THE RIESZ SPACE STRUCTURE OF AN ABELIAN W*-ALGEBRA

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ABSTRACT

Let M be an Abelian W*-algebra of operators on a Hilbert space $\mathcal{H}$. Let M_0 be the set of all linear, closed, densely defined transformations in $\mathcal{H}$ which commute with every unitary operator in the commutant M' of M. A well known result of R. Pallu de Barrière states that if φ is a normal positive linear functional on M, then φ is of the form $T \to (Tx, x)$ for some x in $\mathcal{H}$, where T is in M. An elementary proof of this result is given, using only those properties which are consequences of the fact that ReM is a Dedekind complete Riesz space with plenty of normal integrals. The techniques used lead to a natural construction of the class M_0, and an elementary proof is given of the fact that a positive self-adjoint transformation in M_0 has a unique positive square root in M_0. It is then shown that when the algebraic operations are suitably defined, then M_0 becomes a commutative algebra. If ReM_0 denotes the set of all self-adjoint elements of M_0, then it is proved that ReM_0 is Dedekind complete, universally complete Riesz space, which contains ReM as an order dense ideal. A generalization of the result of R. Pallu de la Barrière is obtained for the Riesz space ReM_0 which characterizes the normal integrals on the order dense ideals of ReM_0. It is then shown that ReM_0 may be identified with the extended order dual of ReM, and that ReM_0 is perfect in the extended sense.

Some secondary questions related to the Riesz space ReM are also studied. In particular it is shown that ReM is a perfect Riesz space, and that every integral is normal under the assumption that every decomposition of the identity operator has non-measurable cardinal. The presence of atoms in ReM is examined briefly, and it is shown that ReM is finite dimensional if and only if every order bounded linear functional on ReM is a normal integral.

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INTRODUCTION

This thesis will be primarily concerned with those properties of an Abelian W*-algebra M which follow from the fact that ReM is a Dedekind complete Riesz space. Of fundamental importance will be the rôle played by the normal integrals on ReM or alternatively, the ultraweakly continuous linear functionals on M. Wherever possible, techniques from the theory of Riesz spaces will be used, although it will be often advantageous to use techniques from the theory of operators.

Part I provides a short summary of background information from the theory of Riesz spaces, together with some results from operator theory. In II, von Neumann algebras are defined and it is shown that a W*-algebra with the Riesz decomposition property is necessarily Abelian. In III, attention is focussed on the order dual of ReM, where M is an Abelian W*-algebra. In particular, it is shown that every integral on ReM is normal except in a very pathological case, and that if M is not finite dimensional, then non-zero singular functionals exist. The presence of atoms in ReM is examined briefly.

The crucial result of R. Pallu de la Barrière is obtained in IV, which characterizes the normal integrals on an Abelian W*-algebra. In V, it is shown that the real part of an Abelian W*-algebra M is a perfect Riesz space; this is used to derive the well known result that M is a dual space as a Banach space, namely, M is the Banach dual of the Banach space of ultraweakly continuous linear functionals on M.

In VI, the space M_0 of (unbounded) closed transformations which "belong" to the Abelian W*-algebra M in a certain sense, are defined. An elementary proof is given that each positive self-adjoint element of M_0 has a unique positive square root in M_0. The algebraic structure of M_0 is examined in VII. It is necessary to give a lemma which replaces the spectral theorem for general self-adjoint transformations so that some crucial results of von Neumann and Murray are available within the framework developed.

It is shown in VIII that ReM_0 may be endowed with a partial order in which it becomes a Riesz space which contains ReM as an order dense ideal. IX shows that ReM_0 is a Dedekind complete, universally complete Riesz space. A generalization of the result of R. Pallu de la Barrière is obtained in X which leads to a characterization of the normal integrals on the order dense ideals of ReM_0. The extended order dual of ReM is examined in XI, and it is shown that ReM_0 is perfect in the extended sense.

Finally in XII, the results obtained in VI are used to give an elementary proof of the fact that any positive self-adjoint transformation in $\mathcal{N}$ has a unique positive self-adjoint square root.

I. PRELIMINARY INFORMATION

Riesz Spaces.

A partially ordered real linear vector space $(L, \leq)$, with elements $f, g, \ldots$, is called an ordered vector space if the partial order on L is compatible with the algebraic structure of L, i.e.,

(i) $f \leq g$ implies $f + h \leq g + h$ for every $h \in L$

(ii) $f \geq 0$ implies $af \geq 0$ for every real $a \geq 0$

An ordered vector space L is called a Riesz space if, for every pair $f, g \in L$, $\sup(f, g)$ exists in L.

If L is an ordered vector space, the subset $L^+ = \{f \in L: f \geq 0\}$ is called the positive cone of L. Elements of L^+ are called positive. If L is a Riesz space, we will write $\sup(f, g) = f \vee g$, $\inf(f, g) = f \wedge g$. $f^+ = f \vee 0$, $f^- = (-f) \vee 0$, $|f| = f \vee (-f)$. We have $f = f^+ - f^-$, $|f| = f^+ + f^-$. If $|f| \wedge |g| = 0$, then f and g are said to be disjoint and this is denoted by $f \perp g$. If D is an arbitrary subset of a Riesz space L the set $D^d = \{f \in L: f \perp D\}$ is called the disjoint complement of D. If ρ is a norm on the Riesz space L such that $\rho(f) \leq \rho(g)$ if $|f| \leq |g|$, then ρ is called a Riesz norm on L. Note that $\rho(f) = \rho(|f|)$ for any Riesz norm ρ on L. A Riesz space L has the Riesz decomposition property: if $0 \leq u \leq z_1 + z_2$, $z_1, z_2 \in L^+$, then there exist $u_1, u_2 \in L^+$ such that $u = u_1 + u_2$, and $u_1 \leq z_1, u_2 \leq z_2$.

The indexed subset $\{f_\tau : \tau \in \{\tau\}\}$ of the ordered vector space L is called directed upwards if for any $\tau_1, \tau_2 \in \{\tau\}$, there exists $\tau_3 \in \{\tau\}$ such that $f_{\tau_3} \geq f_{\tau_1}$, $f_{\tau_3} \geq f_{\tau_2}$ hold simultaneously. This is denoted by $f_\tau \uparrow_\tau$. If $f_\tau \uparrow_\tau$ and $f = \sup f_\tau$ exists in L, we will write $f_\tau \uparrow_\tau f$.

The linear subspace K of a Riesz space L is called a <u>Riesz subspace</u> of L whenever, for every pair f, g in K, the elements $f \vee g$, $f \wedge g$ are also in K. The linear subspace A of L is called <u>an (order) ideal</u> in L if A is solid, i.e., $f \in A$, $g \in L$, and $|g| \leq |f|$ implies $g \in A$. The ideal A in L is called a <u>band</u> whenever it follows from $0 \leq f_\tau \uparrow f$, $f_\tau \in A$ for all τ, that $f \in A$. If D is an arbitrary subset of L, then the intersection of all ideals (bands) containing D is again an ideal (band), in fact the smallest such containing D, and will be called the <u>ideal</u> (<u>band</u>) <u>generated by D.</u> If D consists of a single element f of L, the ideal (band) generated by f will be called the <u>principal ideal</u> (<u>band</u>) generated by f. Any band in the Riesz space L such that $A \oplus A^d = L$ holds is called a <u>projection band.</u>

The Riesz space L is called <u>Dedekind complete</u> if every non-empty subset of L which is bounded from above has a supremum. Equivalently L is Dedekind complete whenever, given the upwards directed set $0 \leq f_\tau \uparrow_\tau \leq g$ in L, it follows that there exists $f \in L$ such that $f_\tau \uparrow_\tau f$ in L. If L is a Dedekind complete Riesz space then every band is a projection band.

<u>The order dual of a Riesz space.</u>

The <u>real linear functional</u> φ on the Riesz space L is said to be <u>positive</u> whenever $\varphi(f) \geq 0$ for all $f \in L^+$. The real linear functional φ on L is said to be <u>order bounded</u> if for every $u \in L^+$, the number $\sup(|\varphi(f)| : |f| \leq u)$ is finite. The set of all order bounded linear functionals is denoted by $L^\sim$. Under the natural definitions of addition and scalar multiplication, $L^\sim$ is a real linear vector space, partially ordered by setting $\varphi_1 \geq \varphi_2$ whenever $\varphi_1 - \varphi_2 \geq 0$. With respect to this partial ordering $L^\sim$ is a Dedekind complete Riesz space.

The order bounded linear functional φ on the Riesz space L is said to be an integral whenever it follows from $0 \le u_n \downarrow 0$ that $\varphi(u_n) \to 0$ as $n \to \infty$. The collection of all integrals, $L_c^\sim$, is a band in $L^\sim$. The element $\varphi \in L^\sim$ will be called a singular functional if $\varphi \perp \psi$ for all $\psi \in L_c^\sim$. The set of all singular functionals $L_s^\sim$ is a band in $L^\sim$ and $L^\sim = L_c^\sim \oplus L_s^\sim$. The order bounded linear functional φ on a Riesz space L is said to be a normal integral if $u_\tau \downarrow_\tau 0$ implies $\inf_\tau |\varphi(u_\tau)| = 0$. The set of all normal integrals on L will be denoted by $L_n^\sim$. $L_n^\sim$ is a band in $L_c^\sim$ and we set $L_c^\sim = L_{c,sn}^\sim \oplus L_n^\sim$.

For any subset A of a Riesz space L, the Riesz annihilator A^o is defined by $A^o = \{\varphi : \varphi \in L^\sim, \varphi(f) = 0 \text{ for all } f \in A\}$. For any subset B in $L^\sim$, the inverse Riesz annihilator ${}^oB$ is defined by ${}^oB = \{f : f \in L, \varphi(f) = 0$ for all $\varphi \in B\}$. If A is an ideal in L, then A^o is a band in $L^\sim$. If B is an ideal in $L^\sim$, then ${}^oB$ is an ideal in L. If (L, ρ) is a normed Riesz space, denoted by L_ρ, L_ρ^* will denote the Banach dual of L_ρ. L_ρ^* is an ideal in $L_\rho^\sim$. For any subset $A \subseteq L_\rho$ the (Banach) annihilator $A^\perp$ is the set of all $\varphi \in L_\rho^*$ satisfying $\varphi(f) = 0$ for all $f \in A$. Similarly for $B \subseteq L_\rho^*$, the inverse annihilator ${}^\perp B$ is the set of all $f \in L_\rho$ satisfying $\varphi(f) = 0$ for all $\varphi \in B$. If A is an ideal in L_ρ, then $A^\perp$ is a band of L_ρ^*. If B is an ideal in L_ρ^*, then ${}^\perp B$ is an ideal in L_ρ.

For any $\varphi \in L^\sim$, set $N_\varphi = \{f \in L : |\varphi|(|f|) = 0\}$. N_φ is always an ideal and is called the null ideal of φ; if Φ denotes the band of $L^\sim$ generated by φ, then $N_\varphi = {}^o\Phi$. Define C_φ, the carrier of φ, by setting $C_\varphi = (N_\varphi)^d$. If L is Dedekind complete, and if $\varphi, \psi \in L_n^\sim$, let Φ, Ψ denote the principal bands generated by φ, ψ; $\Psi \subseteq \Phi$ if and only if ${}^o\Phi \subseteq {}^o\Psi$.

For a more complete discussion of Riesz spaces, the reader is referred to [9], [10], [11].

Topologies on $\mathcal{L}(\mathcal{H})$.

Let $\mathcal{H}$ be a complex Hilbert space with elements x, y, z, . . . ; by $\mathcal{L}(\mathcal{H})$ denote the algebra of all (bounded) linear operators on $\mathcal{H}$ with elements S, T, ... $\mathcal{L}(\mathcal{H})$, equipped with the usual operator norm, is a B*-algebra. $\mathcal{L}(\mathcal{H})$ may be endowed with a variety of locally convex topologies which are important in the study of operator algebras. The coarsest locally convex topology on $\mathcal{L}(\mathcal{H})$ for which the maps $T \to Tx$ of $\mathcal{L}(\mathcal{H})$ into $\mathcal{H}$ is called the strong operator topology. The locally convex topology on $\mathcal{L}(\mathcal{H})$ generated by the family of semi-norms $T \to (Tx, y)$ is called the weak operator topology. Let $x_1, x_2, \ldots$, be a sequence of elements of $\mathcal{H}$ which satisfy $\sum_{i=1}^{\infty} \|x_i\|^2 < \infty$. The collection of all semi-norms of the form $T \to \{\sum_{i=1}^{\infty} \|Tx_i\|^2\}^{\frac{1}{2}}$ defines a locally convex topology on $\mathcal{L}(\mathcal{H})$ called the ultrastrong topology. Similarly, the collection of all semi-norms of the form $T \to |\sum_{i=1}^{\infty} (Tx_i, y_i)|$, where $\sum_{i=1}^{\infty} \|x_i\|^2 < \infty$, $\sum_{i=1}^{\infty} \|y_i\|^2 < \infty$, defines on $\mathcal{L}(\mathcal{H})$, the ultraweak topology. The algebraic structure of $\mathcal{L}(\mathcal{H})$ is not in general compatible with any of these topologies. However, the maps $S \to ST$, $T \to TS$ are continuous in each topology, while the map $T \to T^*$ is continuous in the weak topology and in the ultraweak topology. If $\mathcal{L}_1(\mathcal{H})$ denotes the unit ball of $\mathcal{L}(\mathcal{H})$ in the uniform operator topology, then on $\mathcal{L}_1(\mathcal{H})$, the strong operator topology coincides with the ultrastrong topology and the weak operator topology coincides with the ultraweak topology. For a more complete discussion, see [1].

The spectral theorem for self-adjoint operators

Let A be a self-adjoint operator of $\mathcal{L}(\mathcal{X})$. A will be called positive, written $A \geq 0$, if for every $x \in \mathcal{X}$, $(Ax, x) \geq 0$. To every positive self-adjoint operator A, there corresponds a unique positive self-adjoint operator B such that $B^2 = A$. B is called the positive square root of A, written $A^{\frac{1}{2}}$. If S is any self-adjoint operator which commutes with A, then S also commutes with $A^{\frac{1}{2}}$. For any operator $S \in \mathcal{L}(\mathcal{X})$, denote by N(S) the null space of S and by R(S) the closure in $\mathcal{X}$ of the range of S. If A is a self-adjoint operator in $\mathcal{L}(\mathcal{X})$, set $|A| = (A^2)^{\frac{1}{2}}$, $A^+ = \frac{1}{2}(A + |A|)$. For each α, $-\infty < \alpha < +\infty$, denote by P_α the orthogonal projection on $R((\alpha I - A)^+)$, where I denotes the identity operator in $\mathcal{X}$. The system $\{P_\alpha\}$ is called the spectral family of A, and has the property that each P_α commutes with every self-adjoint operator that commutes with A. If $\epsilon > 0$, a, b are real numbers such that $aI \leq A \leq (b-\epsilon)I$, let $\pi = \pi(\alpha_0, \ldots, \alpha_n)$ be a partition of [a, b] and set $s(\pi;A) = \Sigma_{k=1}^{n} \alpha_{k-1}(P_{\alpha_k} - P_{\alpha_{k-1}})$, $t(\pi;A) = \Sigma_{k=1}^{n} \alpha_k(P_{\alpha_k} - P_{\alpha_{k-1}})$. From the properties of the system $\{P_\alpha\}$, $s(\pi;A) \leq A \leq t(\pi;A)$. Let π_n, n=1, 2,.. be a sequence of partitions of [a, b], each of which is a refinement of its predecessor and such that $|\pi_n| \downarrow_n 0$. Then $\|A - s(\pi_n;A)\| \to 0$, $\|A - t(\pi_n;A)\| \to 0$ as $n \to \infty$. This is the spectral theorem. As a general reference on operator theory see [12]. An elegant exposition of the spectral theorem may be found in [9] where it is shown that if $\mathcal{B}(\mathcal{X})$ denotes the set of all self-adjoint elements of $\mathcal{L}(\mathcal{X})$, then any subset of $\mathcal{B}(\mathcal{X})$ which is an Abelian algebra that is closed in the weak operator topology, and contains the identity I, is a Dedekind complete Riesz space. The spectral theorem is then deduced as a special case of the

Freudenthal spectral theorem, which is valid in any Riesz space which has the property that every principal band is a projection band.

The polar decomposition.

If $T \in \mathfrak{L}(\mathcal{H})$, set $|T| = (T^*T)^{\frac{1}{2}}$. T has a unique decomposition (the polar decomposition of T) of the form $T = U|T|$ where U is a partial isometry whose initial space is $R(|T|)$. The relations $U^*T = |T|$, $|T^*| = U|T|U^*$, $|T| = U^*|T|U$ are valid.

Notation: If $\mathfrak{M}$ is a linear subspace of $\mathcal{H}$, $[\mathfrak{M}]$ will denote the norm closure of $\mathfrak{M}$ in $\mathcal{H}$. If M is any subalgebra of $\mathfrak{L}(\mathcal{H})$, $\mathfrak{D}$ any subset of $\mathcal{H}$, then $E^M_{\mathfrak{D}}$ will denote the projection on the closed subspace in $\mathcal{H}$ generated by all elements of the form $\{Tx: T \in M, x \in \mathfrak{D}\}$.

When necessary, the real numbers will be denoted by $\mathbb{R}$, the complex numbers by $\mathbb{C}$.

II. INTRODUCTORY REMARKS ON VON NEUMANN ALGEBRAS

Let $\mathcal{H}$ be a complex Hilbert space; $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators in $\mathcal{H}$. Let S be an arbitrary subset of $\mathcal{L}(\mathcal{H})$. Let $S' = \{T \in \mathcal{L}(\mathcal{H}): TS = ST \text{ for all } S \in S\}$. S' is called the commutant of S. It is clear that we always have $S \subseteq S''$.

A subalgebra M of $\mathcal{L}(\mathcal{H})$ will be called a $*$ - subalgebra of $\mathcal{L}(\mathcal{H})$ (or a self-adjoint subalgebra of $\mathcal{L}(\mathcal{H})$) if $S \in M$ implies $S^* \in M$.

Definition 2.1: A $*$ - subalgebra M of $\mathcal{L}(\mathcal{H})$ will be called a von Neumann algebra (briefly a W^*-algebra) in $\mathcal{H}$ if and only if $M = M''$.

We summarize briefly those properties of a von Neumann algebra M which will be needed most frequently in the sequel. The proof of these results and a complete list of the fundamental properties of von Neumann algebras may be found in [1].

If M is a von Neumann algebra, we shall denote by Re M the set of all self-adjoint operators in M. Re M is a real linear vector space, partially ordered by defining $A \le B$ for A, $B \in \text{Re } M$ whenever $(Ax, x) \le (Bx, x)$ holds for each $x \in \mathcal{H}$. By $(\text{Re } M)^+$, or simply M^+ we shall denote $\{T \in \text{Re } M: T \ge 0\}$.

(i) If $A \in \text{Re } M$, and if f is any real valued continuous function of a real variable, then f(A) also belongs to Re M. The spectral family of A belongs to Re M.

(ii) Each operator in M is a linear combination of unitary operators in M.

(iii) If M is any $*$ - subalgebra of $\mathcal{L}(\mathcal{H})$ containing the identity operator I then M is a von Neumann algebra if and only if M (or M_1, the unit ball of M in the uniform operator topology) is closed in any one of the weak, strong, ultraweak or ultrastrong topologies.

Remarks: (a) (i) implies that if $A \in (\mathrm{Re}\, M)^+$ then $A^{\frac{1}{2}} \in (\mathrm{Re}\, M)^+$.

(b) It follows from (ii) that an operator $A \in \mathcal{L}(\mathcal{H})$ belongs to M if and only if TU = UT for each unitary operator U in M'.

(c) (iii) provides a purely topological definition of a von Neumann algebra.

The order structure of a W*- algebra

We are primarily interested in the role played by the order structure of a W*- algebra. If $\mathcal{B}$ denotes the set of all self-adjoint operators in $\mathcal{L}(\mathcal{H})$, then it has been shown by Kadison [6] that if A, B are elements of $\mathcal{B}$, then $A \wedge B$ exists in $\mathcal{B}$ if and only if $A \geq B$ or $B \geq A$. On the other hand, if the W*- algebra M is Abelian, then Re M is a Dedekind complete Riesz space. For an elementary proof of this result see [9], Chapter 5. The proof is elementary in that it does not depend on the spectral theorem for bounded operators, which is then derived as a consequence of the Riesz space structure of Re M.

If the hypothesis of commutativity is deleted, then the Riesz space structure disappears. It has been shown by Sherman [18], that if N is a C*-algebra (a uniformly closed self-adjoint subalgebra of $\mathcal{L}(\mathcal{H})$) such that ReN is a lattice, then N is commutative. It is possible to obtain a relatively simple proof of this result in the special case of a W*-algebra.

Theorem 2.2: Let M be a W*-algebra. Assume that ReM has the Riesz decomposition property. Then M is Abelian.

Proof: It is sufficient to show BP = PB, where P is any projection of M, and $B \in \mathrm{Re}\, M$ satisfies $C \leq B \leq I$. Observe that if Q is any projection of M, and $A \in (\mathrm{Re}\,M)^+$, then $A \leq Q$ implies $AQ = QA$. In fact $A \leq Q$ implies $N(A) \supseteq N(Q)$, and $R(A) \subseteq R(Q)$. Thus $N(Q)$, $R(Q)$ are invariant under A so that $AQ = QA$. From $O \leq B \leq I$, follows $B \leq P + I-P$. Thus $B = B_1 + B_2$ with $O \leq B_1 \leq P$, $O \leq B_2 \leq I-P$. Thus $B_1P = PB_1$, $B_2P = PB_2$ hence also $BP = PB$.

Corollary 2.3: Let M be a W*- algebra, and assume that Re M is a Riesz space. Then M is Abelian.

Proof: Since Re M is a Riesz space, it has the Riesz decomposition property. For other results of this nature, see Ogasawara [14], and Fukamiya et. al., [4].

We shall frequently use the following result [1], Appendix II. Let M be any W*- algebra. Let $\mathcal{S} \subseteq \mathrm{Re}\,M$, and assume $\mathcal{S}$ is directed upwards in Re M. Suppose further that there exists a $T \in \mathrm{Re}\,M$ such that $S \leq T$ for each $S \in \mathcal{S}$. Then sup $\mathcal{S}$ exists in Re M.

It should also be observed that if M_p denotes the collection of all projections in a W*-algebra M, then M is the smallest uniformly closed *- subalgebra of $\mathcal{L}(\mathcal{H})$ containing M_p. Further, M_p is a complete lattice under the natural definition of $\wedge$ and $\vee$ — namely, if $\mathfrak{M}_1$, $\mathfrak{M}_2$ are subspaces in $\mathcal{H}$, let $[\mathfrak{M}_1, \mathfrak{M}_2]$ and $\mathfrak{M}_1 \cap \mathfrak{M}_2$ denote respectively the smallest closed subspaces of $\mathcal{H}$ containing $\mathfrak{M}_1$, $\mathfrak{M}_2$ and the intersection of $\mathfrak{M}_1$, $\mathfrak{M}_2$; if $E_{\mathfrak{M}_1}$, $E_{\mathfrak{M}_2}$ denote the orthogonal projection on these subspaces, then $E_{\mathfrak{M}_1} \vee E_{\mathfrak{M}_2} = E_{[\mathfrak{M}_1, \mathfrak{M}_2]}$ and $E_{\mathfrak{M}_1} \wedge E_{\mathfrak{M}_2} = E_{\mathfrak{M}_1 \cap \mathfrak{M}_2}$.

III. LINEAR FUNCTIONALS ON A W*-ALGEBRA

For any von Neumann algebra M, we shall denote by $M^{\#}$ the set of all complex linear functionals on M, by M* the Banach dual of M, and by M_n the set of all ultraweakly continuous linear functionals on M. An element $\varphi \in M^{\#}$ will be called positive and we shall write $\varphi \geq 0$ if $\varphi(T) \geq 0$ for each $T \in (\mathrm{Re}\,M)^{+}$. If $0 \leq \varphi \in M^{\#}$, then it follows immediately that, for each S, $T \in M$

(i) $\varphi(T^*) = \overline{\varphi(T)}$

(ii) $|\varphi(S^* T)|^2 \leq \varphi(S^*S)\,\varphi(T^*T)$ (Cauchy-Schwartz).

In particular each positive linear functional on M is uniformly bounded, with norm $\varphi(I)$ where I denotes the identity operator in $\mathcal{H}$.

Definition 3.1: A positive linear functional φ on M will be called normal if $0 \leq T_\tau \uparrow_\tau T$ in $(\mathrm{Re}\,M)^+$ implies $\sup_\tau \varphi(T_\tau) = \varphi(T)$.

The notion of normality is related to the ultraweak topology of M via

Theorem 3.2: Let φ be a positive linear functional on M. φ is normal if and only if φ is ultraweakly continuous.

For the proof see [1].

If $x, y \in \mathcal{H}$, we shall denote by $\omega_{x,y}(M)$ the canonical linear functional $T \to (Tx, y)$, for $T \in M$. It is clear that the canonical functionals $\omega_{x,x}(M)$ are normal, where $x \in \mathcal{H}$.

