STRUCTURE THEOREMS FOR NONCOMMUTATIVE COMPLETE LOCAL RINGS

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ABSTRACT

If R is a ring with identity, let N(R) denote the Jacobson radical of R. R is local if R/N(R) is an artinian simple ring and $\cap N(R)^i = 0$. It is known that if R is complete in the N(R)-adic topology then R is equal to $(B)_n$, the full n by n matrix ring over B where B/N(B) is a division ring. The main results of the thesis deal with the structure of such rings B. In fact we have the following.

If B is a complete local algebra over F where B/N(B) is a finite dimensional normal extension of F and N(B) is finitely generated as a left ideal by k elements, then there exist automorphisms $g_1, \ldots, g_k$ of B/N(B) over F such that B is a homomorphic image of $B/N[[x_1, \ldots, x_k; g_1, \ldots, g_k]]$ the power series ring over B/N(B) in noncommuting indeterminates x_i, where $x_i b = g_i(b) x_i$ for all $b \in B/N$.

Another theorem generalizes this result to complete local rings which have suitable commutative subrings. As a corollary of this we have the following. Let B be a complete local ring with B/N(B) a finite field. If N(B) is finitely generated as a left ideal by k elements then there exist automorphisms $g_1, \ldots, g_k$ of a v-ring V such that B is a homomorphic image of

$V[[x_1,\ldots,x_k;g_1,\ldots,g_k]]$.

In both these results it is essential to know the structure of N(B) as a two sided module over a suitable subring of B.

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SECTION I

INTRODUCTION

The structure theory of commutative semilocal and local rings has been developed extensively but there has been little success as yet in extending these results to noncommutative rings. In this paper we prove some structure theorems for noncommutative complete semilocal and local rings. In Section III we obtain some information on the structure of complete semilocal rings whose radical is finitely generated as a left ideal. In commutative theory it is known that such rings are direct sums of complete local rings. As the ring of triangular n by n matrices over a field shows, this result is not true for noncommutative complete semilocal rings but we do obtain the following. (Throughout we shall let N(S) denote the Jacobson radical of the ring S.)

Theorem 2. If R is a complete semilocal ring with finitely generated radical, then $R = R_1 \oplus \ldots \oplus R_k$ where R_1 is an algebra over the rationals and for $i > 1$, R_i is a ring with identity e_i such that for distinct primes p_i, $p_i e_i \in N(R_i)$.

It is well known that the completion of a commutative noetherian local ring is again a noetherian local ring. This problem has not been settled for noncommutative local rings. Goldie [6] shows that this would be a strong condition. In any left noetherian ring of course the radical is finitely generated as a left ideal. In the case of commutative complete local rings, finite generation is sufficient to prove the structure theorems and to imply that the completion is noetherian. In the noncommutative case finite generation of the radical is not sufficient to prove that the completion is noetherian as the power series ring F[[x,y]] in noncommuting indeterminates x and y over F shows. However, in the noncommutative structure theory, finite generation of the radical seems to play an important role. In Section III we prove

<u>Theorem 1</u>. The completion of a semilocal ring with finitely generated radical is again a semilocal ring with finitely generated radical.

I.S. Cohen [3] proved that any commutative complete local ring with finitely generated radical is a homomorphic image of a power series ring with a suitable coefficient ring. Some effort has been made toward extending these results to noncommutative complete local rings. Batho [2] for example has shown that if R is a complete noetherian local algebra over a field F such that R/N(R) is finite dimensional and separable over F then R is a homomorphic image of a quasi-cyclic algebra S. This theorem is extended in Section IV to a complete semilocal algebra R over a field F such that R/N(R) is finite dimensional and separable over F. This theorem is the basis of Section IV and the structure theorems for complete local algebras. It is known that if R is a complete local ring then R is the ring of all n by n matrices over a complete local ring B where B/N(B) is a division ring. In Section IV we obtain the following information about the structure of B.

Theorem 7. Let B be a complete local algebra over F where B/N(B) is a finite dimensional and normal division ring over F. If N(B) is finitely generated by t elements then there exists automorphisms $g_1,\ldots,g_t$ of B/N(B) fixing F such that B is a homomorphic image of $B/N(B)[[x_1,\ldots,x_t; g_1,\ldots,g_t]]$, the power series ring over B/N(B) in noncommuting indeterminates x_i where $x_i b = g_i(b)x_i$ for all $b \in B/N(B)$.

Note that in this theorem we do not say that the automorphisms are distinct. We also obtain a similar theorem for complete local algebras B where B/N(B) is a finite dimensional separable extension of F.

In Section V we extend the results of Section IV to complete local rings which have suitable commutative subrings. Suppose B is a complete local ring with finitely generated radical and B/N(B) is a field F. Also suppose that the characteristic of B/N(B) is a prime p and the characteristic of B is not equal to p. The major results are these:

Theorem 15. Let S be a commutative subring of B which maps onto F under the natural map ϕ which takes B onto B/N(B). Let S be a finite module over a subring S' of the center of B, S' be noetherian and F be a normal extension of $\phi(S')$. If N(B) is finitely generated by t elements, then there exists automorphisms $g_1,\ldots,g_t$ of a v-ring V such that B is a homomorphic image of V $V[[x_1,\ldots,x_t;g_1,\ldots,g_t]]$ where V/N(V) is isomorphic to F.

Theorem 16. If F = B/N(B) is a finite field and N(B) is finitely generated by t elements, then there exists automorphisms $g_1,\ldots,g_t$ of a v-ring V such that B is a homomorphic image of $V[[x_1,\ldots,x_t;g_1,\ldots,g_t]]$ where V/N(V) is isomorphic to F.

SECTION II

DEFINITIONS AND NOTATION

This section introduces some of the terminology and definitions used in the structure theory of complete non-commutative local rings. All rings will be assumed to have an identity.

A ring is <u>artinian</u> if the descending chain condition for left ideals of the ring is satisfied. A ring is <u>noetherian</u> if the ascending chain condition for left ideals of the ring is satisfied. The Jacobson radical of the ring R will be designated by N(R) or by N whenever the ring R is clear from context. The set of all n by n matrices over the ring B will be denoted by $(B)_n$.

A <u>semilocal</u> ring R is a ring such that $\cap N^i = 0$ and R/N is artinian. A <u>local</u> ring is a semilocal ring such that R/N is simple.

