On the Cesari Fixed Point Method
in a Banach Space

Thesis by

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Abstract

In a paper published in 1961, L. Cesari [1] introduces a method which extends certain earlier existence theorems of Cesari and Hale ([2] to [6]) for perturbation problems to strictly nonlinear problems. Various authors ([1], [7] to [15]) have now applied this method to nonlinear ordinary and partial differential equations. The basic idea of the method is to use the contraction principle to reduce an infinite-dimensional fixed point problem to a finite-dimensional problem which may be attacked using the methods of fixed point indexes.

The following is my formulation of the Cesari fixed point method:

Let B be a Banach space and let S be a finite-dimensional linear subspace of B. Let P be a projection of B onto S, and suppose $\Gamma \subseteq B$ such that $P\Gamma$ is compact and such that for every x in $P\Gamma$, $P^{-1}x \cap \Gamma$ is closed. Let W be a continuous mapping from Γ into B. The Cesari method gives sufficient conditions for the existence of a fixed point of W in Γ.

Let I denote the identity mapping in B. Clearly $y = Wy$ for some y in Γ if and only if both of the

following conditions hold:

(i) $Py = PWy$.

(ii) $y = (P + (I - P)W)y$.

Definition. The Cesari fixed point method applies to (Γ, W, P) if and only if the following three conditions are satisfied:

(1) For each x in $P\Gamma$, $P + (I - P)W$ is a contraction from $P^{-1}x \cap \Gamma$ into itself. Let $y(x)$ be that element (uniqueness follows from the contraction principle) of $P^{-1}x \cap \Gamma$ which satisfies the equation $y(x) = Py(x) + (I - P)Wy(x)$.

(2) The function y just defined is continuous from $P\Gamma$ into B.

(3) There are no fixed points of PWy on the boundary of $P\Gamma$, so that the (finite-dimensional) fixed point index $i(PWy, \text{int } P\Gamma)$ is defined.

Definition. If the Cesari fixed point method applies to (Γ, W, P) then define $i(\Gamma, W, P)$ to be the index $i(PWy, \text{int } P\Gamma)$.

The three theorems of this thesis can now be easily stated.

Theorem 1 (Cesari). If $i(\Gamma, W, P)$ is defined and

$i(\Gamma, W, P) \neq 0$, then there is a fixed point of W in Γ.

Theorem 2. Let the Cesari fixed point method apply to both (Γ, W, P_1) and (Γ, W, P_2). Assume that $P_2P_1 = P_1P_2 = P_1$ and assume that either of the following two conditions holds:

(1) For every b in B and every z in the range of P_2, we have that $\|b - P_2b\| \leqslant \|b - z\|$.

(2) $P_2\Gamma$ is convex.

Then $i(\Gamma, W, P_1) = i(\Gamma, W, P_2)$.

Theorem 3. If Ω is a bounded open set and W is a compact operator defined on $\bar{\Omega}$ so that the (infinite-dimensional) Leray-Schauder index $i_{LS}(W, \Omega)$ is defined, and if the Cesari fixed point method applies to $(\bar{\Omega}, W, P)$, then $i(\bar{\Omega}, W, P) = i_{LS}(W, \Omega)$.

Theorems 2 and 3 are proved using mainly a homotopy theorem and a reduction theorem for the finite-dimensional and the Leray-Schauder indexes. These and other properties of indexes will be listed before the theorem in which they are used.

It will be useful to begin with a review of some of the properties of finite-dimensional indexes.

<u>Properties of the finite-dimensional fixed point index</u>. Let E^n be the n-dimensional euclidean space. Then for every bounded open set $\Delta \subseteq E^n$ and for every continuous function $G: \overline{\Delta} \to E^n$ such that $Gx \neq x$ for every x on the boundary of Δ, there is defined an integer $i(G, \Delta)$ which can be positive, negative, or zero, called the index of the mapping G. This index has the following properties [9]:

A. If $i(G, \Delta)$ is defined and if $i(G, \Delta) \neq 0$, then there is an x in Δ such that $Gx = x$.

B. (Homotopy theorem) If $G_t(x)$ is a continuous function on $[0,1] \times \overline{\Delta}$ and if $i(G_t, \Delta)$ is defined for every t in $[0,1]$, then $i(G_0, \Delta) = i(G_1, \Delta)$.

C. If $i(G, \Delta_1)$ and $i(G, \Delta_2)$ are both defined, where G is a continuous function defined on $\Delta_1 \cup \Delta_2$, and if $\Delta_1 \cap \Delta_2 = \emptyset$, then $i(G, \Delta_1 \cup \Delta_2) = i(G, \Delta_1) + i(G, \Delta_2)$.

