RELATIVISTIC VELOCITY - POTENTIAL HYDRODYNAMICS AND STELLAR STABILITY

Thesis by
Bernard Frederick Schutz, Jr.

In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California

1972

(Submitted August 10, 1971)

ACKNOWLEDGEMENTS

I want to thank Edward Fackerell, Yavuz Nutku, and James Bardeen for especially helpful conversations during the course of this research. Special appreciation goes to the Kellogg secretaries, especially to Leslie Will for the many many things she finds time to do. I am very grateful to Professor Kip Thorne for both his teaching and his encouragement. And I must thank my wife, Joan, for her patience and good humor during the past few months.

I was an NDEA Title IV fellow during part of this research. The research was supported in part by the National Science Foundation [GP-15911, GP-9114, GP-27304, GP-19887, GP-28027] and the Office of Naval Research [Nonr-220(47)] .

ABSTRACT

The equations of relativistic, perfect-fluid hydrodynamics are cast in Eulerian form using six scalar "velocity-potential" fields, each of which has an equation of evolution. These equations determine the motion of the fluid through the equation

$$U_\nu = \mu^{-1} (\phi_{,\nu} + \alpha\beta_{,\nu} + \theta S_{,\nu}).$$

Einstein's equations and the velocity-potential hydrodynamical equations follow from a variational principle whose action is

$$I = (R + 16\pi\, p)\,(-g)^{1/2}\, d^4x\ ,$$

where R is the scalar curvature of spacetime and p is the pressure of the fluid. These equations are also cast into Hamiltonian form, with Hamiltonian density $-T_0{}^0(-g^{oo})^{-1/2}$.

The second variation of the action is used as the Lagrangian governing the evolution of small perturbations of differentially rotating stellar models. In Newtonian gravity this leads to linear dynamical stability criteria already known. In general relativity it leads to a new sufficient condition for the stability of such models against arbitrary perturbations.

By introducing three scalar fields defined by

$$\rho\,\underset{\sim}{\xi} = \underset{\sim}{\nabla}\lambda + \underset{\sim}{\nabla}\times(\chi \underset{\sim}{i} + \underset{\sim}{\nabla}\times\gamma\underset{\sim}{i})$$

(where $\underset{\sim}{\xi}$ is the vector displacement of the perturbed fluid element, ρ is the mass-density, and $\underset{\sim}{\xi}$ is an arbitrary vector), the Newtonian stability criteria are greatly simplified for the purpose of practical applications. The relativistic stability criterion is not yet in a form that permits practical calculations, but ways to place it in such a form are discussed.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION AND OVERVIEW

The investigations reported in this thesis began two years ago as a search for stability criteria for relativistic stars. Thorne and Campolattaro (1967) had derived the equations governing the evolution of small nonradial perturbations of fully relativistic, spherically symmetric stellar models. These equations have yielded much information about convection (Islam 1970) and about the emmission of and damping by gravitational radiation (Thorne 1969, Ipser 1971), and it was expected that they would also yield information about the stability of stars against such perturbations.

Accordingly, I attempted to use the techniques pioneered in Newtonian gravity by Chandrasekhar and Lebovitz (see the references in the introduction to Chapter 4) to derive stability criteria. Basically, the idea was to find a Lagrangian from which the Thorne-Campolattaro equations could be derived, and then to obtain stability criteria that used the associated Hamiltonian [cf. the theorems of Kulsrud (1968)]. But it soon became apparent that there did not exist any simple polynomial Lagrangian for the Thorne-Campolattaro equations.

We now know the reason for this: Thorne and Campolattaro had incorrectly formulated the perturbation equations as a fifth-order system of coupled partial differential equations. The equations can acturally be formulated as a fourth-order system (Ipser and Thorne 1971), which presumably does admit a Lagrangian (though this has not yet been verified directly). At the time, however, the puzzling absence of a Lagrangian led me to search for alternate ways of deriving stability criteria, independent of the Thorne-Campolattaro computations.

The search has been both interesting and fruitful, with implications that may extend beyond the theory of stellar stability. The search led first to a new formulation of the equations of relativistic perfect-fluid hydrodynamics, based upon the nonrelativistic work of Seliger (1968) and Seliger and Whitham (1968). This "velocity-potential" version of hydrodynamics had actually been developed independently by Schmid in special relativity in a series of papers beginning in 1966 [see Schmid (1970 a,b) for references]. The general-relativistic version of velocity-potential hydrodynamics is explored in Chapter 2, which was published in Phys. Rev. D. 2,2762(1970).

For our purposes the most interesting feature of velocity-potential hydrodynamics is a variational principle from which its equations can be derived. In Chapter 3 the variational principle is used to cast the equations in Hamiltonian (Poisson-bracket) form. This forms the foundation of our approach to stability in Chapter 5. Chapter 3 has been submitted to the Physical Review.

Chapter 4 considers the stability of stars in the framework of Newtonian gravity. The second variation of the Seliger-Whitham Lagrangian is used as the Lagrangian for the perturbations of an arbitrary differentially rotating star. Stability criteria derived from this Lagrangian are identical to those derived by previous workers using the techniques of Chandrasekhar and Lebovitz (see references in Chapter 4).

In Chapter 4 I also present a technique that hopefully will permit a significant simplification of the calculations whereby one tests realistic stellar models for stability.

Chapter 5 investigates the linear pulsations and stability of fully relativistic, differentially rotating stellar models. It and Chapter 4 have been submitted together to the Astrophysical Journal. As in Chapter 4, the second variation of the velocity-potential Lagrangian is used as the Lagrangian for the perturbations. This is equivalent to the Lagrangian I was unable to obtain from the Thorne-Campolattaro equations, with a bonus: it is applicable to rotating stars as well as to the nonrotating models considered by Thorne and Campolattaro (1967).

The sufficient condition for stability that follows from the Lagrangian is the first exact criterion anyone has obtained for the stability of relativistic stars against aspherical perturbations. It is unfortunately not yet in a useful form for astrophysical applications, because of complications introduced by gravitational radiation. At the end of Chapter 5 are discussed the remaining steps that must be performed before the criterion is ready for astrophysical use.

In Chapter 6 I suggest some possible future applications of the results of this thesis, and I mention some problems that must be solved before the theory of the linear dynamical stability of relativistic stars can be considered in good shape.

The appendix is an article that will be published by Springer-Verlag (Lecture Notes in Physics series) in the the Pittsburgh Conference on Relativity, July 1970. Its subject is not really germane to velocity-potential hydrodynamics, but because it is referred to several times in Chapter 3, and because it is not yet readily available in the literature , I have appended it here.

REFERENCES

Ipser, J.R. 1971, Ap. J., 166, 175.

Ipser, J.R., and Thorne, K.S. 1971, paper in preparation.

Islam, J.N. 1970, M.N.R.A.S., 150, 237.

Kulsrud, R.M. 1968, Ap. J., 152 , 1121.

Schmid, L.A. 1970a, in A Critical Review of Thermodynamics, ed. E.B. Stuart, B. Gal-Or, and A.J. Brainard (Baltimore: Mono).

—————————.1970b, Pure and Appl. Chem., **22**, 493.

Seliger, R.L. 1968, unpublished Ph.D. thesis, California Institute of Technology.

Seliger, R.L., and Whitham, G. B. 1968, Proc. Roy. Soc. A., **305**, 1.

Thorne, K.S. 1969, Ap. J., **158**, 1.

Thorne, K.S., and Campolattaro, A. 1967, Ap. J., **149**, 591.

CHAPTER 2

PERFECT FLUIDS IN GENERAL RELATIVITY: VELOCITY POTENTIALS AND A VARIATIONAL PRINCIPLE

Published in the Physical Review D, **2**, 2762 (1970)

Reprinted from The Physical Review D, Vol. 2, No. 12, 2762–2773, 15 December 1970
Printed in U. S. A.

Perfect Fluids in General Relativity: Velocity Potentials and a Variational Principle*

Bernard F. Schutz, Jr.†
California Institute of Technology, Pasadena, California 91109
(Received 9 April 1970; revised manuscript received 22 June 1970)

The equations of hydrodynamics for a perfect fluid in general relativity are cast in Eulerian form, with the four-velocity being expressed in terms of six velocity potentials: $U_\nu=\mu^{-1}(\phi_{,\nu}+\alpha\beta_{,\nu}+\theta S_{,\nu})$. Each of the velocity potentials has its own "equation of motion." These equations furnish a description of hydrodynamics that is equivalent to the usual equations based on the divergence of the stress-energy tensor. The velocity-potential description leads to a variational principle whose Lagrangian density is especially simple: $\mathfrak{L}=(-g)^{1/2}(R+16\pi p)$, where R is the scalar curvature of spacetime and p is the pressure of the fluid. Variation of the action with respect to the metric tensor yields Einstein's field equations for a perfect fluid. Variation with respect to the velocity potentials reproduces the Eulerian equations of motion.

I. INTRODUCTION

In this paper we introduce a velocity-potential representation for the four-velocity of a perfect fluid in general relativity. This representation permits a new formulation of relativistic hydrodynamics, in which the velocity potentials themselves have first-order "equations of motion," and in which the changes of the four-velocity with time are expressed in terms of Eulerian[1] changes in the potentials. Einstein's field equations plus the equations of evolution in this new formulation can in turn be obtained from a variational principle whose Lagrangian density is

$$\mathfrak{L}=(-g)^{1/2}(R+16\pi p), \tag{1.1}$$

where R is the scalar curvature and p is the fluid's pressure.

Velocity potentials are not new to Newtonian hydrodynamics, but they have been of limited usefulness. It is well known that irrotational motions can be derived from a single potential, $\mathbf{v}=\nabla\phi$. In 1859, Clebsch[2] proved that *any* (Newtonian) motion can be represented by three potentials:

$$\mathbf{v}=\nabla\phi+\alpha\nabla\beta. \tag{1.2}$$

The Clebsch representation had the disadvantage that ϕ, α, and β were not physically useful individually; in particular, there were no individual equations of evolution for ϕ, α, and β that could give changes in $\mathbf{v}$ directly, without reference to the usual equations of hydrodynamics.

By contrast, the Newtonian velocity-potential representation introduced by Seliger and Whitham[3] in 1968 avoids this difficulty. By using five potentials (two more than the minimum necessary), Seliger and Whitham were able to give to each potential an equation of evolution and to some an independent physical interpretation. For example, one potential is the entropy; another is the "thermasy" of van Dantzig.[4]

The representation presented in this paper is a relativistic generalization of the one given by Seliger and Whitham. The six velocity potentials (one more than in the Newtonian case because we have a four-velocity rather than a three-velocity) all have equations of evolution that determine how they change with time. These equations constitute an *alternative* to the usual equations of hydrodynamics (i.e., to those based upon the divergence of a stress-energy tensor), rather than simply an adjunct.

Seliger and Whitham derived their equations from a variational principle. We here generalize their principle to include the effects of a general-relativistic gravitational field. In additi n we place the velocity-potential equations of evolution on a firm foundation apart from the variational principle by giving a rigorous proof that they are equivalent to the standard equations of hydrodynamics. If the reader desires a more intuitive feeling for why the fluid's Lagrangian density should be simply the pressure, or for how one originally came to the velocity-potential representation, he is invited to read Seliger and Whitham and the references they cite.

The present paper is divided into two main parts plus four Appendixes. The first part discusses the equations of hydrodynamics, first in their standard form and then in terms of velocity potentials. The proof of equivalence between these two versions of hydrodynamics is left to Appendix B. The second main part presents the variational principle. Appendix A

* Supported in part by the National Science Foundation [GP-15911, GP-9114] and the Office of Naval Research [Nonr-220(47)].
† NDEA Title IV Predoctoral Fellow.

[1] History has mercilessly given us half a dozen different uses for the names of Lagrange and Euler. The adjectives *Lagrangian* and *Eulerian* refer, respectively, to observers comoving with the fluid or fixed with respect to some arbitrary reference frame through which the fluid flows (see also Ref. 9). The functional whose integral is extremized in a variational principle is the *Lagrangian density*. Finally the equations that express the extremal conditions are the *Euler-Lagrange* equations. Because we wish to emphasize the Eulerian nature of the velocity potentials, we shall henceforth speak of their equations of *evolution* rather than of their equations of *motion*.

[2] A. Clebsch, J. Riene Agnew. Math. 56, 1 (1859).

[3] R. L. Seliger and G. B. Whitham, Proc. Roy. Soc. (London) A305, 1 (1968). Their representation was based in part on work by C. C. Lin, in *Liquid Helium* (International School of Physics "Enrico Fermi", course 21), edited by G. Careri (Academic, New York, 1963), p. 93.

[4] D. van Dantzig, Physica 6, 693 (1939).

contains Pfaff's theorem, an old theorem in differential forms that is essential to understanding the velocity-potential representation; we include it here (without proof) because it is not well known to physicists in its most general form. Appendixes C and D discuss in detail questions that may interest only the specialist: respectively, the uniqueness of the velocity-potential representation and an initial-value formulation of the equations of evolution.

A word about conventions: We use "geometrized units," with $c=G=1$. Greek indices run from 0 to 3; Latin from 1 to 3. The metric has positive signature, so that timelike intervals are negative. We define proper time τ by

$$d\tau^2=-ds^2=-g_{\mu\nu}dx^\mu dx^\nu, \tag{1.3}$$

so that $d\tau$ is real and positive for a particle moving forward in time. We adopt the notation that $D/D\tau$ means covariant differentiation along a world line, while $d/d\tau$ means partial differentiation; a semicolon denotes a general covariant derivative and a comma denotes a general partial derivative. Thus, the four-velocity is defined as

$$U^\nu=dx^\nu/d\tau, \tag{1.4}$$

so that we have

$$U^\nu U_\nu=-1. \tag{1.5}$$

Then for any function χ,

$$D\chi/D\tau\equiv U^\nu(\chi)_{;\nu} \tag{1.6}$$

and

$$d\chi/d\tau\equiv U^\nu(\chi)_{,\nu}. \tag{1.7}$$

Finally, four-vectors are written in boldface sans serif: **A**. Three-vectors appear in boldface: **A**.

II. RELATIVISTIC THEORY OF ONE-COMPONENT PERFECT FLUID

A. Standard Version

Thermodynamics of One-Component Perfect Fluid

We consider a perfect fluid composed of baryons. Because baryons can undergo transmutation, the true rest mass of a group of baryons may not be conserved; but their baryon number N is conserved. Hence, we *define* the (conserved) rest mass of a sample of matter containing N baryons to be $m_H N$, where m_H is the mass of a hydrogen atom in its ground state. The difference between the total mass-energy and $m_H N$ is called the internal energy U. Thus U includes the difference between $m_H N$ and the true rest mass of the actual atoms and baryons; and it also includes the energy of electron-positron pairs, of mesons, of photons, of thermal motions, and of "zero-point" Fermi-gas "motions." We denote by ρ_0 the density of rest mass so defined, and by $\Pi\equiv U/m_H N$ the specific internal energy, both as measured in a local inertial frame momentarily at rest in the fluid. Then the density of total mass-energy is $\rho=\rho_0(1+\Pi)$.

We assume an equation of state of the form $p=p(\rho_0,\Pi)$. Such a two-parameter expression is sufficient for any one-component fluid.[5] The applicability of the results of this paper to a real baryonic fluid depends in part on how well a two-parameter equation of state characterizes the fluid.

The amount of energy per unit rest mass, δq, added to the fluid in any quasistatic process is (first law of thermodynamics)

$$\delta q=d\Pi+pd(1/\rho_0). \tag{2.1}$$

Because of the two-parameter equation of state, Pfaff's theorem (Appendix A) implies that there exist functions $S(\rho_0,\Pi)$ and $T(\rho_0,\Pi)$, the specific entropy and the temperature, respectively, such that[6]

$$d\Pi+pd(1/\rho_0)=TdS=\delta q. \tag{2.2}$$

If one now defines the specific inertial mass by[7]

$$\mu=(\rho+p)/\rho_0=1+\Pi+p/\rho_0, \tag{2.3}$$

one can use $d\mu$ to eliminate $d\Pi$ in Eq. (2.2) and obtain

$$d\mu-\rho_0^{-1}dp=TdS. \tag{2.4}$$

We will often use this in the form

$$dp=\rho_0 d\mu-\rho_0 TdS. \tag{2.5}$$

Clearly one can express ρ_0 and Π as functions of μ and S, so that one can put the equation of state in the form

$$p=p(\mu,S). \tag{2.6}$$

Stress-Energy Tensor and Equations of Motion

The relativistic one-component perfect fluid is defined by its equation of state, Eq. (2.6), and by the stress-energy tensor

$$\begin{aligned}T^{\mu\nu}&=(\rho+p)U^\mu U^\nu+pg^{\mu\nu}\\&=\rho_0\mu U^\mu U^\nu+pg^{\mu\nu}.\end{aligned} \tag{2.7}$$

In a locally comoving inertial frame, $T^{\mu\nu}$ is $\mathrm{diag}(\rho,p,p,p)$. Because the fluid is perfect, the stress-energy tensor

---

[5] E. Fermi, *Thermodynamics* (Dover, New York, 1936), p. 91.

[6] For a many-component system (i.e., one whose equation of state has more than two independent parameters), Pfaff's theorem does not suffice to require $\delta q=TdS$, i.e., to ensure an integrating factor for δq. One must then invoke a weak form of the second law of thermodynamics. See S. Chandrasekhar, *An Introduction to the Study of Stellar Structure* (Dover, New York, 1939), Chap. 1. For an isolated one-component fluid, Pfaff's theorem makes the second law a mathematical identity.

[7] The quantity $\rho+p$ plays the role of inertial mass per unit volume in a perfect fluid. See Eq. (2.19) of this paper, or the article by K. S. Thorne in *High-Energy Astrophysics*, edited by C. DeWitt, E. Schatzmann, and P. Véron (Gordon and Breach, New York, 1967), Vol. 3. I thank Professor Thorne for pointing out to me that $m_H\mu$ is also the injection energy at constant entropy: the energy required to create one baryon and place it in the fluid with the same energy ($m_H\Pi$) as neighboring baryons, doing work $m_H p/\rho_0$ to create the same volume (m_H/ρ_0) for it as the other baryons have.

contains no viscosity or energy-transport terms.[8] The conservation of baryon number, rewritten in terms of rest mass ρ_0, is embodied in the equation

$$(\rho_0 U^\nu)_{;\nu}=0. \tag{2.8}$$

Normalization of the four-velocity reads

$$U^\nu U_\nu=-1, \tag{2.9}$$

covariant differentiation of which yields the useful equation

$$U^\nu U_{\nu;\sigma}=0. \tag{2.10}$$

The equations of motion obeyed by the fluid are expressed in conservation form by requiring the stress-energy tensor to be divergence-free:

$$T^{\mu\nu}{}_{;\nu}=0. \tag{2.11}$$

These four equations supplemented by Eqs. (2.8) and (2.9) determine the motion of a fluid whose equation of state is known.

The physical meaning of the four equations (2.11) becomes clearer upon separating out their components parallel and perpendicular to the four-velocity. The equation parallel to **U**,

$$U_\mu T^{\mu\nu}{}_{;\nu}=0, \tag{2.12}$$

reduces [by Eqs. (2.8)–(2.10)] to

$$U^\nu p_{,\nu}-\rho_0 U^\nu \mu_{,\nu}=0. \tag{2.13}$$

By Eq. (2.5) this becomes

$$\rho_0 T U^\nu S_{,\nu}=0. \tag{2.14}$$

Thus, the motions of a perfect fluid conserve the entropy per baryon. Because $\delta q=TdS$, this confirms that no heat flows in or out of any element of the perfect fluid during its motions.

One can construct the three independent equations of motion perpendicular to **U** by using the projection tensor

$$P^\sigma{}_\mu=\delta^\sigma{}_\mu+U^\sigma U_\mu. \tag{2.15}$$

The equations are

$$P^\sigma{}_\mu T^{\mu\nu}{}_{;\nu}=0. \tag{2.16}$$

By using Eqs. (2.8)–(2.10), one can reduce this to

$$-P_\sigma{}^\nu p_{,\nu}=\mu\rho_0 U_{\sigma;\nu}U^\nu \tag{2.17}$$

$$=\mu\rho_0 DU_\sigma/D\tau. \tag{2.18}$$

In a locally comoving intertial frame, $P_\sigma{}^\nu$ picks out the spatial gradient of p. If $\mathbf{v}$ is the (instantaneously zero) spatial part of **U**, Eq. (2.18) becomes

$$-\nabla p=(\rho+p)d\mathbf{v}/dt. \tag{2.19}$$

This is the familiar force law; it justifies calling $(\rho+p)$ the inertial mass per unit volume.

B. Velocity-Potential Version

Velocity-Potential Representation and Equations of Motion

One usually interprets the equations of motion in the "standard version" in a Lagrangian sense. One regards the four-velocity as vector representing the change of a particle's position in proper time. It is a vector "field" only in the continuum approximation, in which one overlooks the fact that the fluid is "really" composed of discrete particles packed very closely together. Because one tends to regard the four-velocity as a little arrow carried along by the particles, one also tends to interpret the equations of motion in terms of what happens to little fluid elements. Thus, Eq. (2.19) describes the response of a fluid element to a pressure gradient, and Eqs. (2.8) and (2.14) require the conservation of the number of baryons and the amount of entropy contained in a fluid element.

The "velocity-potential version" of hydrodynamics, by contrast, lends itself most naturally to an Eulerian interpretation. One regards the four-velocity as a vector field over spacetime. As such it can be represented in terms of scalar fields and their gradients. While the particles move through space, the scalars at a given point of space simply change their amplitudes with time.[9]

According to Pfaff's theorem (Appendix A), four potentials are sufficient to describe the four-velocity:

$$U_\nu=AB_{,\nu}+CD_{,\nu}. \tag{2.20}$$

While four such potentials are guaranteed to exist, they may not be physically useful. In this paper we introduce instead a six-potential representation that has a ready and important physical interpretation. This representation is[10]

$$U_\nu=\mu^{-1}(\phi_{,\nu}+\alpha\beta_{,\nu}+\theta S_{,\nu}). \tag{2.21}$$

The potentials μ and S are just the specific inertial mass and the specific entropy as defined above. The physical significance of the remaining potentials ϕ, α, β, and θ[11] will be explored below.

---

[8] Newtonian perfect fluids permit heat conduction. In relativity, however, conduction leads to a nonzero momentum density and to anisotropic stresses in the rest frame of the baryons; it must therefore be excluded from *perfect* fluids in relativity.

[9] The distinction between Eulerian and Lagrangian coordinates, while useful, is not rigid in general relativity, because all equations are independent of coordinate system. Lagrangian interpretations are valid only in comoving frames. The "Eulerian" equations for the velocity potentials are good in *any* reference frame; in fact, however, they are most easily interpreted in a comoving frame. (See the section on Physical Interpretation below.)

[10] Seliger and Whitham replace the term $\theta S_{,\nu}$ with $-S\theta_{,\nu}$ in their Newtonian representation, and thus achieve the nonrelativistic version of Schmid's representation (Ref. 26). Note also that Eq. (2.21) is a local equation; the existence of a global set of potentials is not guaranteed.

[11] To my knowledge, D. van Dantzig (Ref. 4) was the first to define θ. He called it the "thermasy."

The equations of evolution in this representation are

$$(\rho_0 U^\nu)_{;\nu}=0, \tag{2.22a}$$

$$U^\nu S_{,\nu}=dS/d\tau=0, \tag{2.22b}$$

$$U^\nu \alpha_{,\nu}=d\alpha/d\tau=0, \tag{2.22c}$$

$$U^\nu \beta_{,\nu}=d\beta/d\tau=0, \tag{2.22d}$$

$$U^\nu \phi_{,\nu}=d\phi/d\tau=-\mu, \tag{2.22e}$$

$$U^\nu \theta_{,\nu}=d\theta/d\tau=T. \tag{2.22f}$$

From Eqs. (2.21), (2.22b), (2.22d), and (2.22e) follows the result

$$U^\nu U_\nu=-1. \tag{2.23}$$

There is no equation for μ. Its evolution can be computed from Eqs. (2.22a), (2.22b), and the equation of state.

Appendix B contains the proof that these velocity-potential equations are equivalent to the standard version of the equations of motion.

Physical Significance of Velocity-Potential Version

Circulation. The representation Eq. (2.21) is well suited to Taub's[12] Eulerian analysis of circulation. Taub defines a current vector $\mathbf{V}=\mu\mathbf{U}$,[13] which in our representation is

$$V_\sigma=\mu U_\sigma=\phi_{,\sigma}+\alpha\beta_{,\sigma}+\theta S_{,\sigma}. \tag{2.24}$$

He then defines the circulation tensor $\Omega_{\sigma\lambda}=2V_{[\sigma;\lambda]}$, where square brackets denote the antisymmetric part. In our representation this becomes

$$\Omega_{\sigma\lambda}=2V_{[\sigma;\lambda]}=2\alpha_{[,\lambda}\beta_{,\sigma]}+2\theta_{[,\lambda}S_{,\sigma]}. \tag{2.25}$$

Taub then defines circulation C in the following manner. Consider a spacelike hypersurface Σ through the world lines of the fluid's particles. A closed curve Λ in Σ may in general enclose some circulating fluid. If λ is the ordinary length parameter along Λ, and $l^\alpha=dx^\alpha/d\lambda$ is the tangent vector to Λ in Σ, then the circulation C is defined as the integral

$$C=\oint_\Lambda V_\alpha l^\alpha d\lambda \tag{2.26}$$

around the closed curve. From Eq. (2.24) we see that

$$C=\oint_\Lambda \alpha d\beta_\Lambda+\oint_\Lambda \theta dS_\Lambda, \tag{2.27}$$

where the subscript Λ on $d\beta$ and dS means that the differentials are directed along the curve. Clearly, if α is a function only of β, and if either θ or S is a constant,[14] then C will be zero for any choice of Σ and Λ. In this case, $\Omega_{\sigma\kappa}$ is also zero. One can easily see, then, that C will vanish for *every* curve Λ in *every* hypersurface Σ if and only if $\Omega_{\sigma\kappa}=0$, which is a result Taub also mentions. This establishes the significance of $\Omega_{\sigma\kappa}$ in circulation.

In order to see the roles of α, β, θ, and S more clearly, let us look at the circulation in a momentarily comoving local Lorentz frame, with $g_{\alpha\beta}=\mathrm{diag}(-1, 1, 1, 1)$. Then $V^0=-V_0=\mu$, $V^i=V_i=0$. Define the vorticity vector

$$\begin{aligned} v^\nu&=\tfrac{1}{2}(-g)^{-1/2}\epsilon^{\nu\sigma\lambda\kappa}U_\sigma U_{\lambda;\kappa}\\ &=\tfrac{1}{2}\mu^{-2}(-g)^{-1/2}\epsilon^{\nu\sigma\lambda\kappa}V_\sigma\Omega_{\lambda\kappa}, \end{aligned} \tag{2.28}$$

where $(-g)^{-1/2}\epsilon^{\nu\sigma\lambda\kappa}$ is the completely antisymmetric contravariant tensor. In the comoving frame, because $V_i=0$, v^ν has vanishing time component. In fact, we have

$$\mu v^i=\tfrac{1}{2}\epsilon^{ijk}\Omega_{jk}, \tag{2.29}$$

$$\mu\mathbf{v}=\nabla\alpha\times\nabla\beta+\nabla\theta\times\nabla S. \tag{2.30}$$

That is, if $\nabla S=0$, surfaces of constant α and β intersect along vortex lines, which are carried along with the fluid because $d\alpha/d\tau=d\beta/d\tau=0$.[15] If initially $\alpha=$const, $\beta=$const, but $S\neq$const, then surfaces of constant θ and S determine vortex lines whose orientation with respect to the fluid's particles changes in time because $d\theta/d\tau=T$.

Uniqueness of the velocity-potential representation. Formulation of Eqs. (2.22) in terms of initial values will give us more insight into the velocity-potential representation. The first-order nature of Eqs. (2.22) makes an initial-value approach especially simple for the restricted case of no self-gravitation, i.e., the case where the fluid does not disturb the background geometry of spacetime. The case with self-gravitation, although important, is more difficult and would not add substantially to our understanding of the potential representation itself, so we ignore it here.

The first question—which has nothing to do with self-gravitation—concerns the uniqueness of the representation. Given a physical situation, how much "gauge freedom" does one have to choose the initial values of the potentials without changing the physical situation they describe? If any such freedom exists, Eqs. (2.22) clearly imply that it lies only in the choice of initial values: The evolution of the potentials away from their initial values is fully determined by the physical situation ($\mathbf{U}$, μ, and T). Before we can deal

---

[12] A. H. Taub, Arch. Ratl. Mech. Anal. **3**, 312 (1959).

[13] By Ref. 7, $m_B\mathbf{V}$ is the four-momentum a baryon must have to be injected into the fluid. In the nonrelativistic limit ($\mu\rightarrow 1$), we have $\mathbf{V}\rightarrow\mathbf{U}$. Thus, both $\mathbf{U}$ and $\mathbf{V}$ are relativistic generalizations of the three-velocity $\mathbf{v}$. In circulation it is more useful to deal with $\mathbf{V}$ (see Ref. 12). For example, Bernoulli's equation for nonsteady irrotational isentropic flow generalizes using $\mathbf{V}$ because in that case it can be derived from a single potential ϕ. As another example, the tangential component of $\mathbf{V}$ is conserved across shock fronts. The simple form the circulation equations assume in terms of velocity potentials is another indication of the utility of $\mathbf{V}$.

[14] If θ and S are not both constant, neither can be a function only of the other, because $d\theta/d\tau=T$ while $dS/d\tau=0$. If either of them is constant, the second integral in Eq. (2.27) is zero. Only if $T=0$ can θ be constant.

[15] Circulation due to α and β will not change in a comoving frame. Such circulation may, however, emit gravitational radiation and be damped out as seen by a distant, noncomoving observer.

with this evolution we must resolve the question of gauge freedom among the potentials.

There is no gauge freedom in μ and S because changing them changes the physical situation. The question is whether there are two sets of potentials, $(\phi,\alpha,\beta,\theta)$ and $(\phi',\alpha',\beta',\theta')$, differing in initial values, which give the same $\mathbf{U}$ when substituted into Eq. (2.21) using the same μ and S. Two such sets are said to be equivalent. The equivalence transformations by which one set is obtained from another are discussed in detail in Appendix C. These transformations are essentially contact transformations. The result of interest here is: *The initial value of any one potential may be chosen arbitrarily; the remaining initial values are then constrained by the physical condition of the fluid (by* $\mathbf{U}$*,* μ*,* S*, and the equation of state).*[16]

Let us discuss the physical meaning of the equivalence transformations. Circulation in the fluid is an observable and hence must be preserved by the transformation. In the isentropic case ($S_{,\nu}=0$), circulation proceeds around intersections of surfaces of constant α and β. The effect of the equivalence transformation Eqs. (C21) is to preserve these intersections while changing α and β.

Intersecting surfaces of constant θ and S determine a kind of thermal circulation. Because physical conditions fix S, equivalence transformation on θ but not on α and β must leave $\nabla\theta$ unchanged except for parts parallel to ∇S. This is why requiring any equivalence transformation to leave α and β unchanged leads to the equation $\theta'=\theta+f(S)$.

A general equivalence transformation changes α and β as well as θ, but it keeps the sum $\nabla\alpha\times\nabla\beta+\nabla\theta\times\nabla S$ constant by transferring some circulation from one term to the other. The two types of circulation cannot therefore be separated from each other uniquely on any given spacelike hypersurface; they can be distinguished, however, by the way they change as the fluid moves off that hypersurface.

Restricted initial-value formulation. Suppose one chooses initial values of the velocity potentials on some initial hypersurface; what kind of initial-value information is necessary to determine a unique fluid motion in the background metric? Are the initial values of the six velocity potentials μ, S, ϕ, α, β, and θ sufficient; or are their derivatives off the hypersurface also necessary? Once the set of initial values is chosen, the equivalence transformation of Appendix C can lead to other sets that give the same fluid motion. Nevertheless, each set can be so chosen that it determines one and only one fluid motion. Appendix D presents two different initial-value schemes whereby the four-velocity and thermodynamic state of the fluid are determined throughout spacetime by the specification of certain data on an initial hypersurface. The first scheme shows that specifying values of all six potentials and the equation of state is sufficient. The second scheme shows that specifying the thermodynamic condition (μ and S) is not essential: The equation of state, the initial values of ϕ, α, β, and θ, and the derivatives of any two of those four potentials normal to the hypersurface will fully determine μ, S, and $\mathbf{U}$. Appendix D also leads to an obvious consistency condition on the initial values: *The initial values of* μ*,* S*,* ϕ*,* α*,* β*, and* θ *must be so chosen that the three-space velocity of the fluid parallel to the initial hypersurface nowhere exceeds the velocity of light.*

Once sufficient Cauchy data have been specified, the subsequent evolution of the velocity potentials is most easily discussed from a Lagrangian point of view. From Eqs. (2.22) one can see that the initial values of α, β, S, and baryon number N are carried along by the fluid: Each fluid element sees no change in these four functions. They are therefore "initial-value parameters." By contrast, the functions θ and ϕ are "dynamical variables": Their evolution is determined by the thermodynamic condition of the fluid. Changes in them cause the changes in the motion of the fluid seen in a comoving (Lagrangian) frame. They are dynamical in the sense that the complete history of a fluid element can be given by a plot of θ against ϕ, along which the given values of S, α, β, and N are constant. That there are only two dynamical variables in this sense does not imply that there are only two "degrees of freedom" in the fluid's motion. The question of degrees of freedom is taken up at the end of Appendix C.

III. EULERIAN VARIATIONAL PRINCIPLE

In 1954, Taub[17] gave a variational principle whose Euler-Lagrange equations were the general-relativistic field equations plus the equations of motion for a perfect fluid in what we have called the standard version. An essential feature of any such variational principle is that the world lines of the fluid's particles be among the quantities varied. Consistent with the Lagrangian interpretation of the standard version that we discussed in Sec. II A, in Taub's principle one varies the world lines in a Lagrangian manner: One attaches a label to every particle and directly changes the particle's path by changing the position of its label in spacetime.

The variational principle given in this section uses the velocity-potential version of hydrodynamics and hence is Eulerian. The independent coordinates with respect to which the Lagrangian density is varied are the velocity potentials themselves. Varying the potentials varies the four-velocity and thence implicitly the world lines.

[16] The remaining initial values are constrained but not fully determined by the physics. See Ref. 35. Moveover, the arbitrary choice of the initial value of one potential *may* lead to divergences in others. These divergences will not affect any observables like $\mathbf{U}$ or the circulation. For example, if the term $\alpha\beta_{,\nu}$ is nonzero in one representation, choosing $\alpha'=0$ will not generally eliminate this term; it will only force β' to diverge in order to keep $\alpha'\beta'_{,\nu}$ nonzero and finite.

[17] A. H. Taub, Phys. Rev. 94, 1468 (1954).

The action principle. In step-by-step form,

(1) Select an equation of state for the one-component perfect fluid. Express it in the form

$$p=p(\mu,S). \qquad (3.1)$$

Then Eq. (2.5) follows from basic thermodynamics:

$$dp=\rho_0 d\mu-\rho_0 T dS. \qquad (3.2)$$

(2) Define the four-velocity vector field in terms of six scalar velocity-potential fields:

$$U_\nu=\mu^{-1}(\phi_{,\nu}+\alpha\beta_{,\nu}+\theta S_{,\nu}). \qquad (3.3)$$

Normalization of **U** implies

$$\mu^2=-g^{\mu\nu}(\phi_{,\mu}+\alpha\beta_{,\mu}+\theta S_{,\mu})(\phi_{,\nu}+\alpha\beta_{,\nu}+\theta S_{,\nu}), \qquad (3.4)$$

which defines the functional dependence of μ upon the independent variables of our variational principle, ϕ, α, β, θ, S, and $g^{\mu\nu}$.

(3) Define the action I as

$$I=\int(R+16\pi p)(-g)^{1/2}d^4x, \qquad (3.5)$$

where R is the scalar curvature, taken as a function of $g^{\mu\nu}$ and its derivatives, and where p is the ordinary pressure, which depends on all the independent variables through Eqs. (3.1), (3.2), and (3.4).

(4) Extremize I to obtain the following Euler-Lagrange equations[18]:

$$\delta g^{\mu\nu}:\quad G_{\mu\nu}-8\pi[(\rho+p)U_\mu U_\nu+pg_{\mu\nu}]=0, \qquad (3.6a)$$

$$\delta\phi:\quad (\rho_0 U^\nu)_{;\nu}=0, \qquad (3.6b)$$

$$\delta\theta:\quad U^\nu S_{,\nu}=0, \qquad (3.6c)$$

$$\delta S:\quad U^\nu\theta_{,\nu}=T, \qquad (3.6d)$$

$$\delta\alpha:\quad U^\nu\beta_{,\nu}=0, \qquad (3.6e)$$

$$\delta\beta:\quad U^\nu\alpha_{,\nu}=0. \qquad (3.6f)$$

Equations (3.3), (3.4), (3.6c), and (3.6e) imply

$$U^\nu\phi_{,\nu}=-\mu. \qquad (3.6g)$$

We have thus reproduced Eqs. (2.22) of the velocity-potential representation. This establishes the validity of the variational principle.

Comparison with other action principles. Our variational principle is equivalent to Taub's 1954 principle.[17] To prove this we use a procedure taken from Seliger and Whitham.[3] Taub extremizes the action

$$I_T=\int[R-16\pi(\rho-\rho_0 TS+\lambda g_{\mu\nu}U^\mu U^\nu)]\times(-g)^{1/2}d^4x, \qquad (3.7)$$

where λ is a Lagrange multiplier that ensures normalization of **U**. Since we impose that normalization explicitly in our principle, we can drop the λ term and work with

$$I_T'=\int[R-16\pi(\rho-\rho_0 TS)](-g)^{1/2}d^4x. \qquad (3.8)$$

Taub imposes two explicit constraints upon variations of I_T'. The first is conservation of baryons, and the second is that there exist a field θ such that $U^\nu\theta_{,\nu}=T$ (Taub uses α rather than θ). The second is not a physical constraint, of course, since θ exists for all **U** and T. Nevertheless, it is a mathematical constraint. We can eliminate both constraints by using Lagrange multipliers:

$$I_T''=\int\{R-16\pi[\rho-\rho_0 TS-\phi(U^\nu\rho_0)_{;\nu}-\theta(\rho_0 U^\nu S)_{;\nu}]\}\times(-g)^{1/2}d^4x. \qquad (3.9)$$

Variations of ϕ and θ give the equations of conservation of baryons and entropy. Variation of S gives $T=U^\nu\theta_{,\nu}$. Variation with respect to ρ_0 gives [noting that $(\partial\rho/\partial\rho_0)_S=\mu$]

$$U^\nu\phi_{,\nu}=-\mu. \qquad (3.10)$$

To complete the identification of Taub's principle with ours, we add to the Lagrangian density the divergence

$$Y^\nu_{,\nu}=16\pi[(-g)^{1/2}(U^\nu\rho_0\phi+U^\nu\rho_0\theta S)]_{,\nu}. \qquad (3.11)$$

We obtain

$$I_T'''=\int[R-16\pi(\rho-\rho_0 TS+\rho_0 U^\nu\phi_{,\nu}+\rho_0 SU^\nu\theta_{,\nu})]\times(-g)^{1/2}d^4x, \qquad (3.12)$$

which reduces to

$$I_T'''=\int(R+16\pi p)(-g)^{1/2}d^4x. \qquad (3.13)$$

The modified version of Taub's principle is thus equivalent to ours, except for the technical point that Taub's variations are Lagrangian and do not use velocity potentials, while ours are Eulerian and rely on the velocity potentials.[19]

More recently, Taub published a variational principle expressed in comoving coordinates, in which the action

---

[18] See, e.g., L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley, Reading, Mass., 1962), Sec. 93.

[19] These calculations give the potentials θ and ϕ richer meaning; one might ask if α and β have similar meanings. They do, in a formal way (see Seliger and Whitham, Ref. 3): One can make the transition from Taub's variables to the Eulerian variables complete by requiring "conservation of Lagrangian coordinates" (i.e., once a fluid element is labeled with a comoving coordinate, that coordinate never changes). Let β be such a coordinate and α be its Lagrange multiplier. Then one adds the term $\rho_0\alpha U^\nu\beta_{,\nu}$ into Eq. (3.9); variation of α and β then gives the appropriate equations, without changing either Eq. (3.10) or Eq. (3.13). This device, due orginally to Lin (Ref. 3), is somewhat mysterious, especially since only one Lagrangian coordinate is required, and not all three.

is the same as the present one.[20,21] In fact, by specializing the calculations of this section to a comoving coordinate system, one can show that variation of $g^{\mu\nu}$ is equivalent to Taub's first variation of the action. The price paid for working in a comoving system is that the potential representation is rendered useless while the equations for conservation of rest mass and for normalization of the four-velocity have to be assumed *ad hoc*, because the "comoving" constraint is nonholonomic in the variables ϕ, α, β, θ, and S.

Bardeen[22] has recently obtained an Eulerian action principle for axially symmetric, differentially rotating configurations; we will show that one version of his action principle is equivalent to ours, specialized to such configurations. Bardeen extremizes the action

$$I_2=2\pi\iint[-R/16\pi-T^0{}_0-\Phi\rho_0U^0-\Lambda(\rho+p)U_\varphi U^0]\times(-g)^{1/2}dx^2dx^3, \quad (3.14)$$

where x^2 and x^3 are any two coordinates such that $\partial/\partial x^2$ and $\partial/\partial x^3$ are both orthogonal to the Killing vectors $\partial/\partial t$ and $\partial/\partial\varphi$. The independent variables are the nonzero components of $g_{\alpha\beta}$ and four "internal" variables governing changes in the fluid and its motions: ρ_0, U^φ/U^0, ξ, and η. The variables $\xi(x^2,x^3)$ and $\eta(x^2,x^3)$ are Lagrangian coordinates giving the position of a fluid element in the x^2-x^3 subspace, and are actually redundant: Only one of them is needed to extract the full physical content of the principle.[23] Consequently there are really only three internal variables. The only constraint on the variations is that **U** be normalized. The two Lagrange multipliers $\Phi(\xi,\eta)$ and $\Lambda(\xi,\eta)$ ensure that the baryon number and angular momentum, respectively, of a fluid element be unaffected by variations of $g_{\alpha\beta}$. When the actual values of Φ and Λ are put in ($\Phi=\mu/U^0$, $\Lambda=U^\varphi/U^0$), I_2 reduces to

$$I_2'=-\frac{1}{8}\iint(R+16\pi p)(-g)^{1/2}dx^2dx^3. \quad (3.15)$$

This is the same action as in our principle. Moreover, our principle also has three internal variables: The five variables ϕ, α, β, θ, and S are reduced to three by the relations $U_2=U_3=0$. These three may differ from Bardeen's three, but their Euler-Lagrange equations will be equivalent to his because they are a complete set of variables: a one-component fluid constrained to mover in only the φ direction has three degrees of freedom—two thermodynamic and one kinetic. Since the only constraint on our variational principle is also the normalization of **U**, the two principles are equivalent.

IV. CONCLUDING REMARKS

The work reported in this paper was originally undertaken in the hope of finding stability criteria for self-gravitating masses of fluid. Although that goal is still far off, the existence of an Eulerian variational principle may be a beginning.

What is needed, I believe, is a Hamiltonian principle in a minimum number of variables. The present action principle seems to have "too many" free variables: Witness the existence of equivalence transformations among ϕ, α, β, and θ; witness also the fact that variations of the Lagrangian violate the conservation of ρ_0. Perhaps the methods of Arnowitt, Deser, and Misner[24] or of Dirac[25] can be applied to isolate the "true variables" of the principle. Then one might be able to obtain a self-adjoint variational principle that could lead to stability criteria.

It may also be possible to extend this work to viscous fluids and charged fluids. The key step would be the extension of Theorem 1 of Appendix B to the appropriate case.

Note added in proof. An equivalent set of velocity potentials and a similar variational principle have been obtained independently by Schmid from a very different approach.[26] His potentials nicely illustrate a symmetry of the velocity-potential formulation. He defines ϕ differently: $d\phi/d\tau=-\mu+TS$. Then all the results of this paper carry through if one replaces $\theta S_{,\nu}$ by $-S\theta_{,\nu}$.

ACKNOWLEDGMENTS

I am deeply indebted to Dr. E. D. Fackerell for introducing me to the paper of Seliger and Whitham, upon which so much of this paper is based, and for many helpful discussions. I would also like to thank Professor Whitham himself for discussing his paper with me, Professor A. H. Taub for his correspondence and reprints, and Professor Jürgen Ehlers for many helpful suggestions. Finally I wish to thank Professor K. S. Thorne for his numerous suggestions and criticisms during the writing of this paper that have made it clearer than it would otherwise have been.

[20] A. H. Taub, in *Fluides et Champ Gravitationnel en Relativité Générale* (Centre National de la Recherche Scientifique, Paris, 1969), pp. 57–72.

[21] A. H. Taub, Commun. Math. Phys. **15**, 235 (1969).

[22] James M. Bardeen, Astrophys. J. **162**, 71 (1970).

[23] Variations of ξ and η give the ξ and η components of the (vector) equation of hydrostatic equilibrium. Since the Jacobian $\partial(\xi,\eta)/\partial(x^2,x^3)$ is assumed well behaved, hydrostatic equilibrium in ξ-η space implies equilibrium in x^2-x^3 space. However, since ξ and η are arbitrary functions of x^2 and x^3, the Euler-Lagrange equation for either ξ *or* η is sufficient to guarantee hydrostatic equilibrium everywhere in x^2-x^3 space. This ξ-η redundancy seems closely related to the problem mentioned at the end of Ref. 19, namely, that requiring conservation of only *one* Lagrangian coordinate is sufficient to complete the transformation from Taub's first principle to ours.

[24] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation*, edited by L. Witten (Wiley, New York, 1962), Chap. 7. See also the references cited therein.