We will now assume for the remainder of this chapter that the W*-algebra M is Abelian. Equipped with the usual operator norm, ReM is a normed Riesz space which is Dedekind complete and norm complete. We shall denote the set of all (real) linear functionals on ReM by $(\mathrm{ReM})^{\#}$, the Banach dual of ReM by (ReM)*, the band of normal integrals on ReM by $(\mathrm{ReM})^{\sim}_{n}$, and the order dual of ReM by $(\mathrm{ReM})^{\sim}$. The Riesz space notation and terminology will be as in [10].

Lemma 3.3 $(\mathrm{ReM})^{\sim} = (\mathrm{ReM})^{*}$

Proof: If $\varphi \in (\mathrm{ReM})^{\sim}$, then $\varphi = \varphi_1 - \varphi_2$, $\varphi_1, \varphi_2 \geq 0$. By the Cauchy-Schwartz unequality, φ_1, φ_2 are uniformly bounded, hence $\varphi \in (\mathrm{ReM})^{*}$.

If $\varphi \in (\mathrm{ReM})^{*}$, then $\sup\{ |\varphi(S)| : S \in \mathrm{ReM}, 0 \leq S \leq |T| \} \leq \|\varphi\| \, \|T\|$; hence φ is order bounded.

The following extension theorem will be found useful. We assume that N is a linear space over the complex field and has the following properties:

(i) There exists a map $*$: $N \to N$ which satisfies, for all $f, g \in N$, $\lambda \in \mathbb{C}$

(a) $f^{**} = f$ (b) $(\lambda f)^{*} = \bar{\lambda} f^{*}$ (c) $(f+g)^{*} = f^{*} + g^{*}$.

(ii) If (N, ρ) is a normed linear space, then $\rho(f) = \rho(f^{*})$ for all $f \in N$.

(iii) If (N, τ) is a locally convex linear topological space, then $*$: $f \to f^{*}$ is τ-continuous.

Set $\mathrm{ReN} = \{f \in N : f = f^{*}\}$. Any $f \in \mathrm{ReN}$ will be called self-adjoint. If f is arbitrary in N, set $f = f_1 + i f_2$, $f_1 = \frac{1}{2}(f+f^{*})$, $f_2 = \frac{1}{2i}(f-f^{*})$. We will write $N = \mathrm{ReN} + i\,\mathrm{ReN}$. Denote by $N^{\#}$ (respectively $(\mathrm{ReN})^{\#}$) the set of all $\mathbb{C}$-linear (respectively $\mathbb{R}$-linear) maps $\Phi: N \to \mathbb{C}$ (respectively $\varphi: \mathrm{ReN} \to \mathbb{R}$. If $\Phi \in N^{\#}$, then for $f \in N$ define $\Phi^{*}(f) = \overline{\Phi(f^{*})}$. It follows easily that $\Phi^{*} \in N^{\#}$ and that the map $*$: $N^{\#} \to N^{\#}$ satisfies the conditions of (i) above with N replaced by $N^{\#}$. If $\Phi = \Phi^{*}$, then naturally Φ will be called self-adjoint.

Theorem 3.4: (i) Let Φ in $N^{\#}$ be self adjoint. The restriction of Φ to ReN is an element of $(\mathrm{ReN})^{\#}$. Conversely, if $\varphi \in (\mathrm{ReN})^{\#}$, then φ may be extended uniquely to a self adjoint element Φ of $N^{\#}$.

(ii) <u>If (N, ρ) is a normed linear space, and $\varphi \in (\text{Re}N)^{\#}$ is ρ-bounded, then its self-adjoint extension $\Phi \in N^{\#}$ is ρ-bounded and satisfies $\|\varphi\|\rho = \|\Phi\|\rho$.</u>

(iii) <u>If τ is a locally convex linear topology on N, and $\varphi \in (\text{Re}N)^{\#}$ is τ-continuous, then its self-adjoint extension $\Phi \in N^{\#}$ is also τ-continuous.</u>

<u>Proof:</u> (i) Let $\Phi \in N^{\#}$ be self-adjoint; if $f \in \text{Re}N$ then

$$\Phi(f) = \Phi^*(f) = \overline{\Phi(f^*)} = \overline{\Phi(f)}.$$

Hence the restriction of Φ to ReN is an element of $(\text{Re}N)^{\#}$.

Conversely let $\varphi \in (\text{Re}N)^{\#}$. For $f \in N$ set $\Phi(f) = \varphi(\frac{1}{2}(f+f^*)) - i\,\varphi(\frac{i}{2}(f-f^*))$. It follows easily that $\Phi \in N^{\#}$, Φ is self-adjoint and Φ is an extension of φ. That Φ extends φ uniquely follows from the fact that $N = \text{Re}N + i\,\text{Re}N$.

(ii) If $f \in N$, then $|\Phi(f)| = \Phi(\alpha f)$ where $\alpha = \exp(-i \arg \Phi(f))$. Thus $\Phi(\alpha f)$ is real; $\Phi(\alpha f) = \frac{1}{2}(\Phi(\alpha f) + \overline{\Phi(\alpha f)}) = \frac{1}{2}(\Phi(\alpha f) + \Phi^*(\overline{\alpha} f^*))$

$$= \Phi(\tfrac{1}{2}(\alpha f + \overline{\alpha} f^*)) = \varphi(\tfrac{1}{2}(\alpha f + \overline{\alpha} f^*)).$$

Observe that $\rho(f) \le 1$ implies $\rho(\frac{1}{2}(\alpha f + \overline{\alpha} f^*)) \le 1$. Hence $\|\Phi\|\rho \le \|\varphi\|\rho$ and since the opposite inequality is obvious we have $\|\Phi\|\rho = \|\varphi\|\rho$.

(iii) follows immediately from $\Phi(f) = \varphi(\frac{1}{2}(f+f^*)) - i\,\varphi(\frac{i}{2}(f-f^*))$ and the fact that $*: N \to N$ is τ-continuous.

We will therefore identify each $\varphi \in (\text{Re}N)^{\#}$ with its corresponding self-adjoint extension $\Phi \in N^{\#}$.

<u>Corollary 3.5</u>: <u>Let M be an Abelian W*-algebra: Then</u>

<u>(i) $\text{Re}(M^{\#}) = (\text{Re}M)^{\#}$ (ii) $\text{Re}(M^*) = (\text{Re}M)^* = (\text{Re}M)^{\sim}$</u>

<u>(iii) $\text{Re}(M_n) = (\text{Re}M)^{\sim}_n$ (iv) $\text{Re}((M_n)^*) = ((\text{Re}M)^{\sim}_n)^*$</u>

Integrals and normal integrals in $(\mathrm{Re}\,M)^{\sim}$

Recall that a linear functional $\varphi \in (\mathrm{Re}\,M)^{\sim}$ is called an integral, whenever it follows from $0 \leq T_n \downarrow 0$ in Re M that $\inf_n |\varphi(T_n)| = 0$. $\varphi \in (\mathrm{Re}\,M)^{\sim}$ is called a normal integral if $T_\tau \downarrow_\tau 0$ in Re M implies $\inf_\tau |\varphi(T_\tau)| = 0$. If φ is a normal integral on ReM, then $\varphi = \varphi^+ - \varphi^-$, where φ^+, φ^- are positive normal linear functionals on M. We have the following decomposition:

$$(\mathrm{Re}\,M)^{\sim} = (\mathrm{Re}M)^{\sim}_{s,c} \oplus (\mathrm{Re}M)^{\sim}_{c,\,sn} \oplus (\mathrm{Re}M)^{\sim}_{n}$$

where $(\mathrm{Re}M)^{\sim}_{n}$, $(\mathrm{Re}M)^{\sim}_{c,\,sn}$, $(\mathrm{Re}M)^{\sim}_{s,\,c}$ denote respectively the band of normal integrals on ReM, the band of integrals which are singular with respect to normality, and the band of singular functionals (cf[11] Note XVA). It is natural to ask whether every integral is a normal integral and it is obvious that we may confine our attention to positive functionals. If $0 \leq \varphi \in (\mathrm{Re}M)^{\sim}$, then φ is a normal integral if and only if, for each family pairwise disjoint projections of M, $\{E_i\}_{i \in \mathcal{J}}$ it follows that $\varphi(\sum_{i \in \mathcal{J}} E_i) = \sum_{i \in \mathcal{J}} \varphi(E_i)$ (cf [1] p. 65). From this it follows easily that if M is σ-finite, i. e. if every family of mutually orthogonal projections of M is at most countable, then the notions of normal integral and integral coincide. In particular, if $\mathcal{H}$ is separable, then every integral is normal. It is the aim of this section to show that, except in a certain pathological case, every integral is a normal integral for an Abelian W*-algebra M. This result is hardly surprising in view of the more general result of [7]. However, the proof in the present case is more algebraic in nature.

We first examine some relations that exist between the algebraic ideals of the Abelian W*-algebra M and the order ideals of the

Riesz space ReM. As usual a subset N of M will be called an algebraic ideal in M if N is a linear subspace of M and $S \in N$, $T \in M$ implies $ST \in N$. The notions of order ideal and band in ReM have been defined in Chapter I.

Remark: Let N be an algebraic ideal in M, and let $S \in M$. From the polar decomposition of S, it follows immediately that $S \in N \iff S^* \in N \iff |S| \in N$. It should also be observed that the closure of N in the weak operator topology is again an algebraic ideal in M.

Notation: For any subset D of the Riesz space ReM we shall denote by $\langle D \rangle$, (respectively $\{D\}$), the order ideal, (respectively band), generated by D.

If N is any subset of M, we shall denote by $\overline{N}^w$ the closure of N in the weak operator topology.

Theorem 3.6: (i) If N is an algebraic ideal (respectively weakly closed algebraic ideal) in M, then ReN is an order ideal (respectively band) in ReM.

(ii) If K is an order ideal (respectively band) in ReM, then $K + iK$ is an algebraic ideal (respectively weakly closed algebraic ideal) in M.

(iii) If N is an algebraic ideal in M then $\overline{N}^w = \{ReN\} + i\{ReN\}$.

(iv) If K is an order ideal of ReM, then $\{K\} = Re\overline{(K+iK)}^w$.

(v) If P is any projection of M, then $\langle P \rangle = \{P\}$.

Proof: (i) Let N be an algebraic ideal of M. From the above remark it follows immediately that ReN is a Riesz subspace of ReM. Assume $S \in ReM$ satisfies $0 \le |S| \le T$, where $T \in (ReN)^+$. By a generalization of the polar decomposition for bounded operators ([1] p. 11), there exists a unique element $A \in M$ which satisfies $|S|^{\frac{1}{2}} = AT^{\frac{1}{2}}$. Thus $|S| = (A^*A)T$ so that $|S| \in \mathrm{Re}N$; hence S^+, S^-, $S = S^+ - S^-$ all belong to ReN, so that ReN is an order ideal in ReM. If N is weakly closed then $0 \le T_\tau \uparrow_\tau T$, $T_\tau \in \mathrm{Re}N$, $T \in \mathrm{Re}M$ implies $T \in \mathrm{Re}N$ so that ReN is a band in ReM.

(ii) If K is an order ideal in ReM, then $K + iK$ is certainly a linear subspace of M. Suppose that $T \in K + iK$, and that U is a unitary element of M. From the uniqueness of the square root follows $|UT| = |T| \in K$. From $|\mathrm{Re}\, UT|$, $|\mathrm{Im}\, UT| \le |UT|$ follows $\mathrm{Re}(UT) \in K$, $\mathrm{Im}(UT) \in K$ so that $UT \in K + iK$ so that $K + iK$ is an algebraic ideal of M. If K is also a band in ReM, let N denote $\overline{K+iK}^{w}$. In particular N is an algebraic ideal of M. Let $0 \le T \in N$. From [1], p. 45 there exists a family $0 \le T_\tau \uparrow_\tau T$ with $0 \le T \in K$. Since K is a band, $T \in K$. Thus $K + iK$ is weakly closed.

(iii), (iv) follow readily and the proof will be omitted.

(v) It is sufficient to prove that $\{P\} \subseteq \langle P \rangle$. Let $T \in \{P\}$. By [9], lemma 26.5 $R(|T|) \subseteq R(P)$. Thus $|T| \le \lambda P$ for some constant λ so that $|T| \in \langle P \rangle$, therefore $T \in \langle P \rangle$. Hence $\langle P \rangle = \{P\}$.

<u>Lemma 3.7</u>: <u>Let $0 \le \varphi \in (\mathrm{Re}M)^\sim_c$. Let $N_\varphi = \{T \in \mathrm{Re}M : \varphi(|T|) = 0\}$. Then $\varphi \in (\mathrm{Re}M)^\sim_n$ if and only if N_φ is a band in ReM.</u>

<u>Proof</u>: Identical to [10], Note VIII, Theorem 27.5, and will be omitted.

Suppose now that P is any projection of M. A decomposition of P is a collection of projections $\{P_\alpha\}_{\alpha \in \mathfrak{A}}$ such that $P_\alpha \ne 0$, $P_{\alpha_1} \perp P_{\alpha_2}$ if $\alpha_1 \ne \alpha_2$ and $\vee_\alpha P_\alpha = P$. The cardinal of the index set $\mathfrak{A}$ is called the cardinal of the decomposition. A set X is said to have a measurable cardinal if there exists a countably additive measure ν on the collection of all subsets of X such that $\nu(X) = 1$, and $\nu(F) = 0$ for every finite subset F of X. If such a measure ν does not exist, then X is said to have a non-measurable cardinal. We now have the following theorem:

Theorem 3.8: Let $\varphi \geq 0$ be an integral on ReM and suppose that every decomposition of I, the identity of M, has a non-measurable cardinal. Then φ is a normal integral.

Proof: Let $\mathfrak{B}$ denote the family of all collections $\{P_\beta\}$ of mutually orthogonal projections P_β such that $\varphi(P_\beta) = 0$. $\mathfrak{B}$ is inductively ordered by inclusion so there is a maximal such collection $\{P_\alpha\}$ $\alpha \in \mathfrak{A}$, say. Let $P = \sup_\alpha P_\alpha$; then $\varphi(P) = 0$. If not, then for $A \in 2^{\mathfrak{A}}$ set $\nu(A) = \varphi(\sup_{\alpha \in A} P_\alpha)$ and observe that $\varphi(P) \neq 0$ contradicts the hypothesis that $\mathfrak{A}$ has non-measurable cardinal.

Let $\{P\}$ denote the principal band generated by P in ReM. By theorem 3.6 (v), $\{P\}$ coincides with the principal ideal $\langle P \rangle$ generated by P in ReM. In view of lemma 3.6, it is now sufficient to show that $\{P\} = N_\varphi = \{T \in \text{ReM}: \varphi(|T|) = 0\}$. Observe that if $T \in \{P\} = \langle P \rangle$, then there exists an integer k such that $|T| \leq k P$. Thus $\varphi(|T|) = 0$ so that $\{P\} \subseteq N_\varphi$. On the other hand, assume $T \in \text{ReM}$ and $\varphi(|T|) = 0$. By the spectral theorem, there exists a sequence $S_n = \Sigma_{i=1}^n \alpha_i Q_i^{(n)}$ in ReM with $\alpha_i > 0$, $Q_i^{(n)}$ projections in M such that $S_n \uparrow_n |T|$ in ReM. $\varphi(|T|) = 0$ implies $\varphi(Q_i^{(n)}) = 0$ for each i, n. Thus $Q_i^{(n)} \in \{P\}$ hence $S_n \in \{P\}$ for each n. Since $\{P\}$ is a band, $|T| \in \{P\}$ and so $\{P\} = N_\varphi$, and φ is normal.

Singular linear functionals on ReM

Following the discussion of the preceding section, we will write

$$(\mathrm{Re}\, M)^{\sim} = (\mathrm{Re}\, M)^{\sim}_{s,n} \oplus (\mathrm{Re}\, M)^{\sim}_{n}$$

In this section we shall examine some of the properties of the singular functionals $(\mathrm{Re}\, M)^{\sim}_{s,n}$.

Lemma 3.9: (i) Let $\varphi \in (\mathrm{Re}\, M)^{\sim}_{s,n}$. $N_{\varphi} = \{T \in \mathrm{Re}\, M : |\varphi|(|T|) = 0\}$ is an order dense ideal in Re M.

(ii) Let $(\mathrm{Re}\, M)^{a} = \{T \in \mathrm{Re}\, M : |T| \geq S_n \downarrow_n 0 \Longrightarrow \|S_n\| \downarrow 0\}$ and $(\mathrm{Re}\, M)^{an} = \{T \in \mathrm{Re}\, M : |T| \geq S_{\tau} \downarrow_{\tau} 0 \longrightarrow \|S_{\tau}\| \downarrow_{\tau} 0\}$

Then $(\mathrm{Re}M)^{a} = (\mathrm{Re}\, M)^{an} = {}^{\perp}((\mathrm{Re}\, M)^{\sim}_{s,n})$. Consequently $(\mathrm{Re}\, M)^{a}$ is an ideal in Re M.

Proof: The proof of the lemma is contained in [11] Theorem 50.4 and Theorem 53.7 (ii) of Note XVA.

Definition 3.10: Suppose P is a non zero projection of M. P will be called an atom if, for any projection Q of M, $0 \leq Q \leq P$ implies either $Q = 0$ or $Q = P$.

Lemma 3.11: Let P be an atom in M; then $P \in (\mathrm{Re}\, M)^{an}$.

Proof: For any $S \in \mathrm{Re}\, M$, $0 \leq S \leq P$ implies $S = \lambda P$ for some real λ, $0 \leq \lambda \leq 1$. This follows readily either from the spectral theorem or as in [9] page 55. Now observe that $\lambda_n P \downarrow_n 0$, $0 \leq \lambda_n \leq 1$ if and only if $\lambda_n \downarrow 0$. Thus also $\|\lambda_n P\| \downarrow_n 0$.

Theorem 3.12: (i) Let $P \in (\mathrm{Re}\, M)^{an}$ be a projection; then $P = \Sigma_{i=1}^{n} P_i$ where the P_i are disjoint atoms.

(ii) <u>$T \in (\mathrm{Re}\, M)^{an}$ if and only if $T = \sum_{i=1}^{\infty} \lambda_i P_i$, where the P_i, $i=1,2,\ldots$, are disjoint atoms and $|\lambda_i| \to 0$ as $i \to \infty$.</u>

<u>Proof</u>: (i) Let $P \in (\mathrm{Re}\, M)^{an}$ be a projection. Assume that P does not dominate a single atom. Assume that pairwise disjoint projections $Q_1, \ldots, Q_k$ have been defined satisfying $0 \neq Q_i < P$ and $\sum_{i=1}^{k} Q_i < P$. Let $E_k = P - \sum_{i=1}^{k} Q_i$. Note that $0 \neq E_k < P$. By hypothesis there exists a projection Q_{k+1} satisfying $0 \neq Q_{k+1} < E_k$. It follows immediately that Q_{k+1} is disjoint to Q_i, $i \leq k$ and $\sum_{i=1}^{k+1} Q_i < P$. Observe that $P \geq F_m = \vee_{n \geq m} Q_n$. We have $F_m \neq 0$, $P \geq F_m \downarrow_m 0$. Since $\|F_m\| = 1$ for each m, this contradicts the fact that $P \in (\mathrm{Re}\, M)^{an}$. Therefore, it follows that there exists an atom P_1 satisfying $0 \neq P_1 \leq P$. If $P - P_1 \neq 0$, there exists an atom P_2 satisfying $0 \neq P_2 \leq P - P_1$. The argument in the first section of this paragraph shows that this procedure breaks off after a finite number of steps, and the statement of (i) follows.

(ii) Assume first that $T = \sum_{i=1}^{\infty} \lambda_i P_i$, where the P_i, $i=1,2,\ldots$ are pairwise disjoint atoms and $|\lambda_i| \to 0$ as $t \to \infty$. If $S \in \mathrm{Re}\, M$ satisfies $0 \leq S \leq |T|$, then $S = \sum_{i=1}^{\infty} s_i P_i$ with $0 \leq s_i \leq |\lambda_i|$, and $\|S\| = \sup_i s_i$. It follows readily that $T \in (\mathrm{Re}\, M)^{an} = (\mathrm{Re}\, M)^{a}$. Conversely, assume $0 \leq T \in (\mathrm{Re}\, M)^{an}$. By the spectral theorem and (i) it follows that $T = \sum_{i=1}^{\infty} \lambda_i P_i$ where $0 \leq \lambda_i \leq \|T\|$, and P_i is an atom $i=1,2,\text{--}$. Assume $P_i \neq 0$ $i=1,2,\ldots$ To show $\lambda_i \to 0$ as $i \to \infty$ assume $\overline{\lim}_i \lambda_i > 0$; by choosing a subsequence if necessary we may assume that, for some $\delta > 0$, $\lambda_i \geq \delta > 0$ for $i=1,2,\text{--}$. Set $Q = \sum_{i=1}^{\infty} P_i$. Let N denote the integers, βN the Stone-Cech compactification of N and choose $\alpha \in \beta N - N$

(see for example [5]). For each $S \in ReM$ denote by f_S the element of $\ell_\infty(N)$ defined by $(f_S)_n = (S x_n, x_n)$ $n=1, 2, \ldots$ where $x_n \in \mathcal{H}$ has been chosen to satisfy $P_n x_n = x_n$, $||x_n|| = 1$. Denote by $\hat{f}_S$ the extension of f_S to a continuous function on βN. Define $\varphi \in (ReM)^{\#}$ by setting $\varphi(T) = \hat{f}_S(\alpha)$ for each $S \in ReM$. Observe that φ is linear and that $|(f_S)_n| \leq ||S||$. Then also $|\varphi(S)| \leq ||S||$ so that $\varphi \in (ReM)^* = (ReM)^\sim$. It is also clear that $\varphi \geq 0$, and that $\varphi(T) \geq \delta > 0$. Since $T \in (ReM)^a = (ReM)^{an}$, it is sufficient to show that $\varphi \in (ReM)^\sim_{s,n}$. Observe that $\varphi(Q) = 1 = \varphi(I)$ so that $\varphi(I-Q) = 0$. Write $\varphi = \varphi_s + \varphi_n$ where $0 \leq \varphi_s \in (ReM)^\sim_{s,n}$, $0 \leq \varphi_n \in (ReM)^\sim_n$. Since the P_i are atoms and by the definition of φ, $0 = \varphi(P_1 + \ldots + P_k) = \varphi_s(P_1 + \ldots + P_k) + \varphi_n(P_1 + \ldots + P_k)$ $= \varphi_n(P_1 + \ldots + P_k)$ for $k = 1, 2, \ldots$. Thus $\varphi_n(Q) = 0$; also $0 = \varphi(I-Q)$ $= \varphi_s(I-Q) + \varphi_n(I-Q)$. Hence $\varphi_n(I-Q) = 0$ so that $\varphi_n(I) = 0$. By Cauchy Schwartz $\varphi_n = 0$. Hence $\varphi \in (ReM)^\sim_{s,n}$ and the proof is complete.

Remark: The proof of (ii) of theorem 3.12 shows that if the dimension of M is not finite, then there exist non zero elements of $(ReM)^\sim_{s,n}$. In fact, let $\{P_i\}$ $i=1, 2, \ldots$ be a system of pairwise disjoint non zero projections of M. Let $\varphi \in (ReM)^\sim$ be defined via the P_i as in the proof of (ii) above. Let $F_m = \bigvee_{n \geq m} P_n$. Observe that $\varphi(F_m) = 1$ for each m. Since $F_m \downarrow_m 0$, it follows that $\varphi \notin (ReM)^\sim_n$, so that the singular part of φ is non zero.

Let $\mathcal{A}$ denote the set of atoms in ReM. In general the ideal generated by the atoms of ReM is not equal to $(ReM)^a$. However if $\{\mathcal{A}\}$ denotes the band generated by the atoms in ReM, and $\{(ReM)^a\}$ denotes the band generated by $(ReM)^a$, then we have the following

<u>Corollary 3.13</u>: $\{\mathcal{A}\} = \{(\text{ReM})^{an}\}$

<u>Proof</u>: $\{\mathcal{A}\} \subseteq \{(\text{ReM})^{an}\}$ trivially. On the other hand $\mathcal{A}^d \subseteq (\text{ReM}^a)^d$ by the above theorem. Therefore $\{\mathcal{A}\} = \mathcal{A}^{dd} \supseteq ((\text{ReM})^a)^{dd} = \{(\text{ReM})^a\}$

We may write ReM = $\{\mathcal{A}\} \oplus \{\mathcal{A}\}^d$ where $\oplus$ denotes the Riesz space direct sum. $\{\mathcal{A}\}$ will be called the atomic part of ReM, $\{\mathcal{A}\}^d$ the non-atomic part. It seems appropriate to call M purely atomic if $\{\mathcal{A}\}$ = ReM, and purely non-atomic if $\{\mathcal{A}\} = \{0\}$. Following ([9]), an ideal N in ReM will be called a maximal ideal if N $\neq$ ReM and if there is no ideal in ReM properly contained between N and ReM. In addition if N is a band, then N will be called a maximal band.

