COMBINATORIAL PROPERTIES OF FINITE GEOMETRIC LATTICES

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To my wife, Betsy, I am eternally grapefruit.

ABSTRACT

Let L be a finite geometric lattice of dimension n, and let w(k) denote the number of elements in L of rank k. Two theorems about the numbers w(k) are proved: first, $w(k) \geq w(1)$ for $k = 2, 3, \ldots, n-1$. Second, $w(k) = w(1)$ if and only if $k = n-1$ and L is modular. Several corollaries concerning the "matching" of points and dual points are derived from these theorems.

Both theorems can be regarded as a generalization of a theorem of de Bruijn and Erdős concerning $\lambda = 1$ designs. The second can also be considered as the converse to a special case of Dilworth's theorem on finite modular lattices.

These results are related to two conjectures due to G.-C. Rota. The "unimodality" conjecture states that the w(k)'s form a unimodal sequence. The "Sperner" conjecture states that a set of non-comparable elements in L has cardinality at most $\max_k \{w(k)\}$. In this thesis, a counterexample to the Sperner conjecture is exhibited.

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I. INTRODUCTION

Many combinatorial problems find a natural setting in the theory of finite geometric lattices. By specializing in various directions, one can apply the theory to such diverse subjects as graph coloring, network flows, partitions, finite geometries, matching theory, and Boolean algebras (see [5], [17]). Recently, there has been considerable interest in combinatorial problems involving finite geometric lattices as a general class. This thesis is devoted to several of these problems.

The principal motivation for this work has come from two conjectures which apparently were first made by G.-C. Rota.

The "Unimodality Conjecture": Let L be a finite geometric lattice, and let $w(k)$ denote the number of elements in L of rank k. The conjecture asserts that $w(k) \geq \{w(i), w(j)\}$ for any i and j such that $i \leq k \leq j$. In other words, there exists an integer m such that the first m values of $w(k)$ are nondecreasing and the succeeding values are nonincreasing.

The "Sperner" Conjecture: Let L be a finite geometric lattice, and let $S \subset L$ be a set of pairwise non-comparable elements. Then the conjecture asserts that

$$|S| \leq \max_k \{w(k)\}.$$

(This conjecture derives from a theorem of E. Sperner [20] which states the same result for Boolean algebras.)

The unimodality conjecture can be immediately verified for projective geometries and Boolean algebras by means of explicit formulas for $w(k)$. It can also be shown that unimodality is preserved under direct products, and the unimodal property thus holds for all finite, complemented, modular lattices, by the Birkhoff - Menger decomposition theorem [2, Ch. 4]. In the case of partition lattices, where the w(k)'s are Stirling numbers of the second kind, the unimodal property has been verified by L. H. Harper [11].

The Sperner conjecture is known to be valid for several classes of geometric lattices in addition to Boolean algebras. Recent work by L. H. Harper [12] shows that geometric lattices having a "normalized matching property"[1] also have both the Sperner property and the unimodal property. A theorem of Harper and R. L. Graham [8] shows that this category includes projective and affine geometries and, more generally, any geometric lattice which can be represented as a direct product of lattices which are "regular" (i.e., elements of the same rank cover and are covered by the same number of elements). By extending a proof of Sperner's Theorem due to Lubell [13], K. Baker has shown that "regularity" alone is sufficient even if the structure is not a lattice: any finite, "regular," graded partially ordered set satisfies the conclusion of Sperner's Theorem [1].

Unfortunately, an example constructed by R. P. Dilworth shows

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1. This property is the following: if $S \subset L$ consists of elements of rank k, and S^* denotes the set of elements of rank k+1 covering some element of S, then

$$\frac{|S|}{w(k)} \leq \frac{|S^*|}{w(k+1)}$$

that not every geometric lattice has Harper's "normalized matching property," and, hence, that this approach cannot be used to resolve either conjecture in its full generality.

This thesis consists of several results of a general nature which relate to both of the conjectures discussed above. In Chapter 3, we show that the Sperner conjecture is, in general, false. In Chapter 4, we prove the following theorem:

Theorem 1: *Let L be a finite geometric lattice of dimension* $n > 1$. *Then* $w(1) \leq w(k)$ *for* $k = 2, 3, \ldots, n-1$.

This theorem trivially verifies the unimodality conjecture for geometric lattices of dimension 4, and, by specialization to various sublattices, provides a large number of inequalities which must hold in any geometric lattice. We also obtain several matching theorems which are, in turn, closely related to the Sperner property. In Chapter 5, we prove

Theorem 2: *Let L be a finite geometric lattice of dimension* $n > 1$. *Then* $w(1) = w(k)$ *if and only if* $k = n-1$ *and L is modular*.

The "if" part of this theorem is a special case of a theorem of Dilworth [6]: in any finite, modular lattice, the number of elements covering precisely k elements is equal to the number of elements covered by precisely k elements. In particular ($k = 1$), the number of join-irreducibles is equal to the number of meet-irreducibles. Our Theorem 2 can be regarded as a converse to this theorem in the geometric case.