[25] P. A. M. Dirac, Proc. Roy. Soc. (London) **A246**, 326 (1958).

[26] L. A. Schmid, in *A Critical Review of Thermodynamics*, edited by E. B. Stuart, B. Gal-Or, and A. J. Brainard (Mono, Baltimore, 1970); also L. A. Schmid, in Proceedings of the International Conference on Thermodynamics, 1970 [Pure Appl. Chem. (to be published)].

APPENDIX A: PFAFF'S THEOREM

We have occasion to use Pfaff's theorem several times in this paper; we state it here without proof. An application of the theorem familiar to physicists concerns criteria for the integrability of a so-called "Pfaffian form,"

$$\sum_{i=1}^{N} f_i(x^k)dx^i.$$

These criteria are closely related to the second law of thermodynamics and lead to definitions of entropy and temperature for many-component systems.[6] Pfaff's theorem is much more general than the second law, however. It says[27] that if $f_i(x^k)$ are N functions of N independent variables x^k, then there exist functions $A_\alpha(x^k)$, $B_\alpha(x^k)$, and $C(x^k)$ such that

$$\begin{aligned}\sum_{i=1}^{N} f_i dx^i &= \sum_{\alpha=1}^{N/2} A_\alpha dB_\alpha \quad \text{if } N \text{ is even}\\ &= dC + \sum_{\alpha=1}^{(N-1)/2} A_\alpha dB_\alpha \quad \text{if } N \text{ is odd.}\end{aligned} \tag{A1}$$

Consequently we have

$$f_i = \sum_{\alpha=1}^{N/2} A_\alpha \frac{\partial B_\alpha}{\partial x^i}$$

or (A2)

$$f_i = \frac{\partial C}{\partial x_i} + \sum_{\alpha=1}^{(N-1)/2} A_\alpha \frac{\partial B_\alpha}{\partial x^i},$$

respectively. The number of functions remains the same, but the number of differentials is cut essentially in half. Pfaff's theorem sets a least upper bound on the number of differentials required: One might need fewer but one never needs more. This least upper bound depends only on the number of independent variables. For example, if $\alpha_i(x^k)$ and $\beta_i(x^k)$ are $2N$ functions $(i=1, \ldots, N)$ of $n<N$ independent variables, then

$$\sum_{i=1}^{N} \alpha_i d\beta_i = \sum_{i=1}^{N} \sum_{k=1}^{n} \alpha_i \frac{\partial \beta_i}{\partial x^k} dx^k.$$

The expressions

$$\sum_{i=1}^{N} \alpha_i \frac{\partial \beta_i}{\partial x^k}$$

are n functions of n variables; from Pfaff's theorem we therefore obtain (if, for example, n is even)

$$\sum_{i=1}^{N} \alpha_i d\beta_i = \sum_{\alpha=1}^{n/2} A_\alpha dB_\alpha. \tag{A3}$$

For $N=2$, Eq. (A1) becomes the familiar statement that every differential form in two variables has an integrating factor.

APPENDIX B: EQUIVALENCE OF STANDARD VERSION AND VELOCITY-POTENTIAL VERSION

The proof of equivalence between the equations of the standard version and those of the velocity-potential version rests upon Theorem 1 below. Once the theorem is established it will allow us to show that the equations of each version imply those of the others. Theorem 1 should be regarded as an algebraic identity: No equations are assumed other than those explicitly stated in the theorem.

Theorem 1. Let $\mathbf{U}$ be the four-velocity of a one-component fluid. Define a tensor $T_\nu{}^\sigma$ with components

$$T_\nu{}^\sigma \equiv \rho_0 \mu U_\nu U^\sigma + p\delta_\nu{}^\sigma. \tag{B1}$$

Define the scalar functions ϕ and θ by the differential equations

$$d\phi/d\tau = -\mu, \tag{B2}$$

$$d\theta/d\tau = T. \tag{B3}$$

Define the entropy by the equation

$$TdS = d\mu - \rho_0^{-1}dp. \tag{B4}$$

Require conservation of entropy[28] and baryons during motions of the fluid:

$$dS/d\tau = 0, \tag{B5}$$

$$(\rho_0 U^\nu)_{;\nu} = 0. \tag{B6}$$

Do not impose any other equations of motion. Then the following is an identity:

$$\mathcal{L}_U(\mu U_\nu - \phi_{,\nu} - \theta S_{,\nu}) = \rho_0^{-1} T_\nu{}^\sigma{}_{;\sigma}, \tag{B7}$$

where $\mathcal{L}_U$ denotes the Lie derivative[29] with respect to $\mathbf{U}$.

We note that Theorem 1 is true even if $T_\nu{}^\sigma{}_{;\sigma} \neq 0$, i.e., when $T_\nu{}^\sigma$ as defined by Eq. (B1) is not the complete stress-energy tensor of the fluid. For example, in magnetohydrodynamics Eqs. (B5) and (B6) still hold, so Theorem 1 is still valid.

Proof. The proof of Theorem 1 is an elementary exercise in Lie derivations, whose properties can be found in many references.[30] We simply note that the definitions of θ and ϕ and Eqs. (B4) and (B5) yield

$$\mathcal{L}_U(\mu U_\nu - \phi_{,\nu} - \theta S_{,\nu}) = U^\sigma(\mu U_\nu)_{;\sigma} + \rho_0^{-1} p_{,\nu}. \tag{B8}$$

Similarly, application of Eq. (B6) to the divergence of

[27] See Seliger and Whitham (Ref. 3) or A. R. Forsythe, *Theory of Differential Equations* (Cambridge U. P., London, 1900), Vol. I.

[28] According to Ref. 8, *perfect* fluids must have $\delta q = TdS = 0$ during their motions.

[29] I am indebted to Professor K. S. Thorne for suggesting the use of Lie derivatives in proving equivalence between the two versions.

[30] See, e.g., K. Yano, *The Theory of Lie Derivatives and its Applications* (North-Holland, Amsterdam, 1955), Chap. 1.

Eq. (B1) gives

$$\rho_0^{-1}T_\nu{}^\sigma{}_{;\sigma}=U^\sigma(\mu U_\nu)_{;\sigma}+\rho_0^{-1}p_{,\nu}. \qquad \text{(B9)}$$

Q.E.D.

Let us now turn to the first half of the proof of equivalence: the proof that the equations of the velocity-potential version imply those of the standard version. The velocity-potential representation of **U**, Eq. (2.21), gives

$$\mu U_\nu-\phi_{,\nu}-\theta S_{,\nu}=\alpha\beta_{,\nu}.$$

Therefore, we have

$$\pounds_U(\mu U_\nu-\phi_{,\nu}-\theta S_{,\nu})=\pounds_U(\alpha\beta_{,\nu})=0, \qquad \text{(B10)}$$

where the last equality follows from $d\alpha/d\tau=d\beta/d\tau=0$. Then Theorem 1 gives

$$T_\nu{}^\sigma{}_{;\sigma}=0, \qquad \text{(B11)}$$

which is the standard version of the equation of motion.

The second half of the equivalence proof is the proof that the equations of the standard version imply those of the velocity-potential version. We already have the three equations $d\phi/d\tau=-\mu$, $d\theta/d\tau=T$, and $dS/d\tau=0$ from the requirements of Theorem 1. We need only show that the velocity-potential representation of **U**,

$$U_\nu=\mu^{-1}(\phi_{,\nu}+\alpha\beta_{,\nu}+\theta S_{,\nu}), \qquad \text{(B12a)}$$

and the two remaining equations of evolution,

$$d\alpha/d\tau=0, \qquad \text{(B12b)}$$

$$d\beta/d\tau=0, \qquad \text{(B12c)}$$

follow from Theorem 1 and the standard version's equations of motion,

$$T_\nu{}^\sigma{}_{;\sigma}=0. \qquad \text{(B13)}$$

Equation (B13) and Theorem 1 imply

$$\pounds_U(\mu U_\nu-\phi_{,\nu}-\theta S_{,\nu})=0. \qquad \text{(B14)}$$

This leads to the following theorem.

Theorem 2. There exist functions α, β, and γ such that

$$\mu U_\nu-\phi_{,\nu}-\theta S_{,\nu}=\alpha\beta_{,\nu}+\gamma_{,\nu} \qquad \text{(B15)}$$

and

$$d\alpha/d\tau=d\beta/d\tau=d\gamma/d\tau=0. \qquad \text{(B16)}$$

Proof.[31] Define

$$W_\nu=\mu U_\nu-\phi_{,\nu}-\theta S_{,\nu}. \qquad \text{(B17)}$$

Then W_ν is orthogonal to and Lie-dragged by U^ν (i.e., its Lie derivative along U^ν is zero); expressed in comoving coordinates (τ, y^i such that $U^\nu=\delta_0{}^\nu$) this means $W_0=0$, $W_{i,0}=0$. Then Pfaff's theorem (Appendix A) for $N=3$ implies

$$W_i dy^i=\alpha d\beta+d\gamma, \qquad \text{(B18)}$$

with α, β, and γ functions only of y^i. Consequently, Eqs. (B15) and (B16) are valid in any coordinate system. Q.E.D.

We now note that ϕ was defined only by the differential equation $d\phi/d\tau=-\mu$, so that any function independent of τ can be added to ϕ without changing any of the previous results. Such a function is γ. Consequently we can "absorb" γ into ϕ and obtain from Theorem 2 the velocity-potential representation

$$U_\nu=\mu^{-1}(\phi_{,\nu}+\alpha\beta_{,\nu}+\theta S_{,\nu}). \qquad \text{(B19)}$$

This completes the proof that the equations of the standard version imply the equations of the velocity-potential version. The two versions are equivalent.

By way of relating Theorem 1 to results more familiar in Newtonian hydrodynamics, we establish a corollary that is a generalization of Weber's transformation.[32] Define the spacelike vector separating two neighboring particles in the fluid, δx^ν, in the following manner. Let $(\delta x^\nu)_0$ be their separation on some arbitrary initial spacelike hypersurface. Then let δx^ν be the vector that results when $(\delta x^\nu)_0$ is Lie-dragged off the initial hypersurface by the fluid's four-velocity; i.e., let δx^ν be the separation between the particles after they have advanced equal proper times off the initial hypersurface. Then by construction we have

$$\pounds_U(\delta x^\nu)=0. \qquad \text{(B20)}$$

Consequently, Theorem 1 implies (with $T_\nu{}^\sigma{}_{;\sigma}=0$)

$$\pounds_U[(\mu U_\nu-\phi_{,\nu}-\theta S_{,\nu})\delta x^\nu]=0. \qquad \text{(B21)}$$

But the quantity inside the square brackets in Eq. (B21) is a scalar, and Lie differentiation of a scalar is simply differentiation in proper time:

$$\frac{d}{d\tau}[(\mu U_\nu-\phi_{,\nu}-\theta S_{,\nu})\delta x^\nu]=0. \qquad \text{(B22)}$$

Define $\delta\chi$, the change in any scalar field χ along the vector δx^ν, by

$$\delta\chi=\chi_{,\nu}\delta x^\nu.$$

Then Eq. (B22) implies the following corollary of Theorem 1.

Corollary *(generalized Weber's transformation)*. Let the subscript 0 denote the value of a quantity on some initial spacelike hypersurface, and let the subscript τ denote its value on some hypersurface advanced a proper time τ from the initial hypersurface. Then the equations of hydrodynamics are equivalent to

$$(\mu U_\nu\delta x^\nu)_\tau-(\mu U_\nu\delta x^\nu)_0=(\delta\phi)_\tau-(\delta\phi)_0+(\theta\delta S)_\tau-(\theta\delta S)_0. \qquad \text{(B23)}$$

[31] This proof was kindly suggested by J. Ehlers (private communication).

[32] See H. Lamb, *Hydrodynamics* (Cambridge U. P., London, 1932), Sec. 15, for the Newtonian version of Weber's transformation in the restricted case $p=p(\rho)$. For the general $p=p(\rho,S)$, see J. Serrin, in *Handbuch der Physik* (Springer-Verlag, Berlin, 1959), Vol. 8, Sec. 29A.

APPENDIX C: PHYSICALLY EQUIVALENT REPRESENTATIONS

Two sets of velocity potentials are said to be equivalent if they give the same four-velocity for the same thermodynamic state of the fluid. The purpose of this appendix is to derive the equations of transformation whereby one set of velocity potentials may be obtained from an equivalent one and thereby to determine how much "gauge freedom" one has to choose the potentials arbitrarily.[33]

Equivalent sets by definition have the same μ and S. We therefore seek transformations between two sets of potentials $(\phi,\alpha,\beta,\theta)$ and $(\phi',\alpha',\beta',\theta')$ such that [from Eq. (2.21)]

$$\phi_{,\nu}+\alpha\beta_{,\nu}+\theta S_{,\nu}=\phi'_{,\nu}+\alpha'\beta'_{,\nu}+\theta' S_{,\nu}. \qquad \text{(C1)}$$

The potentials must individually satisfy these equations:

$$d\phi/d\tau=d\phi'/d\tau=-\mu, \qquad \text{(C2a)}$$

$$d\theta/d\tau=d\theta'/d\tau=T, \qquad \text{(C2b)}$$

$$dS/d\tau=d\alpha/d\tau=d\alpha'/d\tau=d\beta/d\tau=d\beta'/d\tau=0. \qquad \text{(C2c)}$$

We write Eq. (C1) in a more useful form:

$$\phi_{,\nu}-\phi'_{,\nu}=\alpha'\beta'_{,\nu}-\alpha\beta_{,\nu}+(\theta'-\theta)S_{,\nu}. \qquad \text{(C3)}$$

In general, ϕ and ϕ' will differ by some scalar field F:

$$\phi-\phi'=F. \qquad \text{(C4)}$$

By Eq. (C2a) we have

$$dF/d\tau=F_{,\nu}U^{\nu}=0. \qquad \text{(C5)}$$

Equation (C3) becomes

$$F_{,\nu}=\alpha'\beta'_{,\nu}-\alpha\beta_{,\nu}+(\theta'-\theta)S_{,\nu}. \qquad \text{(C6)}$$

As we shall see, each different choice of F generates a different equivalence transformation. The only restriction on the choice of F is Eq. (C5). Accordingly, we can take F to be some arbitrary function of any three functions that are independent of τ. Equation (C6) suggests the choice

$$F=F(\beta,\beta',S). \qquad \text{(C7)}$$

Differentiation of F gives

$$F_{,\nu}=\frac{\partial F}{\partial\beta}\beta_{,\nu}+\frac{\partial F}{\partial\beta'}\beta'_{,\nu}+\frac{\partial F}{\partial S}S_{,\nu}. \qquad \text{(C8)}$$

Treating α and α' as independent variables for the moment, we have

$$\partial F/\partial\alpha=0, \qquad \text{(C9a)}$$

$$\partial F/\partial\alpha'=0. \qquad \text{(C9b)}$$

Having chosen some F and found its derivatives, we see that F will generate an equivalence transformation if and only if it satisfies Eq. (C6). Comparison with Eq. (C8) reveals the equations[34]

$$\alpha'=\partial F/\partial\beta', \qquad \text{(C9c)}$$

$$\alpha=-\partial F/\partial\beta, \qquad \text{(C9d)}$$

$$\theta'-\theta=\partial F/\partial S. \qquad \text{(C9e)}$$

Thus the function F generates a transformation from $(\phi,\alpha,\beta,\theta)$ to $(\phi',\alpha',\beta',\theta')$. We include Eqs. (C9a) and (C9b) as a formal device that will enable us to obtain other equivalence transformations in the following paragraphs. Equations (C2b) and (C2c) are clearly fulfilled.

The restriction of F to functions of β, β', and S can be relaxed by a device called the Legendre transformation. For example, define

$$F=F_2(\alpha,\beta',S)-\alpha\beta. \qquad \text{(C10)}$$

The subscript 2 distinguishes this form of F from Eq. (C7). Then Eqs. (C9) become, in terms of F_2,

$$\alpha'=\partial F_2/\partial\beta', \qquad \text{(C11a)}$$

$$0=\partial F_2/\partial\beta, \qquad \text{(C11b)}$$

$$\theta'-\theta=\partial F_2/\partial S, \qquad \text{(C11c)}$$

$$\beta=\partial F_2/\partial\alpha, \qquad \text{(C11d)}$$

$$0=\partial F_2/\partial\alpha'. \qquad \text{(C11e)}$$

From Eq. (C4) we find

$$\phi'-\phi=-F_2+\alpha\partial F_2/\partial\alpha. \qquad \text{(C11f)}$$

Notice that these equations would also follow directly from Eqs. (C6) and (C10). One special case of this type is the identity transformation, generated by

$$F_2=\alpha\beta'. \qquad \text{(C12)}$$

Then Eqs. (C11) give

$$\alpha'=\alpha, \qquad \text{(C13a)}$$

$$\beta'=\beta, \qquad \text{(C13b)}$$

$$\theta'=\theta, \qquad \text{(C13c)}$$

$$\phi'=\phi. \qquad \text{(C13d)}$$

Infinitesimal transformations can be generated by a function G added to the identity generator:

$$F_2=\alpha\beta'+\epsilon G(\alpha,\beta',S), \qquad \text{(C14)}$$

where ϵ is the infinitesimal parameter. The resulting transformation is

$$\alpha'=\alpha+\epsilon\partial G/\partial\beta', \qquad \text{(C15a)}$$

[33] For a brief but similar analysis of the Clebsch representation, see C. Eckart, Phys. Fluids 3, 421 (1960), Appendix. For a review of contact transformations and their use in classical mechanics, see H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, Mass. 1950), Chap. 8.

[34] The physical interpretation of these and other equations of transformation is discussed more fully in Sec. II B.

$$\beta'=\beta-\epsilon\partial G/\partial\alpha, \tag{C15b}$$

$$\theta'=\theta+\epsilon\partial G/\partial S, \tag{C15c}$$

$$\phi'=\phi+\epsilon(\alpha\partial G/\partial\alpha-G). \tag{C15d}$$

By analogy with F_2 we can define two other types of generating functions:

$$F=F_3(\alpha,\alpha',S)-\alpha\beta+\alpha'\beta' \tag{C16}$$

and

$$F=F_4(\beta,\alpha',S)+\alpha'\beta'. \tag{C17}$$

The nontrivial equations of transformation generated by F_3 are

$$\beta'=-\partial F_3/\partial\alpha', \tag{C18a}$$

$$\beta=\partial F_3/\partial\alpha, \tag{C18b}$$

$$\theta'-\theta=\partial F_3/\partial S, \tag{C18c}$$

$$\phi'-\phi=-F_3+\alpha\partial F_3/\partial\alpha+\alpha'\partial F_3/\partial\alpha'. \tag{C18d}$$

The corresponding set for F_4 is

$$\beta'=-\partial F_4/\partial\alpha', \tag{C19a}$$

$$\alpha=-\partial F_4/\partial\beta, \tag{C19b}$$

$$\theta'-\theta=\partial F_4/\partial S, \tag{C19c}$$

$$\phi'-\phi=-F_4+\alpha'\partial F_4/\partial\alpha'. \tag{C19d}$$

The generator $F_4=-\alpha'\beta$ also generates the identity transformation and can serve as a starting point for infinitesimal transformations. A special case of F_4 is

$$F_4=-\alpha' g(\beta), \tag{C20}$$

which generates

$$\beta'=g(\beta), \tag{C21a}$$

$$\alpha'=\alpha(dg/d\beta)^{-1}, \tag{C21b}$$

$$\theta'=\theta, \tag{C21c}$$

$$\phi'=\phi. \tag{C21d}$$

This is the simplest equivalence transformation; it just reshuffles α and β without touching ϕ and θ.

Notice that if β' is not a monotonic function of β in Eqs. (C21), α' will be infinite wherever $d\beta'/d\beta=0$. This divergence is not of course physically observable. In fact, it ensures that the term $\alpha'\beta'_{,\nu}$ in the velocity-potential representation will equal $\alpha\beta_{,\nu}$. This example is an omen: Ill-chosen transformations will introduce divergences into some of the velocity potentials in order to keep the observables of the fluid's motion unchanged under the transformation.

Inconvenient as such divergences are, they do not fundamentally affect the gauge freedom in ϕ, α, β, and θ. Suppose one has a set of velocity potentials that determines the thermodynamic condition and motion of a fluid. An equivalent set can be obtained by choosing the value of any one potential arbitrarily at each point on the initial hypersurface. The equations of transformation then show how the initial values of the other three potentials must be changed in compensation. (Only initial values are affected because $dF/d\tau=0$.) It is not possible to choose a second potential arbitrarily at every point of the initial hypersurface without affecting the value of the first one. None of the transformations that leave one potential invariant have enough freedom to permit choice of a second one arbitrarily at every point. A simple example is Eq. (C21), which transforms α and β but leaves ϕ and θ alone. It permits only transformations that leave surfaces of constant β invariant: Choosing β at one point fixes its value on a whole two-dimensional subspace of the hypersurface. We therefore conclude that the initial value of one and only one potential is completely arbitrary. The remaining initial values are constrained (but not fully determined) by the physical condition of the fluid.[35]

I thank Professor Kip S. Thorne for pointing out that the arbitrariness of one potential is consistent with intuitive ideas of the number of degrees of freedom in a fluid. That is, it should be possible to describe a fluid completely with five functions at each point: two thermodynamic variables (μ and S) and three independent components of velocity. Because we use six potentials to describe the fluid, one and only one of them must be completely arbitrary.

APPENDIX D: RESTRICTED INITIAL-VALUE FORMULATION

Whereas in Appendix C we began with a physical situation and asked what sets of potentials could describe that situation equally well, in this appendix we begin with the potentials and ask what physical situation they determine. Accordingly we present here two different prescriptions for constructing fluid motions from knowledge of the potentials on some initial hypersurface, under the restriction that the background metric remain unchanged by the fluid's motions.

The first prescription requires knowledge only of the potentials on an initial hypersurface, and not of their derivatives off that hypersurface:

(1) Choose an initial spacelike hypersurface Σ with future timelike normal $\mathbf{N}$. On Σ specify the thermodynamic state of the fluid by giving μ and S. Also specify the initial values of ϕ, α, β, and θ on Σ. Say nothing about their derivatives normal to Σ.

(2) From these initial values, find the three components of $\mathbf{U}$ parallel to Σ from Eq. (2.21). Then the equation $\mathbf{U}\cdot\mathbf{U}=-1$ yields a quadratic equation for

[35] The same situation exists in electromagnetism. Choice of the Lorentz gauge (which corresponds to our choosing one potential arbitrarily) does not completely fix the gauge. Other Lorentz gauges may be generated by any function Λ that satisfies the homogeneous wave equation, $\Box\Lambda=0$. Such transformations do not establish an arbitrary gauge at every point because of the restriction on Λ, but they do modify the gauge without changing the physics.

$\mathbf{U}\cdot\mathbf{N}$. If this equation has imaginary solutions anywhere on Σ, then α, β, ϕ, and θ have been chosen wrong: They have yielded a three-space velocity parallel to Σ greater than the speed of light. This is the only consistency requirement on ϕ, α, β, and θ. If the quadratic equation has real solutions for $\mathbf{U}\cdot\mathbf{N}$ everywhere, choose the sign of $\mathbf{U}\cdot\mathbf{N}$ negative. One now has determined $\mathbf{U}$ on Σ.

(3) Using this value for $\mathbf{U}$, proceed to calculate the condition of the fluid on a hypersurface Σ' slightly advanced in time from Σ. Construct this second hypersurface by advancing off the first a proper time $d\tau$ in the direction of $\mathbf{U}$. Points of Σ and Σ' joined by $\mathbf{U}$ we shall call "corresponding points." The values of S, α, and β at corresponding points are equal. The value of θ has increased from any point in Σ to the corresponding point of Σ' by the amount $Td\tau$, while that of ϕ has decreased by $\mu d\tau$.

(4) Finally, use the equation $(\rho_0 U^\nu)_{;\nu}=0$ to relate the (as yet unknown) values of ρ_0 and $\mathbf{U}\cdot\mathbf{N}'$ on Σ' (where $\mathbf{N}'$ is the future timelike normal to Σ'). Use the equation of state to express ρ_0 in terms of S and μ; because S is known on Σ', one now has a relation between μ and $\mathbf{U}\cdot\mathbf{N}'$ there. Equation (2.21) yields a relation between μ and the *spatial part* of $\mathbf{U}$ on Σ', since only derivatives of ϕ, β, and S parallel to Σ' are known. Use the equation $\mathbf{U}\cdot\mathbf{U}=-1$ to get a third relation, this one among μ, the spatial part of $\mathbf{U}$, and $\mathbf{U}\cdot\mathbf{N}'$. Solve these relations simultaneously for μ and the four components of $\mathbf{U}$ on Σ'. One now has enough information to advance to a third hypersurface, and so on.

In step (2) we imposed the consistency requirement that the spatial velocity of the fluid on the initial hypersurface be less than that of light. Are we guaranteed that the solutions in step four for μ and $\mathbf{U}$ on the new hypersurface will satisfy this requirement: Will μ and all the components of $\mathbf{U}$ be real? It is not hard to show that if the initial conditions are so chosen that the spatial part of $\mathbf{U}$ is zero, and if there are no infinite gradients of p, then the relations of step (4) imply that, on the new hypersurface, $\mathbf{U}\cdot\mathbf{N}'=-1+O(d\tau^2)$, the spatial part of $\mathbf{U}$ is $O(d\tau)$, and μ has changed to order $d\tau$: i.e., that the new condition of the fluid is physically acceptable. Moreover, any physical situation that satisfies the consistency requirement of step (2) admits of a choice of initial spacelike hypersurface on which the spatial part of $\mathbf{U}$ is zero. Since the physics cannot be affected by such a choice, and since the equations of motion in the potential representation are not affected by such a choice, we conclude that if the potentials are constructed to be self-consistent on some initial hypersurface, then they will remain self-consistent throughout spacetime if infinite gradients of p do not develop.

The second prescription for constructing the fluid motions from the potentials is more complex. It does not require knowledge of the initial thermodynamic state of the fluid but does require knowledge of the derivatives of ϕ and θ off the initial hypersurface:

(1) On Σ specify $\alpha, \beta, \phi, \phi_{,\nu}, \theta, \theta_{,\nu}$ and the equation of state. Note that μ, S, and the derivatives of α, β, and S normal to Σ are unnecessary.

(2) From the known data, determine $\mathbf{U}$ and the thermodynamic state of the fluid on Σ in the following manner. The equation $\mathbf{U}\cdot\mathbf{U}=-1$ gives a relation between $\bar{\mathbf{U}}$ (the part of $\mathbf{U}$ parallel to Σ) and $\mathbf{U}\cdot\mathbf{N}$; let us write this as $A(\mathbf{U}\cdot\mathbf{N}, \bar{\mathbf{U}})=0$. The equation $U^\nu\theta_{,\nu}=T$ similarly gives a relation of the form $B(\mu,S,\mathbf{U}\cdot\mathbf{N},\bar{\mathbf{U}})=0$ after the equation of state has been used to express T in terms of μ and S. The equation $U^\nu\phi_{,\nu}=-\mu$ gives a third relation: $C(\mu,\mathbf{U}\cdot\mathbf{N},\bar{\mathbf{U}})=0$. We therefore have three relations in six unknowns. They can be solved[36] to express three of the unknowns in terms of the other three. Thus we can write $\mu=f(\bar{\mathbf{U}})$, $S=g(\bar{\mathbf{U}})$, and $\mathbf{U}\cdot\mathbf{N}=h(\bar{\mathbf{U}})$. Finally, we use the potential representation, Eq. (2.21), to determine $\mu\bar{\mathbf{U}}-\theta\nabla S$, a three-vector parallel to Σ. Because we know μ and S in terms of $\bar{\mathbf{U}}$, we can solve for the three components of $\bar{\mathbf{U}}$. From these we determine μ, S, and $\mathbf{U}\cdot\mathbf{N}$.

(3) We now have as much information as at the end of step (2) of the first prescription. To find the condition of the fluid on Σ', follow steps (3) and (4) of the first prescription.

The second prescription distinguishes between what we refer to in the text as initial-value parameters and dynamical variables. The initial data for the dynamical variables ϕ and θ were their values on Σ plus their derivatives off it. By contrast, only the initial values of α and β are required. This breakup of initial data is not unique, however. One could have specified the derivatives of, say, ϕ and α normal to Σ; the calculations would in fact have been easier.

[36] As in the first prescription, if these equations have complex solutions, the initial data have been chosen inconsistently.

CHAPTER 3

THE HAMILTONIAN THEORY OF A RELATIVISTIC PERFECT FLUID

Submitted to the Physical Review

ABSTRACT

The velocity-potential version of the hydrodynamics of a relativistic perfect fluid is put into Hamiltonian form by applying Dirac's method to the version's degenerate Lagrangian. There is only one independent momentum, and the Hamiltonian density is $-T_0{}^0(-g^{00})^{-1/2}$. The Einstein equations for a perfect fluid are then put into Hamiltonian form by analogue with Arnowitt, Deser, and Misner's vacuum Einstein equations. The Hamiltonian density splits into two pieces, which are the coordinate densities of energy and momentum of the fluid relative to an observer at rest on the hypersurface.

INTRODUCTION

The velocity-potential version of perfect-fluid hydrodynamics as formulated by Seliger and Whitham,[1] generalized to relativity by Schutz,[2] and independently discovered by Schmid,[3a,b] can be regarded as a nonlinear relativistic field theory for five coupled scalar fields, whose Lagrangian density is simply the pressure of the fluid. The theory is degenerate: not all the generalized momenta are independent, so they cannot be solved for the generalized velocities. In this paper we use Dirac's[4] algorithm for degenerate theories to cast the equations of perfect-fluid hydrodynamics into Hamiltonian form, whose Hamiltonian density is the energy density of the fluid. We then match the theory to the Arnowitt, Deser, and Misner[5,6] (hereafter referred to as ADaM) canonical theory for the vacuum gravitational field.

The independent variables of the theory are the velocity potentials: five scalar fields ϕ, α, β, θ, and S. Here S is the entropy per baryon, while the others have less obvious interpretations.[2,3a] The fluid's four-velocity is a combination of the potentials and their gradients[7]:

$$U_\nu = \mu^{-1}(\phi_{,\nu} + \alpha\beta_{,\nu} + \theta S_{,\nu}) \quad , \tag{1}$$

where μ is the specific enthalphy of the fluid,

$$\mu = (\rho + p)/\rho_0 \quad . \tag{2}$$

(Here ρ_0 is the rest-mass density, ρ is the density of total mass-energy, and p is the pressure.) Through the equation of state,

$$p = p(\mu, S) \quad , \tag{3}$$

all thermodynamic quantities are expressed in terms of S (one of the velocity potentials) and μ. In its turn, μ is a function of all the velocity potentials through the equation

$$\mu^2 = - g^{\sigma\nu}(\phi_{,\sigma} + \alpha\beta_{,\sigma} + \theta S_{,\sigma})(\phi_{,\nu} + \alpha\beta_{,\nu} + \theta S_{,\nu}) \quad , \tag{4}$$

which is just the normalization constraint on the four-velocity.

The dynamical field equations are five coupled nonlinear first-order equations:

$$U^\nu \phi_{,\nu} = - \mu \quad , \tag{5a}$$

$$U^\nu \alpha_{,\nu} = 0 \quad , \tag{5b}$$

$$U^\nu \beta_{,\nu} = 0 \quad , \tag{5c}$$

$$U^\nu \theta_{,\nu} = T \quad , \tag{5d}$$

$$U^\nu S_{,\nu} = 0 \quad , \tag{5e}$$

(where T is the temperature) plus one nonlinear second-order equation:

$$(\rho_0 U^\nu)_{;\nu} = 0 \quad . \tag{6}$$

There are really only two independent equations among the three Eqs. (5a,c,e) because of Eq. (4), so that there are five independent equations altogether.

These equations follow from extremizing the action

$$I = \int p \sqrt{-g}\, d^4x \quad . \tag{7}$$

First-order changes in p are computed from the equation

$$\delta p = \rho_0\, \delta\mu - \rho_0\, T\, \delta S \quad ,$$

which expresses the first law of thermodynamics. Equation (4) is used to obtain $\delta\mu$ in terms of the independent variations of ϕ, α, β, θ, and S.

When one formulates these equations in terms of a Hamiltonian, one singles out the time coordinate for special attention, thereby destroying the equations' four-dimensional symmetry. In what follows we will therefore use the ADM notation appropriate to such a 3 + 1 dimensional split of spacetime: the four-dimensional metric ${}^4g_{\alpha\beta}$ is replaced by the three-dimensional metric $g_{ij} = {}^4g_{ij}$ (whose inverse is $g^{ij} \neq {}^4g^{ij}$), by the lapse function $N = (-{}^4g^{00})^{-1/2}$, and by the shift functions $N_i = {}^4g_{oi}$. Derivatives covariant with respect to g_{ij} are denoted by ∇_i or by a subscripted slash (e.g. $h_{ij|k}$). Dots (e.g. $\dot{h}_{ij}$) denote partial derivatives in time.

The action (7) becomes

$$I = \int p\, N\, g^{\frac{1}{2}}\, d^3x\, dt \quad ,$$

so the Lagrangian density of the fluid is $L = p\, N\, g^{\frac{1}{2}}$. In all but the last section of this paper, we will treat the metric ${}^4g_{\alpha\beta}$ as a constant, not as part of the dynamics of the fluid. It will suffice until then to take as the fluid Lagrangian density

$$L = p\, N \quad , \tag{8}$$

so that the action can be written in the standard way

$$I = \int L\, d(\text{three-volume})\, dt \quad .$$

CONSTRAINTS ON THE MOMENTA

Let q_a stand for the five fields ϕ, α, β, θ, S. The momenta conjugate to q_a are

$$p^a \equiv \partial L/\partial \dot{q}_a = \partial pN/\partial \dot{q}_a \quad . \tag{9}$$

They are explicitly

$$
\begin{aligned}
p^{\phi} &= -\rho_0\, U^0\, N \quad , \\
p^{\alpha} &= p^{\theta} = 0 \quad , \\
p^{\beta} &= \alpha\, p^{\phi} \quad , \\
p^{s} &= \theta\, p^{\phi} \quad .
\end{aligned}
\tag{10}
$$

Since only one momentum is independent, there are four constraints on the momenta (the Dirac φ-equations):

$$
\begin{aligned}
\varphi_1 &= p^{\alpha} = 0 \quad , \\
\varphi_2 &= p^{\theta} = 0 \quad , \\
\varphi_3 &= p^{\beta} - \alpha\, p^{\phi} = 0 \quad , \\
\varphi_4 &= p^{s} - \theta\, p^{\phi} = 0 \quad .
\end{aligned}
\tag{11}
$$

There are no arbitrary functions of time in velocity-potential hydrodynamics: what gauge freedom exists lies only in the choice of initial values for the potentials. Consequently we do not expect any of these φ's to be first-class: none of them has vanishing Poisson bracket (see Eq. 16) with all the others.

That there is only one independent momentum is surprising. One might expect at least three (for the spatial components of velocity), if not more. The mathematical reason seems to be that, of all the field equations, only Eq. (6) is second order in time derivatives. Equation (6) is obtained by varying ϕ in the Lagrangian, and p^{ϕ} is the only independent momentum.

The physical reason (if one exists) that there is only one independent momentum is not clear. It would be a mistake to conclude that a perfect fluid has only one dynamical "degree of freedom": that such constraints as zero viscosity and conservation of entropy have wiped out the other degrees of freedom. The relationship between independent momenta and degrees of freedom is not well understood. In the velocity-potential representation one must specify six independent functions on an initial Cauchy hypersurface in order to determine the future evolution of the perfect fluid.[2] This indicates the existence of three dynamical degrees of freedom.

What seems to be the case here is that two of the three second-order dynamical equations (one for each component of velocity) have been replaced by four first order equations (the four independent equations among Eqs. [5]). Hidden among the four potentials α, β, θ, and S are two dynamical variables and their momenta. Since all four are treated as coordinates here, they appear to have no independent momenta among them.

There are some tantalizing suggestions that this may be just the hint of a deeper canonical relationship among the potentials. Seliger and Whitham[1] show that one can modify the formalism slightly and introduce a function $\mathcal{H}$ such that $d\alpha/d\tau = \partial\mathcal{H}/\partial\beta$ and $d\beta/d\tau = -\partial\mathcal{H}/\partial\alpha$. Moreover, Schmid[3a] points out that ϕ obeys the relativistic Hamilton-Jacobi equation

$$- g^{\alpha\beta}\, \phi_{,\alpha}\, \phi_{,\beta} + \epsilon^2 = \mu^2 \quad ,$$

where

$$\epsilon^2 = g^{\alpha\beta}(\alpha\beta_{,\alpha} + \theta S_{,\alpha})(\alpha\beta_{,\beta} + \theta S_{,\beta})$$

is positive-definite because the vector $\alpha\beta_{,\alpha} + \theta S_{,\alpha}$ is spacelike (it is orthogonal to U^{α}). We have nothing more to add to these considerations here, so we return to the Dirac method.

THE HAMILTONIAN AND THE EQUATIONS OF MOTION

The Hamiltonian density is defined in the conventional way:

$$H = \Sigma_a \, p^a \, \dot{q}_a - L \qquad (12a)$$

$$= p^\phi(\dot{\phi} + \alpha\dot{\beta} + \theta\dot{S}) - p\,N \qquad (12b)$$

$$= - T^0{}_0 \, N \quad . \qquad (12c)$$

Although $\dot{\phi}$, $\dot{\beta}$, and $\dot{S}$ appear explicitly in H, we still have $(\partial H/\partial\dot{q}_a)_{p,q} = 0$, so that we can differentiate H with respect to p^a and q_a while holding $\dot{q}_a$ constant.

Because of the φ-equations one cannot solve for all the $\dot{q}_a$'s in terms of p^a's. Instead one introduces additional variables λ_a (which Dirac[4] calls u_a) in place of the $\dot{q}_a$'s. If one varies Eq. (12a) with respect to q_a and p^a, the λ_a's serve as Lagrange multipliers that ensure that variations in the q_a's and p^a's maintain the φ-equations. Then one gets

$$\dot{q}_a = \partial H/\partial p^a + \lambda_m \, \partial\varphi_m/\partial p^a \quad , \qquad (13)$$

$$- \partial L/\partial q_a = \partial H/\partial q_a + \lambda_m \, \partial\varphi_m/\partial q_a \quad . \qquad (14)$$

(A sum on m from 1 to 4 is implied here and throughout.) For the perfect fluid, Eqs. (13) can be solved for the λ_m's to give

$$\lambda_1 = \dot{\alpha} \quad , \quad \lambda_2 = \dot{\theta} \quad , \quad \lambda_3 = \dot{\beta} \quad , \quad \lambda_4 = \dot{S} \quad . \qquad (15)$$

Thus in this case the λ's are self-consistent: Eqs. (14) imply nothing new. So the Hamiltonian variables now are p^ϕ, λ_1, λ_2, λ_3, λ_4, ϕ, α, β, θ, S.

The power of the Dirac approach is that the Poisson bracket version of

Hamilton's equations,

$$\dot{q} = [q, H]$$

$$\dot{p} = [p, H] ,$$

can easily be generalized to the degenerate case. Before applying this to fluids, however, we must define a Poisson bracket for fields in a curved three-dimensional space. The conventional definition from particle dynamics,

$$[A,B] = \sum_a \frac{\partial A}{\partial q_a} \frac{\partial B}{\partial p^a} - \frac{\partial A}{\partial p^a} \frac{\partial B}{\partial q_a} ,$$

is not sufficient when A and B are functions of the spatial derivatives of the fields q_a and p^a. In the appendix we generalize this definition to fields. For the perfect fluid (five scalar fields) the result is

$$\begin{aligned} [A,B] = \sum_{a=1}^{5} \Bigg\{ & \frac{\partial A}{\partial q_a} \frac{\delta B}{\delta p^a} - \frac{\partial A}{\partial p^a} \frac{\delta B}{\delta q_a} \\ & + \frac{\partial A}{\partial q_{a,i}} \frac{\partial}{\partial x^i} \frac{\delta B}{\delta p^a} - \frac{\partial A}{\partial p^a{}_{,i}} \frac{\partial}{\partial x^i} \frac{\delta B}{\delta q_a} \\ & + \frac{\partial A}{\partial q_{a,i|j}} \nabla_j \frac{\partial}{\partial x^i} \frac{\delta B}{\delta p^a} - \frac{\partial A}{\partial p^a{}_{,i|j}} \nabla_j \frac{\partial}{\partial x^i} \frac{\delta B}{\delta q_a} + \cdots \Bigg\} , \end{aligned} \tag{16}$$

where A and B are any functions of q_a, p^a, and their spatial derivatives (of any order), and where $\delta B/\delta q_a$ is the <u>spatial</u> variational derivative

$$\frac{\delta B}{\delta q_a} = \frac{\partial B}{\partial q_a} - \nabla_i \frac{\partial B}{\partial q_{a,i}} + \nabla_j \nabla_i \frac{\partial B}{\partial q_{a,i|j}} - + \cdots \quad . \tag{17}$$

In the Poisson bracket all q's and p's are treated as if they were independent: the φ-equations are used only <u>after</u> the Poisson bracket has been computed.

Dirac[8] shows that the time-derivative of any function f of the q's, p's, and their spatial derivatives (and possibly explicitly of time) can be expressed in the form

$$\dot{f} = [f,H] + [f,\lambda_m \varphi_m] + (\partial f/\partial t)_{p,q} \quad . \tag{18}$$

In the second term, one is to regard λ_m as independent of q_a and p^a but dependent upon position. For example, one contribution to that term will be from a term like

$$\frac{\delta \lambda_m \varphi_m}{\delta p^a} = \frac{\partial \lambda_m \varphi_m}{\partial p^a} - \nabla_i \frac{\partial \lambda_m \varphi_m}{\partial p^a{}_{|i}} + - \ldots$$

$$= \lambda_m \frac{\partial \varphi_m}{\partial p^a} - \nabla_i \lambda_m \frac{\partial \varphi_m}{\partial p^a{}_{|i}} + - \ldots \quad .$$

In Eq. (18) one must treat H as a function only of the <u>independent</u> momenta (cf. Eq. [12b]). Contributions to $\dot{f}$ from the other momenta come from the φ brackets. Equation (18) can be stated concisely as

$$\dot{f} = [f,H'] + \partial f/\partial t \tag{19}$$

by defining a generalized Hamiltonian

$$H' = H + \lambda_m \varphi_m \quad . \tag{20}$$

Because the φ's are all zero, H' is numerically equal to H.

The equations of motion are a special case of Eq. (19):

$$
\begin{aligned}
\dot{p}^{\phi} &= (\rho_0\, N\, U^i)_{|i} \quad , \\
\dot{p}^{\alpha} &= -\,\rho_0\, U^{\nu}\, \beta_{,\nu}\, N \quad , \\
\dot{p}^{\beta} &= (\rho_0\, U^i\, \alpha\, N)_{|i} \quad , \\
\dot{p}^{\theta} &= -\,\rho_0\, U^{\nu}\, S_{,\nu}\, N \quad , \\
\dot{p}^{S} &= (\rho_0\, U^i\, \theta\, N)_{|i} - \rho_0\, T\, N \quad .
\end{aligned}
\tag{21}
$$

The first is the continuity equation, Eq. (6). Upon application of the φ-equations and the continuity equation we see that the remaining four equations are the four independent velocity-potential equations among Eqs. (5).

One must also demand that the φ-equations be maintained in time, i.e. that

$$[\varphi_m, H'] = 0 \quad . \tag{22}$$

These equations are just the four independent velocity-potential equations, Eqs. (5): there are no Dirac χ-equations; i.e., there are no equations from Eq. (22) that involve p's and q's without λ's or $\dot{q}$'s, which would thus be constraints like the φ-equations. For example, φ_1 -- the constraint on p^{α} -- is preserved at zero by the equation $U^{\nu}\,\beta_{,\nu} = 0$, which is obtained from the original variational principle by varying α. This equation can be rearranged to read

$$\lambda_3 = N^i\,\beta_{,i} + \frac{\rho_0\, N}{\mu\, p^{\phi}}\, g^{ij}\,\beta_{,i}(\phi_{,j} + \alpha\,\beta_{,j} + \theta\, S_{,j}) \quad . \tag{23}$$

This is not really solved for λ_3 in terms of p^{ϕ} and q_a because ρ_0 and μ on the right-hand side implicitly depend on all the λ's. Nevertheless, all the λ's do have <u>unique</u> solutions (through the velocity-potential equations) in terms of p^{ϕ} and q_a. This means that there are no first-class φ-equations (as we guessed earlier) and no arbitrary functions of time in the solutions.

COUPLING TO GRAVITY

Until now we have treated the metric tensor $g_{\alpha\beta}$ as a constant because we were interested in the canonical theory of the fluid. The fluid is, however, coupled to the gravitational field, and one ought to treat the full dynamical system, fluid plus field.