<u>Theorem 3.14</u>: (i) <u>If N is a band in ReM, then N is a maximal band if and only if there exists an atom P in ReM such that N = $\{P\}^d$.</u>

(ii) <u>If $\{\mathcal{A}\}$ denotes the atomic part of ReM, then $\{\mathcal{A}\}^d = \bigcap$ (N:N is a maximal band).</u>

(iii) <u>$\{\mathcal{A}\}$ = ReM if and only if $\text{ReM}^{\sim}_{s,n} = ((\text{ReM})^a)^{\perp}$.</u>

<u>Proof</u>: (i), (ii) follow exactly as in ([9] p. 57).

(iii) If $(\text{ReM})^{\sim}_{s,n} = (\text{ReM}^a)^{\perp}$, let $P \in \{\mathcal{A}\}^d$, P a projection and assume P $\neq$ 0. Choose $x \in \mathcal{H}$ such that Px = x and consider the canonical normal functional $\omega_{x,x}$. From $\omega_{x,x} \in \mathcal{A}^{\perp}$, $\omega_{x,x} \in \{\mathcal{A}\}^{\perp}$. In particular $\omega_{x,x} \in (\text{ReM}^a)^{\perp}$. Hence $\omega_{x,x} = 0$. It follows that $\{\mathcal{A}\}^d = \{0\}$, hence $\{\mathcal{A}\}$ = ReM.

Conversely, assume $\{\mathcal{A}\}$ = ReM = $\{\text{ReM}^a\}$. Observe that we always have $(\text{ReM})^{\sim}_{s,n} \subseteq (\text{ReM}^a)^{\perp}$. Assume $\varphi \in (\text{ReM}^{\sim})^+$ satisfies $\varphi(T) = 0$ for all $T \in \text{ReM}^a$. Write $\varphi = \varphi_s + \varphi_n$ where $0 \leq \varphi_s \in (\text{ReM})^{\sim}_{s,n}$, $0 \leq \varphi_n \in (\text{ReM})^{\sim}_n$. Since φ_s vanishes on $(\text{ReM})^a$ so also does φ_n. By normality and the assumption that ReM = $\{\text{ReM}^a\}$, it follows that $\varphi_n = 0$. Thus $\varphi = \varphi_s$ and the proof is complete.

We conclude this section with some remarks on the case when M is finite dimensional

Theorem 3.15: The following conditions on an Abelian W*-algebra M are equivalent:

(i) $ReM = (ReM)^{an}$.

(ii) M is finite-dimensional.

(iii) M is a reflexive Banach space.

(iv) $(ReM)^{\sim} = (ReM)^{\sim}_{n}$.

Proof: (i) $\Rightarrow$ (iv) : Assume M satisfies (i). From $ReM = (ReM)^{a} = (ReM)^{an}$ it follows immediately that every positive linear functional on ReM is a normal integral.

(iv) $\Rightarrow$ (i): Since the only singular order bounded functional on ReM is the zero functional, $ReM = ((ReM^{\sim})_{s,n})^{\perp} = (ReM)^{an}$.

(iv) $\Rightarrow$ (ii): It has been observed in the proof of Theorem 3.12, that if the dimension of M is not finite, then there exist non zero elements of $(ReM)^{\sim}_{s,n}$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (iv): Assume M is reflexive as a Banach space. Recall that $M_n \subseteq M^*$, and that M_n is norm-closed in M^* ([1] p. 38). Assume that $M_n \neq M^*$. Let $0 \neq \varphi \in M^*$ satisfy $\varphi \notin M_n$. By the Hahn-Banach theorem there exists $u \in (M^*)^*$ such that $u(\varphi) \neq 0$ but that $u(\omega_{x,y}) = 0$ for each $\omega_{x,y} \in M_n$. Since M is reflexive, there exists $T \in ReM$ such that $u(\omega_{x,y}) = (Tx, y)$ for all $x, y \in \mathscr{H}$; $u(\omega_{x,y}) = 0$ for all $x, y \in \mathscr{H}$ implies $T = 0$. Thus $u(\varphi) = \varphi(T) = 0$. This is a contradiction. Thus $M_n = M^*$, consequently $(ReM)^{\sim}_{n} = ReM^{\sim}$.

IV. NORMAL INTEGRALS ON AN ABELIAN W*-ALGEBRA

If φ is a positive linear functional on an arbitrary W*-algebra M, then it is well known that φ is normal if and only if $\varphi = \Sigma_{i=1}^{\infty} \omega_{x_i, x_i}$ where the system $\{x_i\} \in \mathscr{H}$ satisfies $\sum_{i=1}^{\infty} \|x_i\|^2 < \infty$ (cf [1], p. 54 Theorem 1). On an Abelian W*-algebra, however, the result of R. Pallu de la Barrière [15] states that every normal positive linear functional φ is of the form $\varphi(T) = \omega_{x, x}(T)$ for some $x \in \mathscr{H}$. The original proof of this result depends rather heavily upon representation theory and it is desirable to obtain a proof which uses the order structure of the Abelian W*-algebra more fully. Dye [3] has essentially treated this problem in a more general context and we shall give a short discussion of his results later, [see p. 29]. In the Abelian case, the situation is especially simple and in this section we shall present an elementary proof of the R. Pallu de la Barrière result which will be fundamental in later sections.

The support of a positive normal linear functional

In the following, it will be assumed that M is a general W*-algebra.

Lemma 4. 1: (cf [1], p. 61) Let φ be a normal positive linear functional on M. There exists a unique projection E_φ in M with the following properties:

(i) If E is a projection in M such that $\varphi(E) = 0$, then $E \leq I - E_\varphi$.

(ii) For each $T \in M$, $\varphi(T) = \varphi(TE_\varphi) = \varphi(E_\varphi T) = \varphi(E_\varphi TE_\varphi)$.

(iii) For each projection F in M such that $E_\varphi F E_\varphi \neq 0$, then $\varphi(F) \neq 0$.

(iv) If M is Abelian, let I-P be the component of the identity in the band N_φ of ReM where $N_\varphi = \{T \in ReM : \varphi(|T|) = 0\}$. Then $P = E_\varphi$.

Proof: Let $\mathcal{A} = \{E \in M : E$ is a projection: $\varphi(E) = 0\}$. Let $\{E_i\}_{i \in \mathcal{J}}$ be a maximal set of pairwise disjoint projections of $\mathcal{A}$. Set $I - E_\varphi = \Sigma_{i \in \mathcal{J}} E_i$. By the normality of φ, $I - E_\varphi \in \mathcal{A}$. Suppose that E is any member of $\mathcal{A}$ and that $E \neq I - E_\varphi$. Set $E' = E \vee (I - E_\varphi) - (I - E_\varphi)$. Observe that $E \vee (I - E_\varphi)$ is the projection on $R(E + I - E_\varphi)$. From $\varphi(E + I - E_\varphi) = 0$ and the spectral theorem for positive self adjoint operators it follows that $E \vee (I - E_\varphi) \in \mathcal{A}$. Further $E' \neq 0$, $E' \in \mathcal{A}$ and $E' \perp I - E_\varphi$. This contradicts the maximality of $\{E_i\}_{i \in I}$ and so we have (i). For any $T \in M$ we have $T = TE_\varphi + T(I - E_\varphi)$. By Cauchy-Schwartz $\varphi(T(I - E_\varphi)) = 0$. The rest of (ii) follows similarly. Uniqueness is immediate from (i). If F is any projection of M, then $\varphi(F) = 0$ implies $F \leq I - E_\varphi$ so that $F(I - E_\varphi) = F$. Thus $F E_\varphi = 0$ and also $E_\varphi F E_\varphi = 0$.

Finally, if M is Abelian, let I-P denote the component of the identity in the band $\mathcal{M} = \{T \in \text{Re}M : \varphi(|T|) = 0\}$. Observe that both $I - E_\varphi$, and I-P belong to $\mathcal{M}$. Thus $P = E_\varphi$.

Definition 4.2: If φ is a normal positive linear functional on M, the projection E_φ of the above lemma will be called the support of φ, and will be denoted by supp(φ) or simply, E_φ.

Remark If $\varphi = \omega_{x,x}(M)$ for some $x \in \mathcal{H}$, then $E_\varphi = E_x^{M'}$. In fact, if E is a projection of M such that $\omega_{x,x}(E) = 0$, then $Ex = 0$. Thus $E M'x = 0$ so that $E E_x^{M'} = 0$ so that $(I - E_\varphi) \leq I - E_x^{M'}$. On the other hand $((I - E_x^{M'})x, x) = 0$ so that $I - E_x^{M'} \leq I - E_\varphi$. Thus $E_\varphi = E_x^{M'}$.

Definition 4.3: Let $\varphi, \psi \in M_n^+$. We will say that ψ is absolutely continuous with respect to φ, and write $\psi \prec \varphi$ if $E_\psi \leq E_\varphi$.

Remark In the special case that $\psi \leq \varphi$, it is clear that $\psi < \varphi$. If the W*-algebra is Abelian, then $\psi < \varphi$ if and only if ψ is in the band generated by φ. (cf Chapter I and lemma 4.1, (iv))

Definition 4.4: A projection E of M will be called σ-finite, if, for each family of mutually disjoint projections $\{E_i\}_{i \in \mathcal{J}}$ with $E_i \leq E$ for all i, it follows that $E_i \neq O$ for an at most countable set of indices.

Lemma 4.5: Let φ be a positive normal linear functional on M, E_φ the support of φ. Then E_φ is σ-finite.

Proof: Let $\{E_i\}_{i \in \mathcal{J}}$ be any family of mutually disjoint projections of M with $E_i \leq E_\varphi$ for each $i \in \mathcal{J}$. If $\mathfrak{F}$ is any finite subset of $\mathcal{J}$ then

$$\varphi(\Sigma_{i \in \mathfrak{F}} E_i) \leq \varphi(E_\varphi) < +\infty$$

Consequently the number of indices $i \in \mathcal{J}$ for which $\varphi(E_i) \geq \frac{1}{n}$ is at most finite. Thus for all except a countable set of indices i, $\varphi(E_i) = 0$. By lemma 4.1, it follows that $E_i = 0$ except for an at most countable set of indices.

Lemma 4.6: Let M be an Abelian W*-algebra, φ a positive normal linear functional on M, E_φ the support of φ. There exists $x \in \mathcal{H}$ such that $E_x^{M'} = E_\varphi$.

Proof: This follows from the fact that E_φ is σ-finite and [1] p. 20.

It is immediate from lemma 4.6 that if φ is a positive normal linear functional on an Abelian W*-algebra M, then there exists an $x \in \mathcal{H}$ such that $\varphi < \omega_{x,x}$.

We now prove the following special case of a more general theorem of S. Sakai [16].

Lemma 4.7: Let φ, ψ be positive normal linear functionals on an Abelian W*-algebra M which satisfy $\psi \leq \varphi$. There exists $H \in M$ with $0 \leq H \leq I$ such that $\psi(T) = \varphi(TH)$ for all $T \in M$.

Proof: Let E_φ denote the support of φ. The Abelian W*-algebra ME_φ becomes a Hilbert algebra if we set $(A, B)_\varphi = \varphi(B^*A)$ for $A, B \in ME_\varphi$ ([1], p. 66). For $A, B \in ME_\varphi$ set $(A, B)_\psi = \psi(B^*A)$. From $\psi \leq \varphi$ follows

$$|(A, B)_\psi| = |\psi(B^*A)| \leq (\psi(B^*B))^{\frac{1}{2}} (\psi(A^*A))^{\frac{1}{2}}$$
$$\leq (\varphi(B^*B))^{\frac{1}{2}} (\varphi(A^*A))^{\frac{1}{2}} = \|B\|_\varphi \|A\|_\varphi$$

Thus $(A, B)_\psi$ defines on the Hilbert algebra ME_φ a bounded, positive, sesquilinear Hermitian form. There exists a positive self-adjoint operator, Ω, defined on the completion of ME_φ with respect to $(\ ,\ )_\varphi$, such that $(A, B)_\psi = (\Omega A, B)_\varphi$. For any C in ME_φ, let R_C denote the (right) multiplication operator on ME_φ defined by $R_C(A) = AC$ for $A \in ME_\varphi$. To show that Ω is given by multiplication on the left by some element H of ME_φ, it is sufficient to show that $R_C \Omega = \Omega R_C$ for each $C \in M$ ([1], p. 69, Theorem 1, and p. 57, Prop. 1). Let A, B be arbitrary in ME_φ. Then

$$(R_C \Omega A, B)_\varphi = (\Omega A, R_C^* B)_\varphi = (\Omega A, R_{C^*} B)_\varphi = (\Omega A, BC^*)_\varphi$$
$$= \psi((BC^*)^*A) = \psi(CB^*A) \qquad = \psi(B^*AC)$$
$$= (\Omega AC, B)_\varphi = (\Omega R_C A, B)_\varphi .$$

Thus $R_C \Omega = \Omega R_C$ holds for all C in ME_φ and the proof is complete.

Lemma 4.8: Let M be an Abelian W*-algebra and let $x \in \mathscr{K}$. Then $E_x^{M'} = \vee \{E_z^M : \omega_{z,z} = \omega_{x,x}(M)\}$.

Proof: For each $z \in \mathscr{K}$ such that $\omega_{z,z} = \omega_{x,x}(M)$, $E_z^{M'} = E_x^{M'}$. Thus $E_z^M \leq E_z^{M'} = E_x^{M'}$ so that $\vee \{E_z^M : \omega_{z,z} = \omega_{x,x}(M)\} \leq E_x^{M'}$. On the other hand, let F be any projection of M' which satisfies $F \geq E_z^M$ for each $z \in \mathcal{H}$ such that $\omega_{z,z} = \omega_{x,x}(M)$. Observe that $\omega_{U'x, U'x} = \omega_{x,x}(M)$ for each unitary operator $U' \in M'$. Thus $U'x \in R(F)$ for each unitary operator $U' \in M'$. Thus $R(F) \supseteq [M'x]$. Hence $F \geq E_x^{M'}$ and the lemma follows.

We are now in a position to proceed directly to the result of R. Pallu de la Barrière [15], which is the principal goal of this section.

<u>Theorem 4.9:</u> <u>Let φ be a normal positive linear functional on an Abelian W*-algebra M. There exists $y \in \mathcal{H}$ such that $\varphi = \omega_{y,y}$.</u>

<u>Proof</u>: Let E_φ be the support of φ. Choose $x \in \mathcal{H}$ such that $E_x^{M'} = E_\varphi$. It follows that φ belongs to the band generated by $\omega_{x,x}$. Hence $\varphi = \vee_n(\varphi \wedge n\,\omega_{x,x})$. Observe that $\psi_n = \varphi \wedge n\,\omega_{x,x}$ has the properties that (i) $\psi_n \uparrow_n$ (ii) $\psi_n \leq n\,\omega_{x,x}$ (iii) for each $0 \leq T \in M$ $\psi(T) = \lim_{n\to\infty}\psi_n(T)$. By lemma 4.7, there exists a sequence $\{H_n\}$ of positive operators of M which satisfy $H_n(I-E_x^{M'}) = 0$ such that $\psi_n(T) = (TH_nx, H_nx)$ for each $T \in M$. Let $z \in \mathcal{H}$ satisfy $\omega_{z,z} = \omega_{x,x}$. Then also $\psi_n(T) = (TH_nz, H_nz)$ for each n and for each $T \in M$. Since $\psi_n\uparrow_n$ it follows that $H_nE_z^M \geq H_mE_z^M$ for $n \geq m$. From lemma 4.8 $H_nE_x^{M'} \geq H_mE_x^{M'}$ for $n \geq m$. Thus $H_n\uparrow_n$. From $\psi(I) = \lim_{n\to\infty}\|H_nx\|^2$, it follows that the sequence of real numbers $\|H_nx\|^2$ is a Cauchy sequence. We now show that the sequence $\{H_nx\}$ is a Cauchy sequence in $\mathcal{H}$. In fact, for $n \geq m$,

$$\begin{aligned}\|H_nx-H_mx\|^2 &= \|H_nx\|^2+\|H_mx\|^2-(H_nx, H_mx)-(H_mx, H_nx)\\ &\leq \|H_nx\|^2+\|H_mx\|^2-(H_mx, H_mx)-(H_mx, H_mx)\\ &= \|H_nx\|^2-\|H_mx\|^2\\ &\longrightarrow 0 \text{ as } n, m \to \infty.\end{aligned}$$

There exists $y \in \mathcal{H}$ such that $y = \lim_{n\to\infty} H_n x$. Hence, for each $T \in M$

$$\psi(T) = \lim_{n\to\infty} (TH_n x, H_n x) = (Ty, y)$$
$$= \omega_{y,y}(T)$$

and the proof is complete.

In later sections we will show that if M is Abelian and the normal positive linear functional φ satisfies $\varphi < \omega_{x,x}$, then $\varphi = \omega_{Tx,Tx}$ where the "Radon-Nikodym derivative" T is a closed densely defined transformation which commutes with each unitary operator in M'. In [3], Dye has shown that if M is any W*-algebra, and if the cyclic projection $E_x^{M'}$, $x \in \mathcal{H}$, is finite then any normal positive linear functional φ which satisfies $\varphi < \omega_{x,x}$, is of the form $\varphi = \omega_{Tx,Tx}$ where T is a closed densely defined transformation which commutes with each unitary operator of M'. Dye shows that the finiteness of all cyclic projections is a necessary and sufficient condition for the universal validity of a Radon-Nikodym theorem of the above type, that is, as long as one insists on having closed transformations as "Radon-Nikodym derivatives." However, a partial Radon-Nikodym theorem holds for the class of W*-algebras which have no purely infinite projections. In fact if M is a W*-algebra which contains no purely infinite projections, and if the normal positive linear functional φ satisfies $\varphi < \omega_{x,x}$ for some x in $\mathcal{H}$, then there exists a vector y in $[M'x] \cap [Mx]$ such that $\varphi = \omega_{y,y}$ (see [3]).

V. THE PERFECTNESS OF AN ABELIAN W*-ALGEBRA

Throughout this section, M will always denote an Abelian W*-algebra. If $T \in (\mathrm{ReM})^+$, then T defines an element $\nu(T)$ of $(\mathrm{ReM})^{\sim\sim}_{n,n}$. In fact for each $\varphi \in (\mathrm{ReM})^{\sim}_n$ set $\nu(T)(\varphi) = \varphi(T)$ and observe that $0 \le \varphi_\tau \uparrow \varphi$ in $(\mathrm{ReM})^{\sim}_n$ implies $\nu(T)(\varphi) = \varphi(T) = \sup_\tau \varphi_\tau(T) = \sup_\tau \nu(T)(\varphi_\tau)$. To each $T \in \mathrm{ReM}$, $\varphi \in (\mathrm{ReM})^{\sim}_n$ set $\nu(T)(\varphi) = \varphi(T)$. It is clear that $\nu: \mathrm{ReM} \to (\mathrm{ReM})^{\sim\sim}_{n,n}$ is linear and preserves partial order. Further ν is 1-1, since $\nu(T) = 0$ implies $\nu(T)(\omega_{x,x}) = 0 = (Tx, x)$ holds for each x in $\mathscr{K}$, thus $T = 0$.

Theorem 5.1: The image of ReM under the canonical map ν is a Riesz subspace of $(\mathrm{ReM})^{\sim\sim}_{n,n}$. If $0 \le T_\tau \uparrow_\tau T$ in ReM, then $\nu(T_\tau) \uparrow_\tau \nu(T)$ in $(\mathrm{ReM})^{\sim\sim}_{n,n}$.

Proof: As in [10], Note VII.

Let $0 \le u'' \in (\mathrm{ReM})^{\sim\sim}_{n,n}$. Recall that u'' may be considered as a linear functional on M_n and that any canonical linear functional $\omega_{x,y} \in M_n$, $x, y \in \mathscr{K}$ has a decomposition of the form

$$\omega_{x,y} = \tfrac{1}{4}\{(\omega_{x+y,x+y} - \omega_{x-y,x-y}) + i(\omega_{x+iy,x+iy} - \omega_{x-iy,x-iy})\}.$$

Observe the following properties of $u''(M_n)$:

(i) $\overline{u''(\omega_{x,y})} = u''(\omega_{y,x})$,

(ii) $u''(\omega_{\lambda x,y}) = \lambda u''(\omega_{x,y})$,

(iii) $u''(\omega_{x+y,z}) = u''(\omega_{x,z}) + u''(\omega_{y,z})$,

(iv) $|u''(\omega_{x,y})|^2 \le |u''(\omega_{x,x})|\,|u''(\omega_{y,y})|$,

where λ is any complex number, x, y, z denote elements of $\mathscr{K}$.

(i) follows by noting that $\omega_{x-y,x-y} = \omega_{y-x,y-x}$, $\omega_{x+iy,x+iy} = \omega_{y-ix,y-ix}$, $\omega_{x-iy,x-iy} = \omega_{y+ix,y+ix}$. (ii) and (iii) are similar and (iv) follows in the usual fashion of the Cauchy-Schwartz inequality from the relation $u''(\omega_{x+\lambda y, x+\lambda y}) \geq 0$ for each complex λ.

Theorem 5.2: (i) $\nu(\mathrm{ReM})$ is an ideal in $(\mathrm{ReM})^{\sim\sim}_{n,n}$. In particular if $0 \leq u'' \leq \nu(T)$, $u'' \in (\mathrm{ReM})^{\sim\sim}_{n,n}$ then $u'' = \nu(ST)$, $0 \leq S \leq I$, $S, T \in \mathrm{ReM}$.

(ii) The smallest normal Riesz subspace of $(\mathrm{ReM})^{\sim\sim}_{n,n}$ containing $\nu(\mathrm{ReM})$ is $(\mathrm{ReM})^{\sim\sim}_{n,n}$.

(iii) $\nu(\mathrm{ReM}) = (\mathrm{ReM})^{\sim\sim}_{n,n}$, i.e. ReM is a perfect Riesz space.

(iv) $\nu(\mathrm{ReM})$ and $(\mathrm{ReM})^{\sim\sim}_{n,n}$ are isometric as Banach spaces.

Proof: (i) Let $u'' \in (\mathrm{ReM})_{n,n}$ satisfy $0 \leq u'' \leq \nu(T)$ for some $T \in \mathrm{ReM}^+$. Then for each $x \in \mathscr{H}$, $u''(\omega_{x,x}) \leq (Tx, x)$. By (iv) of the previous page,

$$|u''(\omega_{x,y})| \leq (Tx,x)(Ty,y) \text{ for all } x, y \in \mathscr{H}.$$

It follows that $[T^{\frac{1}{2}}x, T^{\frac{1}{2}}y] = u''(\omega_{x,y})$ defines on the linear subspace $T^{\frac{1}{2}}\mathscr{H}$ a single valued, sesquilinear bounded positive Hermitian form. In view of (i) - (iv) listed above, we need check only the single valuedness. If $T^{\frac{1}{2}}x = T^{\frac{1}{2}}x'$, then

$$|u''(\omega_{x,y}) - u''(\omega_{x',y})|^2 = |u''(\omega_{x-x',y})|^2$$

$$\leq (T(x-x'), (x-x'))(Ty, y) = 0 .$$

Therefore, there exists a positive self-adjoint operator S defined on the closure of $T^{\frac{1}{2}}\mathscr{H}$ such that $u''(\omega_{x,y}) = (T^{\frac{1}{2}}x, ST^{\frac{1}{2}}y)$ for each $x, y \in \mathscr{H}$. We set $Sx = 0$ if $x \in \mathscr{H}$ belongs to the null space of $T^{\frac{1}{2}}$. Let A be any element of M'. Then

$$\begin{aligned}(T^{\frac{1}{2}}x, AST^{\frac{1}{2}}y) &= (T^{\frac{1}{2}}A^*x, ST^{\frac{1}{2}}y) = u''(\omega_{A^*x,y}) \\ &= u''(\omega_{x,Ay}) = (T^{\frac{1}{2}}x, ST^{\frac{1}{2}}Ay) \\ &= (T^{\frac{1}{2}}x, SAT^{\frac{1}{2}}y).\end{aligned}$$

Hence AS = SA on the closure of $T^{\frac{1}{2}}\mathscr{K}$. Since the projection on the null space of $T^{\frac{1}{2}}$ also belongs to M, AS = SA in $\mathscr{K}$. Thus $S \in \mathrm{ReM}$, and it is clear that $0 \le S \le I$.

(ii) follows immediately from [10], Theorem 28.2 (ii) of Note VIII, since the Dedekind complete Riesz space ReM has plenty of normal integrals, i.e.,

$$^{o}((\mathrm{ReM})_n^{\sim}) = \{T \in \mathrm{ReM} : \omega(T) = 0 \text{ for all } \omega \in (\mathrm{ReM})_n^{\sim}\} = \{0\}$$

(iii) In view of (i) and (ii), it is now sufficient to show that $\nu(\mathrm{ReM})$ is a normal subspace of $(\mathrm{ReM})_{n,n}^{\sim\sim}$. Let $0 \le \nu(T_\tau) \uparrow_\tau u''$, $u'' \in (\mathrm{ReM})_{n,n}^{\sim\sim}$. For $x \in \mathscr{K}$, $\nu(T_\tau)(\omega_{x,x}) = (T_\tau x, x) \le u''(\omega_{x,x})$. By the Banach-Steinhaus theorem, $\sup_\tau \|T_\tau\| < +\infty$. Thus, for some constant K, $0 \le T_\tau \uparrow_\tau \le KI$. By the Dedekind completeness of ReM, there exists $T \in \mathrm{ReM}$ such that $0 \le T_\tau \uparrow_\tau T$ in ReM. By Theorem 5.1, $\nu(T_\tau) \uparrow_\tau \nu(T)$ in $(\mathrm{ReM})_{n,n}^{\sim\sim}$. Thus $u'' = \nu(T)$, and so $\nu(\mathrm{ReM})$ is a band in $(\mathrm{ReM})_{n,n}^{\sim\sim}$ and by (ii), $\nu(\mathrm{ReM}) = (\mathrm{ReM})_{n,n}^{\sim\sim}$.