In Chapter 6, we discuss briefly the connection between Theorems 1 and 2 and the well-known theorem of de Bruijn and Erdős [4] which classifies all $\lambda = 1$ designs (c.f. Ryser [19]). Our results are in fact precisely equivalent to the de Bruijn - Erdős theorem if one takes L to be a geometric lattice of dimension 3.

II. DEFINITIONS AND PRELIMINARIES

In all that follows, we will concern ourselves only with finite lattices. If x and y are elements of a lattice, with $x > y$, and $x \geq z > y$ implies $x = z$, we say that x covers y and write $x \succ y$. Elements which cover 0 (the null element) are called points, and elements covered by 1 (the unit element) are called dual points. A lattice in which every element is the join of the points it contains is called a point lattice. A lattice whose dual is a point lattice is called a dual point lattice. A lattice is semimodular if $x \vee y$ covers x whenever y covers $x \wedge y$. Equivalently, a lattice is semimodular if $x \vee y$ covers x and y whenever x and y cover $x \wedge y$. A lattice L is modular if and only if both L and its dual are semimodular. A geometric lattice is a semimodular point lattice.

If L is geometric, then there exists a unique rank function r on L with the property that $r(0) = 0$, $r(x) = r(y) + 1$ if $x \succ y$, and $r(x \vee y) \leq r(x) + r(y) - r(x \wedge y)$ for all x, y in L. The dimension of L is defined to be the rank of the unit element 1. Every geometric lattice is both a point and a dual point lattice. If L is geometric, then so is every interval sublattice x/y in L. For a general discussion of these and other properties, see [2, Ch. 4].

We will make use of the following notation:

P_L = the set of points of L

D_L = the set of dual points in L

$r(L)$ = the dimension of L

$w(k)$ = the number of elements of rank k in L

$\alpha(x)$ = the number of elements covered by x

$\beta(x)$ = the number of elements covering x

$\alpha^*(x)$ = the number of points contained in x

$\beta^*(x)$ = the number of dual points containing x.

There are many different axiom schemes which are equivalent to the above definition of a geometric lattice. These can be expressed in terms of either lattices or abstract "geometric" structures. One of the simplest and most useful is based on the idea of a <u>closure operator</u> with the <u>exchange property</u>. A closure operator on a set S is a map $C: 2^S \longrightarrow 2^S$ satisfying (i) $C(X) \supset X$, (ii) $X \supset Y \implies C(X) \supset C(Y)$, and (iii) $CC(X) = C(X)$, for all X, $Y \subset S$. A subset X is <u>closed</u> if $C(X) = X$. We say that C has the exchange property if, whenever X is a closed subset of S and p and q are elements of S, we have

$$p \in C(X \cup \{q\}) \text{ and } p \notin X \implies q \in C(X \cup \{p\}).$$

It can be shown (see [14], [3]) that the closed sets of any closure operator with the exchange property form a geometric lattice. Conversely, any geometric lattice yields such a closure operator on its set of points if one defines $C(X) = \{p \mid p \leq \bigvee X\}$.

One of the simplest examples of a class of geometric lattices which is, in general, non-modular is given by the following:

<u>Lemma 2.1</u> (Whitney [21]-Birkhoff [3]): <u>Let G = (V, E) be a finite, non-oriented graph with vertex set V and edge set E. Define a closure operator on E as follows: if $X \subset E$, then C(X) consists of all edges whose endpoints are vertices connected by a sequence of</u>

edges in X. Then C has the exchange property, and the C-closed subsets of E form a geometric lattice.

This lattice is sometimes called the bond-lattice of G. The rank function is given by the formula $r(X) = |V| - \eta(X)$, where $\eta(X)$ is the number of components of the subgraph of G determined by the edges in X. When G is the complete graph on its vertices (i.e., E consists of all pairs of vertices in V), the bond-lattice of G is isomorphic to the lattice of partitions of V. In general, the bond-lattice of a graph can be regarded as the lattice of "admissible" partitions of its vertices -- those partitions whose blocks determine connected subgraphs of G. We will return to this example in Chapter 3.

We continue now with a number of elementary lemmas. The first three are standard and well-known (see [2, Ch. 4]).

Lemma 2.2: Let L be a finite geometric lattice, and let $S \subset P_L$. Let $L^*(S)$ denote the set of all joins of subsets of S. Then $L^*(S)$ is a geometric lattice.

Proof: Since $L^*(S)$ is closed under joins, it is a lattice, and every element is clearly the join of the points it contains. Semimodularity follows from the fact that coverings in $L^*(S)$ are coverings in L.

Lemma 2.3: Let L be a finite geometric lattice of dimension n, and let $k \leq n$. Denote by L_k the result of identifying all elements in L of rank $\geq k$. Then L_k is a geometric lattice.

Proof: All properties follow immediately from the conditions on L.

