RELATIVISTIC QUARK MODELS
AND THE CURRENT ALGEBRA

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ABSTRACT

In this thesis we are concerned with finding representations of the algebra of SU(3) vector and axial-vector charge densities at infinite momentum (the "current algebra") to describe the mesons, idealizing the real continua of multiparticle states as a series of discrete resonances of zero width. Such representations would describe the masses and quantum numbers of the mesons, the shapes of their Regge trajectories, their electromagnetic and weak form factors, and (approximately, through the PCAC hypothesis) pion emission or absorption amplitudes.

We assume that the mesons have internal degrees of freedom equivalent to being made of two quarks (one an antiquark) and look for models in which the mass is SU(3)-independent and the current is a sum of contributions from the individual quarks. Requiring that the current algebra, as well as conditions of relativistic invariance, be satisfied turns out to be very restrictive, and, in fact, no model has been found which satisfies all requirements and gives a reasonable mass spectrum. We show that using more general mass and current operators but keeping the same internal degrees of freedom will not make the problem any more solvable. In particular, in order for any two-quark solution to exist it must be possible to solve the "factorized SU(2) problem," in which the currents are isospin currents and are carried by only one of the component quarks (as in the K meson and its excited states).

In the free-quark model the currents at infinite momentum are found using a manifestly covariant formalism and are shown to satisfy the current algebra, but the mass spectrum is unrealistic. We then consider a pair of quarks bound by a potential, finding the

current as a power series in $1/m$ where m is the quark mass. Here it is found impossible to satisfy the algebra and relativistic invariance with the type of potential tried, because the current contributions from the two quarks do not commute with each other to order $1/m^3$. However, it may be possible to solve the factorized SU(2) problem with this model.

The factorized problem can be solved exactly in the case where all mesons have the same mass, using a covariant formulation in terms of an internal Lorentz group. For a more realistic, nondegenerate mass there is difficulty in covariantly solving even the factorized problem; one model is described which almost works but appears to require particles of spacelike 4-momentum, which seem unphysical.

Although the search for a completely satisfactory model has been unsuccessful, the techniques used here might eventually reveal a working model. There is also a possibility of satisfying a weaker form of the current algebra with existing models.

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I. THE CURRENT ALGEBRA

The theory of quantum electrodynamics is "complete" in that it gives an algorithm for calculating (in principle) any scattering amplitude or energy level shift, that we wish, to any order, providing that the process involves only electromagnetic interactions. On the other hand, for strong interactions, which are primarily responsible for the structure and behavior of the hadrons (strongly-interacting particles), we have no complete theory. We can formulate theories analogous to quantum electrodynamics involving field operators and/or Feynman diagrams, but we cannot use them to calculate whatever we want because perturbation theory is inapplicable due to the size of the coupling constants. We can only test general assumptions about the theories, such as unitarity and analyticity, by deriving observable consequence of them and seeing whether they agree with experiment.

In spite of the difficulties of calculating anything involving strong interactions, many predictions have been possible thanks to the symmetries that these interactions have. The isospin groups, SU(2), appears to be an exact symmetry of strong interactions, and SU(3) an approximate one under which the strong interactions seem to have simple transformation properties. There are also SU(6) and higher symmetries, becoming more and more approximate the higher they are. These symmetries have been useful in classifying the known hadrons and predicting new ones, and in describing their masses and quantum numbers.

Here we shall consider the SU(3) algebra [the generators of the group SU(3)], which consists of a set of operators F_a (a = 1, ..., 8) obeying the commutation relations

$$[F_a, F_b] = if_{abc} F_c , \qquad \text{(1a)}$$

(summation over c understood), where the numbers f_{abc} are the structure constants* of the Lie algebra of SU(3). In particular, for a = 1, 2, 3, F_a is just I_a, the a-component of the isotopic spin, and the commutation relations of them alone are those of SU(2), equivalent to the rotation group:

$$[I_a, I_b] = i\epsilon_{abc} F_c .$$

In the SU(3) algebra we also find the hypercharge operator, $Y = \frac{2}{\sqrt{3}} F_8$, the electric charge operator, $Q = F_3 + \frac{1}{\sqrt{3}} F_8$, and other operators which raise and lower the hypercharge or strangeness.

In general the operators F_a could depend on time and also on which Lorentz frame we are in. The F_a's for different times and frames are tied together by the assertion that, for a given time and Lorentz frame, each F_a is the integral over all space of a "charge" density which is the fourth (time) component of a current density 4-vector** $\mathcal{F}_a^{\mu}(x)$:

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* We use the notation of Reference 1 for f_{abc} as well as the SU(3) λ-matrices.

** Our notation for 4-vectors is $\underset{\sim}{a}$ or $a^{\mu} = (\vec{a}, a^o)$, and the scalar product is $a \cdot b = a^{\mu} b_{\mu} = \vec{a} \cdot \vec{b} - a^o b^o$.

$$F_a(t) = \int d^3\vec{x}\, \mathcal{F}_a^{\,o}(\vec{x},\, t). \tag{2a}$$

In the limit of strong interactions only F_1, F_2, F_3 and F_8 are conserved, so that $\partial_\mu \mathcal{F}_a^\mu(x) = 0$ for $a = 1, 2, 3, 8$, but the other F_a are not. As a first approximation, however, we will be proposing models in which all F_a are conserved, so that all particles in an SU(3) multiplet have the same mass.

The currents $\mathcal{F}_a^\mu(x)$ may be associated with measurable quantities because they describe first-order electromagnetic and weak processes. The electromagnetic current is

$$j_{EM}^\mu(x) = \mathcal{F}_3^\mu(x) + \frac{1}{\sqrt{3}}\, \mathcal{F}_8^\mu(x) \tag{3}$$

and, between hadron states, gives the amplitude for emission or absorption of a virtual photon; in other words, it determines the electromagnetic form factors. There is also a weak current, $j_W^\mu(x)$, which describes the weak coupling of the hadrons to the leptons. Under the conserved vector current and universality hypotheses, [2)] the vector part of this current is given by

$$\begin{aligned} j_V^\mu(x) &= [\mathcal{F}_1^\mu(x) + i\mathcal{F}_2^\mu(x)]\cos\theta \\ &\quad + [\mathcal{F}_4^\mu(x) + i\mathcal{F}_5^\mu(x)]\sin\theta\ , \end{aligned} \tag{4a}$$

where θ ($\approx 15^o$) is the Cabibbo angle.

The weak current also has an axial-vector part $j_A^\mu(x)$ which, according to universality, has the same SU(3) transformation properties as $j_V^\mu(x)$. We thus suppose that there is a set of eight axial-vector currents, $\mathcal{F}_a^{5\mu}(x)$, and that

$$j_A^{\mu}(x) = [\mathcal{F}_1^{5\mu}(x) + i\mathcal{F}_2^{5\mu}(x)]\cos\theta + [\mathcal{F}_4^{5\mu}(x) + i\mathcal{F}_5^{5\mu}(x)]\sin\theta, \tag{4b}$$

the total hadronic weak current being

$$j_W^{\mu}(x) = j_V^{\mu}(x) - j_A^{\mu}(x). \tag{4c}$$

The time components of the axial-currents may be regarded as densities which, when integrated over space, give a set of "axial charges" which we call F_a^5:

$$F_a^5(t) = \int d^3\vec{x}\,\mathcal{F}_a^{50}(\vec{x}, t)\,. \tag{2b}$$

In contrast with the vector currents, the axial currents are generally not conserved, so that F_a^5 depends on time as well as the Lorentz frame in which the space integration is carried out.

The axial currents are also related to measurable quantities through the PCAC (partially-conserved axial current) hypothesis,[3] which relates the divergence of $\mathcal{F}_1^{5\mu}$, $\mathcal{F}_2^{5\mu}$, and $\mathcal{F}_3^{5\mu}$ to pion emission and absorption amplitudes.

Since the F_a^5 are postulated to transform as an 8-dimensional vector under SU(3), its commutators with F_a are determined:

$$[F_a, F_b^5] = if_{abc}\,F_c^5\,. \tag{1b}$$

We may then ask what happens when we commute the axial charges with each other. The simplest possibility is that

$$[F_a^5, F_b^5] = if_{abc} F_c , \tag{1c}$$

so that the set of operators $\{F_a, F_a^5\}$ form a closed algebra. This algebra is that of SU(3) × SU(3) since the operators $(F_a \pm F_a^5)/2$ form two commuting SU(3) subalgebras.

The most famous experimental test of assumption (1c) is the Adler-Weisberger sum rule,[4, 5] which, with the help of PCAC, predicts the nucleon axial-vector renormalization constant in terms of pion-nucleon cross sections with good agreement with experiment.

Assuming that the algebra given by (1a), (1b), and (1c) holds for the charges and axial charges, our next step is to ask whether the currents $\mathcal{F}_a^\mu(x)$ and $\mathcal{F}_a^{5\mu}(x)$ satisfy simple commutation relations. Since measurements of currents at two points in space time separated by a spacelike interval should not interfere with each other, $[\mathcal{F}_a^\mu(x), \mathcal{F}_b^\nu(x')] = 0$ when $x - x'$ is spacelike, and similarly for commutators involving the axial currents. However, if $x - x'$ is timelike we expect such commutators to be very complicated because they involve dynamics: one must solve the complete time-dependent problem for the system under consideration in order to relate currents at the same point in space but at different times, for example. So we always consider equal-time commutators such as $[\mathcal{F}_a^\mu(\vec{x}, t), \mathcal{F}_b^\nu(\vec{x}', t)]$, which must vanish when $\vec{x} \neq \vec{x}'$, and therefore must be finite linear combinations of $\delta^3(\vec{x} - \vec{x}')$ and derivatives thereof. We can say the most about these commutators when $\mu = \nu = 0$, because when we integrate over $\vec{x}$ and $\vec{x}'$ we must recover (1a), (1b), and (1c), so the coefficients of $\delta^3(\vec{x} - \vec{x}')$ are restricted. We therefore assume the following equal-time commutation relations for the charge densities:

$$[\mathcal{F}_a^0(\vec{x}, t), \mathcal{F}_b^0(\vec{x}', t)] = if_{abc} \mathcal{F}_c^0(\vec{x}, t)\, \delta^3(\vec{x} - \vec{x}'), \qquad (5a)$$

$$[\mathcal{F}_a^0(\vec{x}, t), \mathcal{F}_b^{50}(\vec{x}', t)] = if_{abc} \mathcal{F}_c^{50}(\vec{x}, t)\, \delta^3(\vec{x} - \vec{x}'), \qquad (5b)$$

$$[\mathcal{F}_a^{50}(\vec{x}, t), \mathcal{F}_b^{50}(\vec{x}', t)] = if_{abc} \mathcal{F}_c^0(\vec{x}, t)\, \delta^3(\vec{x} - \vec{x}'). \qquad (5c)$$

It is, of course, a strong assumption that there are no additional terms with $\nabla\delta^3(\vec{x} - \vec{x}')$ or higher derivatives. Equations (5) do hold in field-theoretic models analogous to quantum electrodynamics.[6] On the other hand, commutators between space and time components of the currents appear to have singular terms with gradients of δ-functions,[7] the significance of which is not completely understood.

Equations (5) (or their Fourier transforms, which will be considered later) form what we will call the "current algebra"; it consists of an SU(3) x SU(3) algebra at each point in space.

Given the current algebra we can proceed in two directions:

(A) By inserting (5) between initial and final particle states, and putting in a complete set of intermediate states between the factors in each term of the commutator, we obtain sum rules relating the form factors of the currents. The sum rules can be used to test the validity of the algebra (by plugging in the experimental values), or to predict new values for certain form factors.

(B) We can look for simple representations of the algebra and thereby classify the particles in a generalization of the SU(3) scheme. If the current algebra does hold, then of course the set of all states in the universe must form a representation of it, but hopefully we can find smaller sets of states which also form representations; in such smaller sets the real continua of many-particle states would be approximated by discrete particles corresponding to the observed resonances. For example, we should have an SU(3) octet and singlet* of $J^P = 0^-$ mesons (corresponding to π, K, $\bar{K}$, η, and X^0), a similar family of 1^- mesons (ρ, K*, $\bar{K}$*, φ, and ω), and higher excited states, the whole representation having an infinite number of such levels. Similarly, we would expect another representation or set of representations to include the baryons and baryon resonances.

Just as the representations of SU(3) predict the quantum numbers of the particles, the representations of the current algebra would predict the form factors for the vector and axial-vector currents, and therefore the first-order amplitudes** for electromagnetic and weak leptonic processes. Using the PCAC approximation we could also find pionic decay amplitudes and thus get some idea of the effective strong coupling constants.

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*Since the current algebra contains SU(3) as a subalgebra, the representations of the current algebra will consist of "levels" (not necessarily of constant mass), each level being a representation of SU(3).

**That is, first order in the electromagnetic or weak interaction, but to all orders in the strong interaction, since the current algebra is assumed to hold exactly.

In searching for a representation which is to describe the world of particles, we shall be imposing conditions of relativistic invariance, namely, that in (5) (which is not manifestly covariant) $\mathcal{F}_a^o$ and $\mathcal{F}_a^{50}$ must be the time-components of 4-vectors. What kind of mass operator we take for our particle system will then be crucial; only certain mass spectra will be allowed. The spectra that we get can be compared with experiment. Furthermore, we can impose a bootstrap condition[6] by requiring that the poles in the vector and axial-vector form factors (which, as we said above, are also predicted) appear at values of the momentum transfer equal to the masses of the vector and axial-vector mesons.

The current algebra actually has a closer connection with reality when we look at its infinite-momentum limit, to be considered in the next chapter. It is in this limit that useful sum rules can be obtained according to Procedure (A), and the representations found in Procedure (B) will be of the current algebra in the infinite-momentum limit.

II. THE INFINITE-MOMENTUM LIMIT

Suppose we wish to obtain sum rules from the current algebra by sandwiching the commutation relations between particle states. First of all it is convenient to work with the Fourier transforms of (I. 5). Let

$$\tilde{F}_a(\vec{k}) = \int d^3\vec{x}\, e^{i\vec{k}\cdot\vec{x}} \mathcal{F}_a^{o}(\vec{x}, 0)\,,$$
$$\tilde{F}_a^5(\vec{k}) = \int d^3\vec{x}\, e^{i\vec{k}\cdot\vec{x}} \mathcal{F}_a^{50}(\vec{x}, 0)\,. \tag{1}$$

In particular, $\tilde{F}_a(\vec{0}) = F_a$, and $\tilde{F}_a^5(\vec{0}) = F_a^5$. Then the current algebra in "momentum variables" is*

$$[\tilde{F}_a(\vec{k}), \tilde{F}_b(\vec{k}')] = if_{abc}\tilde{F}_c(\vec{k}+\vec{k}'), \tag{2a}$$

$$[\tilde{F}_a(\vec{k}), \tilde{F}_b^5(\vec{k}')] = if_{abc}\tilde{F}_c^5(\vec{k}+\vec{k}'), \tag{2b}$$

$$[\tilde{F}_a^5(\vec{k}), \tilde{F}_b^5(\vec{k}')] = if_{abc}\tilde{F}_c(\vec{k}+\vec{k}'). \tag{2c}$$

Let $|P, n\rangle$ be a state with total 4-momentum P and other variables (spin, isospin, etc.) described by n. Then since

$$\langle P', n'|\mathcal{F}_a^o(\vec{x}, 0)|P, n\rangle = e^{-i(\vec{P}'-\vec{P})\cdot\vec{x}}\langle P', n'|\mathcal{F}_a^o(0)|P, n\rangle, \tag{3}$$

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* Note that (2) is equivalent to (I. 5) for any value of t, since $\mathcal{F}_a^o(\vec{x}, t) = e^{iHt}\mathcal{F}_a^o(\vec{x}, 0)e^{-iHt}$, etc., where H is the Hamiltonian.

we have

$$\langle P', n' | \tilde{F}_a(\vec{k}) | P, n\rangle = (2\pi)^3 \delta^3(\vec{P}' - \vec{P} - \vec{k}) \langle P', n' | \mathcal{F}_a^0(0) | P, n\rangle. \tag{4}$$

We shall always use a covariant normalization of states:

$$\langle P', n' | P, n\rangle = \delta(P', P)\delta_{n'n},$$
$$\sum_n \int dw(P) \, |P, n\rangle \langle P, n| = 1, \tag{5}$$

where

$$\delta(P', P) = 2P^0(2\pi)^3 \delta^3(\vec{P}' - \vec{P}),$$
$$\int dw(P) = \int \frac{d^3\vec{P}}{(2\pi)^3 2P^0} \tag{6}$$

and $P^0 = \sqrt{\vec{P}^2 + M^2}$, M being the (rest) mass of the system (which could depend on n).

If we put (2a) between $\langle P', n'|$ and $|P, n\rangle$, insert a complete set of intermediate states $|P'', n''\rangle$, use (4), and factor out $(2\pi)^3 \delta^3(\vec{P}' - \vec{P} - \vec{k} - \vec{k}')$ from each side of the result, we get

$$\sum_{n''}\left[\frac{1}{2P''^0}\langle P', n'|\mathcal{F}_a^0(0)|P'', n''\rangle\langle P'', n''|\mathcal{F}_b^0(0)|P, n\rangle_{\vec{P}''=\vec{P}+\vec{k}'}\right.$$

$$\left.-\frac{1}{2P''^0}\langle P', n'|\mathcal{F}_b^0(0)|P'', n''\rangle\langle P'', n''|\mathcal{F}_a^0(0)|P, n\rangle|_{\vec{P}''=\vec{P}+\vec{k}}\right]$$

$$= if_{abc}\langle P', n'|\mathcal{F}_c^0(0)|P, n\rangle , \tag{7}$$

where P' and P are such that $\vec{P}' - \vec{P} = \vec{k} + \vec{k}'$. Now the matrix element $\langle P', n'|\mathcal{F}_a^0(0)|P'', n''\rangle$ can be expressed as a linear combination of independent amplitudes whose coefficients are form factors depending on the momentum transfer $t = -(P'-P'')^2$. Since $\vec{P}' - \vec{P}'' = \vec{k}$, $P'^0 = \sqrt{\vec{P}'^2 + M_{n'}^2}$, and $P''^0 = \sqrt{\vec{P}''^2 + M_{n''}^2}$, the momentum transfer represented by this matrix element is

$$t = -\vec{k}^2 + \left(\sqrt{\vec{P}'^2 + M_{n'}^2} - \sqrt{(\vec{P}' - \vec{k})^2 + M_{n''}^2}\right)^2 . \tag{8}$$

The other three matrix elements appearing in (7) have momentum transfers given by similar formulas. Therefore when we express everything in terms of form factors we get a sum rule in which the momentum transfers vary with the mass $M_{n''}$ of the intermediate state, in fact, each $t \to \infty$ since $M_{n''} \to \infty$. Sum rules with variable momentum transfer are inconvenient to test experimentally, and their convergence is difficult to check.

This difficulty can be overcome using the method of Fubini and Furlan,[8] which is to let $\vec{P}$ and $\vec{P}'$ approach infinity in some direction, say, the z-direction, while $\vec{k}$ and $\vec{k}'$ remain fixed. Then from (8), the momentum transfer in $\langle P', n'|\mathcal{F}_a^0(0)|P'', n''\rangle$ is

$$t \to -\vec{k}^2 + k_z^2 = -\vec{k}_\perp^2 , \qquad (9)$$

where "$\perp$" will always refer to the components perpendicular to the z-axis. Similarly for the other momentum transfer: they are either $-\vec{k}_\perp^2$ or $-\vec{k}'^2_\perp$ in the infinite-momentum limit. Thus, we obtain sum rules in which the form factors have constant values of the momentum transfer. We can choose momentum transfers for which the form factors are easily measurable; in some cases (e.g., the Adler-Weisberger relation) the sum in the sum rule can be written as a sum of total cross sections in which the momentum transfer is the mass of an incoming particle. Having a fixed momentum transfer also makes it easier to justify the PCAC approximation, which requires that t be near m_π^2, and to demonstrate the convergence of the sum rules by Regge pole theory. Other advantages of using the infinite-momentum limit [6, 9] will be seen later in this chapter.

The Adler-Weisberger relations[4, 5] are derived in the above manner and test portions of the current algebra with $t = 0$. The sum rule derived directly from the current algebra relates the nucleon axial vector renormalization constant to nucleon-neutrino forward scattering cross sections, which are not readily measured experimentally. The use of the partially conserved axial current hypothesis enables the sum rule to be written in terms of total pion-nucleon cross sections, and it is this form that agrees well with experimental results. Sum rules have also been worked out for non-zero momentum transfer[10]. One sum rule, [6, 11] obtained by considering the dipole moments (i.e., first-order in the momentum transfer) of the vector isospin currents, gives a relation between the nucleon isovector charge radius, the anomalous magnetic moment, and total photoproduction cross sections.

Having demonstrated the usefulness of the current algebra in the infinite-momentum limit, let us now find out what this limit is. Consider a matrix element $\langle P', n'|\mathcal{F}_a^0(0)|P, n\rangle$ and suppose we let $\vec{P}$ and $\vec{P}'$ approach ∞ in the z-direction with $\vec{P}' - \vec{P} = \vec{k}$ fixed. Since $\langle P', n'|P, n\rangle = 2P^0(2\pi)^3\delta^3(\vec{P}' - \vec{P})$ which goes like P_z as $P_z \to \infty$, we might expect the matrix elements of the charge densities to do the same. Such in fact is the case, as we can show by expressing everything in terms of "rest" states $|M_n\lambda, n\rangle$, where λ^μ means $(\vec{0}, 1)$ (a notation which we will use throughout this work). We assume the states are defined relative to each other by

$$|P, n\rangle = D(V_{P/M_n \leftarrow \lambda})|M_n\lambda, n\rangle, \tag{10}$$

where $V_{u\leftarrow\lambda}$ is our notation for the velocity transformation sending λ^μ into v^μ (i.e., the state of rest into the state of 4-velocity u) and D is the representation of the Lorentz group appropriate to the system of states. If $\vec{\mathcal{J}}$ and $\vec{\mathcal{K}}$ are the generators of this representation (the angular momentum and boost operators), then

$$D(V_{P/M_n \leftarrow \lambda}) = e^{-i\hat{P}\cdot\vec{\mathcal{K}}\cosh^{-1}(P^0/M_n)} . \tag{11}$$

Using (10) we find

$$\langle P', n'|\mathcal{F}_a^0(0)|P, n\rangle = \langle M_{n'}\lambda, n'|D(V_{\lambda\leftarrow P'/M_{n'}}V_{P/M_n\leftarrow\lambda}) \times$$
$$\times D(V_{\lambda\leftarrow P/M_n})\mathcal{F}_a^0(0)D(V_{P/M_n\leftarrow\lambda})|M_n\lambda, n\rangle . \tag{12}$$

Now as P_z and $P_z' \to \infty$ with $\vec{P}' - \vec{P} = \vec{k}$ one can show (for example, using a 2-dimensional representation of the Lorentz group) that

$$D(V_{\lambda\leftarrow P'/M_{n'}}V_{P/M_n\leftarrow\lambda}) \to e^{-i\mathcal{K}_z\log(M_{n'}/M_n)} \times$$
$$\times e^{i\vec{k}_\perp\cdot(\vec{\mathcal{K}} - \vec{e}_z \times \vec{\mathcal{J}})/M_n} . \tag{13}$$

Also,

$$D(V_{\lambda\leftarrow P/M_n})\mathcal{F}_a^0(0)D(V_{P/M_n\leftarrow\lambda}) = (V_{P/M_n\leftarrow\lambda})^0{}_\nu\mathcal{F}_a^\nu(0)$$
$$\sim \frac{P_z}{M_n}[\mathcal{F}_a^0(0) + \mathcal{F}_a^z(0)] \text{ as } P_z \to \infty . \tag{14}$$

Using (13) and (14) in (12) we see that $\langle P', n'|\mathcal{F}_a^0(0)|P, n\rangle$ does go like P_z in the infinite-momentum limit, and furthermore,

$$\lim_{\substack{P_z\to\infty \\ \vec{P}'-\vec{P}=\vec{k}}} \frac{\langle P', n'|\mathcal{F}_a^0(0)|P, n\rangle}{2P_z} = \langle n'|F_a(\vec{k}_\perp)|n\rangle, \tag{15a}$$

where

$$\langle n'|F_a(\vec{k})|n\rangle = \frac{1}{2M_n}\langle M_{n'}\lambda,\ n'|e^{-i\mathcal{K}_z \log(M_{n'}/M_n)}\, e^{i\vec{k}_\perp\cdot(\vec{\mathcal{K}}-\vec{e}_z\times\vec{\mathcal{J}})}$$

$$[\mathcal{F}_a^{o}(0)+\mathcal{F}_a^{z}(0)]|M_n\lambda,\ n\rangle . \qquad (15b)$$

Note that $\langle n'|F_a(\vec{k})|n\rangle$ depends only on $\vec{k}_\perp = \vec{P}'_\perp - \vec{P}_\perp$ but not on $\vec{P}'_\perp + \vec{P}_\perp$, for example. We can consider it as a matrix element of a "reduced operator" $F_a(\vec{k})$ which acts only on the internal variables of the system (described by n and n') but does not involve the total momenta (which have gone to ∞ in the z-direction).[12]

Similarly we can define $F_a^5(\vec{k}_\perp)$ by

$$\lim_{\substack{P_z\to\infty \\ \vec{P}'-\vec{P}=\vec{k}}} \frac{\langle P',\ n'|\mathcal{F}^{50}(0)|P,\ n\rangle}{2P_z} = \langle n'|F_a^5(\vec{k}_\perp)|n\rangle . \qquad (15c)$$

We shall call $F_a(\vec{k})$ and $F_a^5(\vec{k}_\perp)$ the "currents at infinite momentum".

