply Corollary 3.3. Thus

(20) $$v'_9 \wedge u'_{10} = v'_{11} = v'_{12} = v'_{10} \wedge u'_9$$

By definition $v'_9 = v_2 \wedge z'_7$. By (14) $v'_1 \leq v_2$. Moreover, $z'_7 \geq c \geq v'_1$. Hence

(21) $$v'_9 \geq v'_1$$

Now by (20) and (21) we have

(22) $$v'_{10} \geq v'_{10} \wedge u'_9 = v'_9 \wedge u'_{10} \geq v'_1 \wedge u'_{10}$$

Also $v'_7 \geq v'_3 \geq v'_1$ by their definitions and $v'_7 \geq z'_{10} \geq v'_{10}$ by (16). Thus $v'_7 \geq v'_{10} \vee v'_1$. Hence, by (22), (16) and (18)

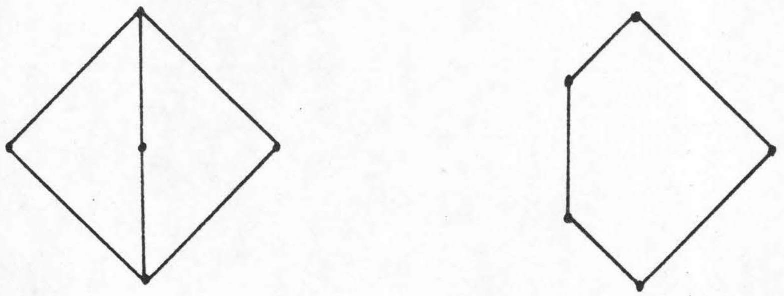
(23) $$\begin{aligned} v'_{10} &= v'_{10} \vee (v'_1 \wedge u'_{10}) \\ &= u'_{10} \wedge (v'_{10} \vee v'_1) \\ &= u'_{10} \wedge v'_7 \wedge (v'_{10} \vee v'_1) \\ &= z'_{10} \wedge (v'_{10} \vee v'_1) = z'_{10} \end{aligned}$$

This last contradiction proves the theorem.

CHAPTER IV

THE MAIN STRUCTURE THEOREM

Let $\mathcal{D}$ be the variety (equatorial class) of all distributive lattices and $\mathcal{M}_4^\infty$ be the variety generated by all modular width four lattices. It is well-known that if $L \notin \mathcal{D}$ then either $M_3 \in S(L)$ or $N_5 \in S(L)$.



M_3 N_5

Figure 4.1

In this chapter we prove an analogous result for $\mathcal{M}_4^\infty$: If $L \notin \mathcal{M}_4^\infty$ then either $A_k \in HS(L)$ for some k, $2 \le k \le 10$ or $N_5 \in S(L)$.

We begin with

<u>Lemma 4.1</u>. Let $D = (v, x, y, z, u)$ be an isometric diamond in L (i.e., $x \succ v$). Let us suppose that $A_4, A_7, A_9 \notin HS(L)$ and that there is another diamond $D_1 = (v_1, x_1, y_1, z_1, u_1)$ such that

(1) $$z/v \nearrow x_1/v_1$$

Then either

(2) $$u/v \nearrow u_1/v_1$$

or

(3) $$u/v \wedge v_1 \nearrow x_1/v_1$$

Proof: Note that $z \leq u_1 \wedge u$. Equality cannot hold, for otherwise Corollary 2.3 would give a contradiction. Since $u > z$ this means $u \leq u_1$. Suppose $u \leq x_1$ as well. Then, since $z \leq u \leq v_1$ would contradict (1), $u \vee v_1 = x_1$, again since $v_1 \prec x_1$. Thus we see that (3) holds in this case.

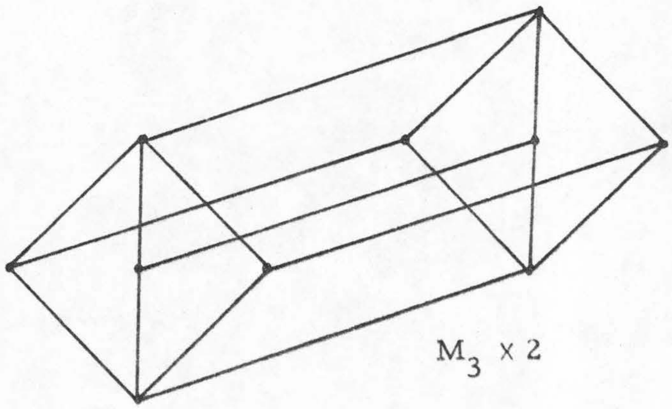
Now suppose that $x_1 \ngeq u$; then $u \wedge x_1 = z$ and $u \vee x_1 = u_1$. Thus, by (1),

(4) $$\begin{aligned} u \vee v_1 &= u \vee z \vee v_1 \\ &= u \vee x_1 = u_1 \end{aligned}$$

(5) $$\begin{aligned} u \wedge v_1 &= u \wedge x_1 \wedge v_1 \\ &= z \wedge v_1 = v \end{aligned}$$

Hence (2) holds in this case.

Theorem 4.2. Let L be a modular, subdirectly irreducible lattice such that $A_2, \ldots, A_{10} \notin HS(L)$. Then $M_3 \times 2 \notin S(L)$.



$M_3 \times 2$

Figure 4.2

Proof: If the conclusion of this theorem fails, then there exist diamonds $D = (v, x, y, z, u)$ and $D' = (v', x', y', z', u')$ such that

$$(1) \qquad D \nearrow_{(1)} D'$$

By Theorem 3.1, L is weakly atomic. Consequently there exist $a, b \in L$ such that $v \leq a \prec b \leq x$. Let $a' = a \vee x'$ and $b' = b \vee x'$. Then $D_{b/a} \nearrow_{(1)} D'_{b'/a'}$, and so $D_{b/a}$ and $D'_{b'/a'}$ form a lattice isomorphic to $M_3 \times 2$. Hence we may assume $v \prec x$. There also must exist e and f such that $v \leq e \prec f \leq v'$. Now the diamonds $(v \vee e, x \vee e, y \vee e, z \vee e, u \vee e)$ and $(v \vee f, x \vee f, y \vee f, z \vee f, u \vee f)$ together form an isometric sublattice isomorphic to $M_3 \times 2$. Hence we assume $v \prec v'$, i.e., D and D' together form an isometric sublattice. Recall that a sublattice L' of L is called isometric if a covers b in L' implies that a covers b in L.

Since L is subdirectly irreducible there is a strongly normal sequence of transposes

$$(2) \qquad b_0/a_0 = v'/v, \ b_1/a_1, \ b_2/a_2, \ \ldots, \ b_n/a_n = z'/v'$$

which satisfies the conditions of Theorem 1.1. Furthermore it may be assumed that

$$(3) \qquad \text{p.d.}(v'/v, z'/v') \leq \min \{\text{p.d.}(v'/v, x'/v'), \ \text{p.d.}(v'/v, y'/v')\}$$

Suppose $v'/v \nearrow b_1/a_1 \searrow b_2/a_2 \nearrow b_3/a_3 = z'/v'$ and $D_1 \searrow_{(1)} D_2^*$. It follows immediately from the definitions of the associated diamonds that $z_2 = v' \wedge x_1$, and $x_1 = z_2 \vee v'$. Thus $z_2 = v' \wedge (z_2 \vee v') = v'$ and $x_1 = z_2 \vee v' = v'$. Thus $x_1 = z_2$ so that $D_1 = D_2^*$.

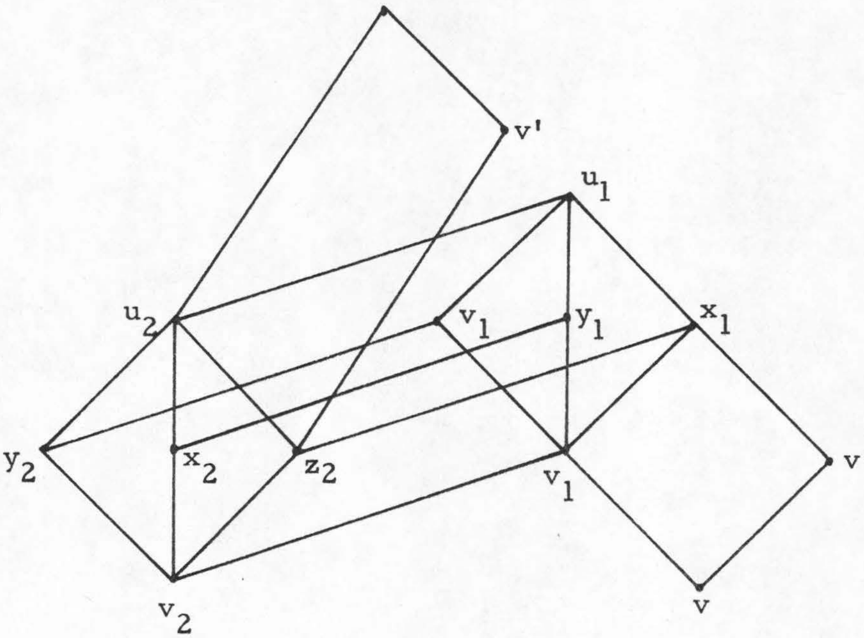


Figure 4. 3

The sequence $v'/v \searrow b_1/a_1 \nearrow b_2/a_2 \searrow b_3/a_3 = z'/v'$ is impossible because $b_1 \leq v'$ and p. d. $(b_1/a_1, z'/v') = 2$ are contradictory.

Recall that if the number under the arrow between D_k and D^*_{k+1} is one, we say D_k transposes to D^*_{k+1}; if it is a two, D_k translates to D_{k+1}.

If D_{n-2} transposes to D^*_{n-1}, then $D_{n-2} = D^*_{n-1}$, provided $n > 3$, since the sequence (2) satisfies the conditions of Theorem 1. 1. The above argument shows that this is the case even if $n = 3$.

Let us suppose that

(4) $$b_{n-1}/a_{n-1} \searrow b_n/a_n = z'/v'$$

Also suppose that

(5) $$D_{n-2} = D^*_{n-1}$$

Lemma 3.4 together with (4) and (5) imply

(6) $$D_{n-3} \underset{(2)}{\searrow} D_{n-2}$$

In fact, since the sequence (2) satisfies the conditions of Theorem 1.1, we have either

(7) $$D_2 \underset{(2)}{\searrow} D_3 = D^*_4,\ D_4 \underset{(2)}{\searrow} D_5 = D^*_6,\ \ldots,\ D_{n-3} \underset{(2)}{\searrow} D_{n-2} = D^*_{n-1}$$

or

(8) $$D_2 = D^*_3,\ D_3 \underset{(2)}{\searrow} D_4 = D^*_5,\ \ldots,\ D_{n-3} \underset{(2)}{\searrow} D_{n-2} = D^*_{n-1}$$

depending on whether n is odd or even. In either case $v_2 \geq v_{n-1} \geq a_n = v'$. Thus $a_2 \geq v_2 \geq v'$. But this contradicts p. d. $(b_2/a_2, v'/v) = 2$. We conclude that (5) cannot hold and hence

(9) $$D_{n-2} \underset{(2)}{\nearrow} D_{n-1}$$

Applying Lemma 4.1 to the diamonds D' and D_{n-1} we conclude that either

(10) $$u'/v' \nearrow u_{n-1}/v_{n-1}$$

or

(11) $$u'/u' \wedge v_{n-1} \nearrow x_{n-1}/v_{n-1}$$

Suppose (10) holds. Consider the set $\{x_{n-1} = v_{n-1} \vee z', y_{n-1}, z_{n-1}, x' \vee v_{n-1}, y' \vee v_{n-1}\}$. By (10) these are all atoms in u_{n-1}/v_{n-1}. If there are four distinct elements in this set then u_{n-1}/v_{n-1} contains a sublattice isomorphic to M_4 which, together with D_{n-2} form a sublattice which has A_8 as a homomorphic image. Thus we may assume $y' \vee v_{n-1} = y_{n-1}$ and $x' \vee v_{n-1} = z_{n-1}$.

An argument dual to one used above shows that (4) implies that $n \geq 4$. Thus

(12) $$z'/v' \nearrow b_{n-1}/a_{n-1} \searrow b_{n-2}/a_{n-2} \nearrow b_{n-3}/a_{n-3} \searrow b_{n-4}/a_{n-4}$$

Since $v_{n-1} = a_n \vee a_{n-2} = v' \vee v_{n-2}$ we have that

(13) $$v_{n-2} \vee x' = v_{n-2} \vee v' \vee x' = v_{n-1} \vee x' = z_{n-1}$$

Thus we may apply the Direct Product Lemma to the sublattices z_{n-1}/x' and z_{n-1}/v_{n-2} to obtain a new diamond $D'_{n-2} = (v_{n-2} \wedge x',$ $x_{n-2} \wedge x', y_{n-2} \wedge x', z_{n-2} \wedge x', u_{n-2} \wedge x') = (v'_{n-2}, x'_{n-2}, y'_{n-2}, z'_{n-2},$ $u'_{n-2})$.