D. (Reduction theorem) Let E^m be a finite-dimensional linear subspace of E^n. Let Δ be a bounded open set in E^n. Let $G: \overline{\Delta} \to E^m$ be continuous and suppose that $i(G, \Delta)$ is defined.

Then $i(G, \Delta) = i(G| \overline{\Delta \cap E^m}, \Delta \cap E^m)$.

As stated, these properties are not sharp enough for the use required of them in this thesis. Since all finite-dimensional Banach spaces of the same dimension are homeomorphic, E^n can be replaced in the statements above by an arbitrary finite-dimensional Banach space F. After making this substitution, property A continues to hold as stated. Property B does not allow for the variation of the set Δ. Applying properties B and D (with only the substitution of F for E^n) to the space $F \times E^1$, the following strengthened form of property B is obtained:

B'. (Strengthened homotopy theorem) Let Σ be an open subset of $F \times E^1$ where F is a finite-dimensional Banach space. Let $G: \overline{\Sigma} \to F \times E^1$ in such a way that $i(G, \Sigma)$ and $i(G| \overline{\Sigma} \cap (F \times \{t\}), \Sigma \cap (F \times \{t\}))$ are defined for every t in $[0,1]$. Then $i(G| \overline{\Sigma} \cap (F \times \{0\}), \Sigma \cap (F \times \{0\})) = i(G| \overline{\Sigma} \cap (F \times \{1\}), \Sigma \cap (F \times \{1\}))$.

This property will not be proved here since the analogous property for infinite-dimensional F is proved in the proof of theorem 3. The following sharpened form of property C will be used [16]:

C'. If $i(G, \Delta_j)$ is defined for $j = 1, \ldots, k$, and if $\Delta \supseteq \Delta_1 \cup \Delta_2 \cup \ldots \Delta_k$, $\bar{\Delta} = \bar{\Delta}_1 \cup \bar{\Delta}_2 \cup \ldots \cup \bar{\Delta}_k$, and $\Delta_i \cap \Delta_j = \phi$ for $i \neq j$, then $i(G,\Delta) = \sum_{j=1}^{k} i(G, \Delta_j)$.

In property D substitute F for E^n and E for E^m, where E is a linear subspace of F.

Now we are ready to present the Cesari fixed point method.

The Cesari fixed point method. Let B be a Banach space and let S be a finite-dimensional linear subspace of B. Let P be a projection of B onto S, and suppose $\Gamma \subseteq B$ such that $P\Gamma$ is compact and such that for every x in $P\Gamma$, $P^{-1}x \cap \Gamma$ is closed. Let W be a continuous mapping from Γ into B. The Cesari method gives sufficient conditions for the existence of a fixed point of W in Γ.

Let I denote the identity mapping in B. Clearly $y = Wy$ for some y in Γ if and only if both of the following conditions hold:

(i) $Py = PWy$.

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Definition. The Cesari fixed point method applies to (Γ, W, P) if and only if the following three conditions are satisfied:

(1) For each x in $P\Gamma$, $P + (I - P)W$ is a contraction from $P^{-1}x \cap \Gamma$ into itself. Let

y(x) be that element (uniqueness follows from the contraction principle) of $P^{-1}x \cap \Gamma$ which satisfies the equation $y(x) = Py(x) + (I - P)Wy(x)$.

(2) The function y just defined is continuous from $P\Gamma$ into B.

(3) There are no fixed points of PWy on the boundary of $P\Gamma$, so that the (finite-dimensional) fixed point index $i(PWy, \text{int } P\Gamma)$ is defined.

Definition. If the Cesari fixed point method applies to (Γ, W, P) then define $i(\Gamma, W, P)$ to be the index $i(PWy, \text{int } P\Gamma)$.

Remark 1. A sufficient condition for condition (2) above to hold is that $(I - P)W$ is a contraction mapping from Γ into B [1].

Remark 2. Often it is not feasible to find the function y exactly, given as it is by a family of contraction mappings. However, the fixed point index is insensitive to small enough changes in the values of the mapping PWy, and thus y need be known only approximately. Estimating the closeness of the approximation consumes a significant portion of Cesari's time in the example he gives in [1].

<u>Theorem 1</u>. (Cesari) If $i(\Gamma, W, P)$ is defined and $i(\Gamma, W, P) \neq 0$, then there is a fixed point of W in Γ.

<u>Proof</u>. $i(PWy, \text{int } P\Gamma) = i(\Gamma, W, P) \neq 0$, so there is an x in int $P\Gamma$ such that $x = PWy(x)$. But since $x = Py(x)$, we have that $Py(x) = PWy(x)$, and thus $y(x)$ satisfies conditions (i) and (ii) above (page 3) and hence is a fixed point of W. Notice that any fixed point of W is in the range of y, for only points in the range of y satisfy condition (ii).