Dividing equation (7) by $2P_z$, taking the infinite momentum limit,* and doing the same for the corresponding equations involving the axial currents, we obtain the "current algebra at infinite momentum":

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*Assuming that the operations of taking the limit and summing over intermediate states can be interchanged, a question to be discussed later.

$$[F_a(\vec{k}_\perp), F_b(\vec{k}'_\perp)] = if_{abc} F_c(\vec{k}_\perp + \vec{k}'_\perp) \tag{16a}$$

$$[F_a(\vec{k}_\perp), F_b^5(\vec{k}'_\perp)] = if_{abc} F_c^5(\vec{k}_\perp + \vec{k}'_\perp) \tag{16b}$$

$$[F_a^5(\vec{k}_\perp), F_b^5(\vec{k}'_\perp)] = if_{abc} F_c(\vec{k}_\perp + \vec{k}'_\perp). \tag{16c}$$

This algebra looks a lot like the one we started with, (2). The difference is that the F's act on a smaller set of variables (only the internal variables) and are functions of only a 2-dimensional momentum transfer.

In the chapters that follow we will be searching for representations of (16) rather than (2) or (I. 5), for the following reasons:

(A) The infinite-momentum limit is more readily tested by sum rules (as we have seen) and thus better established.

(B) We have a better chance of being able to represent the algebra by a relatively small set of states (discrete particles) and still approximate the real world. Consider (7) again with finite momenta. Which states we need to include in our representation depends on what states n'' are necessary in order to "saturate" the sum rule (7). A typical matrix element in the left side of (7) is $\langle P'', n''|\mathcal{F}_a^\mu(0)| P, n\rangle$ (with $\mu = 0$), which is given by the diagram in Figure 1. Here n'' is the intermediate state, $k = P'' - P$, and the wavy line represents the current $\mathcal{F}_a^\mu$. (For the electromagnetic current it actually stands for a particle, the virtual photon, but in general there need not be a particle coupling to every current.) Now if n is a single particle, we could have, in addition to single-particle intermediate states, more complicated ones for n'' obtained by disconnected diagrams such as Figure 2.

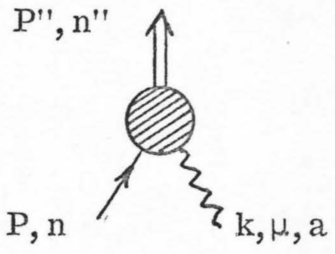

Figure 1. Diagram representing the matrix element of $\mathcal{F}_a^\mu$.

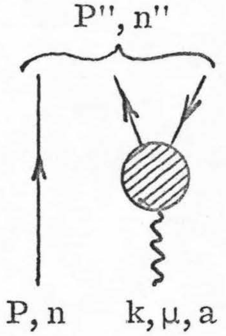

Figure 2. Example of a disconnected diagram.

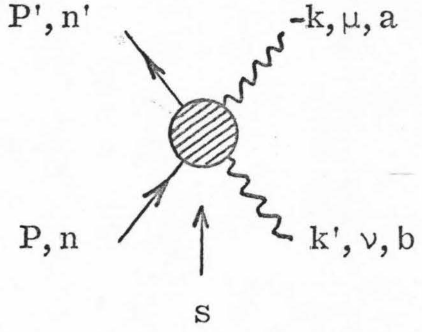

Figure 3. Amplitude involving two currents.

Here the system n'' has three particles; there is a continuum of such n'' because the mass $M_{n''}$ can vary from threshold to infinity. So in addition to single, discrete particles we have to include continua of three-particle states in our representation, if we wish to satisfy (7) as is.

Now suppose we take the infinite-momentum limit $P_z, P_z'' \to \infty$ with $\vec{P}'' - \vec{P} = \vec{k}$ remaining fixed. If the mass $M_{n''}$ of the three-particle state in Figure 2 is kept constant, the diagram cannot contribute because of 4-momentum conservation: The pair created by the current has finite spatial momentum $\vec{k}$ while the free particle has infinite momentum $\vec{P}$, requiring the three-particle state to have infinite mass in this limit. Thus for each fixed n'' (and therefore fixed $M_{n''}$) the contribution from disconnected diagrams vanishes in the infinite-momentum limit. Whether the sum over all n'' in (7) due to disconnected diagrams also vanishes in this limit (without having to put in intermediate states of infinite mass) depends on whether it was legal to interchange the order of limit and sum in going from (7) to (16). This question in turn depends on whether the "scattering amplitude" of Figure 3 satisfies an unsubtracted dispersion relation in the s-channel.[6] The fact that the sum rules converge (as an analysis of Regge trajectories shows) and agree reasonably with experiment where compared seems to indicate that we can interchange the order of limit and summation without having to worry about infinite-mass intermediate states. [One can also check this explicitly by evaluating some simple diagrams contributing to (7).] The currents in the infinite-momentum limit then come from only <u>connected</u> diagrams of the form in Figure 1. In such diagrams n'' could be a member of a continuum of multiparticle states, but we will treat such states as a sequence of discrete resonances in the s-channel. In this approximation we are then assuming that the sum rules may be saturated by single-particle states (stable particles plus resonances), and therefore that we can use such states as a basis for a representation of the current algebra at infinite momentum.

The general problem on which this research is based is to find representations of the current algebra at infinite momentum which are compatible with relativity and include the existing hierarchies of mesons and baryons, from which we should be able to deduce the weak and electromagnetic form factors. To find a relativistically-compatible representation we must find operators $F_a(\vec{k}_\perp)$ and $F_a^5(\vec{k}_\perp)$ which satisfy (16) and are desirable from covariant currents $\mathcal{F}_a^\mu$ and $\mathcal{F}_a^{5\mu}$ through (15a) and (15b). There are two methods of attack which have been used:

(i) ("Non-covariant formalism") We find solutions of (16) in terms of arbitrary operators, and then impose conditions on $F_a(\vec{k}_\perp)$ and $F_a^5(\vec{k}_\perp)$ in order that they be derivable from covariant currents. These conditions (the angular condition of Gell-Mann & Dashen) turn out to be quite restrictive and, for the systems that have been tried, either almost determine the F's or rule out a solution altogether.

(ii) ("Covariant formalism") We write out manifestly covariant expressions defining $\mathcal{F}_a^\mu$ and $\mathcal{F}_a^{5\mu}$ and try them out to see whether the resulting $F_a(\vec{k}_\perp)$ and $F_a^5(\vec{k}_\perp)$, defined by (15a) and (15b), satisfy the algebra (16). This method is more elegant in principle, but in practice it is often hard to show whether or not (16) is satisfied, and, of course, the correct answer (if there is one) for the $\mathcal{F}$'s may not be one which is readily found by inspection.

We conclude this chapter by showing how $F_a(\vec{k}_\perp)$ is related to the "electric" and "magnetic" dipole moments. Assume (as we shall do throughout) that $\mathcal{F}_a^\mu(x)$ is conserved: $\partial_\mu \mathcal{F}_a^\mu(x) = 0$. It follows that F_a, defined by (I. 2a), is time-independent and Lorentz-invariant, and thus connects only states of equal P^μ (in particular, equal mass):

$$\langle P', n'|F_a|P, n\rangle = 2P^0(2\pi)^3\,\delta^3(\vec{P}' - \vec{P})\,\langle n'|F_a|n\rangle,$$
$$\langle n'|F_a|n\rangle \propto \delta_{M_{n'},M_n}\,. \tag{17}$$

From (4) with $\vec{k} = \vec{0}$ it follows that

$$\langle P', n'|\mathcal{F}_a^0(0)|P, n\rangle|_{\vec{P}' = \vec{P}} = 2P^0\,\langle n'|F_a|n\rangle \tag{18}$$

and from (15a) that

$$F_a(0) = F_a\,. \tag{19}$$

From (18) (with $P' = P$) and Lorentz invariance we also find

$$\langle P', n'|\mathcal{F}_a^\mu(0)|P, n\rangle = 2P^\mu\langle n'|F_a|n\rangle. \tag{20}$$

We cannot make such statements about the axial current because it is not conserved.

Now suppose we look at the lowest-order terms (in $\vec{k}_\perp$) of $F_a(\vec{k}_\perp)$:

$$F_a(\vec{k}_\perp) = F_a + i\vec{k}_\perp \cdot \vec{h}_a + O(k_\perp^2), \tag{21}$$

where $\vec{h}_a = (h_{ax}, h_{ay}, 0)$ is hermitian.

Since $\vec{h}_a \propto \partial F_a/\partial \vec{k}_\perp$ we might expect $\vec{h}_a$ to be related to the dipole moments of $\mathcal{F}_a^\mu(x)$. In general the "electric" dipole moment corresponding to the "charge" F_a is

$$\vec{\mathcal{E}}_a = \int d^3\vec{x}\ \vec{x}\mathcal{F}_a^0(\vec{x}, 0), \tag{22}$$

and the "magnetic" dipole moment is

$$\vec{\mathcal{M}}_a = \frac{1}{2}\int d^3\vec{x}\ \vec{x} \times \vec{\mathcal{F}}_a(\vec{x}, 0). \tag{23}$$

Suppose $M_{n'} = M_n = M$. Then from (22) and (3),

$$\langle P', n'|\vec{\mathcal{E}}_a|P, n\rangle = (2\pi)^3 i\nabla\delta^3(\vec{P}' - \vec{P})\langle P', n'|\mathcal{F}_a^0(0)|P, n\rangle,$$

which we separate into two parts using the identity

$$\delta'(x-y)f(x-y) = \delta'(x-y)\frac{f(x,x) + f(y,y)}{2} - \delta(x-y)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)f(x, y).$$

Using (20) we find

$$\begin{aligned}\langle P', n'|\vec{\mathcal{E}}_a|P, n\rangle = {} & (2\pi)^3 i\nabla\delta^3(\vec{P}' - \vec{P})(P'^0 + P^0)\ \langle n'|F_a|n\rangle \\ & + 2P^0(2\pi)^3\delta^3(\vec{P}' - \vec{P})\ \langle n'|\vec{E}_a(P)|n\rangle,\end{aligned} \tag{24}$$

where

$$\langle n'|\vec{E}_a(P)|n\rangle = -\frac{i}{2P^0}\left(\frac{\partial}{\partial \vec{P}'} - \frac{\partial}{\partial \vec{P}}\right)\langle P', n'|\mathcal{F}_a^0(0)|P, n\rangle\Big|_{\vec{P}'=\vec{P}}$$

$$= -\frac{i}{2P^0}\frac{\partial}{\partial \vec{k}}\langle P', n'|\mathcal{F}_a^0(0)|P, n\rangle\Big|_{\vec{k}=\vec{0}} \qquad (25)$$

and $\langle P', n'|\mathcal{F}_a^0(0)|P, n\rangle$ is considered a function of $\vec{k} = \vec{P}' - \vec{P}$ and $\vec{P}' + \vec{P}$. Equation (24) expresses the total dipole moment as an external moment, which would be present even for a spinless, structureless particle,* and an internal moment $\vec{E}_a$. Proceeding similarly with the magnetic moment between states of equal mass, we find

$$\langle P', n'|\vec{\mathcal{M}}_a|P, n\rangle = \frac{1}{2}(2\pi)^3 i\nabla\delta^3(\vec{P}' - \vec{P})\times(\vec{P}' + \vec{P})\langle n'|F_a|n\rangle$$

$$+ 2P^0(2\pi)^3\delta^3(\vec{P}' - \vec{P})\langle n'|\vec{M}_a(P)|n\rangle, \qquad (26)$$

where

$$\langle n'|\vec{M}_a(P)|n\rangle = -\frac{i}{2P^0}\frac{1}{2}\frac{\partial}{\partial \vec{k}}\times\langle P', n'|\vec{\mathcal{F}}_a(0)|P, n\rangle\Big|_{\vec{k}=\vec{0}}. \qquad (27)$$

The electric and magnetic moments are most meaningful when the particle is at rest, i.e., $P^\mu = M\lambda^\mu$. Then (25) and (27) may be used with $\vec{P}' = \vec{k}/2$ and $\vec{P} = -\vec{k}/2$. We can even carry out the differentiation explicitly by writing

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*For such a particle, $\langle P', n'|\mathcal{F}_a^\mu(0)|P, n\rangle = (P' + P)^\mu \times \langle n'|F_a|n\rangle$ [cf. eq. (20), which holds for a general particle].

$$|P', n'\rangle = e^{-i\vec{\eta}\cdot\vec{\mathcal{K}}}|M\lambda, n\rangle, \ |P, n\rangle = e^{+i\vec{\eta}\cdot\vec{\mathcal{K}}}|M\lambda, n\rangle,$$

where $\vec{\eta} = \hat{k}\sinh^{-1}(|\vec{k}|/2M) = \vec{k}/2M + O(k^2)$ as $\vec{k} \to 0$. The result is (since $P^o = M$)

$$\langle n'|\vec{E}_a|n\rangle = \frac{1}{2M^2}\langle M\lambda, n'|\frac{1}{2}\{\vec{\mathcal{K}}, \mathcal{F}_a^o(0)\}|M\lambda, n\rangle \ , \tag{28}$$

and similarly for the magnetic moment. The latter, however, may be simplified using current conservation again. Let $\vec{f}(\vec{k}) = \langle P', n'|\vec{\mathcal{F}}_a(0)|P, n\rangle$ with $\vec{P}' = \vec{k}/2$ and $\vec{P} = -\vec{k}/2$. Then since $(P' - P)^\mu = (\vec{k}, 0)$, current conservation implies $\vec{k}\cdot\vec{f}(\vec{k}) = 0$. Expanding this about $\vec{k} = \vec{0}$ we get

k^o: $0 = 0$,

k^1: $\vec{f}(0) = 0$, already derivable from (20),

k^2: $\partial_i f_j(0) + \partial_j f_i(0) = 0$,

...

In view of the 2nd-order result, the only internal dipole moments of $\vec{\mathcal{F}}_a$ are the anti-symmetric ones, i.e., the components of the magnetic moment (27), and the y-component of (27) may be written as

$$\begin{aligned}\langle n'|M_{ay}|n\rangle &= \frac{i}{2P^o}\frac{\partial}{\partial k_x}\langle P', n'|\mathcal{F}_a^z(0)|P, n\rangle|_{\vec{k}=0} \\ &= -\frac{1}{2M^2}\langle M\lambda, n'|\frac{1}{2}\{\mathcal{K}_x, \mathcal{F}_a^z(0)\}|M\lambda, n\rangle .\end{aligned} \tag{29}$$

Now from (15b) we have an explicit expression for $\vec{h}_a$ defined by (21): for the x-component between states of equal mass M,

$$\langle n'|h_{ax}|n\rangle = \frac{1}{2M^2}\langle M\lambda, n'|(\mathcal{K}_x + \mathcal{J}_y)[\mathcal{F}_a^o(0) + \mathcal{F}_a^z(0)]|M\lambda, n\rangle$$

$$= \frac{1}{2M^2}\langle M\lambda, n'|\tfrac{1}{2}\{\mathcal{K}_x + \mathcal{J}_y, \mathcal{F}_a^o(0) + \mathcal{F}_a^z(0)\}|M\lambda, n\rangle \qquad (30)$$

since $\mathcal{K}_x + \mathcal{J}_y$ commutes with $\mathcal{F}_a^o + \mathcal{F}_a^z$. We next expand the anti-commutator into four terms:

$$\tfrac{1}{2}\{\mathcal{K}_x, \mathcal{F}_a^o\} + \tfrac{1}{2}\{\mathcal{K}_x, \mathcal{F}_a^z\} + \tfrac{1}{2}\{\mathcal{J}_y, \mathcal{F}_a^o\} + \tfrac{1}{2}\{\mathcal{J}_y, \mathcal{F}_a^z\} .$$

The first two give E_{ax} and $-M_{ay}$ by (28) and (29). Because of (20) and since $\mathcal{J}_y = J_y$ on rest states, where J_y is the internal angular momentum acting only on the internal index, the third gives $F_a J_y/M$ and the fourth gives zero. Therefore,

$$\langle n'|h_{ax}|n\rangle = \langle n'|(E_{ax} - M_{ay} + \frac{F_a J_y}{M})|n\rangle$$

$$\text{when } M_{n'} = M_n = M ,$$

or alternatively,

$$h_{ax} = E_{ax} - (M_{ay} - \frac{F_a J_y}{M})$$

$$+ \text{ commutators with } M . \qquad (31)$$

Thus $\vec{h}_a$, the coefficient of $i\vec{k}_\perp$ in the current at infinite momentum, is simply related (between states of equal mass) to the electric and magnetic dipole moments of the system at rest. Note that although the two-dimensional vector $\vec{h}_a$

transforms as a vector only under rotations about the z-axis, it is a sum of terms which are components of a 3-vector ($\vec{E}_a$) and an axial 3-vector ($\vec{M}_a - F_a\vec{J}/M$) under all rotations. We may think of the axial vector as an "anomalous" magnetic moment, the "normal" moment being $2\vec{J}$ times the "Bohr magneton" $F_a/2M$.

III. THE ANGULAR CONDITION AND THE QUARK MODEL

If we choose to use the non-covariant formalism to search for representations of the current algebra (which is understood from now on to be at infinite momentum), the first step is quite easy. For a set of F's satisfying (II. 16) we could take, for example,

$$F_a(\vec{k}_\perp) = \frac{1}{2}\lambda_a e^{i\vec{k}_\perp \cdot \vec{h}} ,$$

$$F_a^5(\vec{k}_\perp) = \frac{1}{2}\lambda_a \omega e^{i\vec{k}_\perp \cdot \vec{h}} , \tag{1}$$

where $\lambda_1, \ldots, \lambda_8$ are the standard SU(3) matrices,[1] h_x, h_y and ω are Hermitian, SU(3)-independent operators which commute with each other, and $\omega^2 = 1$. Or we could use more complicated expressions, to be discussed later.

The hard part comes when we try to make the model consistent with relativity, that is, ensure that the F's are derivable from covariant $\mathcal{F}$'s. Here we will derive a necessary condition for this to be so, called the angular condition because it involves the rotational (angular momentum) properties of the currents.[6,12]

Consider the matrix element $\langle P', n'|\mathcal{F}_a^o(0)|P, n\rangle$. The vectors P and P' define a "preferred" plane in 4-space. If $\mathcal{F}_a^o(0)$ is to be the 4th component of a 4-vector, its matrix element can be defined arbitrarily except that it must transform properly under those (proper) Lorentz transformations which leave P and P' fixed. In the Breit frame, where the spatial momenta are negatives of each other, these Lorentz transformations are spatial rotations about their common direction, and $\mathcal{F}_a^{\,o}(0)$ (which in this frame has a space as well as a time component) must transform as a scalar plus a vector under these rotations.

To translate this derivation of the angular condition from a verbal to an algebraic form, let Λ be a Lorentz transformation which turns the original frame into a Breit frame. That is, ΛP is of the form $(q, 0, 0, E)$ and $\Lambda P'$ of the form $(-q, 0, 0, E)$. Then

$$\begin{aligned} \langle P', n'|\mathcal{F}_a^{\,o}(0)|P, n\rangle &= \langle P', n'|D(\Lambda^{-1})\Lambda^o_{\ \nu}\mathcal{F}_a^{\,\nu}(0)D(\Lambda)|P, n\rangle \\ &= \sum_{\bar{n}', \bar{n}} \mathcal{D}_{n'\bar{n}'}(W'^{-1})\,\langle \Lambda P', \bar{n}'|\Lambda^o_{\ \nu}\mathcal{F}_a^{\,\nu}(0)|\Lambda P, \bar{n}\rangle \mathcal{D}_{\bar{n}n}(W), \end{aligned} \tag{2}$$

where $\mathcal{D}(W)$ is the matrix (acting on the internal index n) corresponding to the Wigner rotation $W = V_{\lambda \leftarrow \Lambda P/M_n}\,\Lambda V_{P/M_n \leftarrow \lambda}$ [recall Eq. (II. 10) defining the states relative to each other], and, similarly, $W' = V_{\lambda \leftarrow \Lambda P'/M_{n'}}\,\Lambda V_{P'/M_{n'} \leftarrow \lambda}$. Inverting the $\mathcal{D}$'s we get

$$\begin{aligned} &\sum_{\bar{n}', \bar{n}} \mathcal{D}_{n'\bar{n}'}(W')\,\langle P', \bar{n}'|\mathcal{F}_a^{\,o}(0)|P, \bar{n}\rangle\,\mathcal{D}_{\bar{n}n}(W^{-1}) \\ &\qquad = \Lambda^o_{\ \nu}\,\langle \Lambda P', n'|\mathcal{F}_a^{\,\nu}(0)|\Lambda P, n\rangle\,. \end{aligned} \tag{3}$$

Let the total angular momentum $\vec{\mathcal{J}}$ be written as an orbital part (acting on P) plus an internal spin $\vec{J}$ (acting on n):

$$\vec{\mathcal{J}} = \left(i\frac{\partial}{\partial\vec{P}} \times \vec{P}\right) + \vec{J}\,. \tag{4}$$

If $\vec{\theta}$ and $\vec{\theta}'$ are the oriented angles of the Wigner rotations W and W', then (3) may be rewritten as

$$\langle P', n'| e^{-i\vec{\theta}' \cdot \vec{J}} \mathcal{F}_a^o(0)\, e^{i\vec{\theta} \cdot \vec{J}} | P, n\rangle$$
$$= \Lambda^o_{\ \nu} \langle \Lambda P', n'| \mathcal{F}_a^{\nu}(0) | \Lambda P, n\rangle \ . \tag{5}$$

Now under a rotation about the x-axis, the right side of (5) transforms as a scalar plus a vector, since $\mathcal{F}_a^{\nu}(0)$ does while ΛP and $\Lambda P'$ are both invariant (i.e., there is no orbital angular momentum in the x-direction). Alternatively, the right side of (5) has $|\Delta J_x| \leq 1$ in the sense that if the initial and final states are eigenstates of J_x, then the matrix element vanishes unless the difference in J_x is 0 or ± 1. The angular condition is therefore that

$$\langle P', n'| e^{-i\vec{\theta}' \cdot \vec{J}} \mathcal{F}_a^o(0) e^{i\vec{\theta} \cdot \vec{J}} | P, n\rangle \text{ has } |\Delta J_x| \leq 1. \tag{6}$$

Dividing by $2P_z$ and taking the infinite-momentum limit, we get a condition on $F_a(\vec{k}_\perp)$. We now find $\vec{\theta}$ and $\vec{\theta}'$ in this limit. Let $\vec{k} = \vec{P}' - \vec{P}$ us before. Since the final results are independent of k_z and $\vec{P}' + \vec{P}$, we will take these components to be zero without loss of generality, and we will take $\vec{k}$ in the x-direction. Thus

$$P^{\mu} = (-\frac{k}{2}, 0, P_z, \sqrt{P_z^2 + \frac{k^2}{4} + M_n^2}\,)$$
$$P'^{\mu} = (\frac{k}{2}, 0, P_z, \sqrt{P_z^2 + \frac{k^2}{4} + M_{n'}^2}\,). \tag{7}$$

We must find* Λ such that ΛP and $\Lambda P'$ are of the form given previously. Let $\Lambda = RV$, where V is the velocity transformation (in the z-direction) taking $P + P'$ into a vector along the time axis, and R rotates (about the y-axis) the space components of VP and VP' so that they are along the $\mp$ x-axis. We could find V and R explicitly, but we can bypass all this by the following trickery: Since the y-axis is unchanged, the Wigner rotation W will be a rotation about the y-axis, i. e., $\vec{\theta} = \theta \vec{e}_y$. Recalling that

$$W = V_{\lambda \leftarrow \Lambda P/M_n} \Lambda V_{P/M_n \leftarrow \lambda}, \tag{8}$$

which leaves $\lambda^{\mu} = (\vec{0}, 1)$ invariant, we look for the effect of W on some other vector. Let

$$a = V_{\lambda \leftarrow P/M_n} P', \quad a' = V_{\lambda \leftarrow \Lambda P/M_n} \Lambda P' . \tag{9}$$

Then W clearly sends a into a'. To find $\vec{\theta}$ we need only find the directions of $\vec{a}$ and $\vec{a}'$. Using the convenient formula for velocity transformations

$$(V_{\lambda \leftarrow \lambda'})^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \frac{(\lambda + \lambda')^{\mu} (\lambda + \lambda')_{\nu}}{1 - \lambda \cdot \lambda'} - 2\lambda^{\mu} \lambda'_{\nu} \tag{10}$$

which holds for any unit timelike vectors λ and λ', one finds

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*Λ is completely determined if we stipulate in addition that the y-axis be left invariant.

$$\vec{a} = \left(k,\ 0,\ -\ \frac{k^2 + M_{n'}^{2} - M_n^2}{2M_n}\right) + O\left(\frac{1}{P_z}\right) \tag{11}$$

as $P_z \to \infty$. Also from (9) one sees that $\vec{a}'$ has only an x-component, which is positive. Therefore,

$$\begin{aligned} \theta &= \tan^{-1}\frac{a_z}{a_x} = -\tan^{-1}\frac{k^2 + M_{n'}^{2} - M_n^2}{2M_n k} \\ &= -\left(\tan^{-1}\frac{k}{M_{n'} + M_n} + \tan^{-1}\frac{M_{n'} - M_n}{k}\right). \end{aligned} \tag{12a}$$

The angle θ' may be obtained from θ by letting $P \leftrightarrow P'$ and $M_n \leftrightarrow M_{n'}$ (and thus $k \to -k$):

$$\theta' = \tan^{-1}\frac{k}{M_{n'} + M_n} - \tan^{-1}\frac{M_{n'} - M_n}{k}\ . \tag{12b}$$

The angular condition for the currents at infinite momentum is then that

$$\begin{aligned} &\langle n' |\exp[-i(\tan^{-1}\frac{k}{M_{n'} + M_n} - \tan^{-1}\frac{M_{n'} - M_n}{k})J_y]F_a(k\vec{e}_x) \\ &\exp[-i(\tan^{-1}\frac{k}{M_{n'} + M_n} + \tan^{-1}\frac{M_{n'} - M_n}{k})J_y]|n\rangle \end{aligned} \tag{13}$$

has $|\Delta J_x| \le 1$.

