SURFACE IMPEDANCE THEORY FOR SUPERCONDUCTORS IN LARGE STATIC MAGNETIC FIELDS

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Richard David Sherman

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ABSTRACT

A theory of electromagnetic absorption is presented to explain the changes in surface impedance for Pippard superconductors ($\xi_o \gg \lambda$) due to large static magnetic fields. The static magnetic field penetrating the metal near the surface induces a momentum dependent potential in Bogolubov's equations. Such a potential modifies a quasiparticle's wavefunction and excitation spectrum. These changes affect the behavior of the surface impedance in a way that in large measure agrees with available observations.

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INTRODUCTION

Over the past twenty years many experimentalists have been measuring the changes in surface impedance of Pippard (or type I) superconductors due to a large static magnetic field near the surface. (See papers by Pippard[1], Spiewak[2], Glosser[3], Lewis[4], Richards[5], Sharvin and Gantmakher[6].) Although the trends in behavior are now obvious, no theory has yet appeared which can explain the patterns of these results. This thesis presents just such a theory.

The principle features observed in the experiments are that the magnetic field increases the surface resistance at "low temperatures" ($T \leq .5\, T_c$) and "high temperatures" ($T \geq .9\, T_c$) but can strongly decrease absorption for "intermediate temperatures". At "low temperatures" the surface reactance is always increased while at "intermediate and high temperatures" the changes can be positive or negative depending on the frequency of the radiation. As usual for solid state problems, everything can change the sizes and proportions of these curves - the actual metal used, surface preparation, impurities in the metal, polarization and frequency of the incident radiation, and orientation of the magnetic field with respect to the crystal axes.

We propose that the above phenomena can be explained as follows: When a superconductor is placed in a magnetic field the elementary excitation properties are strongly modified for quasi-particles near the Fermi surface. In particular, this field creates a momentum

dependent potential for the quasi-particles which can look either like a well or a barrier. The existence of the well gives rise to new discrete states whose excitation energies can be less than the superconducting gap energy. Such states are responsible for the increased surface resistance at "low T". The potential barrier, on the other hand, has the effect of confining quasi-particles near the superconducting gap edge to regions of space where the magnetic field is weak or non-existent. This latter phenomenon accounts for the increased surface reactance at "low T", the decreased surface resistance at "intermediate T", and the complicated frequency dependence of surface reactance at "intermediate and high T".

To calculate surface impedances for this situation is very complicated for the following reasons: (1) The equations of superconductivity (Bogolubov's equations) are two coupled second order differential equations whose solutions must satisfy a self-consistency relation for the energy gap or order parameter function, $\Delta(\vec{r})$. Only one case has been solved - the infinite homogeneous metal which is not the situation here due to the static magnetic field. (2) The relation between the current density, $\vec{j}$ and the electric field, $\vec{E}$ is non-local; current at one point inside the superconductor depends on the electric field strength in the surrounding region, which means that to find the field distribution requires solving a complicated integro-differential equation. And the kernel of this integro-differential equation must be found from (1) above. Once again this can be solved only approximately analytically for an infinite metal (Mattis and Bardeen[7].)

For general spatial variations as created by the static magnetic fields here, some other approximate methods must be used. Once the electric field distribution is known, it is trivial to calculate the surface impedance.

The vehicle for discussing this problem is the Bogolubov canonical transformation from interacting electron states to non-interacting quasi-particle states. This is discussed briefly in Chapter 1. Using the Bogolubov transformation, it is possible to derive a general form of the current versus field relation for spatially-varying situations. This is done in Chapter 2.

In Chapter 3, we solve the Bogolubov equations for the coefficients in the Bogolubov transformation for the situation of a superconductor in a large static magnetic field.

In Chapter 4, we explain a method to calculate the surface impedance while maintaining self-consistency in the field distribution.

In the limit of no static magnetic fields, and vanishing superconducting energy gap, the normal metal case is obtained. We discuss the theory of the anomalous skin effect in normal metals (Chapter 5) to understand the physics of more complicated cases.

In Chapter 6, we use all the previously developed machinery to find the predictions of the theory for $H_{dc} \neq 0$ and compare with available data.

In Chapters 7 and 8, we discuss two left-over topics on the way to finding surface impedances. One is the effect of surfaces themselves on the absorption processes in metals at low temperatures and

the other is the question of whether a large static magnetic field can decrease the superconducting order parameter or energy gap.

For simplicity we have restricted ourselves to radiation fields normally incident upon a plane superconducting surface.

CHAPTER 1

BOGOLUBOV'S EQUATIONS

Bardeen, Cooper, and Schrieffer (BCS)[8] have proposed a theory which explains the behavior of infinite homogeneous superconductors. Bogolubov generalized and extended their theory to cover situations with spatial variations. The Bogolubov method is described by De Gennes[9]. We briefly review the results and some salient features below.

The electrons in the metal are described by the Hamiltonian

$$H = \sum_{\alpha} \int d\vec{r}\, \Psi_{\alpha}^{\dagger}(\vec{r}) \left\{ \frac{1}{2m} \left[-i\nabla - e\vec{A}(\vec{r}) \right]^2 + U_{\text{impurity}}(\vec{r}) - E_f \right\} \Psi_{\alpha}(\vec{r})$$

$$- \frac{1}{2} V_{BCS} \sum_{\alpha,\alpha'} \int d\vec{r}\, \Psi_{\alpha}^{\dagger}(\vec{r})\, \Psi_{\alpha'}^{\dagger}(\vec{r})\, \Psi_{\alpha'}(\vec{r})\, \Psi_{\alpha}(\vec{r}) \qquad (1.1)$$

where the Ψ's are anti-commuting electron field operators, $U_{\text{impurity}}(\vec{r})$ is the impurity scattering potential, E_f is the Fermi energy of the electron ensemble, V_{BCS} is the electron-electron interaction potential, α, α' are spin indices, $\vec{A}(\vec{r})$ is the vector potential of any external fields, and hereafter take units where $\hbar = c = 1$.

If only the first term were present in this Hamiltonian, we would recover the theory of electrons in a normal metal interacting only via the Exclusion Principle and hence obeying Fermi-Dirac statistics.

It is the addition of the second term in the Hamiltonian (1.1) which accounts for superconductivity. This term takes into considera-

tion an extra interaction - that between two electrons due to the exchange of virtual phonons. For two electrons at the Fermi surface whose energies differ by an amount less than the Debye energy of the phonon spectrum, there is a net attraction. And BCS have shown that this attraction leads to the formation of bound pairs of electrons in a metal. Since this interaction occurs over a small range of momenta, it must have a long spatial range. Hence BCS and Bogolubov approximate the interaction potential by a constant, V_{BCS}, independent of range.

Improvements can be made on these assumptions, but for most purposes this is adequate.

The Hamiltonian (1.1) is too difficult to handle and so is replaced by an approximate Hamiltonian, H_{eff},

$$H_{eff} = \int d\vec{r} \left\{ \sum_{\alpha} \left[\Psi_{\alpha}^{\dagger}(\vec{r}) \left(\frac{1}{2m} [-i\nabla - e\vec{A}]^2 + U_{imp}(\vec{r}) - E_f \right) \Psi_{\alpha}(\vec{r}) \right] \right.$$
$$\left. + \Delta(\vec{r}) \Psi_{\uparrow}^{\dagger}(\vec{r}) \Psi_{\downarrow}^{\dagger}(\vec{r}) + \Delta^{*}(\vec{r}) \Psi_{\downarrow}(\vec{r}) \Psi_{\uparrow}(\vec{r}) \right\} \qquad (1.2)$$

where $\Delta(\vec{r})$, the superconducting energy gap or order parameter, is determined later by a self-consistency relation.

Bogolubov diagonalized the Hamiltonian H_{eff} by the transformation

$$\Psi_{\uparrow}(\vec{r}) = \sum_{n} \left[\gamma_{n\uparrow} u_n(\vec{r}) - \gamma_{n\downarrow}^{\dagger} v_n^{*}(\vec{r}) \right] \qquad (1.3a)$$

$$\Psi_{\downarrow}(\vec{r}) = \sum_{n} \left[\gamma_{n\downarrow} u_n(\vec{r}) + \gamma_{n\uparrow}^{\dagger} v_n^{*}(\vec{r}) \right] \qquad (1.3b)$$

where the γ's are anti-commuting annihilation and creation operators.

Then

$$H_{eff} = E_g + \sum_{n,a} \epsilon_n \gamma_{na}^{\dagger} \gamma_{na} \tag{1.4}$$

which is just the Hamiltonian of a gas of non-interacting Fermions. The excitations created by $\gamma_a^{\dagger}$ are the quasi-particles. E_g is the ground state energy of the system of quasi-particles.

The Hamiltonians (1.2) and (1.4) are equivalent if the functions u, v satisfy the set of equations

$$\epsilon_n u_n(\vec{r}) = \left[\frac{1}{2m}(-i\nabla - e\vec{A})^2 + U_{imp.} - E_f\right] u_n(\vec{r}) + \Delta(\vec{r})\, v_n(\vec{r}) \tag{1.5a}$$

$$\epsilon_n v_n(\vec{r}) = -\left[\frac{1}{2m}(i\nabla - e\vec{A})^2 + U_{imp.} - E_f\right] v_n(\vec{r}) + \Delta^*(\vec{r})\, u_n(\vec{r}) \tag{1.5b}$$

which are called Bogolubov's equations.

The order parameter, $\Delta(\vec{r})$, is determined by requiring the free energy to be minimized when calculated with H_{eff}. This leads to

$$\Delta(\vec{r}) = V_{BCS} \sum_n v_n^*(\vec{r})\, u_n(\vec{r}) \left[1 - 2 f_T(\epsilon_n)\right] \tag{1.6}$$

The u, v's must also satisfy a normalization condition

$$\int d\vec{r} \left[|u_n(\vec{r})|^2 + |v_n(\vec{r})|^2 \right] = 1 \tag{1.7}$$

To develop some feel for this method of description, consider the application of Bogolubov's method to a free electron gas. The u, v's are just plane waves and the γ_n^{+}, γ_n are identical to the a_p^{+}, a_p

operators which create and annihilate electrons in plane wave states. If $\gamma_{p\uparrow}^{\dagger}$, $p>p_f$, acts on the ground state of the system, an electron is put above the Fermi surface in the state $u_p(\vec{r}) = e^{i\vec{p}\cdot\vec{r}}$. If $\gamma_{p\downarrow}$, $p<p_f$, acts on the ground state of the system, an electron is removed from the Fermi sea in the state $v_p(\vec{r})=e^{i\vec{p}\cdot\vec{r}}$ which is equivalent to "adding a hole" in the Fermi sea. Hence, we can call the u's wave functions of electron type quasi-particles and the v's wave functions of hole type quasi-particles.

Note that, if $\Delta = 0$, then eqns. (1.5a, b) are just the Schroedinger equations for free electrons and free holes, respectively. u(x) is the electron wavefunction and v(x) is the hole wavefunction. When $u \neq 0$, $v = 0$ and vice versa so that eqn. (1.6) is consistent with $\Delta = 0$.

The new feature of Bogolubov's equations which explains superconductivity is the term with Δ. Electron and hole excitations are no longer independent of each other and the degree of influence depends on Δ, constrained by eqn. (1.6).

CHAPTER 2

$\vec{j}(\vec{A})$ RELATION

Any discussion of electrodynamics inside a metal requires knowledge of a current versus field relation. In this chapter we derive a particular form for the current as a function of the vector potential useful for surface impedance calculations.

Quite generally, the current density operator in second-quantized form evolving in time according to the Heisenberg picture is

$$\tilde{\vec{j}}_{a_{op}}(x)=\frac{ie}{2m}\left(\nabla_{\vec{x}}-\nabla_{\vec{x}'}\right)_{\vec{x}'\to\vec{x}}\tilde{\Psi}_a^{\dagger}(x')\tilde{\Psi}_a(x)-\frac{e^2}{m}\vec{A}(x)\tilde{\Psi}_a^{\dagger}(x)\tilde{\Psi}_a(x) \qquad (2.1)$$

where $x = (\vec{x}, t)$, vector position and time

$a = \uparrow, \downarrow$ the spin index

$\sim$ denotes an operator in the Heisenberg representation.

$\tilde{\Psi}^{\dagger}(x)$ is an electron field creation operator satisfying Fermion anti-commutation rules.

See Abrikosov, Gorkov, and Dzyaloshinski (AGD)[10] for a derivation. Eqn. (2.1) is to be evaluated up to terms first order in $\vec{A}$. The final result will then yield the current density at some point in space and time due to an arbitrary $\vec{A}$ field distribution surrounding this point.

Assume $\vec{A}(x) = \vec{A}(\vec{x}, t)$ to have been turned on adiabatically from some time t in the distant past, $t \to -\infty$. Now switch over to the interaction representation which is the same as the Heisenberg picture for $t \to -\infty$. Any operator O is related between these two

pictures by

$$O_{\text{Heisenberg}} = S^{-1}(t)\ O_{\text{interaction}}\ S(t) \tag{2.2}$$

where S(t) is the S matrix

$$S(t) = T_t\, e^{i\int_{-\infty}^{t} \vec{j}_{op}(y)\cdot\vec{A}(y)\, d^4y} \tag{2.3}$$

where T_t is the time ordering symbol.

Expanding S(t) to terms first order in $\vec{A}$ yields

$$S(t) \cong 1+i\int_{-\infty}^{t} \vec{j}_{op}(y)\cdot\vec{A}(y)\, d^4y \tag{2.4}$$

And hence the following is true to first order in $\vec{A}(x)$.

$$\tilde{\Psi}_a(x) \cong \Psi_a(x) - i\int_{-\infty}^{t} \left[\vec{j}_{op}(y)\,\Psi_a^{\dagger}(x) - \Psi_a(x)\vec{j}_{op}(y)\right]\cdot\vec{A}(y)\, d^4y \tag{2.5}$$

$$\tilde{\Psi}_a^{\dagger}(x') \cong \Psi_a^{\dagger}(x') - i\int_{-\infty}^{t} \left[\vec{j}_{op}(y)\,\Psi_a^{\dagger}(x') - \Psi_a^{\dagger}(x')\,\vec{j}_{op}(y)\right]\cdot\vec{A}(y)\, d^4y \tag{2.6}$$

where $\vec{j}_{op}(y)$ is just the zero order approximation

$$\vec{j}_{op}(y) \cong \frac{ie}{2m}\left(\nabla_{\vec{y}} - \nabla_{\vec{y}'}\right)_{\vec{y}'\to\vec{y}} \sum_{\beta=\downarrow}^{\uparrow} \Psi_\beta^{\dagger}(y')\,\Psi_\beta(y) \tag{2.7}$$

Putting eqns. (2.5), (2.6), and (2.7) into eqn. (2.1) keeping terms first order in $\vec{A}(x)$ and summing over spins yields

$$\tilde{\vec{j}}_{op}(x) = \frac{ie^2}{4m^2}\left(\nabla_{\vec{x}} - \nabla_{\vec{x}'}\right)_{\vec{x}'\to\vec{x}} \int_{-\infty}^{t} \vec{A}(y)\cdot\left(\nabla_{\vec{y}} - \nabla_{\vec{y}'}\right)_{\vec{y}'\to\vec{y}}$$

$$\sum_{\alpha,\beta}\left\{\Psi_\beta^{\dagger}(y')\Psi_\beta(y)\Psi_\alpha^{\dagger}(x')\Psi_\alpha(x) - \Psi_\alpha^{\dagger}(x')\Psi_\alpha(x)\Psi_\beta^{\dagger}(y')\Psi_\beta(y)\right\} d^4y$$

$$- \frac{e^2}{m} \sum_a \Psi_a^\dagger(x)\, \Psi_a(x)\, \vec{A}(x) \tag{2.8}$$

To obtain from eqn. (2.8) a physical current, it is necessary to average $\tilde{\vec{j}}_{op}$ over a grand canonical ensemble.

$$\vec{j}(x) = \langle \tilde{\vec{j}}_{op}(x) \rangle_T = \sum_m e^{(\Omega - \epsilon_m)/T} \langle m | \tilde{\vec{j}}_{op}(x) | m \rangle \tag{2.9}$$

where Ω is defined by

$$\sum_m e^{(\Omega - \epsilon_m)/T} = 1 \tag{2.10}$$

ϵ_m is simply the excitation energy of quasi-particles in the Bogolubov equations. That is, the diagonalized Hamiltonian of the system is

$$H_M = \sum_{n,a} \epsilon_n \gamma_{na}^\dagger \gamma_{na} + \text{ground state energy to be ignored} \tag{2.11a}$$

and

$$H_M | m \rangle = \epsilon_m | m \rangle \tag{2.11b}$$

H_M is the complete Hamiltonian of the system at $t \to -\infty$ when the radiation field is off.

$$\vec{j}(x) = \frac{ie^2}{4m^2} \left(\nabla_{\vec{x}} - \nabla_{\vec{x}'} \right)_{\vec{x}' \to \vec{x}} \int_{-\infty}^{t} \vec{A}(y) \cdot \left(\nabla_{\vec{y}} - \nabla_{\vec{y}'} \right)_{\vec{y}' \to \vec{y}}$$

$$\sum_{a,\beta} \langle \Psi_\beta^\dagger(y') \Psi_\beta(y) \Psi_a^\dagger(x') \Psi_a(x) - \Psi_a^\dagger(x') \Psi_a(x) \Psi_\beta^\dagger(y') \Psi_\beta(y) \rangle_T d^4y - \frac{Ne^2}{m} \vec{A}(x) \tag{2.12}$$

where $N = \sum_a \langle \psi_a^\dagger(x)\, \psi_a(x) \rangle_T$ is the density of electrons in the system. N is written without its spatial dependence since for perturbations of interest here, N is a constant. Impurities, magnetic and electric fields, etc. applied to a metal only change the electron distribution at the Fermi surface, but this is only a very tiny fraction of the total number of electrons present.

Considerable simplification in taking the remaining thermal average occurs if we transform from electron field operators to quasi-particle operators via the Bogolubov canonical variable transformation, which in the Schroedinger representation is

$$\psi_a(\vec{x}) = \sum_j \left[\gamma_{ja} u_j(\vec{x}) + S_{a'}\, \gamma_{ja'}^\dagger\, v_j^*(\vec{x}) \right] \qquad a \neq a' \tag{2.13}$$

where $$S_{a'} = \begin{cases} +1 \text{ if } a' = \uparrow \\ -1 \text{ if } a' = \downarrow \end{cases}$$

$\psi_a^\dagger(\vec{x})$ is obtained from eqn. (2.13) by taking the Hermitian adjoint. In the interaction representation the transformation is then

$$\psi_a(x) = \sum_j \left[\gamma_{ja}(t) u_j(x) + S_{a'}\, \gamma_{ja'}^\dagger(t)\, v_j^*(x) \right] \tag{2.14}$$

where $x = (\vec{x}, t)$

$$\gamma_{ja}(t) = e^{iH_M t}\, \gamma_{ja}\, e^{-iH_M t}$$

Writing $\vec{j}(x)$ in terms of γ's yields sums over many terms each of which is a product of four γ's. But only some of these terms are

non-vanishing upon taking thermal averages. Since the thermal average is a weighted sum of diagonal matrix-elements, only terms of the form $\gamma_i^\dagger \gamma_i \gamma_j^\dagger \gamma_j$ (or permutations in ordering) can contribute. A typical term from $\langle \ \rangle_T$ in eqn. (2.12) is

$$u_\ell^*(\vec{y}')u_m(\vec{y})u_n^*(\vec{x}')u_p(\vec{x}) \langle \gamma_{\ell\beta}^\dagger(t_y)\gamma_{m\beta}(t_y)\gamma_{n\alpha}^\dagger(t_x)\gamma_{p\alpha}(t_x)$$

$$-\gamma_{n\alpha}^\dagger(t_x)\gamma_{p\alpha}(t_x)\gamma_{\ell\beta}^\dagger(t_y)\gamma_{m\beta}(t_y)\rangle_T =$$

$$u \ldots u \left\{\delta_{\alpha\beta}\delta_{\ell p}\delta_{mn}\left[f_T(\varepsilon_\ell)-f_T(\varepsilon_m)\right] e^{i(\varepsilon_\ell-\varepsilon_m)(t_y-t_x)}\right\}$$

where $f_T(\varepsilon_i) = \langle \gamma_i^\dagger \gamma_i \rangle_T = 1 / \left[e^{\varepsilon_i/T} + 1 \right]$ is just the Fermi distribution function for non-interacting Fermions, and the exponential in time results from

$$\langle \ldots \gamma_j(t) \rangle_T =$$

$$\langle \ldots e^{iH_M t} \gamma_j e^{-iH_M t} \rangle_T =$$

$$e^{-i\varepsilon_j t} \langle \ldots \gamma_j \rangle_T .$$

For our adiabatically turned on $\vec{A}$ field of frequency ω

$$\vec{A}(y) = \vec{A}(\vec{y})\, e^{i(\omega-i\delta)t_y} \quad , \quad \delta \to 0+ \tag{2.15}$$

which allows us in each term to trivially perform the time integration

$$\int_{-\infty}^{t_x} dt_y\, e^{iE_{\ell m}(t_x-t_y)} e^{i(\omega-i\delta)t_y} = \frac{\frac{1}{i} e^{i\omega t_x}}{-E_{\ell m}+\omega-i\delta} \tag{2.16}$$

So all terms in $\vec{j}(x) \sim \vec{j}(\vec{x}) e^{i\omega t}$ x .