A topological ring is a ring which is a topological space such that addition and multiplication are continuous. If I is an ideal of the ring R such that $\cap I^n = 0$ then R is a topological ring using $\{I^n\}$ as a base of neighborhoods of $\{0\}$. The topology is called the <u>I-adic topology</u>. A <u>complete</u> ring is a ring which is complete with respect to its N-adic topology. We will designate the completion of R in the N-adic topology by $\hat{R}$.

A <u>v-ring</u> as introduced by Cohen [3] is a complete discrete commutative valuation ring whose maximal ideal is generated by the prime integer p where p = characteristic of $V/N(V)$. That is, a v-ring is a complete commutative local integral domain whose maximal ideal is generated by the prime integer p.

If V is any commutative ring , $g_1, \ldots, g_k$ automorphisms of V then the Hilbert power series ring $V[[x_1, \ldots, x_k; g_1, \ldots, g_k]]$ is defined as the power series ring over V in non-commuting indeterminates $x_1, \ldots, x_k$ such that $x_i v = g_i(v) x_i$ for all $v \in V$.

An algebra R is quasicyclic if 1) R contains a subalgebra A which is mapped isomorphically onto R/N by the natural map, and 2) as a two sided A module R is equal to the complete direct sum $A \oplus N_1 \oplus N_1^2 \oplus \ldots$ where $N_1 \cong N/N^2$.

Let R be semisimple artinian algebra over F. Let the Wedderburn decomposition be $(D_1)_{n(1)} \oplus \ldots \oplus (D_k)_{n(k)}$. Let C_i be the centers of the division rings D_i. Such an algebra R is separable over F if each of the fields C_i is a separable extension of F. A separable algebra is normal over F if each C_i is a normal extension of F and each automorphism of C_i over F is induced by an automorphism of D_i over F.

Finally we next introduce the notion used by Hochschild [8] of a U-regular (V,V) module where V is a ring and U is a subring of V. A U-regular (V,V) module M is a two sided V module with $U \subset \{v \in V : vm = mv \text{ for all } m \in M\}$.

SECTION III

THE COMPLETION OF A SEMILOCAL RING WITH FINITELY GENERATED RADICAL

This section gives us some general information on the completion of a semilocal ring whose radical is finitely generated as a left ideal. We relate semilocal rings with finitely generated radical and projective limits of artinian rings in order to obtain further information.

The following lemma is easily derived from theorems 1.3 and 2.3 of [1].

<u>Lemma 1</u>. If R is a ring with identity such that $\cap N^i = 0$, then $\cap N(\hat{R})^i = 0$, while for $j > 0$, $N(\hat{R})^j \cap R = N(R)^j$ and $R/N(R)^j \cong \hat{R}/N(\hat{R})^j$.

We will now investigate some properties of finite generation of the radical

<u>Lemma 2</u>. If R is a complete semilocal ring with R/N^2 artinian then for each positive integer j, N^j is a finitely generated left ideal.

<u>Proof</u>. We will first show that N is a finitely generated left ideal and use this to obtain the lemma. Let $n \in N$. Since N/N^2 is a finite left R/N module we have $n = \sum_{i=1}^{k} r_i n_i$ mod N^2 where $\{\bar{n}_i\}_{i=1}^{k}$ is a minimal generating set for the

R/N module N/N^2. $n-\sum_{i=1}^{k} r_i n_i = m \in N^2$ but $m+N^3 = \sum_{\text{finite}} ts+N^3$ where $t,s \in N \sim N^2$. Since $t,s \in N \sim N^2$ there exists $\{t_i\}_{i=1}^{k}$ and $\{s_i\}_{i=1}^{k}$ such that $t-\sum_{i=1}^{k} t_i n_i \in N^2$, $s-\sum_{i=1}^{k} s_i n_i \in N^2$ so that $ts-(\sum t_i n_i)s+(\sum t_i n_i)s-(\sum t_i n_i)(\sum s_i n_i) \in N^3$. Hence

$$m+N^3 = \sum_{\text{finite}}(\sum_{i=1}^{k} t_i n_i)(\sum_{i=1}^{k} s_i n_i)+N^3 = \sum_{i(1),i(2)=1}^{k} r^{(2)}_{i(1)i(2)}\ n_{i(1)}n_{i(2)}+N^3$$

because $n_j s_i \in N \sim N^2$ and so is written $\sum_{u=1}^{k} p_u n_u \bmod N^2$. Continuing this process we have $n-\sum_{i=1}^{k} r_i n_i -$

$$\sum_{i(1),i(2)=1}^{k} r^{(2)}_{i(1)i(2)}\ n_{i(1)}n_{i(2)} - \cdots - \sum_{i(1),\ldots,i(u)=1}^{k} r^{(u)}_{i(1)\ldots i(u)}\ n_{i(1)}\cdots n_{i(u)}$$

$\in N^{u+1}$. Since R is complete we have

$$n = \sum_{j=1}^{\infty}\ \sum_{i(1),\ldots,i(j)=1}^{k} r^{(j)}_{i(1)\ldots i(j)} n_{i(1)}\cdots n_{i(j)} \quad \text{where } r^{(1)}_i = r_i.$$

Now we may rearrange the series and collect terms ending in n_q. Thus we have $n = u_1 n_1+\ldots+u_k n_k$ where for example

$$u_1 = r^{(1)}_1 + \sum_{j=2}^{\infty}\ \sum_{i(1),\ldots,i(j-1)=1}^{k} r^{(j)}_{i(1)\ldots i(j-1),1}\ n_{i(1)}\cdots n_{i(j-1)}.$$

Therefore $N = [n_1,\ldots,n_k]$. We will now show that the set of all products $n_{i(1)}\cdots n_{i(j)}$ generate N^j. The statement is true for $j = 1$ so by induction it is sufficient to show that the statement is true for $q+1$ if it is true for q. Let $n \in N^{q+1}$. Therefore $n = \sum_{u=1}^{t} s^{(u)}_1\ldots s^{(u)}_{q+1}$ where $s^{(u)}_i \in N$. Since $s^{(u)}_2\ldots s^{(u)}_{q+1} \in N^q$, we have

$$s^{(u)}_2\ldots s^{(u)}_{q+1} = \sum_{i(2),\ldots,i(q+1)=1}^{k} \hat{r}^{(u)}_{i(2)\ldots i(q+1)}\ n_{i(2)}\cdots n_{i(q+1)} \quad \text{and}$$

$$n = \sum_{u=1}^{t} s^{(u)}_1 \sum_{i(2),\ldots,i(q+1)=1}^{k} \hat{r}^{(u)}_{i(2)\ldots i(q+1)}\ n_{i(2)}\cdots n_{i(q+1)}$$

$$= \sum_{i(2),\dots,i(q+1)=1}^{k} \left(\sum_{u=1}^{t} s_1^{(u)} \hat{r}_{i(2)\dots i(q+1)}^{(u)} \right) n_{i(2)} \cdots n_{i(q+1)} \ .$$ But

$\sum_{u=1}^{t} s_1^{(u)} \hat{r}_{i(2)\dots i(q+1)}^{(u)} \in N$ and hence is equal to

$\sum_{i(1)=1}^{k} r_{i(1)\dots i(q+1)} \ n_{i(1)}$. Therefore

$$n = \sum_{i(1),\dots,i(q+1)}^{k} r_{i(1)\dots i(q+1)} \ n_{i(1)} \cdots n_{i(q+1)}$$

and the lemma is proven.