Now it is easy to check that

(14) $$x'/v' \searrow z'_{n-2}/v'_{n-2} \nearrow u_{n-2}/y_{n-2} \searrow b_{n-4}/a_{n-4}$$

Consequently, p. d. $(x'/v', v'/v) \leq n - 1 < n =$ p. d. $(z'/v', v'/v)$, contradicting (3). Hence we conclude (10) cannot hold and so (11) must hold. As before, if $u' \wedge v_{n-1} \notin \{x', y'\}$ then u'/v' contains M_4 as a sublattice which together with D_{n-1} form a sublattice with A_7 as a homomorphic image. Thus we may assume $v_{n-1} \wedge u' = x'$. Since $D_{n-2} \overset{(2)}{\nearrow} D_{n-1}$ we have $z_{n-1}/v_{n-1} \searrow u_{n-2}/x_{n-2}$. Moreover $v_{n-1} = v_{n-2} \vee v'$. Now as in the proof of Lemma 3.4, D', D_{n-1}, D_{n-2} generate a sublattice with A_3

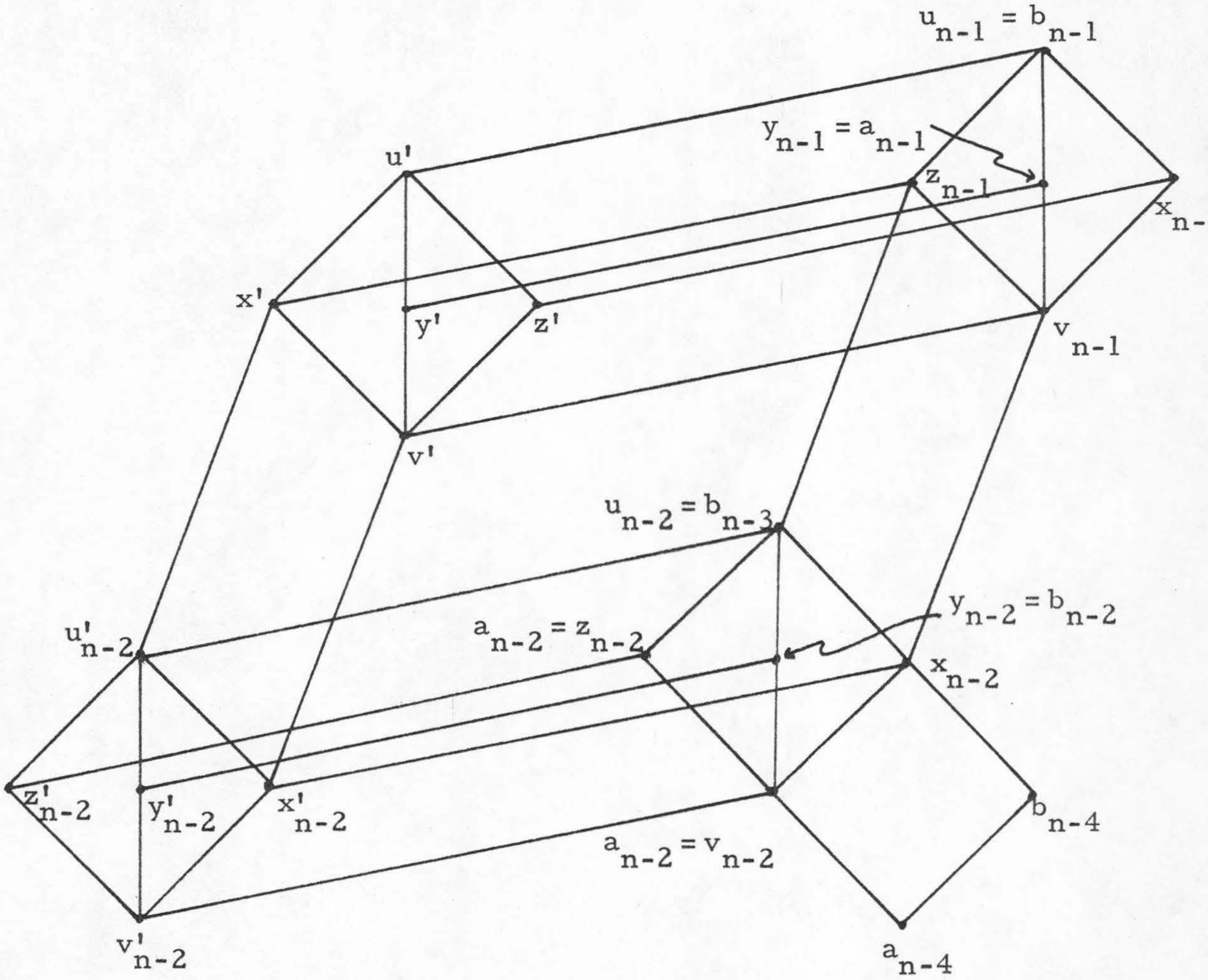


Figure 4. 4

as a homomorphic image. This contradiction shows that (11) cannot hold. It follows that assumption (4) cannot hold. Hence, it may be assumed that

(15) $$b_{n-1}/a_{n-1} \nearrow b_n/a_n = z'/v'$$

This leads to the following four cases

(16a) $$v'/v = b_0/a_0 \searrow b_1/a_1 \nearrow \dots\ b_{n-1}/a_{n-1} \nearrow b_n/a_n = z'/v'$$

with

(16b) $$D_1 \underset{(1)}{\nearrow} D^*_2,\ D_2 \underset{(2)}{\searrow} D_3 = D^*_4,\ D_4 \underset{(2)}{\searrow} D_5 = D^*_6,\ \dots,\ D_{n-2} \underset{(2)}{\searrow} D_{n-1}$$

or

(17a) $$v'/v = b_0/a_0 \searrow b_1/a_1 \nearrow b_2/a_2,\ \dots,\ b_{n-1}/a_{n-1} \nearrow b_n/a_n$$

with

(17b) $$D_1 \underset{(2)}{\nearrow} D_2 = D^*_3,\ D_3 \underset{(2)}{\nearrow} D_4 = D^*_5,\ \dots,\ D_{n-2} = D^*_{n-1}$$

or

(18a) $$b_0/a_0 \nearrow b_1/a_1 \searrow b_2/a_2,\ \dots,\ b_{n-1}/a_{n-1} \nearrow b_n/a_n$$

with

(18b) $$D_1 \underset{(1)}{\searrow} D^*_2,\ D_2 \underset{(2)}{\nearrow} D_3 = D^*_4,\ \dots,\ D_{n-3} \underset{(2)}{\nearrow} D_{n-2} = D^*_{n-1}$$

or

(19a) $$b_0/a_0 \nearrow b_1/a_1 \searrow b_2/a_2,\ \dots,\ b_{n-1}/a_{n-1} \nearrow b_n/a_n$$

with

(19b) $$D_1 \underset{(2)}{\searrow} D_2 = D^*_3,\ D_3 \underset{(2)}{\searrow} D_4 = D^*_5,\ \dots,\ D_{n-2} \underset{(2)}{\searrow} D_{n-1}$$

Let us suppose that the situation of equations (18a) and (18b) holds. If w is any element of L, let $D^*_2 \vee w$ denote $(v_2 \vee w,\ y_2 \vee w,\ z_2 \vee w,\ x_2 \vee w,\ u_2 \vee w)$. Then, since $D_1 \underset{(1)}{\searrow} D^*_2$, $D_1 = D^*_2 \vee v_1$.

Furthermore, as everything in D_2^* is greater than or equal to v_2, $D_2^* \vee v = D_2^* \vee v_2 \vee v = D_2^* \vee v_1 = D_1$, since $v_2 \vee v = a_2 \vee a_0 = v_1$. Now as in the example after Lemma 3.4 (18b) implies $u_2 \leq z'$. Hence, since $D_2^* \vee v = D_1$, $u_1 = u_2 \vee v \leq z'$. Since $v_1 \geq v$, we have $D_1 \leq z'/v$. But the dimension of z'/v is two; thus $u_1 = z'$ and $v_1 = v$. Now the set $\{z, x_1, y_1, z_1\}$ has at least three elements, so we may assume that z, x_1, y_1 are distinct. Then the diamonds (v, z, x_1, y_1, z') and $D = (v, x, y, z, u)$ satisfy the hypotheses of Theorem 2.2, which gives a contradiction.

Now we suppose (17a) and (17b) hold. As before

(20) $$u_2 \leq z'$$

From the definition of the associated diamonds

(21) $$v'/v = b_0/a_0 \searrow u_1/x_1$$

and

(22) $$v \wedge u_2 = a_0 \wedge u_2 = x_1$$

Now if $u_2 \leq v'$, then it would follow from (21) and (22) that $u_2/x_1 \nearrow v'/v$. But $v' > v$ and $x_1 < u_1 < u_2$ by (17b). Hence we have

(23) $$u_2 \not\leq v'$$

Since $v_2 \geq z_1$ and $v \vee z_1 = v'$ and $v_2 \leq u_2 \leq z'$, $z' \geq v \vee v_2 = v \vee z_1 \vee v_2 = v' \vee v_2 \geq v'$. Thus, since $v' \prec z'$, either $v \vee v_2 = v'$ or $v \vee v_2 = z'$. In either case

(24) $$z \vee v_2 = z \vee (v \vee v_2) = z'$$

Thus we may apply the Direct Product Lemma to the sublattices z'/z and z'/v_2 to obtain a new diamond $D_2 \wedge z = (v_2 \wedge z,\ x_2 \wedge z,\ y_2 \wedge z,\ z_2 \wedge z,\ u_2 \wedge z)$. Now $x_1 \vee (v_2 \wedge z) = z \wedge (x_1 \vee v_2) = z \wedge x_2$ and $x_1 \wedge (v_2 \wedge z) = v_1 \wedge z = v_1$. Hence

$$(25) \qquad x_1/v_1 \nearrow x_2 \wedge z/v_2 \wedge z$$

Moreover,

$$(26) \qquad u_1 \wedge (u_2 \wedge z) = u_1 \wedge z = u_1 \wedge v' \wedge z = u_1 \wedge v = x_1$$

By (25) and (26) we may apply Corollary 2.3 to the diamonds $D_2 \wedge z$ and D_1 to arrive at a contradiction.

In both of the two remaining cases we have the following situation:

$$(27) \qquad z'/v' = b_b/a_n \searrow u_{n-1}/z_{n-1}$$

$$(28) \qquad u_{n-1}/x_{n-1} \nearrow z_{n-2}/v_{n-2}$$

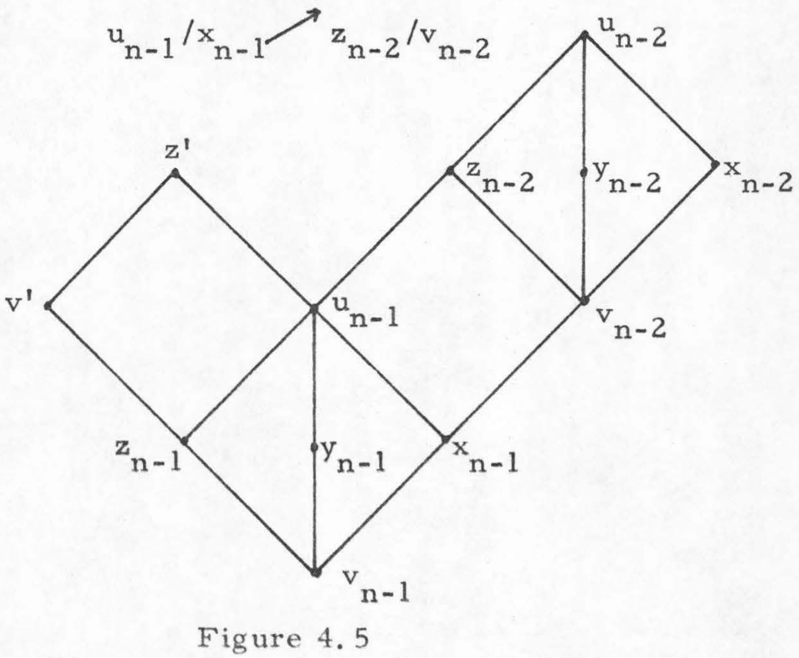


Figure 4.5

We would like to show that D', D_{n-1} and D_{n-2} generate a sublattice with A_2 as a homomorphic image. As pointed out before, in order to do this we must show that $u' \wedge u_{n-2} = u_{n-1}$. By its definition $u_{n-1} = z' \wedge u_{n-2}$. Consider the sublattice L' generated by $x_{n-1} = z' \wedge x_{n-2}$, $y' \wedge x_{n-2}$, $x' \wedge x_{n-2}$. All three pairs of these generators intersect to the least element of the L'. For example, $x' \wedge x_{n-2} \wedge y' \wedge x_{n-2} = v' \wedge x_{n-2} = v_{n-1}$. Also $x_{n-1} \vee (x' \wedge x_{n-2}) = x_{n-2} \wedge (x_{n-1} \wedge x') = x_{n-2} \wedge u'$, the greatest element of L'. Similarly $x_{n-1} \vee (x' \wedge x_{n-2}) = x_{n-2} \wedge u'$ $x_{n-2} \wedge u'$. It follows that L' is a homomorphic image of the lattice diagramed in Fig. 4. 6.

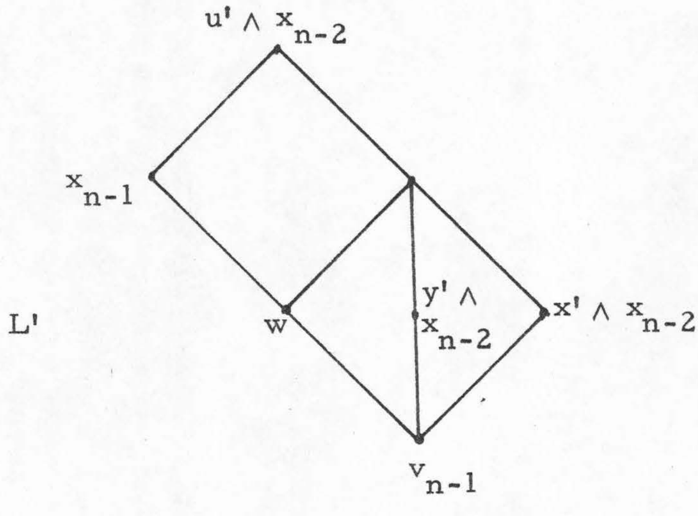


Figure 4. 6

Let $w = [(x' \wedge x_{n-2}) \vee (y' \wedge x_{n-2})] \wedge x_{n-1}$. Since $x_{n-1} > v_{n-1}$ either $w = x_{n-1}$ or $w = v_{n-1}$. If $w = v_{n-1}$ then $x_{n-1} = u' \wedge x_{n-2}$, which implies $u' \wedge u_{n-2} = u' \wedge (u_{n-1} \vee x_{n-2}) = u_{n-1} \vee (u' \wedge x_{n-2}) = u_{n-1} \vee x_{n-1} = u_{n-1}$, the desired conclusion. If $w = x_{n-1}$ then L' is a diamond, which is nontrivial as $x_{n-1} > v_{n-1}$. Moreover, $u_{n-1} \wedge (u' \wedge x_{n-2})$

$= u_{n-1} \wedge x_{n-2} = x_{n-1}$. Hence we can apply Theorem 2.2 to the diamonds L' and D_{n-1}, arriving at a contradiction. This final contradiction proves the theorem.