(iv) follows by observing, for $T \in M$,

$$\begin{aligned}
\|T\| &= \sup(|(Tx, y)| : \|x\| \le 1, \|y\| \le 1) \\
&\le \sup(|\nu(T)(\omega_{x,y})| : \|\omega_{x,y}\| \le 1) \\
&\le \sup(|\nu(T)(\varphi) : \varphi \in (\mathrm{ReM})_n^{\sim}, \|\varphi\| \le 1) \\
&\le \sup(|\varphi(T)| : \|\varphi\| \le 1) \le \|T\|
\end{aligned}$$

By an abuse of notation we shall write $\mathrm{ReM} = (\mathrm{ReM})_{n,n}^{\sim\sim}$.

The Riesz space $(\mathrm{ReM})_n^{\sim}$ is itself norm complete ([1], p. 38). It is easy to show that the norm of the Banach space $(\mathrm{ReM})_n^{\sim}$ is a Riesz norm which is additive on the positive cone of $(\mathrm{ReM})_n^{\sim}$. In fact let $0 \le \varphi_1, \varphi_2 \in (\mathrm{ReM})_n^{\sim}$ satisfy $\varphi_1 \le \varphi_2$. From $\varphi_1(I) \le \varphi_2(I)$

follows immediately that $\|\varphi_1\| \le \|\varphi_2\|$. If φ is any element of $(\mathrm{ReM})_n^{\sim}$ write $\varphi = \varphi^+ - \varphi^-$, $|\varphi| = \varphi^+ + \varphi^-$. Note that $\varphi^+ \wedge \varphi^- = 0$ implies $E_{\varphi^+} \perp E_{\varphi^-}$. It is clear that $\|\varphi\| \le \|\varphi^+\| + \|\varphi^-\|$. Set $E = E_{\varphi^+} - E_{\varphi^-}$. Then $\|\varphi\| \ge \varphi(E) = \varphi^+(E_{\varphi^+}) + \varphi^-(E_{\varphi^-}) = \|\varphi^+\| + \|\varphi^-\|$. Hence $\|\varphi\| = \|\varphi^+\| + \|\varphi\| = \varphi^+(I) + \varphi^-(I) = |\varphi|(I) = \| \, |\varphi| \, \|$. Finally if $\varphi = \varphi_1 + \varphi_2$ with $0 \le \varphi_1, \varphi_2 \in (\mathrm{ReM})_n^{\sim}$, then $\|\varphi\| = \varphi(I) = \varphi_1(I) + \varphi_2(I)$ $= \|\varphi_1\| + \|\varphi_2\|$.

From these remarks it follows that $(\mathrm{ReM})_{n,}^{\sim *} = (\mathrm{ReM})_{n,}^{\sim\sim}$ ([10] Note VII and Note VIII, Theorem 26.4), and that the Banach space $(\mathrm{ReM})_n^{\sim}$ is an abstract L-space. It is well known that every bounded linear functional on an abstract L-space is a normal integral. Consequently $(\mathrm{ReM})_{n,}^{\sim *} = (\mathrm{ReM})_{n,n}^{\sim\sim} = \mathrm{ReM}$. We summarize the above in terms of the Abelian W*-algebra M.

Theorem 5.3: The Banach spaces M, M_n^* are isometrically isomorphic.

Proof: If $T \in M$, $\varphi \in M_n$, define $\sigma(T)(\varphi) = \varphi(T)$. The map σ is clearly an algebraic isomorphism of M into M_n^*. That σ is onto follows from $\mathrm{Re}(M_n{}^*) = (\mathrm{ReM})_{n}^{\sim}{}^{*} = \mathrm{ReM}$. It is clear that $\|\sigma(T)\| \le \|T\|$. On the other hand

$$\begin{aligned} \|\sigma(T)\| &\ge \sup \{ |\sigma(T)(\omega_{x,y})| : \|x\| \le 1, \|y\| \le 1, x, y \in \mathscr{H} \} \\ &= \sup \{ |(Tx, y)| : \|x\| \le 1, \|y\| \le 1, x, y \in \mathscr{H} \} \\ &= \|T\| \end{aligned}$$

Thus $\|\sigma(T)\| = \|T\|$ so that σ is an isometry.

The result of Theorem 5.3 is a well-known property of any W*-algebra (cf[1] p. 40) and S. Sakai [17] has shown that this property may be used to give a non-spatial definition of W*-algebra. More precisely, in [17] a B*-algebra M is called a W*-algebra if there exists a Banach space F such that M = F*. If F is canonically embedded as a norm - closed subspace of M*, then it may be shown that F is generated by the totality of normal positive linear functionals on M. Since normality is determined by the order properties of M only, it follows that if F_1, F_2 are two Banach spaces with the property $F_1^* = F_2^* = M$, then F_1 coincides with F_2 when they are canonically embedded into M*. Further it may be shown that if M is a W*-algebra in the above sense then M may be represented faithfully as a weakly closed *-subalgebra of $\mathcal{L}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and that under such a representation the $\sigma(M, F)$ topology is equivalent to the weak operator topology on bounded spheres.

VI. THE SPACE $\mathrm{Re}\,M_0$

Let M be an Abelian W*-algebra.

Definition 6.1: By M_0 we shall denote the set of all linear densely defined closed transformations T which satisfy $TU \supseteq UT$ for each unitary operator U in M! $\mathrm{Re}M_0$ will denote the set of self-adjoint transformations in M_0. If $T \in M_0$, we shall denote the domain of definition of T by $\mathfrak{D}_T$, and the range of T by $\mathscr{R}_T$. If $T \in \mathrm{Re}M_0$, then T will be called positive, written $T \geq 0$, if $(Tz, z) \geq 0$ for each $z \in \mathfrak{D}_T$. We will write $T \in (\mathrm{Re}M_0)^+$.

In this section we shall show that Theorem 4.7 leads to a natural construction of $\mathrm{Re}M_0$, and we give an elementary proof of the fact that each positive element of $(\mathrm{Re}M_0)^+$ has a unique square root in $(\mathrm{Re}M_0)^+$. The proof is elementary in that it uses only those properties which are consequences of the Riesz space structure of ReM. In later sections it will be shown, that if the algebraic operations are suitably defined, then $\mathrm{Re}M_0$ is itself a Dedekind complete Riesz space, which is, at the same time, a universal completion of the Dedekind complete Riesz space ReM.

We shall frequently use the following useful result:

Lemma 6.2: (cf [13], p. 226) Every linear closed Hermitian transformation $T \in M_0$ is maximal Hermitian and self-adjoint.

Proof: Let V_T denote the Cayley transform of T ([12], p. 74). Since T is closed, $\mathfrak{D}_{V_T} = \mathscr{R}_{T+iI}$, $\mathscr{R}_{V_T} = \mathscr{R}_{T-iI}$ are closed subspaces of $\mathscr{H}$. Since T commutes with every unitary operator in M', it follows that V_T may be extended to a partial isometry in M. Since

M is Abelian $V_T^* V_T = V_T V_T^*$ so that $\mathcal{H}_{T+iI} = \mathcal{H}_{T-iI}$. To show that T is self adjoint, it is sufficient to show that $\mathcal{H}_{T+iI} = \mathcal{H}_{T-iI} = \mathcal{H}$. Assume $z \in \mathcal{H}$ satisfies $((T+iI)x, z) = 0$, $((T-iI)x, z) = 0$ for all $x \in \mathcal{D}_T$. Thus also $(x, z) = 0$ for all $x \in \mathcal{D}_T$, which is dense in $\mathcal{H}$. Thus $z = 0$.

We now obtain a more precise version of Theorem 4. 9, which will lead to the construction of the class ReM_0.

Theorem 6. 3: Let φ be a normal positive linear functional on the Abelian W*-algebra M which satisfies $\varphi < \omega_{x,x}$ for some $x \in \mathcal{H}$. There exists a positive self-adjoint transformation $H \in (ReM_0)^+$ which satisfies $H(I - E_x^{M'}) = 0$, and $\varphi(M) = \omega_{Hx, Hx}(M)$.

Proof: From the proof of Theorem 4. 9, there exists a sequence of positive self-adjoint operators in $(ReM)^+$ which have the following properties:

(i) $0 \leq H_n \uparrow_n$

(ii) $H_n(I - E_x^{M'}) = 0$

(iii) For each T in M, $\varphi(T) = \lim_{n \to \infty} (TH_n x, H_n x)$.

Note that if T' is any element of M', then

$$\|H_n T' x\| \leq \|T' H_n x\| \leq \|T'\| \, \|H_n x\|$$
$$\leq \|T'\| \varphi(I).$$

In particular, if z is any element of $\{M'x\}$, there exists a real constant K(z), independent of n, such that

$$\|H_n z\| \leq K(z)$$

Now suppose $z \in [M'x]$ satisfies $\|H_n z\| \leq K(z)$, where K is a real constant, independent of n; then the sequence $\{H_n z\}$ is actually convergent. In fact, from (i) $\{\|H_n z\|\}$ is a Cauchy sequence of real numbers. Further for $m \geq n$

$$\|H_m z - H_n z\|^2 = (H_n z, H_n z) + (H_m z, H_m z) - (H_m z, H_n z) - (H_n z, H_m z)$$
$$\leq \|H_m z\|^2 - \|H_n z\|^2 \to 0 \text{ as } m,\ n \to \infty$$

We define the linear transformation H as follows:

Let $\mathfrak{M} = \{z \in [M'x] : \|H_n z\| \leq K(z) \text{ for all } n\}$. For $z \in \mathfrak{N}$, set $Hz = \lim_{n\to\infty} H_n z$. For $z \in \mathfrak{N} \ominus [M'x]$, set $Hz = 0$. From $\{M'x\} \subseteq \mathfrak{M} \subseteq [M'x]$, it follows that H is densely defined. By (iii) $\varphi = \omega_{Hx,\,Hx}(M)$. We now show that H is a positive self-adjoint transformation in M_0.

(a) For $z \in \mathfrak{D}_H$ $(Hz, z) \geq 0$ follows immediately from $(H_n z, z) \geq 0$.

(b) H is Hermitian and closed: If $z_1, z_2 \in \mathfrak{D}_H$, then $(Hz_1, z_2) = \lim_{n\to\infty}(H_n z_1, z_2) = \lim_{n\to\infty}(z_1, H_n z_2) = (z_1, Hz_2)$. Thus $z_2 \in \mathfrak{D}_{H^*}$ and $H^* z_2 = Hz_2$ so that $H \subseteq H^*$ and H is Hermitian. In particular H^* is densely defined so that H^{**} is defined. Suppose now that $z_n \in \mathfrak{D}_H$, $z_n \to z$ and $Hz_n \to y$. From $Hz_n \to y$ follows that for some constant K, $\|Hz_n\| \leq K$ for all n. Thus $\|H_m z_n\| \leq \|Hz_n\| \leq K$ holds for all m, n. Fix m and let $n \to \infty$ to obtain $\|H_m z\| \leq K$ for all m. Thus $z \in \mathfrak{D}_H$. Now $H \subseteq H^{**}$, H^{**} is closed and we have z_n, $z \in \mathfrak{D}_{H^{**}}$, $z_n \to z$ and $H^{**} z_n \to y$; hence $H^{**} z = y$, so that $Hz = y$ since $z \in \mathfrak{D}_H$. Thus H is closed.

(c) Let $U \in M'$ be unitary; if $z \in \mathfrak{D}_H$, then $Uz \in \mathfrak{D}_H$ since

$$\|H_n u z\| = \|U H_n z\| = \|H_n z\|.$$

Further, $UHz = U(\lim_{n\to\infty} H_n z) = \lim_{n\to\infty} U H_n z = \lim_{n\to\infty} H_n(Uz)$

$$= HUz$$

Thus $UH \subseteq HU$ for each unitary $U \in M'$. That H is self-adjoint now follows immediately from lemma 6.2.

Remark: Notice that the transformation H of Theorem 6.3 has the special property that $H[I-E_x^{M'}] = 0$. We shall assume for the present that M' has a cyclic vector $x \in \mathcal{K}$, i.e., for some $x \in \mathcal{K}$, $E_x^{M'} = I$. We shall find it convenient to make the following definition:

Definition 6.4: Assume $E_x^{M'} = I$. We shall denote by $M_0(x)$ the set of all elements H in M_0 which satisfy the following conditions:

(i) $x \in \mathcal{D}_H$.

(ii) There exists a sequence $\{H_n\}$, n=1,2.. of elements of M^+ such that $H_n \uparrow_n$, and $z \in \mathcal{D}_H$ if and only if $\|H_n z\| \leq K(z)$, and then $Hz = \lim_{n\to\infty} H_n z$.

From (ii) follows immediately that if $H \in M_0(x)$, then $(Hz, z) \geq 0$ for all $z \in \mathcal{D}_H$, so that $H \geq 0$.

Theorem 6.5: Let $H \in M_0(x)$. There exists a unique element G in $M_0(x)$ which satisfies $G^2 = H$.

Proof: (i) Existence: Let $\{H_n\}_{n=1,2,\ldots,}$ be a sequence of elements of M^+ associated with H as in Definition 6.4. Consider the sequence $\{H_n^{\frac{1}{2}}\}_{n=1,2,\ldots}$. Observe that $H_n^{\frac{1}{2}} \uparrow_n$, $H_n^{\frac{1}{2}} \in M^+$. For each $z \in \mathcal{D}_H$, $\|H_n^{\frac{1}{2}} z\|^2 = (H_n z, z) \leq \|H_n z\| \|z\| \leq \|Hz\| \|z\|$. We define a linear transformation G as follows:

$\mathcal{D}_G = \{z \in \mathcal{K}: \|H_n^{\frac{1}{2}} z\| \leq K(z)$ for $n = 1, 2, \ldots\}$ where K(z) is a constant independent of n. For $z \in \mathcal{D}_G$, we set $Gz = \lim_{n\to\infty} H_n^{\frac{1}{2}} z$. Note that $\mathcal{D}_G \supseteq \mathcal{D}_H$, and that G is a linear, densely defined, self adjoint element of M_0 by precisely the same as the proof of Theorem 6.3, and satisfies $(Gz, z) \geq 0$ for each $z \in \mathcal{D}_G$. In particular then, $G \in M_0(x)$. G has the following properties:

(i) $\mathcal{D}_H \subseteq \mathcal{D}_{G^2}$. In fact, if $z \in \mathcal{D}_H$, there exists $K(z)$ such that $\|H_n z\|^2 \leq \|H_m^{\frac{1}{2}} H_n^{\frac{1}{2}} z\|^2 \leq \|H_m z\|^2 \leq K(z)$ for all $m \geq n$ .... (a)

Noting that $z \in \mathcal{D}_G$ and that (a) implies $\|H_m^{\frac{1}{2}} H_n^{\frac{1}{2}} z\|^2 \leq K(z)$ for all m, n fix m and let $n \to \infty$ to obtain $\|H_m^{\frac{1}{2}} Gz\|^2 \leq K(z)$ for all m. Thus $Gz \in \mathcal{D}_G$ so that $z \in \mathcal{D}_G^2$.

(ii) $\underline{\mathcal{D}_{G^2} \subseteq \mathcal{D}_H}$: For $z \in \mathcal{D}_G$, and each n,

$$H_n^{\frac{1}{2}} G z = H_n^{\frac{1}{2}} \lim_{m\to\infty} H_m^{\frac{1}{2}} z = \lim_{m\to\infty} H_n^{\frac{1}{2}} H_m^{\frac{1}{2}} z = \lim_{m\to\infty} H_m^{\frac{1}{2}} H_n^{\frac{1}{2}} z.$$

Thus $H_n^{\frac{1}{2}} z \in \mathcal{D}_G$ and $G H_n^{\frac{1}{2}} z = H_n^{\frac{1}{2}} Gz$. Thus if $z \in \mathcal{D}_{G^2}$, then for each m, n, $\| H_m^{\frac{1}{2}} H_n^{\frac{1}{2}} z\| \leq \|G H_n^{\frac{1}{2}} z\| = \|H_n^{\frac{1}{2}} G z\| \leq K(z)$.

In particular by setting $m = n$, we obtain that $z \in \mathcal{D}_H$.

(iii) $\underline{G^2 = H}$. From (i) and (ii) $\mathcal{D}_{G^2} = \mathcal{D}_H$. Let $z_1 \in \mathcal{D}_{G^2}$, $z_2 \in \mathcal{D}_G$, then

$$(G^2 z_1, z_2) = (Gz_1, Gz_2) = \lim_{n\to\infty} (H_n^{\frac{1}{2}} z_1, H_n^{\frac{1}{2}} z_2)$$
$$= \lim_{n\to\infty} (H_n z_1, z_2) = (H z_1, z_2)$$

as $\mathcal{D}_G$ is dense, we obtain $G^2 z_1 = Hz_1$.

Before turning to the question of the uniqueness of G, we state two preparatory lemmas

Lemma 6.6: (cf [12], p. 61) Let T be a linear, densely defined closed transformation in $\mathcal{H}$. Then $(I+T^*T)^{-1}$ exists and is equal to a bounded positive self adjoint operator B, $\|B\| \leq 1$. The transformation $C = TB$ is also bounded, $\|C\| \leq 1$. If T' denotes the restriction of T to $\mathfrak{D}_{T^*T}$, then T is the smallest closed linear extension of T'. Consequently $\mathfrak{D}_{T^*T}$ is dense in $\mathfrak{D}_T$, thus in $\mathfrak{X}$.

Lemma 6.7: Let $T \in M_0$ be self adjoint, and $(Tz, z) \geq 0$ for all $z \in \mathfrak{D}_T$. $B = (I+T^2)^{-1} : \mathfrak{X} \to \mathfrak{D}_{T^2}$; then TB is also positive and self adjoint. Further $B, TB \in M$.

Proof: That T^2 is also positive and self adjoint follows as in [12], p. 108. For each $z_1, z_2 \in \mathfrak{X}$

$$(TBz_1, z_2) = (TBz_1, (I+T^2)Bz_2) = (TBz_1, Bz_2) + (TBz_1, T^2Bz_2)$$
$$= (Bz_1, TBz_2) + (T^2Bz_1, TBz_2) = ((I+T^2)Bz_1, TBz_2)$$
$$= (z_1, TBz_2)$$

Hence TB is self adjoint since it is bounded.

Again, for each $z \in \mathfrak{X}$

$$(TBz, z) = (TBz, (I+T^2)Bz) = (TBz_1, Bz) + (TBz, T^2Bz)$$
$$= (TBz, Bz) + (T^2Bz, TBz) \geq 0.$$

Now note that $(I+T^2)B = I = I^* \supseteq B^*(I+T^2)^* \supseteq B(I+T^2)$.

If $U \in M'$ is unitary, then $U = U(I+T^2)B \subseteq (I+T^2)UB$ so that

$$BU \subseteq B(I+T^2)UB \subseteq UB$$

As B is bounded $BU = UB$ so that $B \in M$. We have also that

$$(TB)U = TUB \supseteq U(TB)$$

As TB is bounded we have also that $(TB)U = U(TB)$, thus $TB \in M$.

We now complete the proof of Theorem 6.5.

(ii) Uniqueness of G: Assume $G' \in M_0$ satisfies $G_1^2 = G^2$. Thus $G_1^2 Tx = G^2Tx$ for each $T \in M'$. Set $B = (I+G_1^2)^{-1} = (I+G^2)^{-1}$. Then $(G_1B)^2 Tx = (GB)^2Tx$ holds for each $T \in M'$. Since $E_x^{M'} = I$ and $(G_1B)^2, (GB)^2$ belong to M, follows $(G_1B)^2 = (GB)^2$. Since G_1B, GB

are positive, self adjoint elements of M, the uniqueness of square roots in the bounded case implies that $G_1 B = GB$. In particular then $G_1 = G$ on $\mathscr{H}_B$. Observe $\mathscr{H}_B \supseteq \mathfrak{D}_{G^2} = \mathfrak{D}_{G_1^2}$. If G_1', G' denote the restrictions of G_1, G to $\mathfrak{D}_{G^2} = \mathfrak{D}_{G_1^2}$, then $G_1' = G'$. Lemma 6.6 now implies that $G = G_1$.

We turn now to a related uniqueness problem which will be of use in what follows. If H is any element of $(\mathrm{Re}M_0)^+$ such that $x \in \mathfrak{D}_H$ and $H(I-E_x^{M'}) = 0$, then $Hx \in [M'x]$ so that $[M'Hx] \subseteq [M'x]$. Thus $\omega_{Hx, Hx} < \omega_{x, x}$; by Theorem 6.3, there exists $0 \leq H_0 \in M_0(x)$ such that $(TH_0x, H_0x) = (THx, Hx)$ holds for each $T \in M$. We show that in fact $H = H_0$.

<u>Lemma 6.8:</u> <u>Let $H \in (\mathrm{Re}M_0)^+$ satisfy $x \in \mathfrak{D}_H$ and $H(I-E_x^{M'}) = 0$. Assume that $(THx, Hx) = (TH_0x, H_0x)$ holds for each $T \in M$ where $H_0 \in M_0(x)$. Then $H = H_0$.</u>

<u>Proof</u>: Let $y \in \mathscr{H}$ satisfy $\omega_{y, y} = \omega_{x, x}$. Observe that the restrictions of H, H_0 to $[My]$ are again positive self-adjoint transformations, with domains $\mathfrak{D}_H \cap [My]$, $\mathfrak{D}_{H_0} \cap [My]$ respectively. Note that if $y \in \mathscr{H}$ satisfies $\omega_{y, y} = \omega_{x, x}(M)$, then $y = Ux$ where U is a partial isometry in M', which satisfies $U^*U = E_x^M$, $UU^* = E_y^M$. Consequently $(THy, Hy) = (TH_0y, H_0y)$ holds for all $T \in M$. Alternatively $(Hz_1, Hz_2) = (H_0z_1, H_0z_1)$ for all z_1, z_2 in $\{My\}$. Note that HE_y^M is the smallest closed extension of the restriction of H to $\{My\}$. Let $z \in \mathfrak{D}_H$. Let $\{z_n\}$ in $\{My\}$ satisfy $z_n \to z$, $Hz_n \to Hz$. It follows that $\{H_0z_n\}$ is convergent and thus $z \in \mathfrak{D}_{H_0}$ and $H_0 z_n \to H_0 z$. By symmetry $\mathfrak{D}_H \cap [My] = \mathfrak{D}_{H_0} \cap [My]$; further $(Hz_1, Hz_2) = (H_0z_1, H_0z_2)$ holds for all $z_1, z_2 \in \mathfrak{D}_H \cap [My]$. It follows that $H^2E_y^M = H_0^2 E_y^M$. In fact assume $z \in \mathfrak{D}_{H^2} \cap [My]$; for all $z_1 \in \mathfrak{D}_H \cap [My]$ $= \mathfrak{D}_{H_0} \cap [My]$, $\quad (H_0z_1, H_0z) = (Hz_1, Hz) = (z_1, H^2z)$.

Thus $z \in \mathcal{D}_{H_0^2} \cap [My]$ and $H_0^2 z = H^2 z$. By symmetry, $H_0^2 E_y^M = H^2 E_y^M$.

It follows immediately that $(I+H_0^2)^{-1} E_y^M = (I+H^2)^{-1} E_y^M$. By lemmas 6.6 and 4.8, $(I+H_0^2)^{-1} = (I+H^2)^{-1}$; consequently $H_0^2 = H^2$. Set $G = H(I+H^2)^{-1}$, $G_0 = H_0(I+H_0^2)^{-1}$, and note that $0 \le G, G_0 \in M$. We have $(Gz_1, Gz_2) = (G_0 z_1, G_0 z_2)$ holds for all $z_1, z_2 \in \mathcal{D}_H \cap [My]$. Thus $G^2 E_y^M = G_0^2 E_y^M$ so that by lemma 4.8, $G^2 = G_0^2$. Hence $G = G_0$. In particular it follows that if H', H_0' denote the restrictions of H, H_0 to $\mathcal{D}_{H^2} = \mathcal{D}_{H_0^2}$, then $H' = H_0'$. That $H = H_0$ follows from lemma 6.6.

We now summarize some of the preceding lemmas in

Theorem 6.9: Assume the Abelian W*-algebra M satisfies $E_x^{M'} = I$, for some x in $\mathcal{K}$. Let $H \ge 0$ be an element of ReM_0 with $x \in \mathcal{D}_H$. Then there exists a sequence $\{H_n\}_{n=1,2,\ldots}$ of elements of ReM^+ with the properties

(i) $0 \le H_n \uparrow_n$

(ii) $z \in \mathcal{D}_H$ if and only if $\|H_n z\| \le K(z)$, where $K(z)$ is a constant independent of n, and $Hz = \lim_{n \to \infty} H_n z$. We write $0 \le H_n \uparrow_n H$.