This condition can be put into a more concise form by defining "operators on operators". Let $\mathcal{A}_x$ and $\mathcal{B}_x$ be the operations of commuting and anticommuting with X, respectively. That is, if Y is another operator, then

$$\mathcal{A}_X Y = [X, Y], \quad \mathcal{B}_X Y = \{X, Y\}. \tag{14}$$

Note that $\mathcal{A}_X$ and $\mathcal{B}_X$ are both linear in X as well as their operands (Y).

Let M be the mass operator, defined by $M|n\rangle = M_n|n\rangle$. Since M and J_y commute, the operators $\mathcal{A}_M$, $\mathcal{B}_M$, $\mathcal{A}_{J_y}$, and $\mathcal{B}_{J_y}$ all commute. We can then write the angular condition as follows:

$$\left[\exp i\left(\mathcal{A}_{J_y} \tan^{-1}\frac{\mathcal{A}_M}{k} - \mathcal{B}_{J_y} \tan^{-1}\frac{k}{\mathcal{B}_M}\right)\right] F_a(k\vec{e}_x) \quad \text{has } |\Delta J_x| \leq 1 . \tag{15}$$

That (15) is equivalent to (13) may be seen by putting the former between $\langle n'|$ and $|n\rangle$, noting that $\mathcal{A}_M$ and $\mathcal{B}_M$ may be replaced by $M_{n'} - M_n$ and $M_{n'} + M_n$ respectively,* and using relations of the form

$$\begin{aligned} e^{i\varphi J_y} F e^{-i\varphi J_y} &= e^{i\varphi \mathcal{A}_{J_y}} F, \\ e^{i\varphi J_y} F e^{i\varphi J_y} &= e^{i\varphi \mathcal{B}_{J_y}} F . \end{aligned} \tag{16}$$

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*when (15) is expanded in powers of $\mathcal{A}_M$ and $\mathcal{B}_M$, for example.

The same condition, (15), holds with F_a replaced by F_a^5. The operator in square brackets may be expanded in powers of, e.g., the mass splitting, giving polynomials in the $\mathcal{A}$'s and $\mathcal{B}$'s; each $\mathcal{A}$ and $\mathcal{B}$ then operates on F_a or F_a^5.

Recall that in Chapter II we found an expression, (II. 15b), for the current at infinite momentum in terms of "rest" states $|M_n\lambda, n\rangle$. For $\vec{k}$ in the x-direction it is

$$\langle n'|F_a(k\vec{e}_x)|n\rangle = \frac{1}{2M_n}\langle M_{n'}\lambda, n'|e^{-i\mathcal{K}_z\log(M_{n'}/M_n)}e^{ik(\mathcal{K}_x+\mathcal{J}_y)} \tag{17}$$
$$[\mathcal{F}_a^o(0)+\mathcal{F}_a^z(0)]|M_n\lambda, n\rangle .$$

We can check our angular condition by applying it to this expression. The result (easily found using a spinor representation of the Lorentz group) is that the matrix element in (13) is equal to

$$\frac{1}{2M_n}\langle M_{n'}\lambda, n'|e^{i\alpha\mathcal{K}_x}[\mathcal{F}_a^o(0)+\sin\theta\,\mathcal{F}_a^x(0)+\cos\theta\,\mathcal{F}_a^z(0)]|M_n\lambda, n\rangle, \tag{18}$$

where $$\alpha = 2\sinh^{-1}\sqrt{\frac{k^2+(M_{n'}-M_n)^2}{4M_{n'}M_n}},$$

which indeed has $|\Delta J_x| \le 1$.

Equation (17) can be used to derive other properties of $F(\vec{k}_\perp)$. For example, if P and T are the internal parity and time reversal operators, respectively, then by knowing the behavior of $\mathcal{F}_a^\mu(0)$ under $\mathcal{P}$ and $\mathcal{T}$, the total parity and time reversal, we can find the behavior of $F_a(\vec{k}_\perp)$ under P and T [applied to the rest states in (17), $\mathcal{P}$ = P and $\mathcal{T}$ = T]. Under

$\mathcal{P}$ and $\mathcal{T}$, $\mathcal{F}_a^{\,o}$ is even while $\vec{\mathcal{F}}_a$ is odd, so one finds that $F_a(k\vec{e}_x)$ is invariant under $P e^{i\pi J_y}$ and $T e^{i\pi J_y}$. Since $\mathcal{F}_a^{5\mu}$ has the opposite behavior under $\mathcal{P}$ but the same under $\mathcal{T}$, $F_a^5(k\vec{e}_x)$ is odd under $P e^{i\pi \mathcal{J}_y}$ and even under $T e^{i\pi \mathcal{J}_y}$.

There is also the condition, derivable from (II. 31), that $\partial F_a(k\vec{e}_x)/\partial k$ at $k = 0$ have $|\Delta J| = 1$ between states of equal mass, and other conditions[6, 12] on the higher moments derivable from (17).

Now that we have the relativistic requirements on $F_a(\vec{k}\ )$ and $F_a^5(\vec{k}\ )$, let us see how to satisfy the algebra itself, (II. 16), in the non-convariant formalism. The simplest non-trivial solution is (1) already mentioned. In this case the SU(3) "charges" $F_a(\vec{0})$ are just $\frac{1}{2}\lambda_a$, which generate the representation $\underset{\sim}{3}$ of SU(3). The system therefore consists of single-quark states, the number of states possible for each kind of quark depending on the complexity of the quantum-mechanicam space on which the SU(3)-independent operators $\vec{h}$ and ω act. In other words, (1) describes a single-quark with a number of excited states (provided, of course, we can find $\vec{h}$, ω, and M such that the angular condition is satisfied).

Since we have not found isolated quarks in nature but do have mesons and baryons, we should look for more complicated solutions of (II. 16) which give SU(3) octets, decuplets, and so on. Here we use the quark model, which has met with some success in describing cross sections and masses of the hadrons. In this model the hadrons have an internal structure which is mathematically describable by saying that they are made out of quarks and anti-quarks bound together. Let $\lambda_a^{(i)}$ be the SU(3) matrix for the ith component particle; if this component is a quark the matrix is the usual λ_a acting on the appropriate SU(3) index, while if it is an antiquark the matrix is (with suitable conventions) $-\lambda_a^*$. If we then assume that the current is a sum of contributions from each quark or antiquark,

$$F_a(\vec{k}_\perp) = \sum_i \frac{1}{2} \lambda_a^{(i)} F^{(i)}(\vec{k}_\perp), \tag{19}$$

where the $F^{(i)}(\vec{k}_\perp)$ are SU(3)-independent operators, and similarly for $F_a^5(\vec{k}_\perp)$, and if we require (II. 16) be satisfied,** we find

$$\begin{aligned} F_a(\vec{k}_\perp) &= \sum_i \frac{1}{2} \lambda_a^{(i)} e^{i\vec{k}_\perp \cdot \vec{h}^{(i)}}, \\ F_a^5(\vec{k}_\perp) &= \sum_i \frac{1}{2} \lambda_a^{(i)} \omega^{(i)} e^{i\vec{k}_\perp \cdot \vec{h}^{(i)}}, \end{aligned} \tag{20}$$

where $\vec{h}^{(i)} = (h_x^{(i)}, h_y^{(i)}, 0)$, $\omega^{(i)2} = 1$, and all $h_x^{(i)}$, $h_y^{(j)}$, and $\omega^{(k)}$ commute with each other.

We shall be particularly concerned with the mesons, for which i runs from 1 to 2 in (20) and the one of the components, say #2, is an antiquark. (For brevity, however, we will speak of the "2-quark" model of mesons.) Our representation will then contain a series of levels, each level containing an SU(3) octet and singlet. The number of levels and their spins, parities, etc., will

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**Subject to the "initial condition" $F_a(\vec{0}) = \sum_i \frac{1}{2} \lambda_a^{(i)}$.

depend on what internal structure is given to the 2-quark system, i.e., what other internal variables the SU(3)-independent operators $\vec{h}^{(i)}$ and $\omega^{(i)}$ act on.

Assuming that SU(3) symmetry is unbroken (which of course is only an approximation even with strong interactions), the operator M for the mass of the 2-quark system could be completely SU(3)-independent (in which case each octet has the same mass as the singlet in that level), or it could contain a term proportional to $\lambda_a^{(1)} \lambda_a^{(2)}$ (which removes this degeneracy). As a further approximation we shall assume the first case: that the mass is SU(3)-independent. Applying the angular condition, (15), to (20), we see that each $\frac{1}{2} \lambda_a^{(i)}$ can be factored out, and if we consider its coefficient we find that

$$[\exp i(\mathcal{A}_{J_y} \tan^{-1} \frac{\mathcal{A}_M}{k} - \mathcal{B}_{J_y} \tan^{-1} \frac{k}{\mathcal{B}_M})] \{ {1 \atop \omega^{(i)}} \} e^{ikh_x^{(i)}} \qquad (21)$$

must have $|\Delta J_x| \leq 1$.

To find a representation of the current algebra in the 2-quark model of the mesons (with the assumptions discussed above) which is compatible with relativistic requirements, we must find operators $h_x^{(1)}$, $h_y^{(1)}$, $\omega^{(1)}$, $h_x^{(2)}$, $h_y^{(2)}$, and $\omega^{(2)}$, as well as a mass operator M, such that (21) is satisfied. Recall that all the h's and ω's must commute, and $\omega^{(1)2} = \omega^{(2)2} = 1$. Under rotations about the z-axis, $\vec{h}^{(i)}$ must transform as a vector and $\omega^{(i)}$ as a scalar. Under $Pe^{i\pi J_y}$, $h_x^{(i)}$ must be even and $\omega^{(i)}$ odd, while under $Te^{i\pi J_y}$ both $h_x^{(i)}$ and $\omega^{(i)}$ are even. (M must be invariant under P, T, and $\vec{J}$, of course.) Finally we have requirements due to charge conjugation symmetry: $F_a(\vec{k}_\perp)$ should have the same behavior as F_a under C (namely, $F_a \to +F_a$ for a = 2, 5, 7; $-F_a$ for a = 1, 3, 4, 6, 8), while $F_a^5(\vec{k}_\perp)$ should have the opposite behavior. This amounts to $\vec{h}^{(1)} \leftrightarrow \vec{h}^{(2)}$ and $\omega^{(1)} \leftrightarrow -\omega^{(2)}$ under interchange of the two quarks (i. e., interchange of the non-SU(3) variables).

The model that we will start with is a pair of quarks bound by a potential, each quark having a position in space (and its conjugate momentum) as well as a spin variable. When the center of mass of the system is at rest, the relevant internal operators are $\vec{x}$ (the position of quark #1 relative to #2), $\vec{p}$ (the momentum of #1) and the spins $\vec{\sigma}^{(1)}$ and $\vec{\sigma}^{(2)}$. The internal variable n then means the eigenvalues of a maximal set of commuting operators formed from these. When the center of mass is not at rest (in particular, in the infinite-momentum limit) these operators no longer have the physical significance of position, etc., but they still serve as a set of internal variables. The internal angular momentum is

$$\vec{J} = \vec{x} \times \vec{p} + \frac{1}{2}\vec{\sigma}^{(1)} + \frac{1}{2}\vec{\sigma}^{(2)} \tag{22}$$

regardless of the state of motion of the center of mass.* The internal parity sends $\vec{x} \to -\vec{x}$, $\vec{p} \to -\vec{p}$, and $\vec{\sigma}^{(i)} \to \vec{\sigma}^{(i)}$; while T sends $\vec{x} \to \vec{x}$, $\vec{p} \to -\vec{p}$, and $\vec{\sigma}^{(i)} \to -\vec{\sigma}^{(i)}$, and is antiunitary. "Interchanging the two quarks" means $\vec{x} \to -\vec{x}$, $\vec{p} \to -\vec{p}$, and $\vec{\sigma}^{(1)}$ $\vec{\sigma}^{(2)}$.

What kinds of meson levels do we get in general? Adding the quark spins together gives singlet (s = 0) or triplet (s = 1) states

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*This follows from the fact that if we perform a spatial rotation R on any state $|P, n\rangle$ satisfying (II. 10), the Wigner rotation (applied to n) is just R itself, regardless of P.

and adding the total quark spin s to the orbital angular momentum ℓ gives the total (internal) angular momentum j. We use the standard notation ${}^{2s+1}\ell_j$ (ℓ = S, P, D, F, ...) to describe all of this. Also we consider the parity, $P = -(-)^{\ell}$ (since a fermion and its antiparticle have opposite parities) and the charge conjugation of the neutral member of the level, $C_o = (-1)^{\ell+s}$. We then find the sequence of levels shown in Table 1 (note that in each level there are 9 particles forming an octet and singlet, and each one has $2j+1$ spin states).

The states in a given column all have the same j, P, and C_o, and may be thought of as lying on the same Regge trajectory. We have assumed that the mass operator (which must leave j, P, and C_o invariant) leaves ℓ invariant; if it does not, the two "Odd, -, -" trajectories might be mixed, and so might the two "Even, +, +" trajectories. The pattern in Table 1 will actually occur an infinite number of times, because in addition to the "angular-momentum excitations" shown there are "radial excitations" analogous to the quantum number n for the hydrogen atom (or, rather, $n - \ell$).

Comparing this pattern with the real world,[13] we note that the ${}^1S_0$ level may be identified with the nonet of well-known pseudoscalar mesons π, K, $\bar{K}$, η, and $X^0(958)$ with $J^P_{C_o} = 0^-_+$. (In this case it is not a very good approximation to assume the degeneracy of $\underset{\sim}{8}$ and $\underset{\sim}{1}$.) For ${}^3S_1$ we have the 1^-_- (vector) mesons ρ, K*, $\overline{K^*}$, φ, and ω, and for ${}^3P_2$ the 2^+_+ mesons A2(1300), $K_V(1420)$, $\bar{K}_V$, f(1260), and f'(1514). Other particles are less well determined, but it appears that $\pi_V(1016)$ and $\eta_V(1070)$ may be part of ${}^3P_0$ nonet of 0^+_+ mesons, and that A1(1070, I = 1),

TABLE 1

Spin States Obtained by Adding Two Quark Spins
to the Orbital Angular Momentum

j =	Even	Odd	Odd	Even	Even	Odd	Odd	Even
P =	-	+	-	+	-	+	-	+
C_o =	+	-	-	+	-	+	-	+
	$^1S_0$						$^3S_1$	
		$^1P_1$		$^3P_0$		$^3P_1$		$^3P_2$
	$^1D_2$		$^3D_1$		$^3D_2$		$^3D_3$	
		$^1F_3$		$^3F_2$		$^3F_3$		$^3F_4$
	$^1G_4$		$^3G_3$		$^3G_4$		$^3G_5$	
		⋮		⋮		⋮		⋮
	⋮		⋮		⋮		⋮	

$K_A(1320,\ I = 1/2)$, and $D(1285,\ I = 0)$ may be part of a $^3P_1$ nonet of 1^+_+ mesons, although some of the spins are uncertain. Also uncertain are $B(1220,\ I = 1)$, $K_A(1230,\ I = 1/2)$, and $H(990,\ I = 0)$, which might be 1^+_- and thus fit in the $^1P_1$ level. In addition there seems to be another 0^-_+ meson, $E(1420,\ I = 0)$, which might be part of a radial excitation of $^1S_0$.

The levels predicted by this model are rather numerous, especially when the radial excitations are included, but on the other hand there are many higher resonances found experimentally whose quantum numbers are as yet uncertain and may later be found to fit into the scheme. We could reduce the number of internal variables to get a simpler pattern (for example, connecting the two quarks by a "rigid bar" eliminates the radial excitation). Some such simpler representations will be considered in later chapters. In the next two chapters we shall deal with the model outlined above.

So far we have not yet proposed a specific mass operator. For a system of two free quarks the mass is

$$M = 2\sqrt{m^2 + \vec{p}^{\,2}}\ , \tag{23}$$

where m is the quark mass. To represent bound quarks we can introduce a "potential" $U(\vec{x})$ (assumed spherically symmetric) and write

$$M = 2\sqrt{m^2 + \vec{p}^{\,2} + U(\vec{x})}\ . \tag{24}$$

Given U we can solve the eigenvalue equation for M^2 (which looks like the nonrelativistic Schroedinger equation) and obtain the mass spectrum. A particularly nice potential, for example, would be

$U = \alpha \vec{x}^2$, which (from harmonic oscillator theory) gives values of M^2 rising linearly with the angular momentum, i. e., straight Regge trajectories, which seem to agree with experimental data. More complicated mass operators might include spin-orbit coupling, etc.

In Chapter V we will see what happens when we apply the angular condition with mass given by (24). It will turn out that by expanding (24) and (21) in powers of $1/m$ we can (in principle) almost determine the h's and ω's to each order, if they exist at all. Before doing this, however, it is instructive to consider the special case $U = 0$, i. e., the 2-free-quark system with mass given by (23). Using the covariant formalism and knowing what the current is from a free quark, we can find the h's and ω's exactly, and will do so in Chapter IV.

In case we have trouble (and we will) finding such a 2-quark representation compatible with relativistic requirements, we may ask, first of all, whether we were justified in assuming form (19) for the current. For example, there could be terms with $f_{abc}\lambda_b^{(1)}\lambda_c^{(2)}$ and $d_{abc}\lambda_b^{(1)}\lambda_c^{(2)}$, and then (20) and (21) would have to be modified. Even if (19) is correct, the mass operator could (and should, for a realistic model) depend on SU(3), again modifying (21) since we would no longer be able to collect the coefficients of $\lambda_a^{(i)}$. These two generalizations will be considered in Chapter VI, and there it is found that in an SU(3)-symmetric model they are inessential, in that they will not enable us to find a solution if we could not find one without them.

If we cannot find any suitable 2-quark representation at all, it is worth noting that the single-quark representation given by (1) does have some physical significance (if we can make it relativistic).

We restrict ourselves to the isospin currents only (a = 1, 2, 3), so that $\lambda_a = \tau_a$, and use the model to describe those hadrons which are made of one I = 1/2 (nonstrange) quark (to which the τ_a refers) plus any number of I = 0 (strange) quarks. Thus we would include the K mesons and their excited states, and the Ξ and its excited baryon states. This case is sometimes called the "factorized case" since the current is just the isotopic spin times the momentum-transfer-dependent factor.

IV. THE FREE-QUARK MODEL - COVARIANT FORMALISM

In this chapter we will consider "mesons" which are made of pairs of noninteracting spin-$\frac{1}{2}$ quarks. This model is somewhat unrealistic but it gives an idea of what to expect in the case of bound quarks. Furthermore the currents at infinite momentum can be found exactly using the covariant formalism, and we can show that they satisfy the current algebra, (II. 16).

Let the two quarks* have momenta p_1, p_2, and spins σ_1, σ_2 (the meaning of the spin index will be more precisely defined as we go along). Then according to the usual Dirac** theory the current density 4-vectors are given by

$$\mathcal{F}_a^{\mu}(x) = \frac{1}{2}\lambda_a^{(1)}\mathcal{F}^{(1)\mu}(x) + \frac{1}{2}\lambda_a^{(2)}\mathcal{F}^{(2)\mu}(x)\,, \tag{1}$$

and similarly for $\mathcal{F}_a^{5\mu}(x)$, where, for example, the contribution $\mathcal{F}^{(1)\mu}$ from the first quark is determined by the matrix element

$$\begin{aligned}&\langle p_1', \sigma_1', p_2', \sigma_2'|\mathcal{F}^{(1)\mu}(0)|p_1, \sigma_1, p_2, \sigma_2\rangle \\ &= [\bar{u}(p_1', \sigma_1')i\gamma^{\mu}u(p_1, \sigma_1)][\frac{1}{2m}\bar{u}(p_2', \sigma_2')u(p_2, \sigma_2)\delta_{(m)}(p_2', p_2)],\end{aligned} \tag{2}$$

---

*One of the quarks is actually, of course, an antiquark, but we will treat both as particles (rather than antiparticles), letting the difference be absorbed in the $\lambda_a^{(i)}$.

**Conventions: $\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}$; $\gamma_5 = \gamma^o\gamma^1\gamma^2\gamma^3$. We will use the stand basis, so that $\gamma^{\mu} = (-i\beta\vec{\alpha}, -i\beta)$ with $\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; then $\gamma_5 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$.

and $\mathcal{F}^{(1)5\mu}$ is given by (2) with $i\gamma^{\mu}$ replaced by $\gamma_5\gamma^{\mu}$. Equation (2) in fact represents diagram (a) in Figure 4. The first factor in

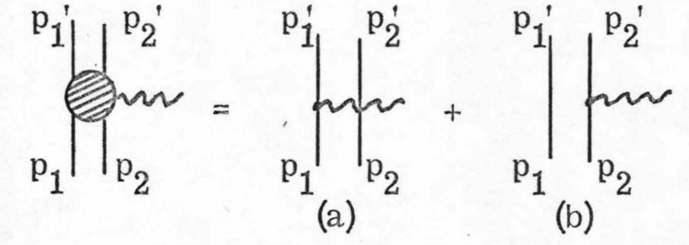

Figure 4. The Current in the Free Quark Model

(2) is the current from quark #1; the second factor is just $\langle p_2', \sigma_2' | p_2, \sigma_2 \rangle$ from the untouched quark, #2. The normalization in momentum space is given by

$$\delta_{(m)}(p_2', p_2) \equiv 2\sqrt{\vec{p}_2^{\,2} + m^2}\,(2\pi)^3 \delta^3(\vec{p}_2{}' - \vec{p}_2)\,, \tag{3}$$

and we have assumed that the spinors are normalized to 2m. (Note, however, that the second factor in (2) will not necessarily be proportional to $\delta_{\sigma_2' \sigma_2}$, because the initial and final spin states will not be defined in quite the same way.)

To get $\mathcal{F}^{(2)\mu}$ we simply interchange 1 and 2 in (2), and for $\mathcal{F}^{(2)5\mu}$ we also replace $i\gamma^{\mu}$ by $-\gamma_5\gamma^{\mu}$. The corresponding diagram is (b) in Figure 4.

The total 4-momentum of the system is

$$P^{\mu} = p_1^{\mu} + p_2^{\mu}\,, \tag{4}$$

and the mass is given by

$$M^2 = -P^2 = -(p_1 + p_2)^2 \ . \tag{5}$$

The mass spectrum is continuous, running from 2m to ∞ .

We must now find $F_a(\vec{k}_\perp)$ and $F_a^5(\vec{k}_\perp)$ and show that they are of the form (III. 20) so that the current algebra at infinite momentum is satisfied.

