LARGE PLANE DEFORMATIONS OF THIN ELASTIC SHEETS OF NEO-HOOKEAN MATERIAL

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ABSTRACT

Large plane deformations of thin elastic sheets of neo-Hookean material are considered and a method of successive substitutions is developed to solve problems within the two-dimensional theory of finite plane stress. The first approximation is determined by linear boundary value problems on two harmonic functions, and it is approached asymptotically at very large extensions in the plane of the sheet. The second and higher approximations are obtained by solving Poisson equations. The method requires modification when the membrane has a traction-free edge.

Several problems are treated involving infinite sheets under uniform biaxial stretching at infinity. First approximations are obtained when a circular or elliptic inclusion is present and when the sheet has a circular or elliptic hole, including the limiting cases of a line inclusion and a straight crack or slit. Good agreement with exact solutions is found for circularly symmetric deformations. Other examples discuss the stretching of a short wide strip, the deformation near a boundary corner which is traction-free, and the application of a concentrated load to a boundary point.

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I. INTRODUCTION

A general theory of plane stress for large elastic deformations of isotropic materials has been developed by Adkins, Green and Nicholas [1] (see also [2, 3]). The theory applies to a thin plane sheet which is stretched by forces in its plane so that it remains plane after deformation, the major surfaces of the sheet being free from traction. Under these conditions a good approximation is obtained if the transverse components of stress associated with elements parallel to the middle plane are assumed to be zero and the variations of the principal extension ratios throughout the thickness of the sheet are neglected. The deformation and stress resultants are then determined by the deformation of the middle surface of the sheet and the theory is reduced to two-dimensional form. The one-dimensional case of circular symmetry was treated earlier by Rivlin and Thomas [4] in order to obtain theoretical solutions for a plane sheet with a circular hole stretched uniformly at infinity. The ordinary differential equation governing the deformation was integrated numerically and the results were compared with experiment. Recently Yang [5] has considered approximate and exact solutions for the symmetric problems of a circular hole, a rigid circular inclusion and inward radial stretching of a sheet with a circular inner boundary.

The two-dimensional equations of the general theory are difficult to solve exactly and a method of successive approximations, with solutions expressed as power series in a real parameter ϵ, has been used to obtain first and second order solutions for unsymmetric problems [1, 2, 3]. Quantities in the first order or infinitesimal solution are $O(\epsilon)$ as $\epsilon \to 0$ and they provide the asymptotic form of the solution for vanishingly small strains. Thus the application of the first and second order solutions is limited to a range of deformations near the undeformed state and is inadequate at large strains. In this work we assume that the elastic material of the plane membrane is incompressible and has the neo-Hookean form for the strain energy function and we develop a method of successive substitutions for the solution of problems involving large strains. The first approximation of the present theory is the asymptotic form of the solution for infinitely large strains in contrast to the method of [1, 2, 3], although the two methods are similar in character. When the extension ratios in the plane of the sheet are appreciably greater than unity, the transverse extension ratio λ is small and the first approximation is obtained by neglecting all terms involving λ in the differential equations and boundary conditions. The coordinates (y_1, y_2) of the deformed state are, according to the first approximation, harmonic functions of the initial coordinates (x_1, x_2) and they satisfy linear boundary conditions. The second approximation is determined by a Poisson equation with the non-linear terms in the differential equations and boundary conditions estimated from the first approximation, and the procedure is repeated for the higher order approximations. In general, the determination of approximations beyond the first will involve considerable labor.

A summary of the basic formulae and equations of the theory of plane elastic membranes is given in Section II. Detailed derivation of these formulae and an exposition of the theory in a general system of coordinates can be found in [1,2], but it is sufficient for our purposes to develop the theory in cartesian coordinates. A method of successive substitutions is outlined in Section III for the particular case of a neo-Hookean material. It is shown that when the principal extension ratios in the plane of the sheet are both of order μ for large μ, the first approximation determines the solution to within terms of order $O(\mu^{-5})$ for large μ; the second approximation determines the solution to within terms of $O(\mu^{-11})$. For principal extension ratios of different orders, the accuracy to which the first approximation determines the solution varies accordingly.

Section IV deals with several problems involving infinite membranes with circular or elliptic inclusions, and the first approximations are obtained. The examples have their analogues in two-dimensional flow of a perfect fluid and two-dimensional electrostatics. The problem of a rigid inclusion in a membrane, for example, is related to flow around a cylinder and the examples on material inclusions correspond to problems of dielectrics in an electric field. For circularly symmetric problems the second approximation can be obtained without difficulty and exact solutions are available through numerical integration. Good agreement was found even for moderate deformations between the analytical approximate solutions and the exact numerical solutions in the problem of a rigid circular inclusion with uniform stretching at infinity.

When the membrane has an edge which is traction free, the method of successive substitution developed in Section III breaks down after the first approximation because the thickness ratio λ calculated from the first approximation becomes infinite as the traction-free edge is approached. Section V provides an alternative way to calculate a better first approximation to the thickness ratio which remains finite at the traction-free edges. Two stress functions are introduced in an intermediate step between the first and second approximations. When second and higher approximations are not required, the modified method is still useful in that it gives accurate values for the thinning of the sheet up to and including the traction-free edges.

Several simple examples involving membranes with traction-free edges are considered in Section VI. In the case of the radially symmetric deformation of an infinite membrane with a circular hole, comparison between approximate and exact numerical solutions is made. Agreement between the solutions is again good for moderate deformations but the agreement is not as good as in the problem of the rigid circular inclusion. Example 3 of Section VI treats a finite membrane under a deformation which is close to a homogeneous state of pure shear. Making use of this circumstance, we obtain a good estimate for the second approximation to the solution even though the problem is two-dimensional. The last example of the section treats the deformation near a boundary which has a corner which is either traction free or under the action of a concentrated load at the vertex of the corner.

The method used here for neo-Hookean materials may be used with obvious modification for materials which have strain energy functions close to the neo-Hookean form over the range of deformation involved in the problem under consideration. The modification occurs in the second approximation because of the additional non-linear terms in the differential equations and in traction boundary conditions arising from the departure from the neo-Hookean form. As for the neo-Hookean materials, the non-linear terms could be estimated by using the first approximation. It is known that the use of the neo-Hookean form can lead to appreciable error for rubber in deformations with extension ratios greater than two or three (see [4, 6], for example). The comparisons with exact solutions for symmetric problems indicate that the first approximations can give results accurate in the range of moderate deformations where the neo-Hookean form provides a fair approximation for rubber-like materials. For larger deformations, where, for example, the Mooney form may be more appropriate, the results based on the first approximation for the neo-Hookean material can still be of value in indicating the main features and characteristics to be expected in the actual deformation of real materials.

II. BASIC EQUATIONS FOR FINITE PLANE STRESS

The theory of plane stress for finite deformations of elastic sheets is summarized in this section. A detailed derivation of the basic equations and formulae can be found in [1].

We suppose that in its initial state the body is a plane sheet of homogeneous elastic material bounded by the surfaces $x_3 = \pm h_0/2$, where (x_1, x_2, x_3) are the coordinates of a particle of the sheet referred to a rectangular cartesian reference frame. The thickness h_0 may depend on x_1, x_2. The sheet undergoes a finite deformation symmetric about the middle plane $x_3 = 0$ and we denote by (y_1, y_2, y_3) the coordinates after deformation of a particle which was at the point (x_1, x_2, x_3) in the unstrained state. The middle plane in the deformed state is $y_3 = 0$ and the major surfaces of the sheet after deformation are given by $y_3 = \pm h/2$, where h is, in general, a function of y_1, y_2. We shall use indicial notation and the summation convention, with Latin indices taking the values 1, 2, 3 and Greek indices the values 1, 2.

In the absence of body forces, the equations of equilibrium are

$$\frac{\partial t_{ij}}{\partial y_i} = 0 \quad ,$$

where t_{ij} are the symmetric components of the stress tensor referred to the rectangular cartesian reference frame. The resultant load on a normal section of the deformed sheet through a curve drawn in the middle plane can be expressed in terms of stress resultants $T_{\alpha\beta}$ defined by

$$T_{\alpha\beta} = \int_{-h/2}^{h/2} t_{\alpha\beta}\, dy_3 \quad .$$

Under the assumption that the major surfaces of the membrane are free from traction, equilibrium requires

$$\frac{\partial T_{\alpha\beta}}{\partial y_\alpha} = 0 \quad .$$

The stress resultants $T_{\alpha\beta}$ may also be considered as functions of the initial coordinates x_α of a particle at y_α in the deformed middle plane and with the identity

$$\frac{\partial}{\partial x_\gamma}\left(J \frac{\partial x_\gamma}{\partial y_\alpha}\right) = 0 \quad ,$$

where

$$J = \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} \quad ,$$

the equilibrium equations become

$$\frac{\partial}{\partial x_\gamma}\left(J \frac{\partial x_\gamma}{\partial y_\alpha} T_{\alpha\beta}\right) = 0 \quad . \qquad (2.1)$$

The load resultant dL_α on a normal section through a line element ds of a curve drawn in the middle plane is given by

$$dL_\alpha = T_{\alpha\beta} n_\beta \, ds \quad , \qquad (2.2)$$

where n_α is the unit normal to the curve. Since

$$n_\beta \, ds = J \frac{\partial x_\gamma}{\partial y_\beta} n^\circ_\gamma \, ds^\circ \quad ,$$

where ds°, n°_α refer to the undeformed state, (2.2) yields

$$dL_\alpha = J \frac{\partial x_\gamma}{\partial y_\beta} T_{\alpha\beta} n^\circ_\gamma ds^\circ \quad . \tag{2.3}$$

In order to relate the stresses to the deformation of the middle surface we assume that the displacement gradients $y_{i,k}$ throughout the thickness are determined with sufficient accuracy for our purposes by their middle plane values. Because of the symmetry of the deformation we assume in particular that

$$\frac{\partial y_3}{\partial x_\alpha} = 0 \quad , \quad \frac{\partial y_\alpha}{\partial x_3} = 0 \quad ,$$

and we shall write

$$\frac{\partial y_3}{\partial x_3} = \lambda \quad ,$$

where $\lambda(x_1, x_2)$ is the extension ratio in the direction normal to the sheet.

For a homogeneous isotropic elastic material, the strain energy W per unit volume of the undeformed body is a function of three independent strain invariants I_1, I_2, I_3 which may be taken to be

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \quad , \quad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2 \quad ,$$

where λ_1, λ_2 and λ_3 are the three principal extension ratios. We shall identify λ_3 with the direction normal to the sheet so that $\lambda_3 = \lambda$ and λ_1, λ_2 are given by

$$\lambda_1^2 + \lambda_2^2 = \frac{\partial y_\alpha}{\partial x_\beta} \frac{\partial y_\alpha}{\partial x_\beta} = K \quad , \quad \lambda_1 \lambda_2 = \left| \frac{\partial y_\alpha}{\partial x_\beta} \right| = J \quad . \tag{2.4}$$

The non-zero stress components are given by

$$t_{\alpha\beta} = (\Phi + \Psi I_1)\frac{\partial y_\alpha}{\partial x_\gamma}\frac{\partial y_\beta}{\partial x_\gamma} - \Psi\frac{\partial y_\alpha}{\partial x_\gamma}\frac{\partial y_\eta}{\partial x_\gamma}\frac{\partial y_\beta}{\partial x_\delta}\frac{\partial y_\eta}{\partial x_\delta} + p\delta_{\alpha\beta} \quad ,$$

$$t_{33} = (\Phi + \Psi I_1)\lambda^2 - \Psi\lambda^4 + p \quad ,$$

where

$$\Phi = \frac{2}{I_3^{\frac{1}{2}}}\frac{\partial W}{\partial I_1} \quad , \quad \Psi = \frac{2}{I_3^{\frac{1}{2}}}\frac{\partial W}{\partial I_2} \quad , \quad p = 2I_3^{\frac{1}{2}}\frac{\partial W}{\partial I_3} \quad .$$

The condition that the major surfaces of the sheet be free from traction is satisfied approximately by requiring t_{33} to be zero, that is

$$p = \Psi\lambda^4 - (\Phi + \Psi I_1)\lambda^2 \tag{2.5}$$

and this equation serves to determine λ in terms of λ_1, λ_2. Substituting for p in the expressions for $t_{\alpha\beta}$ and integrating over the thickness we obtain

$$\begin{aligned} T_{\alpha\beta} &= h\,t_{\alpha\beta} = h_o\lambda\, t_{\alpha\beta} \\ &= h_o\lambda\Big\{(\Phi+\Psi I_1)\frac{\partial y_\alpha}{\partial x_\gamma}\frac{\partial y_\beta}{\partial x_\gamma} - \Psi\frac{\partial y_\alpha}{\partial x_\gamma}\frac{\partial y_\eta}{\partial x_\gamma}\frac{\partial y_\beta}{\partial x_\delta}\frac{\partial y_\eta}{\partial x_\delta} \\ &\qquad + [\Psi\lambda^4 - (\Phi + \Psi I_1)\lambda^2]\,\delta_{\alpha\beta}\Big\} \quad . \end{aligned}$$

The same equation applies when the material is incompressible, but now λ is determined instead by

$$\lambda = \frac{1}{J} = \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} \quad . \tag{2.6}$$

The strain energy W depends on I_1, I_2 only (since $I_3 = 1$) and

$$\Phi = 2\,\frac{\partial W}{\partial I_1} \quad , \qquad \Psi = 2\,\frac{\partial W}{\partial I_2} \quad .$$