Remark. Let L be a modular subdirectly irreducible lattice such that $A_2, \ldots, A_{10} \notin HS(L)$. The dual to the last part of the above proof shows that the following situation cannot occur: L has three isometric diamonds $D_i = (v_i, x_i, y_i, z_i, u_i)$, i = 1, 2, 3 such that

(1) $$u_1/x_1 \nearrow z_2/v_2 \quad \text{and} \quad x_2/v_2 \searrow u_3/z_3$$

and

(2) $$x_1 \vee v_3 = v_2$$

We improve upon this in the next lemma.

Lemma 4.3. Let L be a modular subdirectly irreducible lattice such that $A_2, \ldots, A_{10} \notin HS(L)$. Then L cannot have three isometric diamonds D_i, i = 1, 2, 3, which satisfy (1).

Proof: As remarked we need only show that (2) holds. By (1)

(3) $$(u_1 \vee u_3) \vee v_2 = u_1 \vee v_2 \vee u_3 \vee v_2$$
$$= z_2 \vee x_2 = u_2$$

The Direct Product Lemma, applied to $u_2/u_1 \vee u_3$ and D_2, now yields $M_3 \times 2$ as a sublattice unless $u_2 = u_1 \vee u_3$. Thus by Theorem 4.2 we have $u_2 = u_1 \vee u_3$. Hence

$$(4)\qquad z_3 \vee x_1 = z_3 \vee (v_2 \wedge u_1) = v_2 \wedge (z_3 \vee u_1)$$
$$= v_2 \wedge [(z_2 \wedge u_3) \vee u_1]$$
$$= v_2 \wedge z_2 \wedge (u_1 \vee u_3)$$
$$= v_2 \wedge z_2 \wedge u_2 = v_2$$

Clearly $v_3 \vee x_1 \leq v_2$. Let $w = u_3 \wedge (v_3 \vee x_1)$. Now $v_3 \leq w \leq v_2$. The second inequality shows that $w \neq u_3$, $w \neq x_3$ and $w \neq y_3$. If $w = v_3$ then by the Direct Product Lemma $v_3 \vee x_1/v_3$ and D_3 generate the sublattice $M_3 \times 2$ unless $v_3 = v_3 \vee x_1$. Thus we must have $x_1 \leq v_3$. If $u_1 \leq v_3$ then $u_1 \leq v_3 \leq v_2$ which violates (1). Since $x_1 \prec u_1$, $v_3 \prec u_1 \vee v_3$ by semimodularity. If $u_1 \vee v_3 \leq u_3$ then $u_1 \leq u_1 \vee v_3 \leq u_3 \leq x_2$, again violating (1). Hence, since $v_3 \prec u_1 \vee v_3$, $u_3 \wedge (u_1 \vee v_3) = v_3$. But then $u_1 \vee v_3/v_3$ and D_3 generate $M_3 \times 2$. From this contradiction it follows that $w \neq v_3$. Hence w is an atom in the two-dimensional lattice u_3/v_3. If $w \neq z_3$ then u_3/v_3 contains a copy of M_4 which together with D_1 forms a sublattice with A_7 as a homomorphic image. Thus $z_3 = w = u_3 \wedge (v_3 \vee x_1)$. Hence $z_3 \leq v_3 \vee x_1$, which implies $v_3 \vee x_1 = v_3 \vee z_3 \vee x_1 = z_3 \vee x_1 = v_2$ by (4). Thus (2) holds and the proof is complete.

Theorem 4.4. If L is a subdirectly irreducible modular lattice such that $A_2, \ldots, A_{10} \notin HS(L)$ then $M_{3,3}^+$ is not a sublattice of L, where

$M^{+}_{3,3} =$

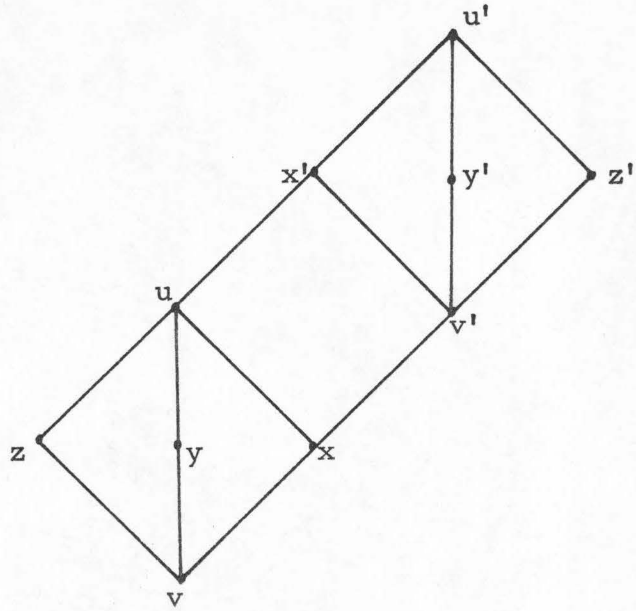


Figure 4.7

<u>Proof:</u> As seen above Theorem 3.1 implies that the existence of a sublattice isomorphic to $M^{+}_{3,3}$ such that both diamonds are isometric sublattices.

Since L is subdirectly irreducible there is a sequence of transposes $x'/v' = b_0/a_0, b_1/a_1, \ldots, b_n/a_n \subseteq v'/x$ which satisfy the conditions of Theorem 1.1. Let us suppose that $b_0/a_0 \nearrow b_1/a_1$. Then $x'/v' \nearrow x_1/v_1$. By Lemma 4.1 either

(1) $$u'/v' \nearrow u_1/v_1$$

or

(2) $$u'/u' \wedge v_1 \nearrow x_1/v_1$$

Suppose that (2) holds. Since $x'/v' \nearrow x_1/v_1$, $x' \nleq v_1$, and so $u' \wedge v_1 \neq x'$.

Trivially $\{y', z'\} - \{u' \wedge v_1\} \neq \emptyset$, let us say that $y' \neq u' \wedge v_1$. Since $x_1 > v_1$, $u' > u' \wedge v_1$ by (2). Thus $v' \prec u' \wedge v_1 \prec u'$. Hence $(v', x', y', u' \wedge v_1, u')$ is a diamond, which together with $D = (v, x, y, z, u)$ and D_1 form a sublattice with A_5 as a homomorphic image. Thus (2) cannot hold.

Now suppose that (1) holds. By Theorem 4.2 we must have $u' = u_1$ and $v' = v_1$. Thus, since $x'/v' \nearrow x_1/v_1$, $x' = x_1$. Furthermore, if $\{y', z'\} \neq \{y_1, z_1\}$ then u'/v' has M_4 as a sublattice which together with D would form a sublattice with A_8 as a homomorphic image. Thus we may assume $y' = y_1$ and $z' = z_1$; that is, $D' = D_1$. Consequently, by Lemma 4.3 it cannot happen that $D_1 \underset{(2)}{\searrow} D_2$. Thus we may assume that $D_1 \underset{(1)}{\searrow} D_2^*$. Theorem 4.2 implies that $D_1 = D_2^*$. By Lemma 3.4 $D_2 \underset{(2)}{\nearrow} D_3 = D_4^*$, $D_4 \underset{(2)}{\nearrow} D_5 = D_6^*, \ldots$. As pointed out before, this implies that $a_{n-2} \geq v' = a_0$. But this contradicts p.d. $(b_{n-2}/a_{n-2}, v'/z)$ = p.d. $(b_{n-2}/a_{n-2}, b_n/a_n) = 2$.

The remaining possibility is that $x'/v' = b_0/a_0 \searrow b_1/a_1$. In this case $x'/v' \searrow u_1/x_1$. Let $s = u \vee u_1$ and $r = s \wedge v'$. Now we have the situation already encountered in Theorem 3.1 (see Fig. 3.21). Exactly as in the proof of Theorem 3.1 we conclude that

(3) $$v \vee u_1 = v_1 \vee u = s$$

But now the Direct Product Lemma yields $M_3 \times 2$ as a sublattice unless $u_1 = u = s$. Then $x = u \wedge v' = u_1 \wedge v' = x_1$. Also $v = v_1$ by Theorem 2.2. Moreover we may assume that $y = y_1$ and $z = z_1$, for otherwise $A_7 \in HS(L)$ as seen several times before. Thus $D = D_1$.

Now either $D_1 \underset{(2)}{\nearrow} D_2$ or $D_1 \underset{(1)}{\nearrow} D_2^*$. Both of these lead to the same contradiction as above when D_1 equaled D'. The proof is complete.

We now introduce the following class of lattices:

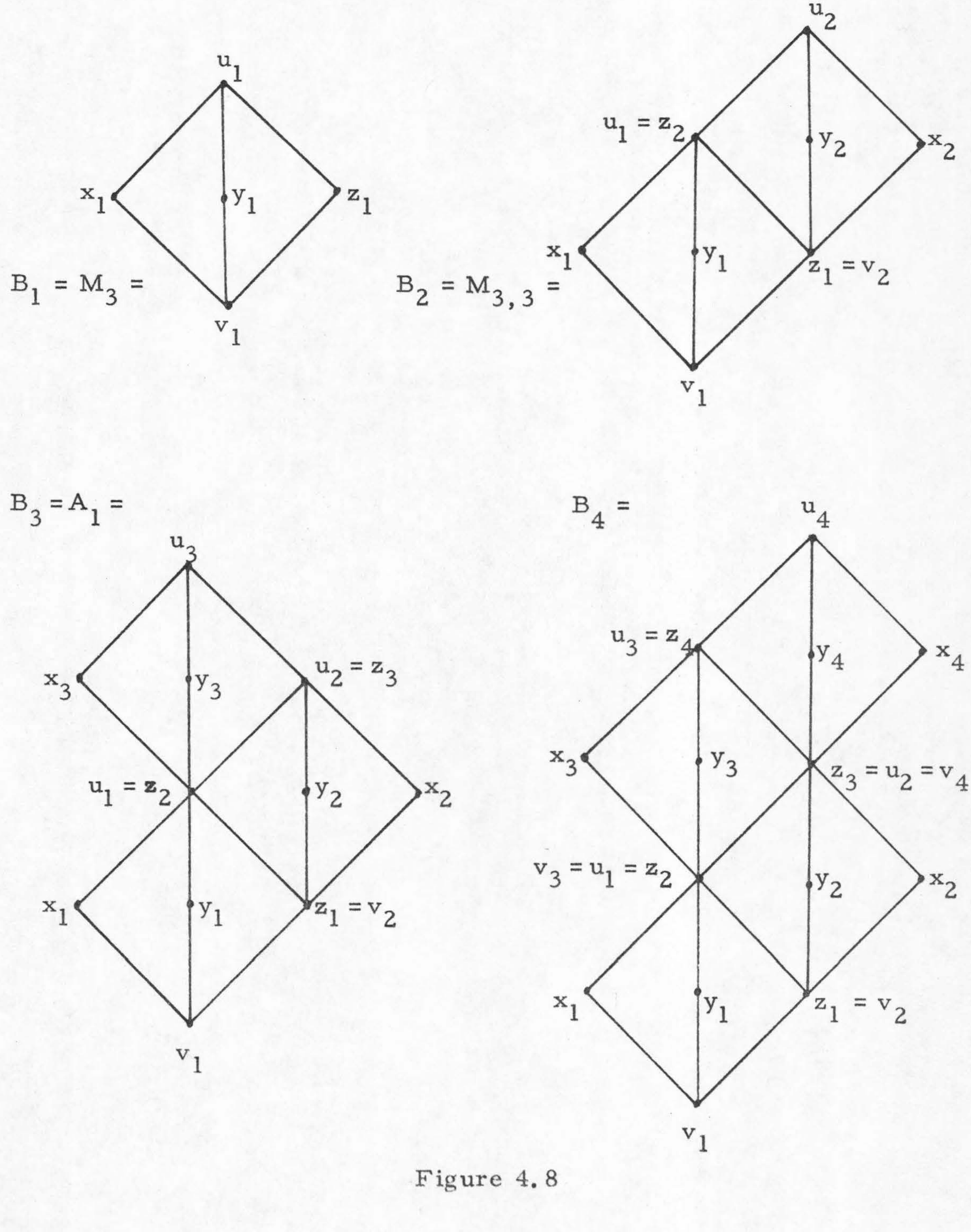


Figure 4.8

$B_5 =$

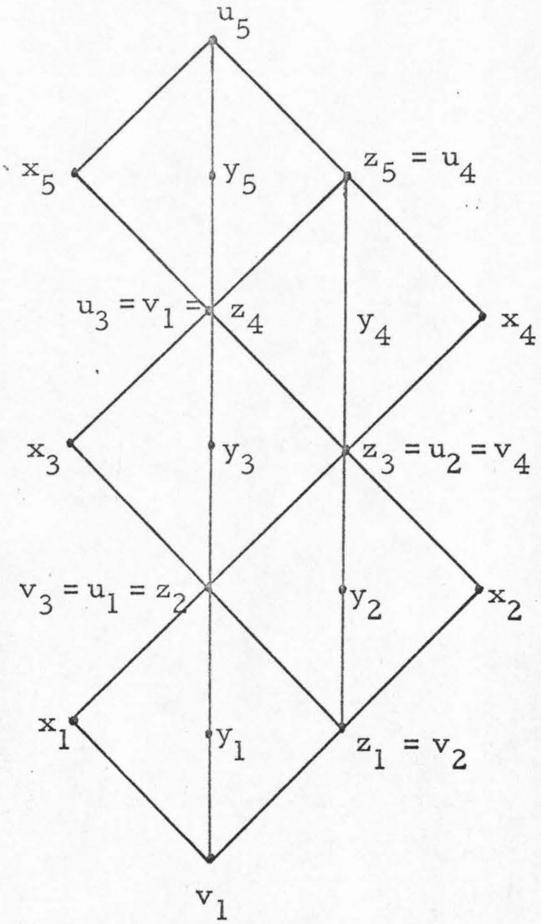


Figure 4. 8 (Continued)

In general B_n consists of n diamonds $D_1, D_2, \ldots, D_n$ such that for $i = 2, \ldots, n-1$

(1) $$u_{i-1} = z_i = v_{i+1}$$

(2) $$z_1 = v_2$$

(3) $$z_n = u_{n-1}$$

B_∞ consists of the diamonds $D_1, D_2, \ldots, D_n, \ldots$ which satisfy (1) and (2). B_∞^d is the dual of B_∞ and B_∞^∞ consists of diamonds $\{D_i \mid i \in Z\}$

satisfying (1). Note that the dimension of B_n is $n+1$ and that $B_n, B_\infty, B_\infty^d, B_\infty^\infty \in \mathfrak{M}_4^\infty$.