If p_1 and p_2 are such that the center of mass is at rest, then

$$\begin{aligned} p_1^\mu &= (\vec{p}, W) \\ p_2^\mu &= (-\vec{p}, W) \\ P^\mu &= (\vec{0}, 2W) = (\vec{0}, M) \end{aligned} \tag{6}$$

where

$$W = \sqrt{\vec{p}^{\,2} + m^2} \ . \tag{7}$$

If the center of mass is not at rest, the variable $\vec{p}$ may still be used to describe the system by writing

$$\begin{aligned} p_1 &= V_{P/M \leftarrow \lambda} (\vec{p}, W) \ , \\ p_2 &= V_{P/M \leftarrow \lambda} (-\vec{p}, W) \ . \end{aligned} \tag{8}$$

Let the 2-quark states be relabeled in terms of the total momentum P and the internal variables $\vec{p}$, σ_1, σ_2:

$$| P, \vec{p}, \sigma_1, \sigma_2 \rangle = \frac{1}{\sqrt{W}} \, | p_1, \sigma_1, p_2, \sigma_2 \rangle \tag{9}$$

where p_1 and p_2 are defined in terms of P and $\vec{p}$ by (8). The new states then have the normalization

$$\langle P', \vec{p}', \sigma_1', \sigma_2' | P, \vec{p}, \sigma_1, \sigma_2\rangle = 2P^0(2\pi)^3\delta^3(\vec{P}' - \vec{P}) \times (2\pi)^3\delta^3(\vec{p}' - \vec{p})\,\delta_{\sigma_1'\sigma_1}\delta_{\sigma_2'\sigma_2}\,. \tag{10}$$

The matrix element of $\mathcal{F}^{(1)\mu}(0)$ between these new states is obtained by dividing (2) by $\sqrt{W'W}$, and from it we will find the infinite-momentum limit,

$$\langle \vec{p}', \sigma_1', \sigma_2' | F^{(1)}(\vec{k}_\perp) | \vec{p}, \sigma_1, \sigma_2\rangle = \lim_{P_z \to \infty} \frac{1}{2P_z} \langle P', \vec{p}', \sigma_1', \sigma_2' | \mathcal{F}^{(1)o}(0) | P, \vec{p}, \sigma_1, \sigma_2\rangle\,. \tag{11}$$

Before we can do this, however, we have to say how the Dirac spinors $u(p_1, \sigma_1)$ and $u(p_2, \sigma_2)$ are defined. The state $|P, \vec{p}, \sigma_1, \sigma_2\rangle$ is obtained from the rest state $|M\lambda, \vec{p}, \sigma_1, \sigma_2\rangle$ by the velocity transformation $V_{P/M \leftarrow \lambda}$, and this rest state is obtained from the state in which both quarks are at rest by applying $V_{(\vec{p}, W)/m \leftarrow \lambda}$ to the first quark and $V_{(-\vec{p}, W)/m \leftarrow \lambda}$ to the second. This is a matter of convention and enables us to separate the total angular momentum into pieces as follows:

$$\vec{\mathcal{J}} = i\frac{\partial}{\partial \vec{P}} \times \vec{P} + \vec{J}\,, \qquad \vec{J} = i\frac{\partial}{\partial \vec{p}} \times \vec{p} + \frac{1}{2}\vec{\sigma}^{(1)} + \frac{1}{2}\vec{\sigma}^{(2)}\,. \tag{12}$$

These transformations must be applied to the rest spinors to get the proper ones to use in (2), namely

$$u(p_1, \sigma_1) = \mathcal{D}(V_{P/M \leftarrow \lambda})\mathcal{D}(V_{(\vec{p},\ W)/m \leftarrow \lambda})\sqrt{2m}\begin{pmatrix} \chi(\sigma_1) \\ 0 \end{pmatrix}, \tag{13}$$

and similarly for $u(p_2, \sigma_2)$ with $\vec{p}$ replaced by $-\vec{p}$, where $\chi(\sigma)$ is a 2-component spinor and $\mathcal{D}$ (V) is the Dirac representation of V. The matrix $\mathcal{D}(V_{\lambda' \leftarrow \lambda})$ can be written explicitly and simply* in terms of the unit vectors λ and λ':

$$(V_{\lambda' \leftarrow \lambda}) = \frac{1}{\sqrt{2}} \frac{1 - \not{\lambda}'\not{\lambda}}{\sqrt{1 - \lambda' \cdot \lambda}}, \tag{14}$$

where $\not{\lambda}$ is the usual notation for $\lambda^\mu \gamma_\mu$.

To find the limit in (11) as $P_z = P_z' \to \infty$ with $\vec{P}' - \vec{P}$ fixed and equal to $\vec{k}_\perp$, consider first the δ-function appearing in (2). Using (8) and (III. 10) one finds that as $P_z \to \infty$,

$$\begin{aligned} \vec{p}_{2\perp} &= \frac{1}{2}(1 - \frac{p_z}{W})\vec{P}_\perp - \vec{p}_\perp + O(\frac{1}{P_z}) \\ p_{2z} &= \frac{1}{2}(1 - \frac{p_z}{W})P_z + O(1) \\ p_2^o &= \frac{1}{2}(1 - \frac{p_z}{W})P_z + O(1) \end{aligned} \tag{15}$$

and

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*Recall also formula (III. 10), which expresses $V_{\lambda' \leftarrow \lambda}$ itself in terms of λ and λ'.

$$\delta_{(m)}(p_2', p_2) = 2p_2^{\,o}(2\pi)^3\delta^3(\vec{p}_2' - \vec{p}_2)$$

$$\rightarrow \frac{2}{1 - p_z/W}(2\pi)^3\,\delta^2\Big(\frac{\vec{p}_\perp'}{1 - p_z'/W'} - \frac{\vec{p}_\perp}{1 - p_z/W} - \frac{\vec{k}_\perp}{2}\Big)\delta\Big(\frac{p_z'}{W'} - \frac{p_z}{W}\Big)\,. \tag{16}$$

We have used $p_z'/W' = p_z/W$ from the z-component of the δ-function to manipulate the xy-components.

The effect of the δ-functions in (16) is to set

$$\vec{\tilde{p}}' = \vec{\tilde{p}} + \frac{\vec{k}_\perp}{2}\,, \tag{17}$$

where

$$\vec{\tilde{p}} = \frac{\vec{p}_\perp}{1 - p_z/W} + g\Big(\frac{p_p}{W}\Big)\vec{e}_z\,. \tag{18}$$

Here g is a function which can be arbitrary (as long as it is one-to-one). Now just as the operator $e^{i\vec{k}_\perp \cdot \vec{x}/2}$ ($\vec{x}$ being the position operator) translates all momenta by $\vec{k}_\perp/2$, the operator $e^{i\vec{k}_\perp \cdot \vec{\tilde{x}}/2}$ will perform the desired transformation, (18), if $\vec{\tilde{x}}$ is the operator conjugate to $\vec{\tilde{p}}$, i. e.,

$$\tilde{x}_i = -i\frac{\partial}{\partial \tilde{p}_i} = \sum_j \frac{\partial p_j}{\partial \tilde{p}_i}\,x_j\,. \tag{19}$$

But $\partial p_j/\partial \tilde{p}_i = (A^{-1})_{ij}$, where $A_{ij} = \partial \tilde{p}_i/\partial p_j$. Carrying out the calculations one obtains $\vec{\tilde{x}}$. Since we always deal with $\vec{k}_\perp \cdot \vec{\tilde{x}}$ we only need $\tilde{x}$ and $\tilde{y}$. The result is

$$\tilde{x} = \frac{1}{2}\{(1 - \frac{p_z}{W})(x + \frac{p_x p_z}{p^2 + m^2} z) + h.c.\} \tag{20}$$

and similarly for $\tilde{y}$. We include the hermitian conjugate since $\tilde{x}$ and $\tilde{y}$ must be hermitian. Note that $\tilde{x}$ and $\tilde{y}$ are independent of the arbitrary function g as they should be.

We can also write as operator relations

$$\begin{aligned} \vec{\tilde{x}} &= T^{-1} \vec{x}\, T \\ \vec{\tilde{p}} &= T^{-1} \vec{p}\, T \end{aligned} \tag{21}$$

where T is a unitary transformation such that

$$T|\vec{p}\rangle = c(\vec{p})|\vec{\tilde{p}}\rangle \tag{22}$$

(ignoring spin indices), $c(\vec{p})$ being a normalization factor. To find $c(\vec{p})$ we impose unitarity of T:

$$(2\pi)^3\delta^3(\vec{p}' - \vec{p}) = \langle\vec{p}'|\vec{p}\rangle = \langle\vec{p}'|T^\dagger T|\vec{p}\rangle = |c(\vec{p})|^2(2\pi)^3\delta^3(\vec{\tilde{p}}' - \vec{\tilde{p}})$$

from which we get

$$c(\vec{p}) = \sqrt{\frac{|g'(p_z/W)|}{W} \cdot \frac{1 + p_z/W}{1 - p_z/W}}\,. \tag{23}$$

Therefore,

$$\delta_{(m)}(p_2', p_2) \to \frac{1 + p_z/W}{2W}\langle\vec{p}'|e^{i\vec{k}_\perp \cdot \vec{\tilde{x}}/2}|\vec{p}\rangle\,. \tag{24}$$

Consider now the Dirac spinors appearing in (2) and given by (13). Since $M = 2W$ and $P^0 = P_z + O(1/P_z)$ as $P_z \to \infty$ we have, using (14),

$$\mathcal{D}(V_{P/M\leftarrow\lambda}) = \frac{1}{2}(1 + \alpha_z)\sqrt{\frac{P_z}{W}} + \frac{1}{2}(1 - \alpha_z + \frac{\vec{P}_\perp \cdot \vec{\alpha}}{W})\sqrt{\frac{W}{P_z}} + O(P_z^{-3/2}) . \tag{25}$$

Also,

$$(V_{(\pm\vec{p}, W)/m\leftarrow\lambda}) = \frac{m + W \pm \vec{p} \cdot \vec{\alpha}}{\sqrt{2m(m + W)}} . \tag{26}$$

Therefore, after some calculation* one finds

$$\bar{u}(p_1', \sigma_1')i\gamma^0 u(p_1, \sigma_1) = P_z(1 + \frac{p_z}{W})\chi^\dagger(\sigma_1')U^\dagger(\vec{p}')U(\vec{p})\chi(\sigma_1) + O(1), \tag{27}$$

where

$$U(\vec{p}) = \frac{m + W + p_z + i(\vec{p} \times \vec{\sigma})_z}{\sqrt{2(m + W)(W + p_z)}} . \tag{28}$$

---

*It is useful to note that between spinors having only upper components an even number of α's may be replaced by the corresponding σ's between 2-component spinors, while an odd number gives zero. If a factor $i\gamma_5$ is also present, interchange "even" and "odd".

The 2×2 matrix $U(\vec{p})$ is unitary: $U^{\dagger}(\vec{p}) = U^{-1}(\vec{p})$. The remaining spinor product in (1) is somewhat harder to calculate because both terms in (25) are needed. Using

$$\vec{P}_{\perp}' - \vec{P}_{\perp} = \frac{2(\vec{p}_{\perp}' - \vec{p}_{\perp})}{1 - p_z/W}, \quad \text{and} \quad \frac{p_z'}{W'} = \frac{p_z}{W},$$

one arrives at the simple result

$$\frac{1}{2m}\,\bar{u}(p_2', \sigma_2')u(p_2, \sigma_2) = \chi^{\dagger}(\sigma_2')U^{\dagger}(-\vec{p}\,')U(-\vec{p})\chi(\sigma_2) + O\left(\frac{1}{P_z}\right), \tag{29}$$

where $U(\vec{p})$ is as defined in (28). For the axial vector current, $i\gamma^{o}$ is replaced by $\gamma_5\gamma^{o}$, the result of which turns out to be an extra factor of σ_z inserted in the right side of (27). Collecting everything together we finally get

$$\langle \vec{p}\,', \sigma_1', \sigma_2' | \left\{ \begin{matrix} F^{(1)}(\vec{k}_{\perp}) \\ F_5^{(1)}(\vec{k}_{\perp}) \end{matrix} \right\} | \vec{p}, \sigma_1, \sigma_2 \rangle$$

$$= \chi^{\dagger}(\sigma_1')U^{\dagger}(\vec{p}\,') \left\{ \begin{matrix} 1 \\ \sigma_z \end{matrix} \right\} U(\vec{p})\chi(\sigma_1) \tag{30}$$

$$\times\ \chi^{\dagger}(\sigma_2')U^{\dagger}(\vec{p}\,')\,U(\vec{p})\chi(\sigma_2)\,\langle \vec{p}\,' | e^{i\vec{k}_{\perp}\cdot\tilde{\vec{x}}/2} | \vec{p} \rangle$$

or, in terms of operators,

$$\left\{ \begin{matrix} F^{(1)}(\vec{k}_{\perp}) \\ F_5^{(1)}(\vec{k}_{\perp}) \end{matrix} \right\} = U^{\dagger}(\vec{p})^{(1)}U^{\dagger}(-\vec{p})^{(2)} \left\{ \begin{matrix} 1 \\ \sigma_z^{(1)} \end{matrix} \right\} e^{i\vec{k}_{\perp}\cdot\tilde{\vec{x}}/2}\, U(\vec{p})^{(1)}U(-\vec{p})^{(2)} \tag{31}$$

where the superscripts (1) and (2) refer to which quark's spin index is being acted on. Therefore,

$$F^{(1)}(\vec{k}_\perp) = e^{i\vec{k}_\perp \cdot \vec{h}^{(1)}} ,$$
$$F_5^{(1)}(\vec{k}_\perp) = \omega^{(1)} e^{i\vec{k}_\perp \cdot \vec{h}^{(1)}} , \tag{32}$$

where

$$h_x^{(1)} = U^\dagger(\vec{p})^{(1)} U^\dagger(-\vec{p})^{(2)} (\tfrac{1}{2}\tilde{x}) U(\vec{p})^{(1)} U(-\vec{p})^{(2)} ,$$

$$\omega^{(1)} = U^\dagger(\vec{p})^{(1)} U^\dagger(-\vec{p})^{(2)} \sigma_z^{(1)} U(\vec{p})^{(1)} U(-\vec{p})^{(2)} \tag{33}$$

$$= U^\dagger(\vec{p})^{(1)} \sigma_z^{(1)} U(\vec{p})^{(1)} .$$

To get $h_x^{(2)}$ and $-\omega^{(2)}$ we let $\vec{p} \to -\vec{p}$, $\vec{x} \to -\vec{x}$, and $\vec{\sigma}^{(1)} \leftrightarrow \vec{\sigma}^{(2)}$ in (33), remembering that $\tilde{x}$ is given by (20). Thus,

$$h_x^{(2)} = U^\dagger(\vec{p})^{(1)} U^\dagger(-\vec{p})^{(2)} \tfrac{1}{2}\tilde{x}' U(\vec{p})^{(1)} U(-\vec{p})^{(2)} ,$$
$$\omega^{(2)} = U^\dagger(\vec{p})^{(1)} U^\dagger(-\vec{p})^{(2)} (-\sigma_z^{(2)}) U(\vec{p})^{(1)} U(-\vec{p})^{(2)} , \tag{34}$$

where

$$\tilde{x}' = -\frac{1}{2}\left\{ \left(1 + \frac{p_z}{W}\right)\left(x + \frac{p_x p_z}{p^2 + m^2} z\right) + \text{h. c.} \right\} . \tag{35}$$

It is easily verified that $\tilde{x}$, $\tilde{y}$, $\tilde{x}'$, and $\tilde{y}'$ commute with each other (and, of course, with $\sigma^{(1)}$ and $\sigma^{(2)}$, so all the h's and ω's commute. Therefore the current algebra is satisfied, $F_a(\vec{k}_\perp)$ and $F_a^5(\vec{k}_\perp)$ being of the form (III. 20).

From (33) we can get explicit (though not very illuminating) expressions for $\vec{h}^{(1)}$ and $\omega^{(1)}$.

$$\vec{h}^{(1)} = \frac{1}{2}\vec{\tilde{x}}_\perp + \frac{W - p_z}{W + p_z}\,\frac{(\vec{p}\times\vec{\sigma}^{(1)})_\perp}{4W(m+W)} + \frac{\vec{p}_\perp}{W+p_z}\,\frac{(\vec{p}\times\vec{\sigma}^{(1)})_z}{4W(m+W)} + \frac{W - p_z}{W+p_z}\,\frac{\vec{e}_z\times\vec{\sigma}^{(1)}}{4W} + \frac{(\vec{p}\times\vec{\sigma}^{(2)})_\perp}{4W(m+W)} + \frac{\vec{p}_\perp}{W+p_z}\,\frac{(\vec{p}\times\vec{\sigma}^{(2)})_z}{4W(m+W)} - \frac{\vec{e}_z\times\vec{\sigma}^{(2)}}{4W}, \tag{36}$$

$$\omega^{(1)} = \frac{m}{W + p_z}\,\sigma_z^{(1)} + \frac{m + W + p_z}{(m + W)(W + p_z)}\,\vec{p}\cdot\vec{\sigma}^{(1)} .$$

It is interesting to note at this point that if we consider mesons as being made of two <u>spinless</u> quarks, replacing (2) by what we would expect from spinless particle theory:

$$\langle p_1', p_2'|\mathcal{F}^{(1)\mu}(0)|p_1, p_2\rangle = (p_1' + p_1)^\mu\,\delta_{(m)}(p_2', p_2)\,, \tag{37}$$

then it turns out that

$$\vec{h}^{(1)} = \frac{1}{2}\,\vec{\tilde{x}}\ , \tag{38}$$

the same as that obtained from (36) by dropping all terms with σ's. In this model there is no axial-vector current (i. e., $\omega^{(i)} = 0$) since a single spinless quark cannot contribute an axial current.

We can use (36) to find the electric and magnetic dipole moments according to (II. 31) and see whether they are what we would expect from a free-quark model.[6] Between states of equal mass (i. e., throwing out terms expressible as a commutator of something with W), we have

$$h_x^{(1)} = \frac{1}{2}\left(x + \frac{zp_x - xp_z}{2W} - \frac{\sigma_y^{(1)}}{2W} + \frac{\sigma_y^{(2)}}{2W}\right),$$

$$h_x^{(2)} = \frac{1}{2}\left(-x + \frac{zp_x - xp_z}{2W} + \frac{\sigma_y^{(1)}}{2W} - \frac{\sigma_y^{(2)}}{2W}\right),$$

so that

$$h_{ax} = \frac{1}{2}\lambda_a^{(1)} h_x^{(1)} + \frac{1}{2}\lambda_a^{(2)} h_x^{(2)} \tag{39}$$

is of the form $E_{ax} + F_a J_y/M - M_{ay}$ with

$$\vec{E}_a = \frac{1}{2}\lambda_a^{(1)} \frac{\vec{x}}{2} + \frac{1}{2}\lambda_a^{(1)} \left(-\frac{\vec{x}}{2}\right)$$

$$\vec{M}_a = \frac{1}{2}\lambda_a^{(1)} \frac{1}{2W}\left(\frac{\vec{x}}{2} \times \vec{p} + \vec{\sigma}^{(1)}\right) + \frac{1}{2}\lambda_a^{(2)} \frac{1}{2W}\left(\frac{\vec{x}}{2} \times \vec{p} + \vec{\sigma}^{(2)}\right) \tag{40}$$

which is just what we would expect for the moments of a pair of independent quarks.

Let us expand $h_x^{(1)}$ and $\omega^{(1)}$ in powers of $1/m$:

$$h_x^{(1)} = \frac{1}{2}x + \frac{1}{2m}[-xp_z - \frac{1}{2}\sigma_y^{(1)} + \frac{1}{2}\sigma_y^{(2)}]$$

$$+ \frac{1}{2m^2}[zp_xp_z + \frac{1}{4}p_z(3\sigma_y^{(1)} - \sigma_y^{(2)}) + \frac{1}{4}p_y(\sigma_z^{(1)} + \sigma_z^{(2)})]$$

$$+ \frac{1}{2m^3}[\frac{1}{2}xp_zp^2 - zp_xp_z^2 - (p_z^2 - \frac{1}{4}p^2)\sigma_y^{(1)} - \frac{1}{4}p^2\sigma_y^{(2)}$$

$$- \frac{1}{2}p_z(p_y\sigma_z^{(1)} - p_z\sigma_y^{(1)}) + \frac{1}{4}p_x(p_x\sigma_y^{(1)} - p_y\sigma_x^{(1)}) \quad (41a)$$

$$+ \frac{1}{4}p_x(p_x\sigma_y^{(2)} - p_y\sigma_x^{(2)})] + \ldots$$

- antihermitian part,

$$\omega^{(1)} = \sigma_z^{(1)} + \frac{1}{m}\vec{p}_\perp \cdot \vec{\sigma}^{(1)} + \frac{1}{m^2}[-\frac{1}{2}p_z\vec{p}_\perp \cdot \vec{\sigma}^{(1)} - \frac{1}{2}p_\perp^2\sigma_z^{(1)}]$$

$$+ \frac{1}{m^3}[-\frac{1}{2}p_\perp^2\vec{p}_\perp \cdot \vec{\sigma}^{(1)} + \frac{1}{2}p_\perp^2 p_z\sigma_z^{(1)}] + \ldots . \quad (41b)$$

This expansion will be useful when we try to find the corresponding operators for the bound-quark system in the next chapter. From (41), or directly from (36), one notices that each successive order in $1/m$ has one more unit of $|\Delta J_x|$. In fact, by inserting δ_{zz} and ϵ_{xyz} (both being equal to 1) when necessary, we can write

$$h_x^{(1)} = A_x + \frac{1}{m}B_{xz} + \frac{1}{m^2}C_{xzz} + \frac{1}{m^3}D_{xzzz} + \ldots$$

$$\omega^{(1)} = a_z + \frac{1}{m}b_{zz} + \frac{1}{m^2}c_{zzz} + \frac{1}{m^3}d_{zzzz} + \ldots . \quad (42)$$

Expanding in powers of $1/m$ turns out to be equivalent to putting in c (the velocity of light) explicitly, and expanding in powers of $1/c$.

To wind up this discussion of the free-quark model we present an alternative form for writing the currents. In the limit $m \to \infty$ the current takes on the "nonrelativistic" limit

$$F_a(\vec{k}) = \frac{1}{2}\lambda_a^{(1)} e^{i\vec{k}_\perp \cdot \vec{x}/2} + \frac{1}{2}\lambda_a^{(2)} e^{-i\vec{k}_\perp \cdot \vec{x}/2} ,$$

$$F_a^5(\vec{k}) = \frac{1}{2}\lambda_a^{(1)} \sigma_z^{(1)} e^{i\vec{k}_\perp \cdot \vec{x}/2} - \frac{1}{2}\lambda_a^{(2)} \sigma_z^{(2)} e^{-i\vec{k}_\perp \cdot \vec{x}/2} , \tag{43}$$

which, of course, satisfies the current algebra, but satisfies the angular condition only to lowest order in $1/m$. The exact current to all orders in $1/m$ can then be written in the form

$$F_a(\vec{k}) = e^{iS^{(1)}} \frac{1}{2}\lambda_a^{(1)} e^{i\vec{k}_\perp \cdot \vec{x}/2} e^{-iS^{(1)}}$$

$$+ e^{iS^{(2)}} \frac{1}{2}\lambda_a^{(2)} e^{-i\vec{k}_\perp \cdot \vec{x}/2} e^{-iS^{(2)}} , \tag{44}$$

and similarly for $F_a^5(\vec{k}_\perp)$ with the same $S^{(1)}$ and $S^{(2)}$. In other words, the contribution from each quark is related to its lowest-order limit by a unitary transformation (which is 1 to lowest order). The operator $S^{(1)}$ is such that

$$\vec{h}^{(1)} = e^{iS^{(1)}} \frac{1}{2}\vec{x}_\perp e^{-iS^{(1)}}$$

$$\omega^{(1)} = e^{iS^{(1)}} \sigma_z^{(1)} e^{-iS^{(1)}} \tag{45}$$

and similarly for $S^{(2)}$ (with appropriate minus signs). From (33),

$$e^{-iS^{(1)}} = e^{-iS_0} U(\vec{p})^{(1)} U(-\vec{p})^{(2)}, \tag{46}$$

where S_0 is a spin-independent hermitian operator satisfying

$$\vec{\tilde{x}}_\perp = e^{iS_0} \vec{x}_\perp e^{-iS_0} . \tag{47}$$

In fact, e^{-iS_0} is just the transformation T appearing in (21) and (22). It follows that S_0 is of the form

$$S_0 = \frac{1}{2}[\vec{f}(\vec{p}) \cdot \vec{x} + \vec{x} \cdot \vec{f}(\vec{p})] , \tag{48}$$

for with such an S_0,

$$e^{-itS_0} | \vec{p} \rangle \propto | \vec{p}(t) \rangle , \tag{49}$$

where $\vec{p}(t)$ satisfies

$$\frac{d\vec{p}(t)}{dt} = - \vec{f}(\vec{p}(t)); \ \vec{p}(0) = \vec{p} . \tag{50}$$

The constant of proportionality in (49) is such as to preserve the normalization, i. e., make the transformation unitary.