Further there exists a unique element $0 \le G \in ReM_0$, which satisfies $G^2 = H$. We have $0 \le H_n^{\frac{1}{2}} \uparrow_n G$.

We shall now proceed to remove the restriction that $E_x^{M'} = I$ for some x in $\mathcal{K}$. We have the following result:

Theorem 6.10: Let M be an Abelian W*-algebra. Let $0 \le H \in ReM_0$. There exists a sequence $\{H_n\} n = 1, 2, \ldots$ of elements of $(ReM)^+$ with the properties that

(i) $0 \le H_n \uparrow_n$

(ii) $z \in \mathcal{D}_H$ if and only if $\|H_n z\| \le K(z)$, where $K(z)$ is a constant independent of n and $Hz = \lim_{n \to \infty} H_n z$.

<u>We shall write $0 \le H_n \uparrow_n H$.</u>

<u>There exists a unique element $p \le G \in ReM_0$, which satisfies $G^2 = H$. We have $0 \le H_n^{\frac{1}{2}} \uparrow_n G$. We shall write $G = H^{\frac{1}{2}}$.</u>

<u>Proof</u>: Let $0 \le H \in ReM_0$. Choose a maximal family $\{x_i\}_{i \in \mathcal{J}}$ of elements of $\mathcal{K}$ with the property that $x_i \in \mathcal{D}_H$ and $E_{x_j}^{M'}$ if $i \ne j$. Then $\Sigma_{i \in \mathcal{J}} E_{x_i}^{M'} = I$. In fact set $E = \Sigma_{i \in \mathcal{J}} E_{x_i}^{M'}$ and if $E \ne I$, there exists $x \in \mathcal{D}_H$ such that $((I-E)x, x) \ne 0$. The linear functional $\omega_{(I-E)x,(I-E)x}(M)$ is non zero, and its support is majorized by I-E contradicting the maximality of the family $\{x_i\}_{i \in \mathcal{J}}$.

Let H_i denote the part of H in $[M'x_i]$ i. e., the restriction of H to $[M'x_i]$. Observe that $0 \le H_i \in ReM_0$. From [12], p. 70, it follows that $H = \prod_{i \in \mathcal{J}} \times H_i$. By theorem 6. 9 above, for each i, there exists $0 \le H_n^{(i)} \in ReM$ with $H_n^{(i)}(I - E_{x_i}^{M'}) = 0$, and $0 \le H_n^{(i)} \uparrow_n H_i$. Set $H_n = \prod_{i \in \mathcal{J}} \times H_n^{(i)}$. It is clear that $H_n \in M$, $0 \le H_n \uparrow_n$. For each $z \in \mathcal{K}$ put $z_i = E_{x_i}^{M'} z$. We have $z \in \mathcal{D}_H$ if and only if $z_i \in \mathcal{D}_{H_i}$ for all i and $\Sigma_{i \in \mathcal{J}} \|H_i z_i\|^2 < +\infty$. If $z \in \mathcal{D}_H$

$$\|H_n z\|^2 = \Sigma_{i \in \mathcal{J}} \|H_n^{(i)} z_i\|^2 \le \Sigma_{i \in \mathcal{J}} \|H_i z_i\|^2 = \|Hz\|^2$$

Observe that for each i, $\|H_i z_i - H_n^{(i)} z_i\| \downarrow_n 0$. In fact let $n \ge m$, then

$$((H_n^{(i)} + H_m^{(i)})z_i, (H_n^{(i)} - H_m^{(i)})z_i) \le 2(H_i z_i, (H_n^{(i)} - H_m^{(i)}) z_i)$$

hence $((H_n^{(i)2} - H_m^{(i)2})z_i, z_i) \le 2(H_i z_i, (H_n^{(i)} - H_m^{(i)})z_i)$

$$\|H_n^{(i)} z_i\|^2 - \|H_m^{(i)} z_i\|^2 \le 2(H_i z_i, (H_n^{(i)} - H_m^{(i)})z_i)$$

$$\|H_n^{(i)} z_i\|^2 + \|H_i z_i\|^2 - (H_i z_i, H_n^{(i)} z_i) - (H_n^{(i)} z_i, H_i z_i)$$

$$\le \|H_m^{(i)} z_i\|^2 + \|H_i z_i\|^2 - (H_i z_i, H_m^{(i)} z_i) - (H_m^{(i)} z_i, H_i z_i)$$

i. e. $\|H_i z_i - H_n^{(i)} z_i\|^2 \le \|H_i z_i - H_m^{(i)} z_i\|^2$. Now

$$\|Hz-H_nz\|^2 = \Sigma_{i\in\mathcal{J}} \|H_iz_i - H_n^{(i)}z_i\|^2$$

$= \Sigma_{m=1}^{\infty} \|H_{i_m} z_{i_m}\|^2$, since in the first sum only countably many terms are different from zero. Given $\epsilon > 0$, choose m_0 such that $\Sigma_{m>m_0} \|H_{i_m} z_{i_m} - H_n^{(i_m)} z_{i_m}\|^2 < \epsilon/2$. Then choose $n_0(\epsilon)$ sucn that $\sum_{n=1}^{m_o} \|H_{i_m} z_{i_m} - H_n^{(i_m)} z_{i_m}\|^2 < \epsilon/2$ for all $n \geq n_0(\epsilon)$. It follows that $\lim_{n\to\infty} H_nz = Hz$.

Let $\mathfrak{M} = \{z\in\mathcal{X}: \|H_nz\| \leq K(z)$ for all $n\}$. We have shown that $\mathcal{D}_H \subseteq \mathfrak{M}$ and if $z \in \mathcal{D}_H$ then $Hz = \lim_{n\to\infty} H_nz$. We define a transformation B by setting $\mathcal{D}_B = \mathfrak{M}$ and for $z\in\mathfrak{M}$, set $Bz = \lim_{n\to\infty} H_nz$. That B is a linear densely defined, closed, Hermitian transformation which commutes with all the unitary operation of M' follows exactly as in Theorem 6.3. It is clear that $B \supseteq H$. Thus $B = H$ by lemma 6.2. Finally the existence and uniqueness of $0 \leq G\in ReM_0$ satisfying $G^2 = H$ is proved exactly as in Theorem 6.5, by setting $\mathcal{D}_G = \{z\in\mathcal{X}: \|H_n^{\frac{1}{2}}z\| \leq K'(z)$, $K'(z)$ independent of $n\}$ and if $z\in\mathcal{D}_G$, set $Gz = \lim H_n^{\frac{1}{2}}z$. It is clear that $H_n^{\frac{1}{2}} \uparrow_n G$.

VII. THE ALGEBRAIC STRUCTURE OF M_0

If A, B are linear closed densely defined transformations in $\mathcal{H}$ it is not true in general that A+B, AB are even densely defined, if A+B, AB denote sum and product in the usual sense of general transformations. In fact there are closed linear densely defined transformations T which satisfy $T \subseteq T^*$ for which $\mathcal{D}_{T^2} = \{0\}$. If M is an Abelian W*-algebra, then this pathology does not occur in M_0. This follows from some results of von Neumann and Murray which state essentially that if M is Abelian, and if $A, B \in M_0$, then A+B, AB have unique extensions in M_0, and these extensions satisfy the proper algebraic relations. The key to the von Neumann-Murray result is based on the concept of essentially dense subspaces; in particular if $A \in M_0$ then $\mathcal{D}_A$ is essentially dense. The proof of this latter statement as given in [13] depends on the spectral representation of general self-adjoint transformations, so it is desirable to obtain a proof which lies within the existing framework developed so far. The relevant definitions and lemma follow. The result will act as a bridge between what has been obtained in the previous sections and the results of von Neumann and Murray on the algebraic properties of M_0.

Definition 7.1 (cf [13], p. 222). Let $\mathfrak{M}$ be an arbitrary linear manifold in $\mathcal{H}$. ($\mathfrak{M}$ is not necessarily closed, nor invariant under M'). If a sequence $\mathfrak{M}_1, \mathfrak{M}_2, \ldots$ of linear closed subspaces exists which has the following properties:

(i) $E_{\mathfrak{M}_i} \in M$ ($E_{\mathfrak{M}_i}$ denotes the projection on $\mathfrak{M}_i$) for all i.

(ii) $\mathfrak{M}_1 \subseteq \mathfrak{M}_2 \subseteq \ldots \subseteq \mathfrak{M}$.

(iii) $[\mathfrak{M}_1, \mathfrak{M}_2, \ldots] = \mathscr{H}$ (i.e. $\bigvee_i E_{\mathfrak{M}_i} = I$)

then $\mathfrak{M}$ is said to be essentially dense.

Lemma 7.2: If $\mathfrak{M}$ is essentially dense, then $\mathfrak{M}$ is also dense.

Proof: $\mathscr{H} = [\mathfrak{M}_1, \mathfrak{M}_2, \ldots] \subseteq [\mathfrak{M}] \subseteq \mathscr{H}$, thus $[\mathfrak{M}] = \mathscr{H}$.

The next lemma is a substitute for the spectral theorem for the unbounded self-adjoint transformations of M_0.

Lemma 7.3: Let H be a positive self-adjoint transformation of M_0. There exists a sequence $\{F_n\}$ of projections in M with the following properties:

(i) $F_n\mathscr{H} \subseteq \mathscr{D}_H$.

(ii) The restriction of H to $F_n\mathscr{H}$ is bounded and belongs to M.

(iii) For $n \geq m$, $F_n \geq F_m$; $\bigvee_n F_n = I$.

Corollary 7.4: $\mathscr{D}_H$ is essentially dense.

Proof of the Corollary: Immediate.

Proof of the lemma: Let H be a positive self-adjoint transformation of M_0 and let $\{H_n\} \in M^+$ satisfy $0 \leq H_n \uparrow_n H$ as in Theorem 6.10. For each n denote by $\{E_\lambda^{(n)}\}$ the spectral family of H_n. We may assume that $0 \leq \lambda < +\infty$ for each n. Note the following properties of the $E_\lambda^{(n)}$

(a) for each n, $E_\lambda^{(n)} \uparrow_\lambda I$ in ReM.

(b) for $n \geq m$, $E_\lambda^{(n)} \leq E_\lambda^{(m)}$; this follows immediately from the fact that for $n \geq m$, $H_n \geq H_m$, and that $E_\lambda^{(n)}$ is the projection on the closure of the range of $(\lambda I - H_n)^+$. Thus for each fixed λ, there exists a projection $F_\lambda \geq 0$, $F_\lambda \in M$ such that $E_\lambda^{(n)} \downarrow_n F_\lambda$. It is clear that F_λ

is the projection on $[\cap_n \text{Range } (E_\lambda^{(n)})]$. We claim that $F_\lambda \uparrow_\lambda I$. That $F_\lambda \uparrow_\lambda$ is obvious. Suppose $z \in \mathscr{K}$, $F_\lambda z = 0$ for all λ. Thus for each $\lambda, E_\lambda^{(n)} z \to 0$ as $n \to \infty$. By the definitions and various properties of $E_\lambda^{(n)}$ (see for example [9], p. 131 ff.)

$$(H_n z, H_n z) \geq (H_n(I-E_\lambda^{(n)})z,\ H_n(I-E_\lambda^{(n)})z) \geq \lambda^2 \,\|(I-E_\lambda^{(n)})z\|^2$$

Fix λ, then choose n such that

$$\|(I-E_\lambda^{(n)})z\|^2 \geq \frac{1}{2}\ \|z\|^2.$$

Thus for each λ, there exists $n(\lambda)$ such that $\|H_{n(\lambda)}z\|^2 \geq \lambda^2 \cdot \frac{1}{2}\|z\|^2$. Thus $z \notin \mathfrak{D}_H$. Thus $\vee_\lambda F_\lambda \neq I$ contradicts the fact that $\mathfrak{D}_H$ is dense. Hence $F_\lambda \uparrow_\lambda I$. In the above we have made use of the fact that

$$H_n(I-E_\lambda^{(n)}) \geq \lambda\, (I-E_\lambda^{(n)}) \qquad \text{implies}$$
$$H_n^2\,(I-E_\lambda^{(n)}) \geq \lambda\, H_n(I-E_\lambda^{(n)}) \geq \lambda^2 (I-E_\lambda^{(n)}).$$

It follows that, for each λ

$$\text{Range } F_\lambda \subseteq \{z \in \mathfrak{D}_H : (Hz, z) \leq \lambda(z, z)\}$$

for if $z \in \text{Range } F_\lambda$, then $z \in \cap_n \text{Range } E_\lambda^{(n)}$, so that for all n $(H_n z, z) \leq \lambda(z, z)$ and also $(H_n z, H_n z) \leq \lambda^2(z, z)$. Thus indeed $z \in \mathfrak{D}_H$, and $(Hz, z) \leq \lambda(z, z)$. From the closed graph theorem, it follows that the restriction of H to $F_\lambda \mathscr{K}$ i.e. HF_λ is bounded. The statement of the lemma follows by taking a suitable sequence $\{\lambda_n\}$.

Lemma 7.5: (cf [15], p. 222 Lemma 16.2.2). Let $\eta_1, \ldots, \eta_n$ be a finite set of essentially dense subspaces of $\mathscr{K}$, then $\cap_{i=1}^{n} \eta_n$ is also essentially dense.

Proof: It is clearly sufficient to consider the case n = 2. Thus suppose that η_1, η_2 are essentially dense linear manifolds of $\mathscr{K}$ and let $\{\mathfrak{M}_{1,n}\}$, $\{\mathfrak{M}_{2,n}\}$ be the associated linear subspaces of the definition.

By $E_{i,n}$ we denote the projection (in M) whose range is the subspace $\mathfrak{M}_{i,n}$, i=1, 2. We have $E_{i,n} \uparrow_n I$. Set $\mathfrak{M}_n = \mathfrak{M}_{1,n} \cap \mathfrak{M}_{2,n}$, let E_n be the projection (in M) with range $\mathfrak{M}_n$. Observe $E_n \uparrow_n$, and that $\mathfrak{M}_n \subseteq \mathfrak{M}_1 \cap \mathfrak{M}_2$. We have that $I-E_{1,n} \wedge E_{2,n} = (I-E_{1,n}) \vee (I-E_{2,n})$ so that for each $z \in \mathfrak{N}$,

$$0 \subseteq ((I-E_{1,n} \wedge E_{2,n})z, z) = ((I-E_{1,n}) \vee (I-E_{2,n})z, z)$$
$$\leq ((I-E_{1,n})z, z) + ((I-E_{2,n})z, z) \to 0 \text{ as } n \to \infty$$

Thus $(I- \vee_n (E_{1,n} \wedge E_{2,n})z, z) = 0$ holds for each $z \in \mathfrak{N}$ so that $\vee_n (E_{1,n} \wedge E_{2,n}) = I$ and observe that the range of projection $E_{1,n} \wedge E_{2,n}$ is just $\mathfrak{M}_{1,n} \cap \mathfrak{M}_{2,n} = \mathfrak{M}_n$. Thus the sequence $\{\mathfrak{M}_n\}$ satisfies the requirements of definition 7. 1 for the linear manifold $\eta_1 \cap \eta_2$.

Via theorem 6. 10, we have available the following polar decomposition for any element $T \in M_0$. We only state the result, the details of the proof are precisely as in [12], p. 108.

Lemma 7. 6 : Every closed linear densely defined transformation T in M_0 can be represented in one and only one way in the form T = VH where H is a positive self-adjoint transformation in M_0 and V is a partial isometry in M. We have $H = +\sqrt{T^*T} = |T|$, and V = projection [Range H].

Lemma 7. 8: Let $\mathfrak{M}$ be an essentially dense linear manifold in $\mathfrak{N}$, and T an arbitrary element of M_0. Then $\mathfrak{M}' = \{z : z \in \mathfrak{D}_T, Tz \in \mathfrak{M}\}$ is essentially dense (cf [13], p. 223).

Proof: From lemma 7.6, let $T = VH$ be the polar decomposition of T. By lemma 7.3, let F_n be a sequence of projections of M which satisfy $F_n \uparrow_n I$, $H F_n \in M$. Let $\mathfrak{M}_n$ satisfy $\mathfrak{M}_n \uparrow_n \subseteq \mathfrak{M}$, $E_{\mathfrak{M}_n} \uparrow I$, $E_{\mathfrak{M}_n} \in M$. Set $\mathfrak{M}^{(n)} = \{z : z \in F_n \mathcal{K},\ VHF_n z \in \mathfrak{M}_n\}$. Note that $\mathfrak{M}^{(n)}$ is a closed linear subspace since $VHF_i \in M$ for all i, and further observe that $\mathfrak{M}^{(n)} \subseteq \mathfrak{M}'$ for all n. Since $z \in F_n \mathcal{K}$ implies $z \in F_{n+1}\mathcal{K}$ we have $\mathfrak{M}^{(n)} \subset \mathfrak{M}^{(n+1)}$ for each n. It is clear from the definition the $E_{\mathfrak{M}}(n)$, the projection on $\mathfrak{M}^{(n)}$, belong to M for all n. Let P_n denote the projection on $\{z : (I - E_{\mathfrak{M}_n}) VHF_n z = 0\}$. We have

$$\mathfrak{M}^{(n)} = F_n \mathcal{K} \cap \{z : VHF_n z \in \mathfrak{M}_n\} = F_n \mathcal{K} \cap \{z : (I - E_{\mathfrak{M}_n}) VHF_n z = 0\}$$

Thus $E_{\mathfrak{M}}(n) = F_n \wedge P_n$. Observe also that

$$\mathfrak{M}_n = \{z : (I - E_{\mathfrak{M}_n}) z = 0\} \subseteq \{z : (I - E_{\mathfrak{M}_n}) VHF_n z = 0\}$$

i.e. $I - E_{\mathfrak{M}_n} \geq I - P_n$. We have

$$\begin{aligned} 0 \leq I - E_{\mathfrak{M}}(n) = I - (F_n \wedge P_n) &= (I - F_n) \vee (I - P_n) \\ &\leq (I - F_n) \vee (I - E_{\mathfrak{M}_n}) \\ &\leq (I - F_n) + (I - E_{\mathfrak{M}_n}) \end{aligned}$$

It follows that $I - E_{\mathfrak{M}}(n) \downarrow 0$, thus $E_{\mathfrak{M}}(n) \uparrow_n I$ and $\mathfrak{M}'$ is essentially dense.

Remark 7.9: If $A, B \in M_0$, then from lemma 3.8, 3.5 it follows that $\mathcal{D}_{AB}$, $\mathcal{D}_{A+B}$ are essentially dense in $\mathcal{K}$.

We now have available the following results of von Neumann and Murray concerning the algebraic properties of M_0, without recourse to the general form of the spectral theorem.

Theorem 7.10: (cf [13], p. 227 ff.).

(1) Let $A, B \in M_0$. If $A \supseteq B$ then $A = B$, i.e. proper extensions do not exist in M_0.

(2) If A, B $\in M_0$, then A+B, AB have unique extensions to elements of M_0. Denote these extensions by [A+B], [AB] respectively.

(3) With addition and multiplication as in (2), the following properties are valid: A, B, C $\in M_0$, a, b are complete numbers.

(i) $[A+B] = [B+A]$

(ii) $[[A+B]+C] = [A+[B+C]]$

(iii) $[a[A+B]] = [a[A]+b[B]]$

(iv) $[(a+b)A] = [aA+bB]$

(v) $[[AB]C] = [A[BC]]$

(vi) $[[aA]B] = [a[AB]]$

(vii) $[a[bA]] = [(ab)A]$

(viii) $[[A+B]C] = [[AC] + [BC]]$

(ix) $[A[B+C]] = [[AB] + [AC]]$

(x) $[aA]^* = [\bar{a} A^*]$

(xi) $[A+B]^* = [A^*+B^*]$

(xii) $[AB]^* = [B^*A^*]$

It should be noted that (2) is proved essentially by showing that $\mathfrak{D}_{(A+B)^*}$, $\mathfrak{D}_{(AB)^*}$ are dense in $\mathfrak{H}$. $(A+B)^{**}$, $(AB)^{**}$ then provide the unique extensions [A+B], [AB] in M_0([12], p. 60). In particular, with addition and multiplication defined as in (2), part (3) states that M_0 is an algebra. We now show that M_0 is a commutative algebra in the sense of the following

Theorem 7.11: If A, B $\in M_0$ then $[AB] = [BA]$.

Proof: Let $0 \le A_n, B_n \in \mathrm{ReM}$ satisfy $A_n \uparrow |A|$, $B_n \uparrow |B|$, in the sense of Theorem 6.10. Observe that $0 \le A_nB_n \uparrow_n$, and for each z in $\mathcal{D}_{|A||B|}$, which is dense in $\mathcal{H}$, we have $\|A_nB_mz\| \le \||A||B|z\|$ for all n, m. It follows that there exists $C \in M_0$ with $A_nB_n \uparrow_n C$, and $\mathcal{D}_C \supseteq \mathcal{D}_{|A||B|}$, and by symmetry $\mathcal{D}_C \supseteq \mathcal{D}_{|B||A|}$. Let $z \in \mathcal{D}_{|A||B|}$, $y \in \mathcal{D}_{|A|}$, then $(|B|z, |A|y) = \lim_{n\to\infty}(A_nB_nz, y) = (Cz, y)$. Since $\mathcal{D}_{|A|}$ is dense, $|A||B|z = Cz$. Thus $|A||B| \subseteq C$, and by symmetry $|B||A| \subseteq C$, hence by lemma 6.2 $[|A||B|] = [|B||A|] = C$.

Now write $A = V_A|A|$, $B = V_B|B|$, where V_A, V_B are partial isometries in M. Observe that

$$V_AV_B|A||B| \subseteq V_A|A|V_B|B| = AB$$

and $$V_AV_B|B||A| \subseteq V_B|B|V_A|A| = BA\,.$$

Consequently
$$\begin{aligned}[AB] &= [V_AV_B|A||B|] = [V_AV_B[|A||B|]] \\ &= [V_AV_B[|B||A|]] = [V_AV_B|B||A|] \\ &= [BA]\;.\end{aligned}$$

VIII. THE RIESZ SPACE STRUCTURE OF ReM_0

We shall denote the set of self-adjoint transformations of M_0 by ReM_0. In this section it will be shown that the natural ordering in ReM may be extended to a partial ordering of ReM_0 so that ReM_0 will then be a (Dedekind complete) Riesz space when the operations $\vee$, $\wedge$ are appropriately defined. It will turn out that ReM is an order dense ideal in ReM_0, which has the property that the band generated by the identity is just ReM_0. In other words, I is a weak unit in ReM_0.

We make the natural definition:

<u>Definition 8.1:</u> If $A \in ReM_0$, we shall say that A is positive and write $A \geq 0$ if and only if $(Az, z) \geq 0$ for all $z \in \mathcal{D}_A$.

That this definition gives a bona-fide partial order on ReM_0 we have:

<u>Lemma 8.2</u>: (cone properties)

(i) <u>$A, B \in ReM_0$, $A \geq 0$, $B \geq 0$, then $[A+B] \geq 0$.</u>

(ii) <u>$A \in ReM_0$, $a \in \mathbf{R}^+$ then $aA \geq 0$.</u>

(iii) <u>$A \in ReM_0$, $A \geq 0$, $-A \geq 0$ then $A = 0$.</u>

<u>Proof</u>: (i) If $z \in \mathcal{D}_{A+B}$ then

$$([A+B]z, z) = ((A+B)z, z) = (Az, z) + (Bz, z) \geq 0$$

If now $z \in \mathcal{D}_{[A+B]}$, there exists $\{z_n\}$, $z_n \in \mathcal{D}_{A+B}$, such that $z_n \to z$ and $(A+B)z_n \to [A+B]z$. Thus

$$([A+B]z, z) = \lim_{n\to\infty} ((A+B)z_n, z_n) \geq 0.$$

(ii) is obvious.

(iii) $A \geq 0$, $-A \geq 0$ implies $(Az, z) = 0$ for all $z \in \mathcal{D}_A$. Thus

$A^{\frac{1}{2}}z = 0$ for all $z \in \mathcal{D}_A$, thus $Az = 0$ for all $z \in \mathcal{D}_A$, thus $A = 0$.

Definition 8.3: If $A, B \in \mathrm{ReM}_0$, set $A \geq B$ if and only if $[A-B] \geq 0$.

From lemma 8.2 it is clear that $(\mathrm{ReM}_0, \leq)$ is an ordered linear vector space.

Theorem 8.4: Let $0 \leq A \in \mathrm{ReM}_0$, $0 \leq B \in \mathrm{ReM}_0$. Let $0 \leq A_n \uparrow_n A$, $0 \leq B_n \uparrow_n B$, $A_n, B_n \in \mathrm{ReM}$, $n=1, 2, \ldots$ as in theorem 6.10, then

(i) $[AB] \geq 0$,

(ii) $A \geq B$ implies $A^2 \geq B^2$,

(iii) $A \geq B$ if and only if $\mathcal{D}_A \subseteq \mathcal{D}_B$ and $(Az, z) \geq (Bz, z)$ for each $z \in \mathcal{D}_B$,

(iv) $A \geq B$ implies $A_n \vee B_n \uparrow_n A$, $A_n \wedge B_n \uparrow_n B$,

(v) $A \geq B$ implies $A^{\frac{1}{2}} \geq B^{\frac{1}{2}}$.

Proof: (i) As in the proof of Theorem 7.11, $0 \leq A_n B_n \uparrow [AB]$. Thus $[AB] \geq 0$.