Alternatively we may write

$$T_{\alpha\beta} = \frac{1}{J}\,\frac{\partial U}{\partial y_{\beta,\gamma}}\,\frac{\partial y_\alpha}{\partial x_\gamma} \quad , \tag{2.7}$$

where U is the strain energy per unit area of the middle surface and is given by

$$U = h_o W(y_{\alpha,\beta}\,;\lambda) = U(y_{\alpha,\beta}) \quad ,$$

in which λ has been expresssed in terms of $y_{\alpha,\beta}$ through (2.5) or (2.6). The strain energy U depends on $y_{\alpha,\beta}$ through $y_{\gamma,\alpha}y_{\gamma,\beta}$ only. For an isotropic material, U is a symmetric function of λ_1, λ_2 and we can write

$$U = U(K, J) \quad .$$

With (2.7) the equilibrium equations (2.1) become

$$\frac{\partial}{\partial x_\alpha}\left(\frac{\partial U}{\partial y_{\beta,\alpha}}\right) = 0 \quad .$$

For the most part we shall confine our discussion to an incompressible material which has the neo-Hookean form for the strain energy function

$$W = C_1(I_1 - 3) \quad ,$$

where C_1 is a material constant. The stress resultants are then

$$T_{\alpha\beta} = 2h_o C_1 \lambda\left(\frac{\partial y_\alpha}{\partial x_\gamma}\,\frac{\partial y_\beta}{\partial x_\gamma} - \lambda^2\delta_{\alpha\beta}\right) \quad . \tag{2.8}$$

and if T_1 and T_2 denote the principal stress resultants in the x_1 - and x_2-directions respectively, we have

$$T_1 = 2h_o C_1 \frac{\lambda_1}{\lambda_2}\left(1 - \frac{1}{\lambda_1^4 \lambda_2^2}\right) \quad , \qquad T_2 = 2h_o C_1 \frac{\lambda_2}{\lambda_1}\left(1 - \frac{1}{\lambda_2^4 \lambda_1^2}\right) .$$

With (2.8) the equilibrium equations (2.1) reduce to

$$\frac{\partial}{\partial x_\gamma}\left(h_o \frac{\partial y_\beta}{\partial x_\gamma}\right) - J \frac{\partial x_\gamma}{\partial y_\beta} \frac{\partial}{\partial x_\gamma}(h_o \lambda^3) = 0 \quad ,$$

or

$$\frac{\partial}{\partial x_\gamma}\left(h_o \frac{\partial y_\beta}{\partial x_\gamma}\right) - J \frac{\partial}{\partial y_\alpha}(h_o \lambda^3) = 0 \quad .$$

For constant initial thickness h_o we have

$$\frac{\partial}{\partial x_\gamma}\left(\frac{\partial y_\beta}{\partial x_\gamma}\right) - 3\lambda \frac{\partial \lambda}{\partial y_\beta} = 0 \quad ,$$

or

$$\nabla^2 y_\beta - \frac{\partial}{\partial y_\beta}\left(\frac{3}{2} \lambda^2\right) = 0 \quad , \tag{2.9}$$

where ∇^2 is the two-dimensional Laplace operator. Equations (2.9) can also be written in the form

$$\left.\begin{aligned} \nabla^2 y_1 &= \frac{\partial \lambda^3}{\partial x_1} \frac{\partial y_2}{\partial x_2} - \frac{\partial \lambda^3}{\partial x_2} \frac{\partial y_2}{\partial x_1} \quad , \\ \nabla^2 y_2 &= \frac{\partial y_1}{\partial x_1} \frac{\partial \lambda^3}{\partial x_2} - \frac{\partial y_1}{\partial x_2} \frac{\partial \lambda^3}{\partial x_1} \quad , \end{aligned}\right\} \tag{2.10}$$

and

$$\frac{\partial y_\beta}{\partial \theta_\alpha} \nabla^2 y_\beta - \frac{\partial}{\partial \theta_\alpha}\left(\frac{3}{2} \lambda^2\right) = 0 \quad ,$$

where θ_α are arbitrary independent variables.

Substituting (2.8) into (2.3) we obtain

$$dL_\alpha = 2h_o C_1 \left(\frac{\partial y_\alpha}{\partial n^\circ} - \lambda^2 \frac{\partial x_\rho}{\partial y_\alpha} n^\circ_\rho \right) ds^\circ \quad .$$

If we use the results

$$\frac{\partial x_\rho}{\partial y_1} n^\circ_\rho \, ds^\circ = \lambda \frac{\partial y_2}{\partial s^\circ} ds^\circ \quad , \quad \frac{\partial x_\rho}{\partial y_2} n^\circ_\rho \, ds^\circ = -\lambda \frac{\partial y_1}{\partial n^\circ} ds^\circ$$

we can write the load components dL_α as

$$\left.\begin{aligned} dL_1 &= 2h_o C_1 \left(\frac{\partial y_1}{\partial n^\circ} - \lambda^3 \frac{\partial y_2}{\partial s^\circ} \right) ds^\circ \quad , \\ dL_2 &= 2h_o C_1 \left(\frac{\partial y_2}{\partial n^\circ} + \lambda^3 \frac{\partial y_1}{\partial s^\circ} \right) ds^\circ \quad , \end{aligned}\right\} \tag{2.11}$$

where the s°-direction is obtained from the n°-direction by an anti-clockwise rotation of amount $\pi/2$. For the special case of a traction-free boundary C° we must have

$$\frac{\partial y_1}{\partial n^\circ} = \lambda^3 \frac{\partial y_2}{\partial s^\circ} \quad , \quad \frac{\partial y_2}{\partial n^\circ} = -\lambda^3 \frac{\partial y_1}{\partial s^\circ} \quad ,$$

on the boundary C° for a neo-Hookean material.

III. SUCCESSIVE SUBSTITUTIONS

As can be seen from (2.10) and (2.11), both the differential equations and the traction boundary conditions are non-linear so that exact solutions will not always be easy to determine. Since the non-linearity in (2.10) and (2.11) comes solely from terms involving λ, a natural first approximation when $\lambda \ll 1$ is obtained by neglecting all such terms in the equations. This is equivalent to using for the strain energy the form

$$U^{(1)} = h_o C_1 (K - 2) \quad , \qquad (3.1)$$

rather than the exact form

$$U = h_o C_1 (K + \lambda^2 - 3) \quad .$$

Superscripted quantities here stand for approximate values, with (1) for the first, (2) for the second and so on.

If λ_1, λ_2 are of the order of μ for large μ throughout the sheet, where μ is a parameter which measures the amount of deformation, then

$$U^{(1)} = O(\mu^2) \quad ,$$

and

$$U = U^{(1)} + O(\mu^{-4}) \quad .$$

If the first approximation provides derivatives $y^{(1)}_{\alpha,\beta}$ which are $O(\mu)$ for large μ, correction terms of order $O(\mu^{-5})$ added to these derivatives will change K and therefore $U^{(1)}$ by terms of order $O(\mu^{-4})$. It is therefore to be expected that, on the average, the first approximation determines $y_{\alpha,\beta}$ to within terms which are $O(\mu^{-5})$ for large μ so that we will have

$$\left.\begin{aligned} y_{\alpha,\beta} &= y^{(1)}_{\alpha,\beta} + O(\mu^{-5}) \\ \text{when} \qquad & \\ y^{(1)}_{\alpha,\beta} &= O(\mu) \quad . \end{aligned}\right\} \tag{3.2}$$

If $\lambda_1 = O(1)$ and $\lambda_2 = O(\mu)$, the approximation will be less good and a similar argument leads to

$$y_{\alpha,\beta} = y^{(1)}_{\alpha,\beta} + O(\mu^{-2}) \quad .$$

When $\lambda_1 = O(\mu^{-\frac{1}{2}})$ and $\lambda_2 = O(\mu)$, so that each element of the sheet is strained close to a state of simple extension, we will have

$$y_{\alpha,\beta} = y^{(1)}_{\alpha,\beta} + O(\mu^{-\frac{1}{2}}) \quad .$$

In this case some of the derivatives $y^{(1)}_{\alpha,\beta}$ can be of the same order, $O(\mu^{-\frac{1}{2}})$, as the correction terms.

Once the first approximation to the solution is known, higher approximations can be obtained by a method of successive substitutions as outlined below. We remark that the first approximation to y_α is exact for a sheet with strain energy U given by (3.1). Such a sheet is isotropic but is stressed in all-around tension in its reference state.

<u>First Approximation</u> Setting $\lambda = 0$ in (2.10) and (2.11) we find that the first approximation satisfies

$$\left.\begin{aligned} \nabla^2 y_1^{(1)} &= 0 \ , \\ \nabla^2 y_2^{(1)} &= 0 \ , \end{aligned}\right\} \text{ in } A^\circ ,$$

with the boundary conditions

$$\left.\begin{aligned} \frac{\partial y_1^{(1)}}{\partial n^\circ}\, ds^\circ &= \frac{1}{2h_o C_1}\, dL_1^* \ , \\ \frac{\partial y_2^{(1)}}{\partial n^\circ}\, ds^\circ &= \frac{1}{2h_o C_1}\, dL_2^* \ , \end{aligned}\right\} \text{on } C_T^\circ \ , \qquad (3.3)$$

and

$$y_\alpha^{(1)} = y_\alpha^* \quad \text{on} \quad C_D^\circ \ .$$

Here A° is the middle plane of the unstrained sheet with boundary C°, C_T° is that part of C° where traction components dL_α^* are prescribed and C_D° is that part where deformed locations y_α^* are given. When A° is infinite, conditions at infinity must also be imposed. For example, if the sheet extends to infinity in all directions and if it is under uniform biaxial extension at infinity with principal extension ratios μ_1 and μ_2 along the x_1- and x_2-axes respectively, the appropriate conditions are, for zero rotation at infinity,

$$\left.\begin{aligned} \frac{\partial y_1^{(1)}}{\partial x_1} &= \mu_1 - \frac{L_1^* \cos\theta}{4\pi h_o C_1 r} + o\left(\frac{1}{r}\right) , \quad \frac{\partial y_1^{(1)}}{\partial x_2} = - \frac{L_1^* \sin\theta}{4\pi h_o C_1 r} + o\left(\frac{1}{r}\right) \\ \frac{\partial y_2^{(1)}}{\partial x_1} &= - \frac{L_2^* \cos\theta}{4\pi h_o C_1 r} + o\left(\frac{1}{r}\right) , \quad \frac{\partial y_2^{(1)}}{\partial x_2} = \mu_2 - \frac{L_2^* \sin\theta}{4\pi h_o C_1 r} + o\left(\frac{1}{r}\right) \end{aligned}\right\} \begin{aligned} &\text{as} \\ &r \to \infty \end{aligned}$$

(3.4)

where L_α^* are the components of the resultant of all external forces acting on the interior boundaries of the membrane and (r, θ) are polar coordinates. In the case when the resultant force is zero, the conditions at infinity become

$$\left.\begin{aligned} \frac{\partial y_1^{(1)}}{\partial x_1} &= \mu_1 + o\left(\frac{1}{r}\right) , \quad \frac{\partial y_1^{(1)}}{\partial x_2} = o\left(\frac{1}{r}\right) , \\ \frac{\partial y_2^{(1)}}{\partial x_1} &= o\left(\frac{1}{r}\right) , \quad \frac{\partial y_2^{(1)}}{\partial x_2} = \mu_2 + o\left(\frac{1}{r}\right) , \end{aligned}\right\} \text{ as } r \to \infty , \tag{3.5}$$

and the logarithmic terms in $y_\alpha^{(1)}$ are excluded.

We note also that to the first approximation, the principal stress resultants are given by

$$T_1^{(1)} = 2h_o C_1 \frac{\lambda_1}{\lambda_2} , \qquad T_2^{(1)} = 2h_o C_1 \frac{\lambda_2}{\lambda_1} ,$$

which are exact for the strain energy function $U^{(1)}$.