To find S_0 we must find a function $\vec{f}$ such that, if we solve (50), $\vec{p}(1) = \vec{\tilde{p}}$. Recall that $\vec{\tilde{p}}$ is given in terms of $\vec{p}$ by (18), and contains an arbitrary function g. Thus S_0 is also partly arbitrary. We can find S_0 as a power series in $1/m$ by writing $\exp(iS_0)\vec{p}\exp(-iS_0) = \tilde{\vec{p}}$ in the form

$$(\exp i\mathcal{A}_{S_0})\vec{p} = \tilde{\vec{p}}, \tag{51}$$

where $\mathcal{A}$ is defined by (III. 14), and expanding both sides in powers of $1/m$ in (51) then determine the nth order term in S_0. The result turns out to be simplest when the arbitrary function is given by

$$g(u) = \frac{mu}{1-u}. \tag{52}$$

Then, and only then, the nth order term in S_0 is of the form $u_{z\ldots z}$ with n z's (or can be made so by inserting $1 = \delta_{zz}$). The result, with g as in (52), is that S_0 is given by (48) with

$$\vec{f}(\vec{p}) = -\frac{1}{m}p_z\vec{p} + \frac{1}{2m^2}p^2p_z\vec{e}_z + \frac{1}{m^3}\left(-\frac{1}{2}p^2p_z^2\vec{e}_z + \frac{1}{4}p^2p_z\vec{p}\right) + \ldots. \tag{53}$$

An expression for $\vec{f}$ in closed form has so far resisted discovery, even in the simplified case where $\vec{p} = 0$. However, there turns out to be a factorization of the form

$$e^{-iS_0} = e^{-iS_2}e^{-iS_1} \tag{54}$$

with S_1 and S_2 expressible in closed form:

$$S_1 = \frac{1}{2}\left[\frac{p_z p^2}{2p_\perp^2}\left(\log\frac{m^2+p_\perp^2}{m^2}\right)z + \text{h. c.}\right],$$
$$S_2 = -\frac{1}{2}\left[\frac{p_z \vec{p}\cdot\vec{x}}{m} + \text{h. c.}\right]. \qquad (55)$$

These were found by writing $\vec{p} \to \tilde{\vec{p}}$ as two successive transformations: $\vec{p} \to \vec{p}\,' = \vec{p} + (mp_z/W)\vec{e}_z$ followed by $\vec{p}\,' \to \tilde{\vec{p}} = \vec{p}\,'/(1 - p_z'/m)$, and finding the $\vec{f}$'s to use in (50) by trial and error. The operator S_0 is related to S_1 and S_2 by

$$S_0 = S_1 + S_2 + \frac{i}{2}[S_1, S_2]$$
$$- \frac{1}{12}[S_1, [S_1, S_2]] - \frac{1}{12}[S_2, [S_2, S_1]]$$
$$+ \dots .$$

V. THE BOUND-QUARK MODEL-NONCOVARIANT FORMALISM

We now set out to find the consequences of imposing the angular condition on the 2-bound-quark model proposed in Chapter III. Rewriting (III. 21), we require that

$$V \equiv e^{iQ} e^{ikh_x^{(1)}} \quad \text{must have } |\Delta J_x| \leq 1 \tag{1a}$$

$$A \equiv e^{iQ} \omega^{(1)} e^{ikh_x^{(1)}} \quad \text{must have } |\Delta J_x| \leq 1 \tag{1b}$$

where

$$Q = \mathcal{A}_{J_y} \tan^{-1} \frac{\mathcal{A}_M}{k} - \mathcal{B}_{J_y} \tan^{-1} \frac{k}{\mathcal{B}_M} , \tag{2}$$

$$\vec{J} = \vec{x} \times \vec{p} + \frac{1}{2}\vec{\sigma}^{(1)} + \frac{1}{2}\vec{\sigma}^{(2)} , \tag{3}$$

$$M = 2\sqrt{m^2 + \vec{p}^2 + U(\vec{x})} . \tag{4}$$

If we find $h_x^{(1)}$, $h_y^{(1)}$ $(= -i[J_z, h_x^{(1)}])$, and $\omega^{(1)}$ satisfying (1), then $h_x^{(2)}$, $h_y^{(2)}$, and $\omega^{(2)}$ are determined by interchanging the two quarks and will automatically satisfy the angular condition since Q is invariant under this interchange. We then have to check whether $h_x^{(2)}$ commutes with $h_x^{(1)}$, etc.

Expanding the mass operator in powers of $1/m$ we obtain

$$M = 2m + \frac{K}{m} - \frac{K^2}{4m^3} + \ldots , \tag{5}$$

where

$$K = \vec{p}^2 + U(\vec{x}) . \tag{6}$$

We then assume that $\vec{h}^{(1)}$ and $\omega^{(1)}$ can be similarly expanded:

$$\begin{aligned} h_x^{(1)} &= h_{ox} + \frac{1}{m} h_{1x} + \frac{1}{m^2} h_{2x} + \dots , \\ \omega^{(1)} &= \omega_o + \frac{1}{m} \omega_1 + \frac{1}{m^2} \omega_2 + \dots , \end{aligned} \tag{7}$$

[the superscript (1) being implied if not given explicitly] and that the lowest-order terms are the same as those for the free-quark system (since M becomes the same as $m \to \infty$), i.e.,

$$\begin{aligned} h_{ox} &= \frac{1}{2} x , \\ \omega_o &= \sigma_z^{(1)} . \end{aligned} \tag{8}$$

To find the higher-order terms we have to expand V and A of (1) in powers of $1/m$ and require that they have $|\Delta J_x| \leq 1$. To expand e^{iQ} we can expand Q and then exponentiate it, or we can expand the expression

$$e^{iQ} = \left(\frac{1 + \frac{i}{k} \mathcal{A}_M}{1 - \frac{i}{k} \mathcal{A}_M} \right)^{\frac{1}{2} \mathcal{A}_{J_y}} \left(\frac{\mathcal{B}_M - ik}{\mathcal{B}_M + ik} \right)^{\frac{1}{2} \mathcal{B}_{J_y}} . \tag{9}$$

[Note, by the way, that $\mathcal{B}_M = 4m + O(1/m)$ while $\mathcal{A}_M$ starts off with order $1/m$.] The result is

$$
\begin{aligned}
e^{iQ} = {} & 1 + \frac{i}{m}\left[\frac{1}{k}\mathcal{A}_K\,\mathcal{A}_{J_y} - \frac{k}{4}\mathcal{B}_{J_y}\right] \\
& - \frac{1}{m^2}\left[\frac{1}{2k^2}\mathcal{A}_K^{\,2}\mathcal{A}_{J_y}^{\,2} - \frac{1}{4}\mathcal{A}_K\,\mathcal{A}_{J_y}\,\mathcal{B}_{J_y} + \frac{k^2}{32}\mathcal{B}_{J_y}^{\,2}\right] \\
& - \frac{i}{m^3}\left[\frac{1}{6k^3}\mathcal{A}_K^{\,3}\,\mathcal{A}_{J_y}(\mathcal{A}_{J_y}^{\,2} + 2) - \frac{1}{8k}(\mathcal{A}_K^{\,2}\mathcal{A}_{J_y}^{\,2}\,\mathcal{B}_{J_y} \right. \\
& - 2\,\mathcal{A}_{K^2}\,\mathcal{A}_{J_y}) + \frac{k}{32}(\mathcal{A}_K\,\mathcal{A}_{J_y}\,\mathcal{B}_{J_y}^{\,2} - 2\mathcal{B}_K\,\mathcal{B}_{J_y}) \\
& \left. - \frac{k^3}{384}\mathcal{B}_{J_y}(\mathcal{B}_{J_y}^{\,2} + 2)\right] + \dots \,.
\end{aligned}
\tag{10}
$$

We also expand $e^{ikh_x^{(1)}}$, obtaining

$$
\begin{aligned}
e^{ikh_x^{(1)}} = {} & [1 + ikh_{ox} - \frac{k^2}{2}h_{ox}^{\,2} - \frac{ik^3}{6}h_{ox}^{\,3} + \dots] \\
& + \frac{1}{m}[ikh_{1x} - \frac{k^2}{2}(h_{ox}h_{1x} + h_{1x}h_{ox}) \\
& - \frac{ik^3}{6}(h_{ox}^{\,2}h_{1x} + h_{ox}h_{1x}h_{ox} + h_{1x}h_{ox}^{\,2}) + \dots] \\
& + \dots,
\end{aligned}
\tag{11}
$$

and, of course, $\omega^{(1)}$ is expanded as in (7). Combining all of this together we obtain expansions of V and A in powers of $1/m$. The coefficient of $1/m^n$ in each contains all powers of k from $-n$ to $+\infty$. For a given n, the coefficient of each power of k must have $|\Delta J_x| \leq 1$. The term that gives the most information is k^1 in the case of V and k^0 in the case of A, for a little inspection shows that the order $(1/m^n)k^1$ in V gives h_{nx} in terms of $h_{ox}, \ldots, h_{(n-1)x}$, and $(1/m^n)\,k^0$ in A gives ω_n in terms of $\omega_o, \ldots, \omega_{n-1}$. However, h_{nx} and ω_n are not completely determined this way, because we can add to V and A any terms with $|\Delta J_x| = 0$ or 1. This indeterminancy is partly resolved by looking at other powers of k, and by imposing commutativity of $h_x^{(1)}$, $h_y^{(1)}$, $h_x^{(2)}$, etc., as well as $\omega^{(1)\,2} = 1$. We also require that when $U = 0$ we obtain the free-quark solution, which we found exactly and expanded in powers of $1/m$ in (IV. 41).

Recall that although $h_x^{(1)}$ is a component of a vector with respect to rotations about the z-axis, it is not with respect to, say, rotations about the x-axis. In fact, in the free-quark model we found that the transformation properties under general rotations were as expressed in (IV. 42), namely, that the coefficient h_{nx} of $1/m^n$ is of the form $T_{xz\,\ldots\,z}$ with n z's. We shall assume this to be true in the bound-quark case also. If it is true for $h_{1x}, \ldots,$ $h_{(n-1)x}$ then by inspection of the angular condition it will also be true for h_{nx} provided that the indeterminate $|\Delta J_x| = 1$ terms have only $\Delta J_x = 0$ for even n and $\Delta J_x = \pm 1$ for odd n, which is true by parity, time reversal, and hermiticity arguments if h_{nx} is a polynomial in $\vec{x}$, $\vec{p}$, and U and does not contain any other constants with dimensions of length.

Starting with h_{ox} and ω_o as in (8) and applying the procedure outlined to order $1/m$ we find that everything is satisfied with h_{1x} and ω_1 equal to their free-quark values. However, there could be undetermined U-dependent terms with $\Delta J_x = \pm 1$ in h_{1x} and with $\Delta J_x = 0$ in ω_1. Requiring $\omega^2 = 1$ to order $1/m$ gives the condition $\{\sigma_z^{(1)}, \omega_1\} = 0$, which eliminates any additional terms with $\Delta J_x = 0$. The ambiguity in h_{1x} can also be eliminated by the angular condition, parity, and the commutativity of the h's and ω's. Thus,

$$h_{1x} = -\frac{1}{2} x p_z - \frac{1}{4} \sigma_y^{(1)} + \frac{1}{4} \sigma_y^{(2)} ,$$
$$\omega_1 = p_x \sigma_x^{(1)} + p_y \sigma_y^{(1)} . \tag{12}$$

Applying the angular condition to second order, and letting

$$U' = \frac{dU}{d(r^2/2)} = \frac{1}{r} \frac{dU}{dr}$$

(so that $\nabla U = \vec{x} U'$; U is assumed to depend only on r), one finds

$$h_{2x} = \text{free-quark result} - \frac{1}{8} x z^2 U' ,$$
$$\omega_2 = \text{free-quark result} + \frac{1}{2} z (x \sigma_x^{(1)} + y \sigma_y^{(1)}) U' , \tag{13}$$

apart from the usual undetermined terms. Without them all other requirements are satisfied to order $1/m^2$. Any additional terms with $|\Delta J_x| \leq 1$ are somewhat restricted, but not completely ruled out [for example, we could add $(\vec{a} \times \vec{\sigma}^{(1)})_z$ to ω_2 where $\vec{a}$ is a pseudovector commuting with $\vec{x}$]. We can assume the absence of them but come back and consider them if something goes wrong later.

The higher-order terms in the expansion of the angular condition become increasingly messy to evaluate, even when the problem is simplified by assuming spinless quarks [look at (10) and (11)]. As an aid in manipulating the operators, particularly with spin included, a set of subroutines written in FORMAC for the IBM 7094 computer was developed. FORMAC is an extension of FORTRAN which deals with algebraic expressions, and while it does not itself deal with noncommuting operators the subroutines were designed to (1) multiply expressions treating them as operators and observing the commutation relations between $\vec{x}$, $\vec{p}$, $\vec{\sigma}^{(1)}$, etc., (2) perform commutations and anticommutations of operators any given number of times, (3) throw out terms with $|\Delta J_x| \leq 1$ when appropriate, since such terms are undetermined in the final results anyway, and (4) print out the answers in readable form. Using these subroutines, a program can easily be written which will calculate h_{3x} and ω_3 in a few minutes (approximately two and five minutes, respectively). The angular condition to 3rd order gives for h_{3x},

$$h_{3x} = \text{free-quark result} + \frac{1}{6} z^3 U' p_x + \frac{1}{4} x z^2 U' p_z + \frac{1}{24} x\, z^3 U'' \vec{x} \cdot \vec{p} + \frac{1}{4} z^2 U' \sigma_y^{(1)} \qquad (14)$$

$$- \text{ antihermitian part } + \text{ terms with } \Delta J_x = \pm 1.$$

At this point we run into trouble. The condition $[h_x^{(1)}, h_y^{(1)}] = 0$ is satisfied to 3rd order, but the condition $[h_x^{(1)}, h_x^{(2)}] = 0$ is not. The latter condition in order $1/m^3$ amounts to

$$[\frac{x}{2}, h_{3x}^{(1)}] + [h_{1x}^{(1)}, h_{2x}^{(2)}] + [h_{2x}^{(1)}, h_{1x}^{(2)}] + [h_{3x}^{(1)}, -\frac{x}{2}] = 0, \tag{15}$$

but an explicit evaluation gives*

$$\frac{i}{6} z^3 U' + \frac{i}{6} x^2 z^3 U'' + \frac{i}{4} x^2 z U' \tag{16}$$

instead of zero. The last term of (16) could probably be removed by adding suitable terms with $\Delta J_x = \pm 1$ to h_{3x}, but the other two have (among other things) $|\Delta J_x| = 3$, and cannot be removed by adding $|\Delta J_x| \leq 1$ terms to h_{3x} or h_{2x}. In other words, h_{3x} is determined enough to show that (15) cannot be satisfied [if $U \propto r^2$ the second term in (16) drops out, but we cannot make the first zero unless U = constant, which is trivial].

Ignoring this dilemma for the moment and going on to ω_3, we find

$$\begin{aligned}\omega_3 = \text{free-quark result} &- \frac{1}{2} \vec{p}_\perp \cdot \vec{\sigma}_\perp^{(1)} U - \frac{1}{2} z\vec{x}_\perp \cdot \vec{p}_\perp \vec{\sigma}_z^{(1)} U' \\ &- \frac{3}{4} z p_z \vec{x}_\perp \cdot \vec{\sigma}_\perp U' - \frac{1}{2} z^2 \vec{p}_\perp \cdot \vec{\sigma}_\perp U' - \frac{1}{8} \vec{\sigma}_\perp^{(1)} \cdot (\vec{x} \times \vec{\sigma}^{(2)})_\perp U' \\ &- \frac{1}{4} z^2 \vec{x} \cdot \vec{p}\, \vec{x}_\perp \cdot \vec{\sigma}_\perp^{(1)} U'' - \text{antihermitian part,}\end{aligned} \tag{17}$$

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*We need only find the U-dependent terms, since we know that the free-quark terms give zero by themselves.

where we have adjusted the $\Delta J_x = 0$ terms so that $\omega^2 = 1$ is satisfied to order $1/m^3$; this determines ω_3 completely (assuming h_{2x} and ω_2 are as stated). The condition $[h_x, \omega]$ is satisfied to this order if we add the following $\Delta J_x = \pm 1$ terms to h_{3x} as given in (14):

$$\frac{1}{8}\sigma_y^{(1)}U - \frac{1}{8}x(\vec{x} \times \vec{\sigma}^{(1)})_z U' + \frac{1}{16}z(\vec{x} \times \vec{\sigma}^{(1)})_x U' . \qquad (18)$$

There could also be terms with $\vec{\sigma}^{(2)}$. We get further restrictions if we require that h_{3x} be of the form $A_x + B_y + [\vec{p}^2 + U^2, \text{something}]$, so that $h_{3x} = A_x + B_y$ between equal-mass states ($\vec{A}$ and $\vec{B}$ being related to the dipole moments). As it stands, h_{3x} does not appear to be of this form.

At this point it should be noted that there is another way to find the h's and ω's to each order by $1/m$, which is in fact the way originally proposed by Gell-Mann and Dashen.[6, 9] We write $h_x^{(1)}$ and ω_1 as a unitary transformation of $\frac{1}{2}x$ and $\sigma_z^{(1)}$ generated by $S^{(1)}$ as in (IV. 45), expand $S^{(1)}$ in powers of $1/m$, and find $S^{(1)}$ to each order by imposing the angular condition. Given $S^{(1)}$ to each order, we can then find $\vec{h}^{(1)}$ and $\omega^{(1)}$ to that order. This method has the advantage that the commutativity of $h_x^{(1)}$, $h_y^{(1)}$ and $\omega^{(1)}$ is automatically satisfied, although commutativity of $h_x^{(1)}$ and $h_x^{(2)}$, etc., still must be checked. On the other hand, the calculation of S for a given order is somewhat messier than that for h_x directly (FORMAC would be a great help here), and there are still problems in determining S exactly to each order. Weyers[14] has found the 3rd-order term of S by this method. Using it to find h_{3x}, one finds the same troublesome term $(z^2U'p_x/6)$ which causes the non-commutativity of $h_x^{(1)}$ and $h_x^{(2)}$. [In fact, we recover (14) and (18) plus some $\sigma^{(2)}$-dependent terms.]

The bound-quark model as presented here, then, does not work. It may be possible to find $\vec{h}^{(1)}$ and $\vec{h}^{(2)}$ to all orders in $1/m$, but they do not commute (this fact first showing up in third order). There appears to be no reasonable way to modify the h's so that they commute to third order even if we retract our assumption that h_{nx} is of the form $T_{xz\ldots z}$ with n z's. (The same problem, incidentally, arises with two bound spinless quarks, and in fact the h's through third order are the same as those found here with the σ-terms dropped.)

Can we change the model to make it solvable? Changing the mass, (5), in third order (i.e., replacing K^2 by something else) will not help because it only contributes $\Delta J_x = \pm 1$ to h_{3x}. We tried replacing (4) with a similar expression with $U(\vec{x})$ outside of the square root, but essentially the same problem arises, appearing this time in order $1/m^2$. We could modify the first-order term in the mass, but it is not clear what to replace it by, and the form is restricted by low-order terms in the angular condition. One reasonable but untried possibility is to add a "spin-orbit" term to K. The calculation of $\vec{h}$ and ω would become more complicated, however, and one would think that if everything worked with the spin-dependent term it would also work without it. Models with different mass operators have been explored using a covariant Lorentz-group formulation (to be discussed in Chapter VIII).

In Chapter VI we investigate the possibility of modifying $F_a(\vec{k}_\perp)$ so that it is no longer of the form (III. 20). The result [at least, for SU(3)] turns out to be that we always end up having to satisfy (III. 21) for <u>some</u> commuting $\vec{h}^{(1)}$ and $\vec{h}^{(2)}$, even though these h's may not be related to the currents in the same way as in (III. 20). This holds even if the mass operator is not SU(3)-invariant (but is still SU(2)-invariant).

As we noted at the end of Chapter III, if we cannot make $\vec{h}^{(1)}$ and $\vec{h}^{(2)}$ commute, but can at least find $\vec{h}^{(1)}$ to all orders, we have a solution of the factorized SU(2) problem of particles containing only one charged quark, describing the K-meson family. It has just recently been suggested that the current algebra itself might be relaxed slightly and that we only require $h_x^{(1)} + i h_y^{(1)}$ and $h_x^{(2)} + i h_y^{(2)}$ to commute (the third order dilemma then disappears). We would be representing a subalgebra with the momentum transfer satisfying $k_y = i k_x$, and since $\vec{k}_{\perp}^2 = 0$ we could describe, for example, amplitudes involving real photons. In either of these applications the operators that we have found to third order could be used as a starting point.

A much more satisfactory way to deal with the bound-quark model would be to write the current 4-vector in covariant form and require that its infinite-momentum limit satisfy the algebra, thus avoiding the angular condition with all of its uncertainties. This is the covariant formalism, which was successfully used in Chapter IV for the free-quark model. However, for bound quarks we do not know how to write down a suitable current covariantly. The Lorentz-group formalism seems to provide a way, but even it has presented problems, as we will see later.

VI. GENERAL NONCOVARIANT FORM OF REPRESENTATIONS IN THE TWO-QUARK MODEL

In searching for representations of the current algebra we have assumed that the current $F_a(\vec{k})$ could be expressed as a sum of two independent terms, and the same for $F_a^5(\vec{k})$, as in (III. 20). That is,*

$$F_a(\vec{k}) = \sum_{i=1}^{2} \frac{1}{2} \lambda_a^{(i)} e^{i\vec{k} \cdot \vec{h}^{(i)}} ,$$
$$F_a^5(\vec{k}) = \sum_{i=1}^{2} \frac{1}{2} \lambda_a^{(i)} \omega^{(i)} e^{i\vec{k} \cdot \vec{h}^{(i)}} \qquad (1)$$

where all of the h's and ω's commute. In the bound-quark model we managed to satisfy (to third order in $1/m$) everything except for commutativity of the (1)-operators with the (2)-operators. Since this commutatively was required by the splitting of F_a and F_a^5 according to (1), it is worth looking at more general expressions for F_a and F_a^5 which satisfy the algebra, (II. 16), and seeing whether we can have any better luck in satisfying the angular condition with them.

In this chapter we will find the most general forms for F_a and F_a^5 (as a function of $\vec{k}$) which satisfy (II. 16), i. e., which represent the current algebra, in a space on which $\lambda_a^{(1)}$ and $\lambda_a^{(2)}$ act (the two-quark model). The results will be useful in that

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*From now on $\vec{k}$ will always mean $\vec{k}_\perp$, i. e., with no z-component.

(a) they will show what operator functions of $\vec{k}$ have to satisfy the angular condition, (b) they will enable us to study systems in which the mass operator is not SU(3)-invariant, and (c) the form in which we write the currents will display the SU(3) properties with respect to the singlet and octet states.

Thus we will find out whether we have previously over-simplified the problem; whether it is essential that we replace (1) by more complicated expressions and/or use an SU(3)-breaking mass in order to be able to satisfy the angular condition. Before doing all this for SU(3), we will solve the simpler SU(2) algebra, in which F_a is the three-component isospin current (a = 1, 2, 3) and the mesons are made of two I = 1/2 nonstrange quarks. We will also see how this solution is modified when we include the extra isoscalar state made from two I = 0 strange quarks. The solution for SU(2) is almost the same as that for SU(3), but there are certain differences which are significant if we try to represent only the SU(2) current algebra.