Theorem 4.5. Let L be a modular subdirectly irreducible lattice such that $A_2, \ldots, A_{10} \notin HS(L)$. If the dimension of L is $n+1$, $1 \leq n < \infty$, then $B_n \in S(L)$; if L is infinite dimensional then either B_∞ or B_∞^d is a sublattice of L.

Proof: Since L is subdirectly irreducible and of dimension at least two, L is nondistributive, hence $B_1 = (v_1, x_1, y_1, z_1, u_1)$ is a sublattice of L, which by Theorem 3.1 we may take to be an isometric sublattice. If the dimension of L is two we are done. Otherwise there exists $s \in L$ such that either $s > u_1$ or $s < v_1$. Let us assume the former. Now with the aid of Theorem 4.4 and the second remark preceding Lemma 3.5 there is a diamond sublattice D_2 such that D_2 and B_1 form B_2. If the dimension of L is three we are done. If not we may assume by duality that there exists $s \in L$ such that $s > u_2$. By the first remark preceding Lemma 3.5 and by Theorem 4.4 there is a diamond D_3 such that B_2 and D_3 form B_3. If there still exists an s in L such that $s > u_3$ then we apply the same procedure to the lattice formed by D_2 and D_3 of B_3. This yields a diamond D_4 such that D_2, D_3, D_4 form a sublattice isomorphic to B_3. This sublattice together with D_1 form B_4. If L is finite dimensional this argument can be repeated to obtain B_n as a sublattice of L with u_n the greatest element of L. By a dual argument and a possible renumbering, it may also be assumed that v_1 is the least element of L. Since B_n is an isometric sublattice of dimension n+1 L must have dimension n+1.

If L is infinite dimensional, then as before, B_1 is an isometric sublattice of L. Either there are elements $s_k \geq u_1$ in L such that the dimension of s_k/u_1 is greater than for all $k > 0$, or there are elements $t_k \leq v_1$ such that the dimension of v_1/t_k is greater than k for all $k \geq 0$. If the former is the case then the process above yields B_∞ as a sublattice of L. If the latter holds B_∞^d is a sublattice of L.

<u>Remark.</u> The above arguments also show that if B_∞ is a sublattice of L then we may assume that either v_1 is the least element of L or that B_∞^∞ is a sublattice of L.

In summary, if L satisfies the conditions of Theorem 4.5 then exactly one of the following four situations occur:

(i) for some n, B_n is a sublattice of L with v_1 and u_n the least and greatest elements of L, respectively;

(ii) B_∞ is a sublattice of L and v_1 is the least element of L;

(iii) the dual situation to (ii);

(iv) B_∞^∞ is a sublattice of L.

We define a <u>core</u> of L, denoted core(L), to be

$$\text{core}(L) = \begin{cases} B_n & \text{if (i) holds} \\ B_\infty & \text{if (ii) holds} \\ B_\infty^d & \text{if (iii) holds} \\ B_\infty^\infty & \text{if (iv) holds} \end{cases}$$

The core of L is to be a specific sublattice of L whose elements are numbered in accordance with equations (1), (2) and (3) preceding

Theorem 4.5. There may be more than one core of L, however, it is easy to see that they are all isomorphic. Core (L) stands for some specific core of L. Actually we will see below that the only lattice satisfying the conditions of Theorem 4.5 with more than one core is M_4. Consequently we will often refer to the core of L.

Lemma 4.6. Let B_n, $n \geq 4$ be a sublattice of L, where L is a modular subdirectly irreducible lattice such that $A_2, \ldots, A_{10} \notin HS(L)$. Then, if $s \in u_n/v_1$ either $s \geq v_2$ or $s \leq u_{n-1}$.

Proof: Let us suppose that $s \ngeq v_2$ and $s \nleq u_{n-1}$. Consider $s \wedge u_1$. Since $u_1 > z_1 = v_2 \nleq s$, $s \wedge u_1 < u_1$. If $s \wedge u_1 = v_1$ then Theorem 4.2 implies that $s = v_1 \leq u_{n-1}$, a contradiction. Hence $s \wedge u_1$ is an atom of u_1/v_1 and $s \wedge u_1 \neq z_1 = v_2$. If $s \wedge u_1 \neq x_1$ or y_1 then u_1/v_1 would contain M_4 as a sublattice which with D_2 would form A_7. Thus we may assume $s \wedge u_1 = x_1$. Dually we may assume $s \vee v_n = x_n$.

It will now be shown that $s \vee v_n = s \vee v_{n-1} = x_n$. First note that

$$(1) \qquad v_{n-1} \leq u_{n-1} \wedge (v_{n-1} \vee s) \leq u_{n-1} \wedge (v_n \vee s) = u_{n-1} \wedge x_n$$
$$= z_n \wedge x_n = v_n$$

Since $s \nleq v_{n-1}$, $s \vee v_{n-1} > v_{n-1}$. Hence, by Theorem 4.2, $u_{n-1} \wedge (v_{n-1} \vee s) \neq v_{n-1}$. Since $v_{n-1} \prec v_n$ (1) now yields $u_{n-1} \wedge (v_{n-1} \vee s) = v_n$; thus $v_{n-1} \vee s \geq v_n$. Hence $s \vee v_{n-1} = s \vee v_n \vee v_{n-1} = x_n$, as desired.

Now $s/s \wedge v_n \nearrow x_n/v_n$. If $s \wedge v_n \leq v_{n-1}$ then $s/s \wedge v_n \nearrow x_n/v_{n-1}$, which is impossible because $s/s \wedge v_n$ has dimension one (since

$s/s \wedge v_n \nearrow x_n/v_n)$ and x_n/v_{n-1} has dimension two. Thus $s \wedge v_n \leq v_n = u_{n-2}$ and $s \wedge v_n \not\leq v_{n-1} = u_{n-3}$. Since $v_n \geq x_1$, we have that $u_1 \wedge (s \wedge v_n) = x_1$. This reduction shows that we may assume n is 5 or 4.

Let us suppose that $n = 4$. Repeating the above argument we obtain $s \wedge v_4 \leq v_4 = u_2$ and $s \wedge v_4 \not\leq u_1$ and $u_1 \wedge (s \wedge v_4) = x_1$. Thus $u_1 \vee (s \wedge v_4) = x_4 = u_2$ since $u_1 < u_2$. Hence $(s \wedge v_4) \vee v_2 = (s \wedge v_4) \vee x_1 \vee v_2 = (s \wedge v_4) u_1 = u_2$. Furthermore, $s \wedge v_4 \neq u_2$ since $u_1 \wedge s \wedge v_4 = x_1$. Hence, by the Direct Product Lemma, $M_3 \times 2$ is a sublattice of L, contradicting Theorem 4.2.

Let $n = 5$. As before we have that $s \wedge v_5 \leq u_3$ and $s \wedge v_5 \not\leq u_2$. Thus $u_1 \not\geq s \wedge v_5$. Since $s \not\geq v_2$, it follows that $s \not\geq v_5$; thus $s \wedge v_5 < v_5 = u_3$. Hence $u_1 \vee (s \wedge v_5) = v_3 \vee (s \wedge v_5) = u_3$, which is again a contradiction by Theorem 4.2. By the argument used several times before we may assume that $u_1 \vee (s \wedge v_5) = x_3$. Since $u_1 \wedge (s \wedge v_5) = x_1$, we have that

(2) $$x_3/s \wedge v_5 \searrow u_1/x_1 \quad \text{and} \quad x_3/u_1 \searrow s \wedge v_5/x_1$$

Since $s \wedge v_5 \leq x_3 \leq v_5$, $s \wedge x_3 = s \wedge v_5$. Thus

(3) $$s \vee x_3/s \searrow x_3/s \wedge v_5 \searrow u_1/x_1 \searrow z_1/v_1$$

As $s \vee x_3 \geq s \not\leq v_5$, $(s \vee x_3) \vee v_5 = s \vee v_5 = x_5$ and $(s \vee x_3) \wedge v_5 = x_3$ since $v_5 = u_3 > x_3$. This together with the first transposition of (3) implies that

(4) $$s/s \wedge v_5 \nearrow s \vee x_3/x_3 \nearrow x_5/v_5 \nearrow u_5/z_5$$

(5) $$x_5/s \vee x_3 \searrow v_5/x_3 = u_3/x_3$$

With the aid of (2), (3), (4) and (5) it is easy to verify that B_5 together with S, $s \vee x_3$ and $s \wedge v_5$ form the sublattice A_{10}. This contradiction proves the lemma.

Lemma 4.7. Let L be a modular, nondistributive subdirectly irreducible lattice such that $A_2, \ldots, A_{10} \notin HS(L)$. Let $s \in L$, then one of the following holds

(i) For some v_k, $u_\ell \in$ core(L) with $0 \leq \ell - k \leq 2$, $v_k \leq s \leq u_\ell$.

(ii) The core(L) is B_∞ or B_∞^∞ and $s \geq u_k$ for all k.

(iii) The core(L) is B_∞^d or B_∞^∞ and $s \leq u_k$ for all k.

Proof: If core(L) = B_n then $v_1 \leq s \leq u_n$ by the remark preceding Lemma 4.6. A straightforward application of Lemma 4.6 gives $v_k, u_\ell \in$ core(L) with $0 \leq \ell - k \leq 2$ and $v_k \leq s \leq u_\ell$.

Hence we may assume the core(L) is B_∞, B_∞^d or B_∞^∞. Suppose also that for some n

(1) $$s \leq u_n \qquad \text{and} \qquad s \nleq u_{n-1}$$

If $s \geq v_k$ for some k then the proof may be completed as above. Thus, in particular, we may assume $s \ngeq v_{n-4}$. Let $t = s \vee v_{n-4}$. Since $s \ngeq v_{n-4}$, $t > s$. By Lemma 4.6, $t \geq v_{n-2} = u_{n-4}$. Now the Direct Product Lemma applied to t/v_{n-4} and t/s yields a sublattice isomorphic to $M_3 \times 2$, which is impossible by Theorem 4.2.

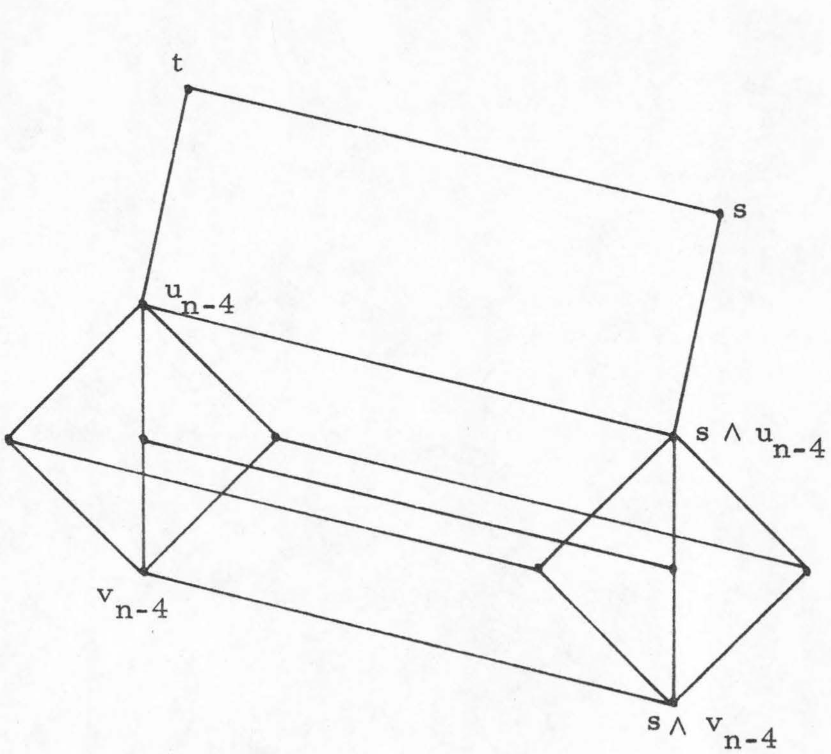


Figure 4.9

We conclude that if (1) holds the lemma is true. Dually if $s \geq v_n$ and $s \ngeq v_{n+1}$ for some n the lemma is true. If core(L) = B_∞ then either this last statement holds or $s \geq v_n$ for all n. In either case the lemma is true. Similarly the lemma is true if core(L) = B_∞^d. Hence it may be assumed that core(L) = B_∞^∞. If $s \leq u_n$ for all n then $s \leq v_n$ for all n. Hence $s \nleq u_{n_0}$ for some n_0. If $s \leq u_{n_1}$ then $n_1 > n_0$ and by choosing the smallest such n, we have $s \leq u_{n_1}$ and $s \nleq u_{n_1 - 1}$. This is the case considered above.