Note that in proving this lemma we showed that any minimal generating set of N/N^2 as an R/N module can be raised to a minimal generating set of N as an ideal and vice versa . Hence from the properties of finitely generated modules over semisimple artinian rings we know that every minimal generating set of N has the same length.

Theorem 1. The completion of a semilocal ring with finitely generated radical is again a semilocal ring with finitely generated radical.

Proof. Since N is a finitely generated left ideal it follows that N/N^2 is a finitely generated left R/N module. Since R/N is artinian by assumption it follows that R/N^2 is artinian. By lemma 1, $\hat{R}/N(\hat{R})^2$ is isomorphic to $R/N(R)^2$ and is therefore artinian. Lemma 2 then proves the corollary.

Proposition 1. If R is a complete semilocal ring with R/N^2 artinian then for each positive integer i, R/N^i is artinian.

Proof. By lemma 2 for any $k > 0$, N^k is a finitely generated ideal. Hence N^k/N^{k+1} is a finitely generated left R/N module. Thus a finite composition series can be obtained from the series $R/N, N/N^2, \ldots, N^{i-1}/N^i$. This implies that R/N^i is artinian.

Some information about the structure of complete semilocal rings can now be gained from our knowledge of artinian rings by using the projective limit. Suppose that $\{R_i\}$, $(i \geq 1)$ is a sequence of rings and $\{\phi_i\}$, $(i \geq 1)$ is a sequence of homomorphisms such that $\phi_{i-1}(R_i) = R_{i-1}$. The projective limit of $\{R_i\}$ is defined to be the subring of the complete direct sum of $\{R_i\}$, consisting of those elements $\{r_i\}$ which satisfy $\phi_{i-1}(r_i) = r_{i-1}$. The projective limit of $\{R_i\}$ will be denoted by $PL(R_i)$. The following is a well known and easily proven fact.

Proposition 2. If R is any ring complete in the N-adic topology where $\cap N^i = 0$ then R is the projective limit of $\{R/N^i\}$ under the natural map $\phi_{i-1} : R/N^i \rightarrow R/N^{i-1}$.

Proof. The map $r \in R \rightarrow (r+N, r+N^2, \ldots)$ is a homomorphism from R into $PL(R/N^i)$. It is 1-1 since $r \rightarrow 0$ implies

$r \in N^i$ for each i which in turn means $r \in \cap N^i = 0$. Suppose $(r_1+N, r_2+N^2, \ldots) \in PL(R/N^i)$. Letting $r_0 = 0$ we have $r_k = \sum_{i=1}^{k}(r_i - r_{i-1})$ is a Cauchy sequence so that $r = \sum_{i=1}^{\infty}(r_i - r_{i-1})$ exists. But $r - r_u = \sum_{j=u+1}^{\infty}(r_j - r_{j-1}) \in N^u$ so that $(r+N, r+N^2, \ldots) = (r_1+N, r_2+N^2, \ldots)$.

<u>Theorem 2</u>. Let R be a complete semilocal ring with finitely generated radical. Then $R = R_1 \oplus \ldots \oplus R_k$ where R is an algebra over the rationals and for $i > 1$, R_i is a ring with identity e_i such that for distinct primes p_i, $p_i e_i \in N(R_i)$.

<u>Proof</u>. We will make use of the fact that $R = PL(R/N^j)$. R/N is artinian and hence is equal to $R_{11} \oplus \ldots \oplus R_{k1}$ where R_{11} is an algebra over the rationals and for $i > 1$, R_{i1} has characteristic p_i for distinct prime integers p_i. Since $1 \in R$ and R/N^j is artinian (Proposition 1) there exist by [5, page 283] rings R_{ij}, $1 \le i \le k$, $1 \le j$ such that $R/N^j = R_{1j} \oplus \ldots \oplus R_{kj}$ where R_{1j} is an algebra over the rationals and for $i > 1$, R_{ij} has characteristic a power of p_i. Because of the distinctness of the characteristics of the components of the decomposition of R/N^j, it is easy to see that this decomposition is unique. From this fact we can deduce that the natural map $\phi_j : R/N^{j+1} \longrightarrow R/N^j$ induces a map $\phi_{ij} : R_{i,j+1} \longrightarrow R_{ij}$. From the decomposition of R/N^{j+1} we have $N/N^{j+1} = N(R_{1j+1}) \oplus \ldots \oplus N(R_{kj+1})$ and $N^j/N^{j+1} = N(R_{1j+1})^j \oplus \ldots \oplus N(R_{kj+1})^j$. Thus ϕ_{ij} is actually the natural

map of R_{ij+1} onto $R_{ij+1}/N(R_{ij+1})^j$. From this it is evident that the diagram

$$\begin{array}{ccc} R/N^j & \xrightarrow{\phi_j} & R/N^{j+1} \\ \downarrow u_{ij} & & \downarrow u_{ij+1} \\ R_{ij} & \xrightarrow{\phi_{ij}} & R_{ij+1} \end{array}$$

commutes where u_{ij} is the projection map of R/N^j onto R_{ij}. We now claim that $R = PL(R/N^j) = PL(R_{1j})\oplus\ldots\oplus PL(R_{kj})$. The commutivity of the diagram guarantees that $u : \{r_i+N^i\} \rightarrow \{r_{1i},\ldots,r_{ki}\}$ is a well defined map. It is easily checked that u is an isomorphism into. Suppose that $\{r_{1j}\} \in PL(R_{1j})$. To show that u is onto we must have a sequence $\{r_i+N^i\} \in PL(R/N^i)$ which maps onto $\{r_{1j},0,\ldots,0\}$ under u. The sequence $\{r_{1j}+N^j\}$ of course does this.
We now need to show that $R_1 = PL(R_{1j})$ contains a copy of the rationals and that $N(R_i) = N(PL(R_{ij}))$ contains p_ie_i. To show the first note that if R_{11} is zero then R_{1j} is zero for every j and of course $PL(R_{1j})$ is zero. Thus R_1 is trivially an algebra over the rationals. Otherwise R_{1j} contains a unique copy of the rationals generated by e_{1j} the identity of R_{1j}. ϕ_{1j-1} maps this copy onto the unique copy in R_{1j-1} so that $PL(R_{1j})$ is an algebra over the rationals. The assertion that $N(R_i) = N(PL(R_{ij})$ contains p_ie_i is clear since p_ie_i is contained in $N(R_{ij})$ and ϕ_{ij-1} maps p_ie_{ij} onto p_ie_{ij-1}. The theorem is thus proven.