<u>Second Approximation</u> In order to get a second approximation $y_\alpha^{(2)}$ for y_α, we use the first approximation $y_\alpha^{(1)}$ to estimate the non-linear terms in (2.10) and (2.11). With $\lambda^{(1)}$ defined by

$$\lambda^{(1)} = \frac{1}{J^{(1)}} = \left[\frac{\partial(y_1^{(1)}, y_2^{(1)})}{\partial(x_1, x_2)}\right]^{-1} , \tag{3.6}$$

the second approximation solution $y_\alpha^{(2)}$ satisfies

$$\left.\begin{aligned}\nabla^2 y_1^{(2)} &= \frac{\partial}{\partial x_1}[\lambda^{(1)}]^3 \frac{\partial y_2^{(1)}}{\partial x_2} - \frac{\partial}{\partial x_2}[\lambda^{(1)}]^3 \frac{\partial y_2^{(1)}}{\partial x_1} \ , \\ \nabla^2 y_2^{(2)} &= \frac{\partial y_1^{(1)}}{\partial x_1} \frac{\partial}{\partial x_2}[\lambda^{(1)}]^3 - \frac{\partial y_1^{(1)}}{\partial x_2} \frac{\partial}{\partial x_1}[\lambda^{(1)}]^3 \ , \end{aligned}\right\} \text{ in } A^\circ \qquad (3.7)$$

with the boundary conditions

$$\left.\begin{aligned}\frac{\partial y_1^{(2)}}{\partial n^\circ} ds^\circ &= \frac{1}{2h_o C_1} dL_1^* + [\lambda^{(1)}]^3 \frac{\partial y_2^{(1)}}{\partial s^\circ} ds^\circ \ , \\ \frac{\partial y_2^{(2)}}{\partial n^\circ} ds^\circ &= \frac{1}{2h_o C_1} dL_2^* - [\lambda^{(1)}]^3 \frac{\partial y_1^{(1)}}{\partial s^\circ} ds^\circ \ , \end{aligned}\right\} \text{ on } C_T^\circ \qquad (3.8)$$

and

$$y_\alpha^{(2)} = y_\alpha^* \qquad \text{on} \quad C_D^\circ \ .$$

The process of successive substitution can be repeated, the approximation $y_\alpha^{(n+1)}$ being determined as the solution to a Poisson equation with inhomogeneous terms in the equation and boundary conditions determined by $y_\alpha^{(n)}$ and the boundary data. The solution $y_\alpha^{(n+1)}$, if it exists, will be unique provided C_D° is non-zero. If $y_\alpha^{(1)}$ and its derivatives are $O(\mu)$ everywhere for μ large and the Jacobian $J^{(1)}$ is such that $\lambda^{(1)} = O(\mu^{-2})$ everywhere, the difference $y_\alpha^{(2)} - y_\alpha^{(1)}$ satisfies a Poisson boundary value problem with inhomogeneous terms which are $O(\mu^{-5})$, in agreement with the earlier estimate (3.2) on the order of error involved in the first approximation. Assuming that the solution $y_\alpha^{(2)} - y_\alpha^{(1)}$ and its derivatives are

$O(\mu^{-5})$, the boundary value problem for the difference $y_\alpha^{(3)} - y_\alpha^{(2)}$ will involve inhomogeneous terms of $O(\mu^{-11})$ for large μ, and so on. Thus a related approach would be to assume that, for large enough μ, the functions y_α/μ can be expanded in an absolutely convergent power series in μ^{-6}, with coefficients which are twice differentiable functions of (x_1, x_2).

When $\lambda_1 = O(1)$ and $\lambda_2 = O(\mu)$ for large μ, the corresponding estimates for $y_\alpha^{(2)} - y_\alpha^{(1)}$ and $y_\alpha^{(3)} - y_\alpha^{(2)}$ are $O(\mu^{-2})$ and $O(\mu^{-5})$, respectively. For a smooth enough first approximation $y_\alpha^{(1)}$ and a smooth enough region A°, it is to be expected, in this case and in the previous case, that the process will converge when μ is large enough. However the convergence of the method is not so apparent when a large region of the sheet is in a state close to simple extension so that the first approximation $y_\alpha^{(1)}$ involves principal extension ratios

$$\lambda_1^{(1)} = O(\mu^{-\frac{1}{2}}) \quad , \quad \lambda_2^{(1)} = O(\mu)$$

for large μ. In this case we will have $\lambda^{(1)} = O(\mu^{-\frac{1}{2}})$ and the terms in (3.7), (3.8) involving $y_\alpha^{(1)}$ are $O(\mu^{-\frac{1}{2}})$. Since the difference $y_\alpha^{(2)} - y_\alpha^{(1)}$ will then be $O(\mu^{-\frac{1}{2}})$, the error in $y_\alpha^{(1)}$ can be of the same order and therefore $y_\alpha^{(1)}$ may not determine the non-linear terms in (3.7), (3.8) correct to $O(\mu^{-\frac{1}{2}})$.

A difficulty arises with the method described in this section when a portion of the boundary is traction free. The first approximation then has a Jacobian $J^{(1)}$ which goes to zero as the unloaded boundary is approached. The terms involving $y_\alpha^{(1)}$ in the equations

(3.7), (3.8) for $y_\alpha^{(2)}$ are then singular on the unloaded part of C° and in fact the singularity is non-integrable in that the solution $y_\alpha^{(2)}$ cannot remain finite. A modification of the method which avoids this difficulty is given later in Section V.

For illustration, several simple problems involving infinite plane sheets are considered in the next section. At infinity the membrane is in a state of uniform biaxial extension with principal extension ratios μ_1 and μ_2, and the membrane contains an inclusion or has an internal boundary which is held fixed. With the method of this section, the first approximations can be readily obtained, either directly or after a conformal transformation. In principle the process can be repeated for the second and higher approximations, but the computations become increasingly involved so that in most cases only the first approximation is derived. For axisymmetric deformations, the symmetry of the problem allows the second approximation to be determined with little difficulty; in fact exact solutions can be determined in this case by numerical integration of an ordinary differential equation. In the axisymmetric deformation of an infinite sheet with a clamped circular hole, first and second approximations are compared with exact numerical solutions. Good agreement is obtained even for moderate extensions and the results support the earlier discussion regarding the error estimates and convergence of the approximate solutions.

IV. SOME INCLUSION PROBLEMS

Several basic inclusion problems are considered in this section. For simplicity the inclusion shape is taken to be a circle or an ellipse, but other geometries can be treated in a similar manner when the appropriate conformal mapping is known.

Example 1 Infinite membrane with clamped circular hole under biaxial extension at infinity.

The infinite membrane contains a circular hole of radius a. The edge of the hole is bonded to a rigid inclusion (or otherwise held fixed) and at infinity the sheet is in a state of biaxial extension with principal extension ratios μ_1 and μ_2 along the x_1- and x_2-axes respectively, the origin being at the center of the hole.

If we use polar coordinates (r, θ) to describe initial locations, the equilibrium equations (2.10) become

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial y_1}{\partial r}\right) + \frac{1}{r^2}\,\frac{\partial^2 y_1}{\partial \theta^2} &= \frac{1}{r}\,\frac{\partial \lambda^3}{\partial r}\,\frac{\partial y_2}{\partial \theta} - \frac{1}{r}\,\frac{\partial \lambda^3}{\partial \theta}\,\frac{\partial y_2}{\partial r}\,, \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial y_2}{\partial r}\right) + \frac{1}{r^2}\,\frac{\partial^2 y_2}{\partial \theta^2} &= \frac{1}{r}\,\frac{\partial y_1}{\partial r}\,\frac{\partial \lambda^3}{\partial \theta} - \frac{1}{r}\,\frac{\partial y_1}{\partial \theta}\,\frac{\partial \lambda^3}{\partial r}\,. \end{aligned}\right\} \qquad (4.1)$$

Applying the approach of Section III, we find that the harmonic functions $y_\alpha^{(1)}$ of the first approximation are determined by the boundary conditions

$$\left.\begin{aligned} \frac{\partial y_1^{(1)}}{\partial x_1} &= \mu_1 + o\left(\frac{1}{r}\right), & \frac{\partial y_1^{(1)}}{\partial x_2} &= o\left(\frac{1}{r}\right), \\ \frac{\partial y_2^{(1)}}{\partial x_1} &= o\left(\frac{1}{r}\right), & \frac{\partial y_2^{(1)}}{\partial x_2} &= \mu_2 + o\left(\frac{1}{r}\right), \end{aligned}\right\} \text{ as } r \to \infty \qquad (4.2)$$

in the case when there is no resultant force on the inclusion, and

$$\left.\begin{aligned} y_1^{(1)} &= a\cos\theta \;, \\ y_2^{(1)} &= a\sin\theta \;, \end{aligned}\right\} \text{ at } r = a \;.$$

We have therefore

$$\left.\begin{aligned} y_1^{(1)} &= \mu_1 r\left(1 - \frac{k_1^2}{r^2}\right)\cos\theta \;, \\ y_2^{(1)} &= \mu_2 r\left(1 - \frac{k_2^2}{r^2}\right)\sin\theta \;, \end{aligned}\right\} \qquad (4.3)$$

where

$$k_\alpha^2 = \left(1 - \frac{1}{\mu_\alpha}\right)a^2 \;,$$

and the extension ratio normal to the sheet is given by

$$\lambda^{(1)} = r\left[\frac{\partial(y_1^{(1)}, y_2^{(1)})}{\partial(r,\theta)}\right]^{-1}$$

$$= \frac{1}{\mu_1\mu_2}\left[\left(1 + \frac{k_1^2}{r^2}\right)\left(1 - \frac{k_2^2}{r^2}\right)\cos^2\theta + \left(1 - \frac{k_1^2}{r^2}\right)\left(1 + \frac{k_2^2}{r^2}\right)\sin^2\theta\right]^{-1} \;. \qquad (4.4)$$

Because $y_\alpha^{(1)}$ are harmonic functions, it follows that $|\nabla y_\alpha^{(1)}|^2$ and hence $K^{(1)}$, $U^{(1)}$ are subharmonic. If we exclude the case of constant strain energy $U^{(1)}$, the sum of the squares of the principal extension ratios Λ_1, Λ_2 obtained from the first approximation must then attain its maximum and minimum at an internal boundary or at infinity.

At the edge of the hole $r = a$, we have

$$[\Lambda_{1,2}]^2 = \frac{1}{2}\left\{[(2\mu_1-1)^2\cos^2\theta + (2\mu_2-1)^2\sin^2\theta + 1]\right.$$
$$\left.\pm\{[(2\mu_1-1)^2\cos^2\theta+(2\mu_2-1)^2\sin^2\theta+1]^2-4[(2\mu_1-1)\cos^2\theta+(2\mu_2-1)\sin^2\theta]^2\}^{\frac{1}{2}}\right\},$$

and

$$\tan\sigma = \frac{\Lambda_1^2 - [(2\mu_1-1)^2\cos^2\theta + \sin^2\theta]}{[(2\mu_1-1)(2\mu_2-1)-1]\sin\theta\cos\theta}$$

$$= \frac{[(2\mu_1-1)(2\mu_2-1)-1]\sin\theta\cos\theta}{\Lambda_1^2 - [(2\mu_2-1)^2\sin^2\theta+\cos^2\theta]} ,$$

where σ denotes the angle between the first principal direction and the positive x_1-axis. We note that at $\theta = 0$,

$$\Lambda_1 = 2\mu_1 - 1 \quad , \quad \Lambda_2 = 1 \quad ,$$

and at $\theta = \pi/2$,

$$\Lambda_1 = 2\mu_2 - 1 \quad , \quad \Lambda_2 = 1 \quad .$$

Thus, for large μ_1, μ_2 principal extension ratios which are close to twice the values at infinity occur near the inclusion.

The first approximation $y_\alpha^{(1)}$ is meaningful only if the Jacobian $J^{(1)} = \Lambda_1\Lambda_2$ is positive everywhere. From (4.4) this condition is seen to require μ_1 and μ_2 to be greater than one-half. However for a neo-Hookean material the principal force resultants are tensile only if $\Lambda_1\Lambda_2^2$ and $\Lambda_1^2\Lambda_2$ are greater than unity. From the values of Λ_1, Λ_2 at the inclusion we see that the first approximation requires μ_1 and μ_2 to be both greater than unity in order to avoid compressive stresses near the inclusion, an indication that wrinkling or folding of the sheet would occur if either of μ_1, μ_2 were less than unity.

If the inclusion is acted upon by a force L^* through the origin at an angle δ to the x_1-axis, the terms

$$-\frac{L^*\cos\delta}{4\pi h_o C_1}\log\frac{r}{a}\quad,\qquad -\frac{L^*\sin\delta}{4\pi h_o C_1}\log\frac{r}{a}$$

must be added to the expressions for $y_1^{(1)}$ and $y_2^{(1)}$, respectively. The extension ratio normal to the sheet at $r = a$ is given by

$$\lambda^{(1)} = \left[(2\mu_1 - 1)\cos^2\theta + (2\mu_2 - 1)\sin^2\theta - \frac{L^*}{4\pi h_o C_1 a}\cos(\theta - \delta)\right]^{-1} .$$

For given μ_1, μ_2, a sufficient condition that the Jacobian $J^{(1)}$ be positive is

$$\frac{L^*}{4\pi h_o C_1 a} < \min\left[(2\mu_1 - 1)\quad,\quad(2\mu_2 - 1)\right] .$$

It is apparent from the nature of the inhomogeneous terms in the differential equations and boundary conditions for $y_\alpha^{(2)}$ that the solution for $y_\alpha^{(2)}$ will not be elementary. Considerable simplification results when the deformation has axial symmetry, that is when $\mu_1 = \mu_2 = \mu$ and the sheet is subjected to an all-around tension at infinity. In this case, $y_\alpha^{(1)}$ and $\lambda^{(1)}$ are given by

$$y_1^{(1)} = \rho^{(1)}\cos\theta\quad,\quad y_2^{(1)} = \rho^{(1)}\sin\theta\quad,\quad \rho^{(1)}(r) = \mu r\left(1 - \frac{k^2}{r^2}\right), \tag{4.5}$$

$$\lambda^{(1)} = \frac{1}{\mu^2}\left(1 - \frac{k^4}{r^4}\right),$$

where

$$k^2 = a^2\left(1 - \frac{1}{\mu}\right) .$$

If $y_\alpha^{(2)}$ are written as

$$y_1^{(2)} = \rho^{(2)}(r)\cos\theta \quad , \quad y_2^{(2)} = \rho^{(2)}(r)\sin\theta \quad ,$$

then $\rho^{(2)}(r)$ satisfies the differential equation

$$\frac{d}{dr}\left(\frac{d}{dr}\,\rho^{(2)} + \frac{\rho^{(2)}}{r}\right) = -\,\frac{12k^4}{\mu^5 r^5}\left[\left(1 + \frac{k^2}{r^2}\right)\left(1 - \frac{k^4}{r^4}\right)^3\right]^{-1} \quad ,$$

and the boundary conditions

$$\rho^{(2)} = a \quad \text{at} \quad r = a \quad ,$$

$$\frac{d\rho^{(2)}}{dr} = \mu + o\left(\frac{1}{r}\right) \quad \text{as} \quad r \to \infty \quad .$$

The solution $\rho^{(2)}(r)$ is found to be

$$\rho^{(2)}(r) = a_1 r + \frac{b_1}{r} + I(r) \quad , \tag{4.6}$$

where

$$I(r) = \frac{3k^2}{32\mu^5}\,\frac{1}{r}\left[\left(5 - \frac{k^2}{r^2}\right)\log\frac{r^2-k^2}{r^2+k^2} + \frac{30\left(\frac{k}{r}\right)^6 - 24\left(\frac{k}{r}\right)^4 + 26\left(\frac{k}{r}\right)^2 + 16}{3\left(\frac{k}{r}\right)^2\left(1 - \frac{k^2}{r^2}\right)\left(1 + \frac{k^2}{r^2}\right)^2}\right] \quad ,$$

$$a_1 = \mu\left(1 - \frac{1}{2\mu^6}\right) \quad , \quad b_1 = a[(1-a_1)a - I(a)] \quad .$$

Comparing (4.5) and (4.6), we note that, as expected when $y_\alpha^{(1)}$ are $O(\mu)$, $\rho^{(1)}(r)$ determines $\rho^{(2)}(r)$ to terms of $O(\mu^{-5})$ and hence for large μ the first approximation describes the deformation closely.