<u>Theorem 2</u>. Let the Cesari fixed point method apply to both (Γ, W, P_1) and (Γ, W, P_2). Assume that $P_2P_1 = P_1P_2 = P_1$ and assume that either of the following conditions holds:

(1) For every b in B and every z in the range of P_2, we have that $\|b - P_2b\| \leq \|b - z\|$.

(2) $P_2\Gamma$ is convex.

Then $i(\Gamma, W, P_1) = i(\Gamma, W, P_2)$.

<u>Proof</u>. For $i = 1, 2$, let S_i be the finite-dimensional subspace which is the range of P_i. The assumption $P_1P_2 = P_2P_1 = P_1$ implies that $S_1 \subseteq S_2$. For $i = 1, 2$, the condition $y_i(x) = P_iy_i(x) + (I - P_i)Wy_i(x)$ is equivalent to the condition $Wy_i(x) - y_i(x) = P_i(Wy_i(x) - y_i(x))$. Thus for x in $P_i\Gamma$, $y_i(x)$ is the only point of $P_i^{-1}x \cap \Gamma$ whose displacement $Wy_i(x) - y_i(x)$ belongs to S_i. Let

$\bar{x}$: $P_1\Gamma \to P_2\Gamma$ be defined by $\bar{x}(x) = P_2y_1(x)$. Then for every x in $P_1\Gamma$ we have that $y_2(\bar{x}(x)) = y_1(x)$, for $Wy_1(x) - y_1(x)$ is in $S_1 \subseteq S_2$. $\bar{x}$ is the composition of two continuous maps and hence is continuous.

We now define an isotopy which moves the graph of $\bar{x}$ into S_1. For every t in $[0,1]$ define $u_t : S_2 \cap P_1^{-1}(P_1\Gamma) \to S_2$ by the formula $u_t(z) = z - t(I - P_1)\bar{x}(P_1z)$. Each u_t is one-to-one and each u_t^{-1} is continuous, for $z = u_t(z) + t(I - P_1)\bar{x}(P_1u_t(z))$. Each u_t is an open mapping taking interior points of $P_2\Gamma$ into interior points of $u_t(P_2\Gamma)$ and boundary points of $P_2\Gamma$ into boundary points of $u_t(P_2\Gamma)$.

For every t in $[0,1]$, let $T_t : u_t(P_2\Gamma) \to S_2$ be defined by $T_t(z) = z + (P_2Wy_2u_t^{-1}(z) - u_t^{-1}(z))$. This one-parameter family of mappings preserves the displacement $P_2Wy_2(r) - r$ of points r in $P_2\Gamma$ as the graph of $\bar{x}$ is carried into S_1. Thus no fixed points of T_t are introduced on the boundary of $u_t(P_2\Gamma)$ during the homotopy, for if $T_t(z) = z$ for some z on the boundary of $u_t(P_2\Gamma)$, then $P_2Wy_2u_t^{-1}(z) = u_t^{-1}(z)$ where $u_t^{-1}(z)$ is on the boundary of $P_2\Gamma$, contradicting the assumption that $i(\Gamma, W, P_2)$ is defined.

Let $\Theta_i = \text{int } P_i\Gamma$ for i = 1, 2. It may easily be verified that the conditions of property B' (strengthened homotopy theorem, page 2) are satisfied, taking Σ to be

$$\{u_0(\Theta_2)\times(-1,0]\} \cup \left[\bigcup_{t\,\varepsilon\,[0,1]} [u_t(\Theta_2)\times\{t\}]\right] \cup \{u_1(\Theta_2)\times[1,2)\}$$

and taking (for (z,t) in $\bar{\Sigma}$, z in S_2, t in $[-1,2]$)

$$G(z,t) = T_0(z)\times\{0\} \text{ for } t \text{ in } [-1,0],$$

$$G(z,t) = T_t(z)\times\{t\} \text{ for } t \text{ in } [0,1], \text{ and}$$

$$G(z,t) = T_1(z)\times\{1\} \text{ for } t \text{ in } [1,2].$$

Thus $i(P_2Wy_2, \Theta_2) = i(T_0, u_0(\Theta_2)) = i(T_1, u_1(\Theta_2))$.