SU(2) Currents

For SU(2), a runs from 1 to 3, and if $\tau_a^{(1)}$ and $\tau_a^{(2)}$ are the isospin matrices for the two quarks, we can write as the most general form for the currents,

$$F_a(k) = F^{(1)}(k) \frac{\tau_a^{(1)}}{2} + F^{(2)}(k) \frac{\tau_a^{(2)}}{2} + F^{(3)}(k) \epsilon_{abc} \frac{\tau_b^{(1)}}{2} \frac{\tau_c^{(2)}}{2} ,$$

$$F_a^5(k) = F^{5(1)}(k) \frac{\tau_a^{(1)}}{2} + F^{5(2)}(k) \frac{\tau_a^{(2)}}{2} + F^{5(3)}(k) \epsilon_{abc} \frac{\tau_b^{(1)}}{2} \frac{\tau_c^{(2)}}{2} . \tag{2}$$

We could plug these expressions directly into the algebra, collect the coefficients of the independent isotopic matrices, and get equations involving $F^{(i)}$ and $F^{5(i)}$ evaluated at $\vec{k}$, $\vec{k}'$, and $\vec{k}+\vec{k}'$. It turns out to be simpler, however, not to use (2) directly but to work with eigenstates of the total isospin, $I_a = \frac{1}{2}\tau_a^{(1)} + \frac{1}{2}\tau_a^{(2)}$. Let the isotriplet states be $|\underset{\sim}{3}, a\rangle$ ($a = 1, 2, 3$) and the isosinglet state $|\underset{\sim}{1}\rangle$ (suppressing the other internal variables). The state with $I = 1$ and $I_3 = 1$, for example, is $-\frac{1}{\sqrt{2}}(|\underset{\sim}{3}, 1\rangle + i|\underset{\sim}{3}, 2\rangle)$. The matrix elements of I_a are then

$$\langle \underset{\sim}{3}, c | I_a | \underset{\sim}{3}, b\rangle = i\,\epsilon_{cab}$$

with all other elements zero; I_a is the only isovector operator connecting the triplet with the triplet. We also define $A_a^{(+)}$ connect the singlet and triplet as follows:

$$\langle 3, b | A_a^{(+)} | 1\rangle = \delta_{ba}$$

with all other matrix elements zero, and define $A_a^{(-)} = A_a^{(+)\dagger}$. In terms of the original τ matrices,

$$A_a^{(\pm)} = \frac{1}{4}\left(\tau_a^{(1)} - \tau_a^{(2)} \pm i\,\epsilon_{abc}\,\tau_b^{(1)}\,\tau_c^{(2)}\right).$$

Any isovector operator must be a linear combination of I_a, $A_a^{(+)}$, and $A_a^{(-)}$, so we can replace (2) by an alternative expression

$$\begin{aligned} F_a(k) &= G^{(I)}(k)\,I_a + G^{(-)}(k)\,A_a^{(+)} + G^{(+)}(k)\,A_a^{(-)}, \\ F_a^{5}(k) &= G^{5(I)}(k)\,I_a + G^{5(-)}(k)\,A_a^{(+)} + G^{5(+)}(k)\,A_a^{(-)}. \end{aligned} \qquad (3)$$

Since the original currents $\mathcal{F}_a^{\mu}(x)$ are hermitian, $F_a(k)^\dagger = F_a(-k)$, so that $G^{(I)}(k)^\dagger = G^{(I)}(-k)$ and $G^{(\pm)}(k)^\dagger = G^{(\mp)}(-k)$. Similar relations hold for the axial operators. When $\vec{k} = 0$, F_a is just the isospin I_a, so that $G^{(I)}(0) = 1$ and $G^{(\pm)}(0) = 0$. We cannot make such a definite statement about the "axial charge", so we just let $G^{5(I)}(0) = \omega^{(I)}$ and $G^{5(\pm)}(0) = \omega^{(\pm)}$.

We now impose the commutation relations using (3) for the F's, and collect the coefficients of the isotopic operators to get relations among the G's. Multiplication involving I_a, $A_a^{(+)}$, $A_a^{(-)}$ is fairly simple, and among all of the products $I_a I_b$, $I_a A_b^{(+)}$, ..., $A_b^{(-)} A_a^{(-)}$ one finds six independent operators. From (II. 16a) we correspondingly obtain six relations involving the G's:

(a) $[G^{(I)}(k), G^{(I)}(k')] = 0$,

(b) $G^{(-)}(k)\, G^{(+)}(k') - G^{(-)}(k')\, G^{(+)}(k) = 0$,

(c) $G^{(+)}(k)\, G^{(-)}(k') - G^{(+)}(k')\, G^{(-)}(k) = 0$,

(d) $$\frac{1}{2}\{G^{(I)}(k), G^{(I)}(k')\} + \frac{1}{2} G^{(-)}(k)\, G^{(+)}(k') + \frac{1}{2} G^{(-)}(k')\, G^{(+)}(k) = G^{(I)}(k+k'), \qquad (4)$$

(e) $G^{(+)}(k)\, G^{(I)}(k') + G^{(+)}(k')\, G^{(I)}(k) = G^{(+)}(k+k')$,

(f) The hermitian conjugate of (e).

Using (a) and (b), we can immediately simplify (d) to

$$G^{(I)}(k)\,G^{(I)}(k') + G^{(-)}(k)\,G^{(+)}(k') = G^{(I)}(k + k') . \qquad (4d')$$

From (II. 16b) we get exactly the same set of equations as (4) but with $G^{(j)}(k')$ replaced by $G^{5(j)}(k')$ on the left and $G^{(j)}(k + k')$ by $G^{5(j)}(k + k')$ on the right $(j = I, +, -)$. Similarly, the consequences of (II. 16c) may be obtained from (4) by replacing $G^{(j)}(k)$ by $G^{5(j)}(k)$ and $G^{(j)}(k')$ by $G^{5(j)}(k')$ on the left.

To solve these equations, define $\vec{h}^{(I)} = (h_x^{(I)}, h_y^{(I)}, 0)$ and $\vec{h}^{(\pm)} = (h_x^{(\pm)}, h_y^{(\pm)}, 0)$ by

$$\begin{aligned} G^{(I)}(k) &= 1 + i\vec{k}\cdot\vec{h}^{(I)} + O(k^2) , \\ G^{(\pm)}(k) &= i\vec{k}\cdot\vec{h}^{(\pm)} + O(k^2) . \end{aligned} \qquad (5)$$

If we know $\vec{h}^{(I)}$ and $\vec{h}^{(+)}$ [note that $\vec{h}^{(-)} = \vec{h}^{(+)\dagger}$], then $G^{(I)}$ and $G^{(+)}$ are determined for all k, because (4d') and (4e) will determine all of the terms in power-series expansions of $G^{(I)}$ and $G^{(+)}$. If in addition we know $\omega^{(I)}$ and $\omega^{(+)}$, the "initial values" of $G^{5(I)}$ and $G^{5(+)}$, then $G^{5(I)}$ and $G^{5(+)}$ are determined for all $\vec{k}$ by letting $\vec{k}' = 0$ in the $[F_a(k), F_b^5(k')]$ analogue of (4d') and (4e). So to find the most general solution for $G^{(I)}$, $G^{(+)}$, $G^{5(I)}$, and $G^{5(+)}$, it suffices to find the constraints on $\vec{h}^{(I)}$, $\vec{h}^{(+)}$, $\omega^{(I)}$, and $\omega^{(+)}$, and then guess (or otherwise find) a solution for the G's satisfying all of the equations in (4) and their axial analogues.

Looking at low orders in $\vec{k}$ and $\vec{k}'$, one finds that the operators $h_i^{(I)}$, $\omega^{(I)}$, $h_i^{(-)}\,h_j^{(+)}$, $h_i^{(-)}\,\omega^{(+)}$, $\omega^{(-)}\,h_i^{(+)}$, and $\omega^{(-)}\,\omega^{(+)}$ form a commuting set, and furthermore,

$$h_i^{(\mp)} h_j^{(\pm)} = h_j^{(\mp)} h_i^{(\pm)} ,$$
$$h_i^{(\mp)} \omega^{(\pm)} = \omega^{(\mp)} h_i^{(\pm)} , \tag{6}$$

$$\omega^{(I)2} + \omega^{(-)} \omega^{(+)} = 1 ,$$
$$\omega^{(+)} \omega^{(I)} = \omega^{(I)} \omega^{(-)} = 0 . \tag{7}$$

Using (6) and appealing to two theorems in the Appendix at the end of this chapter, we can write $h_i^{(+)}$ and $\omega^{(+)}$ in the form $h_i^{(+)} = g\, h_i^{(J)}$ and $\omega^{(+)} = g\, \omega^{(J)}$, where $h_x^{(J)}$, $h_y^{(J)}$, and $\omega^{(J)}$ are hermitian operators commuting with each other and with $h_i^{(I)}$ and $\omega^{(I)}$, and $g^\dagger g = 1$ except possibly on states with $h_i^{(J)} = \omega^{(J)} = 0$. From (7) we obtain the further conditions $\omega^{(I)2} + \omega^{(J)2} = 1$ and $\omega^{(I)}\, \omega^{(J)} = 0$.

Given the h's and ω's, we can now find the (unique) functions $G^{(I)}(k)$, $G^{(\pm)}(k)$, $G^{5(I)}(k)$, and $G^{5(\pm)}(k)$ satisfying (4) by expanding in powers of k, by solving a differential equation, or by simply guessing the solution. The G's in the following paragraph indeed satisfy (4) with no further constraints on the h's and ω's. We give the solutions and summarize the constraints:

The most general two-isospinor-quark solution of the algebra (II. 16) with SU(2) currents is given by (3) with

$$G^{(I)}(k) = e^{i\vec{k} \cdot \vec{h}^{(I)}} \cos \vec{k} \cdot \vec{h}^{(J)} ,$$

$$G^{(+)}(k) = g\, G^{(J)}(k) ,$$

where $$G^{(J)}(k) = i\, e^{i\vec{k} \cdot \vec{h}^{(I)}} \sin \vec{k} \cdot \vec{h}^{(J)} ,$$

$$G^{(-)}(k) = G^{(J)}(k)\, g^{\dagger} \;,$$

$$G^{5(I)}(k) = \omega^{(I)}\, G^{(I)}(k) + \omega^{(J)}\, G^{(J)}(k) \;,$$

$$G^{5(+)}(k) = g\,[\omega^{(I)}\, G^{(J)}(k) + \omega^{(J)}\, G^{(I)}(k)] \;,$$

$$G^{5(-)}(k) = [\omega^{(I)}\, G^{(J)}(k) + \omega^{(J)}\, G^{(I)}(k)]\, g^{\dagger} \;, \tag{8}$$

where $h_x^{(I)}$, $h_y^{(I)}$, $h_x^{(J)}$, $h_y^{(J)}$, $\omega^{(I)}$, and $\omega^{(J)}$ all commute,

$$\omega^{(I)2} + \omega^{(J)2} = 1 \;, \qquad \omega^{(I)}\,\omega^{(J)} = 0 \;, \tag{9}$$

$$g^{\dagger} g = 1 \text{ except possibly on states with } h_x^{(J)} = h_y^{(J)} = \omega^{(J)} = 0.$$

(See the comment at the end of the Appendix.) Note that g need not commute with the h's or ω's.

This solution may, of course, also be described as the most general representation of the SU(2) current algebra containing one isovector particle family and one isoscalar family. Now in the quark model we can form two isoscalar families: one from two $I = 1/2$ quarks as described above, and another from two $I = 0$ quarks. The $I = 0$ pseudoscalar mesons, for example, are η and X^{o}, and the corresponding vector mesons are φ and ω. It is therefore more realistic to represent the isospin current algebra on a space of states containing an isotriplet $\{|\underset{\sim}{3}, a\rangle\}$ and two isosinglets $|\underset{\sim}{1}\rangle$ and $|\underset{\sim}{1}'\rangle$. The most general isospin current is then of the form

$$F_a(k) = G^{(I)}(k)\, I_a + G^{(-)}(k)\, A_a^{(+)} + G^{(+)}(k)\, A_a^{(-)} + G'^{(-)}(k)\, A_a'^{(+)} + G'^{(+)}(k)\, A_a'^{(-)} , \tag{10}$$

and similarly for $F_a^5(k)$, where I_a and $A_a^{(\pm)}$ are as before and $A_a'^{(+)} = A_a'^{(-)\dagger}$ has as its only non-zero element

$$\langle \underset{\sim}{3}, b \mid A_a'^{(+)} \mid \underset{\sim}{1}' \rangle = \delta_{ba} .$$

Imposing the current algebra we obtain equations similar to (4) and can solve them by slightly generalizing the theorems in the Appendix. The result is that $G^{(I)}$, $G^{(\pm)}$, $G^{5(I)}$, and $G^{5(\pm)}$ are still given by (8), and the expressions for $G'^{(\pm)}$ and $G^{5'(+)}$ are the same as for $G^{(\pm)}$ and $G^{5(\pm)}$ except that g is replaced by another operator g'. Conditions (9) remain the same except that $g^\dagger g = 1$ is replaced by $g^\dagger g + g'^\dagger g' = 1$; there are no further conditions on g and g'.

A special case of (10) will appear in our SU(3) solution, since SU(3) contains SU(2) as a subgroup.

SU(3) Currents

For a two-quark system, the most general SU(3) current is of the form

$$F_a(k) = F^{(1)}(k) \frac{\lambda_a^{(1)}}{2} + F^{(2)}(k) \frac{\lambda_a^{(2)}}{2} + F^{(f)}(k)\, f_{abc} \frac{\lambda_b^{(1)}}{2} \frac{\lambda_c^{(2)}}{2} + F^{(d)}(k)\, d_{abc} \frac{\lambda_b^{(1)}}{2} \frac{\lambda_c^{(2)}}{2} , \tag{11}$$

and similarly for $F_a^5(k)$, where a now runs from 1 to 8 and $\lambda_a^{(1)}$ and $\lambda_a^{(2)}$ are the SU(3) matrices for the two quarks. The symmetric and antisymmetric "couplings" d_{abc} and f_{abc} are defined by

$$\lambda_a \lambda_b = \frac{2}{3} \delta_{ab} + (d_{abc} + i f_{abc}) \lambda_c .$$

Here it is important that one quark (say No. 2) be considered an antiquark. If $\lambda_a^{(1)}$ is the usual λ_a matrix, then $\lambda_a^{(2)}$ is, with suitable conventions,[1] $-\lambda_a^*$. As in the isospin case, we prefer to work with eigenstates of the total F-spin, $F_a = \frac{1}{2}\lambda_a^{(1)} + \frac{1}{2}\lambda_a^{(2)}$. Label the octet states $|\underset{\sim}{8}, a\rangle$ (a = 1, ..., 8), and the singlet state $|\underset{\sim}{1}\rangle$. Then

$$\langle \underset{\sim}{8}, c | F_a | \underset{\sim}{8}, b\rangle = i f_{cab}$$

with all other matrix elements of F_a zero. Also define D_a by

$$\langle \underset{\sim}{8}, c | D_a | \underset{\sim}{8}, b\rangle = d_{cab}$$

with all other matrix elements zero, and define $A_a^{(+)}$ and $A_a^{(-)} = A_a^{(+)\dagger}$ by

$$\langle \underset{\sim}{8}, b | A_a^{(+)} | \underset{\sim}{1}\rangle = \langle \underset{\sim}{1} | A_a^{(-)} | \underset{\sim}{8}, b\rangle = \sqrt{\frac{2}{3}}\, \delta_{ab}$$

with all other elements zero. Using F_a, D_a, $A_a^{(+)}$, and $A_a^{(-)}$ as a basis in terms of which any operator transforming like $\underset{\sim}{8}$ can be expressed, we write

$$F_a(k) = G^{(F)}(k)\, F_a + G^{(D)}(k)\, D_a + G^{(-)}(k)\, A_a^{(+)} + G^{(+)}(k)\, A_a^{(-)} \,, \tag{12}$$
$$F_a^5(k) = G^{5(F)}(k)\, F_a + G^{5(D)}(k)\, D_a + G^{5(-)}(k)\, A_a^{(+)} + G^{5(+)}(k)\, A_a^{(-)} \,.$$

The "new basis" is related to the "old basis" as follows:

$$F_a = \frac{1}{2}\, \lambda_a^{(1)} + \frac{1}{2}\, \lambda_a^{(2)} \,,$$

$$D_a = \frac{5}{18}\, (\lambda_a^{(1)} - \lambda_a^{(2)}) + \frac{1}{3}\, d_{abc}\, \lambda_b^{(1)}\, \lambda_c^{(2)} \,, \tag{13}$$

$$A_a^{(\pm)} = \frac{1}{9}\, (\lambda_a^{(1)} - \lambda_a^{(2)}) + \frac{1}{6}\, (-d_{abc} \pm i\, f_{abc})\, \lambda_b^{(1)}\, \lambda_c^{(2)} \,.$$

As with SU(2) we shall use (II. 16) to find all of the G's in terms of the h's and ω's defined by

$$G^{(F)}(k) = 1 + i\,\vec{k} \cdot \vec{h}^{(F)} + O(k^2) \,,$$

$$G^{(D)}(k) = 0 + i\,\vec{k} \cdot \vec{h}^{(D)} + O(k^2) \,,$$

$$G^{(\pm)}(k) = 0 + i\,\vec{k} \cdot \vec{h}^{(\pm)} + O(k^2) \,,$$

$$G^{5(F)}(k) = \omega^{(F)} + O(k) \,, \tag{14}$$

$$G^{5(D)}(k) = \omega^{(D)} + O(k) \,,$$

$$G^{5(\pm)}(k) = \omega^{(\pm)} + O(k) \,.$$

Imposing the algebra turns out to be somewhat more complicated than for SU(2), because there are more independent operators among the products $F_a F_b$, $F_a D_b$, ... It helps to separate out the parts of $[F_a(k), F_b(k')]$ having definite SU(3) transformation properties. The representations that occur in this commutator are those found in $\underset{\sim}{8} \times \underset{\sim}{8}$, namely, $\underset{\sim}{1}$, $\underset{\sim}{8}$, $\underset{\sim}{27}$ (symmetric in a, b) and $\underset{\sim}{8}$, $\underset{\sim}{10}$, $\overline{\underset{\sim}{10}}$ (antisymmetric in a, b). There are two possible operators transforming like $\underset{\sim}{1}$: one connecting only the octet states $|\underset{\sim}{8}, a\rangle$ to themselves and one connecting $|\underset{\sim}{1}\rangle$ to itself. There are four kinds of $\underset{\sim}{8}$ operators (F_a, D_a, $A_a^{(+)}$, $A_a^{(-)}$) giving eight in all* since $\underset{\sim}{8}$ appears twice. Finally, there is one operator each transforming like $\underset{\sim}{10}$, $\overline{\underset{\sim}{10}}$, and $\underset{\sim}{27}$. Hence there are 13 independent operators, giving us 13 equations from (II. 16a):

$$[G^{(F)}(k), G^{(F)}(k')] = 0 \ ,$$

$$[G^{(D)}(k), G^{(D)}(k')] = 0 \ ,$$

$$[G^{(F)}(k), G^{(D)}(k')] = 0 \text{ and the (equivalent) one obtained by interchanging } k \text{ and } k' \ ,$$

$$G^{(-)}(k)\, G^{(+)}(k') - G^{(-)}(k')\, G^{(+)}(k) = 0 \ ,$$

$$G^{(+)}(k)\, G^{(-)}(k') - G^{(+)}(k')\, G^{(-)}(k) = 0 \ , \qquad (15)$$

$$G^{(D)}(k)\, G^{(D)}(k') - G^{(-)}(k)\, G^{(+)}(k') = 0 \ ,$$

$$G^{(+)}(k)\, G^{(D)}(k') - G^{(+)}(k')\, G^{(D)}(k) = 0 \text{ and its hermitian conjugate,}$$

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*I. e., in $[F_a(k), F_b(k')]$ we can have $f_{abc} F_c$, $f_{abc} D_c$, $f_{abc} A_c^{(+)}$, $f_{abc} A_c^{(-)}$, $d_{abc} F_c$, $d_{abc} D_c$, $d_{abc} A_c^{(+)}$, and $d_{abc} A_c^{(-)}$.

$$G^{(F)}(k)\,G^{(F)}(k') + G^{(D)}(k)\,G^{(D)}(k') = G^{(F)}(k+k') ,$$

$$G^{(F)}(k)\,G^{(D)}(k') + G^{(F)}(k')\,G^{(D)}(k) = G^{(D)}(k+k') ,$$

$$G^{(+)}(k)\,G^{(F)}(k') + G^{(+)}(k')\,G^{(F)}(k) = G^{(+)}(k+k') \text{ and its hermitian conjugate.}$$

Here the first few equations have already been used to simplify some of the remaining ones. We also get the same equations with certain G's replaced by G^5's as in Section II.

The method of solving these equations is similar to that of the last section and the solution looks almost the same: the superscript (I) is replaced by (F), and (J) by (D). It is interesting, however, that although the existence of $h^{(J)}$ and $\omega^{(J)}$ had to be shown by Theorem 1 of the Appendix in the SU(2) case, $h^{(D)}$ and $\omega^{(D)}$ are already defined by (14) in the SU(3) case. For SU(3) we therefore only need to use Theorem 2 to show the existence of g. The result is that the most general two-quark solution of the algebra with SU(3) currents is given by (12) with

$$\begin{aligned}
G^{(F)}(k) &= e^{i\vec{k}\cdot\vec{h}^{(F)}} \cos\vec{k}\cdot\vec{h}^{(D)} ,\\
G^{(D)}(k) &= i\,e^{i\vec{k}\cdot\vec{h}^{(F)}} \sin\vec{k}\cdot\vec{h}^{(D)} ,\\
G^{(+)}(k) &= g\,G^{(D)}(k) ,\\
G^{(-)}(k) &= G^{(D)}(k)\,g^{\dagger} ,\\
G^{5(F)}(k) &= \omega^{(F)}\,G^{(F)}(k) + \omega^{(D)}\,G^{(D)}(k) , \qquad (16)\\
G^{5(D)}(k) &= \omega^{(D)}\,G^{(F)}(k) + \omega^{(F)}\,G^{(D)}(k) ,
\end{aligned}$$

$$G^{5(+)}(k) = g\, G^{5(D)}(k) \ ,$$

$$G^{5(-)}(k) = G^{5(D)}(k)\, g^{\dagger} \ ,$$

where

$h_x^{(F)}$, $h_y^{(F)}$, $h_x^{(D)}$, $h_y^{(D)}$, $\omega^{(F)}$, and $\omega^{(D)}$ all commute,

$$\omega^{(F)2} + \omega^{(D)2} = 1, \qquad \omega^{(F)}\,\omega^{(D)} = 0 \ , \tag{17}$$

$g^{\dagger} g = 1$ except possibly on states with $h_x^{(D)} = h_y^{(D)} = \omega^{(D)} = 0$.

Note that this SU(3) solution has exactly the same form as the SU(2) solution in the last section except that in the SU(2) case $G^{(J)}(k)$ does not appear by itself as a "form factor in (3), while in the SU(3) case the corresponding operator $G^{(D)}(k)$ multiplies D_a in (12).

As a check on our SU(3) solution, we may observe that if we find $F_a(k)$ and $F_a^{\,5}(k)$ for SU(3) and restrict a to 1, 2, 3 only, then we have a reducible representation of the SU(2) current algebra. The "non-strange" states $|\underset{\sim}{8}, 1\rangle$, $|\underset{\sim}{8}, 2\rangle$, $|\underset{\sim}{8}, 3\rangle$, $|\underset{\sim}{1}\rangle$, and $|\underset{\sim}{8}, 8\rangle$ are taken into each other and we may identify them* with $|\underset{\sim}{3}, 1\rangle$, $|\underset{\sim}{3}, 2\rangle$, $|\underset{\sim}{3}, 3\rangle$, $|\underset{\sim}{1}\rangle$, and $|\underset{\sim}{1}'\rangle$ of the last section. Restricting F_a, D_a, and $A_a^{(\pm)}$ (a = 1, 2, 3) to this five-dimensional subspace we find $F_a \to I_a$, $D_a \to \sqrt{\frac{1}{3}}\,(A_a'^{(+)} + A_a'^{(-)})$, and $A_a^{(\pm)} \to \sqrt{\frac{2}{3}}\, A_a^{(\pm)}$.

*This identification is partly arbitrary, since the SU(2) states $|\underset{\sim}{1}\rangle$ and $|\underset{\sim}{1}'\rangle$ could correspond to any two orthogonal linear combinations of the SU(3) states $|\underset{\sim}{1}\rangle$ and $|\underset{\sim}{8}, 8\rangle$.

Comparing (12) with (10) and (8), we find $h^{(I)} = h^{(F)}$, $h^{(J)} = h^{(D)}$, $\omega^{(I)} = \omega^{(F)}$, $\omega^{(J)} = \omega^{(D)}$, $g_{SU(2)} = \sqrt{\frac{2}{3}}\, g_{SU(3)}$, and $g'_{SU(2)} = \sqrt{\frac{1}{3}}$. Therefore we have obtained a special case of the representation with one isovector and two isoscalars considered at the end of the last section.

Our SU(3) representation also contains two SU(2) representations with $I = 1/2$ particles. For example, let $|+\rangle = \frac{1}{\sqrt{2}}(|\underset{\sim}{8}, 4\rangle + i\,|\underset{\sim}{8}, 5\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|\underset{\sim}{8}, 6\rangle + i\,|\underset{\sim}{8}, 7\rangle)$ (corresponding to K^+ and K^o). Then with respect to these states $F_a \to \frac{1}{2}\tau_a$, $D_a \to \frac{1}{2}\tau_a$, and $A_a^{(\pm)} \to 0$ $(a = 1, 2, 3)$, so

$$
\begin{aligned}
F_a(k) &\to [G^{(F)}(k) + G^{(D)}(k)]\,\frac{1}{2}\tau_a = e^{i\vec{k}\cdot(\vec{h}^{(F)} + \vec{h}^{(D)})}\,\frac{1}{2}\tau_a\,, \\
F_a^5(k) &\to (\omega^{(F)} + \omega^{(D)})\, e^{i\vec{k}\cdot(\vec{h}^{(F)} + \vec{h}^{(D)})}\,\frac{1}{2}\tau_a\,,
\end{aligned}
\tag{18}
$$

Similarly, with respect to the states $|+'\rangle = \frac{1}{\sqrt{2}}(|\underset{\sim}{8}, 6\rangle - i\,|\underset{\sim}{8}, 7\rangle)$ and $|-'\rangle = -\frac{1}{\sqrt{2}}(|\underset{\sim}{8}, 4\rangle - i\,|\underset{\sim}{8}, 5\rangle)$ (corresponding to $\bar{K}^o$ and $\bar{K}^-$),

$$
\begin{aligned}
F_a(k) &\to e^{i\vec{k}\cdot(\vec{h}^{(F)} - \vec{h}^{(D)})}\,\frac{1}{2}\tau_a\,, \\
F_a^5(k) &\to (\omega^{(F)} - \omega^{(D)})\, e^{i\vec{k}\cdot(\vec{h}^{(F)} - \vec{h}^{(D)})}\,\frac{1}{2}\tau_a.
\end{aligned}
\tag{19}
$$

These two SU(2) representations are examples of the factorized case in which there is only one isospin-carrying quark.