The Hamiltonian density of the free gravitational field is[5,6]

$$H_G = N\, R^0 + N_i\, R^i \qquad (24)$$

with[5]

$$R^i = -2\, \pi^{ij}{}_{|j} \quad , \qquad (25)$$

$$R^0 = -\, g^{\frac{1}{2}}[{}^3R + g^{-1}(\tfrac{1}{2}\, \pi^2 - \pi^{ij}\, \pi_{ij})] \quad , \qquad (26)$$

and

$$\pi^{ij} = N\, g^{\frac{1}{2}}({}^4\Gamma^0{}_{kl} - g_{kl}\; {}^4\Gamma^0{}_{mm}\, g^{mn})g^{ik}\, g^{jl} \quad . \qquad (27)$$

Here ${}^3R$ is the scalar curvature of the hypersurface, and π^{ij} is the momentum canonical to g_{ij}. Since the Lagrangian density of the fluid, $pNg^{\frac{1}{2}}$, does not depend upon time derivatives of the metric, the full Hamiltonian is

$$\mathcal{H} = H_G + 16\pi H' g^{\frac{1}{2}} \quad . \qquad (28)$$

Note that H_G splits into two pieces, with R^0 and R^i independent of N and N_i. Dirac[6] shows that this will also be true of the Hamiltonian density for <u>any</u> field. In our case, we split up $H' g^{\frac{1}{2}}$ in two steps: i) Differentiate with respect to ${}^4g_{\mu\nu}$ while holding p_a and q_a constant,

$$\partial(H'\ g^{\frac{1}{2}})/\partial\left({}^4g_{\mu\nu}\right) = -\ \partial[p(-\ {}^4g^{\frac{1}{2}})]/\partial\left({}^4g_{\mu\nu}\right) = -\ \tfrac{1}{2}\ T^{\mu\nu}(-\ {}^4g)^{\frac{1}{2}} \quad ;$$

and ii) convert derivatives with respect to ${}^4g_{\mu\nu}$ to derivatives with respect to N, N_i, g_{ij} with the formula given by Schutz.[9] We obtain

$$\partial(H'\ g^{\frac{1}{2}})/\partial\ N_i = -\ g^{\frac{1}{2}}\ N(T^{0i} + N^i\ T^{00}) = -\ g^{\frac{1}{2}}\ N\ g^{ij}\ T^0{}_j \tag{29}$$

$$= g^{\frac{1}{2}}\ p_\phi\ g^{ij}(\phi_{,j} + \alpha\,\beta_{,j} + \theta\ S_{,j}) \quad , \tag{30}$$

and

$$\partial(H'\ g^{\frac{1}{2}})/\partial\ N = g^{\frac{1}{2}}\ N^2\ T^{00} = g^{\frac{1}{2}}\ N^2\ {}^4g^{0\mu}\ T^0{}_\mu \quad , \tag{31}$$

$$= \frac{H}{N}\ g^{\frac{1}{2}} - \frac{N_i}{N}\ \frac{\partial\ H'\ g^{\frac{1}{2}}}{\partial\ N_i} \tag{32}$$

Equation (32) implies

$$H' = N\ \frac{\partial\ H'}{\partial\ N} + N_i\ \frac{\partial\ H'}{\partial\ N_i} + \lambda_m\varphi_m \quad . \tag{33}$$

Since $\partial(H'\ g^{\frac{1}{2}})/\partial\ N_i$ is manifestly independent of N and N_i, differentiation of Eq. (33) shows that $\partial(H'\ g^{\frac{1}{2}})/\partial\ N$ is also independent of N and N_i.

The two pieces of H' have straightforward physical interpretations, as is shown by Schutz.[9] Let $\eta^\alpha = -\ N\ {}^4g^{0\alpha}$ be the unit normal to the spacelike hypersurface. Then the two pieces of $H'\ g^{\frac{1}{2}}$ are

$$\partial(H'\ g^{\frac{1}{2}})/\partial\ N = g^{\frac{1}{2}}\ \eta^\alpha\ \eta^\beta\ T_{\alpha\beta} \equiv \mathcal{E} \tag{34}$$

and

$$\partial(H'\ g^{\frac{1}{2}})/\partial\ N_i = g^{\frac{1}{2}}\ g^{ij}\ \eta^\alpha\ T_{\alpha i} \equiv \mathcal{P}^i \quad . \tag{35}$$

They are, respectively, the coordinate densities of energy and momentum measured by an observer at rest in the hypersurface.

By analogy with Eq. (16) we may define a general Poisson bracket for any two functions of π^{ij}, g_{ij}, p^a, and their spatial derivatives (but not of N or N_i, which are arbitrary functions that contain coordinate information but have no dynamical content):

$$[A, B] = \sum_{i,j}\left(\frac{\partial A}{\partial g_{ij}} \frac{\delta B}{\delta \pi^{ij}} - \frac{\partial A}{\partial \pi^{ij}} \frac{\delta B}{\delta g_{ij}} + \frac{\partial A}{\partial g_{ij|k}} \nabla_k \frac{\delta B}{\delta \pi^{ij}} - \frac{\partial A}{\partial \pi^{ij}{}_{|k}} \nabla_k \frac{\delta B}{\delta g_{ij}} + \ldots\right) + \sum_a\left(\frac{\partial A}{\partial q_a} \frac{\delta B}{\delta p^a} - \frac{\partial A}{\partial p^a} \frac{\delta B}{\delta q_a} + \frac{\partial A}{\partial q_{a,i}} \frac{\partial}{\partial x^i} \frac{\delta B}{\delta p^a} - \frac{\partial A}{\partial p^a{}_{,i}} \frac{\partial}{\partial x^i} \frac{\delta B}{\delta q^a} + \ldots\right) . \quad (36)$$

Then the time derivative of any such function that does not depend explicitly on time is

$$\dot{A} = [A, \mathcal{H}] \quad (37)$$

$$= N[A, R^0 + 16\pi\, \mathcal{E}] + N_i[A, R^i + 16\pi\, \mathcal{P}^i] + [A, \lambda_m \varphi_m] . \quad (38)$$

In particular, the ADaM form of the Einstein field equations follows by using g_{ij} and π^{ij} for A:

$$\dot{g}_{ij} = \dot{g}_{ij}(\mathrm{VAC}) \quad , \quad (39)$$

$$\dot{\pi}^{ij} = \dot{\pi}^{ij}(\mathrm{VAC}) + 8\pi\ N\ g^{\frac{1}{2}}(T^{ij} - N^i N^j T^{00}) \quad , \quad (40)$$

where (VAC) indicates the terms that are there in the vacuum case (see ADaM). These must be supplemented by the constraint equations that come from varying the Lagrangian density ($\mathcal{L} = -\mathcal{H} + \pi^{ij}\, \dot{g}_{ij} + \Sigma_a\, p^a\, \dot{q}_a$) with respect to N and N_i (which are not Hamiltonian variables):

$$R^0 + 16\pi\, \mathcal{E} = 0 \quad , \quad (41)$$

$$R^i + 16\pi\, \mathcal{P}^i = 0 \quad . \quad (42)$$

Equations (39)-(42) are identical to those derived by Schutz[9] for a general stress-energy tensor.

The constraints, Eqs. (41) and (42), must be maintained in time; i.e., we must have

$$[R^0 + 16\pi\,\mathcal{E}, \mathcal{H}] = 0 \tag{43}$$

and

$$[R^i + 16\pi\,\mathcal{P}^i, \mathcal{H}] = 0 \quad . \tag{44}$$

In the vacuum case these are the Bianchi identities. In our case the Bianchi identities reduce these to the equations of motion, $T^{\mu\nu}{}_{;\nu} = 0$. These four equations can be used to replace the four independent velocity-potential equations among Eqs. (5),[10] which themselves guaranteed the maintenance of the φ-equations. Therefore the full canonical set of equations is

$$\dot{g}_{ij} = [g_{ij}, \mathcal{H}] \tag{45a}$$

$$\dot{\pi}^{ij} = [\pi^{ij}, \mathcal{H}] \tag{45b}$$

$$\dot{\phi} = [\phi, \mathcal{H}] \tag{45c}$$

$$\dot{p}^{\phi} = [p^{\phi}, \mathcal{H}] \tag{45d}$$

with <u>either</u> the constraints (41) and (42) (maintained in time) <u>or</u> the φ-equations (11) (also maintained in time).

CONCLUDING REMARKS

The direction of any further analysis of these equations must depend upon the application they are intended for. It would in principle be possible to reduce the twelve gravitational variables (π^{ij} and g_{ij}) to four in exactly the same manner as ADaM. Solving the constraint equations would then involve the fluid variables ϕ and p^{ϕ}, but the coordinate conditions would be unaltered (as was pointed out by ADaM).

Methods very similar to these have been used by the author to derive the Hamiltonian density and from it a conserved energy density for the pulsations of and gravitational radiation from a differentially rotating relativistic star. These results will be published elsewhere.

ACKNOWLEDGMENTS

I would like to thank Vincent Moncrief for guiding me through Dirac's theory. I am greatly indebted to my many conversations with Yavuz Nutku, whose assistance was invaluable in preparing this paper.

APPENDIX

POISSON BRACKETS FOR FIELDS IN CURVED SPACES

For a system with n degrees of freedom, the Poisson bracket ("P.b.") of two functions of p^a and q_a is

$$[A, B] = \sum_{a=1}^{n} \left(\frac{\partial A}{\partial q_a} \frac{\partial B}{\partial p^a} - \frac{\partial A}{\partial p^a} \frac{\partial B}{\partial q_a} \right) \quad . \tag{46}$$

A classical field has an infinite number of degrees of freedom, one (or more) for each point in space. Functions like A and B may be functions not only of the fields p^a and q_a, but also of their spatial derivatives. In this case, a simple definition like Eq. (46) above is not sufficient.

Let us suppose that the field variable is a vector field q_i with canonical momentum $p^i \equiv \partial L/\partial \dot{q}_i$. Our results can be extended in a straightforward manner to cases where the field is a higher-rank tensor or a scalar.

Because the field variables at different points are independent, we wish the P.b. of two functions to be nonzero only if they are evaluated at the same point. Accordingly we define the canonical P.b.'s:

$$[q_i(\underset{\sim}{x}), q_j(\underset{\sim}{x}')] = [p^i(\underset{\sim}{x}), p^j(\underset{\sim}{x}')] = 0 \tag{47a}$$

$$[q_i(\underset{\sim}{x}), p^j(\underset{\sim}{x}')] = - [p^j(\underset{\sim}{x}'), q_i(\underset{\sim}{x})] = - \Omega^{j'}{}_i \, \delta^3(\underset{\sim}{x} - \underset{\sim}{x}') \quad . \tag{47b}$$

Here $\Omega^{j'}{}_i$ is the derivative of Synge's <u>world function</u>[11] $\Omega(\underset{\sim}{x}, \underset{\sim}{x}')$ with respect to x^i and $x'^{\,j}$, with the index j' raised by the metric at $\underset{\sim}{x}'$. Because of the delta function the only properties of $\Omega^{j'}{}_i$ that we will need are[11]

(1) its limit as $\underset{\sim}{x}'$ approaches $\underset{\sim}{x}$,

$$\lim_{\underset{\sim}{x}' \to \underset{\sim}{x}} \Omega^{j'}{}_i = - \delta^j{}_i \quad , \tag{48a}$$

and (2) the same limit of its covariant derivatives,

$$\lim_{\underset{\sim}{x}'\to\underset{\sim}{x}} \nabla_{k'} \Omega^{j'}{}_{i} = - \lim_{\underset{\sim}{x}'\to\underset{\sim}{x}} \nabla_{k} \Omega^{j'}{}_{i} \quad , \tag{48b}$$

where $\nabla_{k'}$ is a covariant derivative at $\underset{\sim}{x}'$ and acts only on primed indices, and vice-versa for ∇_k.

The delta function is normalized to proper volume,

$$\int \delta^3(\underset{\sim}{y})\, g^{\frac{1}{2}}\, d^3y = 1 \quad , \tag{49a}$$

and has the usual property

$$\frac{\partial}{\partial x^i} \delta^3(\underset{\sim}{x} - \underset{\sim}{x}') = - \frac{\partial}{\partial x^{i'}} \delta^3(\underset{\sim}{x} - \underset{\sim}{x}') \quad . \tag{49b}$$

Equation (48b) permits us to generalize Eq. (49b) to covariant differentiation:

$$\nabla_k \left\{\Omega^{j'}{}_{i}\, \delta^3(\underset{\sim}{x} - \underset{\sim}{x}')\right\} = - \nabla_{k'} \left\{\Omega^{j'}{}_{i}\, \delta^3(\underset{\sim}{x} - \underset{\sim}{x}')\right\} \quad . \tag{50}$$

We define the differentiated canonical P.b.'s:

$$[q_i(\underset{\sim}{x}),\, \nabla_{k'}\, p^j(\underset{\sim}{x}')] = - \nabla_{k'} \left\{\Omega^{j'}{}_{i}\, \delta^3(\underset{\sim}{x} - \underset{\sim}{x}')\right\} \quad , \tag{51a}$$

$$[\nabla_k\, q_i(\underset{\sim}{x}),\, p^j(\underset{\sim}{x}')] = - \nabla_{k} \left\{\Omega^{j'}{}_{i}\, \delta^3(\underset{\sim}{x} - \underset{\sim}{x}')\right\} \quad , \tag{51b}$$

and so on for higher derivatives. <u>The Poisson bracket</u> [,] <u>is the bilinear antisymmetric two-point differential operator whose domain is all</u> C^1 <u>functions of</u> p^i, q_i, <u>and their covariant derivatives and which obeys relations</u> (47) <u>and</u> (51).

By application of the chain rule we find

$$[A(\underset{\sim}{x}),\ B(\underset{\sim}{x}')] = \frac{\partial A}{\partial q_i}(\underset{\sim}{x})\,[q_i(\underset{\sim}{x}),\ p^j(\underset{\sim}{x}')]\,\frac{\partial B}{\partial p^j}(\underset{\sim}{x}')$$
$$+ \frac{\partial A}{\partial p^i}(\underset{\sim}{x})\,[p^i(\underset{\sim}{x}),\ q_j(\underset{\sim}{x}')]\,\frac{\partial B}{\partial q_j}(\underset{\sim}{x}')$$
$$+ \frac{\partial A}{\partial q_{i|k}}(\underset{\sim}{x})\,[q_{i|k}(\underset{\sim}{x}),\ p^j(\underset{\sim}{x}')]\,\frac{\partial B}{\partial p^i}(\underset{\sim}{x}') + \ldots \quad , \tag{52}$$

$$= \left\{\frac{\partial A}{\partial q_i}(\underset{\sim}{x})\,\frac{\partial B}{\partial p^i}(\underset{\sim}{x}') - \frac{\partial A}{\partial p^i}(\underset{\sim}{x})\,\frac{\partial B}{\partial q_i}(\underset{\sim}{x}')\right\}\delta^3(\underset{\sim}{x} - \underset{\sim}{x}')$$
$$- \frac{\partial A}{\partial q_{i|k}}(\underset{\sim}{x})\,\frac{\partial B}{\partial p^j}(\underset{\sim}{x}')\,\nabla_k\left\{\Omega^{j'}{}_i\,\delta^3(\underset{\sim}{x} - \underset{\sim}{x}')\right\}$$
$$+ \frac{\partial A}{\partial p^i{}_{|k}}(\underset{\sim}{x})\,\frac{\partial B}{\partial q_j}(\underset{\sim}{x}')\,\nabla_k\left\{\Omega^i{}_{j'}\,\delta^3(\underset{\sim}{x} - \underset{\sim}{x}')\right\}$$
$$- \frac{\partial A}{\partial q_i}(\underset{\sim}{x})\,\frac{\partial B}{\partial p^j{}_{|k}}(\underset{\sim}{x}')\,\nabla_{k'}\left\{\Omega^{j'}{}_i\,\delta^3(\underset{\sim}{x} - \underset{\sim}{x}')\right\}$$
$$+ \frac{\partial A}{\partial p^i}(\underset{\sim}{x})\,\frac{\partial B}{\partial q_{j|k}}(\underset{\sim}{x}')\,\nabla_{k'}\left\{\Omega^i{}_{j'}\,\delta^3(\underset{\sim}{x} - \underset{\sim}{x}')\right\}$$
$$+ \ldots \quad . \tag{53}$$

This is the usual definition of a Poisson bracket in classical field theories. But for the purpose of practical calculations it is useful to obtain a one-point P.b. by integrating. <u>The left (right) integrated P.b. is the integral of the P.b. over all $\underset{\sim}{x}'$ ($\underset{\sim}{x}$).</u> We denote these by $_{\underset{\sim}{x}}[\ ,\ ]$ and $[\ ,\ ]_{\underset{\sim}{x}}$, respectively. Integrating Eq. (53) on $\underset{\sim}{x}'$ and using Eq. (50)

gives

$$_{\underset{\sim}{x}}[A, B] = \int_{\text{all space}} [A(\underset{\sim}{x}), B(\underset{\sim}{x}')](g')^{\frac{1}{2}} d^3\underset{\sim}{x}' \qquad (54)$$

$$= \frac{\partial A}{\partial q_i} \frac{\delta B}{\delta p^i} - \frac{\partial A}{\partial p^i} \frac{\delta B}{\delta q_i}$$

$$+ \frac{\partial A}{\partial q_{i|k}} \nabla_k \frac{\delta B}{\delta p^i} - \frac{\partial A}{\partial p^i{}_{|k}} \nabla_k \frac{\delta B}{\delta q_i} + \cdots \quad , \qquad (55)$$

where $\delta B/\delta q_i$ is the variational derivative

$$\frac{\delta B}{\delta q_i} = \frac{\partial B}{\partial q_i} - \nabla_k \frac{\partial B}{\partial q_{i|k}} + \nabla_j \nabla_k \frac{\partial B}{\partial q_{i|k|j}} - + \cdots \quad . \qquad (56)$$

Although one cannot generally integrate a tensor over a curved space, as we have done in Eq. (54), in this case the delta function limits the integration to only one point, so that the integral is unambiguous.

This integrated P.b. is the generalization of the simple P.b., Eq. (46), to which it reduces when neither A nor B depends upon derivatives of q_i and p^i. When such derivatives are involved, the left integrated P.b. is the P.b. of A at the point $\underset{\sim}{x}$ with the entire field B: values of B at other points influence the bracket through the spatial derivatives of B at $\underset{\sim}{x}$. Note also that the integrated P.b.'s are independent of any coordinate system.

The following interesting properties follow directly from the definition of the integrated brackets:

1. $_{\underset{\sim}{x}}[A, B] = - [B, A]_{\underset{\sim}{x}}$; (57a)
2. $_{\underset{\sim}{x}}[A, B] = - {}_{\underset{\sim}{x}}[B, A]$ if and only if both A and B are independent of derivatives of q_i and p^i; (57b)
3. $\int_{\text{all space}} {}_{\underset{\sim}{x}}[A, B]\, g^{\frac{1}{2}} d^3x = \int_{\text{all space}} [A, B]_{\underset{\sim}{x}}\, g^{\frac{1}{2}} d^3x$ (57c)

if ${}_{\underset{\sim}{x}}[A, B]$ is a scalar (if not, the integrals are undefined);

4. $$\nabla_i \, {}_{\underset{\sim}{x}}[A, B] = {}_{\underset{\sim}{x}}[\nabla_i A, B] \quad ; \tag{57d}$$

$$\nabla_i [A, B]_{\underset{\sim}{x}} = [A, \nabla_i B]_{\underset{\sim}{x}} \quad . \tag{57e}$$

The integrated brackets fit into the Hamiltonian theory because the canonical equations are (for a system whose momenta are all independent)

$$\dot{q}_i = \delta H/\delta p^i \quad , \tag{58a}$$

$$\dot{p}^i = \delta H/\delta q_i \quad . \tag{58b}$$

They translate to (from now on we will use only the left integrated brackets):

$$\dot{q}_i = {}_{\underset{\sim}{x}}[q_i, H] \quad , \tag{59a}$$

$$\dot{p}_i = {}_{\underset{\sim}{x}}[p^i, H] \quad . \tag{59b}$$

By property 4 above these imply

$$\dot{q}_{i|k} = {}_{\underset{\sim}{x}}[q_{i|k}, H] \quad , \tag{60a}$$

$$\dot{p}^i{}_{|k} = {}_{\underset{\sim}{x}}[p^i{}_{|k}, H] \quad , \tag{60b}$$

which in turn imply

$$\dot{A} = {}_{\underset{\sim}{x}}[A, H] \tag{61}$$

for any function A (not necessarily a scalar) of q_i, p^i, and their spatial derivatives that does not explicitly depend on time.

Property 2 implies that in general $\dot{H} \neq 0$. This is to be expected: energy can be transferred from point to point. We should only expect that

$$\int_{\text{all space}} \dot{H} \, g^{\frac{1}{2}} \, d^3x = 0 \quad , \tag{62}$$

which is true because of properties 3 and 1. Thus, in general there exists a canonical Poynting vector S^i such that

$$\dot{H} + \nabla_i S^i = 0 \quad .$$

For the simple case where H depends on no derivatives of q_i and p^i higher than first order (which includes almost all physical systems), the Poynting vector is

$$S^i = \frac{\partial H}{\partial q_j} \frac{\partial H}{\partial p^j}_{|i} - \frac{\partial H}{\partial p^j} \frac{\partial H}{\partial q_{j|i}} \quad . \tag{63}$$

For a degenerate system (momenta not all independent) the equations of motion are almost as simple. Dirac[4] shows that for a system with a finite number of degrees of freedom,

$$\dot{q}_i = [q_i, H] + \lambda_m [q_i, \varphi_m] \quad , \tag{64a}$$

$$\dot{p}^i = [p^i, H] + \lambda_m [p^i, \varphi_m] \quad . \tag{64b}$$

For a degenerate field theory these become

$$\dot{q}_i = {}_{\underset{\sim}{x}}[q_i, H] + {}_{\underset{\sim}{x}}[q_i, \lambda_m \varphi_m] \quad , \tag{65a}$$

$$\dot{p}^i = {}_{\underset{\sim}{x}}[p^i, H] + {}_{\underset{\sim}{x}}[p^i, \lambda_m \varphi_m] \quad . \tag{65b}$$

In these equations λ_m appears inside the integrated bracket because it is generally a function of position. To compute a bracket that has λ_m inside, one treats λ_m as a function of $\underset{\sim}{x}$ independent of p^i and q_i. For example, the variational derivative of Eq. (56) is

$$\begin{aligned} \frac{\delta \lambda_m \varphi_m}{\delta q_i} &= \frac{\partial \lambda_m \varphi_m}{\partial q_i} - \nabla_k \frac{\partial \lambda_m \varphi_m}{\partial q_{i|k}} + - \cdots \quad , \\ &= \lambda_m \frac{\partial \varphi_m}{\partial q_i} - \nabla_k \left(\lambda_m \frac{\partial \varphi_m}{\partial q_{i|k}} \right) + - \cdots \quad . \end{aligned} \tag{66}$$

Conservation laws for the Hamiltonian can be derived here, too. They are especially simple in the case where H depends on no derivatives of p^i and only first derivatives of q_i, and where φ_m is independent of any derivatives. The equation maintaining the φ-equations is

$$\dot{\varphi}_n = {}_{\underset{\sim}{x}}[\varphi_n, H] + {}_{\underset{\sim}{x}}[\varphi_n, \lambda_m \varphi_m]$$

$$= {}_{\underset{\sim}{x}}[\varphi_n, H] + \lambda_m \, {}_{\underset{\sim}{x}}[\varphi_n, \varphi_m] = 0 \quad . \tag{67}$$

The time derivative of H is

$$\dot{H} = {}_{\underset{\sim}{x}}[H, H] + {}_{\underset{\sim}{x}}[H, \lambda_m \varphi_m] \quad .$$

Using the properties of the integrated bracket, our assumptions about H and φ_m, and Eq. (67), we can show that this becomes

$$\dot{H} + \nabla_i S^i = 0 \quad , \tag{68}$$

with

$$S^i = -\left(\frac{\partial H}{\partial p^j} + \lambda_m \frac{\partial \varphi_m}{\partial p^j}\right)\frac{\partial H}{\partial q_{j|i}} \quad . \tag{69}$$

But by Eq. (65a) this is just

$$S^i = -\dot{q}_j \frac{\partial H}{\partial q_{j|i}} \quad , \tag{70}$$

which is the canonical flux in the nondegenerate case as well.

In the body of this paper we will consistently use the left-integrated Poisson bracket, which we refer to simply as the Poisson bracket, denoted by [,].

REFERENCES

1 R.L. Seliger and G.B. Whitham, Proc. Roy. Soc. A. 305, 1 (1968).

2 B.F. Schutz, Jr., Phys. Rev. D. 2, 2762 (1970) (Chapter 2 of thesis).

3a L.A. Schmid, in A Critical Review of Thermodynamics, edited by E.B. Stuart, B. Gal-Or, and A.J. Brainard (Mono, Baltimore, 1970). 3b L.A. Schmid, Purè and Appl. Chem. 22, 493 (1970).

4 P. A. M. Dirac, Proc. Roy. Soc. A. 246, 326 (1958).

5 R. Arnowitt, S. Deser, and C.W. Misner in Gravitation, edited by L. Witten (Wiley, New York, 1962), Chap. 7.

6 A similar formulation of Einstein's equations, though not as complete in its discussion, can be found in P.A.M. Dirac, Proc. Roy. Soc. A246, 333 (1958).

7 We follow the notation of Schutz, ref. 2. In particular, Greek indices run from 0 to 3, while Latin indices run from 1 to 3, with these exceptions: a and b label the velocity potentials and run from 1 to 5, while m and n label the Dirac φ-equations and run from 1 to 4.

8 Actually, Dirac's proof (ref. 4) is for a system with finitely many degrees of freedom. We have generalized it to fields using the results of the appendix.

9 B.F. Schutz, Jr. "Non-Vacuum ADaM Field Equations," preprint, to be published in the Proceedings of the Pittsburgh Conference on Relativity (Springer-Verlag's Lecture Notes on Physics series) (Appendix of thesis).

10 See Appendix B of Schutz, ref. 2.

11 J.L. Synge, Relativity: The General Theory (North-Holland, Amsterdam, 1966).

CHAPTER 4

LINEAR PULSATIONS AND STABILITY OF DIFFERENTIALLY ROTATING STELLAR MODELS. I. NEWTONIAN ANALYSIS

Submitted to the *Astrophysical Journal*

ABSTRACT

A systematic method is presented for deriving the Lagrangian governing the evolution of small perturbations of arbitrary flows of a self-gravitating perfect fluid. The method is applied to a differentially rotating stellar model; the result is a Lagrangian equivalent to that of Lynden-Bell and Ostriker (1967). A sufficient condition for stability of rotating stars, derived from this Lagrangian, is simplified greatly by using as trial functions, not the three components of the Lagrangian displacement vector $\underset{\sim}{\xi}$, but three scalar functions defined by

$$\rho\underset{\sim}{\xi} = \underset{\sim}{\nabla}\lambda + \underset{\sim}{\nabla} \times (\chi \underset{\sim}{i} + \underset{\sim}{\nabla} \times \gamma \underset{\sim}{i}),$$

where $\underset{\sim}{i}$ is an arbitrary vector field. This change of variables saves one from integrating twice over the star to find the effect of the perturbed gravitational field.

I. INTRODUCTION AND SUMMARY

There is usually a very close connection between variational principles and stability criteria. If one has a variational principle that gives the dynamical equations for small perturbations of some equilibrium state, he usually can obtain directly a criterion that tells him whether those perturbations will remain small. In fact, Cotsaftis (1968) has shown that it is in principle always possible to derive at least a sufficient condition for stability from the Lagrangian. The most familiar example of this is the use of the Hamiltonian as a Lyapunov function in cases where energy is conserved or dissipated by the perturbations: then positive-definiteness of the Hamiltonian guarantees stability.

In the theory of small pulsations of stellar models made of perfect fluid, the problem of finding a Lagrangian for the pulsational equations has been solved only in the past decade (Chandrasekhar 1964, Chandrasekhar and Lebovitz 1964, Clement 1964, Lynden-Bell and Ostriker 1967, Chandrasekhar and Lebovitz 1968). The Lagrangian for the nonradial pulsations of a nonrotating star was deduced directly from the perturbed equations of motion by Chandrasekhar (1964) and by Chandrasekhar and Lebovitz (1964). Using these same techniques, Lynden-Bell and Ostriker (1967) obtained the Lagrangian for small perturbations of <u>any</u> stationary equilibrium configuration of perfect fluid; and they derived from their Lagrangian a sufficient condition for stability, which is essentially that the conserved Hamiltonian be positive-definite. In principle this nearly solves the stability problem, though in practice the criterion is still very difficult to use.

The purpose of this paper is to show that the Lagrangian can also be deduced in a potentially more powerful way from the general perfect-fluid variational principle of Seliger and Whitham (1968); and to show that the resulting stability criterion can be simplified greatly for the purpose of testing realistic models. The method introduced here is potentially more powerful for two reasons. First, it provides a straightforward, conceptually simple procedure for deducing the Lagrangian for perturbations of any initial flow (not necessarily stationary) with arbitrary boundary conditions on the perturbations (boundary conditions have required special considerations in previous work). Second, it is easily generalized to general-relativistic stellar models, where the pulsational equations (cf. Thorne and Campolattaro 1967) are so complicated that they have defied the earlier techniques. In the second paper in this series (Paper II, Schutz 1971a), we will apply the method illustrated here to fully relativistic, differentially rotating stellar models, starting from the relativistic version of the Seliger-Whitham variational principle obtained by Schutz (1970) (and obtained independently for special relativity by Schmid [1970a,b]). In the present paper we confine ourselves to the Newtonian regime.

The general plan of the paper is as follows. In §II we present the general Lagrangian for the perturbations of any motion of a self-gravitating perfect fluid (not restricted to stationary motions).

It is the second variation of the Seliger-Whitham Lagrangian. In §III we specialize to the case where the unperturbed flow is a differentially rotating stellar model. We reduce the Lagrangian to a function only of the fluid displacement vector, ξ; and we express the action as an integral over the interior of the star plus an integral over the surface of the star (the surface integral permits the perturbation to obey any boundary condition).

In §IV we write down the sufficient condition for stability, first discovered by Lynden-Bell and Ostriker (1967). We then show that a considerable simplification of the criterion can be effected by dealing not with ξ but with three scalar fields from which ξ can be obtained (in complete generality) by the following construction:

$$\rho\xi = \nabla\lambda + \nabla \times (\chi\, e_{\hat{r}} + \nabla \times \gamma\, e_{\hat{r}}).$$

Finally, in §V we examine the special cases of (i) axially symmetric perturbations of a rotating star (as treated by Chandrasekhar and Lebovitz 1968) and (ii) perturbations of a nonrotating star (treated by Chandrasekhar and Lebovitz 1964). We find that the stability criteria for those cases can also be simplified by using the above expression for ξ. In order to preserve the continuity of the discussion, details of the longer calculations have been placed in appendices.

II. PERTURBATIONS OF AN ARBITRARY FLOW

a) The Velocity-Potential Variational Principle

The starting point for our analysis is the variational principle discovered by Seliger and Whitham (1968). It is by no means the only variational principle for perfect fluids, but it is especially well-suited for examining perturbations because it is an Eulerian variational principle. That is, all fluid quantities are expressed in terms of five scalar fields (the velocity potentials $\psi, \alpha, \beta, \theta, S$); one never needs to deal explicitly with "fluid elements" or "particle paths." Perturbations in the flow come from simple Eulerian perturbations of the velocity potentials, and are much easier to deal with than perturbations in particle paths.

The basis of the variational principle is the representation of the velocity field of the perfect fluid in terms of the five velocity potentials:

$$\underset{\sim}{v} = \underset{\sim}{\nabla}\psi + \alpha\underset{\sim}{\nabla}\beta - S\underset{\sim}{\nabla}\theta, \tag{1}$$

where S is the entropy per unit mass. The notation follows that of Schutz (1970), with the definition

$$\psi \equiv \phi + \theta S, \tag{2}$$

where ϕ was used by Schutz (1970) but will not be used here. It turns out to be more convenient in this paper and especially in Paper II to use the set $(\psi, \alpha, \beta, \theta, S)$, rather than $(\phi, \alpha, \beta, \theta, S)$. To convert from this notation to that of Seliger and Whitham (1968), make the replacements $\psi \rightarrow \phi$, $\theta \rightarrow -\eta$. (These are changes in name only: Seliger and Whitham's ϕ is the same as our ψ.)

Each velocity potential obeys a simple "equation of evolution":

$$\frac{\partial\psi}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla}\psi = - h + TS - \Phi + \tfrac{1}{2}\underset{\sim}{v} \cdot \underset{\sim}{v}, \tag{3a}$$

$$\frac{\partial\alpha}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla}\alpha = 0, \tag{3b}$$

$$\frac{\partial\beta}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla}\beta = 0, \tag{3c}$$

$$\frac{\partial S}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla}S = 0, \tag{3d}$$

$$\frac{\partial\theta}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla}\theta = T. \tag{3e}$$

Here T is the temperature; Φ is the gravitational potential,

$$\nabla^2\Phi = 4\pi G\rho; \tag{4}$$

and h is the specific enthalpy,

$$h = (E + p)/\rho, \tag{5}$$

where E is the internal thermodynamic energy density, p is the pressure, and ρ is the mass density. The evolution of the velocity potentials fixes the evolution of $\underset{\sim}{v}$ through equation (1). In order to make this a well-determined set of equations one must add an equation of state,

$$p = p(h, S), \tag{6}$$

and the continuity equation

$$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{v}) = 0. \tag{7}$$

Equations (3), (4), and (7) constitute seven equations for the seven functions Φ, h, S, ψ, α, β, θ. They are completely equivalent to the Euler equation,

$$\frac{\partial \underset{\sim}{v}}{\partial t} + (\underset{\sim}{v} \cdot \underset{\sim}{\nabla})\underset{\sim}{v} = -\frac{1}{\rho}\underset{\sim}{\nabla} p - \underset{\sim}{\nabla} \Phi, \tag{8}$$

supplemented by equations (3d), (4), and (7). A rigorous proof of this equivalence has been given by Schutz (1970) for the relativistic version, but it applies equally well here.

Equations (3), (4), and (7) follow from extremizing the action

$$I = \int (\underset{\sim}{\nabla}\Phi \cdot \underset{\sim}{\nabla}\Phi - 8\pi G p)\, dt\, dV, \tag{9}$$

where the integral is over all space and time (dV is an element of volume). The pressure is taken to be a function of h and S through equation (6), and the enthalpy in turn is defined formally as a function of Φ and of the velocity potentials:

$$h = -\Phi - \psi_{,t} - \alpha\beta_{,t} + S\theta_{,t} - \tfrac{1}{2}(\underset{\sim}{\nabla}\psi + \alpha\underset{\sim}{\nabla}\beta - S\underset{\sim}{\nabla}\theta)^2. \tag{10}$$

Variations in the pressure with respect to the independent variables $(\Phi, \psi, \alpha, \beta, \theta, S)$ are accomplished through the first law of thermodynamics:

$$dp = \rho\, dh - \rho T\, dS. \tag{11}$$

The vanishing of the "first variation"

$$\delta I = \int (2\underset{\sim}{\nabla}\Phi \cdot \underset{\sim}{\nabla}\delta\Phi - 8\pi G\delta p)\, dt\, dV$$

$$= \int (2\underset{\sim}{\nabla}\Phi \cdot \underset{\sim}{\nabla}\delta\Phi - 8\pi G\rho\,\delta h + 8\pi G\rho T\,\delta S)\, dt\, dV \qquad (12)$$

--when δh is expressed in terms of the independent variations $\delta\Phi$, $\delta\psi$, $\delta\alpha$, $\delta\beta$, $\delta\theta$, δS--gives equations (4), (7), and (3b) - (3e). Equation (3a) follows from the rest of equations (3) and equation (10), so it is not an independent Euler-Lagrange equation.

In this paper it is often convenient to use the notation of differential geometry because we wish our expressions to be valid in any curvilinear coordinate system. Thus, we denote partial differentiation by a subscripted comma (as in eq. [10]) and covariant differentiation by a subscripted semicolon. We understand the gradient, $\underset{\sim}{\nabla}$, to be a covariant derivative. We distinguish contravariant components, v^i, from covariant components, v_i; and we raise and lower indices with the metric tensor g_{ij} [which for spherical polar coordinates is just $\mathrm{diag}(1, r^2, r^2\sin^2\vartheta)$]. We always integrate over proper volume, $dV = g^{\frac{1}{2}}\, d^3x$, where $g^{\frac{1}{2}}$--the root of the determinant of the matrix g_{ij}--is the Jacobian of the transformation from Cartesian coordinates to the general curvilinear coordinate system. We are able to integrate by parts because of the identity for any vector $\underset{\sim}{A}$ that $\underset{\sim}{\nabla} \cdot \underset{\sim}{A}\, g^{\frac{1}{2}} = A^j{}_{;j}\, g^{\frac{1}{2}} = (A^j g^{\frac{1}{2}})_{,j}$.

b) The Second Variation

It is well known that the second variation of a Lagrangian serves itself as a Lagrangian for the small perturbations of whatever state of motion causes the first variation to vanish (cf. Taub [1969] for a recent application to the stability of relativistic stars against radial pulsations). The second variation of equation (9) is just the part of I that is quadratic in the variations $\delta\Phi$, $\delta\psi$, $\delta\alpha$, $\delta\beta$, $\delta\theta$, δS. Thus, starting from equation (12), we find[1]

$$\delta^2 I = \int [\, 2\delta\Phi_{,i}\delta\Phi_{,j}g^{ij} - 8\pi G\,\delta\rho\,\delta h - 8\pi G\rho\,\delta^2 h + 8\pi G\,\delta(\rho T)\,\delta S]\,dt\,dV. \quad (13)$$

Now, the second variation in h comes from equation (10):

$$\delta^2 h = -2\delta\alpha\,\delta\beta_{,t} + 2\delta S\,\delta\theta_{,t} - \delta\underset{\sim}{v}\cdot\delta\underset{\sim}{v} - 2\delta\alpha\, v^k\,\delta\beta_{,k} + 2\delta S\, v^k\,\delta\theta_{,k}. \quad (14)$$

Thus, the Lagrangian density for the perturbations is (dividing eq. [13] by $8\pi G$)

$$L_2 = \frac{1}{4\pi G}\, g^{ij}\,\delta\Phi_{,i}\,\delta\Phi_{,j} - \delta\rho\,\delta h + \delta(\rho T)\,\delta S + \rho\,\delta\underset{\sim}{v}\cdot\delta\underset{\sim}{v}$$

$$+ 2\rho\,\delta\alpha\,(\delta\beta_{,t} + v^k\delta\beta_{,k}) - 2\rho\,\delta S\,(\delta\theta_{,t} + v^k\delta\theta_{,k}). \quad (15)$$

This Lagrangian is perfectly general and makes no assumption about the unperturbed state except that it satisfy the unperturbed velocity-potential equations. In the case of the differentially rotating star,

---

[1] Note that we are looking for second-order changes in functions of the potentials when the potentials are perturbed. By definition, then, the second variation of a potential itself is zero; e.g., $\delta^2\Phi = 0$.

the unperturbed motion is steady, so the coefficients of the quadratic perturbation terms in L_2 will be independent of time; this will enable us to obtain stability criteria.

In using L_2 as the Lagrangian density for the perturbed fluid, we have changed the meaning of $\delta\Phi$, $\delta\psi$, $\delta\alpha$, $\delta\beta$, $\delta\theta$, δS. In the first variation, $\delta\Phi$ was a "virtual" change in the gravitational field. Here, $\delta\Phi$ is the real Eulerian change in Φ produced by the perturbed state of the fluid. Extremizing $\int L_2 \, dV \, dt$ with respect to <u>virtual changes in $\delta\Phi$</u> gives the perturbed source equation,

$$\nabla^2 \delta\Phi = 4\pi G \, \delta\rho.$$

Similarly, extremizing $\int L_2 \, dV \, dt$ with respect to virtual changes in the other perturbations gives the Eulerian perturbations of equations (3b - e) and (7). These equations are completely equivalent to the perturbed Euler equation (eq. [8]), which we write down for future reference:

$$\frac{\partial \delta \underset{\sim}{v}}{\partial t} + (\delta \underset{\sim}{v} \cdot \underset{\sim}{\nabla}) \underset{\sim}{v} + (\underset{\sim}{v} \cdot \underset{\sim}{\nabla}) \delta \underset{\sim}{v}$$

$$= - \frac{1}{\rho} \underset{\sim}{\nabla} \delta p + \frac{1}{\rho^2} \delta\rho \underset{\sim}{\nabla} p - \underset{\sim}{\nabla} \delta\Phi. \qquad (16)$$

c) <u>Discussion</u>

For two reasons the Lagrangian density L_2 is not in a form suitable for a stability analysis.

First, the Lagrangian is degenerate. That is, the momenta $\partial L_2 / \partial \, \delta\Phi_{,t}$, $\partial L_2 / \partial \, \delta\psi_{,t}$, ... are not all independent; in fact, three of them are zero and only one of the remaining three is independent. This is partly a reflection of the fact that not all the six variables are dynamical (cf. Schutz [1971b] for further discussion of this point).

Second, the usual criterion for stability is that the perturbations not grow without bound. But even the unperturbed potentials ψ, β, and θ grow in time at any given point (cf. eqs. [3] or [18]), so we can expect that even for a stable, physically bounded perturbation the perturbations $\delta\psi$, $\delta\beta$, and $\delta\theta$ will grow without limit. This presents no physical difficulty because the potentials themselves are not physically observable. But it presents a mathematical difficulty in that the boundedness of the perturbed velocity potentials is neither necessary nor sufficient for stability.

For these reasons we prefer to express L_2 as a function only of the dynamical variable $\underset{\sim}{\xi}$ (the displacement vector of a fluid element).[2] This is accomplished in §III for the case of the differentially rotating star.

It is important to understand that the perturbed action,

$$I_2 = \int L_2 \, dV \, dt, \tag{17}$$

is an integral over all space between two arbitrary moments of time. The reason for this is that the Euler-Lagrange equations extremize

---

[2]An alternative procedure is followed in Paper II: We find the (non-conserved) Hamiltonian from L_2 and construct from it a conserved energy density, whose positive-definiteness ensures stability by Lyaponov's second theorem (La Salle and Lefschetz 1961). The complexity of the relativistic equations makes that the easier procedure; but in the Newtonian case the procedure we follow here is less difficult and physically more satisfying.

I_2 only under the condition that the variables $\delta\Phi$, $\delta\psi$, ... be held fixed at the boundary of the region of integration. The only way to ensure that this represents no physical constraint on the perturbations is to put the spatial boundary at infinity, where all the perturbations must vanish anyway.[3] (Only $\delta\Phi$ is observable outside the star, and it must approach zero at infinity at least as fast as $1/r^2$. The velocity potentials have no physical significance outside the fluid because ρ and p are zero there, but it is convenient to think of them as existing in the exterior and going smoothly to zero at infinity.) In §III, after we have introduced $\underset{\sim}{\xi}$, we will bring the boundary of the region of integration in to just inside the surface of the star, expressing the contribution from the rest of space as a surface integral at the star's surface. In this manner we will ensure that I_2 be an extremum among all perturbations that obey any physically permissible boundary conditions at the star's surface.

---

[3]By contrast, requiring the perturbations to vanish at the endpoints in time is not a physical restriction: it is a direct carry-over from particle mechanics, where it is the heart of Hamilton's principle. In continuum mechanics one cannot demand as well that the variation vanish at some point in space for all time, for that would be a physical constraint.

III. PERTURBATIONS OF DIFFERENTIALLY ROTATING STARS

a) The Unperturbed Equilibrium

From now on we will consider the Lagrangian, equation (15), only in the context of rotating stars. In this Section and the next we make no assumptions about the initial equilibrium except that it be axially symmetric, stationary, and, of course, composed of perfect fluid (no heat flux, no viscosity). In §V we specialize the equilibrium configuration further.

The general stationary axially symmetric flow can be represented by the following set of velocity potentials (r, ϑ, φ are the usual spherical polar coordinates, and t is time):

$$S = \text{arbitrary function of } r \text{ and } \vartheta \tag{18a}$$

$$\Omega = \text{arbitrary function of } r \text{ and } \vartheta \tag{18b}$$

$$\alpha = \Omega r^2 \sin^2 \vartheta = \Omega g_{\varphi\varphi} \tag{18c}$$

$$\beta = \varphi - \Omega t \tag{18d}$$

$$\theta = Tt \tag{18e}$$

$$\psi = (-h + TS - \Phi + \tfrac{1}{2} r^2 \sin^2\vartheta \; \Omega^2)t. \tag{18f}$$

From equation (1) we find

$$v_\varphi = \alpha = \Omega g_{\varphi\varphi}, \tag{19}$$

which means that Ω is the angular velocity,

$$\Omega = v^\varphi = d\varphi/dt. \tag{20}$$

Setting v_ϑ and v_r to zero in equation (1) gives the equation of structure

$$\rho^{-1}p_{,j} + \Phi_{,j} - \tfrac{1}{2}\Omega^2(r^2\sin^2\vartheta)_{,j} = 0, \tag{21a}$$

or

$$\rho^{-1}p_{,j} + \Phi_{,j} - \tfrac{1}{2}\Omega\alpha_{,j} + \tfrac{1}{2}\alpha\Omega_{,j} = 0 . \tag{21b}$$

The source equation for Φ, equation (4), has of course the formal solution

$$\Phi(x) = -\int \frac{G\rho(\underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|}\, dV'. \tag{22}$$

Note that although the velocity potentials are conveniently expressed in terms of the spherical polar coordinates, they are scalars and keep the same values in other coordinate systems.

b) Reduction of L_2

We now eliminate the variables $\delta\Phi$, $\delta\psi$, $\delta\alpha$, $\delta\beta$, $\delta\theta$, and δS from L_2 (eq. [15]), replacing them with $\underset{\sim}{\xi}$. The details of the reduction are given in Appendix A. The essential steps are:

i) Solve the perturbed velocity-potential equations for δS, $\delta\alpha$, and $\delta\beta$ in terms of $\underset{\sim}{\xi}$:

$$\delta S = -\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} S; \tag{23a}$$

$$\delta\alpha = -\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla}\alpha; \tag{23b}$$

$$\delta\beta = -\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla}\beta. \tag{23c}$$

ii) Express $\delta \underset{\sim}{v}$ and $\delta\rho$ in terms of $\underset{\sim}{\xi}$:

$$\delta \underset{\sim}{v} = \frac{\partial \underset{\sim}{\xi}}{\partial t} + (\underset{\sim}{v} \cdot \underset{\sim}{\nabla})\underset{\sim}{\xi} - (\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla})\, \underset{\sim}{v}; \tag{24}$$

$$\delta\rho = - \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{\xi}). \tag{25}$$

iii) Formally solve the perturbed source equation for $\delta\Phi$:

$$\delta\Phi(\underset{\sim}{x}) = - G \int dV' \rho(\underset{\sim}{x}')\underset{\sim}{\xi}(\underset{\sim}{x}') \cdot \underset{\sim}{\nabla}' \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|} . \tag{26}$$

iv) Plug all these expressions into L_2. Perform some integrations by parts so that explicit expressions for $\delta\psi$ and $\delta\theta$ are never needed. Discard all divergences because the integral extends to spatial infinity. Obtain the result

$$\begin{aligned} L_2 = {} & \frac{1}{4\pi G} \delta\Phi_{,i}\delta\Phi_{,j}g^{ij} - \gamma p(\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi})^2 - 2(\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi})(\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} p) \\ & - \frac{1}{\rho}(\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla}\rho)(\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} p) + \tfrac{1}{2}\rho(\alpha\Omega_{,j;k} - \Omega\alpha_{,j;k})\xi^j\xi^k \\ & + \rho g_{jk} v^{\ell} v^m \xi^j{}_{;\ell}\xi^k{}_{;m} + 2\rho g_{jk}\xi^j{}_{,t}\xi^k{}_{;\ell} v^{\ell} \\ & + \rho g_{jk}\xi^j{}_{,t}\xi^k{}_{,t}, \end{aligned} \tag{27}$$

where $\delta\Phi$ is given by equation (26), γ is the adiabatic index

$$\gamma = \frac{\rho}{p}\left(\frac{\partial p}{\partial \rho}\right)_S, \tag{28}$$

and all quantities except $\delta\Phi$ and $\underset{\sim}{\xi}$ have their unperturbed values. This is equivalent to the Lagrangian of Lynden-Bell and Ostriker (1967), specialized to the case of the differentially rotating star.