Now for every t in $[1,2]$, let $T_t: u_1(P_2\Gamma)\rightarrow S_2$ be defined by

(1) $\quad T_t(z) = (1 - (t-1))T_1(z) + (t-1)P_1T_1(z).$

If no fixed points are introduced on the boundary by this homotopy (this question will be investigated later), then by property B (homotopy theorem, page 1) we have that $i(P_2Wy_2, \Theta_2) = i(T_1, u_1(\Theta_2)) = i(T_2, u_1(\Theta_2))$. Since the values of T_2 are all in S_1, property C (reduction theorem, page 1) gives that $i(T_2, u_1(\Theta_2)) = i(T_2|\overline{u_1(\Theta_2)\cap S_1}, u_1(\Theta_2)\cap S_1)$. If x is in $\overline{u_1(\Theta_2)\cap S_1}$, then $u_1^{-1}(x) = \bar{x}(x)$ for u_1 is one-to-one and $u_1(\bar{x}(x)) = x$. For $i = 1, 2$, and for any z in $P_i\Gamma$, $Wy_i(z) - y_i(z) = P_iWy_i(z) - P_iy_i(z)$ as seen before. But $P_iWy_i(z) - P_iy_i(z)$ is equal to $P_iWy_i(z) - z$. We have already proved that for x in $\overline{u_1(\Theta_2)\cap S_1}$ we have that $u_1^{-1}(x) = \bar{x}(x)$. In this case, we have that $T_1(x) =$

$x + (P_2Wy_2\bar{x}(x) - \bar{x}(x)) = x + (Wy_2\bar{x}(x) - y_2\bar{x}(x)) =$
$x + (Wy_1(x) - y_1(x)) = x + (P_1Wy_1(x) - x) = P_1Wy_1(x).$
Thus also $T_2(x) = P_1Wy_1(x)$, and we have $i(P_2Wy_2, \Theta_2) =$
$i(T_2, u_1(\Theta_2)) = i(T_2|\overline{u_1(\Theta_2)\cap S_1}, u_1(\Theta_2)\cap S_1) =$
$i(P_1Wy_1|\overline{u_1(\Theta_2)\cap S_1}, u_1(\Theta_2)\cap S_1)$.

To obtain $i(\Gamma, W, P_1) = i(\Gamma, W, P_2)$ it must be shown that $i(P_1Wy_1|u_1(\Theta_2)\cap S_1, u_1(\Theta_2)\cap S_1) =$ $i(P_1Wy_1, \Theta_1)$. Let us apply property C' (page 2) with $k = 2$, $\Delta_1 = u_1(\Theta_2)\cap S_1$, and $\Delta_2 = \Theta_1 \sim \overline{u_1(\Theta_2)\cap S_1}$. (Note that $u_1(\Theta_2)\cap S_1 \subseteq \Theta_1$, for $u_1(\Theta_2)\cap S_1$ is an open subset of S_1 which is also a subset of $P_1\Gamma$.) If x is in $\overline{\Delta_2}$, then x is not a point of $u_1(\Theta_2)$, so $u_1^{-1}(x) = \bar{x}(x)$ is a boundary point of $P_2\Gamma$. $P_1Wy_1(x) - x = P_2W\, y_2\bar{x}(x) - \bar{x}(x)$ $\neq 0$. Thus $i(P_1Wy_1, \Delta_2)$ is defined, and by property A (page 1), $i(P_1Wy_1, \Delta_2) = 0$. Thus $i(P_2Wy_2, \Theta_2) =$ $i(P_1Wy_1|\overline{u_1(\Theta_2)\cap S_1}, u_1(\Theta_2)\cap S_1) = i(P_1Wy_1|\Delta_1, \Delta_1) =$ $i(P_1Wy_1, \Theta_1)$, as required.

Now that the reason for the study has been made clear, it is time to complete the proof of theorem 2 by showing that either of conditions (1) and (2) (page 5) implies that the homotopy T_t for t in [1,2] (see equation (1), page 7, for the equation giving T_t) introduces no fixed points on the boundary of $u_1(P_2\Gamma)$.