Lemma 2.4: *Let L be a finite semimodular lattice. Let $p \in P_L$ and $d \in D_L$ with $p \not\leq d$. Then the map $x \rightarrow x \vee p$ is an injection from $d/0$ to $1/p$ which takes points of $d/0$ to points of $1/p$ and dual points of $d/0$ to dual points of $1/p$.*

Proof: Let $x, y \in d/0$, and suppose $x \vee p = y \vee p = z$. Then $r(x) = r(y) = r(z) - 1$, and $x \vee y \leq z$. If $x \neq y$, it follows that $x \vee y = z$, which implies that $z \in d/0$. But then $d \geq z \geq p$, a contradiction. Hence $x = y$, and the map is injective. A rank argument shows that the map preserves points and dual points.

Lemma 2.5: *Let L be a finite semimodular lattice. Let $p \in P_L$ and $d \in D_L$ with $p \not\leq d$. Then $\alpha(d) \leq \beta^*(p)$ and $\alpha^*(d) \leq \beta(p)$.*

Proof: The inequalities follow immediately from Lemma 2.4.

Lemma 2.6: *Let L be a finite dual point lattice, and let x be an element of L with $r(x) > 0$. For any set S of elements covered by x, let S^* be the set of dual points which contain some element of S. Then*

$$|S| \leq |S^*| - \beta^*(x) \leq |D_L| - \beta^*(x).$$

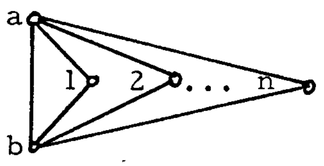
In particular, we have $\alpha(x) \leq |D_L| - \beta^*(x)$.

Proof: For each $y \in S$, pick $d_y \in D_L$ such that $y = x \wedge d_y$. (This can be done since every element is the meet of the dual points which contain it.) The d_y's must be distinct elements of S^*. Since none contains x, the inequalities follow.

III. COUNTEREXAMPLE TO THE SPERNER CONJECTURE

In this chapter, we describe an infinite class of geometric lattices which do not have the Sperner property. These lattices were originally constructed, in a different form, by R. P. Dilworth in order to illustrate another property.[1] We describe them below as bond-lattices of a class of graphs. The smallest counterexample which we exhibit contains 60,073 elements.

Consider the following graph G_n:



Let $L(G_n)$ denote the bond-lattice of G_n. By the remarks in Chapter 2, $L(G_n)$ is a geometric lattice of dimension $n+1$.

If e_0 denotes the edge with vertices a and b, and X is a closed set of edges containing e_0, then $r(X) = (|X| + 1)/2$ and we have

$$(a,i) \in X \Longleftrightarrow (b,i) \in X.$$

Hence the interval $1/e_0$ in $L(G_n)$ is isomorphic to the Boolean algebra of subsets of $\{1, 2, \ldots, n\}$. If $k \leq n+1$, then the number of elements in $L(G_n)$ of rank k containing e_0 is precisely $\binom{n}{k-1}$.

If X is a closed subset of edges not containing e_0, then $r(X) = |X|$ and we have

$$(a,i) \in X \Longrightarrow (b,i) \notin X \quad \text{and} \quad (b,i) \in X \Longrightarrow (a,i) \notin X.$$

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1. They can be used to construct examples of nonisomorphic pairs of geometric lattices having the same structure above rank 1.

Thus sets of this type having rank k are determined by k-subsets of the vertices $\{1, 2, \ldots, n\}$ together with a function which assigns either a or b to each vertex. Hence the number of elements in $L(G_n)$ of rank k which do not contain e_0 is $2^k\binom{n}{k}$, and we have the formula

$$w(k) = \binom{n}{k-1} + 2^k\binom{n}{k}.$$

Let L be any geometric lattice, and suppose that $w(k_m) = \max_k w(k)$. If $p \in P_L$, let $w_p(i)$ denote the number of elements in L of rank i which contain p, and suppose that

$$w_p(k_m - 1) > w_p(k_m).$$

Then the Sperner property cannot hold in L. For, if R(i) denotes the set of elements in L of rank i, and S(i) denotes the set of elements of rank i containing p, then $S(k_m - 1) \cup (R(k_m) - S(k_m))$ is a non-comparable set, and

$$\left| S(k_m - 1) \cup (R(k_m) - S(k_m)) \right| = w_p(k_m - 1) + w(k_m) - w_p(k_m)$$

which is strictly greater than $w(k_m)$. We now observe that, for sufficiently large values of n, this situation occurs in the bond-lattices $L(G_n)$. From the formula for w(k) we obtain the relation

$$(*) \quad w(k) - w(k-1) = \frac{n!}{(k-1)!\,(n-k+1)!}\left[\frac{2^{k-1}(2n-3k+2)}{k} + \frac{(n-2k+3)}{(n-k+2)}\right].$$