In order to obtain the exact solution for the case $\mu_1 = \mu_2 = \mu$, we substitute the expressions

$$y_1 = \rho(r)\cos\theta \quad , \qquad y_2 = \rho(r)\sin\theta \tag{4.7}$$

into the exact equilibrium equations. We then require, for $r > a$,

$$\frac{d^2\rho}{dr^2} + \frac{1}{r}\frac{d\rho}{dr} - \frac{\rho}{r^2} = \frac{3r}{\rho^3\left(\frac{d\rho}{dr}\right)^4}\left[\rho\,\frac{d\rho}{dr} - r\left(\frac{d\rho}{dr}\right)^2 - r\rho\,\frac{d^2\rho}{dr^2}\right] . \tag{4.8}$$

With the boundary conditions

$$\rho = a \quad \text{at} \quad r = a \quad ,$$

$$\frac{d\rho}{dr} = \mu \quad \text{as} \quad r \to \infty \quad ,$$

equation (4.8) can be integrated numerically to yield an exact solution. The integration is straightforward if a value is chosen for the slope $d\rho/dr$ at $r = a$, and corresponding to each initial value of $d\rho/dr$ there is a limiting value for the extension ratio μ at infinity. The integration is terminated when the values of the extension ratios become constant to within some pre-imposed error. For a case of moderate deformation, $\mu = 1.24$, it was found that the first and second approximate solutions given by (4.5) and (4.6) gave values for ρ/r which were within 0.3% of the values obtained from numerical integration of the exact equation (4.8). The choice of the extension ratio at infinity μ was influenced by the numerical work of Rivlin and Thomas [4] on the stretching of a sheet with a circular hole, a problem which will be discussed later.

When the circular inclusion is rotated counterclockwise through an angle β about its center, the sheet being uniformly strained at infinity as before, the first approximation $y_\alpha^{(1)}$ can easily be obtained. It is found that as β is increased the Jacobian $J^{(1)}$ remains positive only until the value β_o is reached, where

$$\beta_o = \cos^{-1}\left(\frac{1}{\mu_1+\mu_2} + \frac{|\mu_1-\mu_2|}{\mu_1+\mu_2}\right) .$$

For a neo-Hookean material the first approximation indicates that the stresses in the sheet at points near the inclusion will cease to be tensile at a value of β somewhat smaller than β_o. For values of β greater than this critical value, folding of the sheet will occur and as β increases the sheet will wrap around the inclusion (assuming it is thicker than the sheet).

For future reference, we write (4.3) in the complex form

$$\left.\begin{aligned} y_1^{(1)} &= \mu_1 \operatorname{Re}\left[z - \frac{k_1^2}{z}\right] , \\ y_2^{(1)} &= \mu_2 \operatorname{Im}\left[z + \frac{k_2^2}{z}\right] , \end{aligned}\right\} \qquad (4.9)$$

where

$$z = x_1 + ix_2 = re^{i\theta} .$$

Example 2 Infinite membrane with an elliptic rigid inclusion under biaxial extension at infinity and limiting case of a line inclusion or splinter.

In this example the rigid inclusion occupies the interior of the ellipse

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 \qquad (a > b) \quad .$$

We introduce a new complex variable $w = Re^{i\phi}$ by means of the conformal transformation

$$w = \frac{1}{2}\left[z + (z^2 - c^2)^{\frac{1}{2}}\right] \quad , \qquad z = w + \frac{c^2}{4w} \quad ,$$

where $c^2 = a^2 - b^2$. If we write

$$z = c\cosh\zeta \quad , \quad \zeta = \xi + i\eta \quad \text{or} \quad x_1 = c\cosh\xi\cos\eta \quad ,$$

$$x_2 = c\sinh\xi\,\sin\eta \quad ,$$

the transformation is equivalent to

$$w = \frac{c}{2}\,e^{\zeta} \quad ,$$

ξ, η being the usual elliptic coordinates. The exterior of the ellipse in the z-plane is mapped onto the exterior of the circle of radius $(a + b)/2$ in the w-plane and the mapping is such that there is no distortion or rotation at infinity. The functions $y_\alpha^{(1)}$ are harmonic in the w-plane and the transformed boundary condition on the internal boundary is

$$\left.\begin{aligned} y_1^{(1)} &= a\cos\phi \quad , \\ y_2^{(1)} &= b\sin\phi \quad , \end{aligned}\right\} \text{ at } R = (a+b)/2 \quad .$$

If the extension ratios at infinity in the deformed sheet are again μ_1, μ_2 along the axes, by comparison with (4.9) we see that

$$y_1^{(1)} = \mu_1 \mathrm{Re}\left[w - \left(1 - \frac{2a}{(a+b)\mu_1}\right)\frac{(a+b)^2}{4w}\right] ,$$

$$y_2^{(1)} = \mu_2 \mathrm{Im}\left[w + \left(1 - \frac{2b}{(a+b)\mu_2}\right)\frac{(a+b)^2}{4w}\right] ,$$

in the case when there is no resultant force on the inclusion. Alternatively we may write

$$\left.\begin{aligned} y_1^{(1)} &= \frac{c\mu_1}{2}\,\mathrm{Re}[e^{\zeta} - \kappa_1 e^{-\zeta}] = \frac{c\mu_1}{2}\,(e^{\xi} - \kappa_1 e^{-\xi})\cos\eta , \\ y_2^{(1)} &= \frac{c\mu_2}{2}\,\mathrm{Im}[e^{\zeta} + \kappa_2 e^{-\zeta}] = \frac{c\mu_2}{2}\,(e^{\xi} - \kappa_2 e^{-\xi})\sin\eta , \end{aligned}\right\} \qquad (4.10)$$

where κ_1 and κ_2 are defined to be

$$\kappa_1 = \left[1 - \frac{2a}{(a+b)\mu_1}\right]\left(\frac{a+b}{a-b}\right) , \qquad \kappa_2 = \left[1 - \frac{2b}{(a+b)\mu_2}\right]\left(\frac{a+b}{a-b}\right) ,$$

and we find that

$$\lambda^{(1)} = \frac{4(\sinh^2\xi + \sin^2\eta)}{\mu_1\mu_2[(e^{\xi}+\kappa_1 e^{-\xi})(e^{\xi}-\kappa_2 e^{-\xi})\cos^2\eta + (e^{\xi}-\kappa_1 e^{-\xi})(e^{\xi}+\kappa_2 e^{-\xi})\sin^2\eta]} .$$

The inclusion boundary is

$$\xi = \xi_o = \frac{1}{2}\log\frac{a+b}{a-b} ,$$

with

$$c\cosh\xi_o = a , \quad c\sinh\xi_o = b .$$

At the ends of the major and minor axes of the ellipse the extension ratios for directions normal to the inclusion have the values

$$\mu_1\frac{(a+b)}{b} - \frac{a}{b} , \quad \mu_2\frac{(a+b)}{a} - \frac{b}{a} ,$$

respectively. As in the case of a circular inclusion, μ_1 and μ_2 must be greater than unity if the force resultants of the first approximation are to be tensile (for a neo-Hookean material).

In the limit $b \to 0$ ($a \to c$, $\xi_o \to 0$), the inclusion degenerates into a line inclusion or splinter of length $2c$ along the x_1-axis and (4.10) simplifies to

$$y_1^{(1)} = c[(\mu_1 - 1)\sinh\xi + \cosh\xi]\cos\eta \quad ,$$

$$y_2^{(1)} = c\mu_2 \sinh\xi \sin\eta = \mu_2 x_2 \quad ,$$

and the transverse extension ratio is given by

$$\lambda^{(1)} = \frac{\sinh^2\xi + \sin^2\eta}{\mu_2[(\mu_1 - 1)\cosh\xi \sinh\xi + \sinh^2\xi \cos^2\eta + \cosh^2\xi \sin^2\eta]} \quad .$$

On the splinter $\xi = 0$ and along its length $\lambda^{(1)}$ has the constant value $1/\mu_2$, independent of the extension ratio μ_1 at infinity parallel to the splinter. The end points ($\xi = 0$, $\eta = 0, \pi$) are singular points and the limiting value of $\lambda^{(1)}$ at the ends varies between 0 and $1/\mu_2$ according to the manner of approach. The principal extension ratios at the splinter are given by

$$[\Lambda_{1,2}]^2 = \frac{1}{2}\{1 + \mu_2^2 + (\mu_1 - 1)^2\cot^2\eta \pm [(\mu_2 - 1)^2 + (\mu_1 - 1)^2\cot^2\eta]^{\frac{1}{2}}$$
$$[(\mu_2 + 1)^2 + (\mu_1 - 1)^2\cot^2\eta]^{\frac{1}{2}}\} \quad ,$$

and the maximum extension ratio Λ_1 varies from μ_2 in the middle of the splinter to infinity at the end points. If σ again denotes the angle between the first principal direction and the positive x_1-axis, then at the inclusion

$$\tan\sigma = \frac{\mu_2(\mu_1 - 1)\cot\eta}{\Lambda_1^2 - \mu_2^2} = \frac{\Lambda_1^2 - 1 - (\mu_1 - 1)^2\cot^2\eta}{\mu_2(\mu_1 - 1)\cot\eta} .$$

On the line $x_1 \geq c$, $x_2 = 0$, the principal extension ratio in the x_2-direction is μ_2 while the principal extension ratio in the x_1-direction is

$$(\mu_1 - 1)\frac{x_1}{(x_1^2 - c^2)^{\frac{1}{2}}} + 1 ,$$

and this tends to ∞ as the end of the splinter is approached.

We now suppose that the major axis of the ellipse was initially inclined at an angle α to the positive x_1-axis ($|\alpha| < \pi/2$) and that the inclusion is allowed to rotate when the sheet is strained. In the deformed state, the angle of inclination of the major axis is β, and α, β are measured positive in the counterclockwise direction. Noting that the transformation

$$w = \frac{1}{2}[z + (z^2 - c^2 e^{2i\alpha})^{\frac{1}{2}}]$$

maps the exterior of an ellipse rotated about its center through an angle α onto the exterior of the circle $R = (a+b)/2$, without distortion or rotation at infinity, we can once again reduce the problem to that of Example 1. The condition at the interior boundary is now

$$\left.\begin{aligned} y_1^{(1)} &= a\cos(\phi - \alpha)\cos\beta - b\sin(\phi - \alpha)\sin\beta , \\ y_2^{(1)} &= a\cos(\phi - \alpha)\sin\beta + b\sin(\phi - \alpha)\cos\beta , \end{aligned}\right\} \text{ at } \quad R = (a+b)/2 .$$

The solution can be written as

$$y_1^{(1)} = \frac{c\mu_1}{2} \mathrm{Re}[e^{\zeta+i\alpha} - (s_1 - is_2)e^{-(\zeta+i\alpha)}]$$

$$= \frac{c\mu_1}{2} [(e^{\xi} - s_1 e^{-\xi})\cos(\eta+\alpha) + s_2 e^{-\xi}\sin(\eta+\alpha)] \quad ,$$

$$y_2^{(1)} = \frac{c\mu_2}{2} \mathrm{Im}[e^{\zeta+i\alpha} + (s_3 + is_4)e^{-(\zeta+i\alpha)}]$$

$$= \frac{c\mu_2}{2} [(e^{\xi} - s_3 e^{-\xi})\sin(\eta+\alpha) + s_4 e^{-\xi}\cos(\eta+\alpha)] \quad ,$$

where

$$s_1 = \frac{a+b}{a-b}\left[1 - \frac{2(a\cos\beta\cos\alpha + b\sin\beta\sin\alpha)}{(a+b)\mu_1}\right] \quad ,$$

$$s_2 = \frac{2}{(a-b)\mu_1}(a\cos\beta\sin\alpha - b\sin\beta\cos\alpha) \quad ,$$

$$s_3 = \frac{a+b}{a-b}\left[1 - \frac{2(a\sin\beta\sin\alpha + b\cos\beta\cos\alpha)}{(a+b)\mu_2}\right] \quad ,$$

$$s_4 = \frac{2}{(a-b)\mu_2}(a\sin\beta\cos\alpha - b\cos\beta\sin\alpha) \quad ,$$

and

$$z = ce^{i\alpha}\cosh\zeta \quad , \quad \zeta = \xi + i\eta \quad .$$

Thus ξ, η are the usual elliptic coordinates associated with the elliptic inclusion in its initial position. As in the case of the rotated circular inclusion, the range of β for given α, μ_1, μ_2 is limited by the requirement of tensile force resultants in the sheet.