In the special case $g = 1$, our SU(3) solution becomes

$$F_a(k) = G^{(F)}(k)\, F_a + G^{(D)}(k)\, (D_a + A_a^{(+)} + A_a^{(-)}) \,,$$
$$F_a^{\,5}(k) = G^{5(F)}(k)\, F_a + G^{5(D)}(k)\, (D_a + A_a^{(+)} + A_a^{(-)}). \tag{20}$$

But from (13) and (16), we find that (20) is equivalent to (1) with

$$\vec{h}^{(1)} = \vec{h}^{(F)} + \vec{h}^{(D)} \,, \qquad \omega^{(1)} = \omega^{(F)} + \omega^{(D)},$$
$$\vec{h}^{(2)} = \vec{h}^{(F)} - \vec{h}^{(D)} \,, \qquad \omega^{(2)} = \omega^{(F)} - \omega^{(D)}. \tag{21}$$

In other words, $g = 1$ gives the form of the currents usually assumed in the quark model.

Significance of the Results

Suppose as an approximation that the mass operator for the two-quark system is SU(3)-independent. Then if the angular condition is to be satisfied by $F_a(k)$ and $F_a^{\,5}(k)$, it must be satisfied by all of the G's, namely (in the SU(3) case) by $G^{(F)}$, $G^{(D)}$, $g\,G^{(D)}$, $G^{5(F)}$, $G^{5(D)}$, and $g\,G^{5(D)}$. Now if all of these operators satisfy the condition, then they satisfy it *a fortiori* with $g = 1$, which means that there exists a simpler solution of the form (1) which obeys the angular condition. In other words, if we cannot make (1) work (and the only success so far has been in the free-quark model), then we cannot make the general solution of the form (12) work either.

If we are content to deal with only SU(2) currents on states with only I = 0 or 1, then the operators which have to satisfy the angular condition are $G^{(I)}$, $g\,G^{(J)}$, and the corresponding axial operators, but $G^{(J)}$ itself need not satisfy it. It is conceivable, therefore, that there might be a solution for non-trivial g which does not continue to satisfy the angular condition when g is replaced by 1; this possibility has not been further investigated.

Our results also have profound implications in tackling the more general case of an SU(3)-breaking mass. In nature we have terms in the mass operator proportional to $\lambda_a^{(1)}\lambda_a^{(2)}$, which is SU(3)-symmetric but splits the singlets from the octets, and other terms (such as D_8) which actually break SU(3). We assume, however, that the SU(3) current algebra still holds (just as the ordinary SU(3) group exists even though it is not an exact symmetry in nature). As we saw in the last section, a general representation of it reduces into (among other things) two I = 1/2 representations of the SU(2) current algebra given by (18) and (20). Now isospin is always conserved in strong interactions, and since the masses of all four I = 1/2 particles in a meson octet are equal, the angular condition applies to (18) and (20) with the same (τ-independent) mass operator for both. Thus, using the notation of (21), $\exp i\vec{k}\cdot\vec{h}^{(j)}$ and $\omega^{(j)}\exp i\vec{k}\cdot\vec{h}^{(j)}$ must satisfy the angular condition for j = 1 and 2. Also, everything must commute by (17). But then (1), with these h's and ω's, satisfies the angular condition with the same SU(3)-independent mass operator (i. e., the operator which originally described the masses of the K and its excited states). In other words, if we cannot make (1) work (i. e., satisfy the angular condition) for an SU(3)-independent mass, then we cannot make a more general solution of the form (11) or (12) work for any realistic mass operator!

It is evident, then, that in looking for a relativistic two-quark* representation of the current algebra, it is not an over-simplification to assume the simple form (1) for the currents or to assume the mass operator SU(3)-independent. Although more complicated currents and masses may approximate nature more closely, it is sufficient to use the simple ones to find out whether we can get any representation at all.

In the next two chapters we will be mainly concerned with finding $\vec{h}$ and ω such that $\exp(i\vec{k}\cdot\vec{h})$ and $\omega\exp(i\vec{k}\cdot\vec{h})$ satisfy the angular condition. This is obviously necessary (from above results) if we expect to represent the SU(3)-algebra, and if it does hold then by multiplying by $\tau_a/2$ we have a solution to the factorized problem and can describe, e.g., the K-meson and its excited states.

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*Our results can also be extended to systems with, e.g., one octet and several singlets in each level, the form of the current being analogous to (10) for the isospin current.

APPENDIX

We prove here two theorems used earlier in this chapter.

Theorem 1: Suppose we have a set of operators $A_n^{(+)} = A_n^{(-)\dagger}$ such that the operators $A_m^{(-)} A_n^{(+)}$ commute with each other for all m and n, and

$$A_m^{(\mp)} A_n^{(\pm)} = A_n^{(\mp)} A_m^{(\pm)} . \qquad \text{(A1)}$$

Then there exists a set of commuting hermitian operators H_n such that

$$A_m^{(-)} A_n^{(+)} = H_m H_n . \qquad \text{(A2)}$$

Proof: Let $H_{mn} = A_m^{(-)} A_n^{(+)}$. Then $H_{mn} = H_{mn}^{\dagger} = H_{nm}$, and $H_{k\ell} H_{mn} = H_{kn} H_{m\ell}$, because of (A1). Now since the H_{mn} all commute, they can be simultaneously diagonalized, so assume that this has been done and let $|\alpha\rangle$ be any eigenstate of H_{mn} with eigenvalue $H_{mn}^{(\alpha)}$. The properties of H_{mn} are reflected in the eigenvalues: $H_{mn}^{(\alpha)} = H_{mn}^{(\alpha)*} = H_{nm}^{(\alpha)}$ and $H_{k\ell}^{(\alpha)} H_{mn}^{(\alpha)} = H_{kn}^{(\alpha)} H_{m\ell}^{(\alpha)}$, or

$$\begin{vmatrix} H_{k\ell}^{(\alpha)} & H_{m\ell}^{(\alpha)} \\ H_{kn}^{(\alpha)} & H_{mn}^{(\alpha)} \end{vmatrix} = 0 .$$

Therefore $H_{mn}^{(\alpha)}$ can be factored: $H_{mn}^{(\alpha)} = H_m^{(\alpha)} H_n^{(\alpha)}$, and we define H_n to be that operator with only the diagonal elements $H_n^{(\alpha)}$. The H_n clearly commute and satisfy (A2).

Note, by the way, that if there are other operators C_p commuting with each other as well as with $A_m^{(-)} A_n^{(+)}$, then we can diagonalize them along with H_{mn}, so that the operators H_n will commute with C_p also.

This theorem was applied in the SU(2) case with $\{A_n^{(+)}\} = \{h_x^{(+)}, h_y^{(+)}, \omega^{(+)}\}$ and $\{C_p\} = \{h^{(I)}, \omega^{(I)}\}$, to define $\{H_n\} = \{h_x^{(J)}, h_y^{(J)}, \omega^{(J)}\}$. We then appealed to the following theorem:

Theorem 2: Suppose (A2) holds for a set of commuting H_n. Then there exists an operator g such that

$$A_n^{(+)} = g\, H_n \qquad \text{(and therefore } A_n^{(-)} = H_n\, g^\dagger\text{)} ,$$

$$g^\dagger g = 1, \qquad \text{except possibly on states where all } H_n = 0. \tag{A3}$$

Proof: Diagonalize all H_n so that $\langle \alpha | H_n | \beta \rangle = H_n^{(\alpha)} \delta_{\alpha\beta}$. We want to define g by $\langle \alpha | g | \beta \rangle = \langle \alpha | A_n^{(+)} | \beta \rangle / H_n^{(\beta)}$ but we have to show that the right side is independent of n, and also worry about $H_n^{(\beta)}$ being zero. If $H_m^{(\beta)}$ and $H_n^{(\beta)}$ are both non-zero, then putting (A2) between $\langle \beta |$ and $| \beta \rangle$, we find

$$\sum_\alpha \left(\frac{\langle \alpha | A_m^{(+)} | \beta \rangle}{H_m^{(\beta)}} \right)^* \left(\frac{\langle \alpha | A_n^{(+)} | \beta \rangle}{H_n^{(\beta)}} \right) = 1 .$$

The same holds, of course, if m is replaced by n or vice versa. Then

$$\sum_\alpha \left| \frac{\langle \alpha | A_m^{(+)} | \beta \rangle}{H_m^{(\beta)}} - \frac{\langle \alpha | A_n^{(+)} | \beta \rangle}{H_n^{(\beta)}} \right|^2 = 1 + 1 - 2 \operatorname{Re} 1 = 0 ,$$

so that

$$\frac{\langle \alpha | A_m^{(+)} | \beta \rangle}{H_m^{(\beta)}} = \frac{\langle \alpha | A_n^{(+)} | \beta \rangle}{H_n^{(\beta)}} \quad \text{whenever } H_m^{(\beta)}, H_n^{(\beta)} \neq 0 .$$

Define g by

$$\langle \alpha | g | \beta \rangle = \begin{cases} \dfrac{\langle \alpha | A_n^{(+)} | \beta \rangle}{H_n^{(\beta)}} & \text{if } H_n^{(\beta)} \neq 0 \quad \text{for some } n , \\ 0 & \text{if } H_n^{(\beta)} = 0 \quad \text{for all } n . \end{cases}$$

Now if $H_n^{(\beta)} = 0$ for some n and β, then $\sum_\alpha |\langle \alpha | A_n^{(+)} | \beta \rangle|^2 = 0$ from (A2), so that $\langle \alpha | A_n^{(+)} | \beta \rangle = 0$. Therefore

$$\langle \alpha | g H_n | \beta \rangle = \langle \alpha | g | \beta \rangle H_n^{(\beta)} = \langle \alpha | A_n^{(+)} | \beta \rangle ,$$

whether or not $H_n^{(\beta)} = 0$, so the first part of (A3) holds. Using (A2) and the definition of g, one also finds

$$\langle \gamma | g^+ g | \beta \rangle = \begin{cases} \delta_{\gamma\beta} \text{ if } H_n^{(\beta)} \neq 0 & \text{for some } n , \\ 0 \text{ if } H_n^{(\beta)} = 0 & \text{for all } n , \end{cases}$$

or $g^+g = 1 - P_o$, where P_o is the projection operator onto the set of states on which all $H_n = 0$. On this set of states, g can be arbitrarily redefined, so it might be (but is not always) possible to make $g^+g = 1$ on all states.

VII. THE DEGENERATE-MASS PROBLEM AND THE INTERNAL-LORENTZ-GROUP FORMULATION

We have tried two models of the mesons, one of which was successful but unrealistic, and the other realistic but unsuccessful. We shall now tackle a model which is both unrealistic and unsuccessful. The mesons will be assumed to all have the same mass, making the angular condition much simpler. We still will not be able to find $\vec{h}^{(1)}$ and $\vec{h}^{(2)}$ which commute with each other, but the model is partly successful in that it enables $\vec{h}^{(1)}$ to be found exactly, thus solving the factorized problem, and leads the way to a formulation which will be useful in more realistic problems.

If M is the constant value of the mass, then $\mathcal{A}_M = 0$ and $\mathcal{B}_M = 2M$, so the angular condition becomes simply that

$$\left\{ \begin{matrix} V \\ A \end{matrix} \right\} \equiv e^{-i\mathcal{B}_{J_y} \tan^{-1} \frac{k}{2M}} \left\{ \begin{matrix} 1 \\ \omega \end{matrix} \right\} e^{ikh_x} \tag{1}$$

must have $|\Delta J_x| \leq 1$. [We will leave off the superscript (1) until we need it.] This condition is easily expanded in powers of k, and doing this to $\bar{V}$ we find $V = V_0 + ikV_1 + (ik)^2 V_2 + \ldots$, where

$$V_0 = 1,$$

$$V_1 = h_x - \frac{1}{M} J_y , \tag{2}$$

$$V_2 = \frac{1}{2} h_x^2 - \frac{1}{2M} \{J_y, h_x\} + \frac{1}{2M^2} J_y^2 = \frac{1}{2} (h_x - \frac{1}{M} J_y)^2 .$$

The condition on V_0 is trivial; on V_1 it implies that h_x must have $|\Delta J_x| \leq 1$ (since J_y does). But we know this already because we found in Chapter II that between equal-mass states (which is always the case in our present model) h_x has the form

$$h_x = E_x - M_y + \frac{1}{M} J_y \tag{3}$$

where $\vec{E}$ and $\vec{M}$ are the electric and magnetic dipole moments. From the condition on V_2, M_y^2 must have $\Delta J_x = 0$, and by rotational invariance this implies

$$\{M_i, M_j\} = 0 \text{ for } i \neq j . \tag{4}$$

We have not yet said anything about what internal variables our system has. If we consider the system as being made of spinless quarks (with no axial current), it is reasonable to suppose that $\vec{M} = 0$, since it is hard to imagine where we would find matrices satisfying (4). Then imposing $[h_x, h_y] = 0$ gives the condition

$$[E_x, E_y] = -\frac{i}{M^2} J_z ,$$

and therefore

$$\vec{E} = \frac{1}{M} \vec{K} , \tag{5}$$

where $\vec{J}$ and $\vec{K}$ form a representation* of the (homogeneous)

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*The representation must be unitary, since $\vec{J}$ and $\vec{K}$ must be hermitian.

Lorentz group:

$$[J_x, J_y] = iJ_z$$

$$[J_x, K_y] = iK_z \tag{6}$$

$$[K_x, K_y] = -iJ_z .$$

The h's are then given by

$$h_x = \frac{1}{M}(K_x + J_y), \quad h_y = \frac{1}{M}(K_y - J_x) . \tag{7}$$

We can now show that the angular condition is satisfied to all orders in k. From (1),

$$V = e^{-iJ_y \tan^{-1} \frac{k}{2M}} e^{i \frac{k}{M}(K_x + J_y)} e^{-iJ_y \tan^{-1} \frac{k}{2M}} .$$

The three exponentials represent operations in the Lorentz group, so we can find their product independently of the representation. The simplest representation to use in evaluating the product is that with $\vec{J} \rightarrow \frac{1}{2}\vec{\sigma}$, $\vec{K} \rightarrow \pm \frac{i}{2}\vec{\sigma}$, and one finds after a simple calculation

$$V = e^{-2iK_x \sinh^{-1} \frac{k}{2M}} , \tag{8}$$

which has $\Delta J_x = 0$. Therefore the angular condition is satisfied.

We can in fact obtain (7) directly from a covariant current by assuming a set of states $|P, \alpha\rangle$, where P is the 4-momentum and α the "internal Lorentz group" index operated on by $\vec{J}$ and $\vec{K}$.

Let the action of $|P, \alpha\rangle$ under a Lorentz transformation Λ be given by

$$D(\Lambda)\,|P, \alpha\rangle = \sum_{\alpha'} |\Lambda P, \alpha'\rangle \mathcal{D}_{\alpha'\alpha}(\Lambda)\,, \tag{9}$$

where $\mathcal{D}$ is the representation generated by $\vec{J}$ and $\vec{K}$. The states are assumed to have the normalization

$$\begin{aligned} \langle P', \alpha'|P, \alpha\rangle &= \delta_{(M)}(P', P)\,\delta_{\alpha'\alpha} \\ &\equiv 2P^{0}(2\pi)^{3}\delta^{3}(\vec{P}' - \vec{P})\,\delta_{\alpha'\alpha}\,. \end{aligned} \tag{10}$$

Let the current contribution from the first quark be given by (apart from a factor of $\frac{1}{2}\lambda_a^{(1)}$)

$$\langle P', \alpha'|\mathcal{F}^{\mu}(0)|P, \alpha\rangle = (P' + P)^{\mu}\,\delta_{\alpha'\alpha}\,. \tag{11}$$

To find $F(\vec{k})$, the infinite-momentum result, we can use (II. 15a), but note that the states must be given in terms of the "rest" states ($P^{\mu} = M\lambda^{\mu}$) by (II. 10) in order for (II. 15a) to be applicable. Thus

$$\begin{aligned} \langle \alpha'|F(\vec{k})|\alpha\rangle &= \lim_{\substack{P_z \to \infty \\ P_\perp' - P_\perp = k}} \frac{1}{2P_a} \langle M\lambda, \alpha'|D^{\dagger}(V_{P'/M \leftarrow \lambda})\mathcal{F}^{\mu}(0)\,\times \\ &\qquad\qquad \times\, D(V_{P/M \leftarrow \lambda})|M\lambda, \alpha\rangle \\ &= \lim_{P_z \to \infty} \frac{P'^{0} + P^{0}}{2P_z} \mathcal{D}_{\alpha'\alpha}(V_{\lambda \leftarrow P'/M}\; V_{P/M \leftarrow \lambda})\,. \end{aligned}$$

The product of the two velocity transformations approaches a finite limit, and we find

$$F(\vec{k}) = e^{i\vec{k}\cdot(\vec{K} - \vec{e}_z \times \vec{J})/M} \tag{12}$$

which is of the form $e^{i\vec{k}\cdot\vec{h}}$ with $\vec{h}$ given by (7).

For the second quark (in a bound pair) we might expect to get $\vec{h}^{(2)}$ by parity: $\vec{J} \to \vec{J}$ and $\vec{K} \to -\vec{K}$. But this gives $h_x^{(2)} = \frac{1}{M}(-K_x + J_y)$ which does not commute with $h_x^{(1)}$ $(= h_x)$. So we have not solved the two-charged-quark problem, but we have a solution to the one-charged-quark problem. The electric dipole moment of a "meson" at rest is $\frac{1}{2}\lambda_a^{(1)}\vec{K}/M$, and the total magnetic moment is zero (i. e., the "anomalous moment" cancels the "natural moment" arising from the spin).

As an example, suppose that the two quarks have the same internal variables ($\vec{x}$ and $\vec{p}$, but no spin) as in the free-quark model, but that the mass is constant instead of $\sqrt{\vec{p}^2 + m^2}$ or $\sqrt{\vec{p}^2 + m^2 + U}$. Using the notation of Chapter IV, we can put this model in covariant form by saying that the quarks have "free" 4-momenta p_1 and p_2 but the momentum of the whole system is $P^\mu = M(p_1^\mu + p_2^\mu)/|p_1 + p_2|$ instead of $p_1^\mu + p_2^\mu$. We then postulate that the current contribution from the first quark is given by

$$\langle p_1', p_2' | \mathcal{J}^\mu(0) | p_1, p_2 \rangle \propto (P' + P)^\mu \, \delta_{(m)}(p_2', p_2) \, , \tag{13}$$

with the factor of proportionality determined by normalization. Using the methods of Chapter IV one finds that h_x is given by (7) with

$$K_x = \frac{1}{2}\{x, \sqrt{\vec{p}^2 + m^2}\} \ . \tag{14}$$

This is quite reasonable because p_2 in (13) corresponds to the index α in (11). When the center of mass is at rest, $p_2 = (-\vec{p}, \sqrt{\vec{p}^2 + m^2})$, and K_x as given by (14) is just the operator which generates velocity transformations of this momentum.

Using the same internal variables, Dashen and Gell-Mann,[12] found a different operator K_x (there called Q_x) which also satisfies the Lorentz-group commutation relations but whose physical significance is not as clear.

The model just discussed gives the same multitude of meson "levels" as in Table 1, Chapter III (with radial excitations included), although they are all degenerate. The representation of the Lorentz group involved is infinitely reducible. We can find simpler models using irreducible (or at least simply reducible) representations of the Lorentz group, so let us look at what representations there are.

A general irreducible representation[15] of the Lorentz group is characterized by the values of the two invariants:

$$\begin{aligned} \vec{J}^2 - \vec{K}^2 &= j_0^2 + j_1^2 - 1 \ , \\ \vec{J} \cdot \vec{K} &= -i j_0 j_1 \ , \end{aligned} \tag{15}$$

where j_0 is a non-negative integer or half-integer, and j_1 any complex number. The representation may be denoted by (j_0, j_1);

in it the angular momentum j takes on the values j_0, $j_0 + 1$, $j_0 + 2, \ldots$, each spin-j representation of the rotation group appearing just once. Now for a unitary representation it turns out that either (a) $j_1 = i\rho$ where ρ is real, or (b) $j_0 = 0$ and $-1 \leq j_1 \leq 1$. In either case the representation is infinite-dimensional unless $j_1 = \pm 1$ which gives the trivial one-dimensional representation. (When $j_0 = 0$, j_1 and $-j_1$ are equivalent.) The reducible representation considered previously contains all values of ρ $(0 \leq \rho < \infty)$ with $j_0 = 0$.

An irreducible representation would describe an infinite set of particles, one for each spin from j_0 on up in steps of unity. Suppose $j_0 = 0$, so that the "lowest" state is a scalar particle; call this state $|0\rangle$. As an example of the use of the formalism we will find the form factor for this scalar particle* assuming the current is given by (11). The form factor f(t) is defined by

$$\langle P', 0|\mathcal{F}^{\mu}(0)|P, 0\rangle = (P' + P)^{\mu} f(t) , \tag{16}$$

where $t = -(P' - P)^2$. Choosing a system such that

$$P'^{\mu} = (\vec{0}, M), \; P^{\mu} = (0, 0, M\sinh\eta, M\cosh\eta)$$

we have

$$t = -2M^2(\cosh\eta - 1) = -4M^2 \sinh^2\frac{\eta}{2} .$$

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*Actually it is an SU(3) octet plus singlet of scalar particles; the form factor should be multiplied by $\frac{1}{2}\lambda_a^{(1)}$.

Then from (16) and (11), and since $|P, 0\rangle$ is obtained from $|M\lambda, 0\rangle$ by a velocity transformation, one finds

$$f(t) = \langle 0| e^{-i\eta K_z} |0\rangle .$$

This matrix element can be evaluated using a scheme in which the Lorentz group is represented on the set of functions on the unit sphere. The result is, for $j_1 = i\rho$,

$$f(t) = \frac{\sin \rho\eta}{\rho \sinh \eta} = \frac{\sin(2\rho \sinh^{-1} \sqrt{-t/4M^2})}{2\rho \sqrt{-t/4M^2} \sqrt{1 - t/4M^2}} . \tag{17}$$

For $j_1 = \frac{1}{2}$, i.e., $\rho = -\frac{i}{2}$ (one of the "Majorana representations") the form factor becomes particularly simple

$$f(t) = \frac{1}{\cosh(\eta/2)} = \frac{1}{\sqrt{1 - t/4M^2}} . \tag{18}$$

The form factor, like the mass spectrum, is unrealistic; it is too singular at the threshold value $t = 4M^2$, for example.

The internal-Lorentz-group formalism has also been used by Fubini[16] and by Bebie and Leutwyler.[17] In the latter reference the representation with $j_0 = \frac{1}{2}$ and $j_1 = 0$ (the other Majorana representation) is used and the form factor for the spin-$\frac{1}{2}$ state calculated.

We have not yet considered the axial current. Letting $A = A_0 + ikA_1 + \ldots$ we have from (1),

$$A_0 = \omega$$

$$A_1 = \omega h_x - \frac{1}{2M} \{ J_y, \omega \} \ . \tag{19}$$

The condition on A_0 says that ω must have $|\Delta J_x| \leq 1$, so let $\omega = a + b_z$ where a is a pseudoscalar and $\vec{b}$ a pseudovector. The further requirement $\omega^2 = 1$ implies that $a^2 + b_z^2 = 1$ and $\{a, b_z\} = 0$, and it also follows that $\{b_i, b_j\} = 0$ when $i \neq j$. It is apparent that $\vec{b}$ is something like a Pauli spin matrix. By analogy with previous models, let us suppose that $a = 0$ and $\vec{b} = \vec{\sigma}$, i.e., $\omega = \sigma_z$, where $\vec{\sigma}$ is some spin associated with the system and may be identified as the spin of the charge-carrying quark. The magnetic moment M can no longer be zero but is now determined by the following arguments: The condition on A_1 implies that $\sigma_z M_y$ must have $\Delta J_x = 0$, and writing $M_i = c_i + d_{ij} \sigma_j$ (c_i and d_{ij} independent of $\vec{\sigma}$) one finds $c_i = 0$ and $d_{ij} \propto \delta_{ij}$. Furthermore, $[\omega, h_x] = 0$ implies $[\sigma_x, E_x] = 0$ and $[M_y, \sigma_z] = i\sigma_x/M$, so that $\vec{M} = \vec{\sigma}/2M$. Letting

$$\vec{L} = \vec{J} - \frac{\vec{\sigma}}{2} \tag{20}$$

and imposing $[h_x, h_y] = 0$, we arrive at the result

$$h_x = \frac{1}{M}(N_x + L_y), \quad h_y = \frac{1}{M}(N_y - L_x), \quad \omega = \sigma_z, \tag{21}$$

where $\vec{L}$ and $\vec{N}$ generate a representation of the Lorentz group. The electric dipole moment is $\vec{E} = \vec{N}/M$ and the magnetic moment is $\vec{M} = \vec{\sigma}/2M$. Note that $\vec{L}$ is the angular momentum of everything except for the spin of the current-carrying quark. The angular conditions on V and A are satisfied to all orders in k, as a direct evaluation of (1) shows.