Corollary 1. For $i > 1$, either R_i has characteristic a power of a prime or R_i contains a copy of the valuation ring $I_{p(i)}$ determined by the prime p_i in the integers.

Proof. $R_i = PL(R_i/N(R_i)^j)$ and each $R_i/N(R_i)^j$ contains $Ie_i/N(R_i)^j$ where I is the integers and e_i the identity of R_i. If there is some integer n such that $p_i e_i \in N(R_i)^j$ for each j then $p_i^n e_i = 0$ and R has characteristic dividing p_i^n. If this is not the case then the localization and completion of the subring $PL(Ie_i/N(R_i)^j)$ of R_i about the ideal generated by $p_i e_i$ is again a subring of R_i and is in fact a copy of $I_{p(i)}$.

Corollary 2. Let $R_i/N(R_i) = (D_1)_{n(1)} \oplus \ldots \oplus (D_t)_{n(t)}$ be the Wedderburn decomposition of the semisimple ring $R_i/N(R_i)$ where $e_j \longrightarrow \bar{e}_j$ is the map of orthogonal idempotents of R_i onto orthogonal idempotents of $R_i/N(R_i)$. Let $M_{uj} = e_u R_i e_j$. If we make the convention that a characteristic which is equal to zero is written as equal to infinity then the characteristic of M_{uj} divides min $\{$characteristic of e_u, characteristic of $e_j\}$.

Proof. M_{uj} is an $(e_u R_i e_u, e_j R_i e_j)$ module. Hence the characteristic of M_{uj} divides the characteristic of e_u and the characteristic of e_j.

SECTION IV

COMPLETE LOCAL AND SEMILOCAL ALGEBRAS

This section deals with the structure of certain complete semilocal and local algebras with finitely generated radical. The techniques used are an extension of some ideas of Hochschild [7,9]. We prove that if R is a complete local algebra over F with finitely generated radical such that R/N is finite dimensional and normal over F then R is a complete matrix ring over the homomorphic image of a Hilbert power series ring. A new type of complete local ring is then defined and we investigate the structure of complete local and separable algebras in relation to this ring.

The following theorem was proven by Curtis [4, page 80].

<u>Theorem 3</u>. Let R be an algebra over a field F where $\cap N^i = 0$ and R is complete in the N-adic topology. If R/N is finite dimensional and separable over F then R contains a subalgebra A such that $R = A + N$ and $A \cap N = 0$.

The next lemma and theorem are found in Hochschild [7, pages 371 and 372].

<u>Lemma 3</u>. If A is a separable algebra over F then every two sided A module M is semisimple, in the sense that every two sided submodule of M has a complement.

Theorem 4. Let R be an algebra over F with R/N finite dimensional and separable over F. If $N^k = 0$ for some k then R is a homomorphic image of $(R/N)\oplus N_1 \oplus \ldots \oplus N_1^{k-1}$ where $N_1 \cong N/N^2$.

We now extend these results to complete semilocal algebras. Batho [2] proved the following theorem but we include a proof here using different terminology and concepts.

Theorem 5. Let R be a complete semilocal algebra over F. If R/N is finite dimensional and separable over F then R is a homomorphic image of a quasicyclic algebra $R/N \oplus N_1 \oplus N_1^2 \oplus \ldots$.

Proof. Proposition 2 gives $R = PL(R/N^i)$. By theorem 3 there is a subalgebra A such that $R = A \oplus N$ as two sided A modules. Since A is a separable algebra lemma 3 guarantees that N^2 has a complement N_1 in N. That is $N = N_1 \oplus N^2$ as two sided A modules. Using this N_1 and A, theorem 4 then implies that R/N^i is a homomorphic image of $C_i = \bar{A} \oplus \bar{N}_1 \oplus \bar{N}_1^2 \oplus \ldots \oplus \bar{N}_1^{i-1}$ where $\bar{A}$ and $\bar{N}_1$ are the appropriate images of A and N_1 in R/N^i. Let σ_i be the homomorphism of C_i onto R/N^i. Let ϕ_i be the natural map from R/N^{i+1} onto R/N^i and ψ_i the natural map from C_{i+1} to C_i. Note that the diagram

$$\begin{array}{ccc} C_{i+1} & \xrightarrow{\psi_i} & C_i \\ \Big\downarrow \sigma_{i+1} & & \Big\downarrow \sigma_i \\ R/N^{i+1} & \xrightarrow{\phi_i} & R/N^i \end{array} \qquad \text{commutes.}$$

Let $(c_1, c_2, \ldots) \in PL(C_i)$. Define $\gamma(c_1, c_2, \ldots) = (\sigma_1(c_1), \sigma_2(c_2), \ldots)$. The commutivity of the diagram implies that this is a map of $PL(C_i)$ onto $PL(R/N^i) = R$. It is clear that γ is a homomorphism. Note that $PL(C_i) = A \oplus N_1 \oplus N_1^2 \oplus \ldots$ as a two sided A module complete direct sum.

In order to gain additional information about complete local algebras, we need to know the structure of N/N^2 as a two sided R/N module. The following theorem of Hochschild [8, page 451] gives us this structure for certain rings R/N. We restate it for our special case.

<u>Theorem 6</u>. Let D be a finite dimensional normal division ring over the field F. Then every K-regular (D,D) space is a sum of simple K-regular spaces. Every simple K-regular (D,D) space has the form Dn where $nd = g(d)n$, $d \in D$, and g is a fixed automorphism of D over F.

If R is a complete local ring then $R = (B)_n$ where B is a complete local ring with B/N(B) a division ring. Hence the structure of R is determined by the structure of B.