One ought to wonder if $L_2(\underset{\sim}{\xi})$ is really still the Lagrangian:

might not the substitutions of step (iv) fundamentally alter its character? The proof that they don't is, of course, that they don't: it is not hard to show that varying L_2 with respect to $\underset{\sim}{\xi}$ gives just the perturbed Euler equation, equation (16), when all perturbed quantities are expressed in terms of $\underset{\sim}{\xi}$.

This is reasonable on general grounds: the action I_2 is an extremum for motions obeying the perturbed versions of equations (3), (4), and (7). If we solve some of these equations for some of the variables in terms of the others and then substitute the solutions back into L_2, then I_2 must still be an extremum for the solution of the rest of the equations. That this is what we have done is evident from equations (23). In the general case, $\underset{\sim}{\nabla} S$, $\underset{\sim}{\nabla}\alpha$, and $\underset{\sim}{\nabla}\beta$ are linearly independent vectors. We have simply relabeled some of the variables by defining $\underset{\sim}{\xi}$ to be a vector whose component on $\underset{\sim}{\nabla} S$ is $-\delta S$, whose component on $\underset{\sim}{\nabla}\alpha$ is $-\delta\alpha$, and whose component on $\underset{\sim}{\nabla}\beta$ is $-\delta\beta$. We then eliminated $\delta\theta$, $\delta\psi$, and $\delta\Phi$ in terms of these three components of $\underset{\sim}{\xi}$. I_2 ought still to be an extremum for whatever δS, $\delta\alpha$, $\delta\beta$ made it an extremum before.

What about uniqueness? It is still possible that our procedure could introduce spurious solutions that extremize the reduced I_2 but not the original. This will in fact happen if one reduces the number of variables in a Lagrangian below the number of true degrees of freedom the system has, because then one has implicitly assumed some relation between one or more degrees of freedom that isn't generally true. As a simple example, consider the free-particle Lagrangian, $\mathcal{L} = \dot{x}^2 + \dot{y}^2$, whose Euler-Lagrange equations have the solution $\dot{x}$ = const, $\dot{y}$ = const. Assume that $\dot{x} = ky$. Substitute this into $\mathcal{L}$: $\mathcal{L} = k^2y^2 + \dot{y}^2$. The Euler-Lagrange equations still have as one solution $y = 0$ ($\Rightarrow \dot{x} = \dot{y} = 0$), but they also have the spurious solution $y = \exp(kt\sqrt{2})$. So in general one must exercise care not to infringe on a system's dynamical freedom. In our case we have not introduced spurious solutions: the three components of $\underset{\sim}{\xi}$ are the only dynamical variables the pulsating star has.

c) Surface Boundary Conditions: Expressing the Action as an Integral over the Interior of the Star Plus a Surface Integral

One generally prefers to express the action as an integral over the interior of the star, where all the dynamics occurs. Our action, $I_2 = \int L_2 \, dV \, dt$, with L_2 from equation (27), includes an integral over all of space. The only contribution outside the star is from the term in $\delta\Phi$. We shall see that it can be expressed as a divergence plus a term that is zero outside the star; thus the integral of L_2 outside the star can be expressed as a surface integral evaluated just above the surface of the star.

The star's surface is defined as that place where $p = 0$. For some equations of state this does not imply $\rho = 0$. Outside the surface we must of course have $\rho = 0$, so that ρ may be discontinuous and the terms in L_2 that contain gradients of ρ may be delta-functions at the surface. Therefore, bringing the limit of integration in I_2 to just inside the star's surface will bring in a surface integral.

We consider separately the two steps: first bringing the limit in to Σ^+, a surface just outside the star's surface Σ; and second, bringing the limit into Σ^-, a surface just inside Σ.

i) The Integral over the Exterior Region

The only nonzero term in I_2 outside the star comes from $\delta\Phi$. Ignoring for the moment the integral on time, we have

$$\int \underset{\sim}{\nabla} \delta\Phi \cdot \underset{\sim}{\nabla} \delta\Phi \, dV = \int [\, \underset{\sim}{\nabla} \cdot (\delta\Phi \underset{\sim}{\nabla} \delta\Phi) - \delta\Phi \nabla^2 \delta\Phi] \, dV$$

$$= -4\pi G \int \delta\Phi \, \delta\rho \, dV + \int \underset{\sim}{\nabla} \cdot (\delta\Phi \underset{\sim}{\nabla} \delta\Phi) \, dV. \quad (28)$$

If the region of integration is all space, the second term in the right-hand side vanishes. But if the region of integration is from Σ^+ outward, then the first term is zero and the second term is a surface integral ($\underset{\sim}{n}$ is the unit <u>outward</u> normal to Σ):

$$\int_{\text{exterior}} \underset{\sim}{\nabla} \delta\Phi \cdot \underset{\sim}{\nabla} \delta\Phi \, dV = - \int_{\Sigma^+} \delta\Phi \underset{\sim}{\nabla} \delta\Phi \cdot \underset{\sim}{n} \, d\sigma.$$

With this, I_2 becomes

$$I_2 = \int_{\substack{\text{out to} \\ \Sigma^+}} L_2 \, dV \, dt - \frac{1}{4\pi G} \int_{\Sigma^+} \delta\Phi \underset{\sim}{\nabla} \delta\Phi \cdot \underset{\sim}{n} \, d\sigma \, dt. \quad (29)$$

ii) The Surface Integral

If we integrate the first term on the right-hand side in equation (29) only out to Σ^-, we omit only an infinitesimal volume of space. Only if L_2 has delta-functions at the surface will this region contribute to I_2. As we mentioned previously, a discontinuity in ρ would give such a delta-function. We do not need to worry about discontinuities in $\underset{\sim}{\xi}$ or Ω: we can perfectly well define fields $\underset{\sim}{\xi}$ and Ω outside the star that are continuous at its surface. They don't affect I_2 because ρ and p are zero outside. Moreover, there can be no discontinuities in p and $\delta\Phi$ at Σ.

One contribution to the integral of L_2 between Σ^- and Σ^+ might come from the term $\nabla \delta\Phi \cdot \nabla \delta\Phi$. This has no delta-functions, so its net contribution is zero. However, from equation (28) we see that this means

$$4\pi G \int_{\Sigma^-}^{\Sigma^+} \delta\Phi \, \delta\rho \, dV = \int_{\Sigma^+} \delta\Phi \nabla \delta\Phi \cdot n \, d\sigma - \int_{\Sigma^-} \delta\Phi \nabla \delta\Phi \cdot n \, d\sigma. \qquad (30)$$

If ρ is discontinuous, the term $\delta\rho = -\nabla \cdot (\rho \xi)$ contributes to the left-hand side, and the result is

$$\int_{\Sigma^+} \delta\Phi \nabla \delta\Phi \cdot n \, d\sigma = \int_{\Sigma^-} \delta\Phi \nabla \delta\Phi \cdot n \, d\sigma + 4\pi G \int_{\Sigma^-} \delta\Phi \rho \xi \cdot n \, d\sigma$$

$$= \int_{\Sigma^-} \delta\Phi (\nabla \delta\Phi + 4\pi G \rho \xi) \cdot n \, d\sigma. \qquad (31)$$

This enables us to move the surface integral in equation (29) from Σ^+ to Σ^-.

The only contribution to the integral of L_2 between Σ^- and Σ^+ comes from the fourth term in equation (27):

$$-\rho^{-1} (\xi \cdot \nabla\rho)(\xi \cdot \nabla p).$$

Its integral is

$$-\int_{\Sigma^-}^{\Sigma^+} \rho^{-1} (\xi \cdot \nabla\rho)(\xi \cdot \nabla p) \, dV = \int_{\Sigma^-} (\xi \cdot \nabla p)(\xi \cdot n) \, d\sigma. \qquad (32)$$

Note that because Σ is a surface of constant pressure, ∇p and n are parallel there. With equations (29), (31), and (32), the action becomes

$$I_2 = \int_{\text{interior}} L_2 \, dV \, dt + \int_{\Sigma^-} (\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} p)(\underset{\sim}{\xi} \cdot \underset{\sim}{n}) \, d\sigma \, dt$$

$$- \int_{\Sigma^-} \delta\Phi(\rho\underset{\sim}{\xi} + \frac{1}{4\pi G} \underset{\sim}{\nabla}\delta\Phi) \cdot \underset{\sim}{n} \, d\sigma \, dt, \qquad (33)$$

where by "interior" we mean the region inside Σ^-.

We should mention that these same surface integrals can be obtained if, instead of integrating L_2 over all space and then bringing the limit of integration in, one always integrates L_2 just over the interior but adds surface terms in order to make Σ a free boundary. This procedure is examined in detail by Courant and Hilbert (1953) under the name "natural boundary conditions." The procedure followed in this section was first suggested to me by Professor Kip Thorne.

IV. STABILITY OF DIFFERENTIALLY ROTATING STARS

a) The Stability Criterion

The Lagrangian density, equation (27), has the form

$$L_2 = \rho \underset{\sim}{\xi}_{,t} \cdot \underset{\sim}{\xi}_{,t} + \mathcal{A}[\underset{\sim}{\xi}, \underset{\sim}{\xi}_{,t}] + \mathcal{C}[\underset{\sim}{\xi}, \underset{\sim}{\xi}], \tag{34}$$

where $\mathcal{A}$ and $\mathcal{C}$ are homogeneous quadratic time-independent operators. Moreover, $\mathcal{A}$ is antisymmetric and $\mathcal{C}$ is symmetric when L_2 is integrated over all space. Note that $\mathcal{C}$ includes all except the last two terms of equation (27). It is easy to show (cf. Kulsrud 1968) that a sufficient condition for stability is (for all $\underset{\sim}{\xi}$ bounded everywhere and zero at infinity)

$$-\int_{\substack{\text{all}\\ \text{space}}} \mathcal{C}[\underset{\sim}{\xi}, \underset{\sim}{\xi}]\, dV > 0. \tag{35}$$

This is sufficient for stability because it guarantees that the "kinetic energy,"

$$K \equiv \int_{\substack{\text{all}\\ \text{space}}} \rho \underset{\sim}{\xi}_{,t} \cdot \underset{\sim}{\xi}_{,t}\, dV, \tag{36}$$

will remain bounded for all time for all perturbations.

Another way of obtaining the same result is to construct the Hamiltonian density

$$\mathcal{H} = \rho \underset{\sim}{\xi}_{,t} \cdot \underset{\sim}{\xi}_{,t} - \mathcal{C}[\underset{\sim}{\xi}, \underset{\sim}{\xi}]. \tag{37}$$

Because the operator $\mathbf{C}$ is time-independent, the total energy

$$\mathcal{E} \equiv \int_{\substack{\text{all} \\ \text{space}}} \mathcal{H}\, dV, \tag{38}$$

is constant, so that $\mathcal{H}$ is a Lyapunov function whose positive-definiteness guarantees stability. Clearly inequality (35) guarantees the positive-definiteness of $\mathcal{H}$. It is this Lyapunov criterion to which we will appeal in Paper II in order to obtain a sufficient condition for the stability of relativistic stars.

For the realistic Newtonian star, inequality (35) is more than just a sufficient condition for stability. According to Lynden-Bell and Ostriker (1967), it is also the condition for secular stability: if friction is introduced, stable modes of pulsation will remain stable if and only if equation (35) is satisfied. It is therefore of great importance to cast the criterion in a form that is easy to test realistic models with. That is the subject of the remainder of this paper. Although the criterion (35) is not new, our way of handling it is.

b) The Transverse and Longitudinal Parts of $\rho\underset{\sim}{\xi}$

The typical procedure for testing a stellar model for stability is to choose a trial function for $\underset{\sim}{\xi}$, which might have some arbitrary parameters in it, and then to plug it into the operator $\mathbf{C}$ and see if inequality (35) is satisfied for all values of the parameters. This procedure is made very difficult by the term $\underset{\sim}{\nabla}\delta\Phi \cdot \underset{\sim}{\nabla}\delta\Phi$. In order to find $\delta\Phi$ at any point inside the star one must integrate $\rho\underset{\sim}{\xi}$ over the entire star (cf. eq. [26]). This is impractical for all but the simplest stellar models and trial functions.

Fortunately we can overcome this difficulty. The source equation for $\delta\Phi$ is

$$\underset{\sim}{\nabla} \cdot (\underset{\sim}{\nabla}\delta\Phi) = -4\pi G\underset{\sim}{\nabla} \cdot (\rho\underset{\sim}{\xi}).$$

This can be integrated to give

$$\underset{\sim}{\nabla}\delta\Phi = -4\pi G\underset{\sim}{\eta}^{L}, \tag{39}$$

where $\underset{\sim}{\eta}^{L}$ is the longitudinal (curl-free) part of the vector field[4]

$$\underset{\sim}{\eta} \equiv \begin{cases} \rho\underset{\sim}{\xi} & \text{inside the star} \\ 0 & \text{outside the star.} \end{cases} \tag{40}$$

Any piecewise differentiable vector field $\underset{\sim}{A}$ that approaches zero at infinity at least as fast as $1/r^2$ can be decomposed into unique longitudinal and transverse parts,

$$\underset{\sim}{A} = \underset{\sim}{A}^{L} + \underset{\sim}{A}^{T}, \tag{41a}$$

where (cf. Phillips 1933)

$$\underset{\sim}{A}^{L} = \underset{\sim}{\nabla} f = \underset{\sim}{\nabla}\left[\frac{1}{4\pi}\int \underset{\sim}{A}(\underset{\sim}{x}') \cdot \underset{\sim}{\nabla}' \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|}\, dV'\right] \tag{41b}$$

and

$$\underset{\sim}{A}^{T} = \underset{\sim}{\nabla} \times \underset{\sim}{F} = \underset{\sim}{\nabla} \times \left[\frac{1}{4\pi}\int \underset{\sim}{A}(\underset{\sim}{x}') \times \underset{\sim}{\nabla}' \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|}\, dV'\right]. \tag{41c}$$

---

[4]A good introduction to longitudinal and transverse parts of vector fields can be found in Phillips (1933).

The function f and the vector $\underset{\sim}{F}$ are the unique continuous scalar and divergence-free vector potentials of the field $\underset{\sim}{A}$. Note that $\underset{\sim}{F}$ is unique only if we demand that it be divergence-free: we can--and later we will--add a divergence to $\underset{\sim}{F}$ without changing $\underset{\sim}{A}^T$.

From equation (26) we see that the scalar potential for $\underset{\sim}{\eta}$ is just $-(4\pi G)^{-1}\delta\Phi$, which proves equation (39). Thus, the gravitational term in $\mathcal{C}$ becomes

$$\frac{1}{4\pi G} \underset{\sim}{\nabla}\delta\Phi \cdot \underset{\sim}{\nabla}\delta\Phi = 4\pi G\underset{\sim}{\eta}^L \cdot \underset{\sim}{\eta}^L. \tag{42}$$

We can achieve a considerable savings of effort in testing a stellar model for stability if instead of choosing a trial function for $\underset{\sim}{\xi}$ we choose one for $\underset{\sim}{\eta}^L$ and one for $\underset{\sim}{\eta}^T$. The search for a suitable curl-free vector for $\underset{\sim}{\eta}^L$ and a suitable divergence-free vector for $\underset{\sim}{\eta}^T$ might still prove difficult, so in the next subsection we will simplify the task even more by introducing three arbitrary scalar functions in place of $\underset{\sim}{\eta}^T$ and $\underset{\sim}{\eta}^L$. But first it is convenient to re-express the stability criterion (35) in terms of $\underset{\sim}{\eta}$.

Inequality (35) has $\mathcal{C}$ integrated over all space. If we bring the limits of integration in to Σ^-, we pick up the identical surface terms as in equation (33). We can therefore write inequality (35) in the form

$$-\int_{\text{interior}} \mathcal{C}[\underset{\sim}{\eta},\underset{\sim}{\eta}]\, dV - \int_{\Sigma^-} \underset{\sim}{D}[\underset{\sim}{\eta},\underset{\sim}{\eta}] \cdot \underset{\sim}{n}\, d\sigma > 0, \tag{43}$$

where $\mathcal{C}[\underset{\sim}{\eta},\underset{\sim}{\eta}] \equiv \mathcal{C}[\underset{\sim}{\xi},\underset{\sim}{\xi}]$, and where (cf. eq. [33])

$$\underset{\sim}{D}[\underset{\sim}{\eta},\underset{\sim}{\eta}] = \rho^{-2}(\underset{\sim}{\eta} \cdot \underset{\sim}{\nabla} p)\underset{\sim}{\eta} - \delta\Phi\underset{\sim}{\eta}^T. \tag{44}$$

It is understood in equation (44) that $\delta\Phi$ is $-4\pi G$ times the scalar potential of η.

The operator $C[\eta,\eta]$ has covariant derivatives of η in it. When doing calculations one must replace covariant derivatives with ordinary partial derivatives and Christoffel symbols. When one does this in spherical polar coordinates one finds (now indices j, k run over r, ϑ, φ)

$$
\begin{aligned}
C[\eta,\eta] = {} & 4\pi G g_{jk}\eta^{(L)j}\eta^{(L)k} + \frac{1}{\rho}\Omega^2 g_{jk}\eta^j{}_{,\varphi}\eta^k{}_{,\varphi} \\
& + \frac{1}{\rho}\Omega^2\left(\frac{\alpha}{\Omega}\right)_{,j}(\eta^j\eta^\varphi{}_{,\varphi} - \eta^\varphi\eta^j{}_{,\varphi}) + \frac{1}{2\rho}(\alpha\Omega_{,jk} - \Omega\alpha_{,jk})\eta^j\eta^k \\
& + \frac{1}{\rho}\Omega^2[(\eta^r)^2 + 2r\sin 2\vartheta\,\eta^\vartheta\eta^r + r^2\cos 2\vartheta\,(\eta^\vartheta)^2] \\
& - \frac{\gamma p}{\rho^2}(\nabla\cdot\eta)^2 + \frac{1}{\rho^3}(\eta\cdot\nabla\rho)(\eta\cdot\mathcal{S}) - \frac{2}{\rho^2}(\nabla\cdot\eta)(\eta\cdot\mathcal{S}).
\end{aligned}
\tag{45}
$$

Here we have defined

$$\mathcal{S} \equiv \nabla p - \frac{\gamma p}{\rho}\nabla\rho,$$

which is the vector Schwarzschild discriminant. For nonrotating stars, $\mathcal{S}_r > 0$ is necessary for stability against convection. Components η^r, η^ϑ, η^φ in equation (45) are components on the <u>unnormalized</u> coordinate basis vectors e_r, e_ϑ, e_φ.

For future reference it is convenient to write down the entire Lagrangian L_2 from equation (27) in terms of η. It is

$$
\begin{aligned}
L_2 = {} & \frac{1}{\rho} g_{jk}\eta^j{}_{,t}\eta^k{}_{,t} + \frac{2}{\rho}\Omega g_{jk}\eta^j{}_{,t}\eta^k{}_{,\varphi} \\
& + \frac{1}{\rho}\Omega\left(\frac{\alpha}{\Omega}\right)_{,j}(\eta^j\eta^\varphi{}_{,t} - \eta^\varphi\eta^j{}_{,t}) + C[\eta,\eta].
\end{aligned}
\tag{46}
$$

c) Scalar Potential for $\rho\underset{\sim}{\xi}$

We have seen that it is possible to reduce the number of integrations necessary to test for stability by replacing $\underset{\sim}{\xi}$ by $\underset{\sim}{\eta}$. We now show that it is possible to express $\underset{\sim}{\eta}$ in terms of three scalars in such a way that the two pieces $\underset{\sim}{\eta}^L$ and $\underset{\sim}{\eta}^T$ separate automatically. Then trial functions may be chosen for the scalars without losing the advantage obtained by separating $\underset{\sim}{\eta}$ into $\underset{\sim}{\eta}^L$ and $\underset{\sim}{\eta}^T$.

Our procedure rests on the following theorem: For any vector fields $\underset{\sim}{A}$ and $\underset{\sim}{i}$ $(\underset{\sim}{i} \cdot \underset{\sim}{i} \neq 0)$ whose Cartesian components are analytic functions of position in the neighborhood of some point, there exist functions κ, χ, γ also analytic in that neighborhood such that

$$\underset{\sim}{A} = \underset{\sim}{\nabla}\kappa + \chi\underset{\sim}{i} + \underset{\sim}{\nabla} \times \gamma\underset{\sim}{i} . \tag{47}$$

The existence of κ, χ, γ follows from the Cauchy-Kowalewski existence theorem for systems of first order partial differential equations (cf. Courant and Hilbert 1962). The restriction to analytic functions is probably not important. In practice one can choose $\underset{\sim}{i}$ to be analytic almost everywhere. Moreover, the functions κ, χ, γ probably exist for most well-behaved but nonanalytic $\underset{\sim}{A}$ as well. Even if they do not exist for some $\underset{\sim}{A}$, it will usually be possible to approximate $\underset{\sim}{A}$ as closely as one wishes with analytic functions, except at isolated points. Note that one might need several "patches" to represent $\underset{\sim}{A}$ in a finite region.

In the previous subsection we showed that there exist λ and $\underset{\sim}{A}$ such that

$$\underset{\sim}{\eta} = \underset{\sim}{\nabla}\lambda + \underset{\sim}{\nabla} \times \underset{\sim}{A} .$$

If we now replace $\underset{\sim}{A}$ by equation (47), we obtain

$$\underset{\sim}{\eta} = \underset{\sim}{\nabla}\lambda + \underset{\sim}{\nabla} \times (\chi \underset{\sim}{i} + \underset{\sim}{\nabla} \times \gamma \underset{\sim}{i}). \tag{48a}$$

Thus, there always exist λ, χ, and γ such that for any analytic, nowhere-zero vector field $\underset{\sim}{i}$

$$\eta^L = \underset{\sim}{\nabla}\lambda, \tag{48b}$$

$$\eta^T = \underset{\sim}{\nabla}\times (\chi \underset{\sim}{i} + \underset{\sim}{\nabla}\times \gamma \underset{\sim}{i}) . \tag{48c}$$

We are still free to choose $\underset{\sim}{i}$ in any way we might wish. In this paper we will choose $\underset{\sim}{i} = \underset{\sim}{e}_r$, which is analytic everywhere but at $r = 0$; this will allow our results to assume a convenient form in the nonrotating, spherical case, where the ϑ - and φ-directions are equivalent. One would therefore expect our results to be well-adapted to the study of modes that have analogues in the nonrotating star; they might do less well on other modes. A variant on this is to choose $\underset{\sim}{i} = \underset{\sim}{\nabla}p/ |\underset{\sim}{\nabla}p|$ (at the surface, $\underset{\sim}{i}$ is the normal), which might do slightly better for isentropic models, where surfaces of ρ and p coincide. On the other hand, for investigations of highly flattened, rapid rotating models, it might be better to choose $\underset{\sim}{i} = \underset{\sim}{e}_{\tilde{\omega}}$, where $\tilde{\omega}$ is the radius in cylindrical polar coordinates $(\tilde{\omega}, \vartheta, z)$.

d) Testing for Stability

We define the trial functions a, b, c by

$$\underset{\sim}{\eta}^L = \underset{\sim}{\nabla} a, \tag{49a}$$

$$\underset{\sim}{\eta}^T = \underset{\sim}{\nabla} \times \underset{\sim}{A}, \tag{49b}$$

$$\underset{\sim}{A} = - r^2 c \, \underset{\sim}{e}_r + \underset{\sim}{\nabla} \times (r^2 b \, \underset{\sim}{e}_r). \tag{49c}$$

Since the star has azimuthal symmetry, we expand

$$a = \sum_{M=0}^{\infty} [\, a_M^+(r, \vartheta, t) \sin M\varphi + a_M^-(r, \vartheta, t) \cos M\varphi] \, , \tag{50}$$

and similarly for b and c. Modes corresponding to different M are orthogonal, but plus and minus modes of the same M are mixed by the equations of motion and variational principle. Appendix B contains the details of the reduction of the stability criterion to a condition on $a^{\pm}$, $b^{\pm}$, $c^{\pm}$ for each M. The expressions are very complicated; we will deal only with special cases from now on.

V. SPECIAL CASES

a) Axially Symmetric Perturbations

Axially symmetric perturbations were examined by Lynden-Bell and Ostriker (1967) and in great detail for uniformly rotating stars by Chandrasekhar and Lebovitz (1968). We do not need the restriction to uniform rotation.

Requiring $\underset{\sim}{\eta}$ to be independent of φ is equivalent to setting $M = 0$ (cf. eq. [B1] of Appendix B). Thus, there is no distinction between plus and minus modes. Representation (B1) for $\underset{\sim}{\eta}$ becomes

$$\underset{\sim}{\eta}^{L} = a_{,r}\underset{\sim}{e}_{\hat{r}} + a_{,\vartheta}\underset{\sim}{e}_{\hat{\vartheta}} \tag{51a}$$

$$\underset{\sim}{\eta}^{T} = - L^{2} b\underset{\sim}{e}_{\hat{r}} + \frac{1}{r}\frac{\partial}{\partial r} r^{2} b_{,\vartheta}\underset{\sim}{e}_{\hat{\vartheta}} + rc_{,\vartheta}\underset{\sim}{e}_{\hat{\varphi}} , \tag{51b}$$

where $\underset{\sim}{e}_{\hat{r}}$, $\underset{\sim}{e}_{\hat{\vartheta}}$ and $\underset{\sim}{e}_{\hat{\varphi}}$ are unit vectors, and where $L^{2} \equiv \left(\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial}{\partial\vartheta} - \frac{M^{2}}{\sin^{2}\vartheta}\right)$ is the angular part of the Laplacian.

Notice that the scalar c separates from the other two: its sole function is to determine the φ-component of $\underset{\sim}{\eta}$. This separation shows up in the equations of motion. The equation for c can be obtained by varying the Lagrangian, equation (46), with respect to η^{φ} after setting derivatives with respect to φ to zero:

$$- \frac{2}{\rho} r^{2}\sin^{2}\vartheta\, \eta^{\varphi}{}_{,tt} - \frac{2}{\rho}\Omega\left(\frac{\alpha}{\Omega}\right)_{,j}\eta^{j}{}_{,t} = 0 \tag{52}$$

This is just the equation for the coriolis acceleration in the azimuthal direction of the displaced fluid element as it is carried around the star. We can integrate this equation:

$$\sin\vartheta\,\eta^{\varphi}{}_{,t} = c_{,\vartheta t} = -\frac{\Omega}{r^2\sin^2\vartheta}(r^2\sin^2\vartheta)_{,j}\eta^j + f(r,\vartheta), \qquad (53)$$

where $f(r,\vartheta)$ is an arbitrary function that represents an "initial" (i.e. when $\tilde{\eta} = 0$) azimuthal velocity perturbation.

Suppose that we take $f \equiv 0$. Then for this restricted class of perturbations we can substitute equation (53) into the Lagrangian density (46), which remains a Lagrangian for η^r and η^{ϑ}, and in which there are no terms linear in time derivatives of $\tilde{\eta}$. From the theorem of Laval, Mercier, and Pellat (1965) we obtain the following necessary and sufficient condition for the stability of the star against our restricted class of perturbations ($f = 0$):

$$\int_{\text{interior}} \left\{\frac{\Omega^2}{\rho r^2\sin^2\vartheta}\left[(r^2\sin^2\vartheta)_{,j}\eta^j\right]^2 - C[\tilde{\eta},\tilde{\eta}]\right\} dV$$

$$- \int_{\Sigma^-} \tilde{D}[\tilde{\eta},\tilde{\eta}] \cdot \tilde{n}\, d\sigma > 0. \qquad (54)$$

Here C and $\tilde{D}$ are the same as in equation (43), reduced to the axially symmetric case.

This condition--as was indicated by Chandrasekhar and Lebovitz (1968)--is only necessary for stability against all axially symmetric perturbations. However, Lynden-Bell and Ostriker (1967) point out that it is nearly sufficient as well, in the following sense: If all the stellar models that can be obtained from the one we are testing by changing Ω slightly satisfy inequality (54), then the model we are testing is stable against all axially symmetric perturbations.

The reason is that a nonzero f in equation (53) means physically that when $\underset{\sim}{\eta} = 0$ the fluid is given an extra angular velocity of $f/\rho \sin \vartheta$. If this mode is unstable for some f then a stellar model differing from the one we are testing by an angular velocity $f/\rho \sin \vartheta$ should be unstable against perturbations with $f = 0$. This argument ignores the effect of the additional angular velocity on the structure (p, ρ) of the equilibrium model, so it is not completely rigorous. Nevertheless it suggests that inequality (54) ought to be an accurate stability criterion, especially for sequences of models. Note that inequality (43) is still a sufficient condition for stability.

By specializing the calculations of Appendix B to $M = 0$, inequality (54) can also be put in a form that makes testing models easier. This is done in Appendix C.

The special choice of trial function made in §III of Chandrasekhar and Lebovitz (1968) corresponds here to setting $b = 0$. They apparently saw the advantage of using scalars and decomposing $\underset{\sim}{\eta}$ into transverse and longitudinal parts, but their trial function with $b = 0$ lacked the generality of our equation (51): its transverse part vanished.

b) The Nonrotating Star

Expressions suitable for analyzing the pulsations of non-rotating stars can be obtained by setting Ω to zero in previous results and expanding a, b, and c in spherical harmonics Y_L^M. Then the representation (49) of $\underset{\sim}{\eta}$ becomes[5]

[5]The b in this section is really $L(L+1)$ times the one in equation (49). Consequently one must set $b = 0$ when $L = 0$.

$$\underset{\sim}{\eta}^L = \sum_{LM} \left(a_{LM,r} Y_L{}^M \underset{\sim}{e}_{\hat{r}} + \frac{1}{r} a_{LM} Y_L{}^M{}_{,\vartheta} \underset{\sim}{e}_{\hat{\vartheta}} + \frac{a_{LM}}{r \sin\vartheta} Y_L{}^M{}_{,\varphi} \underset{\sim}{e}_{\hat{\varphi}} \right) , \quad (55a)$$

$$\underset{\sim}{\eta}^T = \sum_{LM} \left\{ b_{LM} Y_L{}^M \underset{\sim}{e}_{\hat{r}} + \left[\frac{(r^2 b_{LM})_{,r}}{r L(L+1)} Y_L{}^M{}_{,\vartheta} - \frac{r c_{LM}}{\sin\vartheta} Y_L{}^M{}_{,\varphi} \right] \underset{\sim}{e}_{\hat{\vartheta}} \right.$$

$$\left. + \left[\frac{(r^2 b_{LM})_{,r}}{L(L+1) r \sin\vartheta} Y_L{}^M{}_{,\varphi} + r c_{LM} Y_L{}^M{}_{,\vartheta} \right] \underset{\sim}{e}_{\hat{\varphi}} \right\} . \quad (55b)$$

We should note that one can obtain exactly this expression by expanding $\underset{\sim}{\eta}$ in Regge-Wheeler (1957) spherical harmonics, and then separating $\underset{\sim}{\eta}^L$ from $\underset{\sim}{\eta}^T$. That procedure avoids questions of analyticity raised by the theorem proved in §IVc.

Because the underlying star is spherically symmetric, modes belonging to the same L but different M are degenerate, so it suffices to consider the case $M = 0$. Then the action, from equations (46) and (33), becomes

$$I_2 = \int_{\text{interior}} \left[\frac{1}{\rho} \underset{\sim}{\eta}_{,t} \cdot \underset{\sim}{\eta}_{,t} + 4\pi \underset{\sim}{\eta}^L \cdot \underset{\sim}{\eta}^L - \frac{2}{\rho^2} (\underset{\sim}{\nabla} \cdot \underset{\sim}{\eta})(\underset{\sim}{\eta} \cdot \underset{\sim}{\mathcal{S}}) \right.$$

$$\left. + \frac{1}{\rho^3} (\underset{\sim}{\eta} \cdot \underset{\sim}{\nabla} \rho)(\underset{\sim}{\eta} \cdot \underset{\sim}{\mathcal{S}}) - \frac{p\gamma}{\rho^2} (\underset{\sim}{\nabla} \cdot \underset{\sim}{\eta})^2 \right] dV$$

$$+ \int_{\Sigma^-} \frac{1}{\rho^2} p_{,r} (\eta^r)^2 R^2 \sin\vartheta \, d\vartheta \, d\varphi - \int_{\Sigma^-} \delta\Phi \, \eta^{(T)r} R^2 \sin\vartheta \, d\vartheta \, d\varphi , \quad (56)$$

where R is the radius of the star.

Inspection of I_2 shows that c will enter it only in the $\underset{\sim}{\eta}_{,t} \cdot \underset{\sim}{\eta}_{,t}$ term. This is because c generates the "odd parity" (cf. Regge and Wheeler 1957, Thorne and Campolattaro 1967) part of the perturbation, which is a zero-frequency rotational mode. It does not couple to other modes and does not affect the star's stability.

This Lagrangian is equivalent to the variational principle contained in the appendix to Chandrasekhar and Lebovitz (1964). It is interesting that if one varies it with respect to a one gets the divergence of the dynamical equation for $\underset{\sim}{\eta}$, while if one varies it with respect to b and c one gets the two independent parts of the curl of that equation. Since a vector is zero if and only if its divergence and curl are zero, the Euler-Lagrange equations of a, b, and c are equivalent to that of $\underset{\sim}{\eta}$. Thus, the potentials a, b, and c are also good variables for the variational principle! This presumably also holds for the general variational principle for differentially rotating stars.

The theorem of Laval, et al. (1965) applies to the Lagrangian for the nonrotating star and gives a necessary and sufficient condition for stability against pulsations of order L:

$$\int_0^R \left\{ -4\pi\left(a_{,r}{}^2 + \frac{L(L+1)}{r^2} a^2\right) + \frac{2\mathcal{S}}{\rho^2}(a_{,r} + b)\left(\nabla_r{}^2 a - \frac{L(L+1)}{r^2} a\right) - \frac{1}{\rho^3}\rho_{,r}\mathcal{S}(a_{,r}+b)^2 + \frac{p\gamma}{\rho^2}\left(\nabla_r{}^2 a - \frac{L(L+1)}{r^2} a\right)^2 \right\} r^2\, dr$$
$$+ \frac{R^2}{\rho(R)^2} p_{,r}(R)\left[a_{,r}(R) + b(R)\right]^2 + 4\pi G R^2 a(R) b(R) > 0, \qquad (57)$$

where $\mathcal{S} \equiv \mathcal{S}_r = p_{,r} - (\gamma p/\rho)\rho_{,r}$ and where we have defined the operator

$$\nabla_r^{\,2} \equiv \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} ,$$

which is the radial part of the Laplacian. The terms evaluated at R are to be evaluated just inside the star's surface if there are any discontinuities there.

VI. CONCLUSIONS

We have presented a general method for finding the Lagrangian for arbitrary perturbations of arbitrary flows of a perfect fluid; and we have illustrated the method for the case of differentially rotating stars. It enabled us to reproduce the stability criteria of Lynden-Bell and Ostriker (1967), as well as those obtained by other authors for less general cases.

We also showed that the testing of realistic stellar models with these criteria can be greatly simplified by the introduction of three scalar functions in place of the three components of $\underset{\sim}{\xi}$ in such a manner that one need never perform a Green's function integration to determine the perturbed graviational field. We hope that this will prove to be a useful technique in the future.

In Paper II we will extend these results so far as possible to the general-relativistic case.

I am very grateful to Professor Kip S. Thorne for his continued advice and encouragement, and for his many helpful suggestions during the writing of this paper.

APPENDIX A

REDUCTION OF L_2

We wish to transform L_2 from the form

$$L_2 = \frac{1}{4\pi G} g^{ij} \delta\Phi_{,i} \delta\Phi_{,j} - \delta\rho\, \delta h + \delta(\rho T)\, \delta S + \rho\, \delta \underset{\sim}{v} \cdot \delta \underset{\sim}{v}$$

$$+ 2\rho\, \delta\alpha(\delta\beta_{,t} + v^k \delta\beta_{,k}) - 2\rho\, \delta S(\delta\theta_{,t} + v^k \delta\theta_{,k}) \tag{A1}$$

into an expression involving only the unperturbed state of the fluid and $\underset{\sim}{\xi}$, which is defined as the difference between the position of a fluid element in the perturbed state and the position it would have occupied at exactly the same time in the unperturbed flow. As a first step we will express the perturbations themselves in terms of $\underset{\sim}{\xi}$. Then we will substitute them into equation (A1).

a) <u>Expression of the Eulerian Perturbations in Terms of $\underset{\sim}{\xi}$</u>

As mentioned in §IIb, the perturbations are <u>Eulerian</u> perturbations, taken at fixed coordinate and time. The vector $\underset{\sim}{\xi}$, on the other hand, is the <u>Lagrangian</u> displacement of the fluid. The relations among $\underset{\sim}{\xi}$ and the Eulerian perturbations are well known and need not be derived here. One can consult Lynden-Bell and Ostriker (1967) or Lebovitz (1961). The relevant ones are

$$\delta\rho = - \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{\xi}), \tag{A2}$$

$$\delta S = - \underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} S, \tag{A3}$$

$$\delta p = - \gamma p (\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi}) - \underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} p, \tag{A4}$$

and

$$\delta T = \left(\frac{\partial T}{\partial p}\right)_S \delta p + \left(\frac{\partial T}{\partial S}\right)_p \delta S, \tag{A5}$$

supplemented by the Maxwell identity

$$\left(\frac{\partial T}{\partial p}\right)_S = \frac{1}{\rho p \gamma}\left(\frac{\partial p}{\partial S}\right)_\rho = -\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial S}\right)_p . \tag{A6}$$

In equations (A4) and (A6), γ is the adiabatic index,

$$\gamma = \frac{\rho}{p}\left(\frac{\partial p}{\partial \rho}\right)_S . \tag{A7}$$

Moreover, since $\delta\alpha$ and $\delta\beta$ obey the same equation as δS, we have

$$\delta\alpha = -\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla}\alpha + (\delta\alpha)_o, \tag{A8a}$$

$$\delta\beta = -\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla}\beta + (\delta\beta)_o. \tag{A8b}$$

Here $(\delta\alpha)_o$ and $(\delta\beta)_o$ are "initial values" of $\delta\alpha$ and $\delta\beta$: their values when $\underset{\sim}{\xi} = 0$. They are constants of integration in the following sense:

$$\frac{\partial}{\partial t}(\delta\alpha)_o + \underset{\sim}{v} \cdot \underset{\sim}{\nabla}(\delta\alpha)_o = 0,$$

and similarly for $(\delta\beta)_o$. There were no such initial values in equations (A2) - (A5) because we assume that the perturbation is an initial velocity perturbation that does not affect the initial distribution of ρ, p, and S. This does not restrict the generality of our result: changes in the initial perturbed values of ρ, p, and S are equivalent to changes in the unperturbed ρ, p, and S. Instabilities due to such initial conditions will show up in nearby models whose unperburbed ρ, p, and S are the same as those of the original model plus the initial perturbations.

It is not possible to solve explicitly for $\delta\psi$ and $\delta\theta$. We shall need only the equation for $\delta\theta$:

$$\delta\theta_{,t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla}\delta\theta + \delta\underset{\sim}{v} \cdot \underset{\sim}{\nabla}\theta = \delta T, \tag{A9}$$

where the perturbed velocity, $\delta\underset{\sim}{v}$, is (also from Lynden-Bell and Ostriker [1967])

$$\delta\underset{\sim}{v} = \underset{\sim}{\xi}_{,t} + (\underset{\sim}{v} \cdot \underset{\sim}{\nabla})\underset{\sim}{\xi} - (\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla})\underset{\sim}{v} \tag{A10}$$

$$= \underset{\sim}{\xi}_{,t} + \mathcal{L}_{\underset{\sim}{v}}\underset{\sim}{\xi}. \tag{A11}$$

(Here $\mathcal{L}_{\underset{\sim}{v}}$ is the Lie derivative with respect to $\underset{\sim}{v}$.)

With the definition (A10), equations (A2), (A3), (A8a) and (A8b) are equivalent to the perturbed versions of equations (7), (3d), (3b), and (3c), respectively. The last remaining perturbation is $\delta\Phi$, which has the formal solution

$$\delta\Phi = - G \int dV' \rho(\underset{\sim}{x}')\underset{\sim}{\xi}(x') \cdot \underset{\sim}{\nabla}' \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|} . \tag{A12}$$

b) Expression of L_2 in Terms of $\underset{\sim}{\xi}$

In what follows we will often integrate by parts, using the identity mentioned at the end of §IIa; and we will throw away the resulting divergences, since they become surface integrals at infinity. We will also discard total time derivatives (cf. footnote 3).