By this and the dual argument we may assume

(2) $\qquad s \nleq u_n \qquad$ and $\qquad s \ngeq u_n \qquad$ for all n

Suppose

(3) $s \wedge u_n \leq u_k$ and $s \vee u_n \geq u_k$ for all n and k

Let $n > m$. Then $s \wedge u_n \leq u_m$ implies that $s \wedge u_n = s \wedge u_m$. Similarly $s \vee u_n = s \vee u_m$. Then s, $s \wedge u_n$, $s \vee u_n$, u_n, u_m form a sublattice isomorphic to N_5, contradicting modularity. Hence by duality we may assume that for some n and k $s \wedge u_n \nleq u_k$. Since $s \wedge u_n \leq u_n$, k may be chosen such that $s \wedge u_n \nleq u_k$ and $s \wedge u_n \leq u_{k+1}$. But then Lemma 4.6 implies that $u_{k-3} = v_{k-1} \leq s \wedge u_n \leq u_{k+1}$. Then $s \geq u_{k-3}$, contradicting (2). This proves the lemma.

<u>Lemma 4.8.</u> Let L be a modular subdirectly irreducible lattice such that $A_2, \ldots, A_{10} \notin SH(L)$. Let C = core (L) and suppose that the dimension of C is greater than two. Let $s \in L$ such that $v_k \leq s \leq u_{k+1}$ for some $v_k, u_{k+1} \in C$ then $s \in C$. If $v_k \leq s \leq u_{k+2}$, $v_k, u_{k+2} \in C$, then either $s \in C$ or $s \vee u_k \in \{x_{k+2}, y_{k+2}\}$ and $s \wedge u_k \in \{x_k, y_k\}$ (see Fig. 4.10).

<u>Proof:</u> If $s \in u_n/v_n$ for n equal k, k + 1 or k + 2 then $s \in \{v_n, x_n, y_n, z_n, u_n\}$ for otherwise u_n/v_n had M_4 as a sublattice and since core (L) has dimension greater than two, A_7 or $A_8 \in HS(L)$. If $v_k \leq s \leq u_{k+1}$ and $s \notin C$ then $u_k \wedge s$ cannot be $u_k = z_{k+1}$ or $z_k = v_{k+1}$. For then we would have $s \in u_{k+1}/v_{k+1}$, contradicting the above. If $s \wedge u_k = v_k$ then $s = v_k \in C$ by Theorem 4.3. Thus $s \wedge u_k$ is an atom of u_k/v_k which must be either x_k or y_k, for otherwise $A_7 \in HS(L)$. Say that $s \wedge u_k = x_k$. Thus $s \geq x_k$, and therefore $s \vee v_{k+1} =$

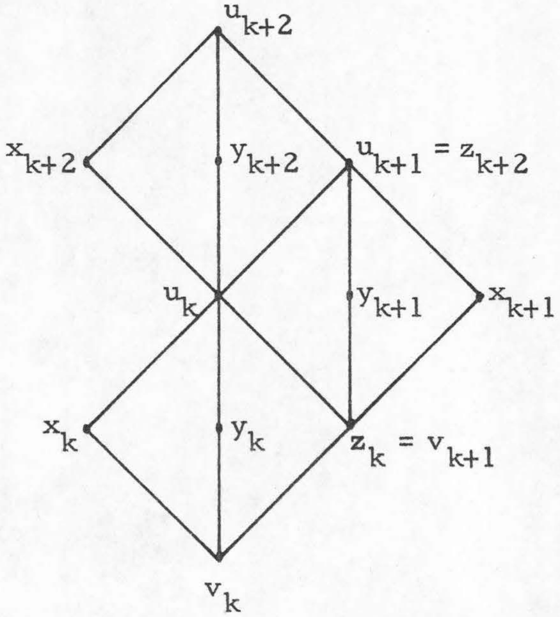


Figure 4.10

$s \vee x_k \vee v_{k+1} = s \vee u_k$. Hence $u_k \leq s \vee v_{k+1} \leq u_{k+1}$. Thus either $s \vee v_{k+1} = u_k$ or $s \vee v_{k+1} = u_{k+1}$. If $s \vee v_{k+1} = u_k$ then $s \in u_k/v_k$, which is the case considered above. If $s \vee v_{k+1} = u_{k+1}$ then by Theorem 4.2, $s = u_{k+1}$, contradicting $s \notin C$.

Now suppose $v_k \leq s \leq u_{k+2}$. If $v_{k+1} \leq s$, then the above applies. Similarly, it may be assumed that $s \nleq u_{k-1}$. An argument similar to that above shows that if $s \notin C$ then $s \vee u_k \in \{x_{k+2}, y_{k+2}\}$ and $s \wedge u_k = \{x_k, y_k\}$.

Now we are ready to prove the main theorem of this thesis.

<u>Theorem 4.9.</u> Let L be a modular lattice such that $A_2, \ldots, A_{10} \notin HS(L)$. Then $L \in \mathcal{M}_4^\infty$.

<u>Proof:</u> Assume $L \notin \mathcal{M}_4^\infty$. L is a subdirect product of subdirectly

irreducible lattices. If all these subdirectly irreducible lattices lie in $\mathfrak{M}_4^\infty$ then $L \in \mathfrak{M}_4^\infty$. Hence it may be assumed that L is subdirectly irreducible. Since $L \notin \mathfrak{M}_4^\infty$ there exist five noncomparable elements s_1, s_2, s_3, s_4, s_5 in L. It follows from Lemma 4.7 that if $s_1 \geq u_k$ for all k then $s_2 \geq u_k$ for all k ($u_k \in \text{core}(L) = C$). Then the nontrivial quotient $s_1/s_1 \wedge s_2$ lies entirely above u_k for all k. Since L is subdirectly irreducible there exists a sequence of transposes $x_1/v_1 = b_0/a_0$, $b_1/a_1, \ldots, b_n/a_n \subseteq s_1/s_1 \wedge s_2$. It will be shown that this is impossible by showing that for some j_i and ℓ_i, $i = 1, \ldots, n$

(1)
$$v_{j_i} \leq a_i \leq u_{j_i+2}$$
$$v_{\ell_i} \leq b_i \leq u_{\ell_i+2} \qquad i = 0, \ldots, n$$

Indeed, $b_n \leq u_{\ell_n+2}$ contradicts $b_n \geq s_1 \wedge s_2 \geq u_k$, for all k. We prove (1) by induction. For $i = 0$, (1) holds with $j_0 = \ell_0 = 1$. Let us suppose that (1) holds for $i = k$ and suppose that $b_k/a_k \nearrow b_{k+1}/a_{k+1}$. Since $b_k \leq u_{\ell_k+2}$ and $a_k \geq v_{j_k}$ this transposition implies $v_{j_k} \leq a_{k+1} \not\geq u_{\ell_k+2}$. It follows from Lemma 4.7 that $v_{j_{k+1}} \leq a_{k+1} \leq u_{j_{k+1}+2}$ for some j_{k+1}. By semimodularity $b_{k+1} \vee u_{j_{k+1}+2}$ is either $u_{j_{k+1}+2}$ or covers $u_{j_{k+1}+2}$. In either case $b_{k+1} \not\geq u_{j_{k+1}+4}$. Since $v_{j_{k+1}} \leq a_{k+1} \leq b_{k+1}$ Lemma 4.7 again implies that (1) holds.

It follows from this that

(2)
$$v_{k_i} \leq s_i \leq u_{k_i+r_i} \qquad 0 \leq r_i \leq 2, \ i = 1, 2, 3, 4, 5$$

Clearly the k_i's may be picked so that

(3)
$$s_i \not\geq v_{k_i+1}$$

Since the s_i's are incomparable, $k_i - 3 \le k_j \le k_i + 3$, $1 \le i, j, \le 5$. Let $k_0 = \min\{k_1, k_2, k_3, k_4, k_5\}$. Then $k_0 \le k_j \le k_0 + 3$, $j = 1, 2, 3, 4, 5$. Hence two of the k_i's are equal, say $k_1 = k_2$. Let us suppose that $s_1 \notin C$. Thus, by Lemma 4.8 it may be assumed that

(4) $$s_1 \vee u_{k_1} = x_{k_1+2} \qquad \text{and} \qquad s_1 \wedge u_{k_1} = x_{k_1}$$

Now suppose $s_2 \notin C$. Then $s_2 \vee u_{k_1} \in \{x_{k_1+2}, y_{k_1+2}\}$ and $s_2 \wedge u_{k_1} = \{x_{k_1}, y_{k_1}\}$. Suppose $s_2 \wedge u_{k_1} = x_{k_1}$ and $s_2 \vee u_{k_1} = y_{k_1+2}$. Since s_1 and s_2 are incomparable $s_1 \wedge s_2 = x_{k_1}$. Since $s_2 > x_{k_1}$, it follows that $s_1 \vee s_2 > s_1$. By a dimension argument $s_1 \vee s_2 < u_{k_1+2}$. But $s_1 \vee s_2 \vee v_{k_1+2} = s_1 \vee s_2 \vee u_{k_1} = x_{k_1+2} \vee y_{k_1+2} = u_{k_1+2}$, which is impossible by Theorem 4.2. Similarly $s_2 \wedge u_{k_1} = y_{k_1}$ and $s_2 \vee u_{k_1} = x_{k_1+2}$ cannot both hold. If $s_2 \wedge u_{k_1} = x_{k_1}$ and $s_2 \vee u_{k_1} = x_{k_1+2}$ then it is easy to see that u_{k_1+2}/v_{k_1} contains A_2 as a sublattice. If $s_2 \wedge u_{k_1} = y_{k_1}$ and $s_2 \vee u_{k_1} = y_{k_1+2}$ then u_{k_1+2}/v_{k_1} contains A_4 as a sublattice. We conclude that one of s_1, s_2 is an element of the core C. By (3) we may assume we have the following situation:

(5) $$s_2 = y_{k_1} \qquad \text{and} \qquad s_1 \wedge u_{k_1} = x_{k_1}$$

Here either $s_1 \notin C$ or $s_1 = x_{k_1}$.

Let us suppose that $k_3 = k_1$ as well. Then $s_3 \wedge u_{k_1} \in \{x_{k_1}, y_{k_1}\}$. Since $y_{k_1} = s_2$ we must have $s_3 \wedge u_{k_1} = x_{k_1}$. If either $s_1 = x_{k_1}$ or $s_3 = x_{k_1}$ then s_1 and s_3 are comparable. Thus $s_1 \ne x_{k_1} \ne s_3$. By (3) $s_1, s_3 \notin C$. But it has already been shown that this leads to a contradiction.

Suppose we have another pair of equal k_i's, say $k_3 = k_4$. Then as before we may assume $s_4 = y_{k_3} = y_{k_4}$. Since s_2 and s_4 are incomparable we must have $k_3 = k_1 \pm 1$. The situation is symmetric so we assume that $k_3 = k_1 - 1$; that is,

(6) $$s_4 = y_{k_1 - 1}$$

Also as before

(7) $$s_3 \wedge u_{k_1 - 1} = x_{k_1 - 1}$$

Since the lattice generated by s_1, s_2, s_3, s_4 and C has width four, $s_5 \notin C$. As pointed out above $k_5 \geq k_1 - 3$ and $k_5 \leq k_3 + 3 = k_1 + 2$. If $k_5 = k_1 - 3$, then by Lemma 4.8 $s_5 \vee u_{k_1 - 3} \in \{x_{k_1 - 1}, y_{k_1 - 1}\}$. Since $s_5 \leq s_5 \vee u_{k_1 - 3}$ and $x_{k_1 - 1} \leq s_3$ and $y_{k_1 - 1} = s_4$, it follows that s_5 is comparable with s_3 or s_4, a contradiction. Similarly $k_5 = k_1 - 2$, implies that s_5 is comparable with s_1 or s_2. If $k_5 \geq k_1 + 1$ then $s_5 \geq v_{k_1 + 1} = u_{k_1 - 1} \geq y_{k_1 - 1} = s_4$. If $k_5 = k_1$ or $k_5 = k_1 - 1$ then we have three equal k_i's, a situation already shown to be impossible.

For the remaining case we have $k_1 = k_2$ and k_1, k_3, k_4, k_5 are distinct. Recall $k_0 = \min\{k_1, k_2, k_3, k_4, k_5\}$ and $k_0 \leq k_1 \leq k_0 + 3$. Thus $\{k_1, k_3, k_4, k_5\} = \{k_0, k_0 + 1, k_0 + 2, k_0 + 3\}$. Also $k_1 \geq k_0 \geq k_1 - 3$. Suppose $k_0 \leq k_1 - 2$. Then one of k_3, k_4, k_5 must be $k_1 - 2$, say $k_3 = k_1 - 2$. By Lemma 4.8 $s_3 \leq s_3 \vee u_{k_3} \in \{x_{k_3 + 2}, y_{k_3 + 2}\} = \{x_{k_1}, y_{k_1}\}$. So s_3 is comparable to s_1 or s_2, contrary to our assumption. Hence $k_0 \geq k_1 - 1$. Then one of k_3, k_4, k_5 must be $k_1 + 2$, say $k_3 = k_1 + 2$. But then $s_3 \geq v_{k_3} = v_{k_1 + 2} = u_{k_1} \geq y_{k_1} = s_2$. This final contradiction proves the theorem.