The components of the resultant force acting on the elliptic inclusion are, from (3.3),

$$L_1^{(1)} = -2h_o C_1 \int_o^{2\pi} \left(\frac{\partial y_1^{(1)}}{\partial \xi}\right)_{\xi=\xi_o} d\eta \quad ,$$

$$L_2^{(1)} = -2h_o C_1 \int_o^{2\pi} \left(\frac{\partial y_2^{(1)}}{\partial \xi}\right)_{\xi=\xi_o} d\eta \quad ,$$

and we can easily verify that they are zero. The torque exerted by the sheet on the inclusion is found to be

$$M^{(1)} = h_o C_1 c^2 \mu_1 \mu_2 \pi (s_4 - s_2) \quad .$$

If the elliptic inclusion is free to move with the membrane when the latter is strained, we must have

$$M^{(1)} = 0 \quad ,$$

and therefore

$$s_4 = s_2$$

or

$$\tan\beta = \frac{a\mu_2 + b\mu_1}{a\mu_1 + b\mu_2} \tan\alpha \quad . \tag{4.11}$$

This equation determines the position of the inclusion in the deformed sheet in terms of its initial position and the principal extension ratios at infinity.

Proceeding to the limiting case of a splinter (b = 0), we have

$$\left.\begin{aligned} y_1^{(1)} &= c[\mu_1 \sinh\xi \cos(\eta+\alpha) + \cos\beta \, e^{-\xi} \cos\eta] \quad , \\ y_2^{(1)} &= c[\mu_2 \sinh\xi \sin(\eta+\alpha) + \sin\beta \, e^{-\xi} \cos\eta] \quad . \end{aligned}\right\} \tag{4.12}$$

Along the splinter where $\xi = 0$ we have

$$\lambda^{(1)} = \frac{\sin\eta}{[\mu_2 \sin(\eta+\alpha)\cos\beta - \mu_1 \cos(\eta+\alpha)\sin\beta]} \tag{4.13}$$

while, as before, the value of $\lambda^{(1)}$ at the ends depends upon the direction of approach. Setting $b = 0$ in (4.11), the condition of zero torque exerted by the sheet on the splinter requires

$$\tan\beta = \frac{\mu_2}{\mu_1} \tan\alpha \quad . \tag{4.14}$$

In the pure strain which the sheet suffers at infinity, a line element initially at an angle α to the x_1-axis becomes inclined at an angle β to x_1-axis with β given by (4.14). Thus the splinter and the line elements at infinity which were initially parallel remain parallel during the deformation, according to the first approximation.

If we use (4.14) in the expression for $\lambda^{(1)}$ we get, on the splinter,

$$\lambda^{(1)} = \frac{1}{\mu_1\mu_2} (\mu_1^2 \cos^2\alpha + \mu_2^2 \sin^2\alpha)^{\frac{1}{2}} \quad ,$$

so $\lambda^{(1)}$ is constant on the splinter (excluding the end points). We note that unless β is given by (4.14), the denominator in the expression (4.13) for $\lambda^{(1)}$ assumes negative values at points on the splinter. The solution is therefore inadmissible except for the value of β in the particular attitude the splinter assumes under zero torque. In consequence, it is to be expected that a small torque applied to the splinter will produce wrinkling of the sheet in the neighborhood of the ends of the line inclusion.

Example 3 Circular material inclusion in an infinite membrane under biaxial extension at infinity.

We suppose that the portion $r \leq a$ of an infinite sheet is composed of a different neo-Hookean material with material constant $\overline{C}_1$ and initial constant thickness $\overline{h}_o$. We use a bar to indicate quantities associated with the inclusion. The functions $\overline{y}_\alpha^{(1)}$, $y_\alpha^{(1)}$ of the first approximation are harmonic in the regions $r < a$, $r > a$ respectively, and they must satisfy the conditions

$$\overline{y}_\alpha^{(1)} = y_\alpha^{(1)} \quad , \quad \overline{h}_o \overline{C}_1 \frac{\partial \overline{y}_\alpha^{(1)}}{\partial n^\circ} = h_o C_1 \frac{\partial y_\alpha^{(1)}}{\partial n^\circ} \quad \text{at} \quad r = a$$

in order to ensure continuity of traction and displacement at the interface $r = a$. At infinity the functions $y_\alpha^{(1)}$ again satisfy (4.2). It is found that

$$y_1^{(1)} = \mu_1 r \left[1 + \frac{(1-m)a^2}{(1+m)r^2} \right] \cos\theta \quad , \quad y_2^{(1)} = \mu_2 r \left[1 + \frac{(1-m)a^2}{(1+m)r^2} \right] \sin\theta \quad , \tag{4.15}$$

$$\overline{y}_1^{(1)} = \frac{2\mu_1}{(1+m)} x_1 \quad , \quad \overline{y}_2^{(1)} = \frac{2\mu_2}{(1+m)} x_2 \quad , \tag{4.16}$$

where m is the product of the ratios of material constants and initial thicknesses,

$$m = \frac{\overline{h}_o \overline{C}_1}{h_o C_1} \quad .$$

We note the following properties of the solution:

(i) The form of the first approximation $\overline{y}_\alpha^{(1)}$ is independent of the size of the inclusion.

(ii) The inclusion material is in a state of homogeneous deformation with principal extension ratios

$$\bar{\Lambda}_1 = \frac{2\mu_1}{1+m} \quad , \quad \bar{\Lambda}_2 = \frac{2\mu_2}{1+m} \tag{4.17}$$

along the x_1 - and x_2 -axes respectively. The principal extension ratios in the material exterior to the inclusion are $(m\bar{\Lambda}_1, \bar{\Lambda}_2)$ and $(\bar{\Lambda}_1, m\bar{\Lambda}_2)$ at the points where the x_1 -and x_2 -axes meet the interface respectively.

(iii) The limiting case $m = 0$ corresponds to a sheet with a circular hole under biaxial tension at infinity. Setting $m = 0$ in (4.15) we obtain

$$\left. \begin{aligned} y_1^{(1)} &= \mu_1 r\left(1 + \frac{a^2}{r^2}\right)\cos\theta \quad , \\ y_2^{(1)} &= \mu_2 r\left(1 + \frac{a^2}{r^2}\right)\sin\theta \quad . \end{aligned} \right\} \tag{4.18}$$

The extension ratio in the direction normal to the sheet is given by

$$\lambda^{(1)} = \frac{1}{\mu_1\mu_2}\left(1 - \frac{a^4}{r^4}\right)^{-1} \quad ,$$

and as the boundary $r = a$ is approached, $\lambda^{(1)}$ becomes infinite. This singular behavior of the first approximation in the presence of traction-free boundaries was discussed in Section 3. We return to this problem in Section VI after developing an alternative approach in the next section for the determination of $\lambda^{(1)}$.

(iv) The limiting case $m \to \infty$ would, for neo-Hookean materials, correspond to a sheet with a rigid circular inclusion. However as $m \to \infty$, expressions (4.15) do not approach the values (4.3) of the solution given previously for a rigid circular inclusion. The discrepancy arises from the circumstance that for m very large the stresses in the inclusion will be small, and the material with strain energy $\overline{U}^{(1)}$ is unstressed only when $\overline{\Lambda}_1, \overline{\Lambda}_2$ are zero. Thus as $m \to \infty$, an inclusion composed of material with strain energy $\overline{U}^{(1)}$ would shrink to the origin. In order for the solution (4.15), (4.16) to be a reasonable approximation for neo-Hookean materials the transverse extension ratios $\overline{\lambda}^{(1)}$ and $\lambda^{(1)}$ in the inclusion and the exterior material must be small compared with unity. From (4.15), (4.16), this implies $\mu_1 \mu_2$ large and

$$m \ll 2\sqrt{\mu_1 \mu_2} - 1 \quad ,$$

and the limiting case $m \to \infty$ lies outside the range of validity of the first approximation.

Example 4 Elliptic material inclusion in an infinite membrane under biaxial extension at infinity.

In this example the inclusion material is again taken to be neo-Hookean with material constant $\overline{C}_1$ and initial thickness $\overline{h}_o$ but now the inclusion occupies the interior of the ellipse

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 \qquad (a > b) \quad . \tag{4.19}$$

The first approximations $y_\alpha^{(1)}$, $\overline{y}_\alpha^{(1)}$ are harmonic in the regions exterior and interior to the ellipse, respectively, and they satisfy the same boundary and continuity conditions of the previous example except that the interface is now the ellipse (4.19).

The region bounded by an ellipse may, like every region bounded by a simple contour, be mapped conformally onto a circle. The corresponding transformation, however, is complicated and moreover it is more convenient to use a single mapping for the regions interior and exterior to the ellipse. If the plane is cut along the segment connecting the foci of the ellipse, the cut, which we denote by AB, may be considered as an ellipse which is confocal with the original one and whose minor axis is zero. Thus the cut plane consists of regions lying between confocal ellipses and the transformation of Example 2 can be used to map them onto regions between concentric circles, with no distortion or rotation at infinity. Hence if we write

$$w = \frac{c}{2} e^{\zeta} \quad ; \quad z = c \cosh \zeta \quad , \quad \zeta = \xi + i\eta \quad ,$$

where $c = (a^2 - b^2)^{\frac{1}{2}}$, the exterior of the ellipse in the z-plane is mapped onto the exterior of the circle $R = (a+b)/2$ in the w-plane while the interior with cut AB is mapped onto the annulus between the two circles $R = (a+b)/2$ and $R = c/2$.

We note that since the origin of the w-plane is excluded, the expansion of the harmonic functions $\overline{y}_\alpha^{(1)}$ as series in $R^n \sin n\phi$ and $R^n \cos n\phi$ will, in general, include negative powers of R. Further, because the points $\frac{c}{2} e^{i\phi}$ and $\frac{c}{2} e^{-i\phi}$ in the w-plane correspond to one and the same points of the segment AB in the z-plane, in order to avoid introducing singularities on AB we must have on $R = c/2$

$$\overline{y}_\alpha^{(1)}(\phi) = \overline{y}_\alpha^{(1)}(-\phi) \quad , \quad \frac{\partial \overline{y}_\alpha^{(1)}}{\partial R}(\phi) = - \frac{\partial \overline{y}_\alpha^{(1)}}{\partial R}(-\phi) \quad .$$

If these conditions are satisfied, the functions $\overline{y}_\alpha^{(1)}$ will be harmonic in the uncut ellipse.

After some calculation, the solutions are found to be

$$\left.\begin{aligned} y_1^{(1)} &= \frac{c\mu_1}{2}\,\mathrm{Re}[e^{\zeta} + s_1 e^{-\zeta}] = \frac{c\mu_1}{2}(e^{\xi} + s_1 e^{-\xi})\cos\eta \ , \\ y_2^{(1)} &= \frac{c\mu_2}{2}\,\mathrm{Im}\,[e^{\zeta} - s_2 e^{-\zeta}] = \frac{c\mu_2}{2}(e^{\xi} + s_2 e^{-\xi})\sin\eta \ , \end{aligned}\right\} \tag{4.20}$$

$$\bar{y}_1^{(1)} = \frac{c\mu_1 s_3}{2}\,\mathrm{Re}[e^{\zeta} + e^{-\zeta}] = \mu_1 s_3 x_1 \ ,$$

$$\bar{y}_2^{(1)} = \frac{c\mu_2 s_4}{2}\,\mathrm{Im}[e^{\zeta} + e^{-\zeta}] = \mu_2 s_4 x_2 \ ,$$

where

$$s_1 = \frac{(1+b/a)(1-mb/a)}{(1-b/a)(1+mb/a)} \ , \qquad s_2 = \frac{(a/b+1)(1-ma/b)}{(a/b-1)(1+ma/b)}$$

$$s_3 = \frac{(1+b/a)}{(1+mb/a)} \ , \qquad s_4 = \frac{(1+a/b)}{(1+ma/b)} \ .$$

It can be seen that the first approximation $\bar{y}_\alpha^{(1)}$ depends on the dimensions of the inclusion only through the ratio a/b. As for the circle, the inclusion material is uniformly strained, the principal extension ratios being $\mu_1 s_3$ and $\mu_2 s_4$. Comments similar to those made in connection with the circular inclusion apply to the two limiting cases $m = 0$ and $m \to \infty$.