<u>Theorem 7</u>. Let B be a complete local algebra over F where B/N is a finite dimensional and normal division ring over F. If N is finitely generated then there exist automorphisms $g_1,\ldots,g_k$ of B/N such that B is a homomorphic image of $B/N[[x_1,\ldots,x_k;g_1,\ldots,g_k]]$, where k is the dimension of N/N^2 as a left B/N space.

<u>Proof</u>. By theorem 5, B is a homomorphic image of $A \oplus N_1 \oplus N_1^2 \oplus \ldots$, $N_1 \equiv N/N^2$. Theorem 6 tells us that $N_1 = (B/N)n_1 \oplus \ldots \oplus (B/N)n_k$ where $n_i b = g_i(b)n_i$ for $b \in B/N$ and automorphisms g_i of B/N. Hence B is a homomorphic image of $B/N[[x_1,\ldots,x_k;g_1,\ldots,g_k]]$. Note that k is the dimension of N/N^2 as a B/N module and hence by the proof of lemma 2 $n_1,\ldots,n_k$ is a generating set of N. This set is of course minimal.

We now consider the case where B/N is a field and is a finite dimensional separable extension of F. As in theorem 7 we need to gain some information on the structure of N/N^2 as a two sided B/N module. Since B/N is a separable extension of B we have $B/N = F(\theta)$ where θ is a root of a separable and irreducible polynomial f(x) contained in F[x]. We now collect the following information from Jacobson [10].

<u>Theorem 8</u>. Let f(x) be a separable irreducible polynomial in F[x]. Let θ be a root of f(x) and let $K = F(\theta)$. If

M is an F-regular (K,K) module which has finite dimension as a left K space then $M = M_1 \oplus \ldots \oplus M_t$ as simple bimodules. With each M_i is associated an irreducible factor

$-a_{0i}-a_{1i}x-\ldots-a_{m(i)-1,i}x^{m(i)-1}+x^{m(i)}$ of f(x) in K[x] such that $M_i = Kx_{1i} \oplus \ldots \oplus Kx_{m(i)i}$ as left K spaces. In M_i multiplication on the right by θ is identical with the linear transformation induced on M_i as a left K space by the companion matrix

$$\begin{pmatrix} 0 & 1 & 0 \ldots & 0 \\ 0 & 0 & 1 \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdot & \cdot & 1 \\ a_{0i} & a_{1i} & \cdot & a_{m(i)-1,i} \end{pmatrix}$$

of $-a_{0i}-a_{1i}x-\ldots-a_{m(i)-1,i}\ x^{m(i)-1}+x^{m(i)}$.

Proof. Multiplication on the right of M by θ induces a K linear transformation T on the left K space M. Since θ satisfies $f(x) \in F[x]$ we have $f(T) = 0$. Hence the minimal polynomial for T in K[x] divides f(x) and is thus separable. The rational canonical form for linear transformations now guarantees that $M = M_1 \oplus \ldots \oplus M_t$ where M_i is invariant and cyclic under T. Thus each M_i is a simple (K,K) module. There is a basis for M_i such that the transformation T_i induced on M_i by T takes the form of the companion matrix for some irreducible factor of f(x) in K[x].

We are now in a position to prove the final theorem of the section. First we shall define a class of complete local algebras. Let $K = F(\theta)$ be a finite dimensional separable extension of F. Let $f(x)$ be the irreducible monic polynomial of θ over $F[x]$. Let T be the companion matrix of an irreducible monic factor of $f(x)$ in $K[x]$. We may suppose that this factor has degree $n(i)$ so that T_i is an $n(i)$ by $n(i)$ matrix. Let us note here that the T_i are not necessarily distinct and do not necessarily include all the companion matrices of the irreducible factors of $f(x)$. The ring $K[[X_1, \ldots, X_t; T_1, \ldots, T_t]]$ is defined to be the power series ring with coefficients from K in $n(1) + \ldots + n(k)$ noncommuting indeterminates x_{ij} where $X_i = \{x_{i1}, \ldots, x_{in(i)}\}$, $x_{ij} f = f x_{ij}$ for all $f \in F$ and $x_{ij}\theta = T_i x_{ij}$ where T_i is considered as a linear transformation on the space $Kx_{i1} \oplus \ldots \oplus Kx_{in(i)}$.

<u>Theorem 9</u>. If B is a complete local algebra over F whose radical is finitely generated, and if $B/N = K$ is a finite dimensional separable extension field of F then there exists matrices T_i such that B is a homomorphic image of $K[[X_1, \ldots, X_t; T_1, \ldots, T_t]]$ where $n(1) + \ldots + n(t) = k$, the dimension of N/N^2 as a left B/N space.

<u>Proof</u>. Since K is finite dimensional and separable extension of F we have $K = F(\theta)$. By theorem 5 B is a homomorphic image of $K \oplus N_1 \oplus N_1^2 \oplus N_1^3 \oplus \ldots$, the complete direct sum. By theorem 8 we have $N = M_1 \oplus \ldots \oplus M_t$ where $M_i = Kn_{i1} \oplus \ldots \oplus Kn_{in(i)}$ and $n_{ij}\theta = T_i n_{ij}$ for the appropriate companion

matrix T_i. Hence B is a homomorphic image of $K[[X_1,\ldots,X_t;T_1,\ldots,T_t]]$ under the map induced by $x_{ij} \rightarrow n_{ij}$.

We may note here that theorem 7 is a direct result of this theorem when B/N is a field but does not follow if B/N is a division ring.

Corollary 3. If B is a complete local ring whose radical is generated by k elements, k minimal, and if B/N = K is a finite dimensional separable extension field of the rationals then there exist matrices T_i such that B is a homomorphic image of $K[[X_1,\ldots,X_t;T_1,\ldots,T_t]]$ where $n(1) + \ldots + n(t) = k$.

Proof. Since the characteristic of B/N is zero, B is an algebra over the rationals. Hence the hypothesis of theorem 9 are satisfied.

SECTION V

COMPLETE LOCAL RINGS WITH SUITABLE COMMUTATIVE SUBRINGS

This section deals with complete local rings B whose radical is finitely generated as a left ideal and which have suitable commutative subrings. We will see that these rings include those complete local rings B with finitely generated radical such that B/N is a field F which is a finite dimensional normal extension of its prime subfield.

If the characteristic of B is equal to the characteristic of B/N then the results of this section reduce to the results of section IV. Therefore in the following we will assume that the characteristic of B/N is a prime p and the characteristic of B is not equal to p.