CHAPTER V

APPLICATIONS

In this chapter we present some applications of Theorem 4.9. We begin with the characterization of the subdirectly irreducible width four modular lattices announced in [11]. Let L be such a lattice. Clearly $A_2, \ldots, A_{10} \notin HS(L)$ so that the previous theorems apply. In particular L has a core. Recall that the core is one of the sublattices B_n, B_∞, B_∞^d, B_∞^∞ and, in some sense, it is the largest such sublattice that will fit in L (see the definition following Theorem 4.5). Recall that B_∞^∞ is a sequence of diamonds $D_i = (v_i, x_i, y_i, z_i, u_i)$ $i \in Z$ such that

(1) $$u_{i-1} = z_i = v_{i+1}$$

and B_n, B_∞, B_∞^d have similar definitions which are given before Theorem 4.5.

We would like to find the elements of L which are not in core(L). With regard to Theorem 4.7, suppose $s \in L$ - core L such that $s \geq u_k$ for all k. If $t \in L$ - core(L), $t \geq u_k$ for all k and $t \neq s$ then, as in the proof of Theorem 4.9, L is not subdirectly irreducible, contrary to assumption. It follows that s must be the greatest element of L. A similar argument shows that if $t \leq u_k$ for all k then t is the least element of L.

It is clear that the only subdirectly irreducible width four modular lattice of dimension two is M_4, and that there is none of dimension one. Hence assume that the dimension of L is greater than two. Let 0 and 1 denote the least and greatest elements of L, if they exist. Now by Lemma 4.7 and Lemma 4.8 it follows that if $s \in L - (\text{core } L \cup \{0,1\})$ then

(2) $v_k \leq s \leq u_{k+2}$, $s \not\geq v_{k+1}$ and $s \not\leq u_{k+1}$ for some k

Lemma 4.8 also tells us that

(3)
$$s \vee u_k \in \{x_{k+2}, y_{k+2}\}$$
$$s \wedge u_k \in \{x_k, y_k\}$$

Thus, for each $s \in L - (\text{core } L \cup \{0,1\})$, there corresponds a $k = k(s)$ such that (2) and (3) hold.

It was shown in the proof of Theorem 4.9 that if $s, t \in L - (\text{core } L \cup \{0,1\})$ and $k(s) = k(t)$ then either A_2 or A_4 is in HS(L). Thus $k(s) = k(t)$ implies $s = t$.

<u>Theorem 5.1</u>. Let L be a modular subdirectly irreducible lattice of width four. Then either

(i) $L = M_4$.

(ii) L has dimension $n + 1 > 2$, L has B_n as a sublattice and for each k, $2 \leq k \leq n - 1$ there is at most one element $w_k \in L - B_n$ dimension k. Also $w_k \vee z_k \in \{x_{k+1}, y_{k+1}\}$ and $w_k \wedge z_k = \{x_{k-1}, y_{k-1}\}$.

(iii) L has B_∞ as a sublattice with v_1 (the least element of B_∞) equal to the least element of L. For each $k \geq 2$ there is at most one element $w_k \in L - B_\infty$ of dimension k, and $w_k \vee z_k \in \{x_{k+1}, y_{k+1}\}$ and

$w_k \wedge z_k \in \{x_{k-1}, y_{k-1}\}$. L may also have a greatest element.

(iv) L is the dual of one of the lattices of (ii).

(v) L has B_∞^∞ as a sublattice. For all k there is at most one element $w_k \in L - B_\infty^\infty$ which is incomparable with z_k and $z_k \vee w_k \in \{x_{k+1}, y_{k+1}\}$ and $z_k \wedge w_k \in \{x_{k-1}, y_{k-1}\}$. L may also have either a top element, a bottom element or both.

Furthermore, all the lattices described in (i)-(v) are subdirectly irreducible modular lattices of width four. Hence this is a complete list of such lattices. All the lattices of (i) and (ii) are simple; all those of (iii) without a greatest element and all those of (iv) without a least element and all those of (v) without a least or a greatest element are simple.

Now we turn to the subject of lattice varieties. If $\mathcal{L}$ is a class of lattices, we let $V(\mathcal{L})$ denote the variety (equational class) generated by $\mathcal{L}$. Also we let $P_u(\mathcal{L})$ denote all ultraproducts of elements of $\mathcal{L}$. The next theorem, which is basic to the study of lattice varieties, is due to B. Jónsson.

Theorem 5.2. Let $\mathcal{L}$ be a class of lattices. Then every subdirectly irreducible member of $V(\mathcal{L})$ is a member of $HSP_u(\mathcal{L})$. Moreover, if $\mathcal{L}$ has only finitely many members each of which is finite then every subdirectly irreducible member of $V(\mathcal{L})$ is a member of $HS(\mathcal{L})$. Furthermore, if V and W are lattice varieties then every subdirectly irreducible member of $V \vee W$, the variety generated by V and W, is a member of either V or W.

A proof of this theorem appears in [15].

If $\mathcal{L}$ is a class of modular lattices, each of which has width at most four, then $P_u(\mathcal{L})$ is a class of modular lattices, each of width at most four. Consequently the subdirectly irreducible members of $\mathcal{M}_4^\infty$, the variety generated by width four modular lattices, are just the subdirectly irreducible lattices of width four or less. The subdirectly irreducible modular lattices of width exactly four are given by Theorem 5.1. M_3 is the only subdirectly irreducible modular lattice of width three. This follows from Theorem 4.5 and is also in [16]. The remaining subdirectly irreducible modular lattices of width less than three are 2 and 1, the lattices with two and one elements, respectively.

Now we answer the problem suggested in the introduction. Let $V_i = \mathcal{M}_4^\infty \vee V(A_i)$, $i = 2, \ldots, 10$ and $V_1 = \mathcal{M}_4^\infty \vee V(N_5)$. Let $\mathcal{M}$ be the variety of all modular lattices and Λ the variety of all lattices.

<u>Theorem 5.3.</u> The quotient sublattice $\Lambda / \mathcal{M}_4^\infty$ of the lattice of all varieties is atomic with atoms $V_1, \ldots, V_{10}$. Consequently $\mathcal{M}_4^\infty$ is finitely based.

<u>Proof:</u> Let W be a variety of modular lattices such that $W \supsetneq \mathcal{M}_4^\infty$. Since every lattice is a subdirect product of subdirectly irreducible lattices, there exists a subdirectly irreducible lattice L in $W - \mathcal{M}_4^\infty$. Hence L has width greater than four. By Theorem 4.9, $A_i \in HS(L)$ for some i, $2 \le i \le 10$. But then $W \supseteq V(L) \supseteq V_i$. It only remains to show that $V_i \succ \mathcal{M}_4^\infty$, $i = 2, 3, \ldots, 10$. Suppose $V_i \subseteq V_j$ for some $i \neq j$ $2 \le i, j \le 10$. Then $A_i \in V_j = V(A_j) \vee \mathcal{M}_4^\infty$. $A_i \notin \mathcal{M}_4^\infty$ and the last part of Theorem 5.2 imply $A_i \in V(A_j)$, but this contradicts the second part of

that same theorem. Hence the varieties $V_2, \ldots, V_{10}$ are incomparable. Now suppose that for some variety V and some j, $2 \le j \le 10$, $V_j \supseteq V \underset{\neq}{\supset} \mathcal{M}_4^\infty$. Then, by the first part of the proof $V \supseteq V_i$ for some $i = 2, \ldots, 10$. By the above $i = j$. Hence $V = V_j$ and $V_j \succ \mathcal{M}_4^\infty$, $j = 2, \ldots, 10$. If W is a variety which contains $\mathcal{M}_4^\infty$ and which is not contained in $\mathcal{M}$, then $N_5 \in W$, thus $W \supseteq V_1$. As above it is easy to see that V_1 is incomparable with $V_2, \ldots, V_{10}$ and that $V_1 \succ \mathcal{M}_4^\infty$.

Since varieties are determined by the identities all of their members satisfy, $A_i \notin \mathcal{M}_4^\infty$, $i = 2, 3, \ldots, 10$, implies there exist identities $\epsilon_2, \epsilon_3, \ldots, \epsilon_{10}$, such that ϵ_i holds in all members of $\mathcal{M}_4^\infty$ but fails in A_i, $i = 2, \ldots, 10$. It follows easily from the first part of the theorem that the modular law together with $\epsilon_2, \ldots, \epsilon_{10}$, determine the variety $\mathcal{M}_4^\infty$. That is, all identities of $\mathcal{M}_4^\infty$ are derivable from the modular identity, $x \wedge (y \vee (x \wedge y)) = (x \wedge y) \vee (x \wedge z)$, and $\epsilon_2, \ldots, \epsilon_{10}$. This completes the proof.

In [2] K. Baker gives an infinite set of identities σ_k, $k = 0, 1, 2, \ldots$, which define $\mathcal{M}_4^\infty$. Let r_{ij} and s_{ij}, $i \le i$, $j \le 5$, $i \ne j$ be the lattice polynomials in the variable $x_i, x_j, z_1^{ij}, z_2^{ij}, \ldots, z_6^{ij}$ given by

(1)
$$r_{ij} = (((((x_i \vee z_1^{ij}) \wedge z_2^{ij}) \vee z_3^{ij}) \wedge z_4^{ij}) \vee z_5^{ij}) \wedge z_6^{ij}$$
$$s_{ij} = (((((x_i \vee x_j \vee z_1^{ij}) \wedge z_2^{ij}) \vee z_3^{ij}) \wedge z_4^{ij}) \vee z_5^{ij}) \wedge z_6^{ij}$$

Then σ_6 is the identity

(2)
$$(\ldots(((u \vee r_{12}) \wedge s_{12}) \vee r_{13}) \wedge s_{13}) \ldots \vee r_{54}) \wedge s_{54}$$
$$= (\ldots(((v \vee r_{12}) \wedge s_{12}) \vee r_{13}) \wedge s_{13}) \ldots \vee r_{54}) \wedge s_{54}$$

The identity holds in all members of $\mathcal{M}_4^\infty$. To see this let L be a lattice of width four. Hence, if $x_1, \ldots, x_5$ are substituted into L, $x_i \le x_j$ for some $i \ne j$, $1 \le i, j \le 5$. But then $r_{ij} = s_{ij}$. It follows easily from this that σ_6 holds in L. Since each member of $\mathcal{M}_4^\infty$ is the subdirect product of width four lattices σ_6 holds in $\mathcal{M}_4^\infty$. It can be checked that σ_6 fails in $A_2, \ldots, A_{10}$ ([2] gives an easy method for this; see also [3]). Hence $\mathcal{M}_4^\infty$ is defined by σ_6 and the modular law.

σ_6 has 127 variables. One might ask what is the least number n such that there exists an identity which together with the modular defines $\mathcal{M}_4^\infty$. The following five variable identity was used by Jónsson in [16] as an example of an identity which holds in M_4 but fails in M_5:

$$(3) \qquad a \wedge \bigwedge_{1 \le i \le j \le 4} (x_i \vee x_j) \le \bigvee_{1 \le i \le 4} (a \wedge x_i)$$

One can show that this identity holds in $\mathcal{M}_4^\infty$ (use the modular law). This identity fails in A_2, A_3, A_5, A_6, A_8, A_9 but holds in A_4 and A_{10}.

J. B. Nation points out that no five variable identity can hold in $\mathcal{M}_4^\infty$ and fail in A_{10}. Indeed, A_{10} has eight elements which are both join and meet irreducible. Thus $n \ge 8$.

Now A_4 is generated by four elements a_1, a_2, a_3, a_4. (See Fig. 5.1.) Let ε_4: $f(z_1, \ldots, z_k) = g(z_1, \ldots, z_k)$ hold in $\mathcal{M}_4^\infty$ but fail in A_4. Then for some substitution $b_i \in A_4$, $i = 1, \ldots, k$, $f(b_1, \ldots, b_k) \ne g(b_1, \ldots, b_k)$. Each $b_i = w_i(a_1, a_2, a_3, a_4)$. Hence the four-variable identity ε_4': $f(w_1(x_1, \ldots, x_4), \ldots, w_k(x_1, \ldots, x_4)) = g(w_1(x_1, \ldots, x_4), \ldots$

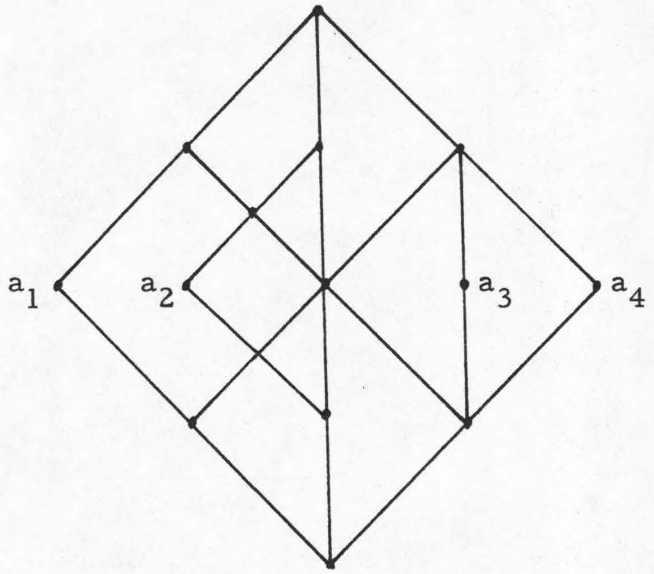


Figure 5.1

$\ldots, w_k(x_1, \ldots, x_4))$ does not hold in A_4. Moreover, since ε'_4 is derived from ε_4, ε'_4 holds in $\mathfrak{M}_4^{\infty}$. Similarly there is an eight-variable identity ε'_{10} which holds in $\mathfrak{M}_4^{\infty}$ and fails in A_{10}. Since for any two lattice identities in r and s variables, respectively, there is a lattice identity in r + s variable equivalent to the conjunction of the first two, we conclude using (3), ε'_4, ε'_8 that $n \leq 17$.