Proceeding in the above manner for all terms in eqn. (2.12), doing the trivial sums over δ functions, then puts the current density $\vec{j}(\vec{x})$ into the following form:

$$\vec{j}(\vec{x}) = \vec{j}_1(\vec{x}) - \frac{Ne^2}{m} \vec{A}(\vec{x}) \tag{2.17}$$

where

$$\vec{j}_1(\vec{x}) = \frac{e^2}{2m^2} \left(\nabla_{\vec{x}} - \nabla_{\vec{x}'} \right)_{\vec{x}' \to \vec{x}} \int d\vec{y}\, \vec{A}(\vec{y}) \cdot \left(\nabla_{\vec{y}} - \nabla_{\vec{y}'} \right)_{\vec{y}' \to \vec{y}} \sum_{\ell m}$$

$$\left\{ \left[f_T(\epsilon_\ell) - f_T(\epsilon_m) \right] \right.$$

$$\left[\frac{u_\ell(\vec{x}) u_m^*(\vec{x}') u_m(\vec{y}) u_\ell^*(\vec{y}') - v_m^*(\vec{x}) v_\ell(\vec{x}') u_m(\vec{y}) u_\ell^*(\vec{y}')}{\epsilon_\ell - \epsilon_m + \omega - i\delta} \right.$$

$$\left. + \frac{u_m(\vec{x}) u_\ell^*(\vec{x}') v_m^*(\vec{y}) v_\ell(\vec{y}') - v_\ell^*(\vec{x}) v_m(\vec{x}') v_m^*(\vec{y}) v_\ell(\vec{x}')}{-(\epsilon_\ell - \epsilon_m) + \omega - i\delta} \right]$$

$$+ \left[1 - f_T(\epsilon_\ell) - f_T(\epsilon_m) \right]$$

$$\left[- \frac{u_m(\vec{x}) v_\ell(\vec{x}') v_m^*(\vec{y}) u_\ell^*(\vec{y}') + u_\ell(\vec{x}) v_m(\vec{x}') v_m^*(\vec{y}) u_\ell^*(\vec{y}')}{\epsilon_\ell + \epsilon_m + \omega - i\delta} \right.$$

$$\left. \left. + \frac{v_m^*(\vec{x}) u_\ell^*(\vec{x}') u_m(\vec{y}) v_\ell(\vec{y}') + v_\ell^*(\vec{x}) u_m^*(\vec{x}') u_m(\vec{y}) v_\ell(\vec{y}')}{-(\epsilon_\ell + \epsilon_m) + \omega - i\delta} \right] \right\} \tag{2.18}$$

Eqn. (2.18) is quite general and can be applied to numerous physical situations. If the u, v functions appropriate for an infinite pure superconductor (no static magnetic or electric fields present) are substituted in eqn. (2.18) and the Fourier transform of $\vec{j}(\vec{x})$ is taken, we obtain the same expression as Abrikosov et.al.[13] who calculated the surface impedance for superconductors. This must be the case, of course. We will refer quite frequently to this paper since as all perturbations are shut off, all results must tend to their case.

We proceed now by considering the geometry in Figure 2.1. A metal surface lies in the y-z plane. It is infinite in y and z, and semi-infinite in the +x-direction. Note that there are two sets of fields to consider - those from the static magnetic fields to be applied to the surface and those from the high frequency weak radiation impingent upon the surface.

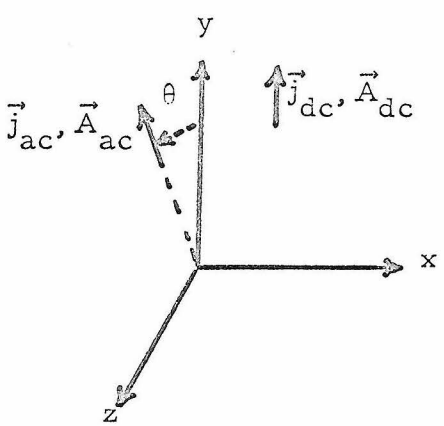


FIGURE 2.1

GEOMETRY OF FIELDS

The vector potential, $\vec{A}_{dc}$, due to the $\vec{H}_{dc}$ defines the y-axis. The vector potential, $\vec{A}_{ac}$, due to the radiation is linearly polarized in the y-z plane and makes an angle θ with respect to the y-axis. Each $\vec{A}$ field has associated with it a current $\vec{j}$ in the metal and these "screening" currents are parallel to their respective $\vec{A}$ field. It is to be noted here that we are treating $\vec{A}_{ac}$ and $\vec{A}_{dc}$ separately. The present chapter considers $\vec{A}_{ac}$ as a small perturbation on the system and the current versus field relations are $\vec{j}_{ac}$ as a function of $\vec{A}_{ac}$. $\vec{A}_{dc}$ is not small and its effects are included in the u, v functions directly through Bogolubov's equations which are solved in Chapter 3.

Getting back to further processing of the $\vec{j}$ versus $\vec{A}$ relation, we write

$$\vec{A}_{ac}(\vec{x}) = (\vec{e}_y \cos\theta + \vec{e}_z \sin\theta)\, A_{ac}(x) \tag{2.19}$$

where $\vec{e}_{y,z}$ are unit vectors in the y or z direction and x refers to the coordinate on the x-axis. (There will no longer be any reference to $x = (\vec{x}, t)$ as a four-vector.) Note that $A_{ac}(x)$ falls off only with x; we have an infinite plane wave impinging upon the surface.

It will be shown later that u, v must be of the form

$$u(\vec{x}) = u(x)e^{i(K_y y + K_z z)} \tag{2.20a}$$

$$v(\vec{x}) = v(x)e^{i(K_y y + K_z z)} \tag{2.20b}$$

If eqns. (2.20a, b) and (2.19) are inserted in eqn. (2.18), the differentiations $\vec{A}\cdot\nabla$ performed, limits taken, integrations performed over the pure exponentials (giving 2π δ-functions), and some simple

rearrangements made depending on ℓ, m being dummy indices, we can obtain the following result:

$$j_1(x) = \int_0^\infty Q_1(x, x')A_{ac}(x')dx' \tag{2.21a}$$

$$Q_1(x, x') = \frac{2e^2}{m^2}(2\pi)^2 \sum_{\ell, m} (K_y^\ell \cos\theta + K_z^\ell \sin\theta)^2$$

$$\Big\{ [f_T(\varepsilon_\ell) - f_T(\varepsilon_m)]\, \delta(K_y^\ell - K_y^m)\, \delta(K_z^\ell - K_z^m)$$

$$\frac{[u_\ell(x)u_m^*(x) + v_\ell(x)v_m^*(x)]\,[u_m(x')u_\ell^*(x') + v_m(x')v_\ell^*(x')]}{\varepsilon_m - \varepsilon_\ell - \omega + i\delta}$$

$$+\tfrac{1}{2}[1 - f_T(\varepsilon_\ell) - f_T(\varepsilon_m)]\, \delta(K_y^\ell + K_y^m)\, \delta(K_z^\ell + K_z^m)$$

$$\Bigg[\frac{[u_\ell(x)v_m(x) - u_m(x)v_\ell(x)]\,[u_\ell^*(x')v_m^*(x') - u_m^*(x')v_\ell^*(x')]}{\varepsilon_m + \varepsilon_\ell + \omega - i\delta}$$

$$+ \frac{[u_m^*(x)v_\ell^*(x) - u_\ell^*(x)v_m^*(x)]\,[u_m(x')v_\ell(x') - u_\ell(x')v_m(x')]}{\varepsilon_m + \varepsilon_\ell - \omega + i\delta} \Bigg] \Big\} \tag{2.21b}$$

$$\vec{j}_1(x) = (\vec{e}_y \cos\theta + \vec{e}_z \sin\theta)\, j_1(x) \tag{2.21c}$$

The last equation says that the current flows in the same direction as the A field's linear polarization.

Eqn. (2.21) for the j versus A relation is now in a form amenable to physical interpretation. Two distinct physical processes are represented: The A field causes thermally excited quasi-

particles to make transitions from one state to another state. The A field causes the creation or destruction of a pair of excitations; this pair is made up of a quasi-particle and an anti-quasi-particle. (These processes are directly analogous to electron-position theory in quantum electrodynamics; the rest mass energy mc^2 plays a role very similar to the superconducting energy gap, Δ.)

Consider first the scattering process. Since the incident photon has momentum only normal to the surface and none in the plane of the surface, only the x-component of a quasi-particle's momentum can change in a transition. So the y and z- component remain the same. This is expressed through the δ-functions $\delta(K_y^{\ell} - K_y^{m})$, $\delta(K_z^{\ell} - K_z^{m})$. If the transition is from a state of lower energy to one of higher energy, there must be some non-zero probability for the lower state to contain a quasi-particle and some non-zero probability for the upper state to have an empty slot available. The net current, though, is proportional to the number of upward transitions minus the number of downward transitions. This fact is expressed through the Fermi factor difference $f_T(\epsilon_{\ell}) - f_T(\epsilon_m)$. (Recall that if $f_T(\epsilon)$ is the probability for ϵ to be occupied, then $1 - f_T(\epsilon)$ is the probability for ϵ to be empty.) Now suppose the incident photon annihilates at x'. There is an amplitude for it to scatter a quasi-particle from state ℓ to state m. But since a state has amplitude u to be particle-like and amplitude v to be anti-particle-like, we must consider the quantum mechanical interference and add the amplitudes of the processes: Particle to particle $u_m(x')u_{\ell}^*(x')$ plus

anti-particle to anti-particle $v_m(x')v_\ell^*(x')$ transitions. Similarly, a current found at x generated by the transition from state ℓ to state m could be due either to a particle-particle transition or an anti-particle - anti-particle transition. Hence the factor $u_\ell(x)u_m^*(x) + v_\ell(x)v_m^*(x)$. The factor $1/[\varepsilon_m - \varepsilon_\ell - \omega + i\delta]$ splits into a real part $P(1/[\varepsilon_m - \varepsilon_\ell - \omega])$ and an imaginary part $-i\pi\delta(\varepsilon_m - \varepsilon_\ell - \omega)$ via a familiar identity. It will be seen later that the imaginary part is related to power absorbed by the metal in which case energy must be conserved in the scattering process. This is expressed via the δ-function part. The real part which does not conserve energy thus considers only virtual processes and these are related to the diamagnetic or screening properties of a material. We discuss this later after developing more formalism.

Next consider the pair processes. Once again transverse momentum must be conserved. If initially there is no pair, transverse momentum is zero; after a pair is created, there is total y-momentum $K_y^\ell + K_y^m$ for the excitations put in states ℓ, m. Thus there are the factors $\delta(K_y^\ell + K_y^m)$, $\delta(K_z^\ell + K_z^m)$. In order to send a pair of excitations into states ℓ, m, these states must be empty or if a pair is to be annihilated from ℓ, m, there must be a pair sitting in ℓ, m. Hence the factor $[1 - f_T(\varepsilon_\ell) - f_T(\varepsilon_m)]$. This includes up minus down transitions, as before. The energy denominators express conservation of energy for absorption processes and yield virtual processes for non-energy conserving processes. Note that since ε_m, ε_ℓ, and ω are positive, always, the term involving $\delta(\varepsilon_m + \varepsilon_\ell + \omega)$ never contributes

to the absorption. In fact, this term more nearly corresponds to the destruction of a pair creating a photon and hence depleting current in the material. A photon annihilating at x^i can put a particle in m and an anti-particle in ℓ or a particle in ℓ and an anti-particle in m. The amplitude for this is $u_m(x^i)\ v_\ell(x^i) - u_\ell(x^i)v_m(x^i)$. And the explanation of the other matrix elements is obvious.

The factor $(K_y^\ell \cos\theta + K_z^\ell \sin\theta)^2$ is the coupling between the photon's polarization and an absorbing quasi-particle's transverse motion. A quasi-particle has the strongest interaction when it is moving exactly parallel or anti-parallel to the direction of linear polarization.

CHAPTER 3

SOLUTIONS OF BOGOLUBOV'S EQUATIONS IN A LARGE STATIC MAGNETIC FIELD

Further evaluation of the j_{ac} versus A_{ac} relation in Chapter 2 requires explicit knowledge of the u, v functions satisfying Bogolubov's equations with large static magnetic fields, $H_{dc} \sim H_c$, the superconductor's critical field. Once again we consider the geometry of Figure 2.1. If $\vec{H}_{dc}$ is along the z-axis, then $\vec{A}_{dc}$ is along the y-axis since $\vec{H} = \nabla \times \vec{A}$ or for our geometry $\vec{H}_{dc} = \vec{e}_z \, \partial A_y(x)/\partial x$.

We are to solve Bogolubov's equations

$$\epsilon_n u_n(\vec{r}) = \left[\frac{1}{2m} \left(-i\nabla - e\vec{A}_{dc}(\vec{r}) \right)^2 - E_f \right] u_n(\vec{r}) + \Delta(\vec{r})\, v_n(\vec{r}) \tag{3.1a}$$

$$\epsilon_n v_n(\vec{r}) = -\left[\frac{1}{2m} \left(i\nabla - e\vec{A}_{dc}(\vec{r}) \right)^2 - E_f \right] v_n(\vec{r}) + \Delta^*(\vec{r})\, u_n(\vec{r}) \tag{3.1b}$$

$$\Delta(\vec{r}) = V_{BCS} \sum_n v_n^*(\vec{r})\, u_n(\vec{r}) \left[1 - 2 f_T(\epsilon_n) \right] \tag{3.1c}$$

for the geometry of Figure 2.1.

Outside the metal ($x < 0$) all the fields are uniform; inside the fields are approximately $e^{-x/\delta}$ or $e^{-x/\lambda}$ where δ is called the ac penetration depth and λ is the static field penetration depth. Although these penetration depths have been calculated only for weak fields it is an experimental fact that the penetration depth is independent of field intensity for field strengths right up to the critical field, H_c.

Thus for $x \geq 0$

$$\vec{H}_{dc} = \vec{e}_z H_{dc} \tag{3.2}$$

$$H_{dc} = He^{-x/\lambda} \tag{3.3}$$

which in the $\nabla \cdot \vec{A} = 0$ or "transverse gauge" is derivable from the vector potential

$$\vec{A}_{dc} = \vec{e}_y A_{dc} \tag{3.4a}$$

$$A_{dc} = -H\lambda e^{-x/\lambda} \tag{3.4b}$$

using $\vec{H} = \nabla \times \vec{A}$.

Since the metal is infinite in y and z, $|u|$, $|v|$, $|\Delta|$ must be constant for translational symmetry. Then we try solutions of the form

$$u(\vec{r}) = u(x)e^{i(K_y y + K_z z)} \tag{3.5a}$$

$$v(\vec{r}) = v(x)e^{i(K_y y + K_z z)} \tag{3.5b}$$

which by eqn. (3.1c) is consistent with a gap function Δ varying only with x.

$$\Delta(\vec{r}) = \Delta(x) \tag{3.5c}$$

Using this converts eqns. (3.1a, b, c) to

$$\left[\varepsilon_n - \xi_t + \frac{1}{2m}\frac{d^2}{dx^2} - V(x)\right] u_n(x) - \Delta(x)v_n(x) = 0 \tag{3.6a}$$

$$\left[\varepsilon_n + \xi_t - \frac{1}{2m}\frac{d^2}{dx^2} - V(x)\right] v_n(x) - \Delta(x)u_n(x) = 0 \tag{3.6b}$$

$$\Delta(x) = V_{BCS} \sum_n v_n^*(x)u_n(x)\left[1 - 2f_T(\varepsilon_n)\right] \tag{3.6c}$$

where $\xi_t \equiv \frac{K_\rho^2}{2m} - \frac{K_f^2}{2m}$

$$V(x) = \frac{eH\lambda}{m} K_y e^{-x/\lambda}$$

$$K_y = K_\rho \cos\theta$$

$$K_z = K_\rho \sin\theta$$

K_f is the Fermi momentum.

K_ρ is the transverse momentum.

$E_f = K_f^2/2m$ so that the model here seems to require a spherical Fermi surface and thus ignores crystal anisotropy. Actually this is not really right; the situation is better as far as surface impedance calculations are concerned. We will elaborate on this point when we discuss the surface impedance calculation for the anomalous skin effect.

We have ignored the A_{dc}^2 term since $eA_{dc} \ll K_y$ for all $K_y \geq 10^{-4}K_f$ and the K_y of interest in all subsequent work is $K_y \cong K_f$.

The quantity designated by V(x) is so labelled because it acts like a potential barrier or well depending on the polarity of K_y. Since λ is pure real for static fields, V(x) is always pure real.

It would now be very helpful if we somehow knew in advance the spatial dependence of the gap function, $\Delta(x)$. For an infinite superconductor considered by BCS, $\Delta(x)$ was a constant making solution of Bogolubov's equations trivial. But in the presence of a large static magnetic field we can no longer be so sure that $\Delta(x)$

is still constant.

Bogolubov's equations are sufficiently complicated that we cannot solve for u and v while simultaneously satisfying the self-consistency relation on Δ, in the presence of large static magnetic fields. Therefore, we shall make a guess at what the final self-consistent Δ might be, solve Bogolubov's equations for the u, v functions, and check through the self-consistency relation just how good was our guess. (This latter point will be considered in Chapter 8.) We shall say that Δ is independent of H_{dc} and spatially constant.

Even with the assumption of a constant gap, Bogolubov's equations are still unmanageable. The trouble lies with V(x). Any step function approximation to the exponential in V(x) will lead to a set of coupled $2^{\underline{nd}}$ order differential equations with constant coefficients and that can be handled reasonably. We shall content ourselves, first, with a single step and thus make the replacement

$$V(x) \rightarrow V_e(x) = \begin{cases} V_e & x \leq \lambda_e \\ 0 & x > \lambda_e \end{cases} \qquad (3.7)$$

V_e, the strength of the effective potential, and λ_e, the extent of the effective potential, are the only parameters in this theory. We shall pick $\lambda_e = 2\lambda$, $0 \leq V_e \leq \Delta$, $V_e \sim HK_y$; this is not unique but seems reasonable to us. If the results are suspect, a many-step approximation to the exponential can be tried; it will not be so sensitive to parameter choices.

In order to solve ordinary constant coefficient differential

equations, we must form solutions in each region and match values and appropriate derivatives across the boundaries, at the $x = 0$ surface, and at the $x = L >>> \lambda$ surface. Take first the problem at the $x = 0$ surface.

At the surface of the metal, the current normal to the surface must vanish. But $j_x \sim u\partial u/\partial x + v\partial v/\partial x$ so either u and v both vanish at $x = 0$, or $\partial u/\partial x$ and $\partial v/\partial x$ vanish there, or both do. To pick the proper set, consider the electron density at the surface, N. $N \sim |u|^2 + |v|^2$. Since the electrons are principally confined to within an angstrom or so of the surface, we imagine our system confined by a large potential barrier at $x = 0$, whence $N \to 0 \Rightarrow u, v \to 0$ at $x = 0$. We thus consider the boundary conditions n, v= 0 while $\partial u/\partial x$, $\partial v/\partial x \neq 0$.

It is to be noted that imposing these boundary conditions corresponds to a situation known as specular reflection. We discuss the physical significance of this in Chapter 7.