The structure of certain U-regular (V,V) modules will be essential in the theorems of this chapter. We use some elementary techniques to obtain the required information in the next series of lemmas. In these lemmas we will assume that B and U are v-rings and V is a finite module over U.

The following lemma is an easy variation of a result of Cohen [3, page 68]. In the version as stated below, we do not need the hypothesis that R is noetherian.

Lemma 4. Let R and S be commutative local rings with $R \subset S$ and R complete. If $S \cdot N(R) = N(S)$ and $S/N(S)$ is a finite algebraic extension of $R/N(R)$ then S is complete and $S = Ra_1 + \ldots + Ra_k$ where $a_1, \ldots, a_k$ is any lifting of a basis $\bar{a}_1, \ldots, \bar{a}_k$ of $S/N(S)$ over $R/N(R)$.

In the remainder of this section we will denote $V/N(V)$ by F and $U/N(U)$ by F_1. We will denote the field of quotients of V by Q and the field of quotients of U by Q_1.

Lemma 5. If $\bar{a}_1, \ldots, \bar{a}_k$ is a basis for F over F_1 and $a_1, \ldots, a_k$ if a set of elements of V mapping onto $\bar{a}_1, \ldots, \bar{a}_k$ under the natural map then $a_1, \ldots, a_k$ is both a basis for V over U and Q over Q_1.

Proof. Lemma 4 guarantees $V = Ua_1 + \ldots + Ua_k$. We must now show independence over U. Suppose $a_1, \ldots, a_k$ are dependent over U. Without loss of generality we may assume $T = \{u \neq 0 : u \in U, 0 = ua_1 + \ldots + u_k a_k\}$ is not empty. Let $S = \{\alpha : \alpha$ is a non-negative integer such that for some $u \in T$ we have $u = p^{\alpha} u'$ where $u' \in U$ and $u \neq p^{\alpha+1} u''$ for any $u'' \in U\}$. Since T is not empty then under the natural map we have $0 = \overline{ua}_1 + \ldots + \bar{v}_u \bar{a}_k$ which implies $\bar{u} = 0$ and $u = pu'$ so that S is not empty. Pick a minimal $\alpha \in S$. Therefore there exists $u_1, \ldots, u_k$ such that $0 = u_1 a_1 + \ldots + u_k a_k$ and u_1 is divisible by p^{α} but not by $p^{\alpha+1}$. In F we have $0 = \bar{u}_1 \bar{a}_1 + \ldots + \bar{u}_k \bar{a}_k$ so that $\bar{u}_i = 0$ for each i. Therefore $u_i \in N(U)$ which implies $u_i = pu_i'$ for each i. Therefore $0 = p(u_1' a_1 + \ldots + u_k' a_k)$ and since V is an integral domain

$0 = u_1'a_1 + \ldots + u_k'a_k$, $u_1' \neq 0$. We have $u_1' = p^{\alpha-1}u'$ and $u_1' \neq p^{\alpha}u''$ for any $u'' \in U$. Since α is minimal, we have $\alpha = 0$, which contradicts the fact that u_1 is divisible by p. Hence each $u_i = 0$ and $a_1, \ldots, a_k$ are linearly independent over U.

Since $Q = \{v/p^{\alpha} : v \in V,\ \alpha$ is a non negative integer$\}$ we have $v/p^{\alpha} = (u_1/p^{\alpha})a_1 + \ldots + (u_k/p^{\alpha})a_k$ so that $Q = Q_1a_1 + \ldots + Q_1a_k$. A linear dependency of the a_i in Q over Q_1 would imply a linear dependency in V over U. Therefore $\{a_1, \ldots, a_k\}$ is indeed a basis for Q over Q_1.

Lemma 6. If F is a normal extension of F_1 then there is an isomorphism between the group of automorphisms of V fixing U and the group of automorphisms of F fixing F_1. If g is an automorphism of V then an isomorphism is given by $g \longrightarrow \bar{g}$ where $\bar{g}(\bar{v}) = \overline{g(v)}$.

Proof. Since F is normal over F_1, F is the splitting field of a separable irreducible polynomial $\bar{f}(x) \in F_1[x]$. Suppose $\bar{f}(x) = (x-\bar{a}_1)\ldots(x-\bar{a}_k)$ in $F[x]$. We may suppose that $\{\bar{a}_1, \ldots, \bar{a}_k\}$ forms a normal basis for F over F_1. Choose $f(x) \in U[x]$ which maps onto $\bar{f}(x)$. By Hensel's lemma $\{\bar{a}_1, \ldots, \bar{a}_k\}$ can be raised to $\{a_1, \ldots, a_k\}$ contained in V such that f(x) splits in $V[x]$ to $f(x) = (x-a_1)\ldots(x-a_k)$. By lemma 5, $\{a_1, \ldots, a_k\}$ is a basis for V over U and Q over Q_1. This implies that $Q = Q_1a_1 + \ldots + Q_1a_k$ is the splitting field of the irreducible polynomial f(x) in

$Q_1[x]$, which proves that Q is a normal extension of Q_1. If g is an automorphism of Q over Q_1 then $g(a_i) = a_{\pi(i)}$ where π is a permutation of $1,\ldots,k$. Hence $g(V) = V$ and any automorphism of Q over Q_1 restricts to an automorphism of V over U. Note that the map $g \longrightarrow \bar{g}$ where $\bar{g}(\bar{v}) = \overline{g(v)}$ is a homomorphism of the Galois group of V over U into the Galois group of F over F_1. Since g induces a permutation of $a_1,\ldots,a_k$, $\bar{g}$ will induce the same permutation on $\bar{a}_1,\ldots,\bar{a}_k$. Hence $g \neq 1$ implies $\bar{g} \neq 1$ and $g \longrightarrow \bar{g}$ is an isomorphism. To see that the isomorphism is onto we note that the Galois group G of V over U has order k as does the Galois group of F over F_1.

We are now in a position to investigate certain U-regular (V,V) modules. We will require that F is normal over F_1, and under this assumption we have

__Theorem 10__. Let M be a U-regular (V,V) module. Let $g_1,\ldots,g_k$ be the automorphisms of V over U. Then M is equal to $M_1 \oplus \ldots \oplus M_k$ as (V,V) submodules where $mv = g_i(v)m$ for all $m \in M_i$.

__Proof__. Since F is normal over F_1, there is as in the proof of lemma 6 an irreducible polynomial $f(x) \in U[x]$ which splits in $V[x]$ and which has a_1 as a root where $V = U(a_1)$. Suppose $f(x) = (x-a_1)\ldots(x-a_k)$ in $V[x]$.