When the major axis of the ellipse is initially inclined at an angle α to the positive x_1-axis $(|\alpha| < \pi/2)$, it is easy to show by superposition that the first approximation inside the ellipse is

$$\left.\begin{aligned}\bar{y}_1^{(1)} &= \mu_1[(s_3\cos^2\alpha + s_4\sin^2\alpha)x_1 + (s_3 - s_4)\sin\alpha\cos\alpha\, x_2] \quad ,\\ \bar{y}_2^{(1)} &= \mu_2[(s_3 - s_4)\sin\alpha\cos\alpha\, x_1 + (s_3\sin^2\alpha + s_4\cos^2\alpha)x_2] \quad ,\end{aligned}\right\} \qquad (4.21)$$

and the deformation of the inclusion material remains homogeneous. In the limit $b \to 0$, one has the special case of a line inclusion of length $2c$. As $b \to 0$ the effect of the inclusion diminishes and the locations y_α tend everywhere to the uniform state $y_1 = \mu_1 x_1$, $y_2 = \mu_2 x_2$.

V. SUCCESSIVE SUBSTITUTIONS FOR TRACTION-FREE BOUNDARIES

When the membrane has an edge which is traction free it was found in Section III that the proposed method of successive approximations breaks down. As the edge is approached $J^{(1)} \to 0$ and the partial differential equations for $y_\alpha^{(2)}$ have a non-integrable singularity when $\lambda^{(1)}$ is taken as the inverse of $J^{(1)}$. We note that the material at a traction-free boundary is under simple tension so that on the boundary

$$\lambda_n = \lambda = (\lambda_s)^{-\frac{1}{2}} = \left[\left(\frac{\partial y_1}{\partial s^\circ}\right)^2 + \left(\frac{\partial y_2}{\partial s^\circ}\right)^2\right]^{-\frac{1}{4}} , \tag{5.1}$$

where λ_s and λ_n denote the principal extension ratios in directions tangential and normal to the edge respectively. The force resultant T_s parallel to the boundary is given by

$$T_s = 2h_o C_1 \lambda_s^{3/2} \left(1 - \frac{1}{\lambda_s^3}\right) = 2h_o C_1 \lambda^3 \left(\frac{1}{\lambda^6} - 1\right)$$

for a neo-Hookean material.

In this section we describe an alternative method for the determination of a first approximation $\lambda^{(1)}$ to λ which remains valid in the neighborhood of a traction-free boundary. In order to develop the method we introduce two stress functions φ and ψ and we show that the locations y_α can be found straightforwardly when φ and ψ are known. Although the functions φ and ψ can be obtained, in principle, by successive approximation, and problems solved in this manner, the main use of the stress functions here is to provide intermediate steps leading to the alternative procedure for determining a first approximation $\lambda^{(1)}$. When $\lambda^{(1)}$ is known it can be used with the first approximation $y_\alpha^{(1)}$, as before, in the equations for $y_\alpha^{(2)}$ and the method of Section III can then proceed as described. If the second and higher approximations are not required, the first approximation $y_\alpha^{(1)}$ being considered sufficiently accurate to describe the deformed geometry, the method here leads to values for $\lambda^{(1)}$ which accurately describe the thinning of the sheet, including both the interior and regions near traction-free edges.

We begin by writing the equilibrium equations (2.10) in the form

$$\frac{\partial}{\partial x_1}\left(\frac{\partial y_1}{\partial x_1} - \lambda^3 \frac{\partial y_2}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial y_1}{\partial x_2} + \lambda^3 \frac{\partial y_2}{\partial x_1}\right) = 0 \quad ,$$

$$\frac{\partial}{\partial x_1}\left(\frac{\partial y_2}{\partial x_1} + \lambda^3 \frac{\partial y_1}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial y_2}{\partial x_2} - \lambda^3 \frac{\partial y_1}{\partial x_1}\right) = 0 \quad .$$

These equations imply the existence of stress functions φ and ψ such that

$$\left.\begin{aligned} \frac{\partial\varphi}{\partial x_1} &= -\frac{\partial y_1}{\partial x_2} - \lambda^3 \frac{\partial y_2}{\partial x_1} \,, & \frac{\partial\psi}{\partial x_1} &= -\frac{\partial y_2}{\partial x_2} + \lambda^3 \frac{\partial y_1}{\partial x_1} \\ \frac{\partial\varphi}{\partial x_2} &= \frac{\partial y_1}{\partial x_1} - \lambda^3 \frac{\partial y_2}{\partial x_2} \,, & \frac{\partial\psi}{\partial x_2} &= \frac{\partial y_2}{\partial x_1} + \lambda^3 \frac{\partial y_1}{\partial x_2} \end{aligned}\right\} \quad (5.2)$$

and we then have, assuming that $\lambda \neq 1$,

$$\left.\begin{aligned}
\frac{\partial y_1}{\partial x_1} &= \frac{1}{1-\lambda^6}\left(\frac{\partial \varphi}{\partial x_2} - \lambda^3 \frac{\partial \psi}{\partial x_1}\right), &\quad \frac{\partial y_2}{\partial x_1} &= \frac{1}{1-\lambda^6}\left(\lambda^3 \frac{\partial \varphi}{\partial x_1} + \frac{\partial \psi}{\partial x_2}\right) \\
\frac{\partial y_1}{\partial x_2} &= \frac{1}{1-\lambda^6}\left(-\lambda^3 \frac{\partial \psi}{\partial x_2} - \frac{\partial \varphi}{\partial x_1}\right), &\quad \frac{\partial y_2}{\partial x_2} &= \frac{1}{1-\lambda^6}\left(-\frac{\partial \psi}{\partial x_1} + \lambda^3 \frac{\partial \varphi}{\partial x_2}\right)
\end{aligned}\right\} \quad (5.3)$$

If we substitute these expressions in the identity

$$\lambda = \left(\frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_2} - \frac{\partial y_1}{\partial x_2}\frac{\partial y_2}{\partial x_1}\right)^{-1},$$

we obtain an equation for λ in terms of φ and ψ, thus

$$\lambda^{12} - j\lambda^7 - 2\lambda^6 - \lambda^4\left(|\nabla\varphi|^2 + |\nabla\psi|^2\right) - j\lambda + 1 = 0\,, \qquad (5.4)$$

where

$$j = \frac{\partial \varphi}{\partial x_1}\frac{\partial \psi}{\partial x_2} - \frac{\partial \varphi}{\partial x_2}\frac{\partial \psi}{\partial x_1}\,.$$

Furthermore, compatibility of equations (5.3) requires

$$\frac{\partial}{\partial x_2}\left[\frac{1}{1-\lambda^6}\left(\frac{\partial \varphi}{\partial x_2} - \lambda^3 \frac{\partial \psi}{\partial x_1}\right)\right] + \frac{\partial}{\partial x_1}\left[\frac{1}{1-\lambda^6}\left(\lambda^3 \frac{\partial \psi}{\partial x_2} + \frac{\partial \varphi}{\partial x_1}\right)\right] = 0,$$

$$\frac{\partial}{\partial x_2}\left[\frac{1}{1-\lambda^6}\left(\lambda^3 \frac{\partial \varphi}{\partial x_1} + \frac{\partial \psi}{\partial x_2}\right)\right] + \frac{\partial}{\partial x_1}\left[\frac{1}{1-\lambda^6}\left(\frac{\partial \psi}{\partial x_1} - \lambda^3 \frac{\partial \varphi}{\partial x_2}\right)\right] = 0,$$

or

$$\left.\begin{aligned}
\nabla^2\varphi &= \frac{\partial\psi}{\partial x_1}\frac{\partial\lambda^3}{\partial x_2} - \frac{\partial\psi}{\partial x_2}\frac{\partial\lambda^3}{\partial x_1} \\
&\quad - \frac{6\lambda^5}{1-\lambda^6}\left[\frac{\partial\lambda}{\partial x_2}\left(\frac{\partial\varphi}{\partial x_2} - \lambda^3\frac{\partial\psi}{\partial x_1}\right) + \frac{\partial\lambda}{\partial x_1}\left(\lambda^3\frac{\partial\psi}{\partial x_2} + \frac{\partial\varphi}{\partial x_1}\right)\right], \\
\nabla^2\psi &= \frac{\partial\lambda^3}{\partial x_1}\frac{\partial\varphi}{\partial x_2} - \frac{\partial\lambda^3}{\partial x_2}\frac{\partial\varphi}{\partial x_1} \\
&\quad - \frac{6\lambda^5}{1-\lambda^6}\left[\frac{\partial\lambda}{\partial x_2}\left(\lambda^3\frac{\partial\varphi}{\partial x_1} + \frac{\partial\psi}{\partial x_2}\right) + \frac{\partial\lambda}{\partial x_1}\left(\frac{\partial\psi}{\partial x_1} - \lambda^3\frac{\partial\varphi}{\partial x_2}\right)\right].
\end{aligned}\right\} \quad (5.5)$$

Equations (5.5) constitute the governing differential equations for the functions φ and ψ.

The components of the total load resultant associated with a curve C drawn in the middle plane which was initially a curve C° between two points P° and Q° are, from (2.11),

$$L_1 = 2h_o C_1 \int_{P^\circ}^{Q^\circ} \left(\frac{\partial y_1}{\partial n^\circ} - \lambda^3 \frac{\partial y_2}{\partial s^\circ}\right) ds^\circ ,$$

$$L_2 = 2h_o C_1 \int_{P^\circ}^{Q^\circ} \left(\frac{\partial y_2}{\partial n^\circ} + \lambda^3 \frac{\partial y_1}{\partial s^\circ}\right) ds^\circ ,$$

where the path of integration is along C°. From (5.2), it is easy to show that the load resultants L_α are given by

$$L_1 = 2h_o C_1 \int_{P^\circ}^{Q^\circ} \frac{\partial\varphi}{\partial s^\circ} ds^\circ , \qquad L_2 = 2h_o C_1 \int_{P^\circ}^{Q^\circ} \frac{\partial\psi}{\partial s^\circ} ds^\circ ,$$

or

$$\left.\begin{aligned} \varphi(P^\circ) - \varphi(Q^\circ) &= \frac{L_1}{2h_o C_1}\,, \\ \psi(P^\circ) - \psi(Q^\circ) &= \frac{L_2}{2h_o C_1}\,, \end{aligned}\right\} \text{ on } C^\circ .$$

In particular, for a closed contour Π° which is free from traction, we have

$$\varphi = \text{const.} = a_1\,, \quad \psi = \text{const.} = b_1 \quad \text{on } \Pi^\circ .$$

If Π° is the only traction-free contour we may set a_1, b_1 equal to zero without loss in generality. When there are N traction-free contours Π_n° $(n = 1, 2, \ldots, N)$ say, then

$$\varphi = a_n, \quad \psi = b_n \quad \text{on } \Pi_n^\circ \quad (n = 1, 2, \ldots, N)\,,$$

where a_n, b_n are constants. Only one of the constant pairs (a_n, b_n) can, in general, be set equal to zero, the others being determined by the condition that the integrals y_α of equations (5.3) be single-valued.

Although traction boundary conditions are simplified by the use of the stress functions, boundary conditions of place are rendered more complex. From (5.3) we see that

$$\frac{\partial y_1}{\partial s^\circ} = \frac{-1}{1-\lambda^6}\left(\frac{\partial \varphi}{\partial n^\circ} + \lambda^3 \frac{\partial \psi}{\partial s^\circ}\right), \quad \frac{\partial y_2}{\partial s^\circ} = \frac{1}{1-\lambda^6}\left(-\frac{\partial \psi}{\partial n^\circ} + \lambda^3 \frac{\partial \varphi}{\partial s^\circ}\right), \tag{5.6}$$

and $\partial y_\alpha / \partial s^\circ$ will be known on a boundary C° where y_α are prescribed.

If the sheet extends to infinity in all directions and if the sheet is in uniform biaxial tension at infinity with extension ratios μ_1, μ_2 along the axes, then we have, from (5.2) and (3.4), as $r \to \infty$

$$\left.\begin{aligned}
\frac{\partial \varphi}{\partial x_1} &= \frac{1}{4\pi h_o C_1 r}\left[L_1^* \sin\theta + \frac{L_2^* \cos\theta}{(\mu_1\mu_2)^3}\right] + o\left(\frac{1}{r}\right), \\
\frac{\partial \varphi}{\partial x_2} &= \mu_1 - \frac{\mu_2}{(\mu_1\mu_2)^3} + \frac{1}{4\pi h_o C_1 r}\left[\frac{L_2^* \sin\theta}{(\mu_1\mu_2)^3} - L_1^* \cos\theta\right] + o\left(\frac{1}{r}\right), \\
\frac{\partial \psi}{\partial x_1} &= \frac{\mu_1}{(\mu_1\mu_2)^3} - \mu_2 + \frac{1}{4\pi h_o C_1 r}\left[L_2^* \sin\theta - \frac{L_1^* \cos\theta}{(\mu_1\mu_2)^3}\right] + o\left(\frac{1}{r}\right), \\
\frac{\partial \psi}{\partial x_2} &= -\frac{1}{4\pi h_o C_1 r}\left[L_2^* \cos\theta + \frac{L_1^* \sin\theta}{(\mu_1\mu_2)^3}\right] + o\left(\frac{1}{r}\right),
\end{aligned}\right\} \quad (5.7)$$

where L_α^* are the components of the resultant of all external forces acting on the internal boundaries of the sheet.