One can see intuitively from this expression that the w(k)'s increase until k is approximately 2n/3. On the other hand, we have

$$w_{e_0}(k) = \binom{n}{k-1}$$

which decreases strictly for $k > n/2 + 1$. To make this argument precise, it suffices to show that $w(k) > w(k-1)$ for $k \leq k_0$, where $k_0 = n/2 + 2$ if n is even and $k_0 = (n+1)/2 + 2$ if n is odd. It will then follow that $k_m \geq k_0$, and hence $w_{e_0}(k_m - 1) > w_{e_0}(k_m)$. Analysis of formula (*) shows that this is true provided that $n \geq 10$ if n is even and $n \geq 13$ if n is odd. (The argument fails for $n < 10$ and $n = 11$.) Hence the analog of Sperner's Theorem fails for the lattices $L(G_n)$ when $n = 10$ and for all $n \geq 12$.

For $L(G_{10})$ we have the following table:

k	w(k)
0	1
1	21
2	190
3	1005
4	3480
5	8274
6	13692
7	15570
8	11640
9	5165
10	1034
11	1

For this example we have

$$\left| S(k_m - 1) \cup (R(k_m) - S(k_m)) \right| = 15570 - 210 + 252 = 15612 > 15570.$$

IV. A RANK INEQUALITY, AND APPLICATIONS

It follows from Lemma 2.3 that any theorem about points and elements of a fixed rank > 0 in a geometric lattice can be reduced to a statement about points and dual points. Such is the case with the main theorem of this chapter:

Theorem 1: Let L be a finite geometric lattice of dimension $n > 1$. Then $w(1) \leq w(k)$ for $k = 2, 3, \ldots, n-1$.

Proof: By Lemma 2.3, it suffices to prove the single inequality $w(1) \leq w(n-1)$. The main tool is the P. Hall Theorem on distinct representatives of sets, and the proof is by induction on the dimension of L. The theorem holds trivially for all lattices L with $r(L) \leq 2$. We now suppose that L is a geometric lattice of dimension n, and that the theorem holds for all lattices of smaller dimension.

The inductive assumption implies that Theorem 1 is true for all proper quotients of L. In particular, we have $\alpha(d) \geq \alpha^*(d)$ for all $d \in D_L$, and $\beta^*(p) \geq \beta(p)$ for all $p \in P_L$. Furthermore, $\alpha^*(d) \leq \alpha(d) \leq \beta^*(p)$ whenever $p \not\leq d$, by Lemma 2.5.

Let S be any subset of P_L with $|S| \leq |D_L|$. We show that there exists an injection $f: S \longrightarrow D_L$ with the property that $f(p) \not\geq p$ for all $p \in S$. For each $p \in S$, let $T(p) = \{d \in D_L \mid d \not\geq p\}$, and let $p_1, p_2, \ldots, p_k$ be k points in S. If U is the union of the sets $T(p_i)$, $i = 1, 2, \ldots, k$, then it is evident that $U = \{d \in D_L \mid d \not\geq u\}$, where $u = p_1 \vee p_2 \vee \cdots \vee p_k$. Thus $|U| = |D_L| - \beta^*(u)$. If $u = 1$, the unit element of L, then $|U| = |D_L| \geq |S| \geq k$. If $u < 1$, then $|U| = |D_L| - \beta^*(u) \geq \alpha(u)$ by Lemma 2.6, and $\alpha(u) \geq \alpha^*(u)$ by the inductive assumption. Thus $|U| \geq \alpha^*(u) \geq k$, since each p_i is contained in u. Hence $|U| \geq k$ in any case, and the sets T(p), $p \in S$, have a system of distinct representatives. This gives the desired function f.

Suppose now that $|P_L| > |D_L|$. Let S be any subset of P_L with $|S| = |D_L|$, and let $f: S \longrightarrow D_L$ be a bijection with $f(p) \ngeq p$ for all $p \in S$. Since $\alpha^*(f(p)) \leq \beta^*(p)$, we have

$$\sum_{p \in S} \beta^*(p) \geq \sum_{p \in S} \alpha^*(f(p)) = \sum_{d \in D_L} \alpha^*(d) .$$

Since there are points in P_L which are not in S, it follows that

$$\sum_{p \in P_L} \beta^*(p) > \sum_{d \in D_L} \alpha^*(d)$$

which is impossible, since both sums give the total number of "lines" in the partially ordered set of points and dual points. Thus $|P_L| \leq |D_L|$ and the theorem follows by induction.

We remark at this point that Theorem 1 is related to a set-theoretic result of T. Motzkin ([16], p. 463, Lemma 4.5): if k subsets of an n-element set have the property that for every set S and point $p \notin S$, p is contained in at least $|S|$ other sets, then (except for certain trivial cases) $k \geq n$. We could use this result to prove Theorem 1 immediately after establishing the relation $\alpha^*(d) \leq \beta^*(p)$ for $p \nleq d$. However, the method used above illustrates techniques which will be used to obtain many of the results which follow. The author is indebted to H. J. Ryser for clarifying certain aspects of the Motzkin result.