In [17] McKenzie raises the following question: For which integers k is there a variety which possesses an independent basis with k elements but not one with k + 1? He shows that such varieties exist for any $k \leq 12$. Let K_n be the lattice B_n with w_2 and w_{n-1} adjoined such that $w_2 \vee z_2 = x_3$, $w_2 \wedge z_2 = x_1$, $w_{n-1} \vee z_{n-1} = x_n$ and $w_{n-1} \wedge z_{n-1} = x_{n-2}$. Let K'_5 be B_5 with w_2 and w_4 adjoined such that $w_2 \vee z_2 = x_3$, $w_2 \wedge z_2 = x_1$, $w_4 \vee z_4 = y_4$ and $w_4 \wedge z_4 = y_3$. Let K''_5 be

B_5 with w_3 adjoined so that $w_3 \vee z_3 = x_4$ and $w_3 \wedge z_3 = x_2$. Then, if $n \geq 5$, $V(K_n)$ is covered by $V(K_n) \vee V(L)$ where L is any member of the set $S = \{M_4, B_{n+1}, K_5', K_5'', A_2, A_3, A_4, A_6, A_7, A_8, A_9, N_5\} \cup \{K_m | 4 \leq m < n\}$. Furthermore if V is any variety properly containing $V(K_n)$ then V contains $V(K_n) \vee V(L)$ for some L in S. To see this let L_0 be a subdirectly irreducible lattice in V but not in $V(K_n)$. If L_0 has width greater than four then one of $A_2, \ldots, A_{10}, N_5$ is in $HS(L_0)$. If $A_5 \in HS(L_0)$ then $M_4 \in HS(L_0)$; if $A_{10} \in HS(L_0)$ then $K_5'' \in HS(L_0)$. If L_0 has width less than four and is modular then it is M_3 or a two-element chain, contrary to L_0 not in $V(K_n)$. If L_0 is modular and has width four then it is one of the lattices described in Theorem 5.1. Now it is easily checked that L_0 not in $V(K_n)$ implies that one of M_4, B_{n+1}, K_5', K_5'', K_4, $K_5, \ldots, K_{n-2}$, or K_{n-1} is a sublattice of L_0. In conclusion, it has been shown that if $n \geq 5$ $V(K_n)$ is covered by exactly $n + 8$ varieties and that any variety properly containing $V(K_n)$ contains one of these $n + 8$ covering varieties.

Now we apply to above result to show that $V(K_n)$ has an independent basis with $n + 8$ equation but no independent basis with more equations. The second part of this statement follows immediately from the fact that all varieties properly containing $V(K_n)$ contain one of $n + 8$ covering carieties. Let $L \in S$, then by Theorem 5.2 L is not in $V((S - L) \vee K_n)$. Consequently there is an equation ε_L which holds in $V((S - L) \vee K_n)$ but fails in L. Now it is easy to verify that $\{\varepsilon_L | L \in S\}$ is an independent basis with $n + 8$ elements.

A lattice is called <u>locally finite</u> if its finitely generated sublattices are finite. A variety is <u>locally finite</u> if all its members are locally finite.

Theorem 5.4. $\mathcal{M}_4^\infty$ is locally finite.

Proof: We must show that finitely generated members of $\mathcal{M}_4^\infty$ are finite. If L is a finitely generated subdirectly irreducible member of $\mathcal{M}_4^\infty$ then it follows from Theorem 5.1 that L is finite. Furthermore suppose that $L = \langle G \rangle$ where $|G| = n$. Since L is finite, it is finite dimensional; say the dimension of L is $m + 1$. By Theorem 5.1 the core of L is B_m, see Fig. 5.2.

The only other possible elements of $L - B_m$ are the elements w_k such that $w_k \vee z_k \in \{x_{k+1}, y_{k+1}\}$ and $w_k \wedge z_k \in \{x_{k-1}, y_{k-1}\}$, $k = 2, \ldots, m-1$. Let $k_1, \ldots, k_r$ be those k's such that $w_{k_i} \in L$, $i = 1, \ldots, r$. Since the w_{k_i} is a join and meet irreducible $w_{k_1}, \ldots, w_{k_1} \in G$. Let $\{j_1, \ldots, j_{m-r-2}\}$ be such that $\{k_1, \ldots, k_r\} \cap \{j_1, \ldots, j_{m-r-2}\} = \phi$ and $\{k_1, \ldots, k_r\} \cup \{j_1, \ldots, j_{m-r-2}\} = \{2, \ldots, m-1\}$. Note that if $w_k \notin L$ then either x_{k+1} or y_{k+1} is both meet and join irreducible; say y_{k+1} is join and meet irreducible. Then $y_{k+1} \in G$, $k = j_1, \ldots, j_{m-r-2}$. Thus there must be at least r plus $m - r - 2$ elements in G. Therefore

$$r + m - r - 2 \leq n$$

Thus

$$\dim(L) = m + 1 \leq n + 3$$

We conclude that if L is a subdirectly irreducible member of $\mathcal{M}_4^\infty$ which is generated by n elements then the dimension of L is less than or equal to $n+3$. Since L has width four or less it follows immediately

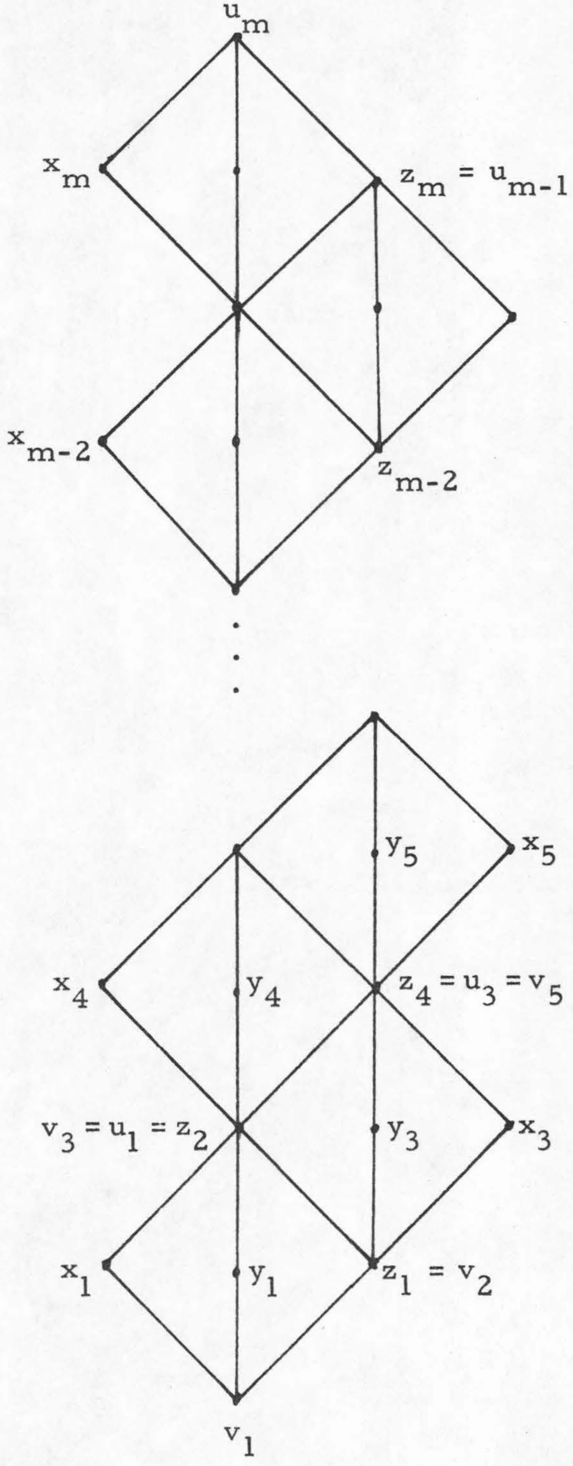


Figure 5.2

from this that $\mathfrak{M}_4^\infty$ has only finitely many subdirectly irreducible lattices with n generators for any fixed n.

Now let L be any member of $\mathfrak{M}_4^\infty$ which is generated by n elements. Then L is a sublattice of L' which is the direct product of subdirectly irreducible lattices L_i, $i \in I$ each of which is a homomorphic image of L. That is, $L' = \prod_{i \in I} L_i$. Since L is generated by n elements each L_i is generated by n elements. Thus, by the above, each L_i is finite and there are only finitely many distinct members of the set $\{L_i \mid i \in I\}$. In order to complete the proof it is sufficient to show that L' is locally finite.

<u>Lemma 5.5.</u> Let $L' = \prod_{i \in I} L_i$ where each L_i is finite and there are only finitely many distinct L_i's. Then L' is locally finite.

<u>Proof:</u> Let $f_1, \ldots, f_n \in \prod_{i \in I} L_i$ and let L be the sublattice generated by $f_1, \ldots, f_n$. Since each L_i is finite and there are only finitely many different L_i's, the set on the n-tuples $\{(f_1(i), \ldots, f_n(i)) \mid i \in I\}$ is finite. Pick $i_1, \ldots, i_\ell$ such that $\{f_1(i), \ldots, f_n(i) \mid i \in I\} = \{f_1(i_k), \ldots, f_n(i_k) \mid k = 1, \ldots, \ell\}$. Let φ be the projection homomorphism from L' to

$$\prod_{k=1}^{\ell} L_{i_k}$$

that is, $\varphi(f) = (f(i_1), \ldots, f(i_\ell))$. To prove the lemma we need to show that φ restricted to L is an isomorphism. It then follows that L is finite and so that L' is locally finite. Pick $i \in I$. Then for some k, $1 \le k \le \ell$, $(f_1(i), \ldots, f_n(i)) = (f_1(i_k), \ldots, f_n(i_k))$. Now let

$f, g \in L = \langle f_1, \ldots, f_n \rangle$. Since f and g are words in $f_1, \ldots, f_n$, $f(i) = f(i_k)$ and $g(i) = g(i_k)$. Consequently, if $\varphi(f) = \varphi(g)$, i.e., if $f(i_j) = g(i_j)$ $j = 1, \ldots, \ell$ then $f(i) = g(i)$ for all i. Thus $f = g$ and so φ restricted to L is one-to-one.

Corollary 5.6. If V is a subvariety of $\mathcal{M}_4^\infty$, then V is determined by its finite members. That is, the variety generated by the finite members of V is V.

Proof: Any variety is determined by its finitely generated members. Since the finitely generated members of V are finite the corollary follows.

We now turn to the problem of showing that there are $2^{\aleph_0}$ distinct subvarieties of $\mathcal{M}_4^\infty$. Recall that B_∞ consists of diamonds $D_i = (v_i, x_i, y_i, z_i, u_i)$, $i = 1, 2, \ldots$ such that $u_{i-1} = z_i = v_{i+1}$, $i = 2, 3, \ldots$ and $z_1 = v_2$. (See Fig. 5.3.)

Let C_∞ be the lattice B_∞ together with elements w_k, $k = 2, 3, \ldots$ such that $w_k \vee z_k = x_{k+1}$ and $w_k \wedge z_k = x_{k-1}$. Let $\mathcal{K}$ be the class of all sublattices of C_∞ obtained by deleting some of the w_k's from C_∞. Let $L \in \mathcal{K}$. We associate with L an infinite sequence $(a_1, a_2, a_3, \ldots)$ of zeros and ones as follows: if $w_k \in L$ then $a_{k-1} = 1$ and $a_{k-1} = 0$ if $w_k \notin L$. This is clearly a one-to-one and onto correspondence. Hence $|\mathcal{K}| = 2^{\aleph_0}$. It will be shown that $|\{V(L) \mid L \in \mathcal{K}\}| = 2^{\aleph_0}$. With each finite sequence of zeros and ones $(a_1, a_2, \ldots, a_n)$ associate the lattice L obtained by appending w_k to B_{n+2} if $a_{k-1} = 1$ in such a way that $w_k \vee z_k = x_{k+1}$ and $w_k \wedge z_k = x_{k-1}$.

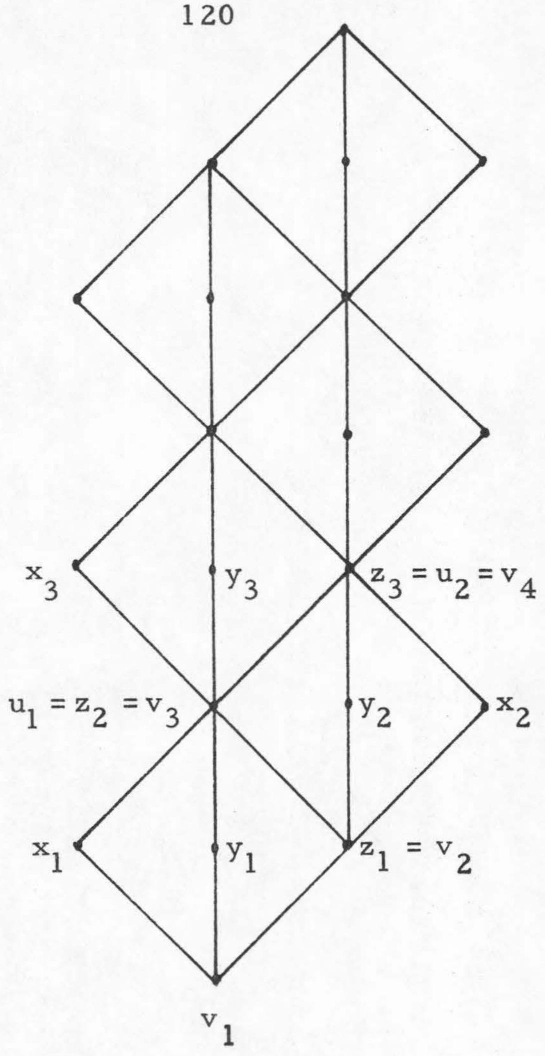


Figure 5. 3

Lemma 5. 7. Suppose L and L' are the lattices associated with $(a_1, a_2, \ldots,)$ and $(b_1, \ldots, b_n)$, respectively. Then $L' \in HSP_u(L)$ if and only if for some k $(b_1, b_2, \ldots, b_n) \le (a_{k+1}, a_{k+2}, \ldots, a_{k+n})$. Here the less than or equal to sign means that $a_i \le b_{k+i}$, $i = 1, \ldots, n$.