If for some t in [1,2] and for some z in the boundary of $u_1(P_2\Gamma)$,

$$z = T_t(z) = (1 - (t-1))T_1(z) + (t-1)P_1T_1(z),$$

then z is on the line segment joining $T_1(z)$ and $P_1T_1(z)$, and $P_1z = P_1T_1(z)$. Since z is a boundary point of $u_1(P_2\Gamma)$, $u_1^{-1}(z)$ is a boundary point of $P_2\Gamma$. Since $P_1z = P_1T_1(z) = P_1(P_1T_1(z))$, and since u_1 is linear on $P_1^{-1}(P_1z) \cap S_2$, $u_1^{-1}(z)$ is on the line segment joining $u_1^{-1}(T_1(z))$ and $u_1^{-1}(P_1T_1(z))$. But $u_1^{-1}(T_1(z)) = T_1(z) + (I - P_1)\bar{x}(P_1T_1(z)) = z + P_2Wy_2u_1^{-1}(z) - u_1^{-1}(z) + (I - P_1)\bar{x}(P_1(z)) = P_2Wy_2u_1^{-1}(z)$ since $u_1^{-1}(z) - z = \bar{x}(P_1(z)) - P_1\bar{x}(P_1(z))$. But we also have that $u_1^{-1}(P_1T_1(z)) = P_1T_1(z) + (I - P_1)\bar{x}(P_1P_1T_1(z)) = P_1z + (I - P_1)\bar{x}(P_1z) = \bar{x}(P_1z)$. Thus to prove the theorem it is only necessary to show that each of conditions (1) and (2) implies that there is no point r on the boundary of $P_2\Gamma$ which is on the line segment joining $P_2Wy_2(r)$ and $\bar{x}(P_1r)$, where each of these three points has the same P_1-projection. Notice that $r \neq \bar{x}(P_1r)$, for if $r = \bar{x}(P_1r)$, then $P_1r = P_1P_2Wy_2(r) = P_1Wy_2\bar{x}(P_1r) = P_1Wy_1(P_1r)$, and thus $P_2Wy_2r - r = P_2Wy_2\bar{x}(P_1r) - \bar{x}(P_1r) = P_1Wy_1(P_1r) - P_1r = 0$ with r on the boundary of $P_2\Gamma$, contradicting the assumption that $i(\Gamma, W, P_2)$ is defined.

<u>To show that condition (1) gives the theorem.</u> Suppose

that condition (1) holds, and assume that the forbidden r exists. Let $x = P_1 r$. Because $P_1 + (I - P_1)W$ is a contraction mapping on $P_1^{-1}x \cap \Gamma$, with fixed point $y_1(x) = y_2(\bar{x}(x))$, and since $y_2(r) \neq y_2(\bar{x}(x))$,

$$A_1 = P_1 y_2(r) + (I - P_1)Wy_2(r)$$

is closer to

$$A_2 = y_1(x) = P_1 y_1(x) + (I - P_1)Wy_1(x)$$

than is

$$A_3 = y_2(r) = r + (I - P_2)Wy_2(r).$$

Now let

$$A_1 = A_{11} + A_{12} + A_{13}$$

$$A_2 = A_{21} + A_{22} + A_{23}$$

$$A_3 = A_{31} + A_{32} + A_{33}$$

where $P_1 A_i = A_{i1}$, $(P_2 - P_1)A_i = A_{i2}$, and $A_i - P_2 A_i = A_{i3}$ for $i = 1, 2, 3$. Clearly $A_{11} = A_{21} = A_{31} = x$, $A_{13} = A_{33} = (I - P_2)Wy_2(r)$, and $\|A_1 - A_2\| < \|A_3 - A_2\|$. We wish to show that $r = A_{31} + A_{32}$ is not on the line segment joining $P_2 W y_2(r) = A_{11} + A_{12}$ and $\bar{x}(P_1 r) = A_{21} + A_{22}$, or equivalently, we wish to show that A_{32} is not on the line segment joining A_{12} and A_{22}.

Assume that $A_{32} = \lambda_0 A_{12} + (1-\lambda_0)A_{22}$ with $0 \leq \lambda_0 \leq 1$. Then $\|A_1 - A_2\| < \|A_3 - A_2\|$ implies that

$\|A_{12} + A_{13} - A_{22} - A_{23}\| < \|\lambda_0 A_{12} + (1-\lambda_0)A_{22} + A_{33} - A_{22}$

$- A_{23}\| = \|\lambda_0(A_{12} - A_{22}) + A_{13} - A_{23}\|$. Let q be a real number between these two norms. Then consider S = $\{b \in B;\ \|b - (A_{23} - A_{13})\| < q\}$. This set contains $A_{12} - A_{22}$ in S_2 and hence it must also contain $P_2(A_{23} - A_{13}) = 0$, by condition (1) of this theorem. Clearly S is convex, so S must contain $\lambda(A_{12} - A_{22})$ for $0 \leq \lambda \leq 1$. However $\lambda_0(A_{12} - A_{22})$ is not in S since $\|\lambda_0(A_{12} - A_{22}) - (A_{23} - A_{13})\| > q$. This contradiction proves theorem 2 assuming condition (1).

<u>To show that condition (2) gives the theorem</u>. It has been shown that $i(P_1Wy_1, \text{int } P_1\Gamma) = i(P_2Wy_2, \text{int } P_2\Gamma)$ if there is no point r on the boundary of $P_2\Gamma$ which lies on the line segment joining $P_2Wy_2(r)$ and $\bar{x}(P_1r)$ where all three have the same P_1-projection. In case hypothesis (2) is satisfied, it is possible to define a homotopy F_t of P_2Wy_2 to a function F which has the following three properties:

(i) There is no point r on the boundary of $P_2\Gamma$ which lies on the line segment joining F(r) and $\bar{x}(P_1r)$, all having the same P_1-projection.