Before solving the differential eqns. (3.6) it is most convenient to renormalize the variables to dimensionless form as follows:

$$\varepsilon \to {}^{\varepsilon}/_{\Delta} \to \varepsilon$$

$$\xi_t \to {}^{\xi_t}/_{\Delta} \to \xi_t$$

$$V \to {}^{V}/_{\Delta} \to V$$

$$x \to K_f x \to x$$

Note that the dimensionless variables are denoted by the same letter as before.

Then eqns. (3.6) read in this normalized form

$$\left[\varepsilon - \xi_t + \frac{E_f}{\Delta}\frac{d^2}{dx^2} - V_e(x)\right] u(x) - v(x) = 0 \tag{3.8a}$$

$$\left[\varepsilon + \xi_t - \frac{E_f}{\Delta}\frac{d^2}{dx^2} - V_e(x)\right] v(x) - u(x) = 0 \tag{3.8b}$$

where we have also dropped the subscript labelling the states temporarily.

Denote the solutions of eqns. (3.8a, b) in the surface region where the effective potential is non-zero generally by the subscript 1 and the solutions in the metal's interior by the subscript 2. Solutions in each region are of the form e^{iqx}. Then straightforward substitution in eqns. (3.8a, b) yields the following general solutions:

$$u_1(x) = A_1\,(e^{ir_+x} - e^{-ir_+x}) + C_1(e^{ir_-x} - e^{-ir_-x}) \tag{3.9a}$$

$$v_1(x) = A_1\,(\varepsilon - \xi_r - V_e)\,(e^{ir_+x} - e^{-ir_+x}) + C_1\,(\varepsilon + \xi_r - V_e)\,(e^{ir_-x} - e^{-ir_-x}) \tag{3.9b}$$

$$u_2(x) = A_2\,e^{ip_+x} + B_2\,e^{-ip_+x} + C_2\,e^{ip_-x} + D_2\,e^{-ip_-x} \tag{3.9c}$$

$$v_2(x) = A_2(\varepsilon - \xi_p)e^{ip_+x} + B_2\,(\varepsilon - \xi_p)\,e^{-ip_+x} + C_2(\varepsilon + \xi_p)\,e^{ip_-x} + D_2(\varepsilon + \xi_p)e^{-ip_-x} \tag{3.9d}$$

where

$$r_\pm = \left\{\frac{\Delta}{E_f}\left[-\xi_t \pm \xi_r\right]\right\}^{\frac{1}{2}} \tag{3.10}$$

$$p_\pm = \left\{\frac{\Delta}{E_f}\left[-\xi_t \pm \xi_p\right]\right\}^{\frac{1}{2}} \tag{3.11}$$

$$\xi_r = \left[(\epsilon - V_e)^2 - 1 \right]^{\frac{1}{2}} \tag{3.12}$$

$$\xi_p = (\epsilon^2 - 1)^{\frac{1}{2}} \tag{3.13}$$

A_1, C_1, A_2, B_2, C_2, and D_2 are constants to be fixed later by imposing boundary conditions on u, v and normalization

$$\int_0^{L>>>\lambda} \left[|u(x)|^2 + |v(x)|^2 \right] dx = 1 \tag{3.14}$$

We have picked the solutions so that the boundary condition at $x = 0$ is automatically satisfied.

Matching the u, v solutions and their derivatives at $x = \lambda_e$ yields four constraints on the coefficients A_1, C_1, A_2, B_2, C_2, D_2.

$$A_1 \left(e^{ir_+\lambda_e} - e^{-ir_+\lambda_e} \right) + C_1 \left(e^{ir_-\lambda_e} - e^{-ir_-\lambda_e} \right) =$$

$$A_2 e^{ip_+\lambda_e} + B_2 e^{-ip_+\lambda_e} + C_2 e^{ip_-\lambda_e} + D_2 e^{-ip_-\lambda_e} \tag{3.15a}$$

$$A_1 r_+ \left(e^{ir_+\lambda_e} + e^{-ir_+\lambda_e} \right) + C_1 r_- \left(e^{ir_-\lambda_e} + e^{-ir_-\lambda_e} \right) =$$

$$A_2 p_+ e^{ip_+\lambda_e} - B_2 p_+ e^{-ip_+\lambda_e} + C_2 p_- e^{ip_-\lambda_e} - D_2 p_- e^{-ip_-\lambda_e} \tag{3.15b}$$

$$A_1 (\epsilon - \xi_r - V_e) \left(e^{ir_+\lambda_e} - e^{-ir_+\lambda_e} \right) + C_1 (\epsilon + \xi_r - V_e) \left(e^{ir_-\lambda_e} - e^{-ir_-\lambda_e} \right) =$$

$$A_2 (\epsilon - \xi_p) e^{ip_+\lambda_e} + B_2 (\epsilon - \xi_p) e^{-ip_+\lambda_e} + C_2 (\epsilon + \xi_p) e^{ip_-\lambda_e} + D_2 (\epsilon + \xi_p) e^{-ip_-\lambda_e} \tag{3.15c}$$

$$A_1(\varepsilon-\xi_r-V_e)r_+\left(e^{ir_+\lambda_e}+e^{-ir_+\lambda_e}\right)+C_1(\varepsilon+\xi_r-V_e)r_-\left(e^{ir_-\lambda_e}+e^{-ir_-\lambda_e}\right)=$$

$$A_2(\varepsilon-\xi_p)p_+e^{ip_+\lambda_e}-B_2(\varepsilon-\xi_p)p_+e^{-ip_+\lambda_e}+C_2(\varepsilon+\xi_p)p_-e^{ip_-\lambda_e}-$$

$$-D_2(\varepsilon+\xi_p)e^{-ip_-\lambda_e} \qquad (3.15d)$$

Using the states given by (3.9) in the normalization equation (3.14) yields the following:

$$\text{Thus } \int_o^{\lambda_e}\left[|u_1(x)|^2+|v_1(x)|^2\right]dx+\int_{\lambda_e}^{\infty}\left[|u_2(x)|^2+|v_2(x)|^2\right]dx=1 \Rightarrow$$

$$i=|A_1|^2\left[1+(\varepsilon-\xi_r-V_e)(\varepsilon-\xi_r^*-V_e)\right]$$

$$\left\{\frac{e^{i(r_+-r_+^*)\lambda_e}-e^{-i(r_+-r_+^*)\lambda_e}}{r_+-r_+^*}-\frac{e^{i(r_++r_+^*)\lambda_e}-e^{-i(r_++r_+^*)\lambda_e}}{r_++r_+^*}\right\}$$

$$+|C_1|^2\left[1+(\varepsilon+\xi_r-V_e)(\varepsilon+\xi_r^*-V_e)\right]$$

$$\left\{\frac{e^{i(r_--r_-^*)\lambda_e}-e^{-i(r_--r_-^*)\lambda_e}}{r_--r_-^*}-\frac{e^{i(r_-+r_-^*)\lambda_e}-e^{-i(r_-+r_-^*)\lambda_e}}{r_-+r_-^*}\right\}$$

$$+A_1C_1^*\left[1+(\varepsilon-\xi_r-V_e)(\varepsilon+\xi_r^*-V_e)\right]$$

$$\left\{\frac{e^{i(r_+-r_-^*)\lambda_e}-e^{-i(r_+-r_-^*)\lambda_e}}{r_+-r_-^*}-\frac{e^{i(r_++r_-^*)\lambda_e}-e^{-i(r_++r_-^*)\lambda_e}}{r_++r_-^*}\right\}$$

$$+A_1^* C_1 \left[1+(\varepsilon-\xi_r^*-V_e)\,(\varepsilon+\xi_r-V_e)\right]$$

$$\left\{ \frac{e^{i(r_+^*-r_-)\lambda_e} - e^{-i(r_+^*-r_-)\lambda_e}}{r_+^*-r_-} - \frac{e^{i(r_+^*+r_-)\lambda_e} - e^{-i(r_+^*+r_-)\lambda_e}}{r_+-r_-^*} \right\}$$

$$+|A_2|^2 \left[1+(\varepsilon-\xi_p)\,(\varepsilon-\xi_p^*)\right] \frac{e^{i(p_+-p_+^*)L} - e^{i(p_+-p_+^*)\lambda_e}}{p_+-p_+^*}$$

$$+A_2 B_2^* \left[1+(\varepsilon-\xi_p)\,(\varepsilon-\xi_p^*)\right] \frac{e^{i(p_++p_+^*)L} - e^{i(p_++p_+^*)\lambda_e}}{p_++p_+^*}$$

$$+A_2 C_2^* \left[1+(\varepsilon-\xi_p)\,(\varepsilon+\xi_p^*)\right] \frac{e^{i(p_+-p_-^*)L} - e^{i(p_+-p_-^*)\lambda_e}}{p_+-p_-^*}$$

$$+A_2 D_2^* \left[1+(\varepsilon-\xi_p)\,(\varepsilon+\xi_p^*)\right] \frac{e^{i(p_++p_-^*)L} - e^{i(p_++p_-^*)\lambda_e}}{p_++p_-^*}$$

$$+B_2 A_2^* \left[1+(\varepsilon-\xi_p)\,(\varepsilon-\xi_p^*)\right] \frac{e^{-i(p_++p_+^*)L} - e^{-i(p_++p_+^*)\lambda_e}}{-p_+-p_+^*}$$

$$+|B_2|^2 \left[1+(\varepsilon-\xi_p)\,(\varepsilon-\xi_p^*)\right] \frac{e^{i(p_+^*-p_+)L} - e^{i(p_+^*-p_+)\lambda_e}}{p_+^*-p_+}$$

$$+B_2 C_2^* \left[1+(\varepsilon-\xi_p)\,(\varepsilon+\xi_p^*)\right] \frac{e^{-i(p_++p_-^*)L} - e^{-i(p_++p_-^*)\lambda_e}}{-p_+-p_-^*}$$

$$+B_2 D_2^* \left[1+(\epsilon-\xi_p)(\epsilon+\xi_p^*)\right] \frac{e^{i(p_-^*-p_+)L} - e^{i(p_-^*-p_+)\lambda_e}}{p_-^*-p_+}$$

$$+C_2 A_2^* \left[1+(\epsilon+\xi_p)(\epsilon-\xi_p^*)\right] \frac{e^{i(p_--p_+^*)L} - e^{i(p_--p_+^*)\lambda_e}}{p_--p_+^*}$$

$$+C_2 B_2^* \left[1+(\epsilon+\xi_p)(\epsilon-\xi_p^*)\right] \frac{e^{i(p_-+p_+^*)L} - e^{i(p_-+p_+^*)\lambda_e}}{p_-+p_+^*}$$

$$+|C_2|^2 \left[1+(\epsilon+\xi_p)(\epsilon+\xi_p^*)\right] \frac{e^{i(p_--p_-^*)L} - e^{i(p_--p_-^*)\lambda_e}}{p_--p_-^*}$$

$$+C_2 D_2^* \left[1+(\epsilon+\xi_p)(\epsilon+\xi_p^*)\right] \frac{e^{i(p_-+p_-^*)L} - e^{i(p_-+p_-^*)\lambda_e}}{p_-+p_-^*}$$

$$+D_2 A_2^* \left[1+(\epsilon+\xi_p)(\epsilon-\xi_p^*)\right] \frac{e^{-i(p_-+p_+^*)L} - e^{-i(p_-+p_+^*)\lambda_e}}{-p_--p_+^*}$$

$$+D_2 B_2^* \left[1+(\epsilon+\xi_p)(\epsilon-\xi_p^*)\right] \frac{e^{i(p_+^*-p_-)L} - e^{i(p_+^*-p_-)\lambda_e}}{p_+^*-p_-}$$

$$+D_2 C_2^* \left[1+(\epsilon+\xi_p)(\epsilon+\xi_p)\right] \frac{e^{-i(p_-+p_-^*)L} - e^{-i(p_-+p_-^*)\lambda_e}}{-p_--p_-^*}$$

$$+|D_2|^2 \left[1+(\epsilon+\xi_p)(\epsilon+\xi_p^*)\right] \frac{e^{i(p_-^*-p_-)L} - e^{i(p_-^*-p_-)\lambda_e}}{p_-^*-p_-} \qquad (3.16)$$

In general, not all of the constants A_2, B_2, C_2, D_2 can be simultaneously non-zero. This arises upon considering the u, v solutions (3.9) for $x \to \infty$. If p_+ and/or p_- has a non-zero imaginary part, the coefficient of the appropriate term must be set to zero for the u, v solutions to be bounded as $x \to \infty$.

Regarding ϵ (the excitation energy), ξ_t (a measure of transverse momentum), and V_e (the effective potential strength) as the independent state naming variables, we consider the consequences of the following possibilities for

$$p_\pm = \left\{ \frac{\Delta}{E_f} \left[-\xi_t \pm (\epsilon^2 - 1)^{\frac{1}{2}} \right] \right\}^{\frac{1}{2}} :$$

(1) $0 \le \epsilon < 1$, any ξ_t and $\epsilon \ge 1$, $-\xi_t + (\epsilon^2 - 1)^{\frac{1}{2}} < 0$

For this case both p_+ and p_- have imaginary parts so that $B_2 = D_2 = 0$ necessarily. But then eqns. (3.15) are inconsistent unless the determinant of the coefficient vanishes. If ξ_t and V_e are fixed, there exist solutions only for certain discrete values of ϵ given by the vanishing of the determinant here.

Since $p_\pm$ are complex, these states are localized near the surface as determined by $e^{ip_\pm x}$ in u, v. Generally the distances here are $\sim 5000\text{Å}$ into the sample. BCS used infinite metals, so, of course, they could not obtain any solutions where p_+ or p_- had any imaginary part. Furthermore, these surface states exist for excitation energies less than the gap width, Δ, a fact which will have important consequences for absorption at "low temperatures" ($kT << \Delta$).

Eqns. (3.15) are sufficiently complicated that results can only be obtained numerically. We have examined, therefore, many cases numerically using a computer and find a few general features to be described below.

Firstly, surface states exist only for $V_e < 0$ and only for $\varepsilon > 1 - |V_e|$; i.e., V_e must appear to be a potential well and the possible states lie above the bottom of the well. The preceding is true irrespective of how V_e and λ_e are chosen.

Secondly, surface states exist only for $-20 \leqslant \xi_t \leqslant 1$ which depends on λ_e. The small values of ξ_t indicate a particular direction of quasi-particle travel to be discussed later.

The spectrum used in further calculations is shown in Figures 3.1 below.

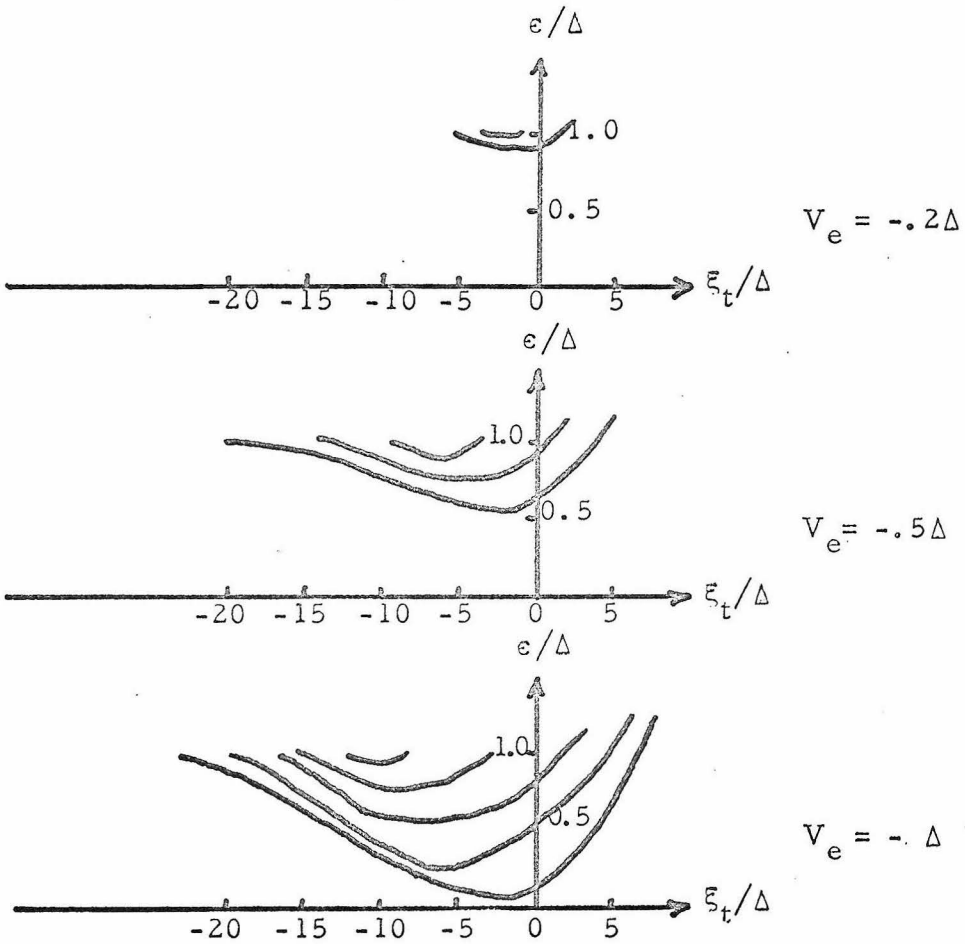


FIGURE 3.1

Typical Surface State Spectrum

$\lambda_e = 2\lambda = 1000\text{Å}$

$\Delta/E_f = 10^{-4}$

On each contour line $p_\pm$ vary continuously with ξ_t.

It might be thought that these surface states depend critically on the surface at $x = 0$ being perfectly flat and smooth. However, as long as the surface irregularities are small compared to a penetration depth of the H_{dc}, this is not so. For then one could make wave packets of sinusoids which would match the boundary conditions, out of sinusoids whose momenta were all peaked sharply about the correct value for a flat surface. In such a case, all the surface state energies would still be nearly the same as before. Thus small scale ($\lesssim 10\mathring{A}$) irregularities are no problem.

(2) $\quad -\xi_t \pm (\varepsilon^2 - 1) > 0, \ \varepsilon \geq 1$

In this realm both p_+ and p_- are pure real and all the constants A_2, B_2, C_2, D_2 can be non-zero, which leaves a dilemma. There are more unknown constants than constraining equations. Two approaches are possible to resolve the problem. We can put the system in a large box of length L_x and require u, v to vanish at $x = L_x$ just as we imposed for u, v at $x = 0$ surface. This yields a situation like the prior case - solutions exist only if the determinant of the coefficients vanishes. However, since L_x is very large these states are very closely spaced in energy and for $L_x \to \infty$ form a continuum. All this is well defined but extremely difficult to handle numerically. Clearly the results should be essentially independent of the size of the box for $L_x >>> \lambda$ and yet in solving for the u, v functions and the locations of the states in energy slight changes in L_x radically affect the numbers involved through fast oscillating terms $\sim e^{ip_\pm L_x}$.

Fortunately another approach is possible that is more easily implemented. We imagine a scattering process. If e^{ipx} represents a quasi-particle incident upon the $x = 0$ surface from deep inside the metal in the state with x-component momentum p, then e^{-ipx} is the reflected quasi-particle; the transverse momentum is conserved and the momentum normal to the surface reverses sign upon collision. Thus it is natural to consider two cases shown below in Figures 3.2a, b.

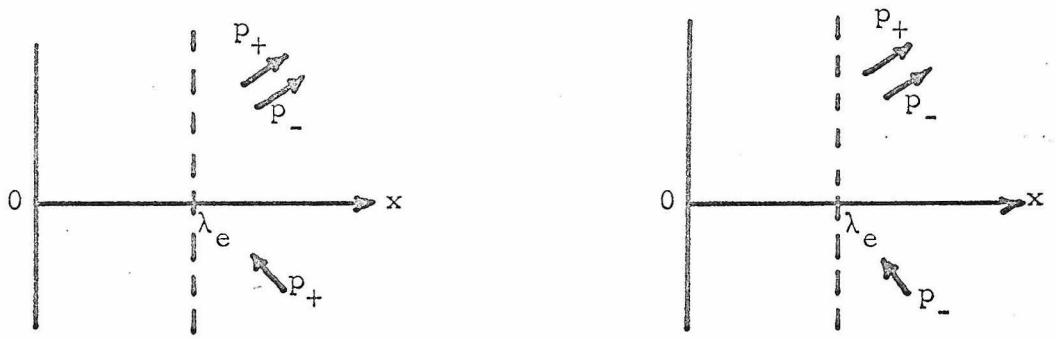


FIGURE 3.2 a, b

SCHEMATIC OF SCATTERING STATES

A quasi-particle can be incident upon the surface in either state p_+ or p_-. By a change of variables to spherical momentum coordinates it can be shown that p_+ states lie <u>above</u> the Fermi momentum while p_- states lie <u>below</u> the Fermi momentum. For each case, there is some amplitude for the quasi-particle to be reflected in its original state with only the direction altered and some amplitude to find the quasi-particle in the other momentum state also with altered direction. This is a new feature of a superconductor when $H_{dc} \neq 0$. In BCS theory

where $H_{dc} = 0$ a quasi-particle is always reflected in the same state as it was incident; no mixing of p_+ and p_- states occurs. Regarding ξ_t, ϵ, and V_e as the independent variables of a state, there are two degenerate solutions which we will refer to as "p_+ incident" and "p_- incident" continuum states. In the former $D_2 = 0$ and in the latter $B_2 = 0$.