Proof: Suppose $L' \in HSP_u(L)$. Then L' is a homomorphic image of L_1 where $L_1 \in SP_u(L)$. Choose an inverse image of each element of L'. Let L_2 be the sublattice of L_1 generated by these inverse images. If we restrict the homomorphism φ which maps L_1 onto L' to L_2 we obtain a homomorphism $\varphi | L_2$ from L_2 into L'. But since L_2 has an inverse image of each element of L', $\varphi | L_2$ maps L_2 onto L'. Since $L_2 \in SSP_u(L) = SP_u(L) \subseteq \mathfrak{M}_4^\infty$ and is finitely generated, L_2 is finite by Theorem 5.4. The fact that L_2 is finite and $L_2 \in SP_u(L)$ imply $L_2 \in S(L)$. Hence L_2 may be regarded as a sublattice of L. In order to avoid confusion we label the elements of L' with primes: $D_i' = (v_i', x_i', y_i', z_i', u_i')$, $i = 1, 2, \ldots, n+2$, and w_i' (if $a_{i-1} = 1$). Since L_2 is finite and φ maps L_2 onto L', there is a smallest element $b \in L_2$ such that $\varphi(b) = u'_{n+2}$, the greatest element of L'. It is easy to verify that φ restricted to the quotient sublattice of elements of L_2 lying below b is onto L'. Hence, by replacing L_2 with this quotient sublattice we may assume that u'_{n+2} has exactly one inverse image in L_2. Now by the dual of this argument we may also assume that v_1', the least element of L', has exactly one inverse image.

Let $\mathcal{V}$ be the class of lattices associated with all the (0, 1)-sequences, $(c_1, c_2, \ldots, c_n)$, for all $n < \omega$ together with the lattices M_3 and $M_{3,3}$ (Fig. 5.4).

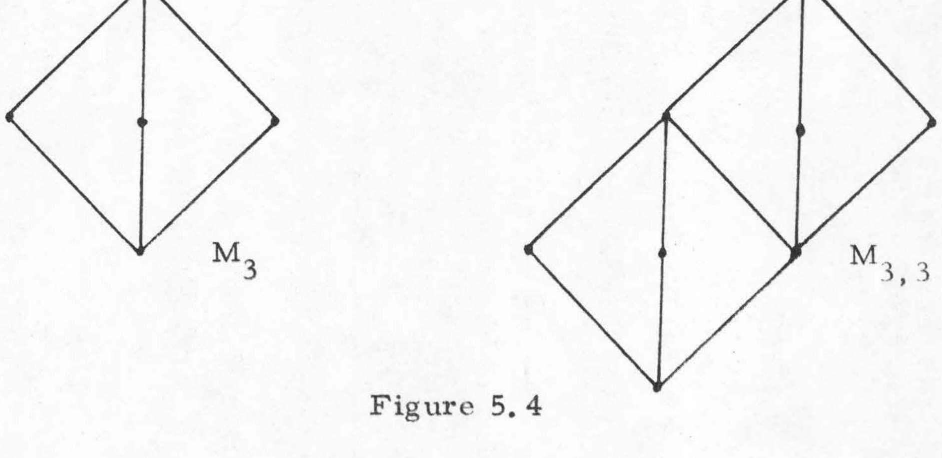


Figure 5.4

Lemma 5.8. Let M be a finite sublattice of the lattice L (of Lemma 5.7), let N $\in \mathscr{K}$ and let ψ be a homomorphism of M onto N. Let N have dimension n + 2, $n \geq 0$ so that v_1' and u_{n+1}' are least and greatest elements of N. Suppose v_1' and u_{n+1}' have unique inverse images under ψ. Then for some k, and r such that $k - r = n$, $\varphi^{-1}(u_{n+1}') = u_k$ and $\varphi^{-1}(v_1') = v_r$. Consequently ψ is an isomorphism and thus $N \cong M$. Furthermore, M is an isometric sublattice of L.

Proof: Let dim X denote the dimension of any finite modular lattice X and let d_L be the dimension function on the elements of L. The first conclusion of the lemma implies that

$$\dim M \leq d_L(u_k) - d_L(v_\ell) = k - r + 2 = n + 2 = \dim N$$

Since N is a homomorphic image of M we must have dim M = dim N and therefore ψ must be one-to-one. Also, the fact that dim M = $d_L(u_k) - d_L(u_r)$ implies that M is an isometric sublattice of L. Hence it only remains to prove the first conclusions of the lemma. We do this by induction on n.

If n = 0 then $N = M_3 = D_1' = (v_1', x_1', y_1', z_1', u_1')$. Let $\overline{v}_1'$, $\overline{x}_1'$, $\overline{y}_1'$, $\overline{z}_1'$, $\overline{u}_1'$ be inverse images of v_1', x_1', y_1', z_1', u_1', respectively. It follows from the uniqueness of $\overline{v}_1'$ and $\overline{u}_1'$ that $\overline{D}_1' = (\overline{v}_1', \overline{x}_1', \overline{y}_1', \overline{z}_1', \overline{u}_1')$ is a diamond sublattice of L. Hence $\overline{D}_1' = D_k$ for some k, which proves the lemma in this case.

Now suppose dim N = n + 2, n > 0. Let $\overline{u_{n+1}'}$ and $\overline{v}_1'$ denote the unique inverse images of u_{n+1}' and v_1'. Let $\overline{u}_n'$ denote the smallest inverse image of u_n'. Applying the induction hypothesis to $\overline{u}_n'/\overline{v}_1'$, u_n'/v_1'

and $\psi|\overline{u'_n}/\overline{v'_1}$ it follows that $\overline{u'_n} = u_m$ and $\overline{v'_1} = v_r$, where $m - r = n - 1$. Now let $\overline{\overline{x'_{n+1}}}$, $\overline{\overline{y'_{n+1}}}$ and $\overline{\overline{u'_n}}$ denote the largest inverse images of x'_{n+1}, y'_{n+1} and $u'_n = z'_{n+1}$. Then $\overline{\overline{x'_{n+1}}}$, $\overline{\overline{y'_{n+1}}}$ and $\overline{\overline{u'_n}}$ are incomparable and are covered by $\overline{u'_{n+1}}$. The only way this can happen in L is $\overline{u'_{n+1}} = u_k$, for some k, and $\{\overline{\overline{x'_{n+1}}}, \overline{\overline{y'_{n+1}}}, \overline{\overline{u'_n}}\} = \{x_k, y_k, z_k = u_{k-1}\}$. Since x'_{n+1}, y'_{n+1} and $z'_{n+1} = u'_n$ are incomparable, $\overline{\overline{x'_{n+1}}}$, $\overline{\overline{y'_{n+1}}}$ and $\overline{u'_n} = u_m$ are incomparable. Thus u_m is incomparable with x_k, y_k. It follows that $k = m+1$ so that $\overline{u'_{n+1}} = u_k$, $\overline{v'_1} = v_r$ and $k - r = m+1 - r = n$, proving the lemma.

Now we return to the proof of Lemma 5.7. By the remarks preceding Lemma 5.8 we may apply that lemma with $M = L_2$, $N = L'$ and $\psi = \varphi$. We conclude that $L_2 \cong L'$ and L_2 is an isometric sublattice of L. Moreover, L_2 is simple, since $L_2 \cong L'$ and L' is simple. Also, for some k, r, $k - r = n+1$, L_2 is a sublattice of u_k/v_r. But the only simple sublattices of u_k/v_r with greatest element u_k and least element v_r are those obtained by possibly deleting some of the w_m's from u_k/v_r. Since $L' \cong L_2$, $(b_1, \dots, b_n)$ describes L_2 as well as L'. Consequently $(b_1, b_2, \dots, b_n) \le (a_{r+1}, \dots, a_{k-1})$, the desired conclusion. The conversion of the lemma is obvious.

Now we return to the problem of showing that there are $2^{\aleph_0}$ varieties generated by single members of $\mathcal{K}$. Recall that $\mathcal{K}$ consists of all sublattices of C_∞ obtained by deleting some of the w_k's and associated with each member of $\mathcal{K}$, a sequence of zeros and ones $(a_1, a_2, \dots)$ such that w_k is in the lattice if and only if $a_{k-1} = 1$.

By a finite block subsequence of $(a_1, a_2, a_3, \dots)$ we mean a subsequence of the form $(a_k, a_{k+1}, \dots, a_{k+r})$. Suppose there exists a set of $\mathcal{S}$ sequences such that if $a = (a_1, a_2, \dots)$ and $b = (b_1, b_2, \dots)$ are in $\mathcal{S}$

then either (i) there exists a finite block subsequence $(a_k, a_{k+1}, \ldots, a_{k+r})$ of $(a_1, a_2, \ldots)$ such that

$$(a_k, a_{k+1}, \ldots, a_{k+r}) \nleq (b_m, b_{m+1}, \ldots, b_{m+r})$$

for all choices of m or (ii) there exists a finite block sequence $(b_k, b_{k+1}, \ldots, b_{k+r})$ of $(b_1, b_2, \ldots)$ such that

$$(b_k, b_{k+1}, \ldots, b_{k+r}) \nleq (a_m, a_{m+1}, \ldots, a_{m+r})$$

for all choices of m. Let L_a and L_b be the members of $\mathcal{K}$ associated with a and b, respectively. Then the above conditions imply that L_a and L_b generate distinct varieties, since, by Lemma 5.7 and Theorem 5.2, the lattice associated with $(a_k, a_{k+1}, \ldots, a_{k+r})$ cannot be in $V(L_b)$ if the first condition holds and the lattice associated with $(b_k, b_{k+1}, \ldots, b_{k+r})$ is not in $V(L_a)$ if the second condition holds. Thus to show the existence of $2^{\aleph_0}$ varieties it is sufficient to construct a set $\mathcal{S}$ which satisfies (i) and (ii) such that $|\mathcal{S}| = 2^{\aleph_0}$. Let

$$s_1 = 1\,0\,0\,1$$
$$s_2 = 1\,0\,0\,0\,0\,0\,0\,1$$
$$\vdots$$
$$s_n = 1\,\overbrace{0 \quad\quad 0}^{(n+1)!\ \text{zeroes}}\,1$$
$$\vdots$$

Let N be the set of positive integers, and let $T = \{i_1, i_2, i_3, \ldots\}$ and $U = \{j_1, j_2, j_3, \ldots\}$ be distinct infinite subsets of N. Assume also that $i_1 < i_2 < i_3 < \cdots$ and $j_1 < j_2 < j_3 < \cdots$. Associate the sequence $s_{i_1} s_{i_2} s_{i_3} \cdots$ with T and the sequence $s_{j_1} s_{j_2} s_{j_3} \ldots$ with U. Here $s_{i_1} s_{i_2} s_{i_3} \cdots$ denote the concatenation of the sequences $s_{i_1} s_{i_2} s_{i_3} \cdots$. We may assume that $T \nsubseteq U$. Let $n \in T$, $n \notin U$. Then s_n is a finite block subsequence of the sequence associated with T. Suppose it is less than or equal to a finite block subsequence $(a_m, a_{m+1}, \dots, a_{m+(n+1)!+2})$ of the sequence associated with U. Then $a_m = a_{m+(n+1)!+2} = 1$, so that a_m and $a_{m+(n+1)!+2}$ must be either the beginning or end of one of the s_j's. It follows that for some $j_r, j_{r+1}, \dots, j_k$, $(a_m, a_{m+1}, \dots, a_{m+(n+1)!+2})$ has one of the following four forms.

$$
(1) \qquad
\begin{array}{l}
s_{j_r}\, s_{j_{r+1}} \dots s_{j_k} \\
1\, s_{j_r}\, s_{j_{r+1}} \dots s_{j_k} \\
s_{j_r}\, s_{j_{r+1}} \dots s_{j_k}\, 1 \\
1\, s_{j_r}\, s_{j_{r+1}} \dots s_{j_k}\, 1
\end{array}
$$

Clearly $j_t < n$, $t = r, r+1, \dots, k$. However, each of these four sequences has length less than or equal to

$$2(k-r+1) + \sum_{t=r}^{k} (j_t+1)! + 2$$

Now if n = 1 then the condition $j_t \le n = 1$ shows that there can be no such j_t's and, in fact, it is clear that s_1 is not a block subsequence of $s_{j_1} s_{j_2} s_{j_3} \dots$ in this case. If $n \ge 2$ then since $j_t < n$

$$2(k-r+1) + \sum_{t=r}^{k} (j_t+1)! + 2$$

$$\le 2(n-1) + \sum_{t=1}^{n-1} (t+1)! + 2$$

$$< (n+1)! + 2$$

The first inequality expresses the fact that the length of the sequences in (1) is not greater than the length of the sequence $1\ s_1 s_2 s_3 \ldots s_{n-1}\ 1$. The second inequality is proved easily by induction. Since $(n+1)! + 2$ is the length of s_n we see that s_n is not less than or equal to a finite block subsequence of the sequence $s_{j_1} s_{j_2} s_{j_3} \ldots$ associated with U. Thus for $\mathcal{S}$ we take the sequences associated with the infinite subsets of N. We have proved the following theorem.

Theorem 5.9. There exist $2^{\aleph_0}$ distinct varieties contained in $\mathfrak{M}_4^{\infty}$.

Since there are only countably many varieties defined by a finite set of equation, Theorem 5.9 has the following corollary, which contrasts Theorem 5.3.

Corollary 5.10. There exist $2^{\aleph_0}$ distinct varieties contained in $\mathfrak{M}_4^{\infty}$ which are not defined by any finite set of identities.

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