(ii) $F_t(\bar{x}(P_1z)) - \bar{x}(P_1z)$ is a positive multiple of $P_1Wy_1(P_1z) - P_1z$ throughout the homotopy, for every z in Γ.

(iii) F_t introduces no fixed points on the boundary of $P_2\Gamma$ during the homotopy.

This then will prove that $i(P_2Wy_2, \text{int } P_2\Gamma) = i(F, \text{int } P_2\Gamma) = i(P_1Wy_1, \text{int } P_1\Gamma)$.

Let $M > \sup\{|P_1Wy_2(z) - P_1z| ; z \varepsilon P_2\Gamma\}$, and let $0 < m < \inf\{|P_1Wy_1(P_1z) - P_1z| ; z \varepsilon P_2\Gamma$ and $\bar{x}(P_1z)$ is a boundary point of $P_2\Gamma\}$. Then for $z \varepsilon P_2\Gamma$ and $t \varepsilon [0,1]$, define

$$F_t(z) = P_2Wy_2(z) + \frac{M}{m} t(P_1Wy_1(P_1z) - P_1z).$$

Clearly F_t is a homotopy. To prove (iii), assume that for some z on the boundary of $P_2\Gamma$ and for some t in $[0,1]$ we have

$$z = F_t(z) = P_2Wy_2(z) + \frac{M}{m} t(P_1Wy_1(P_1z) - P_1(z)),$$

so $P_2Wy_2(z) - z = -\frac{M}{m} t(P_1Wy_1(P_1z) - P_1(z)) \varepsilon S_1$. Thus $z = \bar{x}(P_1z)$. Therefore $P_2Wy_2(z) - z = P_1Wy_1(P_1z) - P_1z$, and thus

$$P_2Wy_2(z) - z = -\frac{M}{m} t(P_1Wy_1(P_1z) - P_1(z)),$$

which is a contradiction unless $P_2Wy_2(z) = z$, and this is impossible since z is on the boundary of $P_2\Gamma$ and $i(\Gamma, W, P_2)$ is defined.

To prove (ii), we note that

$$\begin{aligned} F_t(\bar{x}(P_1z)) - \bar{x}(P_1z) &= P_2Wy_2(\bar{x}(P_1z)) - P_2y_2(\bar{x}(P_1z)) \\ &\quad + \frac{M}{m} t(P_1Wy_1(P_1z) - P_1(z)) \\ &= P_1Wy_1(P_1z) - P_1(z) \\ &\quad + \frac{M}{m} t(P_1Wy_1(P_1z) - P_1(z)). \end{aligned}$$

To prove (i), consider any boundary point r of $P_2\Gamma$ for which (i) is false. If $\bar{x}(P_1r)$ is a boundary point of $P_2\Gamma$, then $0 = P_1F(r) - P_1(r) = P_1Wy_2(r) + \frac{M}{m}(P_1Wy_1(P_1r) - P_1r) - P_1(r)$. But $|P_1Wy_2(r) - P_1(r)| < M$ and $|\frac{M}{m}(P_1Wy_1(P_1(r))| > M$, contradiction. Now consider the remaining case that $\bar{x}(P_1r)$ is not a boundary point of $P_2\Gamma$, and assume as before that r is a point on the boundary of $P_2\Gamma$ which lies on the line segment joining $F(r)$ and $\bar{x}(P_1r)$, all three points having the same P_1-projection. Define $s = \frac{M}{m}(P_1Wy_1(P_1r) - P_1(r))$. Notice that $F(r) = P_2Wy_2(r) + s$ and that $s \varepsilon S_1$. Then $P_1y_2(r) = P_1r = P_1F(r) = P_1Wy_2(r) + s$, and $P_2(P_1y_2(r) + (I-P_1)Wy_2(r)) = P_1y_2(r) + (P_2-P_1)Wy_2(r) = (P_1y_2(r) - P_1Wy_2(r)) + P_2Wy_2(r) = s + P_2Wy_2(r) = F(r)$. By the assumption that $P_1 + (I - P_1)W$ is a map of $P_1^{-1}(P_1r) \cap \Gamma$ into itself (Notice that the proof for condition (1) uses only the contraction assumption and not the onto assumption. Here the situation is reversed.), $P_2(P_1y_2(r) + (I - P_1)Wy_2(r)) = F(r)$ must be in $P_2\Gamma$. But $\bar{x}(P_1r)$ is an interior point of the convex set $P_2\Gamma$, $F(r)$ is in $P_2\Gamma$, r is on the line segment joining them, and $r \neq F(r)$. Thus r is an interior point of $P_2\Gamma$, contradiction.