The above picture does not depend upon the type of particle being scattered. They can be either electron-like or hole-like excitations; indeed, our argument applies equally well to both u(x) and v(x) solutions. Holes or anti-particle quasi-particles exist both above and below the Fermi momentum and electrons or particle quasi-particle do so too. This latter fact holds in superconductors whether H_{dc} is zero or not. For normal metals, though, $u \sim e^{\pm ip_- x}$ which says that electron excitations exist only above the Fermi momentum while hole excitations exist only <u>below</u> the Fermi momentum.

For these plane wave like scattering states, the normalization condition (3.16) is particularly simple; in the limit of $L_x \to \infty$ only diagonal terms in region 2 contribute substantially. Thus for p_+ incident states

$$(|A_2|^2 + |B_2|^2)\,[1+(\epsilon-\xi_p)^2] + |C_2|^2\,[1+(\epsilon+\xi_p)^2] = 1/L_x \qquad (3.17a)$$

and for p_- incident states

$$|A_2|^2\,[1+(\epsilon-\xi_p)^2] + (|C_2|^2 + |D_2|^2)\,[1+(\epsilon+\xi_p)^2] = 1/L_x \qquad (3.17b)$$

The sum over states is also simple

$$\sum \rightarrow \frac{L_x}{2\pi} \sum_{+,-} \int_{-\infty}^{\infty} dp\pm \tag{3.18}$$

which will later be transformed to a more useful independent variable.

(3) $\quad \epsilon \geq 1, \; -\xi_t - (\epsilon^2-1)^{\frac{1}{2}} < 0, \; -\xi_t + (\epsilon^2-1)^{\frac{1}{2}} > 0$

In this realm p_+ is pure real so $e^{\pm ip_+x}$ are perfectly acceptable solutions while p_- is complex and hence e^{-ip_-x} must be discarded, so $D_2 = 0$. BCS also could not have considered this case since it depends on a surface being present and a non-zero V_e.

This case is very similar to the p_+ scattering state previously considered except that for p_+ incident, the p_- reflected solution is very strongly localized near the surface. So all can be applied except that the normalization condition simplifies even more than before to

$$(|A_2|^2 + |B_2|^2)\,[1+(\epsilon-\xi_p)^2\,] = 1/L_x \tag{3.19}$$

We can summarize graphically the above results as to which states exist where in Figure 3.4 below.

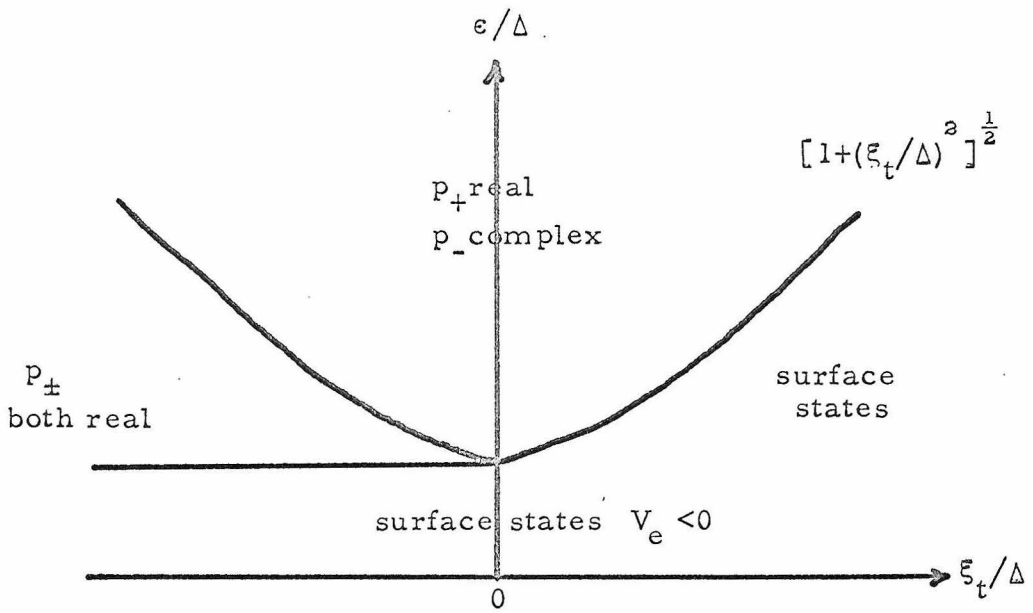


FIGURE 3.4

TYPES OF STATES FOR VARIOUS ξ_t AND ϵ

CHAPTER 4

FORMULAS FOR SURFACE IMPEDANCE

In order to calculate surface impedances, it is necessary to solve Maxwell's equations for the field distributions and then find the complex power dissipated inside the metal. Using the transverse gauge, $\nabla \cdot \vec{A} = 0$, Maxwell's equations read

$$\nabla^2 \vec{A} - \frac{\partial^2 \vec{A}}{\partial t^2} = - 4\pi \vec{j}(\vec{A}) \tag{4.1}$$

$$\vec{B} = \nabla \times \vec{A} \tag{4.2}$$

$$\vec{E} = - \frac{\partial \vec{A}}{\partial t} \tag{4.3}$$

in the vector potential formalism. These equations apply both to the metal and vacuum. Since all fields vary sinusoidally, $A \sim e^{i\omega t}$, eqn. (4.1) can be transformed to

$$\nabla^2 A + \omega^2 \vec{A} = - 4\pi \vec{j}(\vec{A}) \tag{4.4}$$

and eqn. (4.3) to

$$\vec{E} = - i\omega \vec{A} \tag{4.5}$$

Solutions of eqn. (4.4) require the $\vec{j}$ versus $\vec{A}$ relation. In vacuum, $\vec{j} = 0$ everywhere and the $\vec{A}$ field is just a plane traveling wave, $\sim e^{iKx}$. In real metals at room temperature, $\vec{j} = \sigma \vec{E}$; Ohm's law holds and eqn. (4.4) is easily solved. But we are concerned with temperatures near absolute zero in metals and here the $\vec{j}(\vec{A})$ relation is very non-local; the field in a large region determines the

current at some point with this region. We found in Chapter 2 that

$$\vec{j}(\vec{r}) = \int Q(\vec{r},\ \vec{r}')\ \vec{A}(\vec{r}')\, d\vec{r}' \tag{4.6}$$

which means that eqn. (4.4) is an integro differential equation which is difficult to solve generally. One case can be solved in closed form and it is an important result. If the kernel $Q(\vec{r},\ \vec{r}')$ is a function only of the difference $(\vec{r} - \vec{r}')$, then eqn. (4.4) is soluble in closed form. This case corresponds very nearly to the anomalous skin effect in normal metals and superconductors if no magnetic fields are present. We shall consider this in greater detail below.

One further simplification should be mentioned here. The displacement current is negligible in a metal for frequencies of interest since $\omega^2\vec{A} << \nabla^2 \vec{A}$ so that eqns. (4.4) and (4.6) can be combined to read

$$\nabla^2 \vec{A} = -4\pi \int Q(\vec{r},\ \vec{r}')\ \vec{A}(\vec{r}')\, d\vec{r}' \tag{4.7a}$$

or for our planar geometry

$$\frac{\partial^2 A}{\partial_x{}^2} = -4\pi \int_o^\infty Q(x, x')\ A(x')\, dx' \tag{4.7b}$$

Given that solutions of eqn. (4.7) are available, the surface impedance, Z, can then be found from

$$Z = 4\pi\, \frac{\vec{E}(0)}{\int \vec{j}\, d\vec{r}} = -i\,4\pi\omega\ \ \frac{A(0)}{\left.\frac{\partial A}{\partial x}\right|_o} \tag{4.8}$$

The 4π is conventional. $\Big|_o$ means evaluated at $x = 0+$, just inside

the surface.

Although this approach is correct, it is not feasible to calculate the kernel $Q(x, x')$ for our problem. So we develop some other relations useful here.

$$Z^* = i \frac{4\pi\omega}{\left|\frac{\partial A}{\partial x}\right|_o^2} A^*(0) \left.\frac{\partial A}{\partial x}\right|_o = R - iX \tag{4.9}$$

follows trivially from eqn. (4.8).

An integration by parts yields

$$\int_o^\infty A^* \frac{\partial^2 A}{\partial x^2} dx = -A^*(0) \left.\frac{\partial A}{\partial x}\right|_o - \int_o^\infty \left|\frac{\partial A}{\partial x}\right|^2 dx \tag{4.10}$$

which by eqn. (4.7b) is also related to Q.

$$\int_o^\infty A^* \frac{\partial^2 A}{\partial x^2} dx = -4\pi \int_o^\infty Q(x, x') A(x') A^*(x) dx dx' \tag{4.11}$$

Thus

$$Z^* = \frac{i4\pi\omega}{\left|\frac{\partial A}{\partial x}\right|_o^2} \left\{4\pi \int_o^\infty Q A^* A dx dx' - \int_o^\infty \left|\frac{\partial A}{\partial x}\right|^2 dx \right\} \tag{4.12}$$

a form which allows the spatial integrations to be done immediately and Z to be found directly once we know A(x).

We shall now take the following new point of view toward calculating surface impedances. We approximate the A(x) field distribution by an exponential $e^{-x/\delta}$ one parameter, δ, just as for the static field distribution. δ, the ac penetration depth is a

function of H_{dc} and ω. From this assumed form we would change eqn. (4.12) to read

$$Z^* = i4\pi\omega|\delta|^2\left\{4\pi\int_o^\infty e^{-x/\delta^*}e^{-x'/\delta}Q\,dx\,dx' - \int_o^\infty e^{-x/\delta^*}e^{-x/\delta}dx\right\} \tag{4.13}$$

which can now reasonably be done numerically using a computer. But this Z^* must be consistent with the Z^* calculated from eqn. (4.8) which is

$$Z^* = -i4\pi\omega\delta \tag{4.14}$$

Thus it is possible to find the self-consistent δ whence we trivially know the surface impedance.

From eqn. (4.14) we note that surface reactance measurements are in fact a measure of penetration depths

$$X = 4\pi\omega\,\mathrm{Re}(\delta) \tag{4.15}$$

and from eqn. (4.9c) we see that surface resistance measurements determine the amount of power dissipated by the metal.

$$R = 4\pi\omega\,\mathrm{Im}(\delta) \tag{4.16}$$

For superconductors with $T \leq .9T_c$ and $\omega < 2\Delta$, $R << X$ so that $\mathrm{Im}(\delta) << \mathrm{Re}(\delta)$. As a consequence it is really necessary to be concerned with self-consistency only for the surface reactance or penetration depth. Once the correct self-consistent value of $\mathrm{Re}(\delta)$ is known we can calculate directly the surface resistance from eqn. (4.13) using for δ the $\mathrm{Re}(\delta)$. Actually the case for our problem is

even better as far as surface resistance is concerned. The changes in surface resistance due to static magnetic fields are so large ($R_{H_{dc}\neq 0}/R_{H_{dc}=0} > 1$) and the changes in surface reactance sufficiently small (~10%) that using $Re(\delta)$ for $H_{dc} = 0$ is adequate to obtain answers within 20% and of the right sign. This will become more obvious after further formal developments.

The only point here is to say that maintaining self-consistency is less of a problem than might be expected initially.

We now continue to evaluate eqn. (4.13). In Chapter 2, we split Q into a paramagnetic term, Q_1, and a diamagnetic term, $-Ne^2/m$. For the latter, the integrations are immediately done and one finds that

$$Z^* = i\,16\pi^2\omega|\delta|^2\int_0^\infty dx\,dx'\,e^{-x/\delta^*}e^{-x'/\delta}Q_1(x,x') - i4\pi\omega\,\frac{|\delta|^4}{\delta+\delta^*}\left(\frac{4e^2K_f^3}{3\pi m} - \frac{1}{|\delta|^2}\right) \qquad (4.17)$$

Hence, the diamagnetic term contributes only to the surface reactance. (The two last terms are both important.)

The paramagnetic contribution is proportional to $\int_0^\infty dx\,dx'\,e^{-x/\delta^*}e^{-x'/\delta}Q_1(x,x')$. Referring to eqn. (2.21b) we see that this integral is composed of the sum of two quantum mechanical matrix elements magnitude squared.

$$M_{\ell m}^{a} = \int_{o}^{\infty} [u_{\ell}(x)u_{m}^{*}(x)+v_{\ell}(x)v_{m}^{*}(x)\ ]\ e^{-x/\delta^{*}}\ dx \qquad (4.18a)$$

$$M_{\ell m}^{b} = \int_{o}^{\infty} [u_{\ell}(x)v_{m}(x)-u_{m}(x)v_{\ell}(x)\ ]\ e^{-x/\delta^{*}}\ dx \qquad (4.18b)$$

Note that $M_{m\ell}^{b} = -M_{\ell m}^{b}$. $M_{\ell m}^{a}$ is the matrix element for processes scattering a quasi-particle from one state, ℓ, to another state, m. $M_{\ell m}^{b}$ is the matrix element for destruction of a quasi-particle and its anti-quasi-particle from states ℓ and m. $M_{\ell m}^{b*}$ is the same as before except referring to creation of a pair.

Below we calculate $M_{12}^{a} = \int_{o}^{\infty} [u_1(x)u_2^{*}(x)+v_1(x)v_2^{*}(x)]e^{-x/\delta^{*}}dx$ for the most general u, v's.

$$\begin{aligned}
M_{12}^{a} = {} & A_{11}\,A_{12}^{*}[1+(\varepsilon_1-\xi_{r_1}-V_e)\,(\varepsilon_2-\xi_{r_2}^{*}-V_e)]\ I_1\ (r_{+1}, r_{+2}) \\
& + C_{11}\,C_{12}^{*}[1+(\varepsilon_1+\xi_{r_1}-V_e)\,(\varepsilon_2+\xi_{r_2}^{*}-V_e)]\ I_1\ (r_{-1}, r_{-2}) \\
& + A_{11}\,C_{12}^{*}[1+(\varepsilon_1-\xi_{r_1}-V_e)\,(\varepsilon_2+\xi_{r_2}^{*}-V_e)]\ I_1\ (r_{+1}, r_{-2}) \\
& + C_{11}\,A_{12}^{*}[1+(\varepsilon_1+\xi_{r_1}-V_e)\,(\varepsilon_2-\xi_{r_2}^{*}-V_e)]\ I_1\ (r_{-1}, r_{+1}) \\
& + A_{21}\,A_{22}^{*}[1+(\varepsilon_1-\xi_{p_1})\,(\varepsilon_2-\xi_{p2}^{*})]\ I_2\ (p_{+1}, p_{+2}) \\
& + C_{21}\,C_{22}^{*}[1+(\varepsilon_1+\xi_{p_1})\,(\varepsilon_2+\xi_{p_2}^{*})]\ I_2\ (p_{-1}, p_{-2}) \\
& + A_{21}\,C_{22}^{*}[1+(\varepsilon_1-\xi_{p_1})\,(\varepsilon_2+\xi_{p_2}^{*})]\ I_2\ (p_{+1}, p_{-2}) \\
& + C_{21}\,A_{22}^{*}[1+(\varepsilon_1+\xi_{p_1})\,(\varepsilon_2-\xi_{p_2}^{*})]\ I_2\ (p_{-1}, p_{+2})
\end{aligned}$$

$$+ B_{21} B_{22}^{*}\left[1+(\varepsilon_1-\xi_{p_1})(\varepsilon_2-\xi_{p_2}^{*})\right] I_2(-p_{+1}, -p_{+2})$$

$$+ D_{21} D_{22}^{*}\left[1+(\varepsilon_1+\xi_{p_1})(\varepsilon_2+\xi_{p_2}^{*})\right] I_2(-p_{-1}, -p_{-2})$$

$$+ B_{21} D_{22}^{*}\left[1+(\varepsilon_1-\xi_{p_1})(\varepsilon_2+\xi_{p_2}^{*})\right] I_2(-p_{+1}, -p_{-2})$$

$$+ D_{21} B_{22}^{*}\left[1+(\varepsilon_1+\xi_{p_1})(\varepsilon_2-\xi_{p_2}^{*})\right] I_2(-p_{-1}, -p_{+2})$$

$$+ A_{21} B_{22}^{*}\left[1+(\varepsilon_1-\xi_{p_1})(\varepsilon_2-\xi_{p_2}^{*})\right] I_2(p_{+1}, -p_{+2})$$

$$+ A_{21} D_{22}^{*}\left[1+(\varepsilon_1-\xi_{p_1})(\varepsilon_2+\xi_{p_2}^{*})\right] I_2(p_{+1}, -p_{-2})$$

$$+ B_{21} A_{22}^{*}\left[1+(\varepsilon_1-\xi_{p_1})(\varepsilon_2-\xi_{p_2}^{*})\right] I_2(-p_{+1}, p_{+2})$$

$$+ B_{21} C_{22}^{*}\left[1+(\varepsilon_1-\xi_{p_1})(\varepsilon_2+\xi_{p_2}^{*})\right] I_2(-p_{+1}, p_{-2})$$

$$+ C_{21} B_{22}^{*}\left[1+(\varepsilon_1+\xi_{p_1})(\varepsilon_2-\xi_{p_2}^{*})\right] I_2(p_{-1}-p_{+2})$$

$$+ C_{21} D_{22}^{*}\left[1+(\varepsilon_1+\xi_{p_1})(\varepsilon_2+\xi_{p_2}^{*})\right] I_2(p_{-1}, -p_{-2})$$

$$+ D_{21} A_{22}^{*}\left[1+(\varepsilon_1+\xi_{p_1})(\varepsilon_2-\xi_{p_2}^{*})\right] I_2(-p_{-1}, p_{+2})$$

$$+ D_{21} C_{22}^{*}\left[1+(\varepsilon_1+\xi_{p_1})(\varepsilon_2+\xi_{p_2}^{*})\right] I_2(-p_{-1}, p_{-2}) \qquad (4.19)$$

where

$$I_1(r_1, r_2) = \frac{\frac{2}{\delta^*} - ie^{-\lambda_e/\delta^*}\left\{(r_1 - r_2^* - \frac{i}{\delta^*})e^{i(r_1-r_2^*)\lambda_e} - (r_1 - r_2^* + \frac{i}{\delta^*})e^{-i(r_1-r_2^*)\lambda_e}\right\}}{\left(\frac{1}{\delta^*}\right) + (r_1 - r_2^*)^2}$$

$$- \frac{\frac{2}{\delta^*} - ie^{-\lambda_e/\delta^*}\left\{(r_1 + r_2^* - \frac{i}{\delta^*})e^{i(r_1+r_2^*)\lambda_e} - (r_1 + r_2^* + \frac{i}{\delta^*})e^{-i(r_1+r_2^*)\lambda_e}\right\}}{\left(\frac{1}{\delta^*}\right)^2 + (r_1 + r_2^*)^2}$$

(4.20a)

$$I_2(p_1, p_2) = i \frac{e^{i(p_1 - p_2^* + \frac{i}{\delta^*})\lambda_e}}{p_1 - p_2^* + i/\delta^*} \tag{4.20b}$$

Next, we calculate $M_{\ell m}{}^b = \int_o^\infty [u_\ell(x)v_m(x) - u_m(x)v_\ell(x)]e^{-x/\delta^*}dx$ for the most general u, v's.