(ii) From (i), $[A[A-B]] = [A^2-[AB]] \geq 0$,

$$[[A-B]B] = [[AB]-B^2] \geq 0.$$

Hence $\qquad 0 \leq [[A^2-[AB]] + [[AB]-B^2]] = [A^2-B^2]$.

(iii) Let $z \in \mathcal{D}_{A^2} \cap \mathcal{D}_{B^2}$. From (ii), for all n,

$$(B_n z, B_n z) \leq (Bz, Bz) \leq (Az, Az).$$

It follows that, for all n, $(B_n z, B_n z) \leq (A'z, A'z)$ holds now for all $z \in \mathcal{D}_{A'}$, where A' denotes the smallest closed extension of the restriction of A to $\mathcal{D}_{A^2} \cap \mathcal{D}_{B^2}$. By lemma 6.2, $A' = A$ so that $\mathcal{D}_A \subseteq \mathcal{D}_B$. That $(Az, z) \geq (Bz, z)$ for $z \in \mathcal{D}_A$ is trivial. Conversely if $\mathcal{D}_A \subseteq \mathcal{D}_B$, and $(Az, z) \geq (Bz, z)$ holds for $z \in \mathcal{D}_A$, then $[A-B] \geq 0$ follows from the fact that the graph of A-B in $\mathcal{H} \times \mathcal{H}$ is dense in the graph of $[A-B]$.

(iv) $\|A_n z\| \leq \|(A_n \vee B_n)z\| \leq \|A_n z\| + \|B_n z\|$ for all z implies that (a) $A_n \vee B_n \uparrow_n C$ where $C \in ReM_0$, (b) $\mathcal{D}_A = \mathcal{D}_A \cap \mathcal{D}_B \subseteq \mathcal{D}_C \subseteq \mathcal{D}_A$ so that $\mathcal{D}_C = \mathcal{D}_A$, and (c) $A \leq C$. To show $C = A$ it is sufficient to show $C \leq A$. Let $Q_m \uparrow_m I$ be projections in M such that Range $Q_m \subseteq \mathcal{D}_A$ for m=1, 2,..., by lemma 7.3. Observe that AQ_m, BQ_m, CQ_m are elements of M for m=1, 2, ... By the uniqueness of the square root in ReM, $|A_n - B_n|Q_m = |A_nQ_m - B_nQ_m|$. Hence

$$(A_n \vee B_n)Q_m = \tfrac{1}{2}(A_nQ_m + B_nQ_m + |A_n - B_n|Q_m) = \tfrac{1}{2}(A_nQ_m + B_nQ_m + |A_nQ_m - B_nQ_m|)$$
$$= A_nQ_m \vee B_nQ_m .$$

For each $z \in \mathcal{H}$, $(AQ_m z, z) \geq (B_nQ_m z, z)$, $(AQ_m z, z) \geq (A_nQ_m z, z)$. Thus $AQ_m \geq B_nQ_m \vee A_nQ_m = (B_n \vee A_n)Q_m$. Let $m \to \infty$, then for each $z \in \mathcal{D}_A$, we have $(Az, z) \geq ((A_n \vee B_n)z, z)$ for n=1, 2, ... Thus $(Az, z) \geq (Cz, z)$ and $A \geq C$. Thus $A = C$. To show $A_n \wedge B_n \uparrow B$, it is sufficient to show that for each $z \in \mathcal{D}_B$, that $\|A_n \wedge B_n z - B_n z\| \to 0$ as $n \to \infty$. This follows immediately from the fact that $\|A_n \wedge B_n z - B_n z\| = \|A_n z - A_n \vee B_n z\| \to 0$ as $n \to \infty$.

(v) From (iv) we may assume $A_n \uparrow_n A$, $B_n \uparrow B$ satisfy $B_n \leq A_n$ for all n. Hence also $B_n^{\frac{1}{2}} \leq A_n^{\frac{1}{2}}$. Hence $\mathcal{D}_{A^{\frac{1}{2}}} \subseteq \mathcal{D}_{B^{\frac{1}{2}}}$ and $B^{\frac{1}{2}} \leq A^{\frac{1}{2}}$.

The next few paragraphs follow fairly closely the corresponding results for ReM [9], chapter 5. As usual, for $A \in ReM_0$ $|A| = +\sqrt{A^*A} = +\sqrt{A^2}$. Set $A^+ = \frac{1}{2}[A + |A|]$.

<u>Lemma 8.5:</u> If $A \in ReM_0$, $A \leq |A|$, $-A \leq |A|$. Equivalently $A^+ \geq 0$, $A^+ \geq A$.

<u>Proof:</u> Let $A = V|A|$ be the polar decomposition of A. Let $A_n \in (\mathrm{ReM})^+$ satisfy $A_n \uparrow_n |A|$ in the sense of Theorem 6.10.

For all $z \in \mathcal{D}_A = \mathcal{D}_{|A|}$,

$$|(VA_n z, z)| \leq \|V\| \, \|A_n^{\frac{1}{2}} z\|^2 \leq 1 \cdot (A_n z, z)$$

since V is a partial isometry. Let $n \to \infty$ and we obtain

$$|(V|A|z, z)| \leq (|A|z, z),$$

i.e. $(V|A|z, z) \leq (|A|z, z)$, $(-V|A|z, z) \leq (|A|z, z)$,

i.e. $A \leq |A|$, $-A \leq |A|$. The equivalent statement follows immediately from the definition.

<u>Theorem 8.6:</u> <u>Let $A, B \in \mathrm{ReM}_0$ satisfy $B \geq A$, $B \geq 0$. Then $B \geq A^+$. Equivalently $C \geq A$, $C \geq -A$, $C \in \mathrm{ReM}_0$ imply $C \geq |A|$.</u>

<u>Proof:</u> Let $B \geq A$, $B \geq 0$. Set $C = [2B-A]$, then

$$[C-A] = [[2B-A]-A] = [2B + [-2A]]$$
$$= [2B-2A] \geq 0,$$

and

$$[C+A] = [[2B-A]+A]$$
$$= 2B \geq 0.$$

Thus, if we show that $C \geq |A|$ it will follow that $[2B-A] \geq |A|$

i.e. $2B \geq [A + |A|]$ which is the desired result.

By Theorem 8.4, $[[C-A][C+A]] \geq 0$, i.e.,

$$[[C^2 - [AC]] + [[CA] - A^2]] \geq 0.$$

Using the fact that $[AC] = [CA]$ we obtain $[C^2 - A^2] \geq 0$. From $C \geq 0$ and Theorem 8.4(v), it follows that $C \geq +\sqrt{A^2} = |A|$.

It follows immediately from Theorem 8.6 that $(\mathrm{ReM}_0, \leq)$ is a Riesz space, and that $A^+ = \sup(A, 0)$ in ReM_0, $|A| = \sup(A, -A)$ for each $A \in \mathrm{ReM}_0$. If i denotes the inclusion map of ReM into ReM_0

then it is clear that $i(A^+) = (i(A))^+$ for each A in ReM, and that i is one to one. Thus i is a Riesz isomorphism, and in the sequel we make no distinction between ReM and i(ReM).

Lemma 8.7: ReM is an order dense ideal in ReM_0.

Proof: Suppose $0 \le |S| \le T$ where $T \in ReM$, $|S|, S \in ReM_0$. We have by Theorem 8.4 since $[T-|S|] \ge 0$ that $\mathcal{H} = \mathfrak{D}_T \subseteq \mathfrak{D}_{|S|}$. Thus $\mathfrak{D}_S = \mathfrak{D}_{|S|} = \mathcal{H}$ so that $S \in ReM$ by the closed graph theorem. Hence ReM is an ideal in ReM_0. It is clear from the construction of ReM_0 that ReM is order dense in ReM_0.

Lemma 8.8: Let $0 \le S \in ReM_0$ and suppose that $\{S_n\} \uparrow S$ in the sense of Theorem 6.10. Then $S_n \uparrow_n S$ in ReM_0.

Proof: Clearly $0 \le S_n \uparrow_n \le S$. Suppose that $T \in ReM_0$ satisfies $T \ge S_n$ for all n. Hence if $z \in \mathfrak{D}_T$ then

$$(Tz, Tz) \ge (S_n z, S_n z) \text{ for all } n.$$

Thus $\mathfrak{D}_T \subseteq \mathfrak{D}_S$ and for $z \in \mathfrak{D}_T$,

$$(Tz, z) \ge \lim_{n\to\infty}(S_n z, z) = (Sz, z).$$

Thus $T \ge S$ and we have $S_n \uparrow_n S$ in ReM_0.

Theorem 8.9: Let $0 \le S \in ReM_0$. Then $S = \vee_n(nI \wedge S)$. In other words ReM_0 is just the band generated by I.

Proof: Since $0 \le nI \wedge S \le nI$, $nI \wedge S \in (ReM)^+$. Clearly $nI \wedge S \uparrow_n \le S$. By the usual procedure, $\vee_n(nI \wedge S)$ certainly exists in ReM_0 and satisfies $\vee_n(nI \wedge S) \le S$.

On the other hand, lemma 7.3 gives the existence of a sequence of projections $P_n \in M$ such that $P_n \uparrow I$ such that $SP_n \in M$ for each n. We have $SP_n \uparrow_n \leq S$ and again $\vee_n SP_n$ exists in ReM_0. Observe that $\mathcal{D}_{\vee_n SP_n} \supseteq \mathcal{D}_S$ and if $z \in \mathcal{D}_{\vee_n SP_n}$ we have $P_n z \to z$, $SP_n z \to (\vee_n SP_n)z$. However as S is closed, it follows that $z \in \mathcal{D}_S$ and $SP_n z \to Sz$, thus $SP_n \uparrow S$. Now $SP_n \in ReM$ and $SP_n \leq S$ implies $SP_n \leq (m_n I \wedge S)$ for some integer m_n. Hence

$$S = \vee_n SP_n \leq \vee_n (nI \wedge S).$$

Thus $S = \vee_n (nI \wedge S)$.

Remark: In the terminology of Riesz spaces, I is a weak order unit in ReM_0.

IX. THE DEDEKIND COMPLETENESS OF ReM_0

Definition 9.1: If $T \in (ReM_0)^+$, put

$$(Tx, x) = \begin{cases} \|T^{\frac{1}{2}}x\|^2 & \text{if } x \in \mathcal{D}_{T^{\frac{1}{2}}} \\ +\infty \text{ otherwise} \end{cases}$$

Theorem 9.2: Let $0 \le A_\tau$, $S \in (ReM_0)^+$ satisfy $A_\tau \uparrow_\tau \le S$.

(i) $A = \vee_\tau A_\tau$ exists in ReM_0

(ii) For every $\epsilon > 0$ and every $x \in \mathcal{D}_A$, there exists $\tau_{\epsilon, x}$ such that $\|Ax - A_\tau x\| < \epsilon$ for all $A_\tau \ge A_{\tau_{\epsilon, x}}$.

(iii) $A_\tau^{\frac{1}{2}} \uparrow_\tau A^{\frac{1}{2}}$ in ReM_0.

(iv) $0 \le A_\tau \uparrow_\tau A$ in ReM_0 if and only if $A \le A_\tau$ and $(A_\tau x, x) \uparrow_\tau (Ax, x)$. for all $x \in \mathcal{H}$.

Proof: (i) By hypothesis, $\mathcal{D}_S \subseteq \mathcal{D}_{A_\tau}$ for every τ. From $0 \le A_\tau^2 \uparrow_\tau \le S^2$ follows $\|A_\tau x\| \le \|Sx\|$ for each $x \in \mathcal{D}_{S^2}$. Put $\mathfrak{M} = \{x \in \mathcal{H} : \sup_\tau \|A_\tau x\| \le K_x$, for some finite constant $K_x\}$. Note that $\mathcal{D}_{S^2} \subseteq \mathfrak{M}$, so that $\mathfrak{M}$ is dense in $\mathcal{H}$. $\mathfrak{M}$ is clearly a linear manifold in $\mathcal{H}$. If $A_{\tau'} \ge A_\tau$, $x \in \mathfrak{M}$, then

$$\|A_{\tau'}x - A_\tau x\|^2 = \|A_{\tau'}x\|^2 + \|A_\tau x\|^2 - (A_{\tau'}x, A_\tau x) - (A_\tau x, A_{\tau'}x)$$
$$\le \|A_{\tau'}x\|^2 - \|A_\tau x\|^2.$$

For each fixed $x \in \mathfrak{M}$, the upwards directed set of real numbers $\|A_\tau x\|$ has a finite supremum. It follows that for every $\epsilon > 0$, there exists $\tau_{\epsilon, x}$ such that $\|A_{\tau'}x - A_\tau x\| < \epsilon$ for all $A_\tau, A_{\tau'} \ge A_{\tau_{\epsilon,x}}$. In particular, for $n = 1, 2, \ldots$, there exists $\tau_{n,x}$ such that $\|A_\tau x - A_{\tau_{n,x}} x\| < \frac{1}{n}$ for all $A_\tau \ge A_{\tau_{n,x}}$ and we may assume that $A_{\tau_{n+1,x}} \ge A_{\tau_{n,x}}$ for all n. In particular note that $\|A_{\tau_{n,x}}\| \uparrow_n \sup_\tau \|A_\tau x\|$. Thus

$$\|A_{\tau_{m,x}} x - A_{\tau_{n,x}}\| < \frac{1}{n} \quad \text{for all } m \ge n,$$

so that the sequence $\{A_{\tau_{n,x}} x\}$ converges to an element of $\mathcal{H}$ which we shall denote by Ax. Ax is uniquely determined in the sense that if

$\{A_{\tau'_{n,x}}\}$ is another increasing sequence such that $\|A_\tau x - A_{\tau'_{n,x}} x\| < \frac{1}{n}$ for all $A_\tau \geq A_{\tau'_{n,x}}$, then

$$\|A_{\tau'_{n,x}} x - A_{\tau_{n,x}} x\| \leq \|A_{\tau'_{n,x}} x - A_\tau x\| + \|A_\tau x - A_{\tau_{n,x}}\| < \frac{2}{n}$$

for $A_\tau \geq A_{\tau'_{n,x}} \vee A_{\tau_{n,x}}$. It follows easily that A is linear; let $x, y, z \in \mathfrak{D}_A = \mathfrak{M}$, $z = x+y$. Then, for $n = 1, 2, \ldots$

$$\|Ax - A_\tau x\| \leq \frac{2}{n} \text{ for all } A_\tau \geq A_{\tau_{n,x}} \quad \ldots \qquad \text{(i)}$$

$$\|Ay - A_\tau y\| \leq \frac{2}{n} \text{ for all } A_\tau \geq A_{\tau_{n,y}} \quad \ldots \qquad \text{(ii)}$$

$$\|Az - A_\tau z\| \leq \frac{2}{n} \text{ for all } A_\tau \geq A_{\tau_{n,z}}$$

For all $A_\tau \geq A_{\tau_{n,x}} \vee A_{\tau_{n,y}} \vee A_{\tau_{n,z}}$, these inequalities hold simultaneously. $A_\tau z = A_\tau x + A_\tau y$ gives

$$\|Az - Ax - Ay\| \leq \frac{6}{n}$$

which implies $Az = Ax + Ay$.

Let U be unitary in M'; if $x \in \mathfrak{D}_A$, then

$$\|A_\tau U_x\| = \|UA_\tau x\| = \|A_\tau x\| \leq K_x \text{ for all } \tau.$$

It follows that $Ux \in \mathfrak{D}_A$ and that if $A_{\tau_{n,x}} x \to Ax$ then $A_{\tau_{n,x}} Ux \to AUx$. Hence

$$UAx = \lim_{n\to\infty} U A_{\tau_{n,x}} x = \lim_{n\to\infty} A_{\tau_{n,x}} Ux = AUx.$$

Hence $AU \supseteq UA$. If $x, y \in \mathfrak{D}_A$, there exist $A_{\tau_{n,x}} \uparrow_n$, $A_{\tau_{n,y}} \uparrow_n$, such that $A_{\tau_{n,x}} x \to Ax$, $A_{\tau_{n,y}} y \to Ay$. The inequalities (i), (ii), above show that there exists a sequence $A_{\tau_n} \uparrow$ such that $A_{\tau_n} x \to Ax$, $A_{\tau_n} y \to Ay$, so that $(A_{\tau_n} x, y) = (x, A_{\tau_n} y)$ converges to (Ax, y) as well as (x, Ay) i.e., for all $x, y \in \mathfrak{D}_A$ $(Ax, y) = (x, Ay)$. Thus $y \in \mathfrak{D}_{A^*}$ and $A^*y = Ay$. Thus $A \subseteq A^*$, and A is Hermitian. In particular A^{**} exists. To conclude that A is even self-adjoint, it suffices to show that A is closed in view of lemma 6.2. Suppose that $x_n \in \mathfrak{D}_A$, $x_n \to x$, $Ax_n \to y$.

There exists a constant K such that $\|Ax_n\| \le K$ for all n, so that $\|A_\tau x_n\| \le K$ for all τ, n. Let $A_\tau^{(m)} \uparrow_m A_\tau$ where $0 \le A_\tau^{(m)} \in \mathrm{ReM}$. For each m, τ, n, we have $\|A_\tau^{(m)} x_n\| \le K$ hence $\|A_\tau^{(m)} x\| \le K$ holds for each m, τ. This implies that $x \in \mathfrak{D}_{A_\tau}$ for all τ and that $\|A_\tau x\| \le K$. In turn this gives $x \in \mathfrak{D}_A$. Since $A \subseteq A^{**}$ and A^{**} is closed $Ax_n = A^{**}x_n \to A^{**}x = Ax$. Thus A is closed, hence <u>self-adjoint</u>. That $A_\tau^2 \le A^2$ for each τ is immediate from the definition of A. Hence $A_\tau \le A$. Further, for each $x \in \mathfrak{D}_A$, $(A_{\tau_{n,x}} x, x) \uparrow_n (Ax, x)$. This follows from

$$|(Ax, x) - (A_\tau x, x)| \le \|x\| \; \|Ax - A_\tau x\| \le \frac{2}{n} \|x\|$$

for all $A_\tau \ge A_{\tau_{n,x}}$. Thus $(Ax, x) = \sup_\tau (A_\tau x, x)$ holds for each $x \in \mathfrak{D}_A$. It follows that $A = \vee_\tau A_\tau$; if $B \ge A_\tau$ for all τ, then also $B^2 \ge A_\tau^2$ for all τ. By the definition of A, $\mathfrak{D}_B \subseteq \mathfrak{D}_A$; if $y \in \mathfrak{D}_B$, then $(By, y) \ge \sup_\tau (A_\tau y, y) = (Ay, y)$. Thus $B \ge A$.

(iii) Observe that $0 \le A_\tau^{\frac{1}{2}} \uparrow_\tau \le A^{\frac{1}{2}}$. By part (i) $C = \vee_\tau A_\tau^{\frac{1}{2}}$ exists in ReM_0 so that $C \le A^{\frac{1}{2}}$. $C \ge A_\tau^{\frac{1}{2}}$ for all τ implies $C^2 \ge A_\tau$ for all τ, so that $C^2 \ge A$ and $C \ge A^{\frac{1}{2}}$. Therefore $C = A^{\frac{1}{2}}$ and (iii) is proved.

(iv) From parts (iii) and (i), for each $x \in \mathfrak{D}_{A^{\frac{1}{2}}}$, $\|A^{\frac{1}{2}}x\| = \sup_\tau \|A_\tau^{\frac{1}{2}}x\|$; if $x \notin \mathfrak{D}_{A^{\frac{1}{2}}}$ then $\sup_\tau \|A_\tau^{\frac{1}{2}} x\| = +\infty$. Thus $(Ax, x) = \sup_\tau (A_\tau x, x)$ holds for all $x \in \mathfrak{X}$. On the other hand, assume that $0 \le A_\tau$, $A \in \mathrm{ReM}_0$ satisfy $A \ge A_\tau$ for all τ, and $(A_\tau x, x) \uparrow_\tau (Ax, x)$ for every $x \in \mathfrak{X}$. By (i) $A_\tau \uparrow_\tau B \le A$ so that $\mathfrak{D}_A \subseteq \mathfrak{D}_B$. For each $x \in \mathfrak{D}_A$, $(Bx, x) = (Ax, x)$. Since the graph of $[A-B]$ in $\mathfrak{X} \times \mathfrak{X}$ is just the closure in $\mathfrak{X} \times \mathfrak{X}$ of the graph of A-B, it follows that $[A-B] = 0$. By this the theorem is completely proved.

<u>Theorem 9.3: Let $\{A_\sigma\}, \{B_\tau\} \in (\mathrm{ReM}_0)^+$ satisfy $0 \le A_\sigma \uparrow_\sigma A$, $0 \le B_\tau \uparrow_\tau B$. Then $[A_\sigma B_\tau] \uparrow_{\sigma,\tau} [AB]$.</u>

<u>Proof</u>: Without loss of generality, assume A_σ, B_τ belong to $(\mathrm{ReM})^+$. Thus $A_\sigma B_\tau \uparrow_{\tau,\sigma} \le [AB]$, so by Theorem 9.2 (i), there exists $C \in (\mathrm{ReM}_0)^+$ such that $A_\sigma B_\tau \uparrow_{\sigma,\tau} C \le [AB]$. In particular $\mathcal{D}_{[AB]} \subseteq \mathcal{D}_C$. If $x \in \mathcal{D}_{[AB]}$, there exists $x_n \in \mathcal{D}_{AB}$, $x_n \to x$ and $ABx_n \to [AB]x$. It follows that $ABx_n \to Cx$. In fact, $\{ABx_n\}$ converges and so is a Cauchy sequence. Given $\epsilon > 0$, for $n, m \ge n_0(\epsilon)$, $\|AB(x_n - x_m)\| < \epsilon$. Hence, for every $n, m \ge n_0(\epsilon)$, $\sup_\sigma \|A_\sigma B(x_n - x_m)\| = \sup_\sigma \|BA_\sigma(x_n - x_m)\| < \epsilon$ and so $\sup_{\sigma,\tau} \| B_\tau A_\sigma (x_n - x_m)\| < \epsilon$. Let $x_m \to x$ and it follows, since each $B_\tau A_\sigma$ is continuous, that

$$\sup_{\sigma,\tau} \|B_\tau A_\sigma (x_n - x)\| \le \epsilon \text{ for all } n \ge n_0(\epsilon).$$

Now, since $x \in \mathcal{D}_C$, there exist $\sigma(\epsilon), \tau(\epsilon)$ such that

$$\|Cx - A_\sigma B_\tau x\| < \epsilon \text{ for all } A_\sigma B_\tau \ge A_{\sigma(\epsilon)} B_{\tau(\epsilon)}$$

Let y be arbitrary in $\mathcal{D}_{AB}$. From $By \in \mathcal{D}_A$, it follows that there exists $\sigma_{\epsilon,y}$ such that $\|ABy - A_\sigma By\| < \epsilon$ for all $A_\sigma \ge A_{\sigma_{\epsilon,y}}$. Also since $A_\sigma By = BA_\sigma y$, there exists $\tau_{\epsilon,y}$ such that $\|BA_\sigma y - B_\tau A_\sigma y\| = \|A_\sigma By - A_\sigma B_\tau y\| < \epsilon$ for all $B_\tau \ge B_{\tau_{\epsilon,y}}$. Thus, for all $A_\sigma \ge A_{\sigma_{\epsilon,y}}$, $B_\tau \ge B_{\tau_{\epsilon,y}}$

$$\|ABy - A_\sigma B_\tau y\| \le 2\epsilon$$

Hence, given $\epsilon > 0$, for each x_n, there exist $A_{\sigma_{\epsilon,x_n}}$, $B_{\tau_{\epsilon,x_n}}$ such that

$$\|Cx - A_{\sigma_{\epsilon,x_n}} B_{\tau_{\epsilon,x_n}} x\| < \epsilon, \quad \|ABx_n - A_{\sigma_{\epsilon,x_n}} B_{\tau_{\epsilon,x_n}} x_n\| < \epsilon$$

hold simultaneously. Choose $n_0(\epsilon)$ such that $\sup_{\sigma,\tau} \|A_\sigma B_\tau (x_n - x)\| < \epsilon$ for all $n \ge n_0(\epsilon)$. For $n \ge n_0$

$$\|Cx-ABx_n\| \le \|Cx-B_{\tau_{\epsilon,x_n}} A_{\sigma_{\epsilon,x_n}} x\| + \|B_{\tau_{\epsilon,x_n}} A_{\sigma_{\epsilon,x_n}} (x_n - x)\|$$

$$+ \|ABx_n - A_{\sigma_{\epsilon,x_n}} B_{\tau_{\epsilon,x_n}} x_n\| \le 3\epsilon$$

Thus $ABx_n \to Cx$. Thus $Cx = [AB]x$ and so $[AB] \subseteq C$. Therefore $[AB] = C$.