It is convenient to treat separately the following pieces of L_2 (eq. [A1]):

$$A \equiv 2\rho\, \delta\alpha(\delta\beta_{,t} + v^k\, \delta\beta_{,k}), \tag{A13a}$$

$$B \equiv -\, 2\rho\, \delta S(\delta\theta_{,t} + v^k\, \delta\theta_{,k}), \tag{A13b}$$

$$C \equiv \rho\, \delta\underset{\sim}{v} \cdot \delta\underset{\sim}{v}, \tag{A13c}$$

$$D \equiv -\, \delta\rho\, \delta h + \delta(\rho T)\, \delta S. \tag{A13d}$$

i) A. By the perturbed version of equation (3c) we have

$$A = -\, 2\rho\, \delta\alpha\, \beta_{,k}\, \delta v^k.$$

This is the only term in L_2 that explicitly contains $(\delta\alpha)_o$ or $(\delta\beta)_o$. Because the equations derived from L_2 are linear in the perturbations, one should not expect initial values to appear in the Lagrangian. One can in fact show explicitly that

$$A' \equiv -\, 2\rho\, (\delta\alpha)_o\, \beta_{,k}\, \delta v^k$$

is zero to within divergences and time derivatives. The procedure is much the same as that which follows, so we won't go into it explicitly. The remainder of A is

$$A'' = +\, 2\rho(\xi^j\alpha_{,j})\beta_{,k}(\xi^k{}_{,t} + \mathcal{L}_{\underset{\sim}{v}}\, \xi^k),$$

$$= 2\rho[\alpha_{[,j}\beta_{,k]} + \alpha_{(,j}\beta_{,k)}]\, \xi^j\xi^k{}_{,t} + 2\rho\, \alpha_{,j}\beta_{,k}\xi^j\mathcal{L}_{\underset{\sim}{v}}\xi^k\, ,$$

$$= 2\rho\alpha_{[,j}\beta_{,k]}\, \xi^j\xi^k{}_{,t} - \rho\frac{\partial}{\partial t}[\alpha_{(,j}\beta_{,k)}]\, \xi^j\xi^k + 2\rho\alpha_{,j}\beta_{,k}\xi^j\mathcal{L}_{\underset{\sim}{v}}\xi^k.$$

This implies

$$A = 2\rho\alpha_{[\,,j}\beta_{,k]}\,\xi^j\xi^k{}_{,t} + \rho\alpha_{(\,,j}\Omega_{,k)}\xi^j\xi^k$$

$$+ 2\rho\alpha_{,j}\beta_{,k}\xi^j \pounds_{\underset{\sim}{v}}\xi^k. \qquad \text{(A14)}$$

Here and throughout, square brackets around indices denote anti-symmetrization, while round brackets denote symmetrization:

$$\alpha_{[\,,j}\beta_{,k]} \equiv \tfrac{1}{2}\left\{\alpha_{,j}\beta_{,k} - \alpha_{,k}\beta_{,j}\right\},$$

$$\alpha_{(\,,j}\beta_{,k)} \equiv \tfrac{1}{2}\left\{\alpha_{,j}\beta_{,k} + \alpha_{,k}\beta_{,j}\right\}.$$

ii) B. Equation (A9) converts B to

$$B = -\,2\rho\ \delta S\ \delta T + 2\rho\ \delta S(\theta_{,k}\delta v^k).$$

The second term can be handled just as A was to give

$$B = -\,2\rho\ \delta S\ \delta T + 2\,\rho\ \theta_{[\,,j}S_{,k]}\,\xi^j\xi^k{}_{,t}\,,$$

$$+ \rho S_{(\,,j}T_{,k)}\xi^j\xi^k - 2\,\rho S_{,j}\theta_{,k}\xi^j\pounds_{\underset{\sim}{v}}\xi^k. \qquad \text{(A15)}$$

iii) A + B. Before adding A and B, consider the term

$$2\rho\alpha_{,j}\beta_{,k}\xi^j\pounds_{\underset{\sim}{v}}\xi^k = 2\rho\alpha_{,j}\beta_{,k}\xi^j(\xi^k{}_{;\ell}\,v^\ell - v^k{}_{;\ell}\,\xi^\ell).$$

Manipulations similar to those in i) convert this to

$$= 2\,\rho\alpha_{[\,,j}\beta_{,k]}\,\xi^j\xi^k{}_{;\ell}\,v^\ell - 2\rho[\,(\alpha_{(,j}\beta_{,k)})_{;\ell}\,v^\ell$$

$$+ 2\alpha_{,j}\beta_{,\ell}\,v^\ell{}_{;k}]\,\xi^j\xi^k,$$

with a similar expression for the Lie-derivative term in B. Then by adding A and B we get

$$A + B = 2\rho\Omega_{kj}\xi^j(\xi^k{}_{,t} + \xi^k{}_{;\ell}v^\ell) - 2\rho\,\delta S\,\delta T$$
$$+ \rho(T_{,k}S_{,j} + \Omega_{,k}\alpha_{,j})\xi^k\xi^j$$
$$- \rho[(\alpha_{,j}\beta_{,k} - \theta_{,j}S_{,k})_{;\ell}v^\ell + 2(\alpha_{,j}\beta_{,\ell} - S_{,j}\theta_{,\ell})v^\ell{}_{;k}]\xi^j\xi^k \tag{A16}$$

where we have introduced the vorticity tensor (not to be confused with the angular velocity)

$$\Omega_{kj} \equiv v_{[k,j]} = \alpha_{[\,,j}\beta_{,k]} + \theta_{[\,,j}S_{,k]}\,. \tag{A17}$$

Finally, extensive manipulation of the last bracketed term in equation (A16) gives

$$A + B = 2\rho\Omega_{kj}\xi^j(\xi^k{}_{,t} + \xi^k{}_{;\ell}v^\ell) - 2\rho\,\delta S\,\delta T$$
$$+ \rho\,T_{,k}S_{,j}\xi^k\xi^j - 2\rho\Omega_{\ell j}v^\ell{}_{;k}\xi^j\xi^k. \tag{A18}$$

iv) C. From equation (A11) we have

$$\rho\,\delta\underset{\sim}{v}\cdot\delta\underset{\sim}{v} = \rho g_{jk}(\xi^j{}_{,t} + \mathcal{L}_{\underset{\sim}{v}}\xi^j)(\xi^k{}_{,t} + \mathcal{L}_{\underset{\sim}{v}}\xi^k)$$
$$= \rho\underset{\sim}{\xi}_{,t}\cdot\underset{\sim}{\xi}_{,t} + 2\rho g_{jk}\xi^j{}_{,t}(\xi^k{}_{;\ell}v^\ell - v^k{}_{;\ell}\xi^\ell)$$
$$+ \rho g_{jk}(\xi^j{}_{;\ell}v^\ell - v^j{}_{;\ell}\xi^\ell)(\xi^k{}_{;m}v^m - v^k{}_{;m}\xi^m).$$

We treat the last two terms one-by-one:

$$2\rho g_{jk}\xi^j{}_{,t}(\xi^k{}_{;\ell}v^\ell - v^k{}_{;\ell}\xi^\ell)$$

$$= 2\rho g_{jk}\xi^k{}_{,t}\xi^k{}_{;\ell}v^\ell - 2\rho\Omega_{kj}\xi^j\xi^k{}_{,t} \, ;$$

$$\rho g_{jk}(\xi^j{}_{;\ell}v^\ell - v^j{}_{;\ell}\xi^\ell)(\xi^k{}_{;m}v^m - v^k{}_{;m}\xi^m)$$

$$= \rho g_{jk}v^\ell v^m \xi^j{}_{;\ell}\xi^k{}_{;m} - 2\rho\Omega_{kj}\xi^j\xi^k{}_{;\ell}v^\ell$$

$$+ \rho v_{k;j;\ell}v^\ell\xi^k\xi^j + \rho v_{\ell;j}v^\ell{}_{;k}\xi^j\xi^k.$$

Assembling terms, we get

$$\rho\, \delta\underset{\sim}{v} \cdot \delta\underset{\sim}{v} = \rho\underset{\sim}{\xi}_{,t} \cdot \underset{\sim}{\xi}_{,t} - 2\rho\Omega_{kj}\xi^j(\xi^k{}_{,t} + \xi^k{}_{;\ell}v^\ell)$$

$$+ 2\rho g_{jk}\xi^j{}_{,t}\xi^k{}_{;\ell}v^\ell + \rho g_{jk}v^\ell v^m\xi^j{}_{;\ell}\xi^k{}_{;m}$$

$$+ \rho(v_{k;j;\ell}v^\ell + v_{\ell;j}v^\ell{}_{;k})\xi^j\xi^k \, . \qquad \text{(A19)}$$

v) Adding C to A + B gives

$$A + B + C = -2\rho\, \delta S\, \delta T + \rho T_{,k}S_{,j}\xi^k\xi^j + \rho\underset{\sim}{\xi}_{,t} \cdot \underset{\sim}{\xi}_{,t}$$

$$+ 2\rho g_{jk}\xi^j{}_{,t}\xi^k{}_{;\ell}v^\ell + \rho g_{jk}v^\ell v^m\xi^j{}_{;\ell}\xi^k{}_{;m}$$

$$+ \rho(v_{k;\ell}v^\ell)_{;j}\xi^j\xi^k. \qquad \text{(A20)}$$

In spherical polar coordinates, part of the last term becomes

$$v_{k;\ell}v^{\ell} = v_{k,\ell}v^{\ell} - \Gamma^{i}_{k\ell}v^{\ell}v_{i}$$

$$= -\tfrac{1}{2}\, v^{i}v^{\ell} g_{i\ell,k} = -\tfrac{1}{2}\Omega^{2} g_{\varphi\varphi,k}$$

$$= -\tfrac{1}{2}\,\Omega^{2}(\alpha/\Omega)_{,k}$$

$$= \tfrac{1}{2}(\alpha\Omega_{,k} - \Omega\alpha_{,k}).$$

If we differentiate this with respect to j and symmetrize on j and k, we obtain

$$\rho(v_{k;\ell}v^{\ell})_{;j}\,\xi^{j}\xi^{k} = \rho[\tfrac{1}{2}\,\alpha\Omega_{,k;j} - \tfrac{1}{2}\,\Omega\alpha_{,k;j}]\,\xi^{j}\xi^{k}. \tag{A21}$$

vi) D. We add to D two thermodynamic terms from equation (A20) and define

$$E = -\,\delta\rho\;\delta h + \delta(\rho T)\;\delta S - 2\rho\;\delta S\;\delta T + \rho\,T_{,k}S_{,j}\xi^{j}\xi^{k},$$

$$= -\,\frac{1}{\rho}\,\delta\rho\;\delta p - \rho\;\delta T\;\delta S + \rho\;T_{,k}S_{,j}\xi^{j}\xi^{k}. \tag{A22}$$

Upon using equations (A2) through (A7), we find that this reduces to

$$E = -\,\gamma p(\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi})^{2} + \rho^{-1}(\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla}\rho)(\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla}p)$$

$$-\,2(\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi})(\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla}p). \tag{A23}$$

vii) The complete Lagrangian is obtained by substituting equations (A20), (A21), and (A23) into (A1). Equation (27) is the result.

APPENDIX B

TESTING FOR STABILITY

a) The Stability Criterion in Terms of the Scalars a, b, c.

From the definitions of the scalar trial functions, equations (49) and (50), we find (sum on $M \geq 0$ implied)

$$\underset{\sim}{\eta}^L = (a^+_{M,r} \sin M\varphi + a^-_{M,r} \cos M\varphi) \underset{\sim}{e}_{\hat{r}}$$

$$+ (1/r)(a^+_{M,\vartheta} \sin M\varphi + a^-_{M,\vartheta} \cos M\varphi) \underset{\sim}{e}_{\hat{\vartheta}}$$

$$+ (M/r \sin\vartheta)(a^+_M \cos M\varphi - a^-_M \sin M\varphi) \underset{\sim}{e}_{\hat{\varphi}} \quad ; \qquad \text{(B1a)}$$

$$\underset{\sim}{A} = - r^2 (c^+_M \sin M\varphi + c^-_M \cos M\varphi) \underset{\sim}{e}_{\hat{r}}$$

$$+ (rM/\sin\vartheta)(b^+_M \cos M\varphi - b^-_M \sin M\varphi) \underset{\sim}{e}_{\hat{\vartheta}}$$

$$- r(b^+_{M,\vartheta} \sin M\varphi + b^-_{M,\vartheta} \cos M\varphi) \underset{\sim}{e}_{\hat{\varphi}} \quad ; \qquad \text{(B1b)}$$

$$\underset{\sim}{\eta}^T = \Big[\Big(\frac{M^2}{\sin^2\vartheta} b^+_M - \frac{1}{\sin\vartheta} \frac{\partial}{\partial\vartheta} \sin\vartheta\, b^+_{M,\vartheta}\Big) \sin M\varphi$$

$$+ \Big(\frac{M^2}{\sin^2\vartheta} b^-_M - \frac{1}{\sin\vartheta} \frac{\partial}{\partial\vartheta} \sin\vartheta\, b^-_{M,\vartheta}\Big) \cos M\varphi\Big] \underset{\sim}{e}_{\hat{r}}$$

$$+ \Big[\Big(\frac{1}{r} \frac{\partial}{\partial r} r^2 b^+_{M,\vartheta} + \frac{rM}{\sin\vartheta} c^-_M\Big) \sin M\varphi$$

$$+ \Big(\frac{1}{r} \frac{\partial}{\partial r} r^2 b^-_{M,\vartheta} - \frac{rM}{\sin\vartheta} c^+_M\Big) \cos M\varphi\Big] \underset{\sim}{e}_{\hat{\vartheta}}$$

$$+ \Big[\Big(r c^+_{M,\vartheta} - \frac{M}{r\sin\vartheta} \frac{\partial}{\partial r} r^2 b^-_M\Big) \sin M\varphi$$

$$+ \Big(r c^-_{M,\vartheta} + \frac{M}{r\sin\vartheta} \frac{\partial}{\partial r} r^2 b^+_M\Big) \cos M\varphi\Big] \underset{\sim}{e}_{\hat{\varphi}} \,. \qquad \text{(B1c)}$$

Here $\underset{\sim}{e}_{\hat{r}}$, $\underset{\sim}{e}_{\hat{\vartheta}}$, $\underset{\sim}{e}_{\hat{\varphi}}$ are <u>unit</u> vectors.

When these expressions are used in the stability criterion, equation (43), and the integration on φ is performed, modes corresponding to different values of M separate, and we get a separate criterion for each M. In what follows we will accordingly drop the subscript M on the scalars. We will also adopt the notation

$$(a^2 + bc)^{(+-)} \equiv (a^+)^2 + (a^-)^2 + b^+c^+ + b^-c^-, \tag{B2a}$$

$$(ab)^{[+-]} \equiv a^+b^- - a^-b^+ . \tag{B2b}$$

Note that these are <u>not</u> the conventional symmetry and antisymmetry symbols: the plus and minus modes are antisymmetrically coupled by the [+-] operation, but they are not coupled at all by the (+-) operation.

A long but straightforward calculation reduces the stability criterion, equations (43), (44), and (45), to this form: A sufficient condition for stability of the modes of order M is that for any a, b, c (with appropriate boundary conditions -- see below)

$$- \iint\limits_{\text{interior}} C_M r^2 \sin\vartheta \, dr \, d\vartheta - \int\limits_{\Sigma^-} D_M{}^i n_i \, d\sigma > 0 \tag{B3}$$

where in both terms <u>we have already integrated on φ</u>, and where C_M and $D_M{}^i$ are:

$$
\begin{aligned}
-C_M = \Big\{ & -4\pi G\,[a_{,r}{}^2 + \frac{1}{r^2}a_{,\vartheta}{}^2 + \frac{M^2}{r^2\sin^2\vartheta}\,a^2] \\
& + \frac{2}{\rho}(a_{,r} - L^2 b)\,[\,2M^2\Omega^2\sin\vartheta(a + \frac{\partial}{\partial r}r^2 b) + g_r\nabla^2 a] \\
& + \frac{2}{\rho}(\frac{1}{r}a_{,\vartheta} + \frac{1}{r}\frac{\partial}{\partial r}r^2 b_{,\vartheta})[\,M^2\Omega^2\sin 2\vartheta(a + \frac{\partial}{\partial r}r^2 b) + \frac{1}{r}g_\vartheta\nabla^2 a] \\
& + \frac{2}{\rho}\,2M^2\Omega^2 r^3\cos\vartheta\, cc_{,\vartheta} - \frac{1}{\rho}\frac{\Omega^2 M^4}{r^2\sin^2\vartheta}(a + \frac{\partial}{\partial r}r^2 b)^2 + \frac{\gamma p}{\rho^2}(\nabla^2 a)^2 \\
& - \frac{1}{\rho}[\Omega^2 M^2 - 2\Omega\Omega_{,r}r\sin^2\vartheta + \Omega^2\cos^2\vartheta + \frac{1}{\rho^2}\rho_{,r}g_r](a_{,r} - L^2 b)^2 \\
& - \frac{1}{\rho r}[-\Omega\Omega_{,r}r\sin 2\vartheta - 2\Omega\Omega_{,\vartheta}\sin^2\vartheta \\
& \qquad + \frac{1}{r\rho^2}(\rho_{,\vartheta}g_r + \rho_{,r}g_\vartheta)](a_{,r} - L^2 b)(a_{,\vartheta} + \frac{\partial}{\partial r}r^2 b_{,\vartheta}) \\
& - \frac{1}{\rho}[\,\Omega^2 M^2 - \Omega\Omega_{,\vartheta}\sin 2\vartheta + \frac{1}{\rho^2 r^2}\rho_{,\vartheta}g_\vartheta] \\
& \qquad \times[\,(\frac{1}{r}a_{,\vartheta} + \frac{1}{r}\frac{\partial}{\partial r}r^2 b_{,\vartheta})^2 + r^2\frac{M^2}{\sin^2\vartheta}c^2]\Big\}^{(+-)} \\
+ \Big\{ & \frac{4}{\rho}Mr^2\Omega^2\sin^2\vartheta(a_{,r} - L^2 b)c_{,\vartheta} + \frac{2M}{\rho\sin\vartheta}g_\vartheta(\nabla^2 a)c \\
& - \frac{\Omega^2 M^2}{\rho}(\frac{M}{r\sin\vartheta}a + rc_{,\vartheta})^2 \\
& - \frac{rM}{\rho\sin\vartheta}[-\Omega\Omega_{,r}r\sin 2\vartheta - 2\Omega\Omega_{,\vartheta}\sin^2\vartheta + \frac{1}{r\rho^2}(\rho_{,\vartheta}g_r \\
& \qquad + \rho_{,r}g_\vartheta)](a_{,r} - L^2 b)c \\
& - \frac{2M}{\rho\sin\vartheta}[\,\Omega^2 M^2 - \Omega\Omega_{,\vartheta}\sin 2\vartheta \\
& \qquad + \frac{1}{r^2\rho^2}\rho_{,\vartheta}g_\vartheta](a_{,\vartheta} + \frac{\partial}{\partial r}r^2 b_{,\vartheta})c\Big\}^{[+-]}; \qquad \text{(B4)}
\end{aligned}
$$

$$-D_M{}^r = \left\{4\pi G a L^2 b - \frac{1}{\rho^2} p_{,r}(a_{,r} - L^2 b)^2 - \frac{1}{r\rho^2} p_{,\vartheta}(a_{,r} - L^2 b)\left(\frac{1}{r} a_{,\vartheta} + \frac{1}{r}\frac{\partial}{\partial r} r^2 b_{,\vartheta}\right)\right\}^{(+-)} + \left\{-\frac{M}{\rho^2 \sin\vartheta} p_{,\vartheta}(a_{,r} - L^2 b)c\right\}^{[+-]} ; \qquad \text{(B5)}$$

and

$$-D_M{}^\vartheta = \left\{-4\pi G \frac{1}{r^2} a \frac{\partial}{\partial r} r^2 b_{,\vartheta} - \frac{1}{r\rho^2} p_{,r}(a_{,r} - L^2 b)\left(\frac{1}{r} a_{,\vartheta} + \frac{1}{r}\frac{\partial}{\partial r} r^2 b_{,\vartheta}\right) - \frac{1}{r^2\rho^2} p_{,\vartheta}\left(\frac{1}{r} a_{,\vartheta} + \frac{1}{r}\frac{\partial}{\partial r} r^2 b_{,\vartheta}\right)^2 - \frac{M^2}{\rho^2 \sin^2\vartheta} p_{,\vartheta} c^2\right\}^{(+-)}$$
$$+ \left\{-4\pi G \frac{M}{\sin\vartheta} ac - \frac{M}{\rho^2 \sin\vartheta} p_{,r}(a_{,r} - L^2 b)c - \frac{2M}{r\rho^2 \sin\vartheta} p_{,\vartheta}\left(\frac{1}{r} a_{,\vartheta} + \frac{1}{r}\frac{\partial}{\partial r} r^2 b_{,\vartheta}\right)c\right\}^{[+-]} ; \qquad \text{(B6)}$$

and where we have used the notation $L^2 \equiv \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}$ $= \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial}{\partial\vartheta} - \frac{M^2}{\sin^2\vartheta}$; $\mathcal{S}_r = p_{,r} - \frac{\gamma p}{\rho}\rho_{,r}$; $\mathcal{S}_\vartheta = p_{,\vartheta} - \frac{\gamma p}{\rho}\rho_{,\vartheta}$.

While this expression is complicated, it should be reasonably adaptable to computer calculations.

b) Boundary Conditions on a, b, c

Though we have not restricted the perturbation $\underset{\sim}{\xi}$ to have any particular value at the star's surface, there are nevertheless some weak boundary conditions on a, b, and c that arise from the vanishing of $\underset{\sim}{\eta}$ (and $\underset{\sim}{\xi}$) at the star's center and from the vanishing of $\underset{\sim}{\eta}$ (but not $\underset{\sim}{\xi}$) at the star's surface (Σ^-) if ρ vanishes there.

The demand that $\underset{\sim}{\eta}$ vanish at the star's center requires that $\underset{\sim}{\eta}^L = -\underset{\sim}{\eta}^T$, but not that each vanish separately. This implies

$$\left.\begin{aligned} a^+_{,r} &= L^2 b^+ , \\ a^+_{,\vartheta} &= -\frac{\partial}{\partial r} r^2 b^+_{,\vartheta} - \frac{r^2 M}{\sin\vartheta} c^- , \\ a^+ &= -\frac{\partial}{\partial r} r^2 b^+ - \frac{r^2 \sin\vartheta}{M} c^-_{,\vartheta} , \end{aligned}\right\} \text{at the star's center} \tag{B7}$$

plus the conjugate equations (plus and minus interchanged).

At the surface of the star (actually at Σ^-, where the surface integral is evaluated) we demand only that a, b, and c be finite with finite derivatives, except if $\rho = 0$ there. Then again we must have $\underset{\sim}{\eta}^L = -\underset{\sim}{\eta}^T + O(\rho)$; that is, $\underset{\sim}{\eta}$ must vanish at least as fast as ρ near Σ^-. So the same equations (B7) must hold at the surface, to order ρ. (This is also true, of course, anywhere else that ρ vanishes.)

c) Eigenfrequencies of Stable Modes

If condition (B3) is satisfied, the star is stable. In that case the eigenfrequencies of oscillation are the stationary values of the roots of the following quadratic expression (cf. Lynden-Bell and Ostriker 1967):

$$\omega^2 V_1 + \omega V_2 + V_3 = 0, \tag{B8}$$

with

$$V_1 = \frac{1}{\pi} \int\limits_{\text{interior}} \frac{1}{\rho} \underset{\sim}{\eta} \cdot \underset{\sim}{\eta}\, r^2 \sin\vartheta \, dr\, d\vartheta\, d\varphi$$

$$= \iint\limits_{\text{interior}} \frac{1}{\rho} r^2 \sin\vartheta\, dr\, d\vartheta \Big\{ (a_{,r} - L^2 b)^2 + \Big(\frac{1}{r} a_{,\vartheta} + \frac{1}{r}\frac{\partial}{\partial r} r^2 b_{,\vartheta}\Big)^2$$

$$+ \frac{r^2 M^2}{\sin^2\vartheta} c^2 + \frac{M^2}{r^2 \sin^2\vartheta}\Big(a + \frac{\partial}{\partial r} r^2 b\Big)^2 \Big\}^{(+-)}$$

$$+ \iint\limits_{\text{interior}} \frac{1}{\rho} r^2 \sin\vartheta\, dr\, d\vartheta \Big\{ \frac{rM}{\sin\vartheta}(a_{,r} - L^2 b) c$$

$$+ \Big(\frac{M}{r\sin\vartheta} a + r c_{,\vartheta}\Big)^2 \Big\}^{[+-]} ; \tag{B9}$$

$$V_2 = \frac{1}{\pi} \int\limits_{\text{interior}} \frac{2\Omega}{\rho} \Big[g_{jk} \eta^j \eta^k{}_{,\varphi} + \Big(\frac{\alpha}{\Omega}\Big)_{,j} \eta^j \eta^\varphi \Big] r^2 \sin\vartheta\, dr\, d\vartheta\, d\varphi$$

$$= \iint\limits_{\text{interior}} \frac{2\Omega}{\rho} r^2 \sin\vartheta\, dr\, d\vartheta \Big\{ \frac{2M^2}{\sin\vartheta}\Big(a_{,\vartheta} + \frac{\partial}{\partial r} r^2 b_{,\vartheta}\Big) c$$

$$+ 2r \sin\vartheta\, (a_{,r} - L^2 b) c_{,\vartheta} + 2\cos\vartheta \Big(a_{,\vartheta} + \frac{\partial}{\partial r} r^2 b_{,\vartheta}\Big) c_{,\vartheta}$$

$$- \frac{2M^2 \cos\vartheta}{\sin^2\vartheta}\Big(a + \frac{\partial}{\partial r} r^2 b\Big) c \Big\}^{(+-)}$$

$$+ \iint\limits_{\text{interior}} \frac{2\Omega}{\rho} r^2 \sin\vartheta\, dr\, d\vartheta \Big\{ (a_{,r} - L^2 b)^2 + \frac{1}{r^2}\Big(a_{,\vartheta} + \frac{\partial}{\partial r} r^2 b_{,\vartheta}\Big)^2$$

$$- \frac{M^2}{r^2 \sin^2\vartheta}\Big(a + \frac{\partial}{\partial r} r^2 b\Big)^2 + r^2 c_{,\vartheta}{}^2 - \frac{2}{r}(a_{,r} - L^2 b)\Big(a + \frac{\partial}{\partial r} r^2 b\Big)$$

$$- \frac{\cos\vartheta}{r^2 \sin\vartheta}\Big(a_{,\vartheta} + \frac{\partial}{\partial r} r^2 b_{,\vartheta}\Big)\Big(a + \frac{\partial}{\partial r} r^2 b\Big)$$

$$+ \frac{r^2 \cos\vartheta}{\sin\vartheta} c_{,\vartheta} c \Big\}^{[+-]} ; \tag{B10}$$

and

$$V_3 = \iint\limits_{\text{interior}} C_M r^2 \sin\vartheta \, dr \, d\vartheta + \int\limits_{\Sigma^-} D_M^{\,i} n_i \, d\sigma, \tag{B11}$$

where C_M and $D_M^{\,i}$ are given by equations (B4), (B5), and (B6). Thus the trial functions permit estimation of eigenfrequencies for the stable case. Unfortunately one cannot estimate e-folding times for the unstable modes in this manner (see Lynden-Bell and Ostriker 1967).

APPENDIX C

STABILITY OF AXIALLY SYMMETRIC PERTURBATIONS

The necessary condition for stability of axially symmetric perturbations obtained by Chandrasekhar and Lebovitz (1968) and by Lynden-Bell and Ostriker (1967) can be expressed in terms of scalars with the method of Appendix B.

The condition is (eq. [54])

$$\int_{\text{interior}} \left\{ \frac{\Omega^2}{\rho\, r^2 \sin^2\vartheta} [(r^2 \sin^2\vartheta)_{,j} \eta^j]^2 - C[\underset{\sim}{\eta}, \underset{\sim}{\eta}] \right\} dV$$

$$- \int_{\Sigma^-} \underset{\sim}{D}[\underset{\sim}{\eta}, \underset{\sim}{\eta}] \cdot \underset{\sim}{n}\, d\sigma > 0. \qquad \text{(C1)}$$

We can obtain C and $\underset{\sim}{D}$ from Appendix B by setting $M = 0$ in equations (B4), (B5), and (B6). We can then expand the first term in inequality (C1) in terms of a, b, and c, and add it to C. The result is that a necessary condition for stability of a differentially rotating star against axially symmetric perturbations is that, for all a, b, c satisfying the boundary conditions described in Appendix B,

$$- \iint_{\text{interior}} C_A r^2 \sin\vartheta\, dr\, d\vartheta - \int_{\Sigma^-} D_A{}^i n_i\, d\sigma > 0, \qquad \text{(C2)}$$

where

$$-C_A = -4\pi G\left(a_{,r}{}^2 + \frac{1}{2} a_{,\vartheta}{}^2\right) + \frac{2\mathcal{S}_r}{\rho}(a_{,r} - L^2 b)\nabla^2 a$$

$$+ \frac{2\mathcal{S}_\vartheta}{\rho r^2}\left(a_{,\vartheta} + \frac{\partial}{\partial r} r^2 b_{,\vartheta}\right)\nabla^2 a + \frac{1}{\rho^2}(\nabla^2 a)^2$$

$$- \frac{1}{\rho}\left[-2\Omega\Omega_{,r}\, r \sin^2\vartheta + \Omega^2(1 + 3\sin^2\vartheta) + \frac{1}{\rho^2}\rho_{,r}\mathcal{S}_r\right](a_{,r} - L^2 b)^2$$

$$- \frac{1}{\rho r}\left(-\Omega\Omega_{,r}\, r \sin 2\vartheta - 2\Omega\Omega_{,\vartheta}\sin^2\vartheta + 4\Omega^2 \sin 2\vartheta\right)$$

$$\times \left(a_{,\vartheta} + \frac{\partial}{\partial r} r^2 b_{,\vartheta}\right)(a_{,r} - L^2 b)$$

$$- \frac{1}{\rho r^2}\left(-\Omega\Omega_{,\vartheta}\sin 2\vartheta + 4\Omega^2\cos^2\vartheta + \frac{1}{\rho^2 r^2}\rho_{,\vartheta}\mathcal{S}_\vartheta\right)\left(a_{,\vartheta} + \frac{\partial}{\partial r} r^2 b_{,\vartheta}\right)^2; \qquad \text{(C3)}$$

$$-D_A{}^r = 4\pi G a L^2 b - \frac{1}{\rho^2} p_{,r}(a_{,r} - L^2 b)^2$$

$$- \frac{1}{\rho^2 r^2} p_{,\vartheta}(a_{,r} - L^2 b)\left(a_{,\vartheta} + \frac{\partial}{\partial r} r^2 b_{,\vartheta}\right); \qquad \text{(C4)}$$

and

$$-D_A{}^\vartheta = -\frac{4\pi G}{r^2} a \frac{\partial}{\partial r} r^2 b_{,\vartheta} - \frac{1}{\rho^2 r^2} p_{,r}(a_{,r} - L^2 b)\left(a_{,\vartheta} + \frac{\partial}{\partial r} r^2 b_{,\vartheta}\right)$$

$$- \frac{1}{\rho^2 r^2} p_{,\vartheta}\left(\frac{1}{r} a_{,\vartheta} + \frac{1}{r}\frac{\partial}{\partial r} r^2 b_{,\vartheta}\right)^2. \qquad \text{(C5)}$$

REFERENCES

Chandrasekhar, S. 1964, Ap. J., **139**, 664.

Chandrasekhar, S., and Lebovitz, N. R. 1964, Ap. J., **140**, 1517.

__________. 1968, Ap. J., **152**, 267.

Clement, M. 1964, Ap. J., **140**, 1045.

Cotsaftis, M. 1968, J. Inst. Maths. Applics., **5**, 19.

Courant, R. and Hilbert, D. 1953, Methods of Mathematical Physics, Vol. 1 (New York: Interscience).

__________. 1962, Methods of Mathematical Physics, Vol. 2 (New York: Interscience).

Kulsrud, R. M. 1968, Ap. J., **152**, 1121.

Laval, G., Mercier, C., and Pellat, R. 1965, Nucl. Fusion, **5**, 156.

Lebovitz, N. R. 1961, Ap. J., **134**, 500.

La Salle, J., and Lefschetz, S. 1961, Stability of Liapunov's Direct Method (New York: Academic Press).

Lynden-Bell, D., and Ostriker, J. P. 1967, M.N.R.A.S., **136**, 293.

Phillips, H. B. 1933, Vector Analysis (New York: Wiley).

Regge, T., and Wheeler, J. A. 1957, Phys. Rev., **108**, 1063.

Schmid, L. A. 1970a, in A Critical Review of Thermodynamics, ed. E. B. Stuart, B. Gal-Or, and A. J. Brainard (Baltimore: Mono).

__________. 1970b, Pure and Appl. Chem., **22**, 493.

Schutz, B. F. 1970, Phys. Rev. D., **2**, 2762 (Chapter 2 of thesis).

__________. 1971a, following paper (Paper II) (Chapter 5 of thesis).

__________. 1971b, "The Hamiltonian Theory of a Relativistic Perfect Fluid," paper in preparation (Chapter 3 of this thesis).

Seliger, R. L., and Whitham, G. B. 1968, Proc. Roy. Soc. A, **305**, 1.

Taub, A. H. 1969, Commun. Math. Phys., **15**, 235.

Thorne, K. S., and Campolattaro, A. 1967, Ap. J., **149**, 591.

CHAPTER 5

LINEAR PULSATIONS AND STABILITY OF DIFFERENTIALLY ROTATING STELLAR MODELS. II. GENERAL RELATIVISTIC ANALYSIS

Submitted to the Astrophysical Journal

ABSTRACT

The author has previously given a velocity-potential variational principle for relativistic perfect-fluid hydrodynamics. The second variation of the principle is here used as the Lagrangian density for the evolution of small perturbations of fully relativistic, differentially rotating stellar models. Noether's theorem is used to construct a globally conserved angular momentum density, whose integral over a spacelike hypersurface is the second-order correction to be the star's total angular momentum. From the Hamiltonian is constructed a globally conserved energy density, whose integral is the second-order correction to the star's active gravitational mass. By Lyapunov's second theorem, positive-definiteness of the energy density guarantees stability of the star. In the Newtonian limit and in the special case of relativistic radial pulsations, this is equivalent to stability criteria already known. Means are discussed whereby the general criterion might be made more suitable for practical applications.

I. INTRODUCTION AND SUMMARY

The importance of general relativity to so many astrophysical problems makes an analysis of the stability of relativistic systems very desirable. In the Newtonian regime the theory of the stability of perfect fluid stellar models against small dynamical perturbations is well established [cf. Schutz (1971a), preceding paper, hereafter referred to as Paper I; see also the references cited therein]. The corresponding relativistic analysis, however, is complicated by two factors: the existence of ten components of the gravitational field, and the emission of gravitational radiation by the pulsating star.

Only for radial pulsations of spherical systems has a fully relativistic dynamical stability analysis been performed: by Chandrasekhar (1964) for relativistic stars; and by Ipser and Thorne (1968), Ipser (1969), and Fackerell (1970) for relativistic clusters of stars. In addition, Chandrasekhar (1965a,b) has analyzed the nonradial pulsations of stars in the post-Newtonian approximation, which excludes gravitational radiation. Chandrasekhar and Friedman (1971) have also recently investigated criteria for the existence of zero-frequency modes in rigidly rotating stars, where radiation is also negligible. Their work should prove useful in determining the stability of stars that become unstable through zero-frequency oscillations. The equations governing arbitrary nonradial pulsations of fully relativistic nonrotating stars were derived by Thorne and Campolattaro (1967) [see also Ipser and Thorne (1971)]. They are so complicated, however, that -- although they have yielded information about convection (Islam 1970) and about the emission of and damping by gravitational radiation (Thorne 1969, Ipser

1971) -- they have so far given us no information about dynamical stability.

The existence of ten perturbed metric functions rather than just one perturbed gravitational potential is an algebraic complication. It means that in general there will be many coupled equations, which will rarely possess a solution in closed form. It means that relativistic stability analyses will probably have to rely more heavily upon numerical calculations than the corresponding Newtonian analyses do.

The complication of gravitational radiation is more fundamental. It means that realistic pulsations will always have complex frequencies; that normal modes will be replaced by "resonances" of finite width; that self-adjoint equations (standing-wave boundary conditions) will not describe realistic systems; and that a single stability criterion that is both necessary and sufficient is probably not to be hoped for. It is possible to look for necessary conditions for stability by examining standing-wave modes in the zero-frequency limit. This is the approach of Chandrasekhar and Friedman (1971). But such approaches neglect gravitational radiation damping, so they may not pinpoint the onset of instability accurately. It is therefore useful to have sufficient conditions for stability as well.

In Paper I we showed that all known Newtonian dynamical stability criteria could be derived from the velocity-potential variational principle of Seliger and Whitham (1968). That variational principle can be extended to general relativity [Schutz (1970); see also Schmid (1970a,b) for an independent derivation of the special relativistic version]. In this paper we show that methods similar to those we used in Paper I lead us in general relativity to a sufficient condition for the stability of arbitrary pulsations of fully relativistic, differentially rotating stellar models.

We could presumably also derive our criterion from the variational principle of Taub (1954), or from any of the many other relativistic perfect-fluid variational principles. Taub (1969) in fact derived Chandrasekhar's (1964) stability criterion for radial pulsations using a method very similar to the one we use here, but starting from a different variational principle. We have elected to start with the velocity-potential variational principle because it is an Eulerian principle: it does not require us to deal explicitly with "fluid elements" or "particle paths."

The plan of the paper is as follows. In §II we derive the Lagrangian governing arbitrary perturbations of arbitrary flows of a relativistic perfect fluid. This Lagrangian is the second variation of the Lagrangian for the velocity-potential variational principle of Schutz (1970). In §III we specialize the unperturbed state to that of an axially symmetric, differentially rotating star. From Noether's theorem we construct the conserved angular momentum density of the perturbations (including the gravitational waves), and from the Hamiltonian we construct the conserved energy density. Both are quadratic in the perturbations.

We obtain the following results: (i) The total angular momentum and energy (integrals of the densities over a spacelike hypersurface of the unperturbed spacetime) are unique and gauge-invariant. (ii) If the star is stable, and if the "unperturbed" star is defined to be the star that is left behind after the pulsations have damped out, then all first-order contributions to the total angular momentum and energy vanish. (iii) If the star is stable, the total angular momentum and energy are the second-order corrections to the total angular momentum and active gravitational mass of the star. (iv) The gravitational wave parts of the densities of energy and angular momentum become, in the short-wavelength approximation, the appropriate components of the Isaacson (1968) stress-energy tensor for gravitational radiation. (v) In the case of the nonrotating unperturbed star, the energy density reduces in the Newtonian limit to the energy density derived in Paper I.

In §IV we prove that a sufficient condition for stability is that the total energy be positive-definite. Unfortunately, as the energy contains contributions from gravitational radiation, it is not yet in its most practical form for astrophysical applications. A more practical form would be an integral of purely fluid quantities over just the star's interior. We therefore discuss what procedures are most likely to succeed in reducing the stability criterion to such a form. We conclude §IV by demonstrating that our sufficient condition for stability reduces for the case of radial pulsations to the necessary and sufficient condition of Chandrasekhar (1964).

II. PERTURBATIONS OF AN ARBITRARY FLOW

a) The Velocity-Potential Variational Principle

As in Paper I we begin from the Eulerian velocity-potential variational principle, the general-relativistic version of which was obtained by Schutz (1970). [We follow the notation and conventions of Schutz

(1970) throughout. In particular, Greek indices run from 0 to 3, while Latin run from 1 to 3. The metric signature is +2.]

The four-velocity has the representation

$$U_\nu = \mu^{-1} (\psi_{,\nu} + \alpha\beta_{,\nu} - S\theta_{,\nu}). \tag{1}$$

[We find it convenient to deal with $\psi \equiv \phi + \theta S$ rather than with ϕ, which was used by Schutz (1970). This is the only way in which our conventions differ from those of that paper.] In equation (1), S is the specific entropy and μ the specific enthalpy (including rest mass),

$$\mu = 1 + \Pi + p/\rho_0 = (\rho + p)/\rho_0 \; ; \tag{2}$$

Π is the specific internal energy, p the pressure, ρ the density of total mass-energy, and ρ_0 the rest-mass density (number density of baryons times rest mass of one baryon), all as measured in a locally comoving inertial frame.

The velocity potentials obey the equations of evolution

$$U^\nu \psi_{,\nu} = -\mu + TS \;, \tag{3a}$$

$$U^\nu \alpha_{,\nu} = 0 \;, \tag{3b}$$

$$U^\nu \beta_{,\nu} = 0 \;, \tag{3c}$$

$$U^\nu S_{,\nu} = 0 \;, \tag{3d}$$

$$U^\nu \theta_{,\nu} = T \;, \tag{3e}$$

where T is the temperature. Note that equations (1), (3a), (3c), and

(3e) imply

$$U^{\nu} U_{\nu} = -1 \quad . \tag{4}$$

Supplemented by an equation of state,

$$p = p(\mu, S) \quad , \tag{5}$$

and the equation of continuity,

$$(\rho_0 U^{\nu})_{;\nu} = 0 \ , \tag{6}$$

equations (1) and (3) are completely equivalent to the usual hydrodynamical equations: equations (4), (5), (6), and

$$T^{\mu\nu}{}_{;\nu} = 0 \quad , \tag{7}$$

with

$$T^{\mu\nu} = \rho_0 \mu \ U^{\mu} U^{\nu} + p g^{\mu\nu} \ . \tag{8}$$

Equations (3) and (6) plus the Einstein field equations follow from a variational principle whose action is

$$I = \int (R + 16\pi p)(-g)^{\frac{1}{2}} \, d^4x \ , \tag{9}$$

where R is the scalar curvature of spacetime (we set $c = G = 1$). The curvature is varied with respect to $g^{\sigma\nu}$ in the usual manner. The pressure is taken to be a function of μ and S through the equation of state; its variation is found from the first law of thermodynamics:

$$dp = \rho_0 \, d\mu - \rho_0 \, T \, dS \quad . \tag{10}$$

The independent variables of the principle are ψ, α, β, Θ, S, and $g^{\sigma\nu}$.

Equations (1) and (4) combine to give μ as a function of these variables:

$$\mu^2 = - g^{\sigma\nu} (\psi_{,\sigma} + \alpha\beta_{,\sigma} - S\Theta_{,\sigma})(\psi_{,\nu} + \alpha\beta_{,\nu} - S\Theta_{,\nu}). \quad (11)$$

Varying ψ, α, β, Θ, S, and $g^{\sigma\nu}$ gives, respectively, equations (6), (3c), (3b), (3d), (3e), and the field equations

$$R_{\sigma\nu} - \tfrac{1}{2} R\, g_{\sigma\nu} = 8\pi T_{\sigma\nu} , \quad (12)$$

with $T_{\sigma\nu}$ from equation (8). Equation (3a) follows from the rest of equations (3) and equation (11); it is not an independent Euler-Lagrange equation.

b) Gauge Freedom in the Perturbations

A perturbation in the fluid's motion perturbs the geometry of spacetime. If the perturbation is small, it is reasonable to separate it from the "background" unperturbed spacetime and to treat it as a field on the background geometry. We therefore define $h^{\sigma\nu}$ to be the (Eulerian) perturbation in $g^{\sigma\nu}$, and ${g_{(B)}}^{\sigma\nu}$ to be the background unperturbed metric:

$$g^{\sigma\nu}(\text{perturbed spacetime}) = {g_{(B)}}^{\sigma\nu} + h^{\sigma\nu}. \quad (13)$$

Now $h^{\sigma\nu}$ is a tensor on the background spacetime. We can therefore raise and lower its indices with $g_{(B)}$; e.g.,

$$h^{\sigma}{}_{\lambda} \equiv h^{\sigma\nu}\, g_{(B)\nu\lambda} .$$

Our definition of $h^{\sigma\nu}$ is at slight variance with the usual usage, where $h_{\sigma\nu}$ is taken to be the perturbation in $g_{\sigma\nu}$. Here we have

$$h_{\sigma\nu} = h^{\alpha\beta} g_{(B)\alpha\sigma} g_{(B)\beta\nu} = -\delta g_{\sigma\nu} + O(h^2). \tag{14}$$

The "background" geometry is a fiction, however. Because the real spacetime possesses fine structure that is absent from the "background", there is no unique way to identify points in real spacetime with points in the background; thus, there is no unique way to define $h^{\sigma\nu}$ from equation (13). If η^σ generates a point transformation in the perturbed spacetime that is small (i.e., a change in the identification of points between the fictitious background and the real perturbed spacetime that is on the order of the scale of the "fine structure" of the real spacetime) then $h^{\sigma\nu}$ undergoes the change

$$h^{\sigma\nu} \rightarrow h^{\sigma\nu} + \pounds_\eta\, g_{(B)}{}^{\sigma\nu} = h^{\sigma\nu} - \eta^{\sigma;\nu} - \eta^{\nu;\sigma}. \tag{15}$$

Here $\pounds_\eta$ is the Lie derivative along η^σ, and semicolons (throughout this paper) denote derivatives covariant with respect to the unperturbed spacetime.

Under the same point transformation the perturbations in the velocity potentials must also change. For example, we define $\delta\psi$, the Eulerian change in ψ, by the equation

$$\psi(\text{perturbed spacetime}) = \psi_{(B)} + \delta\psi. \tag{16}$$

Then $\delta\psi$ changes by

$$\delta\psi \rightarrow \delta\psi + \pounds_\eta\, \psi_{(B)} = \delta\psi + \psi_{(B),\sigma}\, \eta^\sigma . \tag{17}$$

Similarly, all functions of the perturbed velocity potentials change:

e.g.,

$$\delta U_\nu \to \delta U_\nu + \pounds_\eta U_{(B)\nu} = \delta U_\nu + U_{(B)\nu;\sigma}\, \eta^\sigma + U_{(B)\sigma}\, \eta^\sigma{}_{;\nu} \,. \qquad (18)$$

Equations (15) and (17) together are called a gauge transformation. Most of our expressions -- such as the energy density in the pulsations -- will not be gauge-invariant. Nevertheless, we will see that physically measurable quantities -- such as the total energy -- are gauge-invariant.

In the remainder of this paper we will drop the "(B)" on the background quantities. Quantities such as $g_{\sigma\nu}$, U_ν, ψ, ... are understood to take their unperturbed values.

c) The Second Variation

In the Newtonian case (Paper I) we constructed the Lagrangian density for the perturbations from the second variation of the action, equation (9). The analogous calculations in the relativistic case are complicated by the perturbation in the geometry, so the details have been left to Appendix A. We treat the pressure and curvature parts of the action separately.

i) Second Variation of the Fluid Lagrangian

The fluid Lagrangian density is $p(-g)^{\frac{1}{2}}$. Its second variation is

$$\delta^2 \left[p(-g)^{\frac{1}{2}}\right] = (\delta^2 p)(-g)^{\frac{1}{2}} + 2\delta p\, \delta\left[(-g)^{\frac{1}{2}}\right] + p\, \delta^2\left[(-g)^{\frac{1}{2}}\right] . \qquad (19)$$

In Appendix A we show that

$$\delta^2 p = \delta\rho_0\delta\mu - \frac{\rho_0}{\mu}(\delta\mu)^2 - \delta(\rho_0 T)\,\delta S - 2\rho_0 U_\nu h^{\nu\sigma}\delta V_\sigma$$

$$- \frac{\rho_0}{\mu} g^{\nu\sigma}\,\delta V_\nu \delta V_\sigma - 2\rho_0\, U^\nu\,(\delta\alpha\delta\beta_{,\nu} - \delta S\delta\Theta_{,\nu}), \tag{20}$$

where we let V_ν denote the Taub (1959) current vector

$$V_\nu = \mu U_\nu = \psi_{,\nu} + \alpha\beta_{,\nu} - S\Theta_{,\nu}\,. \tag{21}$$

In equation (20) it is understood that $\delta\rho_0$ is a function (through the equation of state) of $\delta\mu$ and δS, and that $\delta\mu$ is a function of the independent perturbations ($\delta\psi$, $\delta\alpha$, $\delta\beta$, $\delta\Theta$, δS, $h^{\nu\sigma}$) through the perturbed version of equation (11):

$$\delta\mu = -\tfrac{1}{2}\mu h^{\nu\sigma}\, U_\nu U_\sigma - U^\sigma\,\delta\, V_\sigma. \tag{22}$$

From Appendix A we also have

$$\delta\left[(-g)^{\frac{1}{2}}\right] = -\tfrac{1}{2}\, h\, (-g)^{\frac{1}{2}} \tag{23a}$$

and

$$\delta^2\left[(-g)^{\frac{1}{2}}\right] = (\tfrac{1}{4}\, h^2 + \tfrac{1}{2}\, h^{\nu\sigma}\, h_{\nu\sigma})(-g)^{\frac{1}{2}}, \tag{23b}$$

where h is the trace of $h^{\nu\sigma}$:

$$h \equiv h^{\nu\sigma}\, g_{\nu\sigma}\,. \tag{24}$$

If we assemble all these terms and define

$$\mathcal{P} \equiv (-g)^{-\frac{1}{2}}\delta^2\left[p(-g)^{\frac{1}{2}}\right],$$

we have the fluid perturbations' Lagrangian density

$$\mathcal{L} = \delta\rho_0\delta\mu - \frac{\rho_0}{\mu}(\delta\mu)^2 - \delta(\rho_0 T)\delta S - \frac{\rho_0}{\mu} g^{\nu\sigma}\, \delta V_\nu \delta V_\sigma$$

$$- 2\rho_0 U^\nu (\delta\alpha\,\delta\beta_{,\nu} - \delta S\,\delta\Theta_{,\nu}) - 2\rho_0 U_\nu h^{\nu\sigma}\,\delta V_\sigma \tag{25}$$

$$- \delta p\, h + \tfrac{1}{4} p\, h^2 + \tfrac{1}{2} p\, h^{\nu\sigma} h_{\nu\sigma} \; .$$

This is perfectly general: no assumptions have yet been made about the unperturbed spacetime.

ii) Second Variation of the Curvature Lagrangian

The Lagrangian density for the curvature is $R(-g)^{\frac{1}{2}}$. It is simplest to treat it the Palatini way: $R_{\alpha\beta}$ is a function only of the Christoffel symbols,

$$R_{\alpha\beta} = \Gamma^\mu{}_{\alpha\beta,\mu} - \Gamma^\mu{}_{\alpha\mu,\beta} + \Gamma^\mu{}_{\nu\mu}\Gamma^\nu{}_{\alpha\beta} - \Gamma^\mu{}_{\nu\beta}\Gamma^\nu{}_{\alpha\mu} \; . \tag{26}$$

We define the perturbation in $\Gamma^\mu{}_{\alpha\beta}$ to be $\mathcal{S}^\mu{}_{\alpha\beta}$:

$$\Gamma^\mu{}_{\alpha\beta}(\text{perturbed}) = \Gamma^\mu{}_{\alpha\beta}(\text{background}) + \mathcal{S}^\mu{}_{\alpha\beta} \; . \tag{27}$$

It is well known that $\mathcal{S}^\mu{}_{\alpha\beta}$, being the difference between two affine connections, is a tensor on the background spacetime.