Define $f_i(x) = \prod_{j\neq i}(x-a_j)$. Since V is local and $a_i - a_j \notin N(V)$ for $i \neq j$ we have $f_i(a_i) \notin N(V)$. Therefore $(f_i(a_i))^{-1}$ exists in V and we have the identity $1 = \sum(f_i(a_i))^{-1}f_i(x)$ in $V[x]$. We may now note that multiplication on the right of M by a_1 is a V linear transformation T on M as a left V space. Since $f(a_1) = 0$ we also have $f(T) = 0$. Because of the identity in $V[x]$ we have $I = \sum(f_i(a_i))^{-1}f_i(T)$ so that $M = \sum(f_i(a_i))^{-1}f_i(T)M = M_1 + \ldots + M_k$ where $M_i = (f_i(a_i))^{-1}f_i(T)M$. To show that this sum is direct suppose for example that $m \in M_1 \cap (M_2 + \ldots + M_k)$. Since $m \in M_1$ we have $(T-a_1I)m = 0$, and since $m \in M_2 + \ldots + M_k$ we have $(T-a_2I)\ldots(T-a_kI)m = 0$. This implies that $m = Im = \sum(f_i(a_i))^{-1}f_i(T)m = 0$ and the sum is direct. Now for every $m \in M_i$ we have since $f(T) = 0$, $(T-a_iI)m = 0$. Therefore $Tm = a_im$. But $Tm = ma_1$. Since a_i is a root of $f(x)$, as is a_1, there is an automorphism g_i of V over U which maps a_1 onto a_i. Therefore $ma_1 = g_i(a_1)m$ and $mv = m(\sum_j u_ja_1^j) = \sum_j u_jma_1^j = \sum_j u_jg_i^j(a_1)m = g_i(\sum_j u_ja_1^j)m = g_i(v)m$, and the theorem is proven.

<u>Lemma 7</u>. Let S be a subring of a field F. If $F = Sa_1 + \ldots + Sa_k$ then S is a subfield of F.

<u>Proof</u>. Let Q be the field of quotients of S. Since $F = Qa_1 + \ldots + Qa_k$ there is a linearly independent subset $b_1,\ldots,b_m$ over Q with $b_1 = 1$ and $\{b_2,\ldots,b_m\} \subset \{a_1,\ldots,a_k\}$

such that $F = Qb_1 + \dots + Qb_m$. Hence $a_i = \sum_j q_{ij} b_j$ where $q_{ij} \in Q$. Pick $s \in S$ such that $sq_{ij} \in S$ for every $i = 1,\dots,k$, $j = 1,\dots,m$. $f \in F$ implies $f = \sum_i s_i a_i = \sum_i s_i(\sum_j q_{ij} b_j)$. Therefore $sf = \sum_i s_i(\sum_j sq_{ij} b_j) = \sum_j s_j' b_j$ where $s_j' \in S$. This implies $F = sF \subset Sb_1 + \dots + Sb_m$. For $t \in S$ we have $t^{-1} = s_1 + s_2 b_2 + \dots + s_m b_m$, and since $t^{-1} \in Q$ we have $t^{-1} = s_1$ which implies that S is a field.

This lemma tells us that if ϕ is a homomorphism of R onto a field F and $R = R'a_1 + \dots + R'a_k$ where R' is a subring of R then $\phi(R')$ is a subfield of $\phi(R)$.

In proving the main theorems of this section we will need to know the existence of homomorphic images of v-rings within our noncommutative local rings. The existence of these v-rings is determined by using the following theorem of Cohen [3, page 79].

Theorem 11. Let R be a commutative complete noetherian local ring with residue field R/N of characteristic p. Let ϕ be the natural map from R onto R/N. Then R contains a subring S which is a homomorphic image of a v-ring V where if δ is that homomorphism, $\phi(\delta(V)) = R/N$.

We will now investigate the structure of a local ring B of characteristic not equal to p where B/N is a field F of characteristic p. Let C denote the center of B, ϕ the natural map of B onto F and suppose that S is a commutative subring of B which maps onto F under ϕ.

Theorem 12. Let S be a local ring and let S' be a complete noetherian local ring contained in $C \cap S$. If S is a finite module over S' and F is a normal extension of $\phi(S') = F_1$ then there exist automorphisms, $g_1, \ldots, g_k$, of a v-ring V such that B is a homomorphic image of $V \oplus N$ as rings where $N = N(B)$. $N = N_1 \oplus \ldots \oplus N_k$ as (V,V) submodules where $n_i v = g_i(v) n_i$ for all $n_i \in N_i$.

In proving this theorem we need the existence of v-rings $V \supset U$ and a homomorphism δ which maps V into S such that $\delta(U) \subset S'$, $\phi(\delta(V)) = F$ and $\phi(\delta(U)) = F_1$. This will enable us to determine the structure of N(B) as a U-regular (V,V) module. We first need the following lemma found in Curtis [4, theorem 2].

Lemma 8. Let R' be a finite extension of a noetherian semilocal ring R. Then R' is a noetherian semilocal ring. If R is complete in the N(R)-adic topology then R' is complete in the N(R')-adic topology.

Lemma 9. With hypothesis as in theorem 12, there exist v-rings $V \supset U$ and a homomorphism $\delta : V \longrightarrow S$ such that

$\delta(U) \subset S'$, $\phi(\delta(V)) = F$ and $\phi(\delta(U)) = F_1$.

<u>Proof</u>. By applying theorem 11 to the complete noetherian local ring S' we obtain a v-ring U and a homomorphism $\delta : U \longrightarrow S'$ such that $\phi(\delta(U)) = F_1$. Since F is finite dimensional and normal over F_1 we have $F = F_1(\bar{\theta})$ where $\bar{\theta}$ is a root of an irreducible polynomial $\bar{f}(x) \in F_1[x]$. Let $f(x) \in \delta(U)[x]$ be a polynomial which maps onto $\bar{f}(x)$ under the map induced on $\delta(U)[x]$ by ϕ. Since S is a complete noetherian local ring Hensel's lemma guarantees that there is a root $\theta \in S$ such that $f(\theta) = 0$ and $\phi(\theta) = \bar{\theta}$. The ring $\delta(U)(\theta)$ maps onto F. Let $g(x) \in U[x]$ map onto $f(x)$ under the map induced by δ. δ extends naturally to the homomorphism γ which takes $U[x]/(g(x)) = V$ onto $\delta(U)(\theta)$. Since $g(x)$ is irreducible, V is an integral domain. The fact that V is finitely generated over U then implies by lemma 8 that V is a complete local ring. Since $V/pV = F$ we have $N(V) = pV$. Therefore $V \supset U$ are the appropriate v-rings.