<u>Corollary 9.4:</u> <u>If $0 \le \{A_\tau\} \uparrow_\tau A$, $0 \le \{B_\tau\} \uparrow_\tau B$ are indexed by the same index set $\{\tau\}$, then $[A_\tau B_\tau] \uparrow_\tau [AB]$.</u>

<u>Proof</u>: $[A_\tau B_\tau] \uparrow_\tau$ follows immediately from the equidirectedness. It remains to be shown that the systems $[A_\tau B_\tau]$, $[A_\tau B_{\tau'}]$ have the same set of upper bounds. It is obvious that any upper bound of the system $[A_\tau B_{\tau'}]$ is an upper bound for the system $[A_\tau B_\tau]$. Let $[A_{\tau_1} B_{\tau_2}]$ be given choose τ_3 such that $A_{\tau_3} \ge A_{\tau_1}, A_{\tau_2}$; $B_{\tau_3} \ge B_{\tau_1}, B_{\tau_2}$, then $[A_{\tau_3} B_{\tau_3}] \ge [A_{\tau_1} B_{\tau_2}]$. Thus any upper bound of the system $[A_\tau B_\tau]$ is also an upper bound of the system $[A_\tau B_{\tau'}]$.

<u>Theorem 9.5:</u> <u>Let $\{E_i\}_{i \in \mathcal{I}}$ be a system of pairwise disjoint projections of M which satisfies $\Sigma_{i\in\mathcal{I}} E_i = I$. For any element $T \in ReM_0$, set $T_i = TE_i$ and let $\mathfrak{F}$ denote a finite subfamily of the index set $\mathcal{I}$. Then</u>

(i) <u>If $T \in (ReM_0)^+$, $T = \prod_{i\in\mathcal{I}} \times T_i = \vee_i T_i = \vee_{\mathfrak{F}}(\prod_{i\in\mathfrak{F}} \times T_i)$</u>

(ii) <u>If $S \in (ReM_0)^+$, $T \in (ReM_0)^+$, then $[ST] = \vee_i [S_i T_i]$.</u>

<u>Proof</u>: (i) $T = \prod_{i\in\mathcal{I}} \times T_i$ follows from [12], p. 70. Note that $E_i E_j = 0$ implies $T_i \wedge T_j = 0$. If not, there exists $0 \ne A \in (ReM)^+$ such that $A \le T_i$, $A \le T_j$ so that $A = AE_i = (AE_j)E_i = 0$. Hence $T_i \vee T_j = [T_i \oplus T_j]$ if $i \ne j$. Consequently, for each finite subfamily $\mathfrak{F}$ of $\mathcal{I}$, $\vee_{i\in\mathfrak{F}} T_i = \prod_{i\in\mathfrak{F}} \times T_i$. By the Dedekind completeness of ReM_0, $\prod_{i\in\mathfrak{F}} \times T_i \uparrow_{\mathfrak{F}} T' \le T$. Thus

$T' = \vee_{\mathfrak{J}}(\vee_{i\in\mathfrak{J}} T_i) = \vee_{i\in\mathcal{J}} T_i$. $T' \le T$ implies $T'E_i \le TE_i = T_i$. On the other hand $T' \ge T_i$ implies $T'E_i \ge T_i$. Thus $T'E_i = T_i$ and $T' = \prod_{i\in\mathcal{J}}^{\times} T'E_i = \prod_{i\in\mathcal{J}}^{\times} T_i = T$.

(ii) In view of (i), it is sufficient to show that $[ST]E_i = [S_iT_i]$. Note that $S_iT_i = SE_i \cdot TE_i \subseteq [ST]E_i$, so that $[S_iT_i] \subseteq [ST]E_i$. Hence $[S_iT_i] = [ST]_i$, by lemma 6.2.

Let A be any element of ReM_0. R(A) will denote the closure of the range of A, N(A) will denote the null space of A.

Lemma 9.6: If A, B $\in ReM_0$ then

(i) $A \perp B \Longleftrightarrow [AB] = 0 \Longleftrightarrow AB = 0 \Longleftrightarrow R(A) \perp R(B)$.

(ii) If A, B, C $\in ReM_0$, then $A \perp B$ implies $AC \perp BC$.

Proof: (i) Assume first that $A \ge 0$, $B \ge 0$ and set $C = A \wedge B \ge 0$. Let $0 \le A_n \uparrow_n A$, $0 \le B_n \uparrow B$, where $A_n, B_n \in ReM$. Note that $A_n \wedge B_n = 0$ so that $A_nB_n = 0$. Since $[A_nB_n] \uparrow_n [AB]$, $AB = 0$. In the general case, it is clear that $[AB] = 0$ if and only if $AB = 0$. By the uniqueness of the square root in ReM_0, $|[AB]| = [|A||B|]$. Hence $[AB] = 0$ if and only if $|A||B| = 0$, i.e., if and only if $A \perp B$.

If $R(A) \perp R(B)$ then $(Ax, By) = 0$ holds for all $x \in \mathcal{D}_A, y \in \mathcal{D}_B$, so that $(x, ABy) = 0$ for all $x \in \mathcal{D}_A$, $y \in \mathcal{D}_{AB}$. Since $\mathcal{D}_A$ is dense, $AB = 0 = [AB]$. Conversely, if $AB = 0$, then $\mathfrak{M} = \{y' \in \mathcal{D}_B : By' \in \mathcal{D}_A\}$ is essentially dense in $\mathcal{H}$ by lemma 7.8. Hence if $x = Ax'$, $x' \in \mathcal{D}_A$ and $y = By'$ where $y' \in \mathcal{D}_B$, $y \in \mathcal{D}_A$, then $(x, y) = (Ax', By') = (x', ABy') = 0$. To conclude that $R(A) \perp R(B)$, it is sufficient to observe that the closure of the graph in $\mathcal{H} \times \mathcal{H}$ of the restriction of B to $\mathfrak{M}$ is just B.

(ii) If A, B, C $\in ReM_0$, and $A \perp B$, then $AB = 0$. Thus $0 = [[AB]C^2] = [[AC][BC]]$, so that $AC \perp BC$.

<u>Theorem 9.7</u>: <u>Let $0 \le \{T_i\}_{i\in\mathcal{J}}$ be any system of mutually disjoint elements of ReM_0. Then $S = \vee_i T_i$ exists in ReM_0 and satisfies $SE_i = T_i$, where E_i denotes the projection on $R(T_i)$. Consequently ReM_0 is a universally complete Riesz space.</u>

<u>Proof</u>: By lemma 9.6, $T_i \perp T_j$ for $i \neq j$ implies $E_i \perp E_j$ where E_i, E_j respectively denote the projection on $R(T_i)$, $R(T_j)$. Set $S = \prod_{i\in\mathcal{J}} \times T_i$. It is clear that $S = 0$ on $I - \Sigma_{i\in\mathcal{J}} E_i$, and from Theorem 9.5, $S = \vee_{i\in\mathcal{J}} T_i$. Clearly $SE_i \supseteq T_i$ so that $SE_i = T_i$.

X. A GENERALIZATION OF THE THEOREM OF R. PALLU DE LA BARRIÈRE

Definition 10.1: A map ψ: $(\mathrm{Re}M_0)^+ \to [0, +\infty]$ will be called a trace on $\mathrm{Re}M_0$ if for each T_1, $T_2 \in (\mathrm{Re}M_0)^+$, $\lambda \geq 0$, real, we have

$$\psi([T_1+T_2]) = \psi(T_1) + \psi(T_2), \; \psi(\lambda T_1) = \lambda \, \psi(T_1) .$$

ψ will be called a semi-finite trace, if for each $T \in (\mathrm{Re}M_0)^+$, $T \neq 0$, there exists $0 \neq S \in (\mathrm{Re}M_0)^+$ such that $0 < S \leq T$ and $\psi(S) < +\infty$. ψ will be called a normal trace if $0 \leq T_\tau \uparrow_\tau T$ in $(\mathrm{Re}M_0)^+$ implies $\psi(T) = \sup_\tau \psi(T_\tau)$. Finally ψ will be called faithful if $\psi(T) = 0$, $T \in (\mathrm{Re}M_0)^+$, implies $T = 0$.

Definition 10.2: Let $\varphi(M) = \omega_{x,x}(M)$ for some $x \in \mathscr{X}$. For $T \in (\mathrm{Re}M_0)^+$, put

$$\Omega_{x,x}(T) = \begin{cases} \| T^{\frac{1}{2}} x \|^2 & \text{if } x \in \mathfrak{D}_{T^{\frac{1}{2}}}, \\ +\infty & \text{otherwise.} \end{cases}$$

Note that if $x \in \mathfrak{D}_T$, then $\Omega_{x,x}(T) = (Tx, x)$, and if $T \in (\mathrm{Re}M)^+$ then $\Omega_{x,x}(T) = \omega_{x,x}(T)$.

Lemma 10.3: For each $x \in \mathscr{X}$, $\Omega_{x,x}$ is a normal semi-finite trace on $(\mathrm{Re}M_0)^+$. $\Omega_{x,x}$ is faithful if and only if $E_x^{M'} = I$.

Proof: Let $T_1, T_2 \in (\mathrm{Re}M_0)^+$. Let $0 \leq H_n^{(i)} \uparrow_n T_i$, $H_n^i \in (\mathrm{Re}M)^+$, $i = 1, 2; n = 1, 2, \ldots$. It follows that

(i) $H_n^{(1)} + H_n^{(2)} \uparrow_n [T_1 + T_2]$

(ii) $[H_n^{(1)} + H_n^{(2)}]^{\frac{1}{2}} \uparrow_n [T_1 + T_2]^{\frac{1}{2}}$.

For each $y \in \mathscr{X}$, $\| (H_n^{(1)} + H_n^{(2)}) y \|^2 = \| H_n^{(1)} y \|^2 + \| H_n^{(2)} y \|^2 + 2(H_n^{(1)} H_n^{(2)} y, y)$. Since $(H_n^{(1)} H_n^{(2)} y, y) \geq 0$,

(iii) $(\| H_n^{(1)} y \|^2 + \| H_n^{(2)} y \|^2)^{\frac{1}{2}} \leq \| H_n^{(1)} y + H_n^{(2)} y \| \leq \| H_n^{(1)} y \| + \| H_n^{(2)} y \|$;

(iv) $\|(H_n^{(1)}+H_n^{(2)})^{\frac{1}{2}}y\|^2 = \|H_n^{(1)\frac{1}{2}}y\|^2 + \|H_n^{(2)\frac{1}{2}}y\|^2.$

(iii) implies $y \in \mathcal{D}_{[T_1+T_2]}$ if and only if $y \in \mathcal{D}_{T_1} \cap \mathcal{D}_{T_2}$.

(iv) implies $y \in \mathcal{D}_{[T_1+T_2]^{\frac{1}{2}}}$ if and only if $y \in \mathcal{D}_{T_1^{\frac{1}{2}}} \cap \mathcal{D}_{T_2^{\frac{1}{2}}}$.

It follows that $\Omega_{x,x}([T_1+T_2]) = +\infty$ if and only if $\Omega_{x,x}(T_1) = +\infty$ and/or $\Omega_{x,x}(T_2) = +\infty$. Thus $\Omega_{x,x}([T_1+T_2]) < +\infty$ if and only if $\Omega_{x,x}(T_1) < +\infty$ and $\Omega_{x,x}(T_2) < +\infty$. In this case

$$\begin{aligned}\Omega_{x,x}([T_1+T_2)] &= \|(T_1+T_2)^{\frac{1}{2}}x\|^2 = \lim_{n\to\infty}\|(H_n^{(1)}+H_n^{(2)})^{\frac{1}{2}}x\|^2 \\ &= \lim_{n\to\infty}\|H_n^{(1)\frac{1}{2}}x\|^2 + \lim_{n\to\infty}\|H_n^{(2)\frac{1}{2}}x\|^2 \\ &= \Omega_{x,x}(T_1) + \Omega_{x,x}(T_2).\end{aligned}$$

That $\Omega_{x,x}$ is normal follows from theorem 9.2, and the semi-finiteness of $\Omega_{x,x}$ follows immediately from the fact that ReM is order dense in ReM_0, and that $\Omega_{x,x}(T) < +\infty$ for each $T \in (\mathrm{ReM}\ )^+$. If $E_x^{M'} \neq I$, then $\Omega_{x,x}(I-E_x^{M'}) = \omega_{x,x}(I-E_x^{M'}) = 0$. On the other hand, if $E_x^{M'} = I$ then $\Omega_{x,x}$ is faithful; for if $0 \neq T \in (\mathrm{ReM}_0)^+$, then choose $0 \neq S \in (\mathrm{ReM})^+$ with $S \leq T$. Then $E_x^{M'} = I$ implies that $\Omega_{x,x}(S) = \omega_{x,x}(S) \neq 0$.

The next lemma is somewhat in the converse direction.

<u>Lemma 10.4:</u> <u>Let ψ be a normal trace on $(\mathrm{ReM}_0)^+$ such that $\psi(I) < +\infty$, then $\psi = \Omega_{x,x}$ for some $x \in \mathcal{H}$.</u>

<u>Proof:</u> Since ReM is the ideal generated by I, $\psi(T) < +\infty$ for each $T \in (\mathrm{ReM})^+$. The restriction of ψ to $(\mathrm{ReM})^+$ defines a positive normal linear functional on ReM. There exists $x \in \mathcal{H}$ such that $\psi(T) = \Omega_{x,x}(T)$ holds for each $T \in (\mathrm{ReM})^+$. By normality and lemma 8.7, $\psi(T) = \Omega_{x,x}(T)$ holds for each $T \in (\mathrm{ReM}_0)^+$.

Lemma 10.5: If $\{\psi_i\}_{i\in\mathcal{J}}$ is a family of normal traces on $(\text{ReM}_0)^+$, then the map $T \to \Sigma_{i\in\mathcal{J}}\,\psi_i(T)$: $(\text{ReM}_0)^+ \to [0, +\infty]$ is also a normal trace on $(\text{ReM}_0)^+$.

Proof: Set $\psi(T) = \Sigma_{i\in\mathcal{J}}\,\psi_i(T)$, $T \in \text{ReM}_0^+$. ψ is clearly linear. Suppose $0 \le T_\tau \uparrow_\tau T \in \text{ReM}_0^+$. By $\{\mathfrak{F}\}$ denote the family of all finite subsets of the index set $\mathcal{J}$.

$$\begin{aligned}\sup_\tau \Sigma_{i\in\mathcal{J}}\,\psi_i(T_\tau) &= \sup_\tau\,\sup_{\mathfrak{F}}\,\Sigma_{i\in\mathfrak{F}}\,\psi_i(T_\tau)\\ &= \sup_{\mathfrak{F}}\,\sup_\tau\,\Sigma_{i\in\mathfrak{F}}\,\psi_i(T_\tau)\\ &= \sup_{\mathfrak{F}}\,\Sigma_{i\in\mathfrak{F}}\,\Sigma_i(T) \;=\; \Sigma_{i\in\mathcal{J}}\,\psi_i(T).\end{aligned}$$

Thus ψ is a normal trace.

Theorem 10.6: There exists a semi-finite, faithful, normal trace on $(\text{ReM}_0)^+$.

Proof: Let $\omega_{x_i, x_i}(M)$ be a maximal family of positive normal linear functionals on M with the property that their supports $E_{x_i}^{M'} = E_i$ are pairwise disjoint. As usual, $\Sigma_{i\in\mathcal{J}}\,E_i = I$. Let Ω_{x_i, x_i} denote the extension of ω_{x_i, x_i} to a normal trace on $(\text{ReM}_0)^+$. Then $\psi_0 = \Sigma_{i\in\mathcal{J}}\,\Omega_{x_i, x_i}$ has the desired properties. ψ_0 is certainly a normal trace by lemma 10.5. Suppose that for some $T \in (\text{ReM}_0)^+$, $\psi_0(T) = 0$. Let $0 \le T_n \uparrow_n T$, $T_n \in (\text{ReM})^+$. From $\psi_0(T_n) = 0$ follows $\omega_{x_i, x_i}(T_n) = 0$, thus $T_n E_i = 0$, hence $T_n = 0$ since $\Sigma_{i\in\mathcal{J}}\,E_i = I$. Thus T=0, and ψ_0 is faithful. If $T \in \text{ReM}_0^+$, $T \neq 0$ then $TE_i \neq 0$ for at least one index i. Choose $0 \neq S \le TE_i$. Then $\Omega_{x_j, x_j}(S) = 0$ if $i \neq j$. Thus $\psi_0(S) = \Omega_{x_i, x_i}(S) = \omega_{x_i, x_i}(S) < +\infty$, so that ψ_0 is semi-finite.

In the converse direction we have:

Theorem 10.7: If ψ is any normal, faithful semi-finite trace on $(\mathrm{ReM}_0)^+$ then there exists a family $\{x_i\}_{i\in\mathcal{J}}$, $x_i \in \mathcal{K}$ such that $[M'x_i] \perp [M'x_j]$ if $i \neq j$, $\Sigma_{i\in\mathcal{J}} E^{M'}_{x_i} = I$ and $\psi = \Sigma_{i\in\mathcal{J}} \Omega_{x_i, x_i}$.

Proof: An outline of the proof is given, the details follow exactly as in theorem 10.10. Choose a maximal family of pairwise disjoint projections $E_i \in M$ such that $\psi(E_i) < +\infty$. From the semi-finiteness of ψ follows that $\Sigma_{i\in\mathcal{J}} E_i = I$. The restriction of ψ to $(\mathrm{ReM})^+E_i$ defines a positive faithful normal linear functional on ME_i, whence the existence of $x_i \in \mathcal{K}$ such that $E_i x_i = x_i; E^{M'}_{x_i} = E_i$ follows from the faithfulness of ψ. Thus $\psi(TE_i) = \omega_{x_i, x_i}(TE_i)$ holds for each $T \in (\mathrm{ReM})^+$ so that $\psi(TE_i) = \Omega_{x_i, x_i}(TE_i)$ holds for each $T \in (\mathrm{ReM})^+$ by normality. Finally if $\{\mathfrak{F}\}$ denotes the family of all finite subsets of the index set $\mathcal{J}$, then for each $T \in (\mathrm{ReM}_0)^+$,

$$\psi(T) = \sup_{\mathfrak{F}} \psi(\textstyle\prod_{i\in\mathfrak{F}} \times TE_i) = \sup_{\mathfrak{F}} \Sigma_{i\in\mathfrak{F}} \psi(TE_i) = \sup_{\mathfrak{F}} \Sigma_{i\in\mathfrak{F}} \Omega_{x_i, x_i}(TE_i)$$
$$= \sup_{\mathfrak{F}} \Sigma_{i\in\mathfrak{F}} \Omega_{x_i, x_i}(T) = \Sigma_{i\in\mathcal{J}} \Omega_{x_i, x_i}(T) .$$

Lemma 10.8: If $S \in (\mathrm{ReM})^+$, $T \in (\mathrm{ReM}_0)^+$ then

(i) $[TS] = TS$, (ii $(TS)^{\frac{1}{2}} = T^{\frac{1}{2}}S^{\frac{1}{2}}$.

Proof: (i) It is sufficient to notice that $S \in \mathrm{ReM}$, T closed imply TS is closed.

(ii) Observe $[T^{\frac{1}{2}}S^{\frac{1}{2}}T^{\frac{1}{2}}S^{\frac{1}{2}}] = [TS] = TS$. $\therefore (TS)^{\frac{1}{2}} = [T^{\frac{1}{2}}S^{\frac{1}{2}}] = T^{\frac{1}{2}}S^{\frac{1}{2}}$.

Lemma 10.9: Let ψ_0 be a normal faithful semi-finite trace on $(\mathrm{ReM}_0)^+$. Let E be any projection of M such that $\psi_0(E) < +\infty$. Define $P = \vee_x \{E^M_x : \omega_{x,x}(M) = \psi_0(ME)\}$. Then $P = E$.

Proof: Since $\psi_0(E) < +\infty$, the restriction of ψ_0 to ME is a positive normal linear functional on M; by lemma 10.4, there exists (at least one) $x \in \mathcal{H}$ such that $\psi_0(ME) = \omega_{x,x}(M)$. By the faithfulness of ψ_0, $E_x^{M'} = E$. The lemma now follows exactly as in lemma 4.8.

The following theorem is the central result of this section and generalizes the weak Radon-Nikodym theorem of lemma 4.7 and the theorem of R. Pallu de la Barrière.

Theorem 10.10: Let ψ_0 be a semi-finite faithful normal trace on $(\mathrm{ReM}_0)^+$. Let ψ be an arbitrary semi-finite normal trace on $(\mathrm{ReM}_0)^+$. There exists $T \in (\mathrm{ReM}_0)^+$ such that $\psi(S) = \psi_0([TS])$ for all $S \in (\mathrm{ReM}_0)^+$. Conversely if $T \in (\mathrm{ReM}_0)^+$, the map $\psi(S) = \psi_0([TS])$, $S \in (\mathrm{ReM}_0)^+$ is a normal semi-finite trace on $(\mathrm{ReM}_0)^+$. The restriction of ψ to ReM defines a positive normal linear functional on M if and only if $\psi_0(T) < +\infty$. Finally $\psi_0([TS]) \leq \psi_0([T'S])$ holds for all $S \in (\mathrm{ReM}_0)^+$ if and only if $T \leq T'$.

Proof: Consider a maximal family of projections in M, $\{E_i\}_{i \in \mathcal{J}}$, pairwise disjoint with the property that $\psi(E_i) < +\infty$, and $\psi_0(E_i) < +\infty$ for each $i \in \mathcal{J}$. Put $E = \vee_i E_i$, $F = I - E$; if $0 \neq P \leq F$, P a projection in M, then $\psi(P) = +\infty$; for there exists $0 \neq P' \leq P$ such that $\psi_0(P') < +\infty$ by the semi-finiteness of ψ_0. The maximality of the family $\{E_i\}$ then implies $\psi(P') = +\infty$, thus $\psi(P) < +\infty$ contradicting the semi-finiteness of ψ. Thus $F = 0$ and $\vee_i E_i = I$. For $S \in (\mathrm{ReM}_0)^+$ put $\psi_0^i(S) = \psi_0(SE_i)$, $\psi^i(S) = \psi(SE_i)$. By lemma 10.4, ψ_0^i, ψ^i define positive normal linear functionals. ψ_0 faithful implies (support ψ_0^i)(M) $= E_i$.

Thus (support ψ^i)(M) ≤ (support ψ_0^i)(M). Hence there exists $T_i \in (\mathrm{ReM}_0)^+$ such that, for all $S \in (\mathrm{ReM})^+$, $\psi^i(S) = \psi(SE_i) = \psi_0^i([ST_i])$. If we set $S_i = SE_i$, then $\psi(S_i) = \psi_0([T_i S_i])$ holds for each $S \in (\mathrm{ReM})^+$; hence by normality for each $S \in (\mathrm{ReM})^+$. Put $T = \prod_{i \in \mathcal{J}} \times T_i$, and observe that $T \in (\mathrm{ReM}_0)^+$. Let $\{\mathfrak{F}\}$ denote the family of finite subsets of the index set $\mathcal{J}$, and let S be any element of $(\mathrm{ReM}_0)^+$. Note that $\prod_{\mathfrak{F}} \times S_i \uparrow_{\mathfrak{F}} S$ and $\prod_{\mathfrak{F}} \times T_i \uparrow_{\mathfrak{F}} T$ and $\prod_{\mathfrak{F}}^{\times} [S_i T_i] \uparrow_{\mathfrak{F}} [ST]$

$$
\begin{aligned}
\psi(S) &= \sup_{\mathfrak{F}} \psi(\textstyle\prod_{\mathfrak{F}} \times S_i) \\
&= \sup_{\mathfrak{F}} (\Sigma_{\mathfrak{F}}\, \psi(S_i)) \\
&= \sup_{\mathfrak{F}} (\Sigma_{\mathfrak{F}}\, \psi_0([S_i T_i]) \\
&= \sup_{\mathfrak{F}} \psi_0\, (\textstyle\prod_{\mathfrak{F}} \times [S_i T_i]) \\
&= \psi_0([ST]) \text{ using the normality of } \psi, \psi_0.
\end{aligned}
$$

Conversely, if $T \in (\mathrm{ReM}_0)^+$, define $\psi(S) = \psi_0([ST])$, $S \in (\mathrm{ReM}_0)^+$. ψ is clearly linear on $(\mathrm{ReM}_0)^+$. By lemma $0 \le S_\tau \uparrow_\tau S$, $S_\tau, S \in (\mathrm{ReM}_0)^+$ implies $0 \le [S_\tau T] \uparrow_\tau [ST]$ so that ψ is normal. To check the semi-finiteness of ψ, choose $\{P_n\}$ n=1, 2, ... projections in M such that $P_n \uparrow_n I$ and $TP_n \in \mathrm{ReM}$. Let $S \in (\mathrm{ReM}_0)^+$ be given. Choose n such that $[SP_n] \neq 0$ and $S' \in \mathrm{ReM}$ such that $0 \neq S' \le [SP_n] \le S$ and satisfying $\psi_0(S') < +\infty$. Observe $S'P_n = S'$. Since $TP_n \in (\mathrm{ReM})^+$ there exists a constant K such that $TP_n \le KI$. We have

$$
\begin{aligned}
\psi(S') = \psi_0([S'T]) &= \psi_0([S'P_n T]) = \psi_0([S'TP_n]) \\
&\le K\, \psi_0(S') < +\infty. \qquad \text{Thus } \psi \text{ is semi-finite.}
\end{aligned}
$$

That ψ defines a positive normal linear functional on ReM if and only if $\psi_0(T) < \infty$ is an immediate consequence of lemma 10.4 and the fact that $\psi_0(T) < \infty$ if and only if $\psi_0(I) < \infty$.