$$\begin{aligned}
M_{\ell m}{}^b = {} & A_{1\ell}A_{1m}[(\varepsilon_m - \xi_{r_m}) - (\varepsilon_\ell - \xi_{r_\ell})]\, I_3\,(r_{+\ell}, r_{+m}) \\
& + C_{1\ell}C_{1m}[(\varepsilon_m + \xi_{r_m}) - (\varepsilon_\ell + \xi_{r_\ell})]\, I_3\,(r_{-\ell}, r_{-m}) \\
& + A_{1\ell}C_{1m}[(\varepsilon_m + \xi_{r_m}) - (\varepsilon_\ell - \xi_{r_\ell})]\, I_3\,(r_{+\ell}, r_{-m}) \\
& + C_{1\ell}A_{1m}[(\varepsilon_m - \xi_{r_m}) - (\varepsilon_\ell - \xi_{r_\ell})]\, I_3\,(r_{-\ell}, r_{+m}) \\
& + A_{2\ell}A_{2m}[(\varepsilon_m - \xi_{p_m}) - (\varepsilon_\ell - \xi_{p_\ell})]\, I_4\,(p_{+\ell}, p_{+m}) \\
& + C_{2\ell}C_{2m}[(\varepsilon_m + \xi_{p_m}) - (\varepsilon_\ell + \xi_{p_\ell})]\, I_4\,(p_{-\ell}, p_{-m}) \\
& + A_{2\ell}C_{2m}[(\varepsilon_m + \xi_{p_\ell}) - (\varepsilon_\ell - \xi_{p_\ell})]\, I_4\,(p_{+\ell}, p_{-m}) \\
& + C_{2\ell}A_{2m}[(\varepsilon_m - \xi_{p_m}) - (\varepsilon_\ell + \xi_{p_\ell})]\, I_4\,(p_{-\ell}, p_{+m}) \\
& + B_{2\ell}B_{2m}[(\varepsilon_m - \xi_{p_m}) - (\varepsilon_\ell - \xi_{p_\ell})]\, I_4\,(-p_{+\ell}, -p_{+m}) \\
& + D_{2\ell}D_{2m}[(\varepsilon_m + \xi_{p_m}) - (\varepsilon_\ell + \xi_{p_\ell})]\, I_4\,(-p_{-\ell}, -p_{-m}) \\
& + B_{2\ell}D_{2m}[(\varepsilon_m + \xi_{p_m}) - (\varepsilon_\ell - \xi_{p_\ell})]\, I_4\,(-p_{+\ell}, -p_{-m})
\end{aligned}$$

$$+ D_{2\ell}B_{2m}\left[(\varepsilon_m - \xi_{p_m}) - (\varepsilon_\ell + \xi_{p_\ell})\right] I_4\,(-p_{-\ell}, -p_{+m})$$

$$+ A_{2\ell}B_{2m}\left[(\varepsilon_\ell - \xi_{p_\ell}) - (\varepsilon_m - \xi_{p_m})\right] I_4\,(p_{+\ell}, -p_{+m})$$

$$+ A_{2\ell}D_{2m}\left[(\varepsilon_\ell - \xi_{p_\ell}) - (\varepsilon_m + \xi_{p_m})\right] I_4\,(p_{+\ell}, -p_{-m})$$

$$+ B_{2\ell}A_{2m}\left[(\varepsilon_\ell - \xi_{p_\ell}) - (\varepsilon_m - \xi_{p_m})\right] I_4\,(-p_{+\ell}, p_{+m})$$

$$+ B_{2\ell}C_{2m}\left[(\varepsilon_\ell - \xi_{p_\ell}) - (\varepsilon_m + \xi_{p_m})\right] I_4\,(-p_{+\ell}, p_{-m})$$

$$+ C_{2\ell}B_{2m}\left[(\varepsilon_\ell + \xi_{p_\ell}) - (\varepsilon_m - \xi_{p_m})\right] I_4\,(p_{-\ell}, -p_{+m})$$

$$+ C_{2\ell}D_{2m}\left[(\varepsilon_\ell + \xi_{p_\ell}) - (\varepsilon_m + \xi_{p_m})\right] I_4\,(p_{-\ell}, -p_{-m})$$

$$+ D_{2\ell}A_{2m}\left[(\varepsilon_\ell + \xi_{p_\ell}) - (\varepsilon_m - \xi_{p_m})\right] I_4\,(-p_{-\ell}, p_{+m})$$

$$+ D_{2\ell}C_{2m}\left[(\varepsilon_\ell + \xi_{p_\ell}) - (\varepsilon_m + \xi_{p_m})\right] I_4\,(-p_{-\ell}, p_{-m}) \qquad (4.21)$$

$$I_4(p_1, p_2) = i\,\frac{e^{i(p_1 + p_2 + \frac{i}{\delta^*})\lambda_e}}{p_1 + p_2 + i/\delta^*} \qquad (4.22b)$$

$$I_3(r_1, r_2) = \frac{\frac{2}{\delta^*} - ie^{-\lambda_e/\delta}\left\{(r_1 + r_2 - \frac{i}{\delta^*})e^{i(r_1 + r_2)\lambda_e} - (r_1 + r_2 + \frac{i}{\delta^*})e^{-i(r_1 + r_2)\lambda_e}\right\}}{\left(\frac{1}{\delta^*}\right)^2 + (r_1 + r_2)^2}$$

$$- \frac{\frac{2}{\delta^*} - ie^{-\lambda_e/\delta}\left\{(r_1 - r_2 - \frac{i}{\delta^*})e^{i(r_1 - r_2)\lambda_e} - (r_1 - r_2 + \frac{i}{\delta^*})e^{-i(r_1 + r_2)\lambda_e}\right\}}{\left(\frac{1}{\delta^*}\right)^2 + (r_1 - r_2)^2}$$

(4.22a)

In terms of these matrix elements Z^* is just

$$Z^* = i32\pi^2 \frac{e^2}{m^3} \omega |\delta|^2 \sum_{\ell,m} (K_y^{\ell}\cos\theta + K_z^{\ell}\sin\theta)^2$$

$$\left\{ [f_T(\epsilon_\ell) - f_T(\epsilon_m)] \frac{|M^a_{\ell m}|^2}{\epsilon_m - \epsilon_\ell - \omega + i\delta} \right.$$

$$\left. + \frac{1}{2}[1 - f_T(\epsilon_\ell) - f_T(\epsilon_m)] |M^b_{\ell m}|^2 \left[\frac{1}{\epsilon_m + \epsilon_\ell + \omega - i\delta} + \frac{1}{\epsilon_m + \epsilon_\ell - \omega + i\delta} \right] \right\}$$

$$-i\,4\pi\omega \frac{|\delta|^4}{\delta + \delta^*} \left(\frac{4e^2 K_f^3}{3\pi m} - \frac{1}{|\delta|^2} \right) \qquad (4.23)$$

The u, v amplitudes are invariant under the change $K_z \rightarrow -K_z$. Thus $\int_{-\infty}^{\infty} dK_z^{\ell}$ contained in $\sum_{\ell}$ causes the field-particle coupling cross term $K_y^{\ell} K_z^{\ell} \cos\theta\sin\theta$ contribution to vanish in eqn. (4.23). This just expresses an obvious symmetry - the surface impedance is the same for field linear polarization of $\pm\theta$ about the direction of A_{dc}. Because of this

$$(k_y^{\ell}\cos\theta + K_z^{\ell}\sin\theta)^2 \rightarrow (K_y^{\ell})^2 \cos^2\theta + (K_z^{\ell})^2 \sin^2\theta$$

and we can derive the complete angular dependence of the surface impedance by calculating Z for just two angles, 0 and $\pi/2$; i.e.,

$$Z(\theta) = Z(\theta = 0)\cos^2\theta + Z\,(\theta = \pi/2)\sin^2\theta \qquad (4.24)$$

It should be noted that this relation is independent of the long series of approximations made above. We could have derived this long ago.

The sum over states, $\sum_{\ell,m}$, in eqn. (4.23) is actually more than a double sum; this is so since a surface state requires three numbers to fix it - ξ_t, V_e, and the particular branch number, while a continuum state requires four numbers - ξ_t, V_e, ϵ, $p_\pm$ incident. In any case, it's always possible to pull out from the sum

$$\sum_{K_y^{\ell},\,K_z^{\ell}} \rightarrow \int_{-\infty}^{\infty} \frac{dK_y^{\ell}}{2\pi} \int_{-\infty}^{\infty} \frac{dK_z^{\ell}}{2\pi}$$ and express these integrals

in terms of ξ_t and φ ($V_e = V_o \cos\varphi$) which are useful in subsequent work. This is trivially done using the definitions of ξ_t and φ in Chapter 3 and the result is

$$\sum_{\ell,\,m} \rightarrow \frac{m}{2\pi^3} \int_{-E_f}^{\infty} \int_{o}^{\pi} d\varphi \sum_{\ell,\,m}$$

where $\sum_{\ell,\,m}$ on the right hand side now and hereafter refers to the remaining variables necessary to specify the state considered. If only surface states are considered, $\int_o^{\pi} d\varphi \rightarrow \int_{\pi/2}^{\pi} d\varphi$ since there are no surface states for $V_e = 0$. Putting all this together we find Z^*

$$Z^* = i\frac{16e^2\omega|\delta|^3 K_f^2}{m}\int_{-E_f}^{\infty} d\xi_t \int_o^{\pi} d\varphi \sum_{\ell,m}\left(1+\frac{\xi_t}{E_f}\right)$$

$$(\cos^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi)$$

$$\left\{\left[f_T(\epsilon_\ell)-f_T(\epsilon_m)\right]|M_{\ell m}^{\ a}|^2 \frac{1}{\epsilon_m-\epsilon_\ell-\omega+i\delta}\right.$$

$$\left. + \tfrac{1}{2}\left[1-f_T(\epsilon_\ell)-f_T(\epsilon_m)\right]|M_{\ell m}^{\ b}|^2\left[\frac{1}{\epsilon_m+\epsilon_\ell+\omega-i\delta}+\frac{1}{\epsilon_m+\epsilon_\ell-\omega+i\delta}\right]\right\}$$

$$-i4\pi\omega\frac{|\delta|^4}{\delta+\delta^*}\left(\frac{4e^2K_f^3}{3\pi m} - \frac{1}{|\delta|^2}\right) \qquad (4.25)$$

For processes involving surface states, it is most useful to switch from the integration variable φ to the variable V_e. The substitution made is

$$\frac{1}{V_o^2}\int_{-V_o}^{0} dV_e\left\{\cos^2\theta\frac{V_e^2}{(V_o^2-V_e^2)^{\frac{1}{2}}} + \sin^2\theta\,(V_o^2-V_e^2)^{\frac{1}{2}}\right\} \qquad (4.26)$$

Finally, for continuum states, the sum over states is just $\sum_{+,-}\int_{-\infty}^{\infty}\frac{dp_\pm}{2\pi}$ assuming unit volume. $\sum_{+,-}$ means the sum over p_+ and p_- incident states. However, the more useful variable is ϵ. Thus

$$\sum_\ell \rightarrow \frac{1}{\pi}\sum_{+,-}\int_\Delta^{\infty}\frac{\partial p_\pm^{\ \ell}}{\partial\epsilon_\ell}d\epsilon_\ell \qquad (4.27)$$

can be used in eqn. (4.25) whenever considering continuum states.

Assembling all these bits and pieces, we can now write the surface impedance contributions for the various processes in a form that can be directly evaluated numerically on a computer. Let Z_S be the surface impedance of the superconductor and Z_N denote surface impedances of normal metals. Write the total Z_S as $Z_{S_1} + Z_{S_2} + \ldots$. All variables are renormalized to dimensionless form and $\hbar$'s and c's are reinserted so that Z may be found in ohms.

$$\frac{e^2}{\hbar c}\ \frac{1}{\hbar^2 c^2} = 5.62 \cdot 10^{-8}\ \text{ohms}/(\mathring{A} - eV)^2$$

The familiar identity $\frac{1}{x+i\delta} \to P(\frac{1}{x}) - i\pi\delta(x)$ is used in the reduction, where P denotes principle part of the integration.

We merely state the formulas, here, and discuss the consequences later.

(1) Transitions between surface states

$$Z_{S_1}{}^{*} = i8\,\frac{e^2}{\hbar c}\,\frac{1}{\hbar^2 c^2}\,\frac{\omega}{\Delta}\,\frac{\Delta}{E_f}\,E_f^2\,|\delta|^2 \sum_{\substack{\text{branches}\\ \ell,\,m}} \int_{-V_o/\Delta}^{0} d\left(\frac{V_e}{\Delta}\right) \int_{-\infty}^{\infty} d\left(\frac{\xi_t}{\Delta}\right)$$

$$\left\{\cos^2\theta\ \frac{\left(\frac{V_e}{\Delta}\right)^2}{\left(\frac{V_o}{\Delta}\right)^2\left[\left(\frac{V_o}{\Delta}\right)^2 - \left(\frac{V_e}{\Delta}\right)^2\right]^{\frac{1}{2}}} + \sin^2\theta\left[\left(\frac{V_o}{\Delta}\right)^2 - \left(\frac{V_e}{\Delta}\right)^2\right]^{\frac{1}{2}}\right\}$$

$$\left[f_T\left(\frac{\varepsilon_\ell}{\Delta}\right) - f_T\left(\frac{\varepsilon_m}{\Delta}\right)\right] |M^a_{\ell m}|^2 \left[P\left(\frac{1}{\frac{\varepsilon_m}{\Delta} - \frac{\varepsilon_\ell}{\Delta} - \frac{\omega}{\Delta}}\right) - i\pi\delta\left(\frac{\varepsilon_m}{\Delta} - \frac{\varepsilon_\ell}{\Delta} - \frac{\omega}{\Delta}\right)\right]$$

(4.28a)

We shall need later $\int_{\omega_1/\Delta}^{\omega_2/\Delta} R_{S_1}\, d\left(\frac{\omega}{\Delta}\right) \cong R_{S_1}\left(\frac{\omega_2}{\Delta} - \frac{\omega_1}{\Delta}\right)$

so that

$$R_{S_1} \cong \frac{1}{\frac{\omega_2}{\Delta} - \frac{\omega_1}{\Delta}} 8\pi \frac{e^2}{\hbar c} \frac{1}{\hbar^2 c^3} \frac{\Delta}{E_f} E_f^2 |\delta|^2 \sum_{\substack{\text{branches} \\ \ell, m}} \int_{-V_o/\Delta}^{0} d\left(\frac{V_e}{\Delta}\right) \int_{-\infty}^{\infty} d\left(\frac{\xi_t}{\Delta}\right)$$

$$\left\{\cos^2\theta \frac{\left(\frac{V_e}{\Delta}\right)^2}{\left(\frac{V_o}{\Delta}\right)^2 \left[\left(\frac{V_o}{\Delta}\right)^2 - \left(\frac{V_e}{\Delta}\right)^2\right]^{\frac{1}{2}}} + \sin^2\theta \left[\left(\frac{V_o}{\Delta}\right)^2 - \left(\frac{V_e}{\Delta}\right)^2\right]^{\frac{1}{2}}\right\}$$

$$\left(\frac{\epsilon_m}{\Delta} - \frac{\epsilon_\ell}{\Delta}\right)\left[f_T\left(\frac{\epsilon_\ell}{\Delta}\right) - f_T\left(\frac{\epsilon_m}{\Delta}\right)\right] |M_{\ell m}^a|^2 \Bigg|_{\omega_1 \le \epsilon_m - \epsilon_\ell \le \omega_2} \qquad (4.28b)$$

which is further discussed in Chapter 6.

We calculate the matrix element $M_{\ell m}^a$ for integrals (4.28) as follows: Solve the set of simultaneous equations (3.15) with $B_2 = D_2 = 0$, twice. Once for $A_{1\ell}$, $C_{1\ell}$, $A_{2\ell}$, $C_{2\ell}$ and again for A_{1m}, C_{1m}, A_{2m}, C_{2m} where ℓ and m refer to two surface states with the same ξ_t and V_e but different energies, ϵ_ℓ, ϵ_m. These coefficients A_1, C_1, etc. must satisfy the normalization condition (3.16). $M_{\ell m}^a$ is then trivially found from formula (4.19) with $B_{2\ell, m} = D_{2\ell, m} = 0$.

(2) Transitions between surface states and continuum states

$$Z_{S_2}^{*} = i\frac{8}{\pi}\frac{e^2}{\hbar c}\frac{1}{\hbar^2 c^2}\frac{\omega}{\Delta}\frac{\Delta}{E_f}E_f^2\,|\delta|^2\int_{-V_o/\Delta}^{0} d\left(\frac{V_e}{\Delta}\right)\int_{-\infty}^{\infty} d\left(\frac{\xi_t}{\Delta}\right)$$

$$\left\{\sum_{\substack{\text{branches}\\ \ell}}\sum_{\substack{+,-\\ m}}\int_1^{\infty}\frac{\partial(p_{\pm}^{m}/K_f)}{\partial(\epsilon_m/\Delta)}\,d\left(\frac{\epsilon_m}{\Delta}\right)+\sum_{\substack{\text{branches}\\ m}}\sum_{\substack{+,-\\ \ell}}\int_1^{\infty}\frac{\partial(p_{\pm}^{\ell}/K_f)}{\partial(\epsilon_\ell/\Delta)}d\left(\frac{\epsilon_\ell}{\Delta}\right)\right\}$$

$$\left[f_T\left(\frac{\epsilon_\ell}{\Delta}\right)-f_T\left(\frac{\epsilon_m}{\Delta}\right)\right]|M_{\ell m}^{a}|^2\left[P\left(\frac{1}{\frac{\epsilon_m}{\Delta}-\frac{\epsilon_\ell}{\Delta}-\frac{\omega}{\Delta}}\right)-i\pi\delta\left(\frac{\epsilon_m}{\Delta}-\frac{\epsilon_\ell}{\Delta}-\frac{\omega}{\Delta}\right)\right]$$

$$\left\{\cos^2\theta\,\frac{\left(\frac{V_e}{\Delta}\right)^2}{\left(\frac{V_o}{\Delta}\right)^2\left[\left(\frac{V_o}{\Delta}\right)^2-\left(\frac{V_e}{\Delta}\right)^2\right]^{\frac{1}{2}}}+\sin^2\theta\left[\left(\frac{V_o}{\Delta}\right)^2-\left(\frac{V_e}{\Delta}\right)^2\right]^{\frac{1}{2}}\right\} \qquad (4.29)$$

Since the energy of continuum states is always higher than the energy of surface states, the δ - function term contributing to the resistive part contains only the $\sum_{\substack{\text{branches}\\ \ell}}\sum_{\substack{+,-\\ m}}\int\cdots$ terms. The reactance, though, picks up contributions from both continuum and surface states as final states in the interaction and keeps all sums.

We determined $M_{\ell m}^{a}$ for integrals (4.29) as follows: If ℓ is a surface state, then solve eqns. (3.15) with $B_2 = D_2 = 0$ for $A_{1\ell}$, $C_{1\ell}$, $C_{2\ell}$ normalized according to eqn. (3.16). The continuum states, m, are found by solving eqns. (3.15) first with $D_2 = 0$ (for

p_+ incident scattering states) and then with $B_2 = 0$ (for p_- incident scattering states) normalized now according to eqns. (3.17). This determines either the non-zero set A_{1m}, C_{1m}, A_{2m}, B_{2m}, C_{2m} or A_{1m}, C_{1m}, A_{2m}, C_{2m}, D_{2m} respectively and hence $M^a_{\ell m}$ is again fixed by expression (4.19).

(3) Transitions between continuum states

$$Z^*_{S_3} = i\,\frac{8}{\pi^2}\,\frac{e^2}{hc}\,\frac{1}{h^2c^2}\,|\delta|^2\,\frac{\omega}{\Delta}\,\frac{\Delta}{E_f}\,E_f^2\int_{-E_f/\Delta}^{0} d\left(\frac{\xi_t}{\Delta}\right)\int_0^{\pi} d\varphi\int_1^{\infty} d\left(\frac{\epsilon_\ell}{\Delta}\right)$$

$$\int_1^{\infty} d\left(\frac{\epsilon_m}{\Delta}\right)\sum_{\substack{+,-\\ \ell,m}}$$

$$\frac{\partial(p^\ell_\pm/K_f)}{\partial(\epsilon_\ell/\Delta)}\;\frac{\partial(p^m_\pm/K_f)}{\partial(\epsilon_m/\Delta)}\left(1+\frac{\Delta}{E_f}\,\frac{\xi_t}{\Delta}\right)(\cos^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi)$$

$$\left[f_T\left(\frac{\epsilon_\ell}{\Delta}\right) - f_T\left(\frac{\epsilon_m}{\epsilon}\right)\right]|M^a_{\ell m}|^2\left[P\left(\frac{1}{\frac{\epsilon_m}{\Delta}-\frac{\epsilon_\ell}{\Delta}-\frac{\omega}{\Delta}}\right) - i\pi\delta\left(\frac{\epsilon_m}{\Delta}-\frac{\epsilon_\ell}{\Delta}-\frac{\omega}{\Delta}\right)\right]$$

(4.30)

$M^a_{\ell m}$ in eqn. (4.30) is calculated from continuum scattering states solutions to eqns. (3.15), normalized according to eqns. (3.17) as described earlier. However, not all the non-zero terms are kept

in the $M^{a}_{\ell m}$ expression (4.19); we ignore all terms which vary like $e^{i(p_\ell + p_m^*)\lambda_e}$, $e^{i(r_\ell + r_m^*)\lambda_e}$ and keep all terms that vary like $e^{i(p_\ell - p_m^*)\lambda_e}$, $e^{i(r_\ell - r_m^*)\lambda_e}$.