The following "matching" theorem is an immediate corollary to the above proof.

Corollary 4.1: Let L be a finite geometric lattice. Then there exists an injection $f: P_L \longrightarrow D_L$ with the property that $f(p) \not\geq p$ for all $p \in P_L$.

The next result also follows from Theorem 1:

Corollary 4.2: Let L be a finite geometric lattice. Then there exists an injection $g: P_L \longrightarrow D_L$ with the property that $g(p) \geq p$ for all $p \in P_L$.

Proof: We proceed by induction on the size of L, and show that the sets $S(p) = \{d \in D_L \mid d \geq p\}$, for $p \in P_L$, have a system of distinct representatives. Let $p_1, p_2, \ldots, p_k$ be k points of L, and let U be the set of dual points containing some p_i. If $k = w(1)$, then $U = D_L$ and $|U| \geq |P_L| \geq k$ by Theorem 1. If $k < w(1)$, let p_0 be any point not equal to $p_1, p_2, \ldots, p_k$, and let L^* be the geometric lattice obtained by taking the joins of all subsets of points other than p_0 (see Lemma 2.2). If $r(L^*) = r(L)$, then it is clear that $|U| \geq |U \cap L^*| \geq k$, by the inductive assumption. If $r(L^*) < r(L)$, it can easily be shown that L is the direct product of L^* and a two-element chain. In this case, the inductive assumption shows that $p_1, p_2, \ldots, p_k$ are contained in at least k dual points of L^*, which are mapped into distinct dual points of L under the map $x \longrightarrow x \vee p_0$. Thus $|U| \geq k$ in any case, and the function g is given by a system of distinct representatives for the sets $S(p)$, $p \in P_L$.

By modifying the argument used in Theorem 1, and by applying Corollary 4.2, we obtain the following stronger version of Corollary 4.1.

Corollary 4.3: Let L be a finite geometric lattice, and let $S \subset P_L$ with $|S| > 1$. Let $S^* \subset D_L$ denote the set of dual points which contain some point in S. Then there exists an injection $f: S \longrightarrow S^*$ with the property that $f(p) \not\geq p$ for all $p \in S$.

Proof: As in the proof of Theorem 1, we show that the sets $T(p) = \{d \in S^* \mid d \not\geq p\}$, $p \in S$, have a system of distinct representatives. Let $p_1, p_2, \ldots, p_k$ be k points in S, let $u = p_1 \vee \cdots \vee p_k$, and let U be the union of the sets $T(p_i)$. As before, if $u = 1$, then $U = S^*$ and we have $|U| = |S^*| \geq k$, applying Corollary 4.2. If $u < 1$, then $|U| = |S^*| - \beta^*(u)$, since every dual point $\geq u$ is contained in S^*. Suppose that $r(u) > 1$. If R denotes the set of elements covered by u which dominate an element of S, then Lemma 2.6 shows that $|R| \leq |S^*| - \beta^*(u)$. But $k \leq |R|$, by Corollary 4.2. Hence

$$k \leq |S^*| - \beta^*(u) = |U| .$$

To satisfy the conditions for an SDR, it remains to show that $|U| \geq k$ when $k = 1$ -- that is, $|T(p)| \geq 1$ for every $p \in S$. This can fail only if $|T(p)| = 0$, which says that $S^* = \{d \in D_L \mid d \geq p\}$. But S must contain another point $q \neq p$, by hypothesis, and hence

$$\{d \in D_L \mid d \geq q\} \subset \{d \in D_L \mid d \geq p\} ,$$

which implies that $q \geq p$, since L is a dual point lattice. This contradiction completes the proof.

Matching properties such as the one given by Corollary 4.2 are closely related to the Sperner property, as the following result shows.

Corollary 4.4: Let L be a finite geometric lattice, and suppose that $X \subset P_L \cup D_L$ is a set of non-comparable elements. Then $|X| \leq |D_L|$. In particular, the Sperner property holds for geometric lattices of dimension 3.

Proof: It is a standard result in the theory of bipartite graphs that such a property holds whenever a "matching" exists. The argument is as follows: let $X_D = X \cap D_L$, and $X_P = X \cap P_L$. Then $|X_P| \leq |D_L - X_D|$, by Corollary 4.2, since the set of dual points which contain an element of X_P is disjoint from X_D. Hence

$$|X| = |X_D| + |X_P| \leq |X_D| + |D_L - X_D| = |D_L|.$$

In the case when $r(L) = 3$, Theorem 1 shows that $|D_L| = \max_k \{w(k)\}$, and the Sperner property follows.

Thus the Sperner conjecture is true if we restrict our attention to points and dual points. The next corollary shows that non-comparable sets of maximum size ($|D_L|$) are essentially unique.

Corollary 4.5: Let L be a finite geometric lattice, and let $X \subset P_L \cup D_L$ be a non-comparable set with $|X| = |D_L|$. Then $X = D_L$ or (if $|P_L| = |D_L|$) $X = P_L$.