Before beginning theorem 3, it will be useful to review certain properties of the (infinite-dimensional) Leray-Schauder fixed point index.

<u>Properties of the Leray-Schauder fixed point index.</u> Let N be a normed linear space, and let Ω be a bounded open subset of N. Let $W:\overline{\Omega}\to N$ be completely continuous (or compact, to use another terminology), that is, let W be continuous and suppose that $\overline{W(\overline{\Omega})}$ is compact. Suppose that W has no fixed points on the boundary of Ω. Then the Leray-Schauder fixed point index $i_{LS}(W,\Omega)$ is defined. Like the finite-dimensional fixed point index, it is an integer, positive, negative, or zero. In addition, it has the following properties [9]:

A. If $i_{LS}(W,\Omega)$ is defined and if $i_{LS}(W,\Omega)\neq 0$, then there is an $x\in\Omega$ such that $Wx = x$.

B. (Homotopy theorem) If $W_t(x)$ is a continuous function on $[0,1]\times\overline{\Omega}$, continuous in t uniformly for all x in $\overline{\Omega}$, and if $i_{LS}(W_t,\Omega)$ is defined for every t in $[0,1]$, then $i_{LS}(W_0,\Omega) = i_{LS}(W_1,\Omega)$.

C. If $i_{LS}(W,\Omega_1)$ and $i_{LS}(W,\Omega_2)$ are both defined, where W is a completely continuous function defined on $\overline{\Omega_1\cup\Omega_2}$, and if $\Omega_1\cap\Omega_2=\phi$, then $i_{LS}(W,\Omega_1\cup\Omega_2) =$

$i_{LS}(W, \Omega_1) + i_{LS}(W, \Omega_2)$.

D. (Analogue of the reduction theorem) Suppose that $i_{LS}(W, \Omega)$ is defined. Then it is true that $r = \inf\{\|Wy - y\|;$ y is on the boundary of $\Omega\} > 0$, and if $W_{r/2}$ is a continuous function of $\bar{\Omega}$ into a finite-dimensional linear subspace F of N such that $\|W(x) - W_{r/2}(x)\| < r/2$ for every x in $\bar{\Omega}$, then $i(W_{r/2}|\overline{\Omega \cap F}, \Omega \cap F)$ is defined in F, and $i_{LS}(W, \Omega) = i(W_{r/2}|\overline{\Omega \cap F}, \Omega \cap F)$.

These properties are strong enough as stated for the use required of them in theorem 3.

<u>Theorem 3</u>. Let both the Leray-Schauder fixed point index $i_{LS}(W, \Omega)$ and $i(\bar{\Omega}, W, P)$, the number associated with the Cesari fixed point method, be defined. Then $i_{LS}(W, \Omega) = i(\bar{\Omega}, W, P)$.

<u>Proof</u>. Let S be the finite dimensional linear subspace which is the range of P. For every t in $[-1,2]$, define $u_t : P^{-1}(P\bar{\Omega})$ into itself by

$$u_t(z) = z - t(I-P)y(Pz).$$

u_t moves the graph of y into S. Notice that for each t, u_t is one-to-one and u_t^{-1} is continuous, for

$$u_t^{-1}(z) = z + t(I-P)y(Pz).$$

Thus each $u_t(\Omega)$ is open in B, and $u_t(\bar{\Omega}) = \overline{u_t(\Omega)}$.

For each t in $[-1,2]$, define $W_t:u_t(\bar{\Omega})\to B$ by

$$W_t(z) = z + (Wu_t^{-1}(z) - u_t^{-1}(z)).$$

W_t preserves the displacement $Wr - r$ of points r of $\bar{\Omega}$ as they are moved so as to carry the graph of y into S. Since $W_t(z) - z = Wu_t^{-1}(z) - u_t^{-1}(z)$, no fixed points are introduced on the boundary of $u_t(\Omega)$ by the homotopy. Is each W_t a compact transformation? Fix t, then $W_t(\bar{\Omega}) \subseteq \overline{W(\bar{\Omega})} - t(I-P)y(P\bar{\Omega})$. Both $\overline{W(\bar{\Omega})}$ and $t(I-P)y(P\bar{\Omega})$ are clearly compact, so their difference is also. Therefore $\overline{W_t(\bar{\Omega})}$ is compact and $i_{LS}(W_t, u_t(\Omega))$ is defined for all t in $[0,1]$. Is this index constant throughout the homotopy? Answering this question is analogous to proving property B' (page 2) for finite-dimensional fixed point indexes. Consider $B\times E^1$ (where E^1 denotes the real numbers) which has for $b\varepsilon B$ and $r\varepsilon E^1$ the norm $\|(b,r)\| = \|b\| + |r|$. With this norm, $B\times E^1$ is a Banach space. Let