The second variation of $R(-g)^{\frac{1}{2}}$ is

$$\delta^2 \left[R(-g)^{\frac{1}{2}} \right] = \delta^2 \left[g^{\alpha\beta} R_{\alpha\beta}(\Gamma)(-g)^{\frac{1}{2}} \right]$$

$$= 2h^{\alpha\beta} \delta R_{\alpha\beta} (-g)^{\frac{1}{2}} + 2h^{\alpha\beta} R_{\alpha\beta} \delta \left[(-g)^{\frac{1}{2}} \right]$$

$$+ g^{\alpha\beta} \delta^2 R_{\alpha\beta} (-g)^{\frac{1}{2}} + 2g^{\alpha\beta} \delta R_{\alpha\beta} \delta \left[(-g)^{\frac{1}{2}} \right] \qquad (28)$$

$$+ g^{\alpha\beta} R_{\alpha\beta} \delta^2 \left[(-g)^{\frac{1}{2}} \right].$$

In Appendix A we show that

$$\mathcal{R} \equiv (-g)^{-\frac{1}{2}} \delta^2 \left[R(-g)^{\frac{1}{2}} \right]$$

$$= 2\bar{h}^{\alpha\beta} (s^{\mu}{}_{\alpha\beta;\mu} - s^{\mu}{}_{\alpha\mu;\beta}) + 2g^{\alpha\beta} (s^{\mu}{}_{\nu\mu} s^{\nu}{}_{\alpha\beta} - s^{\mu}{}_{\nu\beta} s^{\nu}{}_{\alpha\mu}) \qquad (29)$$

$$- h\, h^{\alpha\beta} R_{\alpha\beta} + R \left(\tfrac{1}{4} h^2 + \tfrac{1}{2} h_{\alpha\beta} h^{\alpha\beta} \right),$$

where we have used the conventional abbreviation

$$\bar{h}^{\alpha\beta} \equiv h^{\alpha\beta} - \tfrac{1}{2} g^{\alpha\beta} h. \qquad (30)$$

Again, we have not yet made any assumption about the background.

iii) Varying the Perturbed Lagrangian

The action for the perturbations is

$$I_2 = \int L_2 (-g)^{\frac{1}{2}} d^4x = \int (\mathcal{R} + 16\pi\mathcal{P})(-g)^{\frac{1}{2}} d^4x. \qquad (31)$$

Extremizing it with respect to $g^{\mu}{}_{\alpha\beta}$ gives the equation

$$0 = \frac{\delta L_2}{\delta g^{\mu}{}_{\alpha\beta}} = \delta^{\beta}{}_{\mu} \left[g^{\nu\sigma} g^{\alpha}{}_{\nu\sigma} + \bar{h}^{\alpha\nu}{}_{;\nu} \right] + g^{\alpha\beta} g^{\nu}{}_{\mu\nu}$$

$$- g^{\alpha}{}_{\nu\mu} g^{\nu\beta} - g^{\beta}{}_{\mu\nu} g^{\alpha\nu} - \bar{h}^{\alpha\beta}{}_{;\mu} . \qquad (32)$$

This is equivalent to

$$g^{\mu}{}_{\alpha\beta} = -\tfrac{1}{2} (h^{\mu}{}_{\alpha;\beta} + h^{\mu}{}_{\beta;\alpha} - h_{\alpha\beta}{}^{;\mu}) \qquad (33)$$

which is of course the correct expression for the perturbation of the Christoffel symbol. (Recall that eq. [14] is responsible for the overall minus sign in eq. [33].)

Extremizing I_2 with respect to $h^{\alpha\beta}$ gives the perturbed field equations:

$$\frac{\delta \mathcal{R}}{\delta h^{\alpha\beta}} = 2 (g^{\mu}{}_{\alpha\beta;\mu} - g^{\mu}{}_{\alpha\mu;\beta}) - g_{\alpha\beta} (g^{\mu\nu}{}_{\nu;\mu} - g^{\mu\nu}{}_{\mu;\nu})$$

$$- R^{\mu\nu} (g_{\alpha\beta} h_{\mu\nu} - h_{\alpha\beta} g_{\mu\nu}) - h (R_{\alpha\beta} - \tfrac{1}{2} R g_{\alpha\beta}) \qquad (34a)$$

$$= 2 (-g)^{-\frac{1}{2}} \delta \left[G_{\alpha\beta} (-g)^{\frac{1}{2}} \right] ; \qquad (34b)$$

$$\frac{\delta \mathcal{P}}{\delta h^{\alpha\beta}} = -\mu U_{\alpha} U_{\beta} \delta\rho_0 + \rho_0 U_{\alpha} U_{\beta} \delta\mu - 2 \rho_0 U_{\alpha} \delta V_{\beta}$$

$$- \delta p \, g_{\alpha\beta} + p h_{\alpha\beta} + \tfrac{1}{2} (\rho_0 \mu \, U_{\alpha} U_{\beta} + p g_{\alpha\beta}) h \qquad (35a)$$

$$= - (-g)^{-\frac{1}{2}} \delta \left[T_{\alpha\beta} (-g)^{\frac{1}{2}} \right] ; \qquad (35b)$$

$$\frac{\delta L_2}{\delta h^{\alpha\beta}} = 2(-g)^{-\frac{1}{2}} \delta \left[(G_{\alpha\beta} - 8\pi T_{\alpha\beta})(-g)^{\frac{1}{2}} \right] = 0 . \qquad (36)$$

Extremizing I_2 with respect to $\delta\psi$, $\delta\alpha$, $\delta\beta$, $\delta\Theta$, and δS gives, respectively, the perturbed versions of equations (6), (3c), (3b), (3d), and (3e). The perturbed version of equation (3a) follows from these and equation (22).

III. PERTURBATIONS OF DIFFERENTIALLY ROTATING STELLAR MODELS

In this section we specialize the Lagrangian density of §II to the case where the background is an axially symmetric, stationary stellar model. For the purpose of a stability analysis, this is hardly any restriction at all. A stability analysis would be very difficult if the unperturbed state were not stationary, and in general relativity -- by contrast with Newtonian theory -- it is very unlikely that non-axially symmetric stationary configurations of perfect fluid can exist. (They would either emit gravitational waves or require anisotropic stresses for their support.)

Up to this point our analysis has followed closely that of Paper I. From now on it will be quite different, however, because of the complications introduced by gravitational radiation. In Newtonian theory, where the gravitational field has no dynamical freedom, we had little difficulty reducing L_2 to a function only of ξ, the Lagrangian displacement of the fluid. We then derived the stability criterion directly from the reduced Lagrangian.

In the relativistic case there are two dynamical degrees of freedom in the gravitational field. In principle it would be possible to choose a gauge, to solve the perturbed initial-value equations, and to be left with two dynamical gravitational variables [e.g., $h_{\mu\nu}^{TT}$, by analogy with Arnowitt, Deser, and Misner (1962) -- hereafter referred to as ADaM]. Then L_2 could be expressed in terms of $\underset{\sim}{\xi}$ and these two gravitational variables. Such a program would be very interesting, and it may well be necessary before a definitive solution of the stability problem is reached. We will discuss this in more detail later. However, there is a simpler way to obtain a stability criterion, and it requires no prior specialization of gauge. In this section we construct the conserved energy density and angular momentum density of the pulsations and discuss some of their properties. In §IV we use the energy density as a Lyapunov function whose positive-definiteness guarantees stability.

a) <u>The Unperturbed Differentially Rotating Star</u>

The asymptotically flat spacetime in which the star sits is characterized by two Killing vectors, $\vec{\xi}_{(t)}$ and $\vec{\xi}_{(\varphi)}$. The four-velocity of the fluid is some timelike normalized linear combination of these:

$$\vec{U} = [\vec{\xi}_{(t)} + \Omega\, \vec{\xi}_{(\varphi)}]/\left|\vec{\xi}_{(t)} \cdot \vec{\xi}_{(t)} + 2\Omega\, \vec{\xi}_{(t)} \cdot \vec{\xi}_{(\varphi)} + \Omega^2\, \vec{\xi}_{(\varphi)} \cdot \vec{\xi}_{(\varphi)}\right|^{\frac{1}{2}} \tag{37}$$

This equation defines Ω: it is the angular velocity as seen from infinity.

We can introduce coordinates t and φ such that $\vec{\xi}_{(t)} = \partial/\partial t$ and $\vec{\xi}_{(\varphi)} = \partial/\partial\varphi$, and two other coordinates y^A (A = 1,2) such that the line element takes the form [cf. Carter (1969) or review by Thorne (1971)]

$$ds^2 = g_{00}\,dt^2 + 2g_{0\varphi}\,dt d\varphi + g_{\varphi\varphi}\,d\varphi^2 + g_{AB}\,dy^A dy^B. \tag{38}$$

However, we will not always want to specialize our coordinates this far; in this section we will usually work with three arbitrary spatial coordinates x^i and with the line element

$$ds^2 = g_{00}\,dt^2 + 2g_{0i}\,dt\,dx^i + g_{ij}\,dx^i dx^j. \tag{39}$$

It is understood, of course, that all $g_{\alpha\beta}$ and all other <u>physically measurable</u> unperturbed quantities are independent of t and φ. (The velocity potentials are <u>not</u> all independent of t and φ, but their physically measurable combinations, such as U_ν, <u>are</u> independent of t and φ.)

The relativistic velocity potentials for this case are similar to the Newtonian potentials:

$$S = \text{arbitrary function independent of } t \text{ and } \varphi\ , \tag{40a}$$

$$\Omega = \text{arbitrary function independent of } t \text{ and } \varphi\ , \tag{40b}$$

$$\alpha = \mu U_\varphi = \mu\vec{\xi}_{(\varphi)} \cdot \vec{U}\ , \tag{40c}$$

$$\beta = \varphi - \Omega t\ , \tag{40d}$$

$$\Theta = Tt/U^0 = Tt\left|\vec{\xi}_{(t)}\cdot\vec{\xi}_{(t)} + 2\Omega\vec{\xi}_{(t)}\cdot\vec{\xi}_{(\varphi)} + \Omega^2\vec{\xi}_{(\varphi)}\cdot\vec{\xi}_{(\varphi)}\right|^{\frac{1}{2}}\ , \tag{40e}$$

$$\psi = (-\mu + TS)t/U^0\ . \tag{40f}$$

That these are the correct velocity potentials is most easily demonstrated in the coordinates of equation (38), where the generating equation for U_ν,

$$U_\nu = \mu^{-1} (\psi_{,\nu} + \alpha\beta_{,\nu} - S\Theta_{,\nu}), \tag{41}$$

reduces to an identity for $\nu = t, \varphi$. Demanding that $U_A = 0$ ($A = 1, 2$) in those same coordinates gives the equation of hydrostatic equilibrium,

$$\frac{1}{\rho_0\mu} p_{,A} - (\ln U^0)_{,A} + U^0 U_\varphi \Omega_{,A} = 0. \tag{42}$$

The velocity potentials are scalars, so they keep their same values in the more general coordinates of equation (39). There one ought to regard φ as a scalar field geometrically defined by $\vec{\xi}_{(\varphi)}$.

b) The Conserved Angular Momentum of Pulsation

i) Noether's Theorem

The existence of a Killing vector $\vec{\xi}_{(a)}$ in the background spacetime makes it possible to define a conserved quantity if the Lagrangian density L_2 is invariant under translations along $\vec{\xi}_{(a)}$ during which the variables $q_r \equiv \{g^\mu{}_{\alpha\beta}, h^{\alpha\beta}, \delta\psi, \delta\alpha, \delta\beta, \delta\Theta, \delta S\}$ are held fixed.[1] Under such conditions Noether's theorem (cf. Trautman 1962, Taub 1970) implies the following conservation law:

$$P_{(a)}{}^{\sigma}{}_{;\sigma} = 0, \tag{43}$$

with

$$P_{(a)}{}^{\sigma} = \sum_r \left(\pounds_{\vec{\xi}_{(a)}} q_r\right) \left(\frac{\partial L_2}{\partial q_{r;\sigma}}\right) - L_2\, \xi_{(a)}{}^{\sigma} . \tag{44}$$

---

[1]More precisely, they are "Lie-dragged" along $\vec{\xi}_{(a)}$, as opposed to being parallel-transported (cf. Yano 1955).

We now show that L_2 is invariant under translations along $\vec{\xi}_{(\varphi)}$ but <u>not</u> along $\vec{\xi}_{(t)}$. Though the unperturbed spacetime is invariant under both, the unperturbed velocity potentials are not. One must look carefully at the way they enter L_2 in order to determine if L_2 is invariant.

The unperturbed velocity potentials enter L_2 only through the term

$$\delta V_\nu = \delta\psi_{,\nu} + \alpha\delta\beta_{,\nu} + \beta_{,\nu}\delta\alpha - S\delta\Theta_{,\nu} - \Theta_{,\nu}\delta S, \tag{45}$$

which contributes to L_2 both implicitly (through $\delta\mu$) and explicitly. Consider how it changes in t and φ if the perturbations are held fixed:

$$\left(\frac{\partial}{\partial t}\delta V_\nu\right)_{q_r} = -\Omega_{,\nu}\delta\alpha - \left(\frac{T}{U^0}\right)_{,\nu}\delta S \neq 0\ ; \tag{46}$$

$$\left(\frac{\partial}{\partial\varphi}\delta V_\nu\right)_{q_r} = \left(\frac{\partial\beta}{\partial\varphi}\right)_{,\nu}\delta\alpha = 0 \quad . \tag{47}$$

<u>So L_2 is φ-invariant but not t-invariant.</u> Note, however, from equation (22) that $\delta\mu$ is t-invariant as well.

This result can be understood as follows: Even if the perturbation eventually dies out completely, $\delta\beta$, $\delta\psi$, and $\delta\Theta$ may continue to change linearly in time at rates that vary across the star, just as β, ψ, and Θ do in the unperturbed state. Therefore, holding $\delta\psi$, $\delta\beta$, $\delta\Theta$ fixed during a translation in time is not the same as holding the physical perturbation fixed. It is not surprising that Noether's theorem fails in our context. Later we will construct the real conserved energy [which must exist because $\vec{\xi}_{(t)}$ exists] in a different manner. First, however, we use the φ-invariance of L_2 to construct the angular momentum.

ii) The Angular Momentum Density

The conservation law (43) can be written in the following form when $\vec{\xi}_{(a)}$ is $\vec{\xi}_{(\varphi)}$:

$$\frac{\partial}{\partial t}\left[\sum_r N q_{r,\varphi} \frac{\partial L_2}{\partial q_{r;0}}\right] + \left[\sum_r N q_{r,\varphi} \frac{\partial L_2}{\partial q_{r;i}} - N L_2 \delta^i{}_\varphi\right]_{|i} = 0. \qquad (48)$$

From now on we use the ADaM notation appropriate to a three-plus-one dimensional split of spacetime. In particular, we define the lapse function $N \equiv (-g^{00})^{-\frac{1}{2}}$; we denote the determinant of the three-dimensional metric by g and that of the four-dimensional metric by ${}^4g$ [which are related by the identity $(-{}^4g)^{\frac{1}{2}} = Ng^{\frac{1}{2}}$]; and we use a slash or a boldface $\underset{\sim}{\nabla}$ to denote differentiation covariant with respect to the three-dimensional metric. Equation (48) implies that if we define

$$\mathcal{J}' \equiv -\frac{1}{32\pi} \sum_r N q_{r,\varphi} \frac{\partial L_2}{\partial q_{r;0}} , \qquad (49a)$$

then the integral of $\mathcal{J}'$ over the entire hypersurface

$$J \equiv \int \mathcal{J}' \, g^{\frac{1}{2}} \, d^3x \qquad (49b)$$

is constant in time. Note that any density differing from $\mathcal{J}'$ by a spatial divergence will likewise be conserved, and will give the same value for J provided the perturbed region of space is of finite extent.

From L_2 as given in §II we find

$$\mathcal{J}' = -\frac{1}{16\pi} N \, g^0{}_{\sigma\nu,\varphi} \, \overline{h}^{\sigma\nu} + \frac{1}{16\pi} N \, g^\nu{}_{\sigma\nu,\varphi} \, \overline{h}^{\sigma 0}$$
$$+ g^{-\frac{1}{2}} \delta(\rho_0 U^0 N g^{\frac{1}{2}})(\delta V_\varphi - \delta\alpha) + N \rho_0 U^0 (\delta\alpha \, \delta\beta_{,\varphi} - \delta S \, \delta\Theta_{,\varphi}). \qquad (50)$$

To cast this in a more familiar form we add the divergence

$$\frac{\partial}{\partial\varphi}\left[\frac{1}{16\pi} N \left(\mathfrak{g}^{0}{}_{\sigma\nu}\, \overline{h}^{\sigma\nu} - \mathfrak{g}^{\nu}{}_{\sigma\nu}\, \overline{h}^{\sigma 0}\right)\right] .$$

We define the result as the <u>angular momentum density:</u>

$$\mathcal{J} \equiv \mathcal{J}_G + \mathcal{J}_F , \tag{51a}$$

where

$$\mathcal{J}_G \equiv \frac{1}{16\pi}\left[\tfrac{1}{2} N \overline{h}^{\sigma\nu}{}_{,\varphi}\, \overline{h}_{\sigma\nu}{}^{;0} - N \overline{h}^{\sigma\nu}{}_{,\varphi}\, \overline{h}^{0}{}_{\sigma;\nu} - \tfrac{1}{4} N \overline{h}_{,\varphi}\, \overline{h}^{,0}\right] \tag{51b}$$

and

$$\mathcal{J}_F \equiv g^{-\frac{1}{2}}\, \delta\left(\rho_0 U^0 N g^{\frac{1}{2}}\right)\left(\delta V_\varphi - \delta\alpha\right) + N \rho_0 U^0 \left(\delta\alpha\, \delta\beta_{,\varphi} - \delta S\, \delta\theta_{,\varphi}\right). \tag{51c}$$

To obtain this form for $\mathcal{J}_G$ we have expressed the $\mathfrak{g}$'s in terms of $\overline{h}$'s from equation (33). The split between $\mathcal{J}_G$ and $\mathcal{J}_F$ is arbitrary. Only their sum is conserved.

The flux associated with $\mathcal{J}$ is

$$\eta^k \equiv \eta_G{}^k + \eta_F{}^k , \tag{52a}$$

with

$$\begin{aligned} \eta_G{}^k \equiv & - \frac{1}{16\pi} N \left[\mathfrak{g}^{k}{}_{\sigma\nu,\varphi}\, \overline{h}^{\sigma\nu} - \mathfrak{g}^{\nu}{}_{\sigma\nu,\varphi}\, \overline{h}^{\sigma k}\right] \\ & + \frac{1}{16\pi} N \delta^k{}_\varphi \left[\tfrac{1}{2} \mathcal{R} - \frac{\partial}{\partial t}\left(\mathfrak{g}^{0}{}_{\sigma\nu}\, \overline{h}^{\sigma\nu} - \mathfrak{g}^{\nu}{}_{\sigma\nu}\, \overline{h}^{\sigma 0}\right)\right] \end{aligned} \tag{52b}$$

and

$$\begin{aligned} \eta_F{}^k \equiv & \; g^{-\frac{1}{2}} \delta\left(\rho_0 U^k N g^{\frac{1}{2}}\right)\left(\delta V_\varphi - \delta\alpha\right) + N \rho_0 U^k \left(\delta\alpha\, \delta\beta_{,\varphi} - \delta S\, \delta\theta_{,\varphi}\right) \\ & + \tfrac{1}{2} N \mathcal{P} \delta^k{}_\varphi . \end{aligned} \tag{52c}$$

Then equation (48) becomes

$$\frac{\partial}{\partial t}(\mathcal{J}_G + \mathcal{J}_F) + (\eta_G{}^k + \eta_F{}^k)_{|k} = 0. \tag{53}$$

Note that since $\mathcal{J}$ differs from $\mathcal{J}'$ by a divergence, the flux η^k differs from $-1/32\pi$ times the flux in equation (48) by the time-derivative

$$-\frac{\partial}{\partial t}\left[\frac{1}{16\pi} N \, (g^0{}_{\sigma\nu}\, \overline{h}^{\sigma\nu} - g^\nu{}_{\sigma\nu}\, \overline{h}^{\sigma 0})\, \delta^k{}_\varphi\right].$$

iii) Heuristic Interpretation of $\mathcal{J}$

The terms in $\mathcal{J}$ may be interpreted heuristically (and incompletely) as follows:

i) The terms called $\mathcal{J}_G$ may be defined as the angular momentum in the gravitational waves. The reasonableness of this definition becomes apparent in the short-wavelength limit (wavelength small compared to the radius of curvature of the background spacetime). There the average of $\mathcal{J}_G$ over a few wavelengths in the hypersurface and over a few cycles of time is just the angular-momentum component of the Isaacson (1968) stress-energy tensor for gravitational radiation, $T^{(GW)0}{}_\varphi$. [More precisely, the average is the "Brill-Hartle" average (cf. Isaacson 1968) of $\mathcal{J}_G/N$.] The short-wavelength limit is most easily calculated using the expressions for $T^{(GW)}{}_{\mu\nu}$ given by Misner, Thorne, and Wheeler (1972):

$$\langle \mathcal{J}_G/N \rangle_{BH} = \frac{1}{16\pi} \langle \tfrac{1}{2}\, \overline{h}^{\sigma\nu}{}_{;\varphi}\, \overline{h}_{\sigma\nu}{}^{;0} - \overline{h}^{\sigma\nu}{}_{;\nu}\, \overline{h}^0{}_{\sigma;\varphi}$$

$$- \tfrac{1}{4}\, \overline{h}_{,\varphi}\, \overline{h}^{,0} \rangle_{BH} \tag{54}$$

$$= T^{(GW)0}{}_\varphi ,$$

independent of any gauge.

We emphasize, however, that the dominant radiation from a pulsating relativistic star may not be of short wavelength near the star. If most of the radiation from a star of mass M has frequency greater than some ω_0, then the short-wavelength approximation is good only in the region

$$r \gg (4\pi M/\omega_0)^{\frac{1}{2}} = (4\pi GM/\omega_0 c)^{\frac{1}{2}}. \qquad (55)$$

For a typical neutron star in quadrupole oscillation as studied by Thorne (1969) ($M \simeq 0.7\ M_\odot$, $\omega_0 \simeq 2\times 10^4\ \text{sec}^{-1}$, $R \simeq 9$ km) this becomes

$$r \gg 14\ \text{km},$$

which puts r well outside the star.

Our expression for $\mathcal{J}$ is only one of many that reduce to the Isaacson tensor in the short-wavelength limit. Only in the radiation zone far from the star can we relate $\mathcal{J}_G$ to the density of angular momentum being lost by the star, because only there is that density truly well defined and measurable.

ii) The angular momentum in the fluid per unit coordinate volume, $T^0{}_\varphi\,(-{}^4g)^{\frac{1}{2}}$, can be written as $\rho_0 U^0 V_\varphi N g^{\frac{1}{2}}$. Now V_φ is the angular momentum per particle per unit rest mass:

$$V_\varphi = \mu\, U_\varphi = \frac{\rho + p}{\rho_0} U_\varphi .$$

Thus, the angular momentum density is the product:

(ang. mom. density) = (rest-mass density) $\times$ (angular momentum per particle per unit rest mass),

$$T^{0}_{\varphi}(-^{4}g)^{\frac{1}{2}} = (\rho_0 U^0 N g^{\frac{1}{2}}) \times (V_{\varphi}) .$$

When the fluid is perturbed, part of the second-order change in this is, from equation (51), $\delta(\rho_0 U^0 N g^{\frac{1}{2}})(\delta V_{\varphi} - \delta\alpha)$. The term $\delta(\rho_0 U^0 N g^{\frac{1}{2}})$ is easy to understand. The term $\delta V_{\varphi} - \delta\alpha$ can be related to the Lagrangian change in the angular momentum per particle per unit rest mass as follows. If j is the angular momentum per particle per unit rest mass, if Δ denotes a Lagrangian change, and if $\underset{\sim}{\xi}$ is defined as the Lagrangian displacement vector of the fluid element (not to be confused with the Killing vectors), then we have

$$\begin{aligned} \Delta j &= \delta j + \underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} j \\ &= \delta V_{\varphi} + \underset{\sim}{\xi} \cdot \underset{\sim}{\nabla}\alpha \end{aligned} \tag{56}$$

because in the unperturbed state $j = V_{\varphi} = \alpha$. But in Appendix B we show that $\delta\alpha = - \underset{\sim}{\xi} \cdot \underset{\sim}{\nabla}\alpha + (\delta\alpha)_0$, where $(\delta\alpha)_0$ is the "initial value" of $\delta\alpha$: its value when $\underset{\sim}{\xi}$ is zero. Therefore we have

$$\Delta j - (\delta\alpha)_0 = \delta V_{\varphi} - \delta\alpha. \tag{57}$$

iii) The final term in $\mathcal{L}_F$ is $N \rho_0 U^0 (\delta\alpha\, \delta\beta_{,\varphi} - \delta S\, \delta\Theta_{,\varphi})$. This is the same as $\frac{1}{2} N \rho_0 U^0 \delta^2 V_{\varphi}$, the contribution from the second-order change in V_{φ}. Because we lack an explicit expression for $\delta\Theta_{,\varphi}$ in terms of $\underset{\sim}{\xi}$, we have been unable to express this term entirely in terms of $\underset{\sim}{\xi}$.

c) The Conserved Energy of Pulsation

i) Calculating the Energy Density

Although Noether's theorem does not give us a conserved energy, we can construct one from the Hamiltonian. The calculations required to do this appear in Appendix C. The essential steps are summarized here:

i) Define the Hamiltonian density,

$$H_2 = \sum_r N\, q_{r,0} \frac{\partial L_2}{\partial q_{r,0}} - N L_2 \ , \tag{58}$$

where $q_r = \{g^\mu{}_{\alpha\beta},\ h^{\alpha\beta},\ \delta\psi,\ \delta\alpha,\ \delta\beta,\ \delta\theta,\ \delta S\}$. It is degenerate: not all the momenta $\partial L_2/\partial q_{r,0}$ are independent.

ii) Find the time-derivative of H_2 using the method of Dirac (1958a) for degenerate theories. Find that

$$\frac{\partial H_2}{\partial t} = \left[\sum_r q_{r,0} \frac{\partial H_2}{\partial q_{r|\ell}}\right]_{|\ell} - N\left(\frac{\partial L_2}{\partial t}\right)_{\text{holding all } q_r,\ q_{r,0} \text{ fixed}} \ , \tag{59}$$

$$= - f^\ell{}_{|\ell} - 32\pi N \frac{\rho_0}{\mu} \delta V^i \left[\Omega_{,i}\, \delta\alpha + \left(\frac{T}{U^0}\right)_{,i} \delta S\right] . \tag{60}$$

Thus, the Hamiltonian is not conserved. We should expect this from the failure of Noether's theorem.

iii) Express the last term in equation (60) in terms of $\underset{\sim}{\xi}$. Define the redshifted temperature,

$$\mathcal{T} \equiv T/U^0 \ , \tag{61a}$$

and a symmetric (for proof see Appendix C) tensor

$$M_{ij} = \alpha_{,i}\, \Omega_{,j} + S_{,i}\, \mathcal{T}_{,j} \ . \tag{61b}$$

Find that

$$- 32\pi N \frac{\rho_0}{\mu} \delta V^i \left[\Omega_{,i}\, \delta\alpha + \mathcal{T}_{,i}\, \delta S \right]$$

$$= \frac{\partial}{\partial t} \left[16\pi N \rho_0 U^0 M_{ij} \xi^i \xi^j - 32\pi N \rho_0 U^0 \Omega_{,i} \xi^i (\delta\alpha)_0 \right] \tag{62}$$

$$+ \left[16\pi N \rho_0 U^\ell M_{ij} \xi^i \xi^j - 32\pi N \rho_0 U^\ell \Omega_{,i} \xi^i (\delta\alpha)_0 \right]_{|\ell} \quad ,$$

where $(\delta\alpha)_0$ is the "initial value" of the perturbation in α. The time derivative can be brought over to the left-hand side of equation (60) and the divergence absorbed into the divergence of f^i. This defines a conservation law,

$$\frac{\partial \mathcal{E}'}{\partial t} + \mathcal{F}'^{\ell}{}_{|\ell} = 0, \tag{63}$$

for a globally conserved energy density,

$$\mathcal{E}' \equiv \frac{1}{16\pi} H_2 - N \rho_0 U^0 M_{ij} \xi^i \xi^j + 2 N \rho_0 U^0 \Omega_{,i} \xi^i (\delta\alpha)_0, \tag{64}$$

and its flux,

$$\mathcal{F}'^{\ell} \equiv \frac{1}{16\pi} f^\ell - N \rho_0 U^\ell M_{ij} \xi^i \xi^j + 2 N \rho_0 U^\ell \Omega_{,i} \xi^i (\delta\alpha)_0. \tag{65}$$

iv) The energy density is defined only to within a spatial divergence. Subtract a divergence from $\mathcal{E}'$ and the appropriate time derivative from $\mathcal{F}'^i$ to arrive at a form of the energy density that is quadratic in derivatives of $h^{\alpha\beta}$. Write the result as:

$$\frac{\partial}{\partial t} (\mathcal{E}_G + \mathcal{E}_F) + (\mathcal{F}_G{}^k + \mathcal{F}_F{}^k)_{|k} = 0, \tag{66}$$

with

$$\mathcal{E}_G = \frac{1}{8\pi} N \left[g^{\alpha\beta} \left(g^{\mu}{}_{\nu\mu} \, g^{\nu}{}_{\alpha\beta} - g^{\mu}{}_{\nu\beta} \, g^{\nu}{}_{\alpha\mu} \right) - \bar{h}^{\alpha\beta}{}_{,0} \, g^{0}{}_{\alpha\beta} + \bar{h}^{\alpha 0}{}_{,0} \, g^{\mu}{}_{\alpha\mu} \right], \quad (67)$$

$$\begin{aligned} \mathcal{E}_F = & -2g^{-\frac{1}{2}} \, \delta(\rho_0 U^0 N g^{\frac{1}{2}}) \, (\delta V_0 + \Omega \delta\alpha + \mathcal{T} \delta S) \\ & + \frac{\rho_0}{\mu} N g^{\nu\sigma} \delta V_\nu \delta V_\sigma + 2 N \rho_0 U_\sigma h^{\sigma\nu} \delta V_\nu - N \delta \rho_0 \delta\mu \\ & + N \frac{\rho_0}{\mu} (\delta\mu)^2 + N \delta(\rho_0 T) \, \delta S + 2 N \rho_0 U^i (\delta\alpha \, \delta\beta_{,i} - \delta S \, \delta\theta_{,i}) \qquad (68) \\ & - N \rho_0 U^0 (\alpha_{,i} \Omega_{,j} + S_{,i} \mathcal{T}_{,j}) \xi^i \xi^j + 2 N \rho_0 U^0 \Omega_{,i} \xi^i (\delta\alpha)_0 \\ & + N h \delta p + \frac{1}{16\pi} N h h^{\alpha\beta} R_{\alpha\beta} - N \left(\frac{1}{16\pi} R + p\right)\left(\tfrac{1}{4} h^2 + \tfrac{1}{2} h^{\alpha\beta} h_{\alpha\beta}\right), \end{aligned}$$

$$\mathcal{F}_G{}^k = \frac{1}{8\pi} N \left(-\bar{h}^{\alpha\beta}{}_{,0} \, g^{k}{}_{\alpha\beta} + \bar{h}^{\alpha k}{}_{,0} \, g^{\mu}{}_{\alpha\mu} \right), \quad (69)$$

and

$$\begin{aligned} \mathcal{F}_F{}^k = & - 2g^{-\frac{1}{2}} \, \delta(\rho_0 U^k N g^{\frac{1}{2}}) (\delta V_0 + \Omega \delta\alpha + \mathcal{T} \delta S) \\ & - 2N \rho_0 U^k (\delta\alpha \, \delta\beta_{,0} - \delta S \, \delta\theta_{,0}) - N \rho_0 U^k (\alpha_{,i} \Omega_{,j} + S_{,i} \mathcal{T}_{,j}) \xi^i \xi^j \\ & + 2N \rho_0 U^k \Omega_{,i} \xi^i (\delta\alpha)_0 . \qquad (70) \end{aligned}$$

The split between $\mathcal{E}_G$ and $\mathcal{E}_F$ (and between $\mathcal{F}_G{}^k$ and $\mathcal{F}_F{}^k$) is arbitrary: only their sum, $\mathcal{E} \equiv \mathcal{E}_G + \mathcal{E}_F$, is conserved. As we shall see in the next subsection, $\mathcal{E}$ is really twice what one would normally call the energy density.

ii) Heuristic Interpretation of $\mathcal{E}$ and $\mathcal{F}^k$

Because of the great number of terms in $\mathcal{E}$ it is difficult to identify different kinds of energy. We have split off ε_G because it is the only nonvanishing part in vacuum, and because it contains all the terms that have derivatives of $h^{\alpha\beta}$.

i) In the short-wavelength limit in the vacuum region outside the star, the Brill-Hartle average of ε_G/N is proportional to the Isaacson energy density. Outside the star the wave equation is (cf. eq. [34a])

$$g^{\mu}{}_{\alpha\beta;\mu} - g^{\mu}{}_{\alpha\mu;\beta} = 0. \tag{71}$$

Then by the identity mentioned in Appendix C (eq. [C15]) we have

$$g^{\alpha\beta}\,(g^{\mu}{}_{\nu\mu}\,g^{\nu}{}_{\alpha\beta} - g^{\mu}{}_{\nu\beta}\,g^{\nu}{}_{\alpha\mu}) = \tfrac{1}{2}\,(\bar{h}^{\mu\nu}\,g^{\alpha}{}_{\mu\nu} - \bar{h}^{\nu\alpha}\,g^{\mu}{}_{\nu\mu})_{;\alpha}\,.$$

This divergence does not contribute to the Brill-Hartle average of ε_G/N, so we obtain in the short-wavelength limit

$$\langle\varepsilon_G/N\rangle_{BH} = -\,2\,T^{(GW)0}{}_{0}. \tag{72}$$

This is in accord with our previous remark that ε is twice the energy density.

ii) The interpretation of ε_F is made difficult by the presence of the term

$$2\,N\,\rho_0\,U^i\,(\delta\alpha\,\delta\beta_{,i} - \delta S\,\delta\theta_{,i}) = N\,\rho_0\,U^i\,\delta^2\,V_i. \tag{73}$$

As with a similar term in $\mathcal{J}$, we have not been able to express this in terms of $\underset{\sim}{\xi}$. Therefore we will not be able to make a comparison of the Newtonian limit of ε with the Newtonian energy density derived in

Paper I.[2] However, this term is not present if the unperturbed star is nonrotating, so in that case there is no problem showing that $\mathcal{E}$ reduces to the Newtonian expression derived in Paper I. We will do that later (§IIIe, ii). For now we simply note that the similarity between this term and one in $\mathcal{J}_F$ permits us to rewrite $\mathcal{E}_F$ in the form

$$\mathcal{E}_F = -2\,g^{-\frac{1}{2}}\delta\,(\rho_0 U^0 N g^{\frac{1}{2}})\,(U^\nu \delta V_\nu + T\delta S)/U^0 + 2\,\Omega\,\mathcal{J}_F \quad + \text{remainder,} \tag{74}$$

where "remainder" means all but the term (73) and the first term of $\mathcal{E}_F$ in equation (68). So the kinetic energy associated with the fluid's angular momentum makes an explicit contribution to the total energy density.

iii) We can get some feeling for the nature of $\mathcal{E}$ by looking at its flux, which tells us how energy leaves a volume. The flux of gravitational energy, $\mathcal{F}_G{}^k$, can be averaged over a few wavelengths and cycles of time to give (in the short-wavelength limit)

$$\langle \mathcal{F}_G{}^k/N \rangle_{BH} = -2\,T^{(GW)k}{}_0 . \tag{75}$$

Therefore, far from the star this is twice the physically measurable flux of energy in the gravitational waves.

---

[2] This is a Newtonian term and even prevents a direct comparison of the Newtonian energy density derived by analogue with the present procedure with that derived in Paper I. It is difficult to see how they could be different, considering especially that in the nonrotating case one can show that they are equal.

iv) The flow of fluid energy across some surface is

$$\left\{\begin{array}{l}\text{transport of fluid energy}\\ \text{in the hypersurface across}\\ \text{a two-surface } \Sigma \text{ with}\\ \text{unit normal } n_k\end{array}\right\} = \int_\Sigma \mathcal{F}_F{}^k\, n_k\, d\sigma \,.$$

If the surface Σ is parallel to the unperturbed streamlines ($U^k n_k = 0$), this becomes

$$\left\{\begin{array}{l}\text{transport of fluid energy}\\ \text{across the unperturbed}\\ \text{streamlines}\end{array}\right\} = -2 \int_\Sigma N(\delta V_0 + \Omega\delta\alpha + \mathcal{T}\delta S)\rho_0\, \delta v^k\, n_k\, d\sigma, \quad (76)$$

where by v^k we mean the coordinate velocity U^k/U^0 (not to be confused with $V^k \equiv \mu U^k$). It can be shown that

$$-(\delta V_0 + \Omega\delta\alpha + \mathcal{T}\delta S) = \frac{1}{\rho_0 U^0}\delta p + \frac{\mu}{2U^0} U_\alpha U_\beta h^{\alpha\beta} + \Omega(\delta V_\varphi - \delta\alpha). \quad (77)$$

Thus the energy carried by the perturbations across the unperturbed streamlines is heuristically of three types: (a) work done (or gained) because of local changes in pressure; (b) "gravitational potential energy" (note that in the Newtonian limit, $\frac{1}{2}U_\alpha U_\beta h^{\alpha\beta} \rightarrow \frac{1}{2}h^{00} \rightarrow \delta\Phi$, the change in the Newtonian potential); and (c) rotational kinetic energy (recall that $\delta V_\varphi - \delta\alpha$ is related to the Lagrangian change in j by eq. [57]).

iii) The Outgoing-Energy Boundary Condition

Far from the star, where the short-wavelength approximation is valid for all but a negligible part of the gravitational energy, it is possible to formulate a physically meaningful condition that the net flux of energy be away from the star. On a closed surface Σ in the short-wavelength region, the net flux of energy will not be inward if

$$\int_{\Sigma} T^{(GW)k0} \, n_k \, d\sigma \geq 0. \tag{78}$$

By equation (75) this is equivalent to

$$\int_{\Sigma} \langle \mathcal{F}_G{}^k/N \rangle_{BH} \, n_k \, d\sigma \geq 0. \tag{79}$$

From this and equation (66) follows the important result: The total energy of pulsation ($\int \mathcal{E} g^{\frac{1}{2}} d^3x$) inside Σ never increases if the radiation satisfies the outgoing-energy boundary condition on Σ.

Note that this is a very weak condition compared to the usual outgoing-wave boundary condition, which requires that the flux be outward at every point of Σ. For our purposes we will need only the weak condition, equation (79).

d) The Total Energy and Angular Momentum

Three conclusions help us understand the physical meaning of the total energy, $E \equiv \int \mathcal{E} g^{\frac{1}{2}} d^3x$, and the total angular momentum, $J \equiv \int \mathcal{J} g^{\frac{1}{2}} d^3x$:

(i) E and J are gauge-independent. This follows from reasoning similar to that used to prove the coordinate-independence of pseudotensor energies (cf. Landau and Lifshitz 1962). Briefly, assume that E or J is different in two different gauges. Choose a third gauge that matches the first on one hypersurface and goes smoothly into the second on a later hypersurface. Then conservation of E and J in every gauge contradicts the assumption. This does not imply that the densities $\mathcal{E}$ and $\mathcal{J}$ are gauge-invariant. Conservation of E and J is fundamental to the argument, and the conservation law is valid only if the perturbations satisfy the initial-value equations on every hypersurface. Therefore the argument implies only that under a gauge transformation $\mathcal{E}$ and $\mathcal{J}$ change by terms that become spatial divergences after the initial-value equations are applied.

(ii) Suppose that a distant observer (outside the furthest wavefront) measures the active gravitational mass M^* and total angular momentum L^* of the pulsating star. Suppose also that the star is stable, so that the pulsations eventually die out and leave behind a star of mass M and angular momentum L. <u>For a stable star, the differences $M^* - M$ and $L^* - L$ are at most second order in the perturbations.</u>

The difference $M^* - M$ is conserved at all orders. If there were a first order piece in $M^* - M$, it would have to be radiated away as the stable star's pulsations damp out. It could not remain localized inside or near the star because by assumption M is the mass left behind. On the other hand, the work of Isaacson (1968) shows that there can be no first-order radiation of physically measurable energy on the stationary background far from the star. Therefore the first-order contribution to $M^* - M$ must vanish. The same argument applies to $L^* - L$.

This result is similar to the theorem of Bardeen (1970) that the equilibrium configuration of a rotating star extremizes the active gravitational mass of all nearby momentarily stationary configurations with the same total baryon number, angular momentum, and entropy that satisfy the initial-value equations. [This was proved for nonrotating stars by

Cocke (1965) and Harrison, Thorne, Wakano, and Wheeler (1965).] Where Bardeen compares momentarily stationary configurations with different masses but identical angular momenta, we compare momentarily stationary configurations whose masses and angular momenta are related by the requirement that one configuration can be obtained from another by the emission or absorption of gravitational radiation. (The configuration with mass M^* can be considered to be momentarily stationary at the moment the perturbation is applied, just before it begins to emit gravitational waves.)

(iii) In the notation of (ii), the following equations are correct to second order in the perturbations:

$$M^* = M + \tfrac{1}{2} E \tag{80a}$$

$$L^* = L + J , \tag{80b}$$

where the background star is the star of mass M and angular momentum L that is left behind. This result follows from three properties of E and J: (a) They are unique apart from additive and multiplicative constants because they depend only on the Killing vectors $\vec{\xi}_{(t)}$ and $\vec{\xi}_{(\varphi)}$. (b) They vanish when the perturbation vanishes. (c) The change in $\int \mathcal{E}\, g^{\frac{1}{2}}\, d^3x$ and $\int \mathcal{J}\, g^{\frac{1}{2}}\, d^3x$ inside any fixed surface surrounding the star and far from it is determined solely by the physically measurable fluxes $T^{(GW)k}{}_0$ and $T^{(GW)k}{}_\varphi$.

If there were any other second-order contribution to M^* or L^*, it would have to be globally conserved. By (c) it would also have to be confined forever within a closed surface at some large but finite distance from the star. The use of (b) and of arguments similar to those of (ii) above then implies equations (80).

e) The Spherically Symmetric, Nonrotating Star

i) Expressions for the Energy and Flux

We turn now to a special case in which our expressions simplify considerably: the nonrotating star. In curvature coordinates the background metric is

$$ds^2 = -e^{\nu}\,dt^2 + e^{\lambda}\,dr^2 + r^2\,(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2). \tag{81}$$

Then we have in the background

$$\begin{aligned} N &= -U_0 = 1/U^0 = e^{\nu/2}\,, \\ g^{\frac{1}{2}} &= r^2 \sin^2\vartheta\, e^{\lambda/2}\,, \\ U^i &= \Omega = 0\,. \end{aligned} \tag{82}$$

In Appendix D we simplify $\mathcal{E}_F$ for this case as much as possible by substituting for the perturbed fluid quantities their expressions in terms of $\underset{\sim}{\xi}$ (cf. Appendix B). The result is

$$\begin{aligned} e^{-\nu/2}\,\mathcal{E}_F = {} & \rho_0 \mu\, e^{-\nu}\, \underset{\sim}{\xi}_{,0} \cdot \underset{\sim}{\xi}_{,0} + \gamma p\,(\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi})^2 + \rho_0^{-1}\,(\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla}\rho_0)\,(\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} p) \\ & + 2(\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi})(\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} p) - \tfrac{1}{2}\rho_0 T(\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} S)(\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla}\nu) - \mu\,\ell\,\underset{\sim}{\nabla} \cdot (\rho_0\, \underset{\sim}{\xi}) \\ & + \rho_0\,\ell\,\delta\mu + \delta p(h^j{}_j - \ell) + \tfrac{1}{8}(\rho + 3p)\,\ell^2 + \tfrac{1}{4}(\rho + 3p)\,\ell\, h^j{}_j \\ & - \tfrac{1}{2} e^{-\nu}(\rho + 3p)\, k_j k^j - \tfrac{1}{4}(\rho - p)\, h^{jk} h_{jk} + \tfrac{1}{8}(\rho - p - 2\gamma p)(h^j{}_j)^2. \end{aligned} \tag{83}$$

In this expression we have defined

$$\ell \equiv e^{-\nu} h_{00} \tag{84a}$$

and

$$k_j \equiv e^{-\nu/2} h_{0j} \; ; \tag{84b}$$

and we mean by $\delta\mu$ and δp

$$\delta\mu = -\frac{\gamma p}{\rho_0} (\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi} - \tfrac{1}{2} h^j{}_j) - \underset{\sim}{\xi} \cdot \underset{\sim}{\nabla}\mu \; , \tag{85a}$$

$$\delta p = -\gamma p (\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi} - \tfrac{1}{2} h^j{}_j) - \underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} p \; . \tag{85b}$$

The flux $\mathcal{F}_F{}^k$ is especially simple in this case. Equation (76) applies becasue <u>all</u> surfaces are orthogonal to the unperturbed streamlines:

$$e^{-\nu/2} \mathcal{F}_F{}^k = e^{\nu/2} (\delta p + \tfrac{1}{2} \rho_0 \mu \ell) \xi^k{}_{,0} \; . \tag{86}$$

The energy density and flux of gravitational waves do not simplify very much from their full form (eqs. [67] and [69]) so we will not reproduce them here.