When $\lambda \ll 1$ a procedure similar to that of Section III can be used to find successive approximations to the stress functions φ and ψ. The first approximations $\varphi^{(1)}, \psi^{(1)}$ are taken to be solutions to (5.5) with λ set equal to zero. Like $y_\alpha^{(1)}$, they are harmonic functions in the domain in question. At a boundary C_T° where the traction is prescribed, the tangential derivatives of $\varphi^{(1)}, \psi^{(1)}$ are prescribed through

$$\frac{\partial \varphi^{(1)}}{\partial s^{\circ}}\, ds^{\circ} = \frac{dL_1^*}{2h_o C_1} \;, \quad \frac{\partial \psi^{(1)}}{\partial s^{\circ}}\, ds^{\circ} = \frac{dL_2^*}{2h_o C_1} \quad \text{on } C_T^{\circ} \;.$$

In contrast to the functions $y_\alpha^{(1)}$, the functions $\varphi^{(1)}$, $\psi^{(1)}$ satisfy the exact conditions on φ and ψ at a boundary where the traction is given. At a boundary C_D° where y_α are prescribed to be the functions y_α^*, we can approximate to the boundary conditions on φ,ψ by taking λ to be zero in (5.6) and requiring

$$\frac{\partial \varphi^{(1)}}{\partial n^{\circ}} = -\frac{\partial y_1^*}{\partial s^{\circ}} \;, \quad \frac{\partial \psi^{(1)}}{\partial n^{\circ}} = -\frac{\partial y_2^*}{\partial s^{\circ}} \quad \text{on } C_D^* \;.$$

With these boundary conditions on the harmonic functions $\varphi^{(1)}$, $\psi^{(1)}$, they will be harmonic conjugates of $y_1^{(1)}$, $y_2^{(1)}$, respectively, when $y_\alpha^{(1)}$ satisfy (3.3) on C_T° and $y_\alpha^{(1)} = y_\alpha^*$ on C_D°. This is in agreement with setting $\lambda = 0$ in (5.2) which gives

$$\left.\begin{aligned} \frac{\partial y_1^{(1)}}{\partial x_1} &= \frac{\partial \varphi^{(1)}}{\partial x_2} \;, & \frac{\partial y_2^{(1)}}{\partial x_1} &= \frac{\partial \psi^{(1)}}{\partial x_2} \;, \\ \frac{\partial y_1^{(1)}}{\partial x_2} &= -\frac{\partial \varphi^{(1)}}{\partial x_1} \;, & \frac{\partial y_2^{(1)}}{\partial x_2} &= -\frac{\partial \psi^{(1)}}{\partial x_1} \;, \end{aligned}\right\} \qquad (5.8)$$

but it should be noted that $(\varphi^{(1)}, y_1^{(1)})$ and $(\psi^{(1)}, y_2^{(1)})$ are conjugate functions only if the approximations made to obtain boundary conditions on $\varphi^{(1)}, \psi^{(1)}$ and on $y_\alpha^{(1)}$ are consistent.

A first approximation $\lambda^{(1)}$ to the transverse extension ratio λ is determined by using $\varphi^{(1)}$, $\psi^{(1)}$ for φ, ψ in (5.4), that is, by the appropriate root of the algebraic equation

$$\lambda^{12} - j^{(1)}\lambda^7 - 2\lambda^6 - \lambda^4 [\, |\nabla\varphi^{(1)}|^2 + |\nabla\psi^{(1)}|^2] - j^{(1)}\lambda + 1 = 0, \quad (5.9)$$

where

$$j^{(1)} = \frac{\partial\varphi^{(1)}}{\partial x_1}\frac{\partial\psi^{(1)}}{\partial x_2} - \frac{\partial\varphi^{(1)}}{\partial x_2}\frac{\partial\psi^{(1)}}{\partial x_1} .$$

If $\varphi^{(1)}$, $\psi^{(1)}$ and $y_\alpha^{(1)}$ are related through (5.8), we can write the equation as

$$\lambda^{12} - J^{(1)}\lambda^7 - 2\lambda^6 - \lambda^4 [\, |\nabla y_1^{(1)}|^2 + |\nabla y_2^{(1)}|^2] - J^{(1)}\lambda + 1 = 0, \quad (5.10)$$

where

$$J^{(1)} = \frac{\partial y_1^{(1)}}{\partial x_1}\frac{\partial y_2^{(1)}}{\partial x_2} - \frac{\partial y_1^{(1)}}{\partial x_2}\frac{\partial y_2^{(1)}}{\partial x_1} ,$$

and $\lambda^{(1)}$ can now be determined directly from the first approximation $y_\alpha^{(1)}$. When $\lambda \ll 1$ and $J^{(1)}$ is not small, terms of λ^4 and higher in equation (5.10) can be neglected for our purposes and we have

$$J^{(1)}\lambda^{(1)} - 1 = 0 \quad \text{or} \quad \lambda^{(1)} = \frac{1}{J^{(1)}} ,$$

which is the value given for $\lambda^{(1)}$ in (3.6) of Section III. Near a traction-free edge of the membrane, however, $J^{(1)}$ becomes small and vanishes on the edge and the term in λ^4 must be retained in the equation even though λ is still much smaller than unity. Thus the equation

$$\lambda^4[\,|\nabla y_1^{(1)}|^2 + |\nabla y_2^{(1)}|^2] + J^{(1)}\lambda - 1 = 0 \tag{5.11}$$

will determine $\lambda^{(1)}$ with sufficient accuracy both on the boundary and inside the sheet. When $J^{(1)}$ is positive equation (5.11) has only one positive root. On the traction-free edge $\lambda^{(1)}$ is determined by

$$[\lambda^{(1)}]^4[\,|\nabla y_1^{(1)}|^2 + |\nabla y_2^{(1)}|^2] - 1 = 0 \quad \text{or} \quad \lambda^{(1)} = [\,|\nabla y_1^{(1)}|^2 + |\nabla y_2^{(1)}|^2]^{-\frac{1}{4}}.$$

Since the normal derivatives of $y_\alpha^{(1)}$ vanish at the edge, we see that on the edge

$$\lambda^{(1)} = \left[\left(\frac{\partial y_1^{(1)}}{\partial s^\circ}\right)^2 + \left(\frac{\partial y_2^{(1)}}{\partial s^\circ}\right)^2\right]^{-\frac{1}{4}} ,$$

in agreement with (5.1). It is because $\varphi^{(1)}$ and $\psi^{(1)}$ satisfy exact boundary conditions where traction is prescribed that equation (5.11) for $\lambda^{(1)}$ yields reliable results up to and including a traction free boundary. We remark that the approximation (5.11) to equation (5.10) will not apply near a point in the sheet which is unstressed. At such a point $\lambda = 1$ and the derivatives of φ, ψ vanish, equation (5.10) being satisfied. Such stress-free points occur at projecting corners in a traction-free portion of the boundary.

When the sheet extends to infinity and $\varphi^{(1)}$, $\psi^{(1)}$ are harmonic conjugates of $y_1^{(1)}$, $y_2^{(1)}$, respectively, the conditions at infinity corresponding to conditions (3.4) on $y_\alpha^{(1)}$ are

$$\left.\begin{aligned} \frac{\partial\varphi^{(1)}}{\partial x_1} &= \frac{L_1^*\sin\theta}{4\pi h_o C_1 r} + o\left(\frac{1}{r}\right), \quad \frac{\partial\varphi^{(1)}}{\partial x_2} = \mu_1 - \frac{L_1^*\cos\theta}{4\pi h_o C_1 r} + o\left(\frac{1}{r}\right), \\ \frac{\partial\psi^{(1)}}{\partial x_1} &= -\mu_2 + \frac{L_2^*\sin\theta}{4\pi h_o C_1 r} + o\left(\frac{1}{r}\right), \quad \frac{\partial\psi^{(1)}}{\partial x_2} = -\frac{L_2^*\cos\theta}{4\pi h_o C_1 r} + o\left(\frac{1}{r}\right), \end{aligned}\right\} \text{ as } r \to \infty \tag{5.12}$$

For moderate values of μ_1, μ_2 the difference between the conditions (5.12) and the exact conditions (5.7) is appreciable, and this leads to appreciable error in the values of λ determined by (5.11) at large distances. A better procedure in these cases is to determine the harmonic functions $\varphi^{(1)}, \psi^{(1)}$ so as to satisfy the exact conditions at infinity rather than to take $\varphi^{(1)}, \psi^{(1)}$ conjugate to $y_\alpha^{(1)}$. A first approximation $\lambda^{(1)}$ which will tend to the exact value as $r \to \infty$ is then determined by using $\varphi^{(1)}, \psi^{(1)}$ in equation (5.9). For our purposes it is sufficient to determine $\lambda^{(1)}$ as the positive root of

$$\lambda^4[\,|\nabla\varphi^{(1)}|^2 + |\nabla\psi^{(1)}|^2] + j^{(1)}\lambda - 1 = 0 \,. \tag{5.13}$$

It may be noted that the stress functions φ, ψ introduced in this section are directly related to the Airy stress function used for two-dimensional stress fields. It is easy to show that, save for a multiplicative factor,

$$\frac{\partial\varphi}{\partial y_1} = -T_{12}, \qquad \frac{\partial\psi}{\partial y_1} = -T_{22},$$

$$\frac{\partial\varphi}{\partial y_2} = T_{11}, \qquad \frac{\partial\psi}{\partial y_2} = T_{21} = T_{12},$$

and

$$\varphi = \frac{\partial \chi}{\partial y_2} \quad , \qquad \psi = - \frac{\partial \chi}{\partial y_1} \quad ,$$

where χ is the usual Airy stress function.

VI. SOME PROBLEMS WITH TRACTION-FREE BOUNDARIES

In this section several simple examples involving membranes with traction-free edges are considered, and the application of the modified method is illustrated. In the first two examples, the membrane is under biaxial extension at infinity and has a traction-free interior boundary which is either a circle or an ellipse. Example 3 concerns a finite membrane with mixed boundary conditions, and the last example treats a boundary with a corner which is either traction free or acted upon by a concentrated load at the vertex.

Example 1 Circular hole in an infinite membrane under biaxial extension at infinity.

The deformation of an infinite membrane with a circular hole of radius a subjected to uniform biaxial extension at infinity was considered in Section IV as a special case of Example 3. The first approximation shows that after deformation the hole becomes an ellipse with major and minor axes of lengths $2\mu_1 a$ and $2\mu_2 a$, respectively, μ_1 being the larger of the two extension ratios.

The harmonic stress functions $\varphi^{(1)}$, $\psi^{(1)}$ which satisfy exact boundary conditions at $r = a$ and at infinity are found to be

$$\varphi^{(1)} = \gamma_1 r \sin\theta \left(1 - \frac{a^2}{r^2}\right) , \qquad \psi^{(1)} = -\gamma_2 r \cos\theta \left(1 - \frac{a^2}{r^2}\right) ,$$

where

$$\gamma_1 = \mu_1 - \frac{\mu_2}{(\mu_1\mu_2)^3} , \qquad \gamma_2 = \mu_2 - \frac{\mu_1}{(\mu_1\mu_2)^3} .$$

From equation (5.13), the transverse extension ratio $\lambda^{(1)}$ is the positive real root of the equation

$$\lambda^4\left[\left(\gamma_1^2+\gamma_2^2\right)\left(1+\frac{a^4}{r^4}\right)-2a^2\left(\gamma_1^2-\gamma_2^2\right)\frac{\cos 2\theta}{r^2}\right]+\lambda\left[\gamma_1\gamma_2\left(1-\frac{a^4}{r^4}\right)\right]-1=0,$$

and at the edge of the hole where $r = a$, we find

$$\lambda^{(1)}=\left\{2\left[\gamma_1^2+\gamma_2^2-\cos 2\theta\,(\gamma_1^2-\gamma_2^2)\right]\right\}^{-\frac{1}{4}}, \tag{6.1}$$

which has a maximum value $(2\gamma_2)^{-\frac{1}{2}}$ at $(\theta = 0, \pi)$ and a minimum value $(2\gamma_1)^{-\frac{1}{2}}$ at $(\theta = \pi/2, 3\pi/2)$.

When the material is under simple tension at infinity, the extension ratios μ_1, μ_2 are related by

$$\mu_2^2 = \frac{1}{\mu_1} = \frac{1}{\mu}.$$

We then have, from (4.18)

$$\left.\begin{aligned} y_1^{(1)} &= \mu r\left(1+\frac{a^2}{r^2}\right)\cos\theta\,, \\ y_2^{(1)} &= \frac{1}{\sqrt{\mu}}\, r\left(1+\frac{a^2}{r^2}\right)\sin\theta\,. \end{aligned}\right\} \tag{6.2}$$

In agreement with the discussion in Section III, $y_2^{(1)}$ is of the same order, $O(\mu^{-\frac{1}{2}})$, as the neglected terms in (2.10), (2.11) and therefore may not be a good approximation to y_2. On the other hand, the difference $y_1^{(2)} - y_1^{(1)}$ is $O(\mu^{-2})$ and so, for large μ, $y_1^{(1)}$ is a good approximation to y_1. From (6.2) the maximum principal extension ratio at the edge of the hole is given by

$$\Lambda_1 = [\,2(2 - \cos 2\theta)]^{\frac{1}{2}}\mu \quad ,$$

correct to $O(\mu^{-2})$.