CHAPTER 5

THEORY OF ANOMALOUS SKIN EFFECT

In order to understand the modifications that will be produced by applying a static magnetic field to a superconductor, it is necessary to know what happens when $H_{dc} = 0$. The theory of surface impedances in such superconductors is, in turn, based upon the theory of surface impedances in normal metals at low temperatures which can be found masquerading under the name anomalous skin effect. (The understanding of these phenomena also represents the direction of historical progress in discussing the more complicated situations.) All this can be accommodated in the previously developed formalism. By shutting off H_{dc}, we obtain the surface impedance of ordinary superconductors and by letting the gap, Δ, vanish, we can find the normal metal absorption properties. Since all this work is thoroughly described in the literature, we merely review the basic features necessary for our problem.

Begin by considering a normal metal at room temperature. The mean free path of conduction electrons ($\leq 10\mathring{A}$) is much less than the penetration depth of the electromagnetic field ($\sim 5000\mathring{A}$). For such situations, the fields felt by an electron over its free motion between collisions can be approximated as a constant. Then the net current density, $\vec{j}(\vec{r})$, generated by a collection of electrons is proportional to the field strength at that point, which just expresses Ohm's familiar law, $\vec{j}(\vec{r}) = \sigma \vec{E}(\vec{r})$. σ is called the conductivity and depends

on the frequency of electron scatterings of the given temperature. This simple point relation means that Maxwell's equations inside the metal have a trivial solution, $A \sim e^{-x/\delta}$.

Now let's see what happens as the temperature is lowered to cryogenic levels. The electron mean free path is generally much larger than typical field penetration depths ($\sim 1000\mathring{A}$) so that an electron between collisions will usually see a field which is anything but constant. It makes no sense to talk about the Ohmic conductivity, σ, relating $\vec{j}$ and $\vec{E}, \vec{A}$ at a point. A non-local relation holds, $\vec{j}(\vec{r}) = \int_0^\infty Q(\vec{r}, \vec{r}')\, \vec{A}(\vec{r}')\, d\vec{r}'$, and solution of Maxwell's equations is more difficult than before but still possible. The solution to this problem has been carried out by Reuter and Sondheimer[11], Chambers[12], Mattis and Bardeen[7], and Abrikosov, Gorkov, and Khalatnikov AGK[13] The AGK paper is most closely related to concepts used here. All the solutions, however, for the surface impedance were made for an infinite, homogeneous, isotropic metal. In view of the long mean free paths, this makes such results very suspect. We have investigated the question and resolved the matter in Chapter 7. Using an infinite metal is acceptable and we continue the discussion here on that basis.

For an infinite medium $Q(\vec{r}, \vec{r}')$ is a function only of the difference coordinates $|\vec{r}-\vec{r}'|$; AGK[13] have calculated the Fourier transform of the kernel and find that for penetration depths much less than the mean free path (extreme anomalous limit)

$$Q(\vec{k}) = \frac{i}{|\vec{k}|} \frac{e^2 K_f^2 \omega}{4\pi} \tag{5.1}$$

One can then use the Fourier transformed Maxwell's equations to solve for the $\vec{A}$ field distribution and obtain the surface impedance of a normal metal $Z_N = R_N + iX_N$. The AGK[13] result is

$$R_N = \frac{4\pi}{3^{\frac{3}{2}}} \left(\frac{\omega}{eK_f}\right)^{\frac{2}{3}} \tag{5.2a}$$

$$X_N = \frac{4\pi}{3} \left(\frac{\omega}{eK_f}\right)^{\frac{2}{3}} \tag{5.2b}$$

To find R_N in ohms, use

$$R_N = 6.75 \cdot 10^{-12} \left(\frac{\omega}{K_f}\right)^{\frac{2}{3}} \tag{5.2c}$$

with ω in radians/sec.
and K_f in $\mathring{A}^{-1}$.

The formulas for surface impedance, here, are provided mainly for reference. Our main consideration is the physical mechanism by which absorption takes place. In the anomalous skin effect realm, a new feature enters -- only electrons traveling very nearly parallel to the surface of the metal play an active role in interacting with the electromagnetic field. Electrons that collide with the surface at a large angle with respect to the surface run in and out of the field too quickly to absorb any sizeable amount of

energy. We are, of course, only considering electrons at the Fermi surface, as usual. Let us refer our discussion to the usual geometry in this work. The surface is in the y - z plane and the $\vec{A}$ field is polarized along the y-axis. The electron has x-component of momentum p and transverse momentum K_ρ. $K_y = K_\rho \cos\varphi$, $K_z = K_\rho \sin\varphi$. So the effective electrons here have very small p compared to K_f and the maximum angle for non-parallelism with the y - z plane is of the order $p/K_f << 1$. This gets smaller as K_f increases which in turn means that the effective number of electrons interacting with the field is decreased.

Almost everything said about the normal metal at low temperatures applies directly to the superconductor with no static magnetic fields present. This first became apparent about twenty years ago when Pippard slightly modified Chambers' results on normal metals to explain the magnetic properties of superconductors.

By this we do not mean that the physics of both situations is the same; rather that the formalism adapted for anomalous skin effect is applicable also to superconductors. Whereas a normal metal allows a static magnetic field to penetrate completely, a superconductor confines a static magnetic field to within about 1000Å of the metal's surface. However, both normal and superconducting metals use only the electrons in a thin slice of Fermi surface to screen the ac fields from the interior region. (Superconductors do this also for dc fields.)

Thus in superconductors one can calculate surface impedances using infinite medium kernels, Q, just for the same reasons as in normal metals.

It turns out that the kernel, Q, for superconductors has for its Fourier transformed behavior the same dependence of k, viz., $Q(k) \sim {}^{1}/_{|\vec{k}|}$. So in this way one can see the relation of normal and superconducting metals through the kernel, Q, after having separated out the common spatial behavior. In the normal metal, Im Q is different from zero and Re Q is zero. Also Im Q vanishes for $\omega = 0$ which means that static fields permeate the metal. In a normal metal Q is temperature independent for the range of anomalous skin effect. For a superconductor Q has a much more complicated behavior. Most results are only available numerically. When $T = 0$, $\omega < 2\Delta$, Re $Q \neq 0$, Im $Q = 0$. It is Re $Q \neq 0$ which gives rise to the Meissner effect - the expulsion of static fields from the bulk interior of a superconductor. Let's point out an important difference between normal and superconducting metals here. Suppose $T = 0$ and $\omega \sim \Delta$, a frequency which generally falls somewhere in the microwave region. Then the field in both cases is confined to the surface region since $|Q|$ in both cases is about the same order of magnitude. But in the normal metal it is Im $Q \neq 0$, Re $Q = 0$ while in the superconductor Re $Q \neq 0$, Im $Q = 0$. So whereas both metal surfaces are highly reflecting the mechanism for each is quite different. It has been pointed out earlier that Im $Q \neq 0$ corresponds to power absorption while Re $Q \neq 0$ is related

to reactive or non-dissipative processes. Thus the normal metal expels the field by just being a very good absorber while the superconductor expels its ac field without any power absorption at all. This can all be understood in terms of the excitation spectrum for quasi-particles in normal and superconducting metals. In normal metals an arbitrarily small amount of energy can create excitations absorbing power out of an electromagnetic field. For superconductors, though, there is a gap in the excitation spectrum of magnitude Δ. The minimum energy necessary for creating a quasi-particle is Δ. And at absolute zero where no thermal excitations are possible, the only way to absorb energy is through the creation of a quasi-particle anti-quasi-particle pair which costs an amount of energy at least 2Δ. Since we are considering $\omega \sim \Delta$, therefore, no power can be absorbed. However, virtual pair creation processes do occur and it is these which give Re $Q \neq 0$. As London has suggested, the superconductor wavefunction has a certain rigidity which is not destroyed by these perturbations.

Feynman[14] says that there is a general rule about the reflectivity of materials. Whenever a material gets to be a very good absorber at some frequency, it also becomes highly reflecting. Our example above for a normal metal in the anomalous skin effect is a perfect case. The larger Im Q, the higher the reflectivity. But, the rule is inappropriate for superconductors. There is no absorption at all and yet the surface is very highly reflecting!

The most extreme difference between normal and superconducting metals occurs at $T = 0$. As T is raised the superconductor tends to behave progressively more like a normal metal until at some critical temperature, T_c, the superconductor turns into a normal metal. This happens for two reasons: higher temperatures thermally excite more quasi-particles across the gap into the continuum of states and higher temperatures cause the gap function to decrease. The thermally excited quasi-particles behave much like the quasi-particles in the normal metal. It is for this reason that a superconductive state used to be considered a mixture of "normal and superconducting fluids". We should mention that the gap function actually changes very little except in the region $T \gtrsim .95\, T_c$. In terms of the kernel, Q, the transformation from superconducting to normal state occurs as follows: $\mathrm{Re}\, Q \to 0$ as T rises while $\mathrm{Im}\, Q$, which was zero for a superconductor, tends toward $\mathrm{Im}\, Q$ for a normal metal as T runs from zero to T_c. Since the gap holds constant until $T \simeq .95\, T_c$, $\mathrm{Re}\, Q$ also stays strong until this point, too, dropping rapidly to zero only for $T \gtrsim .95\, T_c$. Which is to say that virtual pair processes are the dominant mode of flux expulsion throughout the widest range of temperature.

All the above and several other interesting cases can be found discussed by AGK[13].

CHAPTER 6

SURFACE IMPEDANCE THEORY FOR SUPERCONDUCTORS WITH $H_{dc} \neq 0$ AND COMPARISON TO EXPERIMENTAL DATA

We now have developed enough formalism to discuss the implications of this surface impedance theory for $H_{dc} \neq 0$. Since everything must be calculated numerically on a computer ranging from single to quadruple integrals, our results are only for a few cases. These are done in realms where the individual processes are separated as clearly as possible from one another. The experimental data are also rather spotty so a complete comparison of theory and experiment is impossible at this time.

Wherever numerical results of calculations are quoted, the following data have been assumed for the model which corresponds roughly to tin and tantalum:

$$K_f = 1 \ \mathring{A}^{-1}$$

$$E_f = 5 \text{ eV}$$

$$\Delta/E_f = 10^{-4} \Rightarrow \Delta \cong 150\text{gc}$$

$$\delta(T = 0) = 500\mathring{A}$$

$$\lambda_e = 1000\mathring{A}$$

Some important experiments have been performed by

(1) Pippard[1] in tin at 9.4gc

(2) Spiewak[2] in tin at 1 gc

(3) Richards[5] in tin at 3 gc

(4) Lewis[4] in tin at 24 gc

(5) Glosser[3] in tantalum at 9 gc

(6) Sharvin and Gantmakher[6] in tin at 2 mc.

The number measured by the experimentalist is usually the surface impedance for some size of H_{dc}, $Z_S(H) = R_S(H) + iX_S(H)$, divided by the normal state surface resistance, R_N. R_N is temperature independent and for the effects considered here also independent of H_{dc}. (R_N varies with H_{dc} only when in the many kilogauss range.) As we shall soon see, normalizing the superconductor surface resistance changes to R_N can be quite misleading in many instances since large changes divided by an even larger number make the effects appear small. (It is necessary to measure this surface resistance ratio because the power losses in the cavity walls usually far exceed the sample's power losses of interest.)

In Chapter 5 it was pointed out that the effective quasi-particles had to be traveling very nearly parallel to the surface of the metal. That angle was said to be about p/K_f. In actually performing the integrations for surface impedances, p was not found to be a convenient independent variable. A better measure of the angle for our purposes is $\xi_t = K_\rho^2/2m - K_f^2/2m$, the transverse momentum. In terms of ξ_t, then, the angle between quasi-particle trajectories and the surface is about $\left(\xi_t/E_f\right)^{\frac{1}{2}}$. We shall refer

occasionally to this angle.

The reader should have some feeling for the sizes of the effects involved here. Surface reactances are to be interpreted in terms of penetration depths which are about 1000Å. Surface resistances are related to power dissipation in a metal and hence to reflectivity of the metal surfaces. We are concerned with surface resistances $\sim 10^{-4}$ ohms or less. These are extremely "shiny" surfaces reflecting 99.99+% of the incident radiation. So while we shall be talking about "big and small" changes in the surface resistance, the actual changes in reflectivity are very very tiny, indeed!

We shall begin by considering the superconducting surface resistance at "low temperatures", $0 < T \leq .5T_c$, and for electromagnetic radiation frequencies, $\omega \sim .1\Delta$. If $H_{dc} = 0$, R_s (and R_S/R_N) is negligible. This is well understood since for such temperatures there are negligibly few quasi-particles in the continuum to absorb power. If $H_{dc} \leq H_c \sim 500$ guass (H_c is the critical magnetic field where the superconductor reverts to a normal metal), however, there is a sizeable power absorption, $.1\% \leq R_S/R_N \leq 10\%$, which depends on every conceivable factor involved here. Thus H_{dc} has changed the surface resistance by several orders of magnitude. What could be the reason? Within our model, there is only one possible candidate for an explanation --the surface states. As described in Chapter 3, the H_{dc} field sets up a potential well within which there are discrete energy levels that range from $\epsilon \simeq 0$ to $\epsilon \leq \Delta$.

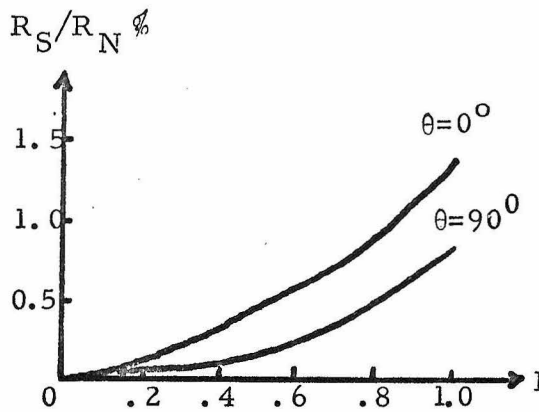
They exist only for $|\xi_t| \leq 20\Delta$ which according to earlier formulas indicate quasi-particle trajectories even more parallel to the surface than the effective electrons in the anomalous skin effect. $|\xi_t| \sim 10\Delta$ corresponds to an angle $\sim 2^o$.

Since there exist states whose excitation energies are much less than the gap, even "low temperatures" yield sufficient thermal populations so that a photon can kick a quasi-particle from a lower branch into another higher energy branch conserving transverse momentum (initial and final states both have same ξ_t and V_e).

Using the surface state spectrum described in Chapter 3, we have calculated the surface resistance at "low temperatures" for various T, ω, H_{dc}, and θ. (Recall that θ is the orientation of the radiation field polarization with the y-axis.) By confining ourselves to $T \leq .5T_c$ and $\omega \leq .2\Delta$, we can be sure that the surface resistance is totally dominated by transitions between surface states.

We must do this calculation numerically, which according to eqn. (4.28a) amounts to just a single integral over either V_e or ξ_t as one chooses. However, doing the integral over the $\delta(\epsilon_m - \epsilon_\ell - \omega)$ introduces either $1/|\partial(\epsilon_m - \epsilon_\ell)/\partial V_e|$ or $1/|\partial(\epsilon_m - \epsilon_\ell)/\partial \xi_t|$ according to which variable is used for integration. And both of these factors are singular for a typical energy spectrum. Although the integrals are well-behaved, it is too difficult to handle such a situation numerically on a computer. A more reasonable approach is to integrate eqn. (4.28a) over a narrow frequency range from ω_1 to ω_2, where

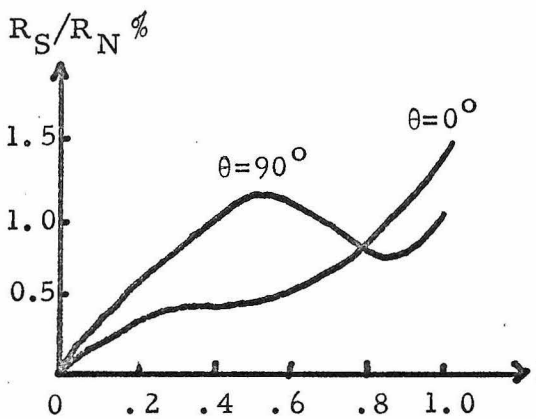
$\omega_2 - \omega_1 << \Delta$, and obtain the surface resistance per unit frequency range from a double integral over V_e and ξ_t. We assume that the surface resistance varies continuously and smoothly with frequency so that for $\omega_2 - \omega_1 \simeq .01\Delta$, we are really calculating the surface resistance at each point in frequency, except for $\omega_{1,2} \leq$ minimum energy separation of the nearest pair of surface states $\simeq .01\Delta$. (Obviously, there is no absorption when ω is less than the minimum energy separation of two surface states, so for some ω there is a discontinuous change in the surface resistance to aero.) The formula then used for calculating the surface resistance is given by eqn. (4.28b). Although there is much extra work in doing a double integral, one is rewarded by obtaining the surface resistance over a wide frequency range, not just at one frequency, in doing the double integral once! Some results of this calculation are summarized graphically in Figures 6.1a, b, c below.



$T = .1\ T_c$

$\omega = .1\ \Delta$

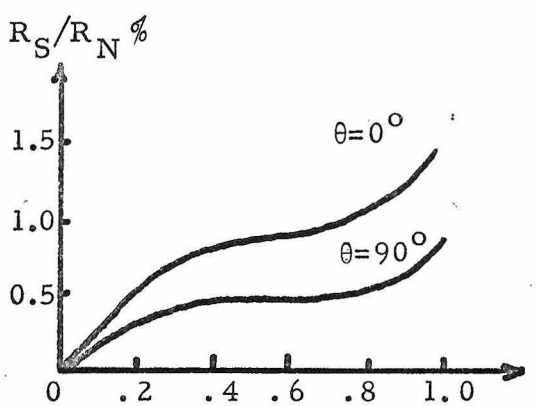
Figure 6.1a



$T = .4\ T_c$

$\omega = .1\ \Delta$

Figure 6.1b



$T = .5\ T_c$

$\omega = .3\ \Delta$

Figure 6.1c

SURFACE RESISTANCE DUE TO SURFACE

STATES AT "LOW T"

It should be mentioned first that the requirement of self-consistency on δ, the penetration depth, is least stringent for this case. The $\mathrm{Im}\,\delta << \mathrm{Re}\,\delta$ here. $\mathrm{Re}\,\delta$ changes by about 10% as H_{dc} varies over its range while R_S/R_N runs from zero to about 1%. So we can use, to very high accuracy, $\delta = \mathrm{Re}\,\delta$ only. And using δ for $H_{dc} = 0$ merely makes an overall consistent error in R_S/R_N of about 20%. The results in Figures 6.1 use δ at $H_{dc} = 0$ and $T = 0$. (It is well known that penetration depths are essentially independent of T up to $T \sim .5T_c$.) Such approximations are adequate for the qualitative behavior desired.

Figure 6.1a displays results which can be most closely related to experimental data available. Glosser[3] has measured R_S/R_N at $\theta = 0^\circ$ and 90° as a function of H_{dc} for $T \cong .2\ T_c$, $\omega \cong .07\Delta$. His results are consistent with ours in that R_S/R_N increases monotonically with H_{dc}, R_S/R_N is larger for $\theta = 0^\circ$ than for 90°, and R_S/R_N is about a maximum of 1.5%.