$$\Psi = \{(b,t)\varepsilon B\times(-1,2);\ b\varepsilon u_t(\Omega)\},$$

an open set. For every t in $[0,1]$, let $P_t:B\times E^1\to B\times\{t\}$ be the obvious projection. For t in $[0,1]$ define $Z_t:\bar{\Psi}\to B\times\{t\}$ by $Z_t(b,r) = (W_t b,t)$. There are no fixed points of Z_t on the boundary of Ψ, and $\overline{Z_t(\bar{\Psi})}$ is compact, since $(b,r)\varepsilon\overline{Z_t(\bar{\Psi})}$ implies that $b\varepsilon W(\bar{\Omega}) - \{t(I-P)y(P\bar{\Omega});$ t is in $[0,1]\}$, and since $\{t(I-P)y(P\bar{\Omega});$ t is in $[0,1]\}$

is the continuous image of the compact set $P\bar{\Omega}\times[0,1]$ and hence is compact. Thus each Leray-Schauder fixed point index $i_{LS}(Z_t,\Psi)$ is defined, and this index is constant for t in $[0,1]$ by property B (homotopy theorem, page 14) of the Leray-Schauder fixed point index. But then by property D (analogue of the reduction theorem, page 15) of the Leray-Schauder fixed point index, $i_{LS}(W_0,\Omega) = i_{LS}(Z_0,\Psi) = i_{LS}(Z_1,\Psi) = i_{LS}(W_1,\ u_1(\Omega))$. Thus the index is invariant throughout the homotopy.

Now for t in $[1,2]$ define $W_t:u_1(\bar{\Omega})\rightarrow B$ (redefining W_t on $[1,2]$) by

$$W_t(x) = (1 - (t-1))W_1(x) + (t-1)PW_1(x).$$

This is a homotopy of compact transformations, uniformly continuous in t. Moreover, it introduces no fixed points on the boundary of $u_1(\bar{\Omega})$, because if for some t in $[1,2]$ and some x in the boundary of $u_1(\bar{\Omega})$ we had $W_t(x) = x$, then we would have that $Px = PW_1(x)$ and x is on the line segment joining $W_1(x)$ and $PW_1(x)$. Thus $z = u_1^{-1}(x)$ is a boundary point of Ω on the line segment joining $W(z)$ and $y(Pz)$, and $PWz = Pz$. Now $z \neq y(Pz)$, because if $z = y(Pz)$, then $Pz + (I - P)Wz = z$ and $PWz = Pz$, so z is a fixed point of W on the boundary of Ω, contradicting the assumption that $i_{LS}(W,\Omega)$ is defined. But $z \neq y(Pz)$ implies that $Pz + (I - P)Wz =$

Wz is closer to y(Pz) than is z, since y(Pz) is the fixed point of the contraction mapping $P + (I - P)W$ on $P^{-1}(Pz) \cap \bar{\Omega}$, contradiction. Thus $i_{LS}(W, \Omega) = i_{LS}(W_1, u_1(\Omega)) = i_{LS}(W_2, u_1(\Omega))$. But the range of W_2 is a subset of the finite-dimensional linear space S. Thus by property D (analogue of the reduction theorem, page 15), $i_{LS}(W_2, u_1(\Omega)) = i(W_2 | \overline{S \cap u_1(\Omega)}, S \cap u_1(\Omega))$. But for x in $\overline{S \cap u_1(\ )}$, $u_1^{-1}(x) = y(x)$ and $W_2(x) = PWy(x)$. Moreover, if x is in $\overline{P\Omega \sim \overline{S \cap u_1(\Omega)}}$ $(S \cap u_1(\Omega) \subseteq P\Omega)$, then x is not in $u_1(\Omega)$ and hence $u_1^{-1}(x) = y(x)$ is a boundary point of Ω. Thus for x in $\overline{P\Omega \sim \overline{S \cap u_1(\Omega)}}$, $PWy(x) \neq x$, and therefore property C' for finite-dimensional fixed point indexes (page 2) gives us that $i_{LS}(W, \Omega) = i_{LS}(W_2, u_1(\Omega)) = i(W_2 | S \cap u_1(\Omega), S \cap u_1(\Omega)) = i(PWy | \overline{S \cap u_1(\Omega)}, S \cap u_1(\Omega)) = i(PWy, P\Omega) = i(\bar{\Omega}, W, P)$.

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