Proof: The result is trivial if $r(L) \leq 2$, so we may assume that $r(L) > 2$. Let $X_P = X \cap P_L$, and $X_D = X \cap D_L$. Since $|X| = |D_L|$, we have $|X_P| = |D_L - X_D|$, from the proof of Corollary 4.4. Denote by X_P^* the set of dual points containing elements of X_P. Since

$|X_P^*| \geq |X_P|$ (by Corollary 4.2), and since $X_P^* \cap X_D$ is empty, we must have $X_P^* = D_L - X_D$, and thus $|X_P^*| = |D_L - X_D| = |X_P|$. We cannot have $|X_P| = 1$, since then $|X_P^*| = 1$, contrary to the fact that every point is contained in at least two dual points. If $|X_P| = 0$, we are done, since then $X = X_D = D_L$. Thus it remains to show that $|X_P| > 1$ implies $X = P_L$. In this case, we can apply Corollary 4.3 and obtain an injection $f: X_P \longrightarrow D_L - X_D$ such that $f(p) \not\geq p$ for all p in X_P. Since $|X_P| = |D_L - X_D|$, f must be surjective. We also have $\alpha^*(f(p)) \leq \beta^*(p)$ for all p in X_P, as in the proof of Theorem 1, so that

$$\sum_{d \in D_L - X_D} \alpha^*(d) = \sum_{p \in X_P} \alpha^*(f(p)) \leq \sum_{p \in X_P} \beta^*(p) .$$

On the other hand,

$$\sum_{p \in X_P} \beta^*(p) \leq \sum_{d \in X_P^*} \alpha^*(d)$$

since $d \in X_P^*$ whenever $d \geq p$ and $p \in X_P$. Hence

$$\sum_{p \in X_P} \beta^*(p) = \sum_{d \in X_P^*} \alpha^*(d) .$$

This relation states that every point less than an element of X_P^* is itself in X_P. Now let $q \in P_L$. Since $|X_P| \neq 0$, by hypothesis, there exists a point $q_1 \in X_P$. Also, since $r(L) \geq 3$, there exists a $d \in D_L$ such that $d \geq q \vee q_1$. But then $d \in X_P^*$ and hence $q \in X_P$. Thus

$X_P = P_L$, which implies $X_P^* = D_L$. Hence X_D is empty and we have $X = X_P = P_L$.

V. EQUALITY

The inequalities given by Theorem 1 can be regarded as essentially independent statements about the bipartite graphs consisting of points and elements of a fixed rank. On the other hand, the main theorem of this chapter shows that the corresponding equalities have implications throughout the lattice.

Theorem 2: Let L be a finite geometric lattice of dimension $n > 1$. Then $w(1) = w(k)$ if and only if $k = n-1$ and L is modular.

Proof: As remarked in Chapter 1, it follows from a well-known theorem of R. P. Dilworth that $w(1) = w(n-1)$ for any finite, modular geometric lattice of dimension n. To prove the converse as stated above, it is first necessary to reduce the equality $w(1) = w(k)$ to the case where $k = n-1$. We isolate this result as a special lemma:

Lemma 5.1: Let L be a finite geometric lattice of dimension n, and let L_k be the geometric lattice obtained by identifying all elements in L of rank $\geq k$. If L_k is modular, then $k = 0, 1, 2,$ or n.

(In other words, L_k is modular only in trivial cases.)

Proof: Assume that $k \neq n$. Since L_k is modular, the interval sublattices $x/0$ are modular for each x in L of rank k. This implies that $\alpha(x) = \alpha^*(x)$ whenever $r(x) = k$. If we denote by $\beta_k^*(p)$ the number of elements in L of rank k containing p, we have $\beta(p) \leq \beta_k^*(p)$ for all $p \in P_L$, by Theorem 1. (Note that this assumes $k \neq n$.) Thus

$$\sum_{r(x)=k} \alpha(x) = \sum_{r(x)=k} \alpha^*(x) = \sum_{p \in P_L} \beta_k^*(p) \geq \sum_{p \in P_L} \beta(p) .$$

Since L_k is modular, we have $w(1) = w(k-1)$. By Corollary 4.2, there is a bijection g which maps P_L onto the elements of rank k-1 and has the property that $g(p) \geq p$ for all $p \in P_L$. Furthermore, we have $\beta(p) > \beta(g(p))$ if $k-1 > 1$, since β is a strictly decreasing function on L. Hence

$$\sum_{r(x)=k} \alpha(x) = \sum_{r(y)=k-1} \beta(y) = \sum_{p \in P_L} \beta(g(p)) < \sum_{p \in P_L} \beta(p) ,$$

in conflict with the previous inequality. Hence $k-1 \leq 1$ and the lemma follows.