<u>Proof of Theorem 12</u>. Define $B' = V \oplus N(B)$ where $vnw = \gamma(v)n\gamma(w)$ for $v, w \in V$, $n \in N(B)$ and γ is the homomorphism from V into S. Since $\gamma(U) \subset C$ we have $\{v \in V : vn = nv$ for all $n \in B'\} \supset U$. Therefore $N(B)$ is a U-regular (V,V) module and B' is a ring which is mapped onto B by the homomorphism $\gamma'(v + n) = \gamma(v) + n$ for $v \in V$ and $n \in N(B)$.

By theorem 10 there exist automorphisms $g_1,\ldots,g_k$ of V over U such that $N(B) = N_1 + \ldots + N_k$ as U-regular (V,V) submodules where $nv = g_i(v)n$ for all $n \in N_i$. This completes the proof of theorem 12.

Note that in theorem 12, the ring B had certain complete subrings but B itself was not required to be complete nor was its radical required to be finitely generated. If these additional assumptions on B are made we have the following

Theorem 13. Let B be a complete local ring whose radical is finitely generated by t elements. If B satisfies the hypothesis of theorem 12 then there exist automorphisms $g_1,\ldots,g_t$ of a v-ring V such that B is a homomorphic image of $V[[x_1,\ldots,x_t;g_1,\ldots,g_t]]$ where $V/N(V)$ is isomorphic to $B/N(B) = F$.

Proof. By theorem 12 B is a homomorphic image of $B' = V \oplus N(B)$, where $N(B) = N_1 \oplus \ldots \oplus N_k$. Label elements in $N(B)/N(B)^2$ with a bar. Hence $\overline{N(B)} = \bar{N}_1 \oplus \ldots \oplus \bar{N}_k$. $\overline{N(B)}$ is a vector space over F so we may pick a basis $\{\overline{n_{ij}}\}$ such that $\{\overline{n_{i1}},\ldots,\overline{n_{ij(i)}}\}$ spans $\bar{N}_i$. Let n_{ij} be elements of N_i which map onto $\overline{n_{ij}}$. Hence $n_{ij}v = g_i'(v)n_{ij}$. Renumbering the n_{ij} as $n_1,\ldots,n_r$ and the g_i' as $g_1,\ldots,g_r$ (these in general will not be distinct) we have $n_iv = g_i(v)n_i$. From the proof of lemma 2 $\{n_1,\ldots,n_r\}$ is a minimal generating set for N(B). Therefore B' is a homomorphic image of $V[[x_1,\ldots,x_r;g_1,\ldots,g_r]]$. This in turn implies that B is a homomorphic image of $V[[x_1,\ldots,x_r;g_1,\ldots,g_r]]$. We of course have that $r \leq t$ and so B is also a homomorphic image of $V[[x_1,\ldots,x_t;g_1,\ldots,g_t]]$ where $g_{r+1},\ldots,g_t$ may be any homomorphisms of V over U.

The condition that B has a complete commutative local subring S mapping onto F with S' a complete noetherian local subring in the center such that S is a finite module over S' appears to be very restrictive. Theorem 14 will show that if B is complete we can omit the hypothesis that S and S' are complete and local.

Theorem 14. Let B be a complete local ring. Let S be a commutative subring of B such that $\phi(S)$ is the field B/N where ϕ is the natural map. If S_1 is a noetherian ring contained in the center C and S is a finite module over S_1 then there exists a commutative local subring S' of B which is a finite module over a complete commutative local ring S_1' contained in C and where $\phi(S') = F$ $\phi(S_1') = \phi(S_1)$.

Proof. Lemma 7 and the fact that $S = S_1a_1 + \ldots + S_1a_k$ means that $\phi(S_1)$ is a field. Since all elements of B outside N(B) are invertible we can localize in S_1 about the ideal $N(B) \cap S_1$. We may now complete in the radical topology (which is identical to the topology induced by the powers of $N(B)$) to obtain a ring S_1'. Note that S_1' is contained in the center C since multiplication is continuous in the $N(B)$-adic topology. Define S' to be the module $S_1'a_1 + \ldots + S_1'a_k$ over S_1'. Note that S' is actually a ring since $a_i a_j$ is contained in S'. By lemma 8 S' is a complete noetherian semilocal ring. Since the only idempotents in B are 0 and 1 this means that S' is in fact local.

Combining theorem 13 and 14 we obtain

Theorem 15. Let B satisfy the hypothesis of theorem 14. If F is finite dimensional and normal over $\phi(S_1)$, and if $N(B)$ is finitely generated by t elements, then B is a homomorphic image of $V[[x_1,\ldots,x_t;g_1,\ldots,g_t]]$ where V is a v-ring mapping onto F and g_i are automorphisms of V.

If B/N is a finite field we can drop the assumptions about the existence of S.

Theorem 16. Let B be a complete local ring with B/N a finite field F. If $N(B)$ is generated by t elements then there exist automorphisms $g_1,\ldots,g_t$ of a v-ring V such that B is a homomorphic image of $V[[x_1,\ldots,x_t;g_1,\ldots,g_t]]$ where $V/N(V) = F$.

Proof. The identity of B generates a subring S in C the center of B. Since F is finite $F = Z_p(\overline{\theta})$ where Z_p is the integers mod p. Let θ be a representative of $\overline{\theta}$ in B. The subring $S[\theta]$ is commutative and noetherian. In $S[\theta]$ localize and complete about $S[\theta] \cap N(B)$ to obtain S'. S' is of course a complete local commutative ring which maps onto F. By theorem 11 there is a v-ring V and a homomorphism δ such that $\delta(V) \subset S'$ and $\phi(\delta(V)) = F$. Since $\delta(V)$ contains the identity it also contains S and the localization and completion S" of S about N(B). S" is of course in C and $N(S") = pS"$. Since $N(\delta(V)) = p\delta(V) = pS"\delta(V)$ the conditions of lemma 4 are satisfied. Hence $\delta(V)$ is a finite module over S" and the conditions of theorem 13 are fulfilled implying that B is a homomorphic image of $V[[x_1,\dots,x_t;g_1,\dots,g_t]]$.

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