It is clear that, if $T, T' \in (ReM_0)^+$ satisfy $T \leq T'$ then $\psi_0([ST]) \leq \psi_0([ST'])$ holds for each $S \in (ReM_0)^+$. Thus assume that $\psi_0([ST]) \leq \psi_0([ST'])$ holds for each S in $(ReM_0)^+$. Choose a maximal family of pairwise disjoint projections $\{E_i\}_{i\in\mathscr{I}}$ of M with the property that $\psi_0(E_i) < +\infty$, $\psi_0([E_i T']) < +\infty$. In particular $\psi_0([E_i T]) < \infty$ for each $i \in \mathscr{I}$. As usual $\Sigma_{i\in\mathscr{I}} E_i = I$ and for each i there exists (at least one) $x_i \in \mathscr{H}$ such that $E_i x_i = x_i$, $E^{M'}_{x_i} = E_i$ and $\psi_0(SE_i) = \Omega_{x_i, x_i}(S_i)$ holds for each S in $(ReM_0)^+$. From $\psi_0([E_i T']) < +\infty$ it follows that $\Omega_{x_i x_i}([S_i T_i]) \leq \Omega_{x_i x_i}([S_i T_i']) < +\infty$ holds for all $S \in (ReM)^+$. By lemma 10.4 and the definition of Ω_{x_i, x_i}, it follows that $x_i \in \mathscr{D}_{T_i^{\frac{1}{2}}}, \mathscr{D}_{T_i'^{\frac{1}{2}}}$, and for each $S \in (ReM)^+$, we have

$$\text{(i)} \quad (T_i^{\frac{1}{2}} S_i^{\frac{1}{2}} x_i, T_i^{\frac{1}{2}} S_i^{\frac{1}{2}} x_i) = (S_i T_i^{\frac{1}{2}} x_i, T_i^{\frac{1}{2}} x_i) \leq (S_i T_i'^{\frac{1}{2}} x_i, T_i'^{\frac{1}{2}} x_i).$$

Suppose first that T, T′ actually belong to $(ReM)^+$. From (i) follows

$(T_i x, x) \leq (T_i' x, x)$ for all $x \in \{M x_i\}$, and hence by continuity for all $x \in [M x_i]$. Thus $T_i E^M_{x_i} \leq T_i' E^M_{x_i}$. By lemma 10.9, $E_i = \vee_{x_i} \{E^M_{x_i} : \psi_0(ME_i) = \omega_{x_i, x_i}(M)\}$. Thus $T_i E_i \leq T_i' E_i$ or $T_i \leq T_i'$.

In the general case, choose projections $\{P_n\}, \{P_n'\}$ in M such that $P_n \uparrow_n I$. $P_n' \uparrow_n I$, such that $T P_n \in (ReM)^+$, $T' P_n' \in (ReM)^+$. For each i, $T_i P_n \in (ReM)^+$, $T_i' P_n' \in (ReM)^+$, and observe that $Q_n = P_n \wedge P_n' \uparrow_n I$, and $T_i Q_n$, $T_i' Q_n$, $T_i^{\frac{1}{2}} Q_n$, $T_i'^{\frac{1}{2}} Q_n \in (ReM)^+$. In relation (i) replace S by SQ_n. For each $S \in (ReM)^+$

$$(S Q_n T_i^{\frac{1}{2}} x_i, T_i^{\frac{1}{2}} x_i) = (S T_i Q_n x_i, x_i) \leq (S T_i' Q_n x_i, x_i).$$

Thus $T_i Q_n \leq T_i' Q_n$, so that $T_i \leq T_i'$. Hence $T \leq T'$.

Corollary 10.11: Let ψ, φ be two normal semi-finite traces on $(\mathrm{ReM}_0)^+$ which satisfy $\psi(T) \leq \varphi(T)$ for each $T \in (\mathrm{ReM}_0)^+$. There exists $S \in (\mathrm{ReM})^+$ with $0 \leq S \leq 1$ such that $\psi(T) = \varphi([ST])$ for all $T \in (\mathrm{ReM}_0)^+$.

Before proving Corollary 10.11 we give the following slight generalization of [1], p. 11, lemma 2.

Lemma 10.12: Let T_1, $T_2 \in (\mathrm{ReM}_0)^+$ satisfy $T_1 \leq T_2$. There exists $S \in (\mathrm{ReM})^+$ with $0 \leq S \leq 1$ such that $T_1 = T_2 S$.

Proof: From $T_1 \leq T_2$ we have $\mathcal{D}_{T_2^{\frac{1}{2}}} \subseteq \mathcal{D}_{T_1^{\frac{1}{2}}}$. If $x \in \mathcal{D}_{T_2}$ then $\| T_1^{\frac{1}{2}} x \|^2 \leq \| T_2^{\frac{1}{2}} x \|^2$. The map $T_2^{\frac{1}{2}} x \to T_1^{\frac{1}{2}} x$ may be extended uniquely to a continuous linear map B: $[\text{Range } T_2^{\frac{1}{2}}] \to \mathcal{H}$. Set $B = 0$ on $\mathcal{H} \ominus [\text{Range } T_2^{\frac{1}{2}}]$. It follows easily that $B \in M$, $0 \leq B^*B \leq 1$ and that $T_1^{\frac{1}{2}} = [B\, T_2^{\frac{1}{2}}]$. Thus $T_1 = T_2 S$, with $S = B^*B$.

Proof of Corollary 10.11: Let ψ_0 be a normal faithful semi-finite trace on $(\mathrm{ReM}_0)^+$. By theorem 10.10, there exist $T_1, T_2 \in (\mathrm{ReM}_0)^+$ such that $\psi(T) = \psi_0([T\, T_1]) \leq \psi_0([T\, T_2]) = \varphi(T)$ holds for each T in $(\mathrm{ReM}_0)^+$. Thus $T_1 \leq T_2$ and by lemma 10.12 there exists $0 \leq S \leq 1$, $S \in (\mathrm{ReM})^+$ with $T_2 S = T_1$. Thus

$$\varphi([ST]) = \varphi(TS) = \psi_0([T\, S\, T_2]) = \psi_0([T\, T_1]) = \psi(T).$$

XI. THE EXTENDED ORDER DUAL OF ReM_0

In this section we shall show that the family of semi-finite traces introduced in X may be endowed with a Riesz space structure. This leads immediately to a representation of the elements of ReM_0 as normal integrals defined on an order dense ideal of ReM. The notation and terminology is essentially the same as in [8].

By $\mathfrak{J}$ we shall denote the family of all order dense ideals of ReM_0. Then $\mathfrak{J}$ is a filter basis. Let $\Phi = \cup\{I_n^{\sim} : I \in \mathfrak{J}\}$. If $\varphi \in \Phi$ we shall denote by I_φ its domain of definition. Thus $I_\varphi \in \mathfrak{J}$ and $\varphi \in (I_\varphi)_n^{\sim}$ for all $\varphi \in \Phi$.

We may define the following relation on Φ: $\varphi_1 \equiv_{\mathfrak{J}} \varphi_2$ whenever

$$\{T \in ReM_0 : \varphi_1(T) = \varphi_2(T)\}$$

contains an order dense ideal of ReM_0. Since $\mathfrak{J}$ is a filter basis, the relation $\equiv_{\mathfrak{J}}$ is an equivalence relation. The set of classes of equivalent elements will be denoted by $\Gamma(ReM_0)$ and its elements denoted by $[\varphi]$. $\Gamma(ReM)$ is defined similarly.

$\Gamma(ReM_0)$ is given a Riesz space structure as follows. For all real a, and all $[\varphi] \in \Gamma(ReM_0)$, set $a[\varphi] = [a\varphi]$; $[\varphi_1]+[\varphi_2] = [\varphi_3]$ whenever there exist $\varphi_1' \in [\varphi_1]$, $\varphi_2' \in [\varphi_2]$ and $\varphi_3' \in [\varphi_3]$ such that $\{T : \varphi_1'(T) + \varphi_2'(T) = \varphi_3'(T)\}$ contains an order dense ideal of ReM_0. That the linear operations are well defined follow from the fact that $\mathfrak{J}$ is a filter basis.

We set $[\varphi] \geq 0$ whenever there exists $\varphi' \in [\varphi]$ such that $\{T \in ReM_0 : \varphi'(T) \geq 0\} \in \mathfrak{J}$. The set of non-negative elements forms a cone in $\Gamma(ReM_0)$; if we set $[\varphi_1] \leq [\varphi_2]$ whenever $[\varphi_2 - \varphi_1] = [\varphi_1] - [\varphi_2] \geq 0$, the order structure defined on $\Gamma(\mathrm{ReM}_0)$ is compatible with its linear structure. For every $[\varphi]\in\Gamma(\mathrm{ReM}_0)$ we have $[\varphi] \leq [\varphi^+]$, and if $[\psi] \geq 0$ is such that $[\psi] \geq [\varphi]$ then $[\varphi^+] \leq [\psi]$. Thus $[\varphi]^+$ exists and equals $[\varphi^+]$. Hence $\Gamma(\mathrm{ReM}_0)$ is a Riesz space.

For $0 \leq \varphi \in \Phi$, set $\overline{\varphi}(T) = \sup\{\varphi(S): 0 \leq S \leq T,\ S \in I_\varphi\}$ for $T \in (\mathrm{ReM}_0)^+$. Theorem 1.1 of [8] asserts that if $0 \leq \varphi,\ \psi \in \Phi$, then $[\varphi]=[\psi]$ if and only if $\overline{\varphi} = \overline{\psi}$ on $(\mathrm{ReM}_0)^+$. $\overline{\varphi}$ has the following properties

(i) $\overline{\varphi}([T_1+T_2])=\overline{\varphi}(T_1)+\overline{\varphi}(T_2)$, $\overline{\varphi}(aT_1)=a\overline{\varphi}(T_1)$ for each real $a \geq 0$, $T_1, T_2 \in (\mathrm{ReM}_0)^+$.

(ii) $0 \leq T_1 \leq T_2$ in ReM, then $\overline{\varphi}(T_1) \leq \overline{\varphi}(T_2)$.

(iii) $\overline{\varphi}$ is semi-finite in the sense of section X.

(iv) $0 \leq T_\tau \uparrow_\tau T$ in ReM_0, then $\overline{\varphi}(T) = \sup_\tau \overline{\varphi}(T_\tau)$.

Thus $\overline{\varphi}$ is the minimal monotone additive extension of φ to $(\mathrm{ReM}_0)^+$ with values in $[0,\infty]$. It is clear that $\overline{\varphi}$ is a normal semi-finite trace on $(\mathrm{ReM}_0)^+$, and that the restriction of $\overline{\varphi}$ to $(\mathrm{ReM})^+$ is a normal semi-finite trace on ReM in the usual sense ([1] p. 79).

For each $\varphi\in\Phi$, we shall write $\mathfrak{D}_\varphi = \{S\in\mathrm{ReM}_0: |\overline{\varphi}|(|S|) < +\infty\}$. Then $\mathfrak{D}_\varphi \supseteq I_\varphi$, and $\mathfrak{D}_\varphi$ is an order dense ideal, in fact the largest on which such that $|\varphi|$ can be extended finitely.

Let ψ_0 be a faithful normal semi-finite trace on $(\mathrm{ReM}_0)^+$, and let $0 \leq \varphi\in\Phi$, $\overline{\varphi}$ the extension of φ to $(\mathrm{ReM}_0)^+$. By theorem 10.7 there exists $T\in(\mathrm{ReM}_0)^+$ such that $\overline{\varphi}(S) = \psi_0([TS])$ holds for each $S\in(\mathrm{ReM}_0)^+$. Conversely an element $T\in(\mathrm{ReM}_0)^+$ defines an element of $\Gamma(\mathrm{ReM}_0)^+$ as follows: set $I_T = \{S\in\mathrm{ReM}_0: \psi_0([|S|T]) < +\infty\}$. Then $I_T \in \Phi$ and if $S, S_1, S_2 \in I_T$, $S = S_1 - S_2$ with $S_1, S_2 \geq 0$, then $\varphi_T(S)=\psi_0([S_1T])-\psi_0([S_2T])$

defines uniquely φ_T as an element of $(I_T)_n^\sim$. By normality $\overline{\varphi}_T(S) = \psi_0([ST])$ holds for each $S \in (\mathrm{Re}M_0)^+$. By theorem 10.7 $\overline{\varphi}_T = \overline{\varphi}_{T'}$ implies $T = T'$ for $T, T' \in (\mathrm{Re}M_0)^+$. Hence if $T, T' \in (\mathrm{Re}M_0)^+$ then $[\varphi_T] = [\varphi_{T'}]$ if and only if $T = T'$.

We may now define a map $m:(\mathrm{Re}M_0)^+ \to \Gamma(\mathrm{Re}M_0)^+$ by setting $m(T) = [\varphi_T]$ for $T \in (\mathrm{Re}M_0)^+$. The preceding remarks show that m is onto and 1-1. It is obvious that $0 \le T_1 \le T_2$ imply that $m(T_1) \le m(T_2)$. Further m is linear on $(\mathrm{Re}M_0)^+$. In fact let $T_1, T_2 \in (\mathrm{Re}M_0)^+$, then for all $S \in (\mathrm{Re}M_0)^+$,

$$\overline{\varphi}_{T_1+T_2}(S) = \psi_0([S(T_1+T_2)]) = \psi_0([ST_1]) + \psi_0([ST_2])$$
$$= \overline{\varphi}_{T_1}(S) + \overline{\varphi}_{T_2}(S)$$

If $I_{T_1+T_2} = \{S \in \mathrm{Re}M_0 : \overline{\varphi}_{T_1+T_2}(|S|) < +\infty\}$, then $I_{T_1+T_2} \in \Phi$, $I_{T_1+T_2} \subseteq I_{T_1} \cap T_{T_2}$ and for all $S \in I_{T_1+T_2}$

$$\varphi_{T_1+T_2}(S) = \varphi_{T_1}(S) + \varphi_{T_2}(S)$$

Hence $[\varphi_{T_1+T_2}] = [\varphi_{T_1} + \varphi_{T_2}] = [\varphi_{T_1}] + [\varphi_{T_2}]$

so that m is linear. We will show that m may be extended to a Riesz isomorphism of $\mathrm{Re}M_0$ onto $\Gamma(\mathrm{Re}M_0)$.

<u>Theorem 11.1:</u> <u>The Riesz spaces $\mathrm{Re}M_0$, $\Gamma(\mathrm{Re}M_0)$, $\Gamma(\mathrm{Re}M)$ are isomorphic.</u>

<u>Proof:</u> Let $m:(\mathrm{Re}M_0)^+ \to \Gamma(\mathrm{Re}M_0)^+$ be defined as above. For $T = [T_1 - T_2]$ $T_1, T_2 \in (\mathrm{Re}M_0)^+$, set $m(T) = m(T_1) - m(T_2)$. That $m:\mathrm{Re}M_0 \to \Gamma(\mathrm{Re}M_0)$ is well defined, linear and 1-1 follows immediately from the linearity and 1-1-ness of m on $(\mathrm{Re}M_0)^+$. Let $[\varphi] \in \Gamma(\mathrm{Re}M_0)$, $[\varphi] = [\varphi_1] - [\varphi_2]$ with $[\varphi_1] \ge 0$, $[\varphi_2] \ge 0$. There exist $T_1, T_2 \in (\mathrm{Re}M_0)^+$ with $m(T_1) = [\varphi_1]$,

$m(T_2) = [\varphi_2]$ so that $m([T_1 - T_2]) = m(T_1) - m(T_2) = [\varphi_1] - [\varphi_2] = [\varphi]$. Thus m is onto.

To show that m is a Riesz isomorphism of ReM_0 onto $\Gamma(ReM_0)$ it is sufficient to show that $m(T^+) = m(T)^+$ or alternatively $[\varphi_{T^+}] = [\varphi_T^+]$. If $S \geq 0$, $S \in \mathfrak{D}_{\varphi_T}$, then

$$\varphi_T^+(S) = \sup\{\varphi_T(S'): 0 \leq S' \leq S\}$$

In particular $\varphi_T^+(S) \leq \varphi_{T^+}(S)$.

For each $A \in ReM_0^+$, define $\psi_{T^+}(A) = \lim_{n\to\infty} \varphi_S(A \wedge n\, T^+)$.

It follows immediately that $\mathfrak{D}_{\psi_{T^+}} \supseteq \mathfrak{D}_{\varphi_S}$, that $\psi_{T^+} \leq \varphi_S$ on $\mathfrak{D}_{\varphi_S}$ and that $0 \leq \overline{\psi}_{T^+} \leq \overline{\varphi}_S$. Since the restrictions of $\overline{\psi}_{T^+}$, $\overline{\varphi}_S$ to ReM are normal semi-finite traces on ReM, by Corollary 10.11 there exists $0 \leq S_{T^+} \leq 1$, $S_{T^+} \in ReM$ such that $\overline{\psi}_{T^+} = \overline{\varphi}_{SS_{T^+}}$. Note that $0 \leq S S_{T^+} \leq S$ and that $\psi_{T^+}(T) = \varphi_{SS_{T^+}}(T) = \varphi_T(SS_{T^+}) = \varphi_S(T^+) = \varphi_{T^+}(S)$. Hence $\varphi_T^+(S) = \varphi_{T^+}(S)$ and so $[\varphi_T^+] = [\varphi_{T^+}]$ and m is a Riesz isomorphism. That $\Gamma(ReM)$, $\Gamma(ReM_0)$ are isomorphic Riesz spaces follows immediately from the fact that ReM is an order dense ideal in ReM_0 and [8], Theorem 2.6.

A Riesz space L is said to be perfect in the extended sense if it satisfies $L = \Gamma(\Gamma(L))$. From [8],p.491 , if L is any Archimedean Riesz space, then $\Gamma(L)$ is perfect in the extended sense. Combining this remark with Theorem 11.1, we have as a generalization of Theorem 5.2:

Theorem 11.2: The Riesz space ReM_0 is perfect in the extended sense.

XII. THE SQUARE ROOT OF AN ARBITRARY POSITIVE SELF-ADJOINT TRANSFORMATION

In preceding sections, it has been shown that an Abelian W*-algebra M may be extended to a class M_0 of closed densely defined linear transformations which commute with every unitary operator in M'. The following question arises naturally. If T is a given self-adjoint transformation on the Hilbert space $\mathcal{H}$, does there exist an Abelian W*-algebra M such that $T \in M_0$? If T is bounded, then $\{I, T\}''$ trivially satisfies the requirements.

Let T be a self-adjoint transformation on the Hilbert space $\mathcal{H}$. Note in particular that T is densely defined, linear and closed.

<u>Lemma 12.1</u>: <u>Let $M_1 = \{S \in \mathcal{L}(\mathcal{H}): ST \subseteq TS\}$. M_1 is a W*-algebra.</u>

<u>Proof</u>: It is clear that M_1 is a linear space. Suppose that $S_1, S_2 \in M_1$, then $S_1 S_2 T \subseteq S_1 TS_2 \subseteq TS_1 S_2$ so that M_1 is an algebra. If $S \in M_1$, then $S^*T \subseteq (TS)^* = TS^*$. Thus M_1 is a *- subalgebra of $\mathcal{L}(\mathcal{H})$. It is now sufficient to show that $M_1'' = M_1$. To this end observe that the bounded self-adjoint operators $(I+T^2)^{-1}$, $T(I+T^2)^{-1}$ belong to $M_1 \cap M_1'$. In fact, from

$$T(I+T^2)^{-1} = (T(I+T^2)^{-1})^* \supseteq (I+T^2)^{-1} T$$

it follows that both $T(I+T^2)^{-1}$, $(I+T^2)^{-1}$ belong to M_1. If further, $S \in M_1$, from $ST \subseteq TS$ it follows that $ST^2 \subseteq T^2S$ and

$$S(T^2+I) = ST^2+S \subseteq T^2S+S = (T^2+I)S$$

implies $(I+T^2)^{-1}S = S(I+T^2)^{-1}$

and $\quad T(I+T^2)^{-1}S = TS(I+T^2)^{-1} \supseteq ST(I+T^2)^{-1}$

so that equality holds since $T(I+T^2)^{-1}S$, $ST(I+T^2)^{-1}$ are bounded.

Now suppose that $S \in M_1''$. In particular

$$S(I+T^2)^{-1} = (I+T^2)^{-1} S$$

and
$$ST(I+T^2)^{-1} = T(I+T^2)^{-1} S = TS(I+T^2)^{-1}$$

Hence, if $x \in \mathcal{D}_{T^2} = \mathcal{R}_{(I+T^2)^{-1}}$, then $Sx \in \mathcal{D}_T$ and $TSx = S\,Tx$. Let $x \in \mathcal{D}_T$. Since the graph of the restriction of T to $\mathcal{D}_{T^2}$ is dense in the graph of T (in $\mathcal{H} \times \mathcal{H}$), there exist $x_n \in \mathcal{D}_{T^2}$ such that $x_n \to x$, $Tx_n \to Tx$. Thus $Sx_n \in \mathcal{D}_T$, $Sx_n \to Sx$ and $TSx_n = S\,Tx_n \to ST_x$. Thus $Sx \in \mathcal{D}_T$ and $T(Sx) = STx$ since T is closed. Thus $ST \subseteq TS$ and $S \in M_1$. Hence $M_1 = M_1''$ and the lemma is proved.

<u>Lemma 12. 2</u>: <u>M_1' is an Abelian W*-algebra.</u>

<u>Proof</u>: We show that $M_1' \subseteq M_1 = (M_1')'$. It is sufficient to note that $(I+T^2)^{-1}$, $T(I+T^2)^{-1}$ belong to M_1. If $S \in M_1'$ $ST \subseteq$ follows exactly as in lemma 12. 1.

<u>Theorem 12. 3: Let T be a self-adjoint linear transformation on a Hilbert space $\mathcal{H}$. There exists a unique, positive self-adjoint linear transformation S such that $S^2 = T$.</u>

<u>Proof</u>: By lemmas 12. 1, 12. 2, there exists an Abelian W*-algebra M such that T commutes with every unitary operator in M'. The statement of the theorem now follows from Theorem 6. 10.

BIBLIOGRAPHY

[1] Dixmier, J., Les Algèbres d'Opérateurs dans L'Espace Hilbertien (Gauthier-Villars, Paris, 1957).

[2] Dixmier, J., Sur certains espaces considérés par M. H. Stone, Summa Brasil. Math., t. 2, fasc.II, (1951), p. 151-182.

[3] Dye, H. A., The Radon-Nikodym theorem for finite rings of operators, Trans. Amer. Math. Soc. 72 (1952), p. 243-280.

[4] Fukamiya, M., Misonou, M., and Z. Takeda, On order and commutativity in B*-algebras. Tôhoku Math. M., t. 6, (1954), p. 89-93.

[5] Gillman, L. and M. Jerrison, Rings of Continuous Functions, (van Nostrand, 1960).

[6] Kadison, R. V., Order properties of bounded self-adjoint operators, Proc. Amer. Math. Soc., vol. 2, (1951), p. 505-510.

[7] Luxemburg, W. A. J., Is every integral normal?, Bull. Amer. Math. Soc., vol. 73, (1967), p. 685-688.

[8] Luxemburg, W. A. J., and J. J. Masterson, An extension of the concept of the order dual of a Riesz Space, Can. J. Math., vol. 19, (1967), p. 488-498.

[9] Luxemburg, W. A. J., and A. C. Zaanen, Riesz Spaces, Part I, (Preprint).

[10] Luxemburg, W. A. J., and A. C. Zaanen, Notes on Banach Function Spaces, Proc. Acad. Sci., Amsterdam Note VI, A66, p. 655-668 (1963); Note VII, A66, p. 669-681 (1963); Note VIII, A67, p. 104-119 (1964); Note X, A67, p. 493-506 (1964); Note XI, p. 507-518 (1964); Note XII, A67, p. 519-529 (1964); Note XIII, A67, p. 530-543 (1964).

[11] Luxemburg, W. A. J., Notes on Banach Function Spaces, Proc. Acad. Sci., Amsterdam; Note XIVA, A68, p. 230-239 (1965); Note XIVB, A68, p. 240-248 (1965); Note XVA, A68, p. 416-429; Note XVB, A68, p. 430-446, (1965); Note XVIA, A68, p. 646-657 (1965); Note XVIB, A68, p. 659-667 (1965).

[12] Nagy, B. v. Sz., Spektraldarstellung linearer Transformationen des Hilbertschen Raumes, Berlin, 1942.

[13] Murray, F. and J. von Neumann, On rings of operators. Ann. of Math. vol. 37 (1936), p. 116-229.

[14] Ogasawara, T., A theorem on operator algebras, J. Sc. Hiroshima Univ. t. 18, (1955), p. 307-309.

[15] Pallu de la Barrière, R., Sur les algèbres d' opérateurs dans les espaces hilbertiens, Bull. Soc. Math. Fr., t. 82, (1954), p. 1-51.

[16] Sakai, S., A Radon-Nikodym theorem in W*-algebras, Bull. Amer. Math. Soc., vol. 71, (1965), p. 149-151.

[17] Sakai, S., The Theory of W*-algebras, Yale University, 1962.

[18] Sherman, S., Order in operator algebras, Amer. J. Math., t. 73, (1951), p. 227-232.