Theorem 2 now depends only on showing that L is modular when $w(1) = w(n-1)$. For it follows from this that L_{i+1} is modular whenever $w(1) = w(i)$. If $1 < i < n$, Lemma 5.1 implies that $i = n-1$.

The main part of the proof now proceeds by induction on the dimension of L. The theorem is trivial when $r(L) \leq 2$, so we assume that L has dimension $n > 2$, and that the result holds for all lattices of smaller dimension.

From Theorem 1 and Lemma 2.5 we have the inequalities $\alpha^*(d) \leq \alpha(d) \leq \beta^*(p)$ and $\alpha^*(d) \leq \beta(p) \leq \beta^*(p)$, holding for all $p \in P_L$, $d \in D_L$, with $p \not\leq d$. Let $f\colon P_L \longrightarrow D_L$ be a bijection with $f(p) \not\geq p$ for all $p \in P_L$. (This is guaranteed by Corollary 4.1). Then

$$\sum_{p \in P_L} \beta^*(p) \geq \sum_{p \in P_L} \alpha^*(f(p)) = \sum_{d \in D_L} \alpha^*(d) .$$

Since the sums on the right and left are equal, it follows that

$\alpha^*(f(p)) = \beta^*(p)$ for all $p \in P_L$. Hence $\alpha^*(f(p)) = \alpha(f(p))$ and $\beta^*(p) = \beta(p)$ for all $p \in P_L$, by the above inequalities. Hence, by hypothesis, the intervals $1/p$ and $d/0$ are modular for all $p \in P_L$ and $d \in D_L$.

Suppose x and y are covered by $x \vee y$. If $x \vee y \neq 1$, then x and y cover $x \wedge y$ by the modularity of $x \vee y/0$. If $x \wedge y \neq 0$, the same result follows from the modularity of $1/x \wedge y$. Thus, to show that L is modular, it suffices to show that $x \wedge y \neq 0$ for all $x, y \in D_L$. But this will follow if we can show that $\alpha^*(d) = \beta^*(p)$ whenever $p \in P_L$, $d \in D_L$, and $p \not\leq d$. For, suppose that x and y are distinct dual points of L, and let p be a point such that $p \leq x$, $p \not\leq y$. Then, since $\alpha(y) = \alpha^*(y) = \beta^*(p)$, the map $z \longrightarrow z \vee p$ takes the set $\{a \mid a \prec y\}$ onto the set $\{b \in D_L \mid b \geq p\}$, by Lemma 2.4. Since $x \geq p$, there must be an $a \prec y$ such that $x = a \vee p$. It then follows that $x \wedge y = a > 0$, since $r(L) > 2$.

Thus, in order to complete the proof of Theorem 2, we need only show that if $1 \succ d_0 \not\geq p_0 \succ 0$, the mapping f of Corollary 4.1 can be chosen so that $f(p_0) = d_0$. For then we have $\alpha^*(d_0) = \alpha^*(f(p_0)) = \beta^*(p_0)$ as was shown earlier.

If $p \in P_L - \{p_0\}$, let $S(p) = \{d \in D_L \mid d \not\geq p\}$, and let $T(p) = \{d \in D_L - \{d_0\} \mid d \not\geq p\}$. We wish to find distinct representatives for the sets $T(p)$, $p \in P_L - \{p_0\}$. Let $p_1, p_2, \ldots, p_k$ be k points in $P_L - \{p_0\}$, and let

$$U_1 = \bigcup_1^k S(p_i) \quad \text{and} \quad U_2 = \bigcup_1^k T(p_i).$$

Since the S(p)'s have an SDR, we have $|U_1| \geq k$, so that $|U_2| \geq k-1$. If $|U_2| = k-1$, then $d_0 \in U_1$ and $|U_1| = k$. We show that this leads either to a conflict or to the desired result.

Suppose that $|U_2| = k-1$, and let $u = p_1 \vee p_2 \vee \cdots \vee p_k$. Note that $d_0 \not\geq u$, for otherwise $d_0 \notin U_1$ and $|U_2| = |U_1| \geq k$. Also $u \neq 1$, since if $u = 1$, then $U_1 = D_L$ and $|U_2| = |U_1| - 1 = |P_L| - 1 \geq k$, contrary to assumption. Hence $u/0$ is contained in a proper interval of L, and is modular. Thus $\alpha^*(u) = \alpha(u)$. In fact, $\alpha^*(u) = \alpha(u) = k$, for if $\alpha^*(u) > k$ we have

$$\left| \bigcup_{q \leq u} S(q) \right| = |U_1| = k < \alpha^*(u)$$

and the SDR condition for the S(q)'s is violated. Since $\alpha(u) = k = |D_L| - \beta^*(u)$, the map $y \longrightarrow d_y$ of Lemma 2.6 maps $\{y \mid y \prec u\}$ onto $\{d \in D_L \mid d \not\geq u\}$. Thus $d \wedge u \prec u$ for all $d \not\geq u$, and $d_1 \neq d_2$ implies that $d_1 \wedge u \neq d_2 \wedge u$ for all $d_1, d_2 \not\geq u$. The proof now splits into several cases.