Glosser's work[3] has the only data taken on the angular (θ) dependence and then only for $H_{dc} \cong .9\,H_c$, near the critical field. In Figure 6.1b we have obtained some very interesting results regarding the θ dependence on R_S/R_N for $\omega \cong .1\Delta$, $T \cong .4\ T_c$ as a function of H_{dc} for which there is no experimental check, as yet. For $H_{dc} \cong H_c$ R_S/R_N is larger for $\theta = 0^\circ$ than for $\theta = 90^\circ$. But for $H_{dc} \cong .5H_c$, the surface resistance is larger for $\theta = 90^\circ$ than for $\theta = 0^\circ$! And we believe this conclusion to be generally true if surface states exist with the only proviso being that the superconductor show

little or no anisotropy in the gap, Δ, for $H_{dc} = 0$ cases - a basic assumption of this model. This effect mostly depends on the delicate interplay of occupation probability, density of states in K_y, K_z space and the coupling of field to quasi-particle motion but not critically upon matrix elements, or spectrum shape! It is governed principally by the terms

$$\left[f_T(\epsilon_m) - f_T(\epsilon_\ell)\right]\left\{\cos^2\theta \frac{V_e^2}{V_o^2(V_e^2 - V_o^2)^{\frac{1}{2}}} + \sin^2\theta\,(V_o^2 - V_e^2)^{\frac{1}{2}}\right\}$$

The "$\theta = 0^o$ piece" is $V_e^2/V_o^2(V_e^2 - V_o^2)^{\frac{1}{2}}$ and the "$\theta = 90^o$ piece "is $(V_o^2 - V_e^2)^{\frac{1}{2}}$ which are plotted in Figure 6.2.

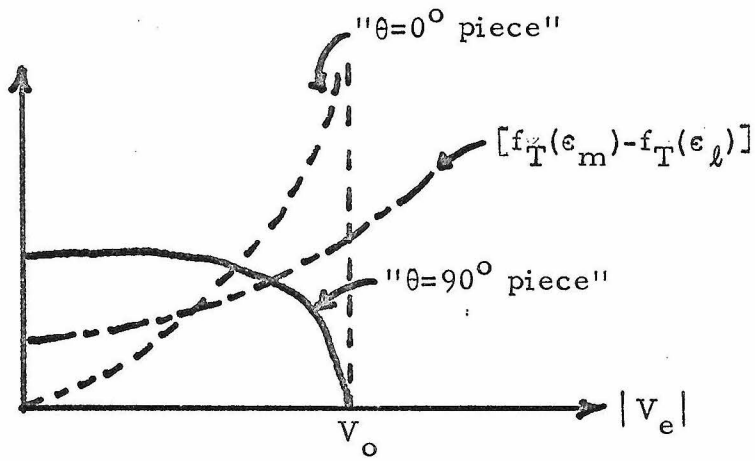


FIGURE 6.2

Although the "$\theta = 0$ piece" has a singularity at V_o, there is no problem since the area for finite width V_e is finite. We can interpret the curves. If the field is polarized along the y-axis ($\theta = 0^o$), the

quasi-particles traveling most nearly parallel to the y-axis absorb the most energy; those traveling along the z-axis absorb none and conversely for polarization along the z-axis ($\theta = 90^{o}$). Suppose $V_o \sim \Delta$. The states for $\epsilon \sim 0$ have a strong occupation probability and couple strongly to the $\theta = 0^{o}$ case but weakly to the $\theta = 90^{o}$ case, since $|V_e| \sim \Delta$ means $K_y \sim K_f$ and $K_z \sim 0$. The states higher in energy have such low occupation probability that their effects barely modify the results, $\theta = 0^{o}$ polarization dominates $\theta = 90^{o}$ polarization. Suppose $V_o \sim \Delta/2$, $0 \leq |V_e| \leq \Delta/2$. Now probability occupation is low <u>but</u> all transitions have about the same value so in each case (0^{o}, 90^{o}) we must consider the whole range of $|V_e|$. But in general there are more states coupling to the $\theta = 90^{o}$ case (small $|V_e|$'s) than to the $\theta = 0^{o}$ case (large $|V_e|$'s). Hence $\theta = 90^{o}$ dominates over $\theta = 0^{o}$ polarization.

Next we consider the superconducting surface resistance at "intermediate T", $.5\ T_c \leq T \leq .8\ T_c$, and frequencies $\omega \leq .1\Delta$. All the investigators have observed that $H_{dc} \sim H_c$ causes the surface resistance to decrease anomalously. Pippard[1] first noticed this in 1949 and it has remained a mystery since that time. This suppression effect ranges in size $-20\% \leq [R_S(H_{dc} \sim H_c) - R_S(H_{dc} = 0)\ ]/R_N \leq -1\%$. It was pointed out earlier that normalizing surface impedance to R_N could be misleading and the particular case is the worst. At "intermediate T", $R_S(H_{dc} = 0) \leq .1\, R_N$ so that the surface impedance variations are more substantial than one might expect from looking at the data. It would be more useful if

experimentalists plotted $[R_S(H_{dc}\neq 0)-R_S(H_{dc}=0)]/R_S(H_{dc}=0)$. Typically this latter quantity would be 50% to 1000%, a rather large change in behavior.

At "intermediate T" and $\omega \leq .1\Delta$, there seem to be three mechanisms at work to consider, but two are unimportant. First, there are transitions between surface states which increase the surface resistance, but R_{S_1}/R_N is only about 1%. Then there are transitions from surface states to continuum states, but once again for small frequencies, $\omega << \Delta$, $R_{S_2}/R_N \sim +1\%$. This contribution is progressively smaller as ω is lesser. Finally there are transitions of thermally excited quasi-particles between states in the continuum. (We have had to estimate the sizes of these three different mechanisms since the Fermi factor is sufficiently large for $T \geq .6\ T_c$ to guarantee occupancy of all types of states.) This latter case would exist if $H_{dc} = 0$, whereas the previous two cases depend upon $H_{dc} \neq 0$. It is the onset of continuum occupancy by thermally excited quasi-particles at $T \simeq .6\ T_c$ which starts the process whereby a superconductor gradually turns into a normal metal at $T = T_c$. And our calculations indicate that $H_{dc} \neq 0$ has such a marked effect as to account for the suppressed surface resistance solely through modifications of the continuum transitions mechanism. $R_{S_3}/R_N >> R_{S_2}/R_N$ so that we may ignore the other two processes involving surface states. (Throughout this Chapter we are picking ω, T, etc. so as to isolate the various processes from one another.) So let us consider the expression for R_{S_3}/R_N; we do the integral over ϵ_2

to drop the δ -function in eqn. (4.30) and are left with a non-trivial triple integral that must be evaluated numerically. There are a number of features to be considered here. Once again there is the question of which δ to use for the self-consistency relation. $\mathrm{Re}\,\delta \geq 10\ \mathrm{Im}\,\delta$ and $\mathrm{Re}\,\delta$ changes by about 10% due to H_{dc}. However, R_S changes by about 100% so that we are once again justified in using $\mathrm{Re}\,\delta$ for $H_{dc} = 0$ to deduce the proper qualitative behavior.

The reader is reminded that this triple integral is no more than the anomalous skin effect in superconductors, and normal metals if $\Delta = 0$. As mentioned in Chapter 5, only quasi-particles traveling nearly parallel to the surface are effective in absorption. Hence only "small ξ_t" are important which for our case turns out to be $-10^3\Delta \leq \xi_t \leq 2\Delta$. ($\xi_t \leq 2\Delta$ since for $\xi_t \geq 0$ the minimum $\epsilon = (\Delta+\xi_t^2)^{\frac{1}{2}}$ and $\epsilon \geq 3\Delta$ can be ignored due to the Fermi factors.) That only "small ξ_t" are involved is guaranteed by the convergence of our integral being very rapid outside this range.

For continuum states, there are many terms in the matrix element, $M^a_{\ell m}$; each term has an exponential factor which either varies as $e^{i(p_\ell+p_M)\lambda}$ or $e^{i(p_\ell-p_m)\lambda}$. The former class oscillate rapidly as ξ_t runs over its range while the latter class have slowly varying phases. Therefore we keep only these terms and drop the other which make little contribution. (It was not necessary to do this for surface state problems since the range of ξ_t integration is two orders of magnitude less.)

The triple integral is so expensive to do that only a few important cases were done; viz. $\omega = 0.1\Delta$, $T = .6\,T_c$, $.7\,T_c$, $.8T_c$, $H_{dc} = H_{c-}$. The result is in excellent agreement with the data of Glosser[3] in that the surface resistance falls monotonically with T over this range and is within 30% of the measured values. Furthermore, for $H_{dc} \to 0$, our results are within 10% of the Glosser data. This serves as a check on the method.

The results are displayed below for comparison with Glosser's[3] data.

T/T_c	$R_S(H_{dc} = 0)/R_N$ %		$[R_S(H_{dc} = H_{c-}) - R_S(H_{dc} = 0)]/R_N$ %	
	theo.	exp.	theo.	exp.
.6	2.9	3	1.2-.45 = .75	1
.7	6.9	7	.8-.75 = .05	0
.8	9.8	10	.3-3.8= - 3.5	-4

In the fourth column we have written our result as the sum of two numbers to emphasize that the two effects of surface state transitions and continuum state transitions compete. The first number is surface state data.

The experimental data originally provided the clue to understanding the negative shifts in surface resistance. $R_S(H_{dc} = 0)$ does not start to rise substantially from zero until $T \cong .6\,T_c$. This is, however, also the same point where $R_S(H_{dc} \neq 0) - R_S(H_{dc} = 0)$,

which is fairly flat with T up to this point, begins to develop a negative slope. Hence, it is likely that H_{dc} is modifying that mechanism responsible for absorption when $T \geq .6\, T_c$; viz., continuum transitions.

But what is the reason for this negative shift in surface resistance? When $H_{dc} = 0$, there is uniform probability to find a quasi-particle anywhere in the metal. If $H_{dc} \neq 0$, though, the situation is vastly different. The probability to find quasi-particles, then, varies with space and even the spatial behavior varies with ϵ, ξ_t and φ in a complicated way. H_{dc} sets up the effective potential $V_e = V_o \cos\varphi$ described in Chapter 3. ($V_o \sim H_{dc}$) The principle effect of V_e is to repel quasi-particles from the surface region $\sim\lambda_e$ for certain important values of the parameters. If $|\xi_t| \leq 20\Delta$, $V_e > 0$, and $\epsilon \leq \Delta + V_e$, $r_\pm$ are complex so that the probability to find quasi-particles falls exponentially in the surface region. But it is the quasi-particles in the surface region for $|\xi_t| \leq 20\Delta$ which are the most important absorbers of energy; a significant fraction of quasi-particles no longer plays a role in interacting with the electromagnetic field. For $|\xi_t| \geq 20\Delta$, $r_\pm \rightarrow p_\pm$ which are usually pure real. But this does not mean that the u, v amplitudes are spatially invariant. Even here there is a smaller probability to find quasi-particles in the surface region, but only fractionally less - not by orders of magnitude as when $r_\pm$ are complex. This phenomenon is related to the induced static currents created by H_{dc} and could probably be

obtained if the u, v functions were calculated for an infinite medium carrying a uniform current along the y-axis. That is, it is due to transforming the infinite medium excitation spectrum from the coordinate system at rest with respect to the screening current to the lab frame in which the superconductor is mounted.

In any case, the net effect of H_{dc} is to make most of the matrix elements, $M^{a}_{\ell m}$, smaller as H_{dc} increases. There are no resonance effects whereby $M^{a}_{\ell m}$ increases substantially.

That this occurs is very important for it will allow us to explain several important features of the reactance behavior without resort to further calculations.

Finally, there is the superconducting surface resistance at "high T", $T \geq .8\, T_c$, and low frequencies $\omega \leq .1\Delta$. Here, the experimental data show that surface resistance starts to increase again with H_{dc} so that at $T \sim .95\, T_c$, H_{dc} once more causes $R_S(H_{dc} \neq 0) > R_S(H_{dc} = 0)$. Finally for $T \geq .95\, T_c$, $R_S(H_{dc} \neq 0) \rightarrow R_S(H_{dc} = 0) \rightarrow R_N$. This last realm is the easiest to understand since $\Delta \rightarrow 0$ rapidly and the superconductor is turning into a normal metal. But why does the shift in R_S change sign and become positive for sufficiently high T? First there is the obvious fact that H_{dc} or V_o have decreasing maximum values as T increases. [$H_c(T) \cong H_c(T=0)(1 - T^2/T_c^2)$] Thus the potential V_e weakens for higher T and is less effective in keeping quasi-particles away from the surface region. Also quasi-particles of larger energy are more important

at higher T and these feel the effect of V_e much less than those of small ϵ. Hence the suppression effect doesn't work as well at "high T". Thus the suppression effect can no longer overwhelm the positive effect of transitions from surface states to continuum states. In fact the latter mechanism dominates once again. We have only estimated the size of this effect but it is sufficient ($R_{S_3}/R_N \sim 2\%$) to give a net positive shift in R_S for large $H_{dc} \sim H_c$. Only one property of this mechanism is important to discuss here related to the experimental data. The data of the experimentalists clearly show that the maximum positive shift in R_S at "high T" shifts to lower T as ω increases. Lewis[4] summarizes all the cases and finds a nice monotonic behavior. For any ω, only surface states within ω of the lower edge of the continuum can contribute. As ω grows then more surface states absorb giving a larger surface resistance. But further, higher ω's use surface states from deeper in the gap and these have higher population/differences at any given T which also makes R_S grow. Thus the higher ω's begin to make a certain size R_{S_2} at lower T than the smaller ω's. When added to R_{S_3} the net result is as observed.

Since the above depends on two competing processes cancelling in a proper manner, a more direct way of observing the surface state to continuum state transition is desirable. We propose the following experiment be performed which, if successful, will be another verification to the existence of surface states. Let $T >> T_c$ and $\Delta \leq \omega \leq 2\Delta$ in tin, tantalum, indium, or aluminum. If $H_{dc} = 0$,

$R_S = 0$ since $\omega < 2\Delta$ means there is insufficient energy for pair creation. Now raise H_{dc} from zero. If R_S suddenly becomes non-zero for some $H_{dc} < H_c$, we are observing pair creation with one excitation going to the continuum edge and the other excitation going to the lowest lying surface state energy. Furthermore, $\omega \geq \Delta$ should require $H_{dc} \lesssim H_c$ and $\omega \leqslant 2\Delta$ should require $H_{dc} \gtrsim 0$ to observe the discontinuity in R_S. This is so since stronger H_{dc} pushes the lowest branch of the surface state spectrum to lower energy. Higher T's will smear the R_S discontinuity so every attempt should be made to keep $T \to 0$.

Our next order of business is the surface reactance behavior of superconductors when $H_{dc} \neq 0$. While surface resistance processes are susceptible to a simple physical interpretation, surface reactance phenomena are more elusive. Surface resistance is just another name for power absorption; a photon annihilates, kicking a quasi-particle from one state to another, conserving energy in the process. Surface reactance was shown to be a measure of field penetration and depends primarily on processes that always violate energy conservation, although staying within the bounds of the Heisenberg Uncertainty Principle. Nevertheless we have devised a picture that simplifies understanding surface reactance and is consistent with the predictions of the formalism heretofore developed.

Consider only virtual processes. Suppose that the initial configuration of quasi-particles changes to another arrangement in the presence of some A field at a particular frequency ω. If the

total energy of the final state is greater than the energy of the initial state plus the energy, $\hbar\omega$, then the effect of such processes is to reduce the penetration depth of the A field. And conversely, if the final state has less energy than the initial plus $\hbar\omega$, the penetration depth increases. Crudely speaking, we can imagine that if there is insufficient energy to make the transition from one state to another, the extra energy is extracted from the A field so that the penetration depth is decreased; whereas if there is more than enough energy, the surplus energy is returned to the A field increasing the penetration depth.

Let us view this in terms of the two particular processes -- scattering and pair creation.

For scattering the relevant terms in the j(A) relation are $[f_T(\epsilon_2) - f_T(\epsilon_1)]/(\epsilon_2 - \epsilon_1 - \omega)$. If $\epsilon_2 > \epsilon_1$, the thermal factors favor a net transition of quasi-particles upward from ϵ_1 to ϵ_2. When $\epsilon_2 > \epsilon_1 + \omega$, the generated current causes a field which is in such a direction as to oppose the original A field. For $\epsilon_2 < \epsilon_1 + \omega$ ($\epsilon_2 > \epsilon_1$), the opposite holds and the original A field is enhanced. If $\epsilon_2 < \epsilon_1$ (ϵ_2 always less than $\epsilon_1 + \omega$, of course), the current opposes the field. Although this transition is energetically favorable, it is statistically undesirable since the final state has higher population probability than the initial state. Thus, for $\epsilon_2 < \epsilon_1$, one might say that it is an anti-particle type quasi-particle which is making the transition and such a particle causes a current that is opposite of particle type quasi-particle currents.

Note that in the limit of $\omega \to 0$, all the currents always oppose the field so that penetration depth is decreased for any transition.

For pair processes, we are concerned with the terms

$-[1-f_T(\epsilon_\ell) - f_T(\epsilon_m)]\{1/(\epsilon_\ell+\epsilon_m+\omega) + 1/(\epsilon_\ell+\epsilon_m-\omega)\}$ in the j(A) relation. The energies ϵ_ℓ, ϵ_m here refer to the energies of a particle and anti-particle type quasi-particle either before or after interaction with the A field. When $\epsilon_\ell+\epsilon_m > \omega$, the only case of interest presently, virtual creation of particle and anti-particle type quasi-particle into states of energy ϵ_ℓ, ϵ_m always tend to make currents which further reduce the A field. Virtual annihilation of the two quasi-particles from states ϵ_ℓ, ϵ_m would yield current tending to increase the A field if it were possible to have greater population probability in ϵ_ℓ, ϵ_m than in the "vacuum state" - but this cannot be. Thus creation and annihilation always tend to screen the field from the interior when $\omega < \epsilon_\ell + \epsilon_m$.

In discussing surface resistance we carefully checked the necessity of maintaining self-consistency on δ and concluded that this was not too important. That, however, is not true when considering surface reactance changes; we must be very careful here. Suppose using eqn. (4.30) we found that $X_S \sim \delta^2 (1+H_{dc})$; it would not imply that X_S shifts positive with H_{dc} because $X_S \sim \delta$ from eqn (4.14) and $\delta \sim 1/(1+H_{dc})$ which means $X_S \sim 1/(1+H_{dc})$. X_S shifts negative with H_{dc}!! This type of situation generally pre-

vails for superconductors.

The discussion of surface resistance for $\omega < \Delta$ above required consideration only of the terms involving scattering of a single quasi-particle from one state to another; the pair terms could make no effects. However, for surface reactance or penetration depth problems, it is the pair terms which are the most important.

Consider the "low temperatures", $T \leqslant .5\, T_c$. All the Fermi factors, $f_T(\epsilon)$, are much less than one for continuum states, $\epsilon \geqslant \Delta$. Hence only the pair terms contribute substantially. And if $\omega < 2\Delta$, only the reactive part of this remains. But that is sufficient to explain the Meissner Effect -- the expulsion of the A field from the superconductor's interior. The penetration depth of a field due to the pair mechanism has a weak frequency dependence. In fact AGK have shown that at $T = 0$, δ varies by about 7% for $0 \leq \omega < 2\Delta$. For $\omega \leqslant .1\Delta$, $\epsilon_\ell + \epsilon_m >> \omega$, so that $\omega = 0$ results are perfectly adequate in this "low temperature" realm. All this, so far, applies irrespective of H_{dc}. A static magnetic field affects only the matrix-elements, $M_{\ell m}{}^b$; the effect, tho', is identical to what happened to continuum states at "intermediate T" when surface resistance was discussed -- H_{dc} reduces the size of most matrix elements. Thus the current of virtual pair processes tending to expel the A field is reduced and so the surface reactance or penetration depth increase as H_{dc} rises. Another way to say this is that H_{dc} creates an effective potential, V_e, which shoves quasi-

particles into the metal's interior. Hence the A field must penetrate further to interact with them. And this is what every experimenter has seen. The data at 700kc, 2mc, 1 gc, 3gc, 8.8gc, 24.5gc in various metals show a positive shift of surface reactance for "low T" with no evident frequency behavior.

When $T \geq .5\,T_c$, the surface reactance shifts with H_{dc} are much more complicated. At 700kc and 2mc, the shift is positive. At 1gc and 3gc, the shift is negative. At 8.8gc and 24.5gc, the shift is again positive. This pattern, too, has an explanation based strictly upon the reduction of matrix elements through H_{dc}.