This result can also be obtained from the following covariant currents, analogous to (11):

$$\langle P', \sigma', \alpha' | \left\{ \begin{matrix} \mathcal{F}^{\mu}(0) \\ \mathcal{F}^{5\mu}(0) \end{matrix} \right\} | P, \sigma, \alpha \rangle = \bar{u}(P', \sigma') \left\{ \begin{matrix} i\gamma^{\mu} \\ \gamma_5 \gamma^{\mu} \end{matrix} \right\} u(P, \sigma) \, \delta_{\alpha'\alpha} . \quad (22)$$

The set of levels in this model is obtained by adding the spin $\frac{1}{2}$ of the first quark to whatever angular momenta ℓ appear in the representation of the Lorentz group. If we take a representation with $j_0 = \frac{1}{2}$, the values of ℓ are $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ with alternating parities, so the levels are as follows (choosing the initial parity appropriately):

$\ell =$				
1/2	0^-		1^-	
3/2		1^+		2^+
5/2	2^-		3^-	
7/2		3^+		4^+
	⋮		⋮	
⋮		⋮		⋮

Thus we have four trajectories on which to find particles. This system is much simpler than the one given in Chapter III (Table 1) and still accounts for the most well-determined multiplets. Actually, of course, it does not account for anything very well as it stands, since the masses are degenerate, but it is a starting point for a search for covariant Lorentz-group representations with non-degenerate mass spectra, which we consider in the next chapter.

VIII. LORENTZ-GROUP FORMALISMS WITH A NONDEGENERATE MASS SPECTRUM

We would like to extend the results of the last chapter to describe more realistic models in which the mass spectrum is nondegenerate and runs from some minimum value to infinity.

To begin we will show how the noncovariant approach can be used, based on an expansion of the angular condition. In Chapter V we expanded about the case where all the masses are equal and infinite; this time we will expand about the case where all masses are equal but finite, i. e., the case solved in the last chapter. It is convenient to expand in powers of k first. To do this we rewrite (III. 21) by factoring out a rotation by $\pi/2$ about the y-axis, giving the condition that

$$[\exp -i(\mathcal{A}_{J_y} \tan^{-1} \frac{k}{\mathcal{A}_M} + \mathcal{B}_{J_y} \tan^{-1} \frac{k}{\mathcal{B}_M})] \{ {1 \atop \omega} \} e^{ikh_x} \tag{1}$$

must have $|\Delta J_z| \leq 1$. Now the terms of order k^n in (1) have $|\Delta J_z| \leq n$, i. e., up to n x's or y's. Let us consider only those terms of order k^n with $\Delta J_z = n$. Then (1) becomes the so-called "stretched" angular condition:

$$[\exp - \frac{k}{2} (\frac{1}{\mathcal{A}_M} \mathcal{A}_{J_+} + \frac{1}{\mathcal{B}_M} \mathcal{B}_{J_+})] \{ {1 \atop \omega} \} e^{ikh_+/2} \tag{2}$$

must have $|\Delta J_z| \leq 1$, where $J_+ = J_x + iJ_y$ and $h_+ = h_x + ih_y$. The operator in square brackets is also equal to $\exp(-k\mathcal{A}_{M J_+}/\mathcal{A}_M{}^2)$. The condition on (2) implies that the coefficient of k^n must vanish for $n \geq 2$. Writing these out for the vector condition one finds[18] that this infinite sequence of conditions (one for each $n \geq 2$) is equivalent to just the two conditions

$$[M^2, h_+^2] + 4i[MJ_+, h_+] = 0,$$
$$\mathcal{A}_{h_+}^3 M^2 = 0 . \tag{3}$$

There is also a sequence of conditions on the axial current which, however, does not simplify so nicely.

We now expand in powers of a parameter ϵ about the case of degenerate mass:

$$M = M_0 + \epsilon M_1 + \epsilon^2 M_2 + \ldots ,$$
$$h_+ = h_{0+} + \epsilon h_{1+} + \epsilon^2 h_{2+} + \ldots , \tag{4}$$

where M_0 is just a number (not an operator) and therefore h_{0+} has solutions given by the results of the last chapter, namely,

$$h_{0+} = \frac{1}{M_0} (N_+ - iL_+) , \tag{5}$$

where $\vec{L}$ and $\vec{N}$ generate a representation of the Lorentz group. Using (3) we get conditions which help to determine h_+ to each order in ϵ, or we can write

$$h_+ = e^{iS} h_{0+} e^{-iS}$$

and find S to each order in ϵ . This is done for the first few orders in Reference 18, considering the axial operator ω also. One of the most striking results is that the mass operator is quite restricted. For example, the second condition in (3) in first order implies that

$$[N_+, [N_+, [N_+, M_1]]] = 0 . \tag{6}$$

As an example, consider the representations (0, 1/2) and (1/2, 0), the Majorana Representations.[19] These are the only irreducible representations of the Lorentz group on which there exists 4-vector operator. Let Γ^μ be this operator; suitably normalized $\Gamma^0 = \ell + 1/2$ on states of spin ℓ. [Thus $L^2 = \ell(\ell + 1) = (\Gamma^0)^2 - 1/4$.] We can describe an infinite set of mesons using (0, 1/2), with $\vec{J} = \vec{L}$ and no axial current (which is like having a pair of scalar quarks with no radial degree of freedom), or we can use (1/2, 0) with $\vec{J} = \vec{L} + \vec{\sigma}/2$. Let us consider the first case. The mass M must be a scalar under rotations, and the only such scalars are functions of ℓ, i.e., functions of Γ^0. It is easily verified that (6) is satisfied with $M_1 = \Gamma^0$ or $(\Gamma^0)^2$ but no higher powers of Γ^0. Thus

$$\begin{aligned} M_1 &= a + b\Gamma^0 + c(\Gamma^0)^2 \\ &= a + b(\ell + 1/2) + c(\ell + 1/2)^2 \text{ on states of spin } \ell . \end{aligned} \tag{7}$$

In other words, the first-order mass splitting must be at most quadratic in ℓ . (The first-order splitting in M^2, by the way, has the same form since $M^2 = M_0^2 + 2\epsilon M_0 M_1 + \ldots$).

Let us turn to covariant formalisms analogous to those discussed in the last chapter. Since the mass is to be variable and obtained by finding the eigenvalues of the mass operator, we will initially use not the 4-momentum P to label the states, but rather the 4-velocity u as well as the internal index α. The mass operator acting on $|u, \alpha\rangle$ is $M(u)|u, \alpha\rangle$, where $M(u)$ acts only on α and is Lorentz-invariant: $\mathcal{D}(\Lambda)M(u)\mathcal{D}(\Lambda^{-1}) = M(\Lambda u)$. When $u = \lambda \equiv (\vec{0}, 1)$ we can (in principle) find the eigenvalues of $M \equiv M(\lambda)$:

$$M|n\rangle = M_n|n\rangle \,. \tag{8}$$

Then the state

$$|P, n\rangle = \frac{1}{M_n}\mathcal{D}(V_{P/M_n \leftarrow \lambda})\sum_{\alpha} |\lambda, \alpha\rangle \langle\alpha|n\rangle \tag{9}$$

has momentum P and is suitably normalized.

The problem is to find a covariant expression for a conserved current, $\langle u', \alpha'|\mathcal{J}^{\mu}(0)|u, \alpha\rangle$, such that the $F(\vec{k})$ obtained from it is of the form $\exp(i\vec{k} \cdot \vec{h})$, or equivalently, $F(\vec{0}) = 1$ and

$$F(\vec{k})\,F(\vec{k}') = F(\vec{k} + \vec{k}') \,. \tag{10}$$

If we could do this much we would have a solution to the factorized problem of one current-carrying quark, and might then investigate as to whether the model could be extended to a two-charged quark model.

If M(u) is independent of u, then the mass is constant over each irreducible component of the representation and we get infinitely many particles of the same mass. Suppose instead that

$$M(u) = -u \cdot V, \tag{11}$$

where V^μ is a 4-vector operator. The mass spectrum is then the eigenvalue spectrum of V^o. Consider the following current

$$\begin{aligned} &\langle u', \alpha' | \mathcal{F}^\mu(0) | u, \alpha\rangle \\ &= -2\langle \alpha' | M(u')\, (V^\mu + V \cdot u u^\mu + V \cdot u' u'^\mu) M(u) | \alpha\rangle, \end{aligned} \tag{12}$$

which is motivated by the facts that (a) it is conserved, and (b) the two-free-spinless quark current, (IV. 37), turns out to be of this form, where $\alpha = p_2$ and $V^\mu = 2p_2{}^\mu$ (using the notation of Chapter IV and writing everything in terms of P and p_2).

Finding $\mathcal{F}^\mu(0)$ between momentum eigenstates and taking the infinite-momentum limit, one arrives at

$$\begin{aligned} \langle n' | F(\vec{k}) | n\rangle &= \langle n' | e^{-iK_z \log(M_{n'}/M_n)} \\ &\cdot e^{i\vec{k} \cdot (\vec{K} - \vec{e}_z \times \vec{J})/M_n} (1 - V_z/M_n) | n\rangle . \end{aligned} \tag{13}$$

It is not clear whether this satisfies the multiplicative property, (10), or not. It is not even obvious that $F(\vec{0}) = 1$ (but that can be verified using the fact that $F(\vec{0})$ connects only states of equal mass due to current conservation). If $[V^o, V_z] = 0$ it can be shown that (10) is satisfied, but then we are back to the free-quark solution (or a direct product of them). A more interesting case would be $V^{\mu} = 2M_o \Gamma^{\mu}$, where the representation is (0, 1/2) with generators J and K and 4-vector Γ^{μ}; then the mass spectrum would be $M_j = (2j + 1)M_o$. However, I have not been able to show that the algebra is satisfied with such a representation.

This model does not seem to be readily comparable to those obtained by expanding in powers of the mass splitting, since there is no arbitrary additive constant in the mass.

There is another model which comes even closer to satisfying the algebra.[18] We consider both vector and axial vector currents in a system whose internal variables include a Lorentz-group representation along with an extra spin 1/2 (thus $\vec{J} = \vec{L} + \vec{\sigma}/2$ and the Lorentz-group representation is generated by $\vec{L}$ and $\vec{N}$). To write everything covariantly the spin 1/2 is treated by the Dirac formalism. We choose for the mass operator (11) with $V^{\mu} = i\gamma^{\mu}$, i.e.,

$$M(u) = -i \not{u} \mathcal{M} , \qquad (14)$$

where $\mathcal{M}$ is invariant under Lorentz transformations of all internal variables (i.e., under $\vec{J}$ and $\vec{K} = \vec{N} + i\vec{\alpha}/2$) and $\bar{\mathcal{M}} = \mathcal{M}$. The mass spectrum can be obtained by finding the positive eigenvalues of $M = \beta\mathcal{M}$ (which is hermitian):

$$\beta\mathcal{M}\psi(n) = M_n\psi(n) , \qquad (15)$$

where $\psi(n)$ is a Dirac spinor as well as a vector in "α-space", and $\psi^{\dagger}(n')\psi(n) = \delta_{n'n}$. Then

$$\psi(P,\ n) \;=\; \sqrt{2M_n}\,\mathcal{D}\,(V_{P/M_n \leftarrow \lambda})\,\psi(n) \tag{16}$$

is the appropriate "spinor" for momentum $P = M_n u$, where $\mathcal{D}$ is the representation generated by J and K. Equation (11) is motivated by the fact that $\psi(P,\ n)$ satisfies

$$(i\not{P} + \mathcal{M})\psi \;=\; 0 \tag{17}$$

which is like the usual Dirac equation except that $\mathcal{M}$ is an operator. For the currents we propose $i\gamma^\mu$ and $\gamma_5\gamma^\mu$ between spinors; that is,

$$\langle P',\ n'|\left\{\begin{matrix}\mathcal{F}^{\mu}(0)\\ \mathcal{F}^{5\mu}(0)\end{matrix}\right\}|P,\ n\rangle = \bar{\psi}(P',\ n')\left\{\begin{matrix}i\gamma^{\mu}\\ \gamma_5\gamma^{\mu}\end{matrix}\right\}\psi(P,\ n). \tag{18}$$

If $\mathcal{M}$ is independent of Dirac matrices we have the degenerate mass case, (VII. 22). For any $\mathcal{M}$ (which is assumed to be independent of u), $\mathcal{F}^{\mu}$ (but not $\mathcal{F}^{5\mu}$) is a conserved current.

From (18) we can find $F(\vec{k})$ and $F^5(\vec{k})$ by dividing by $2P_z$ and taking the infinite-momentum limit. The current algebra will be satisfied if we can show that (10) is satisfied (with analogous relations involving the axial current). Now

$$\langle n'|F(\vec{k})|n\rangle \;=\; \lim_{P_z\to\infty} \frac{1}{2P_z}\,\psi^{\dagger}(P',n')\,\psi(P,n) \tag{19}$$

and to show (10) is sufficient to show that

$$\frac{1}{2P''_z}\sum_{n''} \psi(P'',\ n'')\,\psi^{\dagger}(P'',\ n'')\Big|_{\vec{P}''=\text{const.}} \;\to\; 1 \tag{20}$$

between states $\psi^\dagger(P', n')$ and $\psi(P, n)$, as $P_z = P'_z = P''_z \to \infty$.

By manipulating equation (17) one notices that $\psi(P'', n'')$ is an eigenvector of

$$H(\vec{P}'') \equiv \vec{\alpha} \cdot \vec{P}'' + \beta \mathcal{M} \qquad (21)$$

with eigenvalue $P''^{o} = \sqrt{\vec{P}''^{2} + M_{n''}^{2}}$. Thus with $\vec{P}''$ held constant the spinors $\psi(P'', n'')$ form an orthogonal set as n'' runs over all its values, and with their normalization (20) would be satisfied as the completeness relation if these spinors formed a complete set. However, the $\psi(P'', n'')$ do not form a complete set because $H(\vec{P}'')$ can have negative energy eigenvalues. But (20) does hold if we sum over <u>all</u> eigenspinors of $H(\vec{P}'')$. Now as $P_z \to \infty$, $H(\vec{P}'') \sim \alpha_z P_z$ and $P''^{o} \sim |P_z|$. Thus in this limit the eigenspinors of $H(\vec{P}'')$ are eigenspinors of α_z with eigenvalue +1 for positive energies and -1 for negative energies, so in this limit all positive-energy eigenspinors are orthogonal to all negative-energy eigenspinors, even for different values of $\vec{P}_\perp$. Therefore (20) does seem to be satisfied between the positive energy states $\psi^\dagger(P', n')$ and $\psi(P, n)$. This is the method used in Reference 18 to show that the current algebra is satisfied; we may summarize it by saying that the negative-energy eigenvectors of $H(\vec{P})$ do not couple to the positive-energy eigenvectors of infinite momentum.

Before we point out what is wrong with all this, let us find, as an example,[18] the mass spectrum using the Majorana representation (1/2, 0) and

$$= m_o - ic_1 \gamma_\mu \Gamma^\mu + \frac{1}{2} c_2 \sigma_{\mu\nu} L^{\mu\nu}, \qquad (22)$$

where $L^{\mu\nu}$ is the angular momentum tensor containing $\vec{L}$ and $\vec{N}$. (Models of this kind with $c_1 = 0$ have been studied for other reasons.[20]) The resulting spectrum is

$$M = (c_1 + bc_2)(j + \frac{1}{2}) \pm \sqrt{(c_1 + bc_2)^2 j(j+1) + [M_o - (c_1 + bc_2)/2]^2} \tag{23}$$

where j is the total (internal) angular momentum, b takes on the values +1 and -1, and $M_o = c_1 + \frac{3}{2} bc_2 - bm_o$. In general some of these masses will be negative, but they can be forgotten because they lead to negative-energy states which, as we saw, can be ignored.

The mass of the $j = 0$ state is equal to M_o if the latter is positive. The forms of the mass spectra depend on the relative values of m_o, c_1, and c_2. After considering all cases (some of which give masses which are positive but approach 0 as $j \to \infty$), we find that the most physically reasonable case is with $c_1 > |c_2|$. Then we get two "trajectories", each approaching a linear trajectory as $j \to \infty$ and approaching $j = -\infty$ as $M \to 0$, one of which has particles starting with $j = 0$ and the other starting with $j = 1$. Such sequences of particles are what we would expect from adding spin $\frac{1}{2}$ to the spins in (1/2, 0). However, the behavior of these Regge trajectories continued to $m = 0$ is unphysical, and so is the form factor $F(t) = (1 - t/4M_o^2)^{-3/2}$ which one finds for the scalar particle.

The case considered here appears to agree[18] with the first few terms in a perturbative expansion using the angular condition. Using other representations of the Lorentz group we might expect to recover other solutions which we laboriously sought in noncovariant form.

It looks like we have finally found a semi-reasonable representation of the current algebra, at least for the factorized single-charged-quark case. But again there is something wrong, a flaw in the above reasoning. Our currents actually do not satisfy the algebra except in trivial cases. The reason is that a complete set of eigenvectors of $H(P'')$ defined by (21) will in general include states with $|P''^0| < |P''|$, i.e., states with spacelike 4-momentum. Those with negative energy may be ignored at infinite momentum, but there is no such way to eliminate the possibility of the spacelike solutions (of the eigenvalue equation) coupling with the ordinary timelike solutions $\psi(P, n)$ even at infinite momentum. Thus equation (20) will not in general be satisfied.

A similar model has been proposed by Leutwyler,[21] who found an exact solution using the noncovariant formalism, and that M^2 (which turned out to be related to the Hamiltonian for the nonrelativistic hydrogen atom) had negative as well as positive eigenvalues. An expansion in powers of the mass splitting, however, does not reveal these negative-M^2 states.

To show that there will almost always be spacelike solutions in our model,[22] we note that for a given spatial momentum the energy is given by (21), so that the mass operator is given by

$$\begin{aligned} M^2 &= (\vec{\alpha} \cdot \vec{P} + \beta m)^2 - \vec{P}^2 \\ &= \{\vec{\alpha} \cdot \vec{P}, \beta m\} + (\beta m)^2 . \end{aligned} \tag{24}$$

For no spacelike solutions, M^2 must be positive definite so that $\overline{\psi^\dagger M^2 \psi > 0}$ for all ψ. Since this inequality must hold for all $\vec{P}$, we must have $\psi^\dagger \{\vec{\alpha}, \beta\mathcal{M}\}\psi = 0$ for all ψ, i.e., $\{\vec{\alpha}, \beta\mathcal{M}\} = 0$, which implies $[\vec{\alpha}, \mathcal{M}] = 0$. This implies that $\mathcal{M}$ is invariant under $\sigma_{\mu\nu}$ (the Lorentz transformations of the Dirac index), and therefore under $L_{\mu\nu}$ as well, so $\mathcal{M}$ is constant within an irreducible representation, and we have lost our infinite mass spectrum.

The two-free-quark model, which we worked out in gory detail in Chapter IV, can also be expressed as a special case of our present formulation by writing $\mathcal{M} = -i\not{p}_2 + m$. There are, of course, spacelike solutions, and in fact M^2 runs from $4m^2$ to ∞ and from 0 to $-\infty$. However, the current algebra is still satisfied; the spacelike solutions decouple from the timelike ones at infinite momentum. But such a decoupling does not take place in general. It has been shown, for example,[23] that the spacelike and timelike solutions remain coupled at infinite momentum whenever $\mathcal{M}$ is a linear combination of 1 and $\sigma_{\mu\nu} L^{\mu\nu}$. At present the possibility of nontrivially satisfying the current algebra using the Dirac + Lorentz-group formalism seems doubtful.

One way out of this difficulty might be to include the states of spacelike momentum and find a physical explanation for them, such as virtual resonant states in the t-channel (or something). Or we might give up and just ignore the spacelike solutions, admitting that the current algebra is not quite satisfied and that the discrepancy is due to the discrete-resonance approximation of the continuum being poor at high energies.[21]

One indication that we may not be on the right track in trying to use these particular models is that the mass spectrum of $M = V^0$ is generally linear in the spin for high spins (a general feature of the time-component of a 4-vector operator if we have only one or a few irreducible representations of the Lorentz group involved[15]), whereas the observed mesons and meson resonances as well as data on the Regge trajectories seem to favor the square of the mass being linear in the spin. Perhaps, then, we should look for a covariant formalism with $M = \sqrt{-u \cdot V}$. The current will no longer be of the same form as before; we have to modify it at least to make it conserved using the new mass operator. Models of this type remain to be investigated.

IX. CONCLUSIONS

We have tried in various ways to find relativistically compatible representations of the current algebra at infinite momentum using the two-quark model of the mesons. The only success has been in the free-quark model, which is physically unrealistic. The model with two quarks bound by a potential would (with a suitable potential) give a realistic mass spectrum, but it cannot (with the type of potential tried) be made to satisfy the current algebra and relativistic requirements at the same time. Even if we consider the factorized case in which only one quark carries the current, it seems difficult to find a model with a reasonable mass spectrum.

As we saw, it was not an oversimplification to assume the simplest type of quark model in which the mass was SU(3)-independent and the current was a sum of independent contributions from each quark. That is, making it more complex would not make the problem any more solvable.

If we cannot find any two-charged-quark model that satisfies the algebra, we might ask whether we have tried to put too much into the model, more than can be represented by such a simple system of discrete single-particle states. We expected our requirements in the model to almost uniquely determine the forms of the mass operator and currents; perhaps they are overdetermined and have no solution. One way to relax our requirements was mentioned at the end of Chapter V: we might require that the current algebra be satisfied only for momentum transfers of the form $k = (k, ik, 0)$. Preliminary calculations by M. Gell-Mann indicate that the problems formerly encountered in the bound-quark model are no longer present and that $h^{(1)}$ and $h^{(2)}$ can be found uniquely to each order in $1/m$. We would be able to predict form factors with $t = -k^2 = 0$, which are useful in describing real photon processes, certain lepton-hadron scattering processes, and approximate amplitudes involving pions, since $t = 0$ is near the pion pole where PCAC best applies to the axial current divergence. This relaxation of the current algebra, then, is something worth investigating.

Proceeding in another direction, if we insist that the full current algebra at infinite momentum must be obeyed, then we may have to consider models which are not so simple but more like the real world. In order to have bound quark states, for example, we may have to have a complete field theory for the particles that bind the quarks. We also may have to have a field theory for the quarks themselves* (like the Lagrangian quark model[6] which provided a motivation for the algebra in the first place), so that mesons might be made partly of a quark and an antiquark, partly of two quarks and two antiquarks, and so on. We would also have an "SU(3) explosion" with arbitrarily large SU(3) multiplets appearing[18] (instead of just 8 and 1 as we had previously). Whether or not we have a complete field theory, an infinite sequence of increasing SU(3) multiplets could be obtained by using a noncompact group which contains SU(3) (just as a nontrivial unitary representation of the Lorentz group must contain an infinite sequence of spins).

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*On the other hand, we do not want to have free, single quarks in the theory, since we have not seen any in nature. A harmonic-oscillator potential between two quarks (which would keep them from ever coming apart) is rather difficult to describe in terms of exchanged particles.

We then have to worry about how to assign the observed particles and resonances to the resulting patterns.

Even though we have not found a realistic representation of the current algebra, we have shown what can and what cannot be done with certain models, and have obtained some general results on the nature of two-quark representations which should be useful in formulating future models. From a mathematical point of view this work has been interesting on account of the different approaches and techniques involved and how they are related to each other; from a physical point of view it has been a challenge to take properties of the real world which we believe to hold to at least some extent (relativity, the current algebra, the quark model, reasonable mass spectra, etc.) and try to incorporate them into a simple, idealized system. As with other models of the universe[24] we may be overidealizing the real world, but if we can find a model of the form outlined it should serve as an approximation to the true (probably complicated) theory of strong interactions which we hope will be found someday.

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