Case 1. If $r(u) > 2$, then for any pair of distinct dual points d_1 and d_2, $d_1 \wedge u$ and $d_2 \wedge u$ are covered by or equal to u, and hence $r(d_1 \wedge d_2) \geq r((d_1 \wedge u) \wedge (d_2 \wedge u)) \geq 1$ by the modularity of $u/0$. Hence $d_1 \wedge d_2 > 0$ and the theorem follows.

Case 2. If $r(u) = 1$, we have $k = 1$, $u = p_1$, and $\beta^*(p_1) = |D_L| - 1$. We are assuming that $|U_2| = |T(p_1)| = k-1 = 0$, so that $p_1 \not\leq d_0$ and $|S(p_1)| = 1$. Thus the function f of Corollary 4.1 must map p_1 onto d_0. Hence $\alpha^*(d_0) = \beta^*(p_1) = |D_L| - 1 = |P_L| - 1$. Since $p_1 \not\leq d_0$ and $p_1 \neq p_0$, it follows that $d_0 \geq p_0$, which contradicts the original assumption.

<u>Case 3</u>. If $r(u) = 2$, we show again that $d_0 \geq p_0$. The function f given by Corollary 4.1 must map the points less than u onto the dual points $\not\geq u$, since $\alpha(u) = k = |D_L| - \beta^*(u)$. Since $d_0 \not\geq u$, there must be a point $q \prec u$ with $f(q) = d_0$. Let d_1 be a dual point such that $q = d_1 \wedge u$. Then $\beta^*(q) = \beta^*(d_1 \wedge u) \geq |D_L| - k + 1$, since $\beta^*(u) = |D_L| - k$. If $\beta^*(d_1 \wedge u) > |D_L| - k + 1$, there exists a dual point d_2 distinct from d_1 such that $d_2 \geq d_1 \wedge u$ and $d_2 \not\geq u$. But then $d_1 \wedge u = d_2 \wedge u$, which we have shown cannot hold. Hence

$$\beta^*(q) = \alpha^*(f(q)) = \alpha^*(d_0) = |D_L| - k + 1 = |P_L| - k + 1.$$

Now d_0 contains at most one of the points less than u, since, if it contained two, it would contain their join, which is u. Thus d_0 must contain all of the $|P_L| - k$ points not less than u. These include p_0, so that $d_0 \geq p_0$, contrary to assumption.

We may now conclude that the sets $T(p)$, $p \neq p_0$, have a system of distinct representatives. Thus there exists a function $g\colon P_L \longrightarrow D_L$ with the property that $g(p_0) = d_0$ and $g(p) \not\geq p$ for all $p \in P_L$. As shown above, this completes the proof of Theorem 2.

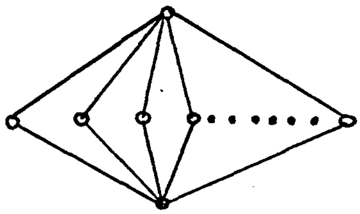
VI GEOMETRIC LATTICES OF DIMENSION 3.

Let L be a geometric lattice of dimension 3. If we associate with each dual point d of L the set $S_d = \{p \in P_L \mid p \leq d\}$, then the following properties hold: (i) every pair of points in P_L are contained in exactly one S_d, (ii) each S_d contains at least two points, and (iii) no S_d contains all the points. Conversely, any collection of subsets $S_1, \ldots, S_m$ of a set T satisfying (i) - (iii) determines a geometric lattice of dimension 3. Condition (i) insures that the system (with a 1 and 0 adjoined) represents a lattice, and also that the lattice is semimodular. Condition (ii) insures that every element of rank 2 is a join of points, and condition (iii) insures that the unit element is a join of points.

A theorem of deBruijn and Erdős [4] makes the following assertions about configurations $S_1, S_2, \ldots, S_m$, T satisfying conditions (i) - (iii): (1) $m \geq |T|$, and (2) if $m = |T|$, then the configuration is either a projective plane or one of the degenerate planes represented by

$$S_1 = \{2, 3, \ldots, m\}, \quad S_2 = \{1, 2\}, \quad S_3 = \{1, 3\}, \ldots, \quad S_m = \{1, m\}.$$

Assertion (1) is equivalent to our Theorem 1 for geometric lattices of dimension 3. Assertion (2) follows from Theorem 2 by means of the Birkhoff-Menger decomposition theorem for complemented, modular lattices [2; Ch. 4, § 7]. This theorem states that a finite-dimensional, complemented, modular lattice can be expressed as a direct product of simple lattices which are either two-element chains or projective geometries (possibly degenerate). Thus a finite, modular geometric lattice of dimension 3 must be either a projective plane or the product of a 2-element chain with a lattice of the following type:



The latter possibility corresponds to the degenerate case of the deBruijn - Erdős theorem.

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