The pair term works as at "low T" causing frequency independent positive reactance shifts so this can be regarded as a constant background. But in addition $T \geq .5\,T_c$ makes the Fermi factors large enough so that reactance effects caused by scattering quasi-particles from one level to another are important. This was discussed qualitatively above. It was noted that for $\omega = 0$ the virtual transitions lead only to further flux expulsion; so if the matrix element is squashed by H_{dc}, then A penetrates further. The mechanism is modified for $\omega \neq 0$. Transitions from ϵ_1 to ϵ_2 where $\epsilon_1 < \epsilon_2 < \epsilon_1 + \omega$ lead to "screening" currents that suck the A field further into the metal as shown earlier. That contribution becomes more important as ω increases. If H_{dc} suppresses the matrix element, then the penetration depth or surface reactance shifts negative which is what the experiments at 1 and 3 gc show. When ω is

further increased, though, the processes tend to cancel; there are as many transitions above resonance as below. Which in turn means that the pair processes again dominate and the surface reactance shifts positive. This corroborates the data at 8.8gc and 24.5gc.

Throughout the reactance analysis we have consistently ignored the effects of surface states. Their total contributions are much too small to make any difference.

All the reactance changes can be explained by reference only to the modification of continuum states wavefunctions.

CHAPTER 7

SURFACE EFFECTS

Early in the research questions arose about the effect of a surface on absorption in metals at low temperatures. We discuss the matter in this chapter.

Consider a normal metal at room temperature. The mean free path of the electrons is several angstroms long. So the current at some point inside the metal is determined by the electric field distribution only in a region of several angstroms radius about this point. And as long as these points are more than a few angstroms from the surface of the metal, there can be no surface effects, hence it makes no difference whether we use an infinite metal or one which makes explicit mention of a boundary. A typical electric field in this case penetrates the surface to about 5000 angstroms depth. Thus, the vast amount of absorption occurs in the interior away from any surfaces. Negligible error is made in computing surface impedances using infinite extent models.

Now, however, suppose that the temperature is lowered to near absolute zero. The mean free path of the electrons can be very long $\sim$ 100,000 angstroms. Then, the current at a point is determined by the field distribution in a huge region surrounding that point. For typical fields penetrating about 1000 angstroms, most of the electrons arriving at some point will have had an interaction with the surface. The surface effects seem to be playing a large role in determining

absorption processes. This phenomenon, the anomalous skin effect, yet, has been treated in the past entirely by considering only infinite medium models! (See work of Reuters and Sondheimer[11], Chambers[12], Mattis and Bardeen[7].) An exactly analogous situation prevails for superconductors and there, too, infinite medium results have been applied. (See Mattis and Bardeen[7], Pippard[1], AGK[13].) These calculations agree very well with the data so what is the paradox? We consider here only the case of the normal metal; the superconductor is handled similarly.

Bogolubov's equations adequately describe a normal metal under present circumstances if the gap, Δ, is set to zero. This is then just a free electron gas; the elementary excitations are just electrons and holes. We consider the same geometry as always with the surface of the metal in the y-z plane and extending infinitely in the +x direction only. For such a case the u, v functions are simply

$$u = \tfrac{1}{2} \sin px\,, \quad \xi > 0 \tag{7.1a}$$

$$= 0\,, \quad \xi < 0 \tag{7.1b}$$

$$v = \tfrac{1}{2} \sin px, \quad \xi < 0 \tag{7.1c}$$

$$= 0\,, \quad \xi > 0 \tag{7.1d}$$

where $\xi = \frac{p^2}{2m} + \frac{K_y^2}{2m} - \frac{K_z^2}{2m} - E_f$

The u, v here satisfy all the boundary conditions imposed earlier in Chapter 2 and 3.

Using these u, v functions, the electromagnetic kernel $Q(x, x')$, defined by $j(x) = \int_0^\infty Q(x, x')\, A(x')\, dx'$, as in Chapter 2 can be grouped into three terms

$$Q(x, x') = Q_\infty(x-x') + Q_\infty(x+x') + Q_S(x, x') \tag{7.2}$$

Those terms that depend only on the difference $(x-x')$ have a functional behavior called Q_∞; those depending on the sum $(x+x')$ also behave according to Q_∞; while the remaining terms are lumped into Q_S. It is not important presently to display any of those Q's more explicitly. The subscripts on the Q's refer to their physical significance which will become obvious later. Q_∞ is the infinite medium kernel while Q_S is the correction kernel for surface effects.

Suppose for the moment that Q_S can be ignored. Then the current at x, $j(x)$, produced by the field distribution, $A(x)$, is given by

$$j(x) = \int_0^\infty [Q_\infty(x-x') + Q_\infty(x+x')]\, A(x')\, dx' \tag{7.3}$$

Now suppose we define $A(x)$ for $x < 0$ to be $A(-x) = A(x)$; i.e., $A(x)$ is symmetrical about the plane at $x = 0$. Then a simple change of variables in the second tern of eqn. (7.3) yields after some rearrangement

$$j(x) = \int_{-\infty}^\infty Q_\infty(x, x')\, A(x')\, dx' \tag{7.4}$$

This transformation and final result is of large significance and in

part resolves the paradox of surface effects mentioned above because $Q_\infty(x-x')$ turns out to be identical to the total electromagnetic kernel calculated in an infinite medium! If we had calculated the kernel $Q(x, x')$ using infinite medium u, v functions

$$u = e^{ipx} \quad , \xi > 0 \tag{7.5a}$$

$$u = 0 \quad , \xi < 0 \tag{7.5b}$$

$$v = e^{ipx} \quad , \xi < 0 \tag{7.5c}$$

$$v = 0 \quad , \xi > 0 \tag{7.5d}$$

we would have found a $Q(x, x')$ that depends only on the difference $(x-x')$ and which is the same function as Q_∞ determined with u, v's in a semi-infinite metal. And that is why we labelled Q_∞ with the subscript ∞.

Eqn. (7.3) and its mathematical equivalent eqn. (7.4) can be neatly interpreted with the aid of some simple physical pictures. Refer to Figures 7.1a, b below.

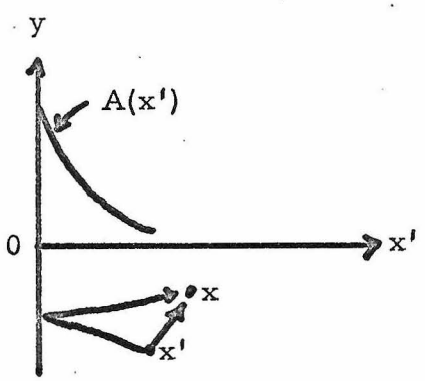


FIGURE 7.1a

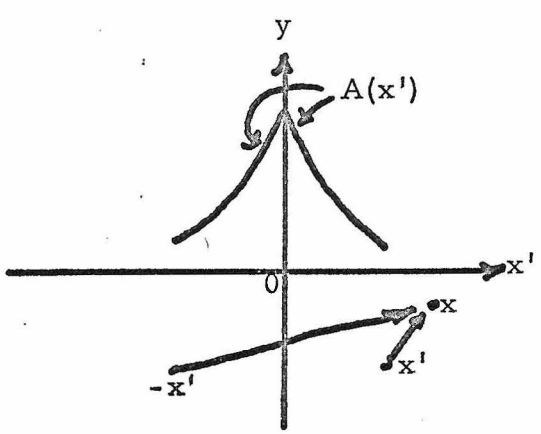


FIGURE 7.1b

Figure 7.1a is a diagram of eqn. (7.3) and Figure 7.1b is a diagram of eqn. (7.4). Consider the former. A photon annihilates at x' creating an excitation which produces a current at x. This excitation could arrive at x via two different journeys -- it could go directly from x' to x, a distance (x-x'), or it could get to x after a collision with the surface of the metal, a distance (x+x'). But the same effect is achieved if the photon annihilates at x' and goes directly to x in an infinite medium through a symmetrized A(x) field. In other words, an observer at x cannot distinguish whether the received excitation at x was generated at x' in the semi-infinite metal or at -x' in the infinite metal. Yet still another way -- an excitation going from x' to x via the indirect path feels the same forces as one traveling from -x' to x directly.

The processes just described are referred to as specular reflection and depends on the surface being perfectly flat. An opposite extreme is the perfectly rough surface so that an incident excitation comes off in any random direction thus making no net contribution to the current. For such a case, $Q_\infty(x+x')$ is ignored in eqn. (7.3) and the current is

$$j(x) = \int_0^\infty Q_\infty(x-x')\, A(x')\, dx' \qquad (7.6)$$

a situation referred to as diffuse reflection. Reuter and Sondheimer have calculated the surface impedance for specular and diffuse reflection in normal metals. The amazing result is that they differ only by about 10% for those two extreme cases.

Thus, it is plausible that infinite medium kernels are capable of yielding sensible surface impedance results if the terms lumped into $Q_S(x, x')$ are somehow small compared to those in Q_∞. That question, though, has not been dealt with heretofore. Q_S is far too difficult to calculate and even if available, the integro-differential equation for $\vec{A}$ would be hard to solve. We therefore use an iterative approach. Using the surface impedance determined from Q_∞, we find a δ to use in $A \sim e^{-x/\delta}$ as explained in Chapter 4. Then with this A distribution, we calculate a new surface impedance but include in the matrix element $M_{\ell m}^{a}$ the surface terms which appear in Q_S. ($M_{\ell m}^{b}=0$ in a normal metal) The complete matrix element $M_{\ell m}^{a}$ looks like

$$M_{\ell m}^{a} \sim \int_o^\infty \sin p_\ell x \sin p_m x\, e^{-x/\delta^*} dx$$

$$\sim \frac{1}{p_\ell - p_m + i/\delta^*} - \frac{1}{p_\ell - p_m - i/\delta^*} + \frac{1}{p_\ell + p_m - i/\delta^*} - \frac{1}{p_\ell + p_m + i/\delta^*}$$

The first two terms are just those that would appear for an infinite medium while the last two depend strictly upon the existence of a surface at $x = 0$.

Further evaluation of the above requires the sizes of typical variables.

$$\frac{1}{\delta} \sim 10^{-3}$$

$$p_{\ell,m} \sim 10^{-2} \left[-\xi_t \pm |\epsilon_{\ell,m}| \right]^{\frac{1}{2}}$$

and the important range of integrations is

$$-1000 \leqslant \xi_t \leqslant 0$$

$$0 \leqslant \epsilon_{\ell,m} \leqslant 2$$

so over most of the range of integration

$$p_\ell - p_m \sim 10^{-2} \frac{|\epsilon_\ell| - |\epsilon_m|}{(-\xi_t)^{\frac{1}{2}}}$$

$$p_\ell + p_m \sim 10^{-2} (-\xi_t)^{\frac{1}{2}}$$

Thus usually

$$|p_\ell - p_m| << |\frac{1}{\delta^*}|$$

$$|p_\ell + p_m| >> |\frac{1}{\delta^*}|$$

So the first two terms in $M^a_{\ell m}$ are each about $\delta^* \sim 10^3$; while the last two terms are about $1/p_\ell + p_m \sim 10^2/(-\xi_t)^{\frac{1}{2}}$ and subtract from each other. The conclusion is that the surface terms make a negligible contribution to the surface impedance!

The only possible way to see a surface effect is for penetration depths $\delta \leqslant 50\text{Å}$. Then the range of ξ_t integration is much smaller since the quasi-particles must make extremely small angles of non-parallelism with the surface to be effective. But no such metal is known where δ is so small.

Although the above treatment concerned normal metals, the

reasoning and conclusions are identical for pure superconductors with no static magnetic fields present.

When a superconductor with a static magnetic field is involved, none of the above conclusions are relevant. In fact, as has been seen, it is the spatial inhomogeneity which gives use to the many interesting features.

CHAPTER 8

GAP SELF-CONSISTENCY

A critical requirement for this theory is that the gap function, $\Delta(x)$, be independent of H_{dc} and spatially constant. This can be checked by seeing if the u, v wavefunctions for a step potential and constant gap are self-consistent. Of course, within a few angstroms of the surface the gap rapidly falls to zero, since u, v do, but this isn't important. We are concerned with the changes over 1000 Å distances.

The self-consistency relation was last encountered in Chapter 3 and we found that

$$\Delta(x) = V_{BCS} \sum_n v_n^*(x)\, u_n(x)\, [1-2f_T(\epsilon_n)] \qquad (8.1)$$

V_{BCS}, the electron-electron interaction, is independent of H_{dc} and T and so can be ignored here.

Whereas the expressions for surface impedance involved only excitations traveling nearly parallel to the surface, the gap relation eqn. (8.1) is strongly sensitive to excitations traveling in every direction. Hence surface states make very little effect, here; only continuum states are important and we make little error in considering continuum states in the approximation $-10^3\Delta \leq \xi_t \leq 10\Delta$ where $r_+ \cong r_- \cong p_+ \cong p_-$, all pure real. The ξ_t range is wide since we are no longer trying to couple to a radiation field; the electron-electron interaction is mediated by an isotropic phonon exchange

mechanism.

With these simplifications in mind it is useful to consider the product $v^*(x)\,u(x)$ for various regions and types of scattering states.

(1) $x \geq \lambda$, p+ incident scattering states

$$\begin{aligned} v^*(x)u(x) &= A_2 A_2^*(\epsilon-\xi_p) + B_2 B_2^*(\epsilon-\xi_p) + C_2 C_2^*(\epsilon+\xi_p) \\ &+ B_2 A_2^*(\epsilon-\xi_p)e^{-i2p_+x} + C_2 A_2^*(\epsilon-\xi_p)e^{i(p_- - p_+)x} \\ &+ A_2 B_2^*(\epsilon-\xi_p)e^{i2p_+x} + C_2 B_2^*(\epsilon-\xi_p)e^{i(p_+ + p_-)x} \\ &+ A_2 C_2^*(\epsilon+\xi_p)e^{i(p_+ - p_-)x} + B_2 C_2^*(\epsilon+\xi_p)e^{-i(p_+ + p_-)x} \end{aligned} \tag{8.2}$$

Integration over the fast oscillating terms will give a negligible contribution so eqn. (8.2) is simplified to

$$\begin{aligned} v^*(x)u(x) &= A_2 A_2^*(\epsilon-\xi_p) + B_2 B_2^*(\epsilon-\xi_p) + C_2 C_2^*(\epsilon+\xi_p) \\ &+ C_2 A_2^*(\epsilon-\xi_p)e^{i(p_- - p_+)x} + A_2 C_2^*(\epsilon+\xi_p)e^{i(p_+ - p_-)x} \end{aligned} \tag{8.3}$$

Around $x \sim \lambda$, the last two terms are very slowly varying. But deep inside the metal, $x >>> \lambda$, even the last two terms are fast varying; thus they can be neglected implying that the equilibrium gap is spatially constant as expected and determined in part by the terms

$$v^*(x)u(x) = (A_2 A_2^* + B_2 B_2^*)(\epsilon-\xi_p) + C_2 C_2^*(\epsilon+\xi_p) \tag{8.4}$$

(2) $x \geq \lambda$, p_- incident scattering states

Using the same approximations as before for $x >> \lambda$

$$v^*(x)u(x) = A_2 A_2^*(\varepsilon-\xi_p) + (C_2 C_2^* + D_2 D_2^*)(\varepsilon+\xi_p) \tag{8.5}$$

(3) $x \leq \lambda$, $p\pm$ incident

Using continuum states for the surface region and disregarding fast oscillating terms, which requires staying several angstroms away from the surface, gives

$$\begin{aligned} v^*(x)u(x) = {} & A_1 A_1^*(\varepsilon-\xi_r-V_e)2 + C_1 C_1^*(\varepsilon+\xi_r-V_e)2 \\ & + C_1 A_1^*(\varepsilon-\xi_r-V_e)\left[e^{i(r_--r_+)x} + e^{i(r_+-r_-)x}\right] \\ & + A_1 C_1^*(\varepsilon+\xi_r-V_e)\left[e^{i(r_+-r_-)x} + e^{i(r_--r_+)x}\right] \end{aligned} \tag{8.6}$$

In the surface region for the range of integration concerned, the exponentials can sometimes be expanded and only the lowest order terms retained; hence

$$\begin{aligned} v^*(x)u(x) \cong 2 \{ & [A_1 A_1^* + C_1 A_1^*](\varepsilon-\xi_r-V_e) \\ & + [C_1 C_1^* + A_1 C_1^*](\varepsilon+\xi_r-V_e)\} \end{aligned} \tag{8.7}$$

which is spatially constant. Let us consider this last transformation carefully.

For $\xi_t \leq -10\Delta$, $r_+-r_- \cong p_+-p_- \cong K_f(\Delta/E_f)^{\frac{1}{2}}(\Delta/-\xi_t)$. Whereas most superconductors have penetration depths, $\lambda \sim 500\text{Å}$-1000Å, the

parameter Δ/E_f has a range of a couple orders of magnitude. Typically $\Delta/E_f \sim 10^{-4}$ for tin, $\Delta/E_f \sim 10^{-5}$ for aluminum. Consider the exponentials in eqn. (8.6). The phase varies most rapidly at $x \sim \lambda$ and least rapidly as $x \to 0$. At $x \sim \lambda$, the phase varies in tin from about 1 radian to 0.1 radians as ξ_t runs its full range; and $A_1 C_1^*$, $C_1 A_1^*$ are largest when the phase is largest (near small ξ_t). So the neglect of spatial variation in eqn. (8.7) is somewhat questionable. That was for tin; but in aluminum the phase varies from about 1/3 radians to 0.03 radians and it is a good approximation to neglect spatial behavior. The smaller Δ/E_f and the smaller λ, the less is the maximum range of the phase; but furthermore, for smaller Δ/E_f and λ, the smaller is the range of ξ_t over which $A_1 C_1^*$, $C_1 A_1^*$ are non-zero so the spatial variation terms can be forgotten irrespective of the behavior. Conversely, superconductors with large Δ/E_f and λ have large phase variations and large ranges of ξ_t where $A_1 C_1^*$, $C_1 A_1^*$ are non-zero. Thus it is likely that the gap is really constant, independent of H_{dc} for superconductors with $\Delta/E_f \leq 10^{-5}$, $\lambda \sim 10^3 \text{Å}$. And that the gap has strong spatial variations due to H_{dc} if $\Delta/E_f \geq 10^{-3}$, $\lambda \geq 10^3 \text{Å}$.

From the above discussion, one might suspect that Δ/E_f is of some fundamental significance in the theory of superconductors. The original BCS theory of infinite homogeneous superconductors contained a quantity, ξ_o, called the "coherence length" which was identified as the average distance between two paired electrons in the momentum-spin states $\vec{k}\uparrow$, $-\vec{k}\downarrow$. Just as the electron is the

basic entity in the normal metal, the "pair" of size $\xi_o \sim {}^{K_f}/_{m}\,\Delta$ is the analogous particle in a superconductor. Thus our ${}^{\Delta}/_{E_f} \sim {}^{1}/_{K_f \xi_o}$; typically $\xi_o \sim 3 \cdot 10^3$ Å in tin and $\xi_o \sim 2 \cdot 10^4$ Å in aluminum. When the trend in gap variation for various ${}^{\Delta}/_{E_f}$ values quoted above is translated into the coherence length language, the behavior is immediately comprehensible. Δ varies little if $\xi_o >> \lambda$ and Δ varies strongly if $\xi_o << \lambda$. In the former case the A(x) field is strong only in a region much smaller than the size of a pair so the pairing is almost totally unaffected and Δ remains unchanged from its unperturbed value. In the latter case A(x) is uniform across a pair so pairing is modified in proportion to the strength of A(x) and hence Δ has strong spatial variations where A(x) has strong spatial variations. This result is not new. Caroli[15] has shown using the Landau-Gingburg equations that Δ does indeed have this behavior with H_{dc}, ξ_o, and λ. However, the L-G equations only apply to situations where $\xi_o << \lambda$ which is the same as requiring local electrodynamics to be true. Our surface impedance theory is primarily concerned with the opposite limit, $\xi_o >> \lambda$, non-local electrodynamics. Hence Caroli's results are only useful for $\xi_o << \lambda$. The self-consistent method used in this thesis does not suffer from any restrictions on the relative sizes of ξ_o and λ, and so provides an independent test of how various perturbations affect the gap, Δ.

That the L-G equations yield the correct qualitative behavior here is still quite a mystery.

In conclusion, we simply say that for the metals considered in this thesis, tin, indium, aluminum, tantalum, ξ_o/λ is sufficiently large that we expect the qualitative results of our surface impedance theory to always be valid.

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