Our previous remark that ε is really twice the energy density is again verified by the "kinetic energy" term in equation (83), which has the form mv^2.

ii) The Newtonian Limit

The Newtonian limit of ε for the nonrotating star is obtained by neglecting p and $\rho\Phi$ compared to ρ (Φ is the Newtonian gravitational potential). In equation (83), the fifth, seventh, and subsequent terms are all of post-Newtonian order or higher. In the Newtonian limit we have $\ell = 2\,\delta\Phi$, so that $\mathcal{E}_F$ becomes

$$(\mathcal{E}_F)_{NEWT} = \rho\, \underset{\sim}{\xi}_{,0} \cdot \underset{\sim}{\xi}_0 + \gamma p\, (\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi})^2 + \rho^{-1}\, (\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla}\rho)(\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} p)$$
$$+ 2\, (\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi})(\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} p) - 2\,\delta\Phi\, \underset{\sim}{\nabla} \cdot (\rho\, \underset{\sim}{\xi}). \tag{87}$$

The perturbed source equation for $\delta\Phi$ (analogue of relativistic initial-value equation) is (cf. Paper I)

$$\nabla^2\, \delta\Phi = 4\pi\, \delta\rho = -\, 4\pi\, \underset{\sim}{\nabla} \cdot (\rho\, \underset{\sim}{\xi}). \tag{88}$$

Therefore the last term in $(\mathcal{E}_F)_{NEWT}$ becomes

$$-\, 2\,\delta\Phi\, \underset{\sim}{\nabla} \cdot (\rho\, \underset{\sim}{\xi}) = -\, \frac{2}{4\pi}\, \underset{\sim}{\nabla}\, \delta\Phi \cdot \underset{\sim}{\nabla}\, \delta\Phi + (\text{divergence}). \tag{89}$$

We will discard the divergence. By comparison with equation (27) of Paper I, we see that $\mathcal{E}_F$ differs from the Newtonian energy density only in that the term in equation (89) is twice as large as it should be. We therefore expect the Newtonian limit of $\mathcal{E}_G$ to be $(4\pi)^{-1}\, \underset{\sim}{\nabla}\, \delta\Phi \cdot \underset{\sim}{\nabla}\, \delta\Phi$.

Rather than find the Newtonian limit of $\mathcal{E}_G$ for arbitrary nonradial pulsations, we will restrict ourselves at first to the case of radial pulsations, for which we have explicitly calculated the relativistic expressions (Appendix D). We will then argue that the nonradial Newtonian limit differs from the radial limit in no important respects.

For relativistic radial pulsations we can choose a gauge such that the only two nonzero metric perturbations are

$$\delta\nu = -\, h^0{}_0, \tag{90a}$$

$$\delta\lambda = -\, h^r{}_r. \tag{90b}$$

In terms of the fluid perturbations these are

$$\delta\lambda = -8\pi r e^{\lambda} \rho_0 \mu \xi , \tag{91a}$$

$$\delta\nu' = 8\pi r e^{\lambda} \left[\delta p - \rho_0 \mu \left(\nu' + \frac{1}{r}\right) \xi\right], \tag{91b}$$

where primes denote differentiation with respect to r. The Newtonian limits of these expressions are

$$\delta\lambda = -8\pi r \rho \xi , \tag{92a}$$

$$\delta\nu' = -8\pi \rho \xi . \tag{92b}$$

From equation (88) applied to the radial case we see that indeed $\delta\nu = 2\,\delta\Phi$. Moreover it is clear that $\delta\lambda$ is of the same order as $\delta\nu$.

The energy $\mathcal{E}_G$ for radial pulsations is

$$(\mathcal{E}_G)_{\text{RADIAL, RELATIVISTIC}} = \frac{e^{\nu/2-\lambda}}{8\pi} \left[\tfrac{1}{4}\nu' (\delta\nu - \delta\lambda)(\delta\nu' - \delta\lambda') + \frac{1}{r}\delta\lambda\,(\delta\lambda' + \delta\nu')\right] . \tag{93}$$

The first term is post-Newtonian compared to the second ($\nu' \ll 1/r$). From equations (92) we find the Newtonian limit to be

$$(\mathcal{E}_G)_{\text{RADIAL, NEWT}} = \frac{1}{8\pi} \left\{ 2(\delta\nu')^2 + \tfrac{1}{2} r \left[(\delta\nu')^2\right]' \right\} . \tag{94}$$

If we add the divergence

$$-\frac{1}{16\pi} g^{-\frac{1}{2}} \left[g^{\frac{1}{2}} r\,(\delta\nu')^2\right]' = -\frac{1}{16\pi r^2}\left[r^3\,(\delta\nu')^2\right]' , \tag{95}$$

we obtain

$$(\mathcal{E}_G)_{\text{RADIAL, NEWT}} = \frac{1}{16\pi}(\delta\nu')^2 = \frac{1}{4\pi}\delta\Phi'\,\delta\Phi' \,. \tag{96}$$

This is exactly what we require to make $\mathcal{E} = \mathcal{E}_F + \mathcal{E}_G$ reduce to the Newtonian energy density for radial pulsations.

We should expect the same result for nonradial pulsations. The nonradial case is made difficult because the appropriate limiting values of $h^{\alpha\beta}$ depend upon the gauge. Even in the radial case we saw that $\delta\lambda$ was comparable in size to $\delta\nu$. Nevertheless, the Newtonian limit of $\mathcal{E}_G$ cannot depend upon the gauge. It should be possible to construct a gauge in which the only two metric perturbations that have nonzero Newtonian limits will be $h^0{}_0$ and $h^r{}_r$. Dragging of inertial frames (given by $h^0{}_i$) and the nonexistence of intrinsically spherical two-surfaces (due to $h^\vartheta{}_\varphi$ and $h^\vartheta{}_\vartheta - h^\varphi{}_\varphi$) are physically of post-Newtonian order. Moreover, gauge freedom can be used to make $h^\vartheta{}_\vartheta$, $h^\vartheta{}_r$, and $h^\varphi{}_r$ of post-Newtonian order, leaving only $h^0{}_0$ and $h^r{}_r$ at the Newtonian level. In such a gauge $\mathcal{E}_G$ will have a Newtonian limit substantially like equation (94), only with three-dimensional gradients replacing r-derivatives. Then $\mathcal{E}$ will limit to the correct Newtonian energy density.

IV. STABILITY

a) The Sufficient Condition

The energy density $\mathcal{E}$ has three properties that qualify it as a Lyapunov function [see, e.g., La Salle and Lefschetz (1961)]: i) it is homogeneous and quadratic in the perturbation variables; ii) it is globally conserved; and iii) its integral over the interior of a large but finite sphere surrounding the star must decrease if the radiation satisfies a physically meaningful outgoing wave boundary condition on the sphere. Therefore a sufficient condition for stability is that $\mathcal{E}$ be positive-definite, i.e., that the integral of $\mathcal{E}$ over the interior of the large sphere be positive for all nontrivial physically acceptable perturbations.

By "physically acceptable" we mean that the perturbation and its time derivative must be consistent with the perturbed initial-value equations. If one specifies $\underset{\sim}{\xi}$ and $\underset{\sim}{\xi}_{,0}$ on the hypersurface, one is not free to specify all ten $h^{\alpha\beta}$ and their derivatives. The initial-value equations (perturbed versions of $G_{\mu 0} - 8\pi\, T_{\mu 0} = 0$) set four restrictions on the twenty functions $h^{\alpha\beta}$ and $h^{\alpha\beta}{}_{,0}$. In addition the choice of a gauge sets twelve more restrictions: The gauge completely determines four of the $h^{\alpha\beta}$ throughout spacetime (four conditions on $h^{\alpha\beta}$ and four conditions on $h^{\alpha\beta}{}_{,0}$ on the hypersurface), plus it permits solving for the four perturbed lapse and shift functions in terms of the remaining variables [cf. ADaM (1962) or Wheeler (1964)]. Another way to do this counting is to realize that the perturbed geometry is completely specified by giving the twelve functions h_{ij} and $h_{ij,0}$ on the hypersurface, though coordinate (gauge) arbitrariness off the hypersurface leaves some indeterminacy in $h^{\alpha\beta}$ off the hypersurface. Then imposing a gauge in the hypersurface (four conditions) and solving the four initial-value equations in the hypersurface reduce the number of free functions to four. Thus, $\mathcal{E}$ must be positive-definite for arbitrary values of the six functions ξ^i and $\xi^i{}_{,0}$ plus the four independent functions among $h^{\alpha\beta}$ and $h^{\alpha\beta}{}_{,0}$. (Unfortunately one is not likely to be able to prove $\mathcal{E}$

positive-definite without imposing the initial-value equations, as we show in the next paragraph.)

b) Obstacles to the Application of this Condition

Both the solution of the initial-value equations and the imposition of a gauge appear to be crucial before the sufficient condition can be used. In Newtonian theory the analogue of the initial-value equations is the source equation for the gravitational potential, $\nabla^2\Phi = 4\pi\rho$. The contribution of the perturbed potential, $\delta\Phi$, to the energy of pulsation is negative-definite (cf. Paper I). Only by solving for $\delta\Phi$ as a Green's functions integral over $\underset{\sim}{\xi}$, or in terms of the longitudinal part of $\rho\underset{\sim}{\xi}$ (as was done in Paper I), can the entire pulsation energy be shown to be positive-definite.

The imposition of a gauge is important because ε is not gauge-invariant (though its integral over the hypersurface is). It may happen that even after solving the initial-value equations one may be able to prove the positive-definiteness of the energy density easily only in some gauges. Thus part of the problem is to find a gauge in which ε_G (or ε_G plus some of the terms in ε_F that are quadratic in $h^{\alpha\beta}$) is manifestly positive-definite in the four free gravitational variables that remain. If such a gauge can be found then the contribution to ε from ε_G can be discarded, and the sufficient condition reduced to an integral just over the interior of the star (plus possible surface integrals, as in Paper I). In that form, with the remaining energy a function only of $\underset{\sim}{\xi}$, the condition will be tractable and ready for application to realistic stellar models.

We should remark that the gauge problem can probably be solved without going to a specific stellar model. The purpose of the gauge is to prove that the "free" gravitational waves -- those that can be specified on the hypersurface independently of the star's perturbation $\underset{\sim}{\xi}$ -- have positive energy. We should also remember that the gauge that solves the radiation problem may not be the same gauge that makes the dynamical equations simple [e.g., the Regge-Wheeler gauge used by Thorne and Campolattaro (1967) for the nonradial pulsations of spherical stars]. Generally, one might expect the dynamical fluid equations to be simplest in the "near-zone" or "Coulomb"-type gauge, which might be poorly behaved at spatial infinity. The gauge that proves the gravitational wave energy to be positive-definite, on the other hand, is likely to be a "radiation" or "Lorentz"-type gauge. This conflict may pose no problem since one need never solve the dynamical fluid equations to use the criterion: one need only prove that a certain functional of $\underset{\sim}{\xi}$ is positive-definite.

c) An Example: Radial Pulsation

To illustrate the procedure outlined above on a problem whose solution is known, we evaluate $\mathcal{E}$ for the radial pulsations of a spherical star. We will find that $\mathcal{E}$ reduces to the same functional whose positive-definiteness Chandrasekhar (1964) proved was necessary and sufficient for stability.[3] The details of the calculations are contained in

---

[3]Taub (1969) derived Chandrasekhar's criterion from the second variation of a variational principle of his own. This appears to be the first application of the second variation to stability problems in relativistic astrophysics.

Appendix D.

i) Choice of a Gauge

The unperturbed metric is given by equation (81). For radial pulsations it is possible to choose a gauge in which the only nonzero metric perturbations are $\delta\nu = -h^0{}_0$ and $\delta\lambda = -h^r{}_r$ [see, for example, Landau and Lifshitz (1962)]. Both can be made to vanish outside the star.

ii) Eliminating Non-Dynamical Gravitational Variables

Since there are no gravitational waves, both $\delta\lambda$ and $\delta\nu$ are determined completely by the fluid perturbations. The two "initial-value equations" that are relevant are

$$r^{\frac{1}{2}}\,(re^{-\lambda})' - \frac{1}{r^2} = 8\pi\; T^0{}_0 \tag{97a}$$

and

$$e^{-\lambda}\,\left(\frac{1}{r}\,\nu' + \frac{1}{r^2}\right) - \frac{1}{r^2} = 8\pi\; T^r{}_r \tag{97b}$$

(where primes denote $\partial/\partial r$). Following Chandrasekhar (1964), the perturbed versions of these equations can be solved to give

$$\delta\lambda \;=\; -\,8\pi r e^{\lambda}\,\rho_0\,\mu\,\xi \tag{98a}$$

and

$$\delta\nu' = 8\pi r e^{\lambda}\left[\delta p - \rho_0\,\mu\,(\nu' + \frac{1}{r})\,\xi\right]\,. \tag{98b}$$

We will not need the last equation for $\delta\nu$ because ε will contain only $\delta\lambda$.

iii) Calculating the Energy Density

In Appendix D we show that the two parts of the gravitational energy density, equation (67), are

$$g^{\alpha\beta}\,(g^{\mu}{}_{\nu\mu}\, g^{\nu}{}_{\alpha\beta} - g^{\mu}{}_{\nu\beta}\, g^{\nu}{}_{\alpha\mu})$$
$$= \tfrac{1}{4}\nu' e^{-\lambda}\,(\delta\nu - \delta\lambda)(\delta\nu' - \delta\lambda') + \frac{1}{r} e^{-\lambda}\delta\lambda\,(\delta\lambda' + \delta\nu') \tag{99}$$

and

$$-\,\bar{h}^{\alpha\beta}{}_{,0}\, g^{0}{}_{\alpha\beta} + \bar{h}^{\alpha 0}{}_{,0}\, g^{\mu}{}_{\alpha\mu} = 0.$$

Adding to the energy density the divergence

$$-\frac{1}{64\pi}\, g^{-\frac{1}{2}}\left[g^{\frac{1}{2}} N\,\nu' e^{-\lambda}(\delta\nu - \delta\lambda)^2 + 4 g^{\frac{1}{2}} N\,\frac{1}{r}\, e^{-\lambda}\delta\lambda^2\right]' \quad ,$$

we obtain

$$\begin{aligned} e^{-\nu/2}\,(\varepsilon_G)_{\text{radial}} &= \frac{1}{8\pi r} e^{-\lambda}\delta\lambda\delta\nu' - \frac{1}{64\pi} e^{-\lambda}\delta\nu^2\,(\nu'' - \tfrac{1}{2}\lambda'\nu' + \tfrac{1}{2}\nu'^2 + \tfrac{2}{r}\nu') \\ &+ \frac{1}{32\pi} e^{-\lambda}\delta\nu\delta\lambda\,(\nu'' - \tfrac{1}{2}\lambda'\nu' + \tfrac{1}{2}\nu'^2 + \tfrac{2}{r}\nu') \\ &- \frac{1}{64\pi} e^{-\lambda}\delta\lambda^2\,(\nu'' - \tfrac{1}{2}\lambda'\nu' + \tfrac{1}{2}\nu'^2 + \tfrac{4}{r}\nu' - \tfrac{2}{r}\lambda' + \tfrac{4}{r^2}). \end{aligned} \tag{100}$$

In Appendix D we also show that ε_F becomes

$$\begin{aligned} e^{-\nu/2}\,(\varepsilon_F)_{\text{radial}} &= -\frac{1}{8}(\rho - p + 2\gamma p)\,\delta\lambda^2 - \tfrac{1}{4}(\rho + 3p)\,\delta\lambda\delta\nu + \frac{1}{8}(\rho + 3p)\,\delta\nu^2 \\ &- \delta p\,\delta\lambda + \rho_0 T\delta S\delta\nu - \mu\delta\nu\,\underset{\sim}{\nabla}\cdot(\rho_0\xi) + \rho_0\mu\, e^{\lambda-\nu}(\xi_{,0})^2 \\ &+ \gamma p(\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi})^2 + 2(\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi})p'\xi + \rho_0^{-1} p'\rho_0'\xi^2 - \tfrac{1}{2}\rho_0 T S'\nu'\xi^2. \end{aligned} \tag{101}$$

When ε_G is added to ε_F, and a convenient divergence added as well, the coefficients of all terms containing $\delta\nu$ vanish by virtue of equation (98a) and the unperturbed field equations. When $\delta\lambda$ is expressed in terms of ξ from equation (98a) and another divergence added, the resultant expression can be simplified to

$$(\varepsilon)_{radial} = \rho_0 \mu\, e^{\lambda - \frac{\nu}{2}} (\xi_{,0})^2 + p\,\gamma\, e^{\nu/2} \chi^2 - \frac{e^{\nu/2}}{\rho_0\,\mu} (p')^2\, \xi^2 + \frac{4}{r} e^{\nu/2}\, p'\, \xi^2 + 8\pi\, e^{\lambda + \frac{\nu}{2}} \rho_0 \mu\, p\, \xi^2, \qquad (102)$$

where χ stands for

$$\chi \equiv \frac{1}{r^2} e^{\nu/2} (r^2 e^{-\nu/2} \xi)' \; . \qquad (103)$$

Then positive-definiteness of the total energy,

$$E_{radial} = \int_0^\infty \varepsilon\, 4\pi r^2\, e^{\lambda/2}\, dr, \qquad (104)$$

for all possible ξ and $\xi_{,0}$ guarantees stability.

Chandrasekhar (1964) proved that the positive-definiteness of this ε integrated from $r = 0$ to $r = R$ (surface of the star) is necessary and sufficient for stability. Since ε is zero for $r > R$ and contains no delta functions at $r = R$, we see that our results demonstrate the sufficiency of Chandrasekhar's criterion. In the next section we use our methods to show that his criterion is also necessary.

iv) Lagrangian for Radial Pulsation

The radial pulsations of a relativistic star are very similar to Newtonian pulsations: there is no gravitational radiation, and the perturbed gravitational field ($\delta\lambda$ and $\delta\nu$) can be expressed entirely in terms of ξ on a given hypersurface, without reference to the dynamics on previous hypersurfaces (cf. eqs. [98]). It is therefore possible to follow the procedure of Paper I here: one can substitute ξ directly into the Lagrangian density, equation (31), and use the resultant expression as the reduced Lagrangian density for the radial pulsations. The calculations are very similar to those required to reduce ε. The result is

$$(L_2)_{\text{radial}} = -\rho_0 \mu e^{\lambda-\nu} (\xi_{,0})^2 + p\gamma\chi^2 - \frac{1}{\rho_0\mu}(p')^2 \xi^2 + \frac{4}{r} p' \xi^2 + 8\pi e^{\lambda} \rho_0 \mu p \xi^2 , \tag{105}$$

where χ was defined by equation (103). Clearly the energy density ε is the Hamiltonian density associated with this Lagrangian density.

The theorem of Laval, Mercier, and Pellat (1965) applies to this case and implies that the positive-definiteness of E_{radial} (eq. [104]) is necessary and sufficient for stability. This demonstrates how Chandrasekhar's theorem can be obtained with our approach. Needless to say, Chandrasekhar's own methods are much better for such a simple case. We used ours only to illustrate the more general procedure.

V. OUTLOOK

The stability criterion derived in this paper is only the first step in what promises to be a difficult but rewarding search for a useful stability criterion for relativistic stars. I have already discussed what steps may be needed before the goal is achieved. The most promising approach seems to me to be the analogue of the ADaM approach to the full field equations: choose a transverse-traceless gauge and solve the initial-value equations. There may be other workable approaches, however. In Appendix C is derived the rate of transfer of energy from $\mathcal{E}_F$ to $\mathcal{E}_G$; it may happen that with the "outgoing-energy" boundary condition and a careful choice of gauge, the initial-value equations imply that this rate is positive. Then $\mathcal{E}_F$ itself must decrease in time and so <u>its</u> positive-definiteness alone would guarantee stability. Both these approaches are under investigation.

Moreover, the Lagrangian, equation (31), has applications beyond the derivation of the sufficient criterion of this paper. It should be possible to derive from it the results of Chandrasekhar and Friedman (1971) in the zero-frequency approximation. It should also be possible to derive from it general criteria for the stability of standing-wave modes. Such criteria might well be less complicated than the one presented in this paper, and might serve as reasonably good indicators of the stability of realistic, outgoing-wave pulsations. The Lagrangian may prove to be an even more useful tool than the sufficient criterion for stability.

I would like to thank Sandor Kovacs and especially James Bardeen for many helpful conversations. I am also deeply grateful to Kip S. Thorne for his remarks on this paper and for his continued advice and support during the past three years.

APPENDIX A

THE SECOND VARIATION OF THE VELOCITY-POTENTIAL LAGRANGIAN

The full velocity potential Lagrangian is (Schutz 1970)

$$\mathcal{L} = (R + 16\pi p)(-g)^{\frac{1}{2}} \ . \qquad (A1)$$

Its second variation is the part that is quadratic in the perturbations when the full perturbed values of the independent variables (Palatini style: $g^{\sigma\nu}$, $\Gamma^{\gamma}{}_{\sigma\nu}$, ψ, α, β, Θ, S) are substituted into equation (A1). By definition, the second variation of any of the independent variables themselves is zero. We treat the two parts of $\mathcal{L}$ separately.

a) Second Variation of the Fluid Lagrangian

The fluid Lagrangian is $p(-g)^{\frac{1}{2}}$. Its second variation is

$$\delta^2\left[p(-g)^{\frac{1}{2}}\right] = \delta^2 p\ (-g)^{\frac{1}{2}} + 2\ \delta p\, \delta\left[(-g)^{\frac{1}{2}}\right] + p\delta^2\left[(-g)^{\frac{1}{2}}\right] \ . \qquad (A2)$$

Now, the middle term is easy:

$$\delta\left[(-g)^{\frac{1}{2}}\right] = -\tfrac{1}{2}(-g)^{\frac{1}{2}}\, g_{\sigma\nu}\, h^{\sigma\nu} = -\tfrac{1}{2}(-g)^{\frac{1}{2}} h \quad , \qquad (A3)$$

and

$$\delta p = \rho_0\, \delta\mu - \rho_0\, T\, \delta S \quad , \qquad (A4)$$

with

$$\delta\mu = \delta\left[(-g^{\nu\sigma}\ V_\nu V_\sigma)^{\frac{1}{2}}\right] = -\frac{1}{2\mu} h^{\nu\sigma} V_\nu V_\sigma - \frac{1}{\mu} g^{\nu\sigma} V_\sigma \delta V_\nu \quad , \qquad (A5)$$

where δV_ν stands for

$$\delta V_\nu = \delta\psi_{,\nu} + \alpha\, \delta\beta_{,\nu} + \delta\alpha\ \beta_{,\nu} - S\ \delta\Theta_{,\nu} - \delta S\ \Theta_{,\nu} \ . \qquad (A6)$$

The second variation of $(-g)^{\frac{1}{2}}$ is also not hard to find:

$$\delta^2\left[(-g)^{\frac{1}{2}}\right] = \delta\left[\delta(-g)^{\frac{1}{2}}\right] = (-g)^{\frac{1}{2}}\left(\tfrac{1}{4}h^2 + \tfrac{1}{2}h^{\mu\sigma}h_{\mu\sigma}\right). \tag{A7}$$

The second variation of p comes from equation (A4):

$$\delta^2 p = \delta(\rho_0\,\delta\mu - \rho_0 T\,\delta S) = \delta\rho_0\,\delta\mu + \rho_0\,\delta^2\mu - \delta(\rho_0 T)\,\delta S \tag{A8}$$

(recall that $\delta^2 S \equiv 0$). From equation (A5) we can compute $\delta^2\mu$:

$$\begin{aligned}\delta^2\mu = &-\frac{1}{\mu}(\delta\mu)^2 - \frac{2}{\mu}h^{\sigma\nu}\,V_\sigma\,\delta V_\nu - \frac{1}{\mu}g^{\nu\sigma}\,\delta V_\sigma\,\delta V_\nu \\ &-\frac{1}{\mu}g^{\nu\sigma}\,V_\sigma\,\delta^2 V_\nu\,.\end{aligned} \tag{A9}$$

Finally, we can find $\delta^2 V_\nu$ from equation (A6):

$$\delta^2 V_\nu = 2\,\delta\alpha\,\delta\beta_{,\nu} - 2\,\delta S\,\delta\theta_{,\nu}. \tag{A10}$$

Equations (A8)-(A10) combine to give

$$\begin{aligned}\delta^2 p = &\,\delta\rho_0\,\delta\mu - \delta(\rho_0 T)\,\delta S - \frac{\rho_0}{\mu}(\delta\mu)^2 - 2\,\rho_0\,h^{\sigma\nu}U_\sigma\,\delta V_\nu \\ &-\frac{\rho_0}{\mu}g^{\nu\sigma}\,\delta V_\sigma\,\delta V_\nu - 2\,\rho_0\,U^\nu\,(\delta\alpha\,\delta\beta_{,\nu} - \delta S\,\delta\theta_{,\nu}).\end{aligned} \tag{A11}$$

This equation plus equations (A3), (A4), and (A7) when substituted into equation (A2) give equation (25) in the body of this paper.

b) Second Variation of the Curvature Lagrangian

In the Palatini method, the curvature Lagrangian is $g^{\alpha\beta}\,R_{\alpha\beta}(\Gamma)(-g)^{\frac{1}{2}}$.

Its second variation is

$$\delta^2 \left[R(-g)^{\frac{1}{2}}\right] = 2\, h^{\alpha\beta}\, \delta R_{\alpha\beta}\, (-g)^{\frac{1}{2}} + 2\, h^{\alpha\beta}\, R_{\alpha\beta}\, \delta \left[(-g)^{\frac{1}{2}}\right]$$

$$+ g^{\alpha\beta}\, \delta^2(R_{\alpha\beta})(-g)^{\frac{1}{2}} + 2\, g^{\alpha\beta}\, \delta R_{\alpha\beta}\, \delta \left[(-g)^{\frac{1}{2}}\right] \quad \text{(A12)}$$

$$+ g^{\alpha\beta}\, R_{\alpha\beta}\, \delta^2 \left[(-g)^{\frac{1}{2}}\right] .$$

The only terms here that we have not yet computed are

$$\delta R_{\alpha\beta} = \delta \left[\Gamma^{\mu}{}_{\alpha\beta,\mu} - \Gamma^{\mu}{}_{\alpha\mu,\beta} + \Gamma^{\mu}{}_{\nu\mu}\, \Gamma^{\nu}{}_{\alpha\beta} - \Gamma^{\mu}{}_{\nu\beta}\, \Gamma^{\nu}{}_{\alpha\mu}\right]$$

$$= g^{\mu}{}_{\alpha\beta;\mu} - g^{\mu}{}_{\alpha\mu;\beta} \;, \quad \text{(A13)}$$

and

$$\delta^2 R_{\alpha\beta} = 2\, g^{\mu}{}_{\nu\mu}\, g^{\nu}{}_{\alpha\beta} - 2\, g^{\mu}{}_{\nu\beta}\, g^{\nu}{}_{\alpha\mu} \;. \quad \text{(A14)}$$

It is straightforward to plug equations (A3), (A7), (A13), and (A14) into (A12) to obtain equation (29) in the body of this paper.

APPENDIX B

EULERIAN PERTURBATIONS

In this paper we often have occasion to convert from $\delta\psi$, $\delta\alpha$, ... to the fluid displacement, ξ. We shall write down the necessary expressions. More details can be found in Lebovitz (1961) or Lynden-Bell and Ostriker (1967). We use the language of the 3 + 1 split of the background spacetime: ξ is the displacement of the fluid in the hypersurface of constant time, whose metric is g_{ij}. The determinant of g_{ij} is g. Covariant derivatives in the hypersurface are denoted by ∇ or by a subscripted slash, "|".

Because baryons are conserved, the change in rest mass inside a coordinate volume equals the transport of rest mass across its surface:

$$\delta\ (\rho_0 U^0 N g^{\frac{1}{2}}) = -g^{\frac{1}{2}}\ \nabla \cdot (\rho_0 U^0 N\ \xi) \tag{B1}$$

Because entropy per baryon is conserved, $\rho_0 S$ obeys the same equation as ρ_0. Together with equation (B1) this implies

$$\delta S = -\ \xi \cdot \nabla S\ . \tag{B2}$$

The velocity potentials α and β obey the same equation as S, so their perturbations are

$$\delta\alpha = -\ \xi \cdot \nabla \alpha\ +\ (\delta\alpha)_0\ , \tag{B3a}$$

$$\delta\beta = -\ \xi \cdot \nabla \beta\ +\ (\delta\beta)_0\ , \tag{B3b}$$

where $(\delta\alpha)_0$ and $(\delta\beta)_0$ are the values of $\delta\alpha$ and $\delta\beta$ when $\xi = 0$. They represent an initial velocity perturbation. They are "constants" of integration in the following sense:

$$U^{\nu}\left[(\delta\alpha)_0\right]_{,\nu} = U^{\nu}\left[(\delta\beta)_0\right]_{,\nu} = 0. \tag{B4}$$

Note that for δS the constant of integration is zero (cf. Paper I). The potentials $\delta\psi$ and $\delta\Theta$ do not have equations as nice as equations (B3) because they are not "conserved" in the way α, β, and S are.

The changes in p, μ, ρ, T, ... can be computed from equations (B1) and (B2) and the equation of state. We obtain

$$\delta p = -\gamma p\,(\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi} + g^{-\frac{1}{2}}\,\delta g^{\frac{1}{2}}) - \underset{\sim}{\xi}\cdot\underset{\sim}{\nabla}p - \frac{\gamma p}{U^0N}\left[\delta(U^0N) + \underset{\sim}{\xi}\cdot\underset{\sim}{\nabla}(U^0N)\right], \tag{B5}$$

$$\delta\mu = -\frac{\gamma p}{\rho_0}(\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi} + g^{-\frac{1}{2}}\,\delta g^{\frac{1}{2}}) - \underset{\sim}{\xi}\cdot\underset{\sim}{\nabla}\mu - \frac{\gamma p}{\rho_0U^0N}\left[\delta(U^0N) + \underset{\sim}{\xi}\cdot\underset{\sim}{\nabla}(U^0N)\right], \tag{B6}$$

$$\delta T = \left(\frac{\partial T}{\partial p}\right)_S \delta p + \left(\frac{\partial T}{\partial S}\right)_p \delta S\,, \tag{B7}$$

with the Maxwell identity

$$\left(\frac{\partial T}{\partial p}\right)_S = \frac{1}{\rho_0 p\gamma}\left(\frac{\partial p}{\partial S}\right)_{\rho_0} = \frac{1}{\rho_0{}^2}\left(\frac{\partial\rho_0}{\partial S}\right)_p. \tag{B8}$$

If we define the three-dimensional coordinate velocity, $\underset{\sim}{v}$, by the equation

$$v^i = U^i/U^0, \tag{B9}$$

then we have

$$\delta v^i = \xi^i{}_{,0} + \pounds_{\underset{\sim}{v}}\,\xi^i\,, \tag{B10a}$$

$$= \xi^i{}_{,0} + \xi^i{}_{|j}\,v^j - v^i{}_{|j}\,\xi^j\,. \tag{B10b}$$

This equation and equation (B2) render the perturbed entropy equation,

$$\delta\left(\frac{1}{U^0} U^\nu S_{,\nu}\right) = \delta S_{,0} + \delta v^i S_{,i} + v^i \delta S_{,i} = 0, \qquad \text{(B11)}$$

an identity, and similarly for the α, β, and ρ_0 equations.

APPENDIX C

THE ENERGY OF PULSATION

a) The Hamiltonian

The generalized momenta of the problem, $\partial L_2/\partial q_{r,0}$, are not all independent, so one cannot solve for the velocities in terms of the momenta. Dirac (1958a,b) has developed an algorithm for expressing the equations of motion in Hamiltonian form in such situations, and Schutz (1971b) has applied the method to the relativistic perfect fluid, starting from the full velocity-potential Lagrangian, equation (9). The only result we will need here is a result demonstrated in the appendix to Schutz (1971b) for the time derivative of the Hamiltonian.

The Hamiltonian is

$$H_2 = \sum_r N\, q_{r,0} \frac{\partial L_2}{\partial q_{r,0}} - N\, L_2 \,, \tag{C1}$$

where $q_r \equiv \{g^{\mu}{}_{\alpha\beta},\ h^{\alpha\beta},\ \delta\psi,\ \delta\alpha,\ \delta\beta,\ \delta\theta,\ \delta S\}$. The overall factor of $N \equiv (-g^{00})^{-\frac{1}{2}}$ in H_2 arises from our abandoning general covariance: The action is to be expressed in the form

$$I_2 = \int \left(\sum_r p^r\, q_{r,0} - H_2\right) g^{\frac{1}{2}}\, d^3x\, dt \,. \tag{C2}$$

In order that this should be the same as

$$I_2 = \int L_2\, (-{}^4g)^{\frac{1}{2}}\, d^4x = \int L_2 N\, g^{\frac{1}{2}}\, d^3x\, dt, \tag{C3}$$

we need to include the factor of N in H_2 and in the generalized momenta, p^r.

By the theorem from Schutz (1971b), the time derivative of H_2 is

$$\frac{\partial H_2}{\partial t} = \left[\sum_r q_{r,0} \frac{\partial H_2}{\partial q_{r|i}}\right]_{|i} - \frac{\partial}{\partial t}(NL_2)_{\text{holding } q_r,\ q_{r,0} \text{ fixed}} \quad . \tag{C4}$$

This is the same as for a nondegenerate Hamiltonian.

If L_2 did not depend explicitly on time, then H_2 would be globally conserved. However, L_2 does depend upon time. From the remarks in §IIIb,i we find that

$$\frac{\partial}{\partial t} NL_2 = -32\pi N \frac{\rho_0}{\mu} (g^{\sigma\nu}\, \delta V_\sigma + \mu U_\sigma\, h^{\sigma\nu})(\Omega_{,\nu}\, \delta\alpha + \mathcal{T}_{,\nu}\, \delta S). \tag{C5}$$

Here we have defined the "redshifted" temperature,

$$\mathcal{T} \equiv T/U^0. \tag{C6}$$

The first parenthesis in equation (C5) is just δV^ν. In terms of the coordinate velocity, $v^i \equiv U^i/U^0$, equation (C5) becomes

$$\frac{\partial}{\partial t} NL_2 = -32\pi N \rho_0 U^0 \delta v^i\, (\Omega_{,i}\, \delta\alpha + \mathcal{T}_{,i}\, \delta S). \tag{C7}$$

In obtaining this we used the fact that Ω and $\mathcal{T}$ are independent of t and φ.

We can express δv^i, $\delta\alpha$, and δS in terms of $\underset{\sim}{\xi}$ by using equations (B10), (B2), and (B3a). Then manipulations similar to those of Appendix A of Paper I can simplify equation (C7) considerably. The crucial idea in the manipulations is that the quantity

$$M_{ij} \equiv \alpha_{,i}\, \Omega_{,j} + S_{,i}\, \mathcal{T}_{,j} \tag{C8}$$

is symmetric; its antisymmetric part is

$$\frac{\partial}{\partial t} V_{[i;j]} = \frac{\partial}{\partial t} (\alpha_{[,j}\, \beta_{,i]} - S_{[,j}\, \Theta_{,i]})$$

$$= M_{[ij]} ,$$

which must vanish because the unperturbed flow is stationary. The final result of the manipulations is

$$-32\pi N \rho_0 U^0 \delta v^i (\Omega_{,i}\, \delta\alpha + \mathcal{J}_{,i}\, \delta S) =$$

$$\frac{\partial}{\partial t}\left[16\pi N \rho_0 U^0 M_{ij} \xi^i \xi^j - 32\pi N \rho_0 U^0 \Omega_{,i} \xi^i (\delta\alpha)_0\right] \qquad \text{(C9)}$$

$$+ \left[16\pi N \rho_0 U^0 M_{ij} \xi^i \xi^j v^\ell - 32\pi N \rho_0 U^0 \Omega_{,i} \xi^i (\delta\alpha)_0 v^\ell\right]_{|\ell} .$$

Notice that the initial perturbation in α appears explicitly.

From this equation we see that the term that prevents H_2 from being conserved is itself a time-derivative plus a divergence! We can therefore rewrite equation (C4) in the form

$$\frac{\partial}{\partial t}\left[H_2 - 16\pi N \rho_0 U^0 M_{ij} \xi^i \xi^j + 32\pi N \rho_0 U^0 \Omega_{,i} \xi^i (\delta\alpha)_0\right]$$

$$= \left[\sum_r q_{r,0} \frac{\partial H_2}{\partial q_{r|\ell}} + 16\pi N \rho_0 U^0 M_{ij} \xi^i \xi^j v^\ell - 32\pi N \rho_0 U^0 \Omega_{,i} \xi^i (\delta\alpha)_0 v^\ell\right]_{|\ell} . \qquad \text{(C10)}$$

b) The Energy and its Flux

We may tentatively identify the energy density of the pulsations as

$$\mathcal{E}' \equiv H_2 - 16\pi N \rho_0 U^0 M_{ij} \xi^i \xi^j + 32\pi N \rho_0 U^0 \Omega_{,i} \xi^i (\delta\alpha)_0 . \qquad \text{(C11)}$$

Its uniqueness and gauge properties are discussed in §IIId. Here we are interested in evaluating $\mathcal{E}'$ and its flux.

From the Lagrangian $L_2 = \mathcal{R} + 16\pi\mathcal{P}$ given in equations (25) and (29) we find

$$N \frac{\partial L_2}{\partial g^{\mu}{}_{\alpha\beta,0}} = 2N\bar{h}^{\alpha\beta}\,\delta^{0}{}_{\mu} - 2N\bar{h}^{\alpha\beta}\,\delta^{\beta}{}_{\mu} \quad , \tag{C12a}$$

$$N \frac{\partial L_2}{\partial h^{\alpha\beta}{}_{,0}} = 0 \quad , \tag{C12b}$$

$$N \frac{\partial L_2}{\partial \delta\psi_{,0}} = -\,32\pi\, g^{-\frac{1}{2}}\,\delta(\rho_0 U^0 N g^{\frac{1}{2}}) \quad , \tag{C12c}$$

$$N \frac{\partial L_2}{\partial \delta\alpha_{,0}} = 0 \quad , \tag{C12d}$$

$$N \frac{\partial L_2}{\partial \delta\beta_{,0}} = -\,32\pi\, g^{-\frac{1}{2}}\,\delta(\rho_0 U^0 \alpha N g^{\frac{1}{2}}) \quad , \tag{C12e}$$

$$N \frac{\partial L_2}{\partial \delta S_{,0}} = 0 \quad , \tag{C12f}$$

$$N \frac{\partial L_2}{\partial \delta\theta_{,0}} = +\,32\pi\, g^{-\frac{1}{2}}\,\delta(\rho_0 U^0 S N g^{\frac{1}{2}}) \quad . \tag{C12g}$$

These imply that H_2 is

$$H_2 = 2N\bar{h}^{\alpha\beta} \mathcal{S}^0{}_{\alpha\beta,0} - 2N\bar{h}^{\alpha 0} \mathcal{S}^\mu{}_{\alpha\mu,0}$$

$$- 32\pi\, g^{-\frac{1}{2}} \delta(\rho_0 U^0 N g^{\frac{1}{2}}) (\delta\psi_{,0} + \alpha\, \delta\beta_{,0} - S\, \delta\theta_{,0}) \tag{C13}$$

$$- 32\pi N \rho_0 U^0 (\delta\alpha\, \delta\beta_{,0} - \delta S\, \delta\theta_{,0})$$

$$- N\Re - 16\pi N\mathcal{P}.$$

Consider the gravitational part first:

$$H_{2(G)} \equiv 2N\bar{h}^{\alpha\beta} \mathcal{S}^0{}_{\alpha\beta,0} - 2N\bar{h}^{\alpha 0} \mathcal{S}^\mu{}_{\alpha\mu,0} - N\Re. \tag{C14}$$

This would appear to contain second time derivatives of $h^{\alpha\beta}$. Actually it does not, as we can see with the help of an identity that follows from the definition of $\mathcal{S}^\mu{}_{\alpha\beta}$ in terms of $h^{\alpha\beta}$ (eq. [33]):

$$\bar{h}^{\alpha\beta} (\mathcal{S}^\mu{}_{\alpha\beta;\mu} - \mathcal{S}^\mu{}_{\alpha\mu;\beta}) \equiv (-{}^4g)^{\frac{1}{2}} \left[(-{}^4g)^{\frac{1}{2}} (\bar{h}^{\mu\nu} \mathcal{S}^\alpha{}_{\mu\nu} - \bar{h}^{\nu\alpha} \mathcal{S}^\mu{}_{\nu\mu}) \right]_{,\alpha}$$

$$- 2g^{\alpha\beta} (\mathcal{S}^\mu{}_{\nu\mu} \mathcal{S}^\nu{}_{\alpha\beta} - \mathcal{S}^\mu{}_{\nu\beta} \mathcal{S}^\nu{}_{\alpha\mu}). \tag{C15}$$

This identity converts $\Re$ (eq. [29]) to

$$\Re = - 2g^{\alpha\beta} (\mathcal{S}^\mu{}_{\nu\mu} \mathcal{S}^\nu{}_{\alpha\beta} - \mathcal{S}^\mu{}_{\nu\beta} \mathcal{S}^\nu{}_{\alpha\mu})$$

$$+ 2(\bar{h}^{\alpha\beta} \mathcal{S}^0_{\alpha\beta} - \bar{h}^{\alpha 0} \mathcal{S}^\mu{}_{\alpha\mu})_{,0} + \frac{2}{N} \left[N\bar{h}^{\mu\nu} \mathcal{S}^i{}_{\mu\nu} - N\bar{h}^{\nu i} \mathcal{S}^\mu{}_{\nu\mu} \right]_{|i} \tag{C16}$$

$$- hh^{\alpha\beta} R_{\alpha\beta} + R(\tfrac{1}{4} h^2 + \tfrac{1}{2} h_{\alpha\beta} h^{\alpha\beta}).$$

With this, $H_{2(G)}$ becomes

$$H_{2(G)} = -2N\bar{h}^{\alpha\beta}{}_{,0}\, S^{0}{}_{\alpha\beta} + 2N\bar{h}^{\alpha 0}{}_{,0}\, S^{\mu}{}_{\alpha\mu}$$

$$+ 2N(S^{\mu}{}_{\nu\mu} S^{\nu}{}_{\alpha\beta} - S^{\mu}{}_{\nu\beta} S^{\nu}{}_{\alpha\mu}) + N h h^{\alpha\beta} R_{\alpha\beta} - NR(\tfrac{1}{4}h^2 + \tfrac{1}{2}h_{\alpha\beta}h^{\alpha\beta}) \quad \text{(C17)}$$

$$+ 2\left[N\bar{h}^{\mu\nu} S^{i}{}_{\mu\nu} - N\bar{h}^{\nu i} S^{\mu}{}_{\nu\mu}\right]_{|i} \; .$$

This is quadratic in derivatives of $h^{\alpha\beta}$ after we throw away the divergence (we must remember to discard the appropriate time derivative from the flux to compensate this divergence).

We make no modification of the rest of H_2 except to note that

$$\delta\psi_{,0} + \alpha\,\delta\beta_{,0} - S\,\delta\theta_{,0} = \delta V_0 - \beta_{,0}\,\delta\alpha + \theta_{,0}\,\delta S$$

$$= \delta V_0 + \Omega\,\delta\alpha + \mathcal{T}\,\delta S. \quad \text{(C18)}$$

When all terms are assembled and divided by 16π, the result is equations (67) and (68).

The energy flux (Poynting vector) is, from equation (C10),

$$\mathcal{F}'^{\ell} = -\sum_r q_{r,0} \frac{\partial H_2}{\partial q_{r|\ell}} - 16\pi N \rho_0 U^0 M_{ij}\,\xi^i \xi^j v^{\ell} + 32\pi N \rho_0 U^0 \Omega_{,i}\,\xi^i (\delta\alpha)_0 v^{\ell}.$$

(C19)

From expressions similar to equations (C12) we find that

$$- \sum_r q_{r,0} \frac{\partial H_2}{\partial q_{r|\ell}} = \sum_r N\, q_{r,0} \frac{\partial L_2}{\partial q_{r|\ell}}$$

$$= 2N\bar{h}^{\alpha\beta} g^{\ell}{}_{\alpha\beta,0} - 2N\bar{h}^{\alpha\ell} g^{\mu}{}_{\alpha\mu,0}$$

$$- 32\pi g^{-\frac{1}{2}} \delta(\rho_0 U^0 N g^{\frac{1}{2}})(\delta\psi_{,0} + \alpha\delta\beta_{,0} - S\,\delta\theta_{,0})$$

$$- 32\pi N \rho_0 U^0 (\delta\alpha\, \delta\beta_{,0} - \delta S\, \delta\theta_{,0})$$

$$= - 2N\bar{h}^{\alpha\beta}{}_{,0}\, g^{\ell}{}_{\alpha\beta} + 2N\bar{h}^{\alpha\ell}{}_{,0}\, g^{\mu}{}_{\alpha\mu}$$

$$- 32\pi N g^{-\frac{1}{2}} \delta(\rho_0 U^0 N g^{\frac{1}{2}}) (\delta V_0 + \Omega\, \delta\alpha + \mathcal{T}\, \delta S) \tag{C20}$$

$$- 32\pi N \rho_0 U^0 (\delta\alpha\, \delta\beta_{,0} - \delta S\, \delta\theta_{,0})$$

$$+ (2N\bar{h}^{\alpha\beta} g^{\ell}{}_{\alpha\beta} - 2N\bar{h}^{\alpha\ell} g^{\mu}{}_{\alpha\mu})_{,0} \,.$$

The last term in this equation is exactly the one required to cancel the divergence in equation (C12)! So when we discard it and divide by 16π we get equations (69) and (70) for the flux.

For completeness we write down what the first three terms of $H_{2(G)}$ (eq. [C17]) become if we substitute for the $g^{\mu}{}_{\alpha\beta}$'s their expressions in terms of $h^{\alpha\beta}$. This is what in the body of the paper we call

$16\pi\ \mathcal{E}_G$:

$$16\pi\ \mathcal{E}_G = -\tfrac{1}{2} N \bar{h}^{\alpha\beta}{}_{,0}\, \bar{h}_{\alpha\beta}{}^{;0} + \tfrac{1}{2} N \bar{h}^{\alpha\beta}{}_{;i}\, \bar{h}_{\alpha\beta}{}^{;i} - N \bar{h}^{\alpha\beta}{}_{;i}\, \bar{h}^{i}{}_{\alpha;\beta}$$

$$+ N \bar{h}^{\alpha\beta}{}_{,0}\, \bar{h}^{0}{}_{\alpha;\beta} + N \bar{h}_{,0}\, \bar{h}^{,0} + \tfrac{1}{2} N \bar{h}^{,i}\, \bar{h}_{,i} \qquad \text{(C21)}$$

$$+ \tfrac{1}{2} N \left(\Gamma^{\alpha}{}_{\sigma 0}\, \bar{h}^{\sigma\beta} + \Gamma^{\beta}{}_{\sigma 0}\, \bar{h}^{\alpha\sigma}\right)\left(\bar{h}_{\alpha\beta}{}^{;0} - 2\, \bar{h}^{0}{}_{\alpha;\beta}\right).$$

Similarly, the gravitational part of the flux (first two terms of eq. [C20]) becomes

$$16\pi\ \mathcal{F}_G{}^{\ell} = - N \bar{h}^{\alpha\beta}{}_{,0}\, \bar{h}_{\alpha\beta}{}^{;\ell} + 2 N \bar{h}^{\alpha\beta}{}_{,0}\, \bar{h}^{\ell}{}_{\alpha;\beta} + \tfrac{1}{2} N\, \bar{h}_{,0}\, \bar{h}^{,\ell}\ . \qquad \text{(C22)}$$

c) Transfer of Energy Between Fluid and Radiation

The Hamiltonian formalism permits us to calculate not only the rate of change of the total energy density $\mathcal{E}$, but also the rate at which different parts of $\mathcal{E}$ change. In the body of this paper we define

$$\mathcal{E}_F = \mathcal{E} - \mathcal{E}_G$$

$$= \frac{1}{16\pi} H_{2(F)} - N \rho_0 U^0 M_{ij}\, \xi^i \xi^j + 2 N \rho_0 U^0 \Omega_{,i}\, \xi^i (\delta\alpha)_0 \qquad \text{(C23)}$$

$$+ \frac{1}{16\pi} N h h^{\alpha\beta} R_{\alpha\beta} - \frac{1}{16\pi} N R \left(\tfrac{1}{4} h^2 + \tfrac{1}{2} h^{\alpha\beta} h_{\alpha\beta}\right),$$

where $H_{2(F)}$ is the Hamiltonian obtained just from the Lagrangian $\mathcal{P}$:

$$\frac{1}{16\pi} H_{2(F)} = -2 g^{-\frac{1}{2}} \delta(\rho_0 U^0 N g^{\frac{1}{2}}) (\delta V_0 + \Omega\, \delta\alpha + \mathcal{J}\, \delta S) - 2 \rho_0 U^0 N (\delta\alpha\, \delta\beta_{,0} - \delta S\, \delta\Theta_{,0}) - \mathcal{P}. \tag{C24}$$

The time derivative of $\mathcal{E}_F$ can be found in this manner:

The time derivative of $H_{2(F)}$ is of three parts: a part due to the time derivatives of the fluid variables, a part due to the time derivatives of the gravitational variables, and a part due to its explicit time dependence. The last part is cancelled by the time derivative of the second and third terms in equation (C23) (by the construction of the previous section!). The first part is just a divergence because $H_{2(F)}$ is the Hamiltonian that governs the time-derivatives of the fluid variables. Thus we have

$$\frac{\partial}{\partial t}\left[\frac{1}{16\pi} H_{2(F)} - N \rho_0 U^0 M_{ij}\, \xi^i \xi^j + 2 N \rho_0 U^0 \Omega_{,i}\, \xi^i (\delta\alpha)_0\right] = -\mathcal{F}_F{}^k{}_{|k} + \frac{1}{16\pi} h^{\alpha\beta}{}_{,0} \frac{\partial H_{2(F)}}{\partial h^{\alpha\beta}} + \frac{1}{16\pi} g^{\mu}{}_{\alpha\beta,0} \frac{\partial H_{2(F)}}{\partial g^{\mu}{}_{\alpha\beta}}, \tag{C25}$$

where $\mathcal{F}_F{}^k$, which is defined in the body of this paper (eq. [70]), represents the energy carried out of some volume by the fluid itself. Now $H_{2(F)}$ does not depend upon $g^{\mu}{}_{\alpha\beta}$; from equation (35) we find

$$\frac{1}{16\pi}\frac{\partial H_{2(F)}}{\partial h^{\alpha\beta}} = -N\frac{\partial \mathcal{P}}{\partial h^{\alpha\beta}} = g^{-\frac{1}{2}}\,\delta\left[T_{\alpha\beta}\,(-{}^{4}g)^{\frac{1}{2}}\right], \tag{C26a}$$

$$= N\left[\mu U_\alpha U_\beta \delta\rho_0 + \rho_0 U_\alpha U_\beta \delta\mu + 2\rho_0 \mu U_\alpha \delta U_\beta + \delta p\, g_{\alpha\beta} - p h_{\alpha\beta} - \tfrac{1}{2} T_{\alpha\beta}\, h\right]. \tag{C26b}$$

[From eq. (C24) one might conclude that $H_{2(F)}$ depends on $h^{\alpha\beta}$ not only through $\mathcal{P}$ but through the first term, which includes $\delta(\rho_0 U^0 N g^{\frac{1}{2}})$. This is not true: $-2g^{-\frac{1}{2}}\delta(\rho_0 U^0 N g^{\frac{1}{2}})$ is the momentum conjugate to $\delta\psi, \partial L_2/\partial\delta\psi_{,0}$. It is a fluid variable, and its time rate of change is included in $\mathcal{F}_F{}^k$.]

Since the last two terms in $\mathcal{E}_F$ also depend only on $h^{\alpha\beta}$, we can write down $\partial\mathcal{E}_F/\partial t$ immediately:

$$\frac{\partial\mathcal{E}_F}{\partial t} + \mathcal{F}_F{}^k{}_{|k} = N\,h^{\alpha\beta}{}_{,0}\left[\mu U_\alpha U_\beta\,\delta\rho_0 + \rho_0 U_\alpha U_\beta \delta\mu + 2\rho_0 \mu U_\alpha\,\delta U_\beta + \delta p\, g_{\alpha\beta} - \tfrac{1}{2}(\rho - p)\,\bar{h}_{\alpha\beta} + \tfrac{1}{2}\rho_0 \mu\, g_{\alpha\beta} U_\mu U_\nu h^{\mu\nu}\right]. \tag{C27}$$

Since the divergence of $\mathcal{F}_F{}^k$ represents transport of energy by the fluid, the total rate of transfer of energy from $\mathcal{E}_F$ to $\mathcal{E}_G$ is negative of the integral of the right-hand side of equation (C27) over the entire star.

APPENDIX D

THE NONROTATING STAR

a) Arbitrary Pulsations

The nonrotating star has the background metric

$$ds^2 = - e^{\nu} dt^2 + e^{\lambda} dr^2 + r^2 (d\vartheta^2 + \sin^2\vartheta\, d\varphi^2). \tag{D1}$$

From the equation $U^{\alpha} U_{\alpha} = -1$ we find

$$\delta U^0 = - \tfrac{1}{2} e^{-3\nu/2} h_{00} \tag{D2a}$$

and

$$\delta U_0 = - \tfrac{1}{2} e^{-\nu/2} h_{00} \quad . \tag{D2b}$$

From Appendix B we learn

$$\delta S = - \underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} S = - \xi^r S_{,r} \tag{D3a}$$

$$g^{-\frac{1}{2}} \delta(\rho_0 U^0 N g^{\frac{1}{2}}) = g^{-\frac{1}{2}} \delta(\rho_0 g^{\frac{1}{2}}) = - \underset{\sim}{\nabla} \cdot (\rho_0 \underset{\sim}{\xi}) \tag{D3b}$$

$$\delta v^i = \xi^i_{\ ,0} \ . \tag{D3c}$$

In order to put $\mathcal{E}_F$ in terms only of $\underset{\sim}{\xi}$ and $h^{\alpha\beta}$ it is convenient to treat separately the following pieces of $\mathcal{E}_F$:

$$A \equiv - 2g^{-\frac{1}{2}} \delta(\rho_0 U^0 N g^{\frac{1}{2}}) (\delta V_0 + \mathcal{T} \delta S), \tag{D4a}$$

$$B \equiv \frac{\rho_0}{\mu} N g^{\nu\sigma} \delta V_{\nu} \delta V_{\sigma} + 2 N \rho_0 U_{\sigma} h^{\sigma\nu} \delta V_{\nu} + N \frac{\rho_0}{\mu} (\delta\mu)^2, \tag{D4b}$$

$$C \equiv - N \delta\rho_0 \delta\mu + N \delta(\rho_0 T) \delta S - N \rho_0 U^0 S_{,i} \mathcal{T}_{,j} \xi^i \xi^j \quad , \tag{D4c}$$

$$D \equiv N h \delta p + \frac{1}{16\pi} N h h^{\alpha\beta} R_{\alpha\beta} - N (\frac{1}{16\pi} R + p)(\tfrac{1}{4} h^2 + \tfrac{1}{2} h^{\alpha\beta} h_{\alpha\beta}). \tag{D4d}$$

i). A. From the above equations and Appendix B, we find

$$\delta(\rho_0 U^0 N g^{\frac{1}{2}}) = - g^{\frac{1}{2}} \underset{\sim}{\nabla} \cdot (\rho_0 \underset{\sim}{\xi}) ,$$

$$\delta V_0 + \mathcal{T} \delta S = - e^{\nu/2} \delta\mu - \tfrac{1}{2} \mu e^{-\nu/2} h_{00} + e^{\nu/2} T \delta S ,$$

and

$$A = - 2e^{\nu/2} \underset{\sim}{\nabla} \cdot (\rho_0 \underset{\sim}{\xi}) \left[\rho_0^{-1} \delta p + \tfrac{1}{2} \mu e^{-\nu} h_{00}\right] . \qquad \text{(D5)}$$

ii) B. This term contains the kinetic energy of the fluid:

$$\begin{aligned} \delta V_\sigma &= U_\sigma \delta\mu + \mu \delta U_\sigma \\ &= U_\sigma \delta\mu + \mu \delta(g_{\sigma\nu} U^\nu) \\ &= U_\sigma \delta\mu + \mu g_{\sigma 0} \delta U^0 + \mu U^0 g_{\sigma i} \delta v^i - \mu U^\nu h_{00} \\ &= - \delta^0{}_\sigma e^{\nu/2} (\delta\mu + \tfrac{1}{2} \mu e^{-\nu} h_{00}) + \delta^j{}_\sigma \mu e^{-\nu/2} (g_{ij} \delta v^i - h_{j0}). \end{aligned} \qquad \text{(D6)}$$

From this we find

$$g^{\sigma\nu} \delta V_\sigma \delta V_\nu = - (\delta\mu + \tfrac{1}{2} \mu e^{-\nu} h_{00})^2 + \mu^2 e^{-\nu} \left[g_{ij} \xi^i{}_{,0} \xi^j{}_{,0} - 2h_{0i} \xi^i{}_{,0} + h_0{}^j h_{0j}\right]$$

and

$$U_\sigma h^{\sigma\nu} \delta V_\nu = e^{-\nu} (\delta\mu + \tfrac{1}{2} \mu e^{-\nu} h_{00}) h_{00} + \mu e^{-\nu} h_{0i} \xi^i{}_{,0} - \mu e^{-\nu} h_{0j} h_0{}^j .$$

These combine to give

$$\begin{aligned} B = \rho_0 \mu e^{-\nu/2} \underset{\sim}{\xi}_{,0} \cdot \underset{\sim}{\xi}_{,0} + \rho_0 e^{-\nu/2} h_{00} \delta\mu - \rho_0 \mu e^{-\nu/2} h_0{}^j h_{0j} \\ + \tfrac{3}{4} \rho_0 \mu e^{-3\nu/2} h_{00}{}^2 . \end{aligned} \qquad \text{(D7)}$$

iii) C. If we add to C the first term of A from equation (D5) and call the result E, we get

$$E = N\rho_0^{-1} g^{-\frac{1}{2}} \delta(\rho_0 g^{\frac{1}{2}})\, \delta p + N\rho_0 \delta T \delta S - N\rho_0 S_{,i}\, T_{,j}\, \xi^i \xi^j$$
$$+ N\,\delta p\, g^{-\frac{1}{2}}\, \delta(g^{\frac{1}{2}}) - N\rho_0 U^0 \left(\frac{1}{U^0}\right)_{,j} T S_{,i}\, \xi^i \xi^j \;. \qquad \text{(D8)}$$

The first three terms of this can be written as

$$N\, g^{-\frac{1}{2}}\, \delta(g^{\frac{1}{2}})\, \delta p + N\, \rho_0^{-1}\, \delta\rho_0\, \delta p + N\, \rho_0\, (\Delta T)\, \delta S, \qquad \text{(D9)}$$

where ΔT is the Lagrangian change in T,

$$\Delta T = \left(\frac{\partial T}{\partial p}\right)_S \Delta p$$
$$= -\frac{1}{\rho_0^{\,2}} \left(\frac{\partial \rho_0}{\partial S}\right)_p \Delta p \;. \qquad \text{(D10)}$$

By writing the second term in expression (D9) as

$$N\, \rho_0^{-1}\, \delta\rho_0\, \Delta p - N\, \rho_0^{-1}\, (\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} p)\, \delta\rho_0 \qquad \text{(D11)}$$

and using equation (D10), we find that E becomes

$$E = N\, \rho_0^{-1} \left(\frac{\partial \rho_0}{\partial p}\right)_S \Delta p\, \delta p - N\, \rho_0^{-1}\, \delta\rho_0\, (\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} p)$$
$$+ 2N g^{-\frac{1}{2}}\, \delta(g^{\frac{1}{2}})\, \delta p - \tfrac{1}{2} N \rho_0 T (\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} S)(\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} \nu) \;. \qquad \text{(D12)}$$

But Appendix B tells us that

$$\Delta p = -\,\gamma p \left[\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi} + g^{-\frac{1}{2}}\, \delta(g^{\frac{1}{2}})\right] \;. \qquad \text{(D13a)}$$

Moreover, the definition of γ is

$$\gamma = \frac{\rho_0}{p}\left(\frac{\partial p}{\partial \rho_0}\right)_S \quad . \tag{D13b}$$

Therefore E becomes

$$E = e^{\nu/2}\gamma p(\nabla \cdot \xi)^2 + 2e^{\nu/2}(\nabla \cdot \xi)(\xi \cdot \nabla p) + e^{\nu/2}\rho_0^{-1}(\xi \cdot \nabla p)(\xi \cdot \nabla \rho_0)$$

$$- \tfrac{1}{2}e^{\nu/2}\rho_0 T(\xi \cdot \nabla S)(\xi \cdot \nabla \nu) - \tfrac{1}{4}e^{\nu/2}\gamma p\,(h^j{}_j)^2. \tag{D14}$$

iv) D. Using the unperturbed Einstein equations, we obtain

$$D = e^{\nu/2}h\,\delta p + \tfrac{1}{2}e^{-\nu/2}\rho_0\mu h_{00}h + \frac{1}{8}e^{\nu/2}(\rho - p)h^2 - \tfrac{1}{4}e^{\nu/2}(\rho - p)h^{\alpha\beta}h_{\alpha\beta}. \tag{D15}$$

If we assemble all these terms we obtain equation (83).

b) Radial Pulsations

If Nature is reasonable, the stability criterion proved in this paper ought to reduce to Chandrasekhar's (1964) necessary and sufficient condition for stability against radial pulsations. In this section we show that $\mathcal{E}$ does indeed reduce to Chandrasekhar's variational function.

We can choose a gauge such that the only two nonzero metric perturbations are (see, e.g., Landau and Lifshitz 1962)

$$\delta\nu = e^{-\nu}h_{00} = -h^0{}_0 \tag{D16a}$$

and

$$\delta\lambda = -e^{-\lambda}h_{rr} = -h^r{}_r. \tag{D16b}$$

In this gauge we have ($\tilde{\xi}$ has only an r-component)

$$\delta\mu = -\frac{\gamma p}{\rho_0}(\tilde{\nabla}\cdot\tilde{\xi} + \tfrac{1}{2}\delta\lambda) - \tilde{\xi}\cdot\tilde{\nabla}\mu \tag{D17a}$$

$$\delta p = -\gamma p\,(\tilde{\nabla}\cdot\tilde{\xi} + \tfrac{1}{2}\delta\lambda) - \tilde{\xi}\cdot\tilde{\nabla}p. \tag{D17b}$$

Since there is no dynamical freedom in the gravitational field (no spherical gravitational waves), we ought to be able to express $\delta\nu$ and $\delta\lambda$ in terms of $\tilde{\xi}$. We use the $\binom{0}{0}$ and $\binom{r}{r}$ Einstein equations:

$$\frac{1}{r^2}(r\,e^{-\lambda})' - \frac{1}{r^2} = 8\pi\, T^0{}_0 \tag{D18a}$$

and

$$e^{-\lambda}\left(\frac{1}{r}\nu' + \frac{1}{r^2}\right) - \frac{1}{r^2} = 8\pi\, T^r{}_r \tag{D18b}$$

(where primes denote $\partial/\partial r$). Their perturbed versions can be solved to give (cf. Chandrasekhar 1964)

$$\delta\lambda = -8\pi r\, e^{\lambda}\rho_0\mu\,\xi \quad , \tag{D19a}$$

$$\delta\nu' = 8\pi r\, e^{\lambda}\left[\delta p - \rho_0\mu\left(\nu' + \frac{1}{r}\right)\xi\right] \quad . \tag{19b}$$

We will never need $\delta\nu$ itself; we will only need to substitute for $\delta\lambda$.

To calculate ε_G we need the following $\mathcal{S}$'s [which can be read off the table of Christoffel symbols in Landau and Lifshitz (1962), §97]

$$\begin{aligned} &\mathcal{S}^0{}_{00} = \tfrac{1}{2}\delta\nu_{,0} && \mathcal{S}^r{}_{00} = \tfrac{1}{2}e^{\nu-\lambda}\,[\delta\nu' + \nu'\,(\delta\nu - \delta\lambda)] && \\ &\mathcal{S}^0{}_{0r} = \tfrac{1}{2}\delta\nu' && \mathcal{S}^r{}_{0r} = \tfrac{1}{2}\delta\lambda_{,0} && \mathcal{S}^r{}_{\vartheta\vartheta} = r\,e^{-\lambda}\delta\lambda \\ &\mathcal{S}^0{}_{rr} = \tfrac{1}{2}e^{\lambda-\nu}\,\delta\lambda_{,0} && \mathcal{S}^r{}_{rr} = \tfrac{1}{2}\delta\lambda' && \mathcal{S}^r{}_{\varphi\varphi} = r\sin^2\vartheta\, e^{-\lambda}\delta\lambda. \end{aligned} \tag{D20}$$

All others that cannot be obtained from these by the symmetry

$S^{\mu}{}_{\alpha\beta} = S^{\mu}{}_{\beta\alpha}$ are zero.

With these we find

$$g^{\alpha\beta}\,(S^{\mu}{}_{\nu\mu}\,S^{\nu}{}_{\alpha\beta} - S^{\mu}{}_{\nu\beta}\,S^{\nu}{}_{\alpha\mu})$$

$$= \tfrac{1}{4}\,\nu'\,e^{-\lambda}\,(\delta\nu - \delta\lambda)(\delta\nu' - \delta\lambda') + \frac{1}{r}\,e^{-\lambda}\,\delta\lambda\,(\delta\lambda' + \delta\nu'), \tag{D21}$$

and

$$-\,\bar{h}^{\alpha\beta}{}_{,0}\,S^{0}{}_{\alpha\beta} + \bar{h}^{\alpha 0}{}_{,0}\,S^{\beta}{}_{\alpha\beta} = 0. \tag{D22}$$

Then from equation (67) ε_G is

$$\varepsilon_G = \frac{\nu'}{32\pi}\,e^{\nu/2-\lambda}\,(\delta\nu - \delta\lambda)(\delta\nu' - \delta\lambda') + \frac{1}{8\pi r}\,e^{\nu/2-\lambda}\,\delta\lambda(\delta\lambda' + \delta\nu'). \tag{D23}$$

By adding the divergence

$$-\,\frac{1}{64\pi}\,g^{-\frac{1}{2}}\left\{g^{\frac{1}{2}}\,e^{\nu/2-\lambda}\left[\nu'(\delta\nu - \delta\lambda)^2 + \frac{4}{r}\,\delta\lambda^2\right]\right\}' , \tag{D24}$$

we can eliminate almost all terms that have derivatives of $\delta\lambda$ and $\delta\nu$. [Note that the factors of $g^{\frac{1}{2}}$ in eq. (D24) ensure that the expression will be a divergence when integrated over proper volume in the hypersurface, $g^{\frac{1}{2}}\,d^3x$.] The result is equation (100).

To calculate ε_F we begin with equation (83). We shall need the following field equations:

$$\frac{1}{16\pi}\,R_{00} = \frac{1}{32\pi}\,e^{\nu-\lambda}\left[\nu'' - \tfrac{1}{2}\nu'\lambda' + \tfrac{1}{2}\,(\nu')^2 + \frac{2}{r}\,\nu'\right] = \tfrac{1}{4}e^{\nu}\,(\rho + 3p), \tag{D25}$$

$$\frac{1}{16\pi}\,R_{rr} = \frac{1}{32\pi}\left[-\,\nu'' + \tfrac{1}{2}\nu'\lambda' - \tfrac{1}{2}\,(\nu')^2 + \frac{2}{r}\,\lambda'\right] = \tfrac{1}{4}\,e^{\lambda}\,(\rho - p) \quad , \tag{D26}$$

$$\frac{1}{16\pi} R = -\frac{1}{16\pi}\left[\nu'' - \tfrac{1}{2}\nu'\lambda' + \tfrac{1}{2}(\nu')^2 + \frac{2}{r}(\nu' - \lambda') + \frac{2}{r^2}(1 - e^{\lambda})\right] = \tfrac{1}{2}(\rho - 3p). \tag{D27}$$

Equation (83) becomes

$$\begin{aligned} e^{-\nu/2}\,\mathcal{E}_F = {}& \rho_0 \mu e^{\lambda-\nu} (\xi_{,0})^2 + \gamma p(\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi})^2 + 2(\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi})\, p'\,\xi + \rho_0^{-1} p'\rho_0'\,\xi^2 \\ & - \tfrac{1}{2}\rho_0 T S'\nu'\xi^2 - \mu\,\delta\nu\,\underset{\sim}{\nabla}\cdot(\rho_0\underset{\sim}{\xi}) + \rho_0 T\,\delta S\,\delta\nu - \delta p\,\delta\lambda \\ & + \frac{1}{8}(\rho + 3p)\,\delta\nu^2 - \tfrac{1}{4}(\rho + 3p)\,\delta\nu\,\delta\lambda - \frac{1}{8}(\rho - p + 2\gamma p)\,\delta\lambda^2. \end{aligned} \tag{D28}$$

By adding to $\mathcal{E}_F$ the divergence

$$g^{-\frac{1}{2}}\left[\mu\, e^{\nu/2} g^{\frac{1}{2}} \rho_0\, \xi\, \delta\nu\right]' , \tag{D29}$$

and by adding $\mathcal{E}_F$ to $\mathcal{E}_G$, we obtain for $\mathcal{E}$

$$\begin{aligned} e^{-\nu/2}\,\mathcal{E} = {}& \frac{-1}{16\pi r^2} e^{-\lambda} (1 + r\nu' + 4\pi r^2\gamma p)\,\delta\lambda^2 - \delta\lambda\,\delta p \\ & + \rho_0\mu\, e^{\lambda-\nu} (\xi_{,0})^2 + \gamma p\,(\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi})^2 + 2(\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi})\,p'\,\xi \\ & + \rho_0^{-1}\, p'\,\rho_0'\,\xi^2 - \tfrac{1}{2}\rho_0 T\, S'\,\nu'\,\xi^2. \end{aligned} \tag{D30}$$

All terms containing $\delta\nu$ have cancelled out by virtue of equations (D19a), (D25)-(D27), and the equation of hydrostatic equilibrium,

$$p' = -\tfrac{1}{2}\rho_0\mu\,\nu'. \tag{D31}$$

Now we define

$$\chi \equiv \frac{1}{r^2} e^{\nu/2} (r^2 e^{-\nu/2}\xi)' = \underset{\sim}{\nabla}\cdot\underset{\sim}{\xi} + \tfrac{1}{2}\delta\lambda. \tag{D32}$$

The last step follows from equation (D19a) and the equation

$$\nu' + \lambda' = 8\pi\, r\, \rho_0 \mu\, e^{\lambda} \,. \tag{D33}$$

This equation and the useful identity

$$\nu' + \frac{1}{r} = \frac{1}{r}\, e^{\lambda}\, (1 + 8\pi r^2 p) \tag{D34}$$

both follow from the unperturbed Einstein equations. From the definition of χ and equations (D31) and (D34) we obtain for $\mathcal{E}$

$$\begin{aligned} e^{-\nu/2}\, \mathcal{E} = {} & \rho_0 \mu\, e^{\lambda-\nu}\, (\xi_{,0})^2 + \gamma p\, \chi^2 + 2p'\, \xi\, \chi \\ & - \frac{1}{16\pi}\left(\frac{1}{r^2} + 8\pi p\right) \delta\lambda^2 + \frac{1}{\rho_0}\, p'\, \rho_0{}'\, \xi^2 - \tfrac{1}{2}\, \rho_0\, T\, S'\, \nu'\, \xi^2 . \end{aligned} \tag{D35}$$

If we now substitute equation (D19a) for $\delta\lambda$, add to $\mathcal{E}$ the divergence

$$- g^{-\frac{1}{2}} \left[g^{\frac{1}{2}}\, p'\, e^{\nu/2}\, \xi^2 \right]' \,, \tag{D36}$$

and use the unperturbed TOV equation, we find that $\mathcal{E}$ simplifies to

$$\begin{aligned} e^{-\nu/2}\, \mathcal{E} = {} & \rho_0 \mu\, e^{\lambda-\nu}\, (\xi_{,0})^2 + p\, \gamma\, \chi^2 - \frac{(p')^2}{\rho_0 \mu}\, \xi^2 \\ & + \frac{4p'}{r}\, \xi^2 + 8\pi\, e^{\lambda}\, \rho_0 \mu\, p\, \xi^2 \,. \end{aligned} \tag{D37}$$

This is exactly the function whose positive-definiteness Chandrasekhar (1964) proved was necessary and sufficient for stability. Our "energy density" $\mathcal{E}$ differs from Chandrasekhar's function by the "redshift" factor $e^{\nu/2}$, which arises from our $3+1$ split of spacetime. Our "total energy" is the same as his: his is the integral of equation (D37) over $(-{}^4g)^{\frac{1}{2}}\, d^3x = 4\pi e^{(\nu+\lambda)/2}\, r^2\, dr$, while ours is the integral of $\mathcal{E}$ over $g^{\frac{1}{2}}\, d^3x = 4\pi\, e^{\lambda/2}\, r^2\, dr$.

REFERENCES

Arnowitt, R., Deser, S., and Misner, C. W. 1962, in Gravitation, ed. L. Witten (New York: John Wiley & Sons, Inc.).

Bardeen, J. M. 1970, Ap. J., **162**, 71.

Carter, B. 1969, J. Math. Phys., **10**, 70.

Chandrasekhar, S. 1964, Ap. J., **140**, 417.

__________ 1965a, Phys. Rev. Letters, **14**, 241.

__________ 1965b, Ap. J., **142**, 1488.

Chandrasekhar, S., and Friedman, J. L. 1971, Phys. Rev. Letters, **26**, 1047.

Cocke, W. J. 1965, Ann. Inst. Henri Poincaré, **2**, 283.

Dirac, P. A. M. 1958a, Proc. Roy. Soc. A, **246**, 326.

__________ 1958b, ibid., **246**, 333.

Fackerell, E. D. 1970, Ap. J., **160**, 859.

Harrison, B. K., Thorne, K. S., Wakano, M., and Wheeler, J. A. 1965, Gravitation Theory and Gravitational Collapse (Chicago: University of Chicago Press).

Ipser, J. R. 1969, Ap. J., **156**, 509.

__________ 1971, ibid., **166**, 175.

Ipser, J. R., and Thorne, K. S. 1968, Ap. J., **154**, 251.

__________ 1971, paper in preparation.

Isaacson, R. A. 1968, Phys. Rev., **166**, 1272.

Islam, J. N. 1970, M.N.R.A.S., **150**, 237.

Landau, L. D., and Lifshitz, E. M. 1962, The Classical Theory of Fields, 2nd ed. (Reading, Massachusetts: Addison-Wesley Press).

La Salle, J., and Lefschetz, S. 1961, Stability by Liapunov's Direct Method (New York: Academic Press).

Laval, G., Mercier, C., and Pellat, R. 1965, Nucl. Fusion, 5, 156.

Lebovitz, N. R. 1961, Ap. J., 134, 500.

Lynden-Bell, D., and Ostriker, J. P. 1967, M.N.R.A.S., 136, 293.

Misner, C. W., Thorne, K. S., and Wheeler, J. A. 1972, Gravitation (San Francisco: Freeman and Co.) in press.

Schmid, L. A. 1970a, in A Critical Review of Thermodynamics, eds. E. B. Stuart, B. Gal-Or, and A. J. Brainard (Baltimore: Mono Press).

__________ 1970b, Pure and Appl. Chem., 22, 493.

Schutz, B. F. 1970, Phys. Rev. D., 2, 2762 (Chapter 2 of thesis).

__________ 1971a, preceding paper (Paper I) (Chapter 4 of thesis).

__________ 1971b, "The Hamiltonian Theory of a Relativistic Perfect Fluid," paper in preparation (Chapter 3 of thesis).

Seliger, R. L., and Whitham, G. B. 1968, Proc. Roy. Soc. A., 305, 1.

Taub, A. H. 1954, Phys. Rev., 94, 1468.

__________ 1959, Arch. Rat. Mech. Anal., 3, 312.

__________ 1969, Commun. Math. Phys., 15, 235.

__________ 1970, Bressanone Lectures.

Thorne, K. S. 1969, Ap. J., 158, 1.

__________ 1971, in Proceedings of the International School of Physics, "Enrico Fermi," Course 47 (New York: Academic Press) in press.

Thorne, K. S., and Campolattaro, A. 1967, Ap. J., 149, 591.

Trautman, A. 1962, in Gravitation, ed. L. Witten (New York: John Wiley & Sons, Inc.).

Wheeler, J. A. 1964, in Relativity, Groups, and Topology, eds. C. DeWitt and B. DeWitt (New York: Gordon and Breach).

Yano, K. 1955, The Theory of Lie Derivatives and its Applications (Amsterdam: North-Holland Publishing Co.).

CHAPTER 6

SUGGESTED LINES OF FUTURE RESEARCH

I would like to list some interesting questions and research problems suggested by the results contained in this thesis.

Chapters 2 and 3.

1. Van Dantzig (1940), Schmid (1967 a,b,c), and Seliger (1968) have shown that the velocity-potential formalism can be extended to charged fluids. Can the results of this thesis thereby be generalized to stars with strong magnetic fields?

2. The remarks in Chapter 3 that there seems to be a deeper canonical relationship between α and β and between θ and S raises several possibilities. In the relativistic version, can one find some H such that $\partial H/\partial\alpha = d\beta/d\tau$ and $\partial H/\partial\beta = -d\alpha/d\tau$? If so, can viscosity be introduced into the fluid by modifying H (perhaps by making it time-dependent)? Can heat conduction be handled by modifying a Hamiltonian for θ and S in a similar manner?

3. Do the velocity potentials have a foundation in statistical mechanics, *i.e.*, are they the continuum-approximation limits of some physically meaningful functions of statistical mechanics? If so, does this shed light on the canonical relationships among the potentials?

4. Can the full velocity-potential variational principle be used to investigate nonlinear wave propagation inside stars, and the coupling of perfect fluids to gravitational waves?

Chapters 4 and 5

5. The problems that need to be solved before the sufficient condition for stability can be made useful have been discussed in Chapter 5.

6. When the energy functional of Chapter 5 is reduced to a

function only of the dynamical variables, the arguments of Low (1961) can almost certainly be used to show that an instability will arise in a sequence of models only through modes that make the energy vanish. The question then arises whether such modes need always have zero frequency. If so, the test for stability reduces to the search for zero-frequency modes.

7. In a sequence of models, do standing-wave zero-frequency modes occur in exactly the same model as do realistic (outgoing-wave) zero-frequency modes? In other words, does the complex frequency of a realistic mode in a sequence of stellar models always approach the origin along a curve tangent to the real axis (ratio $\mathrm{Im}\omega/\mathrm{Re}\omega$ approaches zero: mode equivalent to standing-wave mode)? If so, the self-adjoint standing-wave problem can be used as a good test for the stability of modes that become unstable at zero frequency. If the curve is not tangent to the real axis, is its slope a measure of the accuracy of the standing-wave approximation in pinpointing the onset of instability?

REFERENCES

Low, F.E. 1961, Phys. Fluids, **4**, 842.

Schmid, L.A. 1967a, Nuovo Cim., **47**, 1.

————. 1967b, Nuovo Cim., **52B**, 288.

————. 1967c, Nuovo Cim., **52B**, 313.

Seliger, R.L. 1968, unpublished Ph.D. thesis, California Institute of Technology.

van Dantzig, D. 1940, Kon. Akad. Wet. Amsterdam, **43**, 387.

APPENDIX

NON-VACUUM ADaM FIELD EQUATIONS

To be published in the *Proceedings of the Pittsburgh Conference on Relativity* (Springer-Verlag, *Lecture Notes on Physics* series)

The canonical version of the vacuum Einstein field equations formulated ten years ago by Arnowitt, Deser, and Misner (ADaM)[1] has stimulated several attempts to quantize certain cosmological models, most notably Misner's so-called Mixmaster Universe.[2] Some researchers have begun recently to extend these methods to non-vacuum spacetimes; for example, Nutku earlier at this conference described the canonical theory of a scalar field in Schwarzschild spacetime. The purpose of this talk is to generalize the ADaM field equations to include an arbitrary stress-energy tensor. This is not a "first step" toward a canonical formulation of the full non-vacuum field equations; rather, it is simply a possible starting point.

Essentially, the ADaM field equations are a linear combination of Einstein's $G_{\mu\nu} = 0$ equations that is particularly well suited to a "three-plus-one split" of spacetime, i.e., a division of spacetime into three-dimensional spacelike sections labelled by the parameter time. The metric of each section is the spacelike part of the metric for all of spacetime:

$$g_{ij} \equiv {}^4g_{ij} \quad . \tag{1a}$$

(Superscript "4" denotes quantities referred to the full four-dimensional spacetime, while no superscript implies three-dimensional quantities. Latin indices run from 1 to 3, Greek from 0 to 3. Signature is - 2.) ADaM replace the remaining four metric components - which give information on how one hypersurface fits into the next[3] - with: a three-scalar

$$N \equiv (- {}^4g^{00})^{-\frac{1}{2}} \tag{1b}$$

and a covariant three-vector

$$N_i \equiv {}^4g_{oi} \quad . \tag{1c}$$

The ADaM field equations are derived from the usual variational principle,

$$\delta \int {}^4R(-{}^4g)^{\frac{1}{2}}\, d\,{}^4x = 0 \; . \tag{2}$$

Were one to use $\{{}^4g^{\mu\nu}\}$ as the set of independent variables, one would obtain $G_{\mu\nu} = 0$ from Eq. (2).[4] Using the ADaM variables $\{N, N_i, g_{ij}\}$, on the other hand, gives the ADaM equations.

To obtain the non-vacuum equations, let L be the Lagrangian for the non-gravitational fields. Then Eq. (2) generalizes to

$$\delta I = \delta \int ({}^4R + 2\,\kappa\, L)(-{}^4g)^{\frac{1}{2}}\, d\,{}^4x = 0 \; . \tag{3}$$

Using $\{{}^4g^{\mu\nu}\}$ as the variables gives[5]

$$G_{\mu\nu} = \kappa\, T_{\mu\nu} \; , \tag{4}$$

where

$$T_{\mu\nu} = L\,{}^4g_{\mu\nu} - 2\,\frac{\partial L}{\partial\,{}^4g^{\mu\nu}} + \frac{2}{(-{}^4g)^{\frac{1}{2}}}\left[(-{}^4g)^{\frac{1}{2}}\,\frac{\partial L}{\partial\,{}^4g^{\mu\nu}{}_{,\beta}}\right]_{,\beta} \; . \tag{5}$$

The non-vacuum ADaM equations follow from Eq. (3) if one uses the set $\{a_{\alpha\beta}\}$ of ADaM variables, defined by

$$a_{oo} \equiv (-{}^4g^{oo})^{-\frac{1}{2}} \; ; \; a_{oi} \equiv {}^4g_{oi} \; ; \; a_{io} \equiv {}^4g_{io} \; ; \; a_{ij} \equiv {}^4g_{ij} \; . \tag{6}$$

It is convenient in what follows to ignore the symmetry of $a_{\alpha\beta}$ and ${}^4g_{\mu\nu}$. For instance, variations of a_{oi} will be taken while holding a_{io} fixed. The final results will, of course, by symmetrized.

Because the transformation from $\{{}^4g^{\mu\nu}\}$ to $\{a_{\alpha\beta}\}$ is nonsingular and does not involve derivatives of ${}^4g^{\mu\nu}$ or explicit dependence upon the spacetime coordinates, the equations obtained from varying $a_{\alpha\beta}$ will be the linear combination

$$0 = \frac{\delta I}{\delta a_{\alpha\beta}} = \frac{\partial\, {}^4g^{\mu\nu}}{\partial a_{\alpha\beta}} \frac{\delta I}{\delta\, {}^4g^{\mu\nu}} \tag{7}$$

of the equations obtained from varying ${}^4g^{\mu\nu}$. We therefore need only find $\partial\, {}^4g^{\mu\nu}/\partial a_{\alpha\beta}$, in which it is understood that the derivative is taken holding all other $a_{\gamma\delta}$ fixed. This is the key to the difference between Einstein and ADaM: it means, for example, that $\partial\, {}^4g^{o1}/\partial a_{o1}$ is not the same as $\partial\, {}^4g^{o1}/\partial\, {}^4g_{o1} = -\, {}^4g^{oo}\, {}^4g^{11}$, because in the first case one holds $\{{}^4g^{oo}, {}^4g_{o2}, {}^4g_{o3}, {}^4g_{ij}\}$ fixed while in the second case one holds $\{{}^4g_{oo}, {}^4g_{o2}, {}^4g_{o3}, {}^4g_{ij}\}$ fixed. Bearing this in mind, we write down the equations of transformation:

$$\frac{\partial\, {}^4g^{\mu\nu}}{\partial a_{ij}} = -\, {}^4g^{\mu i}\, {}^4g^{\nu j} + {}^4g^{o\mu}\, {}^4g^{o\nu} N^i N^j \;; \tag{8a}$$

$$\frac{\partial\, {}^4g^{\mu\nu}}{\partial a_{oi}} = -\, {}^4g^{o\mu}\, {}^4g^{\nu i} - {}^4g^{o\mu}\, {}^4g^{o\nu} N^i \;; \tag{8b}$$

$$\frac{\partial\, {}^4g^{\mu\nu}}{\partial a_{io}} = -\, {}^4g^{\mu i}\, {}^4g^{\nu o} - {}^4g^{o\mu}\, {}^4g^{o\nu} N^i \;; \tag{8c}$$

$$\frac{\partial\, {}^4g^{\mu\nu}}{\partial a_{oo}} = 2\, {}^4g^{o\mu}\, {}^4g^{o\nu} N \;. \tag{8d}$$

It is straightforward to use Eqs. (7) and (8) to find the non-vacuum ADaM field equations. (Here π^{ij} is the momentum canonical to g_{ij}, defined by Eq. (9c) below. Indices on it and N_i are raised and lowered by the three-dimensional metric, covariant differentiation with respect to which is denoted by a slash, "|".)

$$-\, g^{\frac{1}{2}}[{}^3R + g^{-1}(\tfrac{1}{2}\pi^2 - \pi^{ij}\pi_{ij})] = -\, 2\kappa N^2 g^{\frac{1}{2}} T^{oo} \;; \tag{9a}$$

$$-\, \pi^{ij}{}_{|j} = \kappa N g^{\frac{1}{2}} (T^{oi} + N^i T^{oo}) \;; \tag{9b}$$

$$\partial_t g_{ij} = 2Ng^{-\frac{1}{2}}(\pi_{ij} - \tfrac{1}{2}g_{ij}\pi) + N_{i|j} + N_{j|i} \quad ; \tag{9c}$$

$$\begin{aligned}\partial_t \pi^{ij} = &- Ng^{\frac{1}{2}}({}^3R^{ij} - \tfrac{1}{2}g^{ij}\,{}^3R) + \tfrac{1}{2}Ng^{-\frac{1}{2}}g^{ij}(\pi^{mn}\pi_{mn} - \tfrac{1}{2}\pi^2)\\ &- 2Ng^{-\frac{1}{2}}(\pi^{im}\pi_m{}^j - \tfrac{1}{2}\pi\pi^{ij}) + g^{\frac{1}{2}}(N^{|ij} - g^{ij}N^{|m}{}_{|m})\\ &+ (\pi^{ij}N^{|m})_{|m} - N^i{}_{|m}\pi^{mj} - N^i{}_{|m}\pi^{mi}\\ &+ \kappa Ng^{\frac{1}{2}}(T^{ij} - T^{oo}N^iN^j) \quad .\end{aligned} \tag{9d}$$

I wish to remark on a few features of these equations. First, as we would expect, they do not contain L, since they are simply a linear combination of Eqs. (4). This means they can be used even if a Lagrangian is not available. Second, Eqs. (9) are instructive in understanding even the ADaM vacuum equations, since the particular linear combination used by ADaM is manifest. And third, the equations contain $T^{\mu\nu}$, the contravariant components of the four-dimensional stress-energy tensor. In many situations (e.g., scalar field) one might feel that the covariant components, $T_{\mu\nu}$, are physically more meaningful in a 3 + 1 split, in which case one can rewrite the equations as follows. Using the unit normal to the three-hypersurface, $\eta^\alpha = - N\,{}^4g^{o\alpha}$, one can define a "preferred" energy and momentum density for the matter:

$$\mathcal{E} = \eta^\alpha\eta^\beta\,{}^4T_{\alpha\beta} \quad , \tag{10a}$$

$$\mathcal{P}_i = \eta^\alpha\,{}^4T_{\alpha i} \quad . \tag{10b}$$

Then the stress tensor in the hypersurface is

$$\mathcal{T}_{ik} = {}^4T_{ik} \quad . \tag{10c}$$

In terms of these quantities, the relevant parts of Eqs. (9) become

$$-2\kappa N^2 g^{\frac{1}{2}} T^{00} = -2\kappa g^{\frac{1}{2}} \mathcal{E} \quad ; \tag{11a}$$

$$\kappa N g^{\frac{1}{2}} (T^{0i} + N^i T^{00}) = -\kappa g^{\frac{1}{2}} \mathcal{P}^i \quad ; \tag{11b}$$

$$\kappa N g^{\frac{1}{2}} (T^{ij} - N^i N^j T^{00}) = \kappa g^{\frac{1}{2}} (N \mathcal{J}^{ij} + N^i \mathcal{P}^j + N^j \mathcal{P}^i) \quad , \tag{11c}$$

where all indices on $\mathcal{P}$ and $\mathcal{J}$ are raised by the three-dimensional metric.

Steps toward a full canonical theory could well begin here. One method would be to specify in advance the motion of the matter in terms of the metric tensor (e.g., homogeneous cosmology), and then to solve the constraint Eqs. (9a,b) by analogy with vacuum ADaM. A more general approach must include a canonical formulation for the fields present in spacetime. In any case, the basic gravitational constraints and dynamical equations will be Eqs. (9).

FOOTNOTES

1. The best introduction to ADaM is the article by R. Arnowitt, S. Deser, and C.W. Misner in Gravitation, edited by L. Witten (John Wiley and Sons, New York, 1962), Chap. 7.

2. For the Mixmaster Universe, see C.W. Misner, Phys. Rev. Lett. **22**, 1071 (1969). For its quantization, see C.W. Misner, "Quantum Cosmology I" (preprint).

3. J.A. Wheeler in Relativity, Groups, and Topology, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1964) p. 346.

4. L. Landau and E. Lifshitz, The Classical Theory of Fields, (Addison-Wesley, Reading, Massachusetts, 1962) §95.

5. ibid., §94