Another special case of interest is the axi-symmetric deformation of an infinite sheet containing a circular hole. In this case $\mu_1 = \mu_2 = \mu$ and equations (4.18), (6.1) assume the simple forms

$$y_1^{(1)} = \rho^{(1)}(r)\cos\theta \,, \quad y_2^{(1)} = \rho^{(1)}(r)\sin\theta \,; \quad \rho(r) = \mu r\left(1 + \frac{a^2}{r^2}\right) \,, \qquad (6.3)$$

$$\lambda^4\left[2\gamma^2\left(1 + \frac{a^4}{r^4}\right)\right] + \lambda\left[\gamma^2\left(1 - \frac{a^4}{r^4}\right)\right] - 1 = 0 \,, \qquad (6.4)$$

where

$$\gamma = \mu\left(1 - \frac{1}{\mu^6}\right) \,.$$

Because of the symmetry of the deformation, the exact formulation itself can be greatly simplified. As in Example 1, if we assume y_α to be given by (4.7), then $\rho(r)$ satisfies the ordinary differential equation (4.8). The condition on $d\rho/dr$ at infinity remains unchanged but at $r = a$, the traction-free condition requires

$$\frac{d\rho}{dr} = \lambda^3\,\frac{\rho}{r} \,,$$

or

$$\left(\frac{d\rho}{dr}\right)^2 \frac{\rho}{r} = 1 \,, \qquad (6.5)$$

since

$$\lambda = r\Big/\left(\rho\,\frac{d\rho}{dr}\right)$$

in this case.

Equations equivalent to (4.7), (4.8) were obtained by Rivlin and Thomas [4] and, by successive application of Taylor series expansions starting from the edge of the hole, they were able to find numerical solutions for given values of the circumferential extension ratio ρ/r at the edge of the hole $r = a$. A more direct method is to integrate equation (4.8) numerically, as in Example 1. A value for ρ/r at $r = a$ is chosen, and the slope $d\rho/dr$ at the same point is then obtained from (6.5). With these initial values, the integration is straightforward. Corresponding to each initial value of ρ/r there is a limiting value for the extension ratio at infinity μ. Figure 1 compares the exact solution ρ/r thus obtained and the approximate solution calculated from (6.3) for all-round extension to moderate amount $\mu = 1.62$ at infinity, the corresponding circumferential extension ratio at the hole being 3.0. A discrepancy of 10% occurs at the edge of the hole but the difference diminishes rapidly as r increases and at a distance four times the radius of the hole, the difference is slight. The transverse extension ratio λ is plotted in Figure 2 against the radius, the approximate values determined from (6.4) being shown as circled points near the curve for the exact values. It can be seen that (6.4) provides good estimates for λ over the whole range of r. In contrast, values for λ determined from $y_\alpha^{(1)}$ through (5.11) are much less accurate and they are shown as crosses in the figure.

Comparisons between the exact (numerical) solution and the first approximation were also made for the case $\mu = 3.03$. Since figures for $\mu = 3.03$ corresponding to Figures 1 and 2 for $\mu = 1.62$ would show no difference between the exact and approximate solutions for either ρ/r or λ, the results are not shown here (the variation of ρ/r with r for $\mu = 3.03$ is given in [4]).

Example 2 Elliptic hole in an infinite membrane under biaxial extension at infinity.

As noted earlier, the deformation of an infinite membrane with a traction-free elliptic hole of semi-axes a and b subjected to biaxial extension at infinity parallel to the axes of the ellipse can be obtained by setting m equal to zero in (4.20) which then becomes

$$\left.\begin{aligned} y_1^{(1)} &= \frac{c\mu_1}{a-b}(a\cosh\xi - b\sinh\xi)\cos\eta\ , \\ y_2^{(1)} &= \frac{c\mu_2}{a-b}(a\cosh\xi - b\sinh\xi)\sin\eta\ . \end{aligned}\right\} \tag{6.6}$$

According to (6.6), the hole is again an ellipse in the deformed state with semi-axes of lengths $(a+b)\mu_1$ and $(a+b)\mu_2$. When $\mu_1 = \mu_2 = \mu$, the hole is always deformed into a circle of radius $(a+b)\mu$.

Stress functions $\varphi^{(1)}, \psi^{(1)}$ which satisfy exact boundary conditions at infinity and on the ellipse are readily determined and equation (5.13) assumes the form

$$\begin{aligned} \lambda^4[\,&(a\sinh\xi - b\cosh\xi)^2(\gamma_1^2\cos^2\eta + \gamma_2^2\sin^2\eta) \\ &+(a\cosh\xi - b\sinh\xi)^2(\gamma_1^2\sin^2\eta + \gamma_2^2\cos^2\eta)] \\ &+\lambda\gamma_1\gamma_2(a\sinh\xi - b\cosh\xi)(a\cosh\xi - b\sinh\xi) - (a-b)^2(\sinh^2\xi + \sin^2\eta) = 0\ , \end{aligned}$$

where the constants γ_1, γ_2 are as defined for Example 1. At the edge of the hole where $\xi = \xi_o$ and

$$c \cosh \xi_o = a \,, \quad c \sinh \xi_o = b,$$

the equation for $\lambda^{(1)}$ reduces

$$\lambda^4 (a+b)^2(\gamma_1^2 \sin^2\eta + \gamma_2^2 \cos^2\eta) - (a^2 \sin^2\eta + b^2\cos^2\eta) = 0.$$

It can be shown that the first approximation $\lambda^{(1)}$ to the transverse extension ratio λ attains its extremum values

$$\left[\frac{a}{(a+b)\gamma_1}\right]^{\frac{1}{2}} , \quad \left[\frac{b}{(a+b)\gamma_2}\right]^{\frac{1}{2}} ,$$

at the ends of the major and minor axes of the ellipse. When $a\mu_2 = b\mu_1$, the hole is deformed into an ellipse of similar shape and since $a\gamma_2$ and $b\gamma_1$ are nearly equal the edge of the deformed hole has nearly constant thickness.

If $a\gamma_2 > b\gamma_1$, the hoop stress resultant is greatest at the two ends of the minor axis and, for μ_1 large, has the approximate value

$$T_s = 2h_o C_1 \left(1 + \frac{b}{a}\right)^{\frac{3}{2}} \mu_1^{\frac{3}{2}} ,$$

which is correct to terms of $O(\mu_1^{-\frac{3}{2}})$. When $a\gamma_2 < b\gamma_1$, the maximum is attained at the two ends of the major axis instead and T_s has the approximate value

$$T_s = 2h_o C_1 \left(1 + \frac{a}{b}\right)^{\frac{3}{2}} \mu_2^{\frac{3}{2}}$$

for large μ_2.

In the limit as b goes to zero, the hole degenerates into a crack or slit of length 2c along the x_1-axis. Setting $b = 0$ in (6.6) we have

$$\left.\begin{aligned} y_1^{(1)} &= \mu_1 c \cosh \xi \cos \eta = \mu_1 x_1 , \\ y_2^{(1)} &= \mu_2 c \cosh \xi \sin \eta , \end{aligned}\right\} \tag{6.7}$$

and we see that the crack $\xi = 0$ becomes an ellipse with semi-axes $\mu_1 c$ and $\mu_2 c$ in the deformed state. It may be noted also that the transverse extension ratio $\lambda^{(1)}$ has the unique limit zero as the tip of the crack is approached. Equations (6.7) can be written as

$$y_1^{(1)} = \mu_1 x_1 , \quad y_2^{(1)} = \frac{\mu_2}{\sqrt{2}} \left\{ c^2 - x_1^2 + x_2^2 + \left[(c^2 - x_1^2 - x_2^2)^2 + 4x_2^2 \right]^{\frac{1}{2}} \right\}^{\frac{1}{2}} . \tag{6.8}$$

Figure 3 indicates the deformation (6.8) for the sheet with a crack when $\mu_1 = \mu_2 = 2.0$. Because of symmetry, only the first quadrant of the plane is shown. The solid lines initially formed a square grid of lines distant 0.2 units apart. Initially the crack extended from -1.0 to 1.0 unit on the x_1-axis, and it is deformed into a circle of radius 2.0 units. Vertical grid lines remain vertical and the deformation is most severe at the tip of the crack, as expected.

When the principal axes of strain at infinity are inclined to the x_1-, x_2-axes, we require

$$y_\alpha = c_{\alpha\beta} x_\beta + o\left(\frac{1}{r}\right) \quad \text{as} \quad r \to \infty, \tag{6.9}$$

where $c_{\alpha\beta}$ are known constants. If we define the harmonic functions z_α through

$$z_1 = \frac{c}{a-b}(a\cosh\xi - b\sinh\xi)\cos\eta, \quad z_2 = \frac{c}{a-b}(a\cosh\xi - b\sinh\xi)\sin\eta,$$

we see that $z_\alpha = x_\alpha$ at infinity and the normal derivatives of z_α vanish on the ellipse (4.19). The first approximation $y_\alpha^{(1)}$ to y_α for the sheet with an elliptic hole and the deformation (6.9) at infinity is then

$$y_\alpha^{(1)} = c_{\alpha\beta} z_\beta \; .$$

Under the deformation in which a particle at the point x_α goes to z_α, the elliptic hole with semi-axes a, b becomes a circle of radius (a+b). The transformation in which z_α goes to $y_\alpha^{(1)}$ subjects the whole plane to the deformation at infinity. Thus according to the first approximation, under all orientations the elliptic hole becomes an ellipse with semi-axes of lengths $(a+b)\mu_1$, $(a+b)\mu_2$ parallel to the principal directions of strain at infinity, where μ_1 and μ_2 are the principal extension ratios at infinity.

For a sheet with N holes bounded by contours $\overset{\circ}{\Pi}_n$, we introduce harmonic functions z_α which have zero normal derivatives on $\overset{\circ}{\Pi}_n$ and which are such that $z_\alpha = x_\alpha$ at infinity. The first approximation to the deformation when (6.9) holds at infinity will then be $y_\alpha^{(1)} = c_{\alpha\beta} z_\beta$. If the contours Π_n' obtained from $\overset{\circ}{\Pi}_n$ by the transformation in which x_α goes to z_α are drawn on the undeformed sheet at infinity, the holes in the stretched sheet will assume the same shape, orientation and relative position as the contours Π'_n drawn on the sheet at infinity.

Example 3 Stretching of a short wide strip between clamps (pure shear).

In order to produce a state of pure shear experimentally [6,7], a short wide strip of rubber is stretched between clamps applied to the long edges of the sheet. The extension ratio in the direction of the width of the sheet is then almost unity in the deformation produced by moving the clamps apart, and the sheet is in a state of pure shear if the volume remains unchanged. When the width to height ratio is large the small amount of non-uniformity in strain due to the traction-free edges does not affect the force-extension relation significantly.

If the strip has width a and height b and the origin is taken at the center of the sheet with the x_1-axis along the width of the sheet, the harmonic functions $y_\alpha^{(1)}$ must satisfy the boundary conditions

$$\frac{\partial y_\alpha^{(1)}}{\partial x_1} = 0 \quad \text{on } x_1 = \pm \frac{a}{2},$$

and

$$\left.\begin{array}{l} y_1^{(1)} = x_1 , \\ y_2^{(1)} = \mu x_2 , \end{array}\right\} \quad \text{on } x_2 = \pm \frac{b}{2},$$

where μb is the height of the deformed strip. We therefore obtain

$$
\left.
\begin{aligned}
y_1^{(1)} &= \frac{4a}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \frac{\cosh \frac{2n+1}{a} \pi x_2}{\cosh \frac{2n+1}{2a} \pi b} \sin \frac{2n+1}{a} \pi x_1 = x_1 - F(x_1, x_2), \\
y_2^{(1)} &= \mu x_2 , \\
\text{where} & \\
F(x_1, x_2) &= \frac{4b}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \frac{\sinh \frac{2n+1}{b} \pi x_1}{\cosh \frac{2n+1}{2b} \pi a} \cos \frac{2n+1}{b} \pi x_2 .
\end{aligned}
\right\} \quad (6.10)
$$

According to (6.10), lines of the sheet initially horizontal remain so after deformation and the location $y_1^{(1)}$ is independent of μ. The shortening S of the line midway between the clamped edges is given by

$$S = 0.742 \frac{b}{a} \times 100\,\% , \qquad (6.11)$$

which decreases with b/a but is independent of μ. For b/a small, the extension ratio λ_1 in the direction parallel to the clamped edges of the sheet is substantially unity, and in the limit

$$\lambda_1 = 1, \quad \lambda_2 = \mu, \quad \lambda = \frac{1}{\mu} ,$$

a state of pure shear.

Since the material at the free edges is in simple extension with extension ratio μ approximately, a better first approximation for y_1 is the harmonic function which satisfies the boundary conditions

$$y_1^{(1)} = x_1 \quad \text{on} \quad x_2 = \pm \frac{b}{2} \ ,$$

$$\frac{\partial y_1^{(1)}}{\partial x_1} = \frac{1}{\sqrt{\mu}} \quad \text{on} \quad x_1 = \pm \frac{a}{2} \, .$$

Hence we have

$$y_1^{(1)} = x_1 - \left(1 - \frac{1}{\sqrt{\mu}}\right) F(x_1, x_2) \ , \tag{6.12}$$

and the shortening S of the middle line is now given by

$$S = 0.742 \, \frac{b}{a} \left(1 - \frac{1}{\sqrt{\mu}}\right) \times 100\,\% \ . \tag{6.13}$$

If we write the second approximation $y_1^{(2)}$ to y_1 as

$$y_1^{(2)} = y_1^{(1)} + \omega \ ,$$

in which $y_1^{(1)}$ is given by (6.12), then ω satisfies the Poisson equation

$$\nabla^2 \omega = -2\pi\rho \ ,$$

with

$$\rho = \frac{3\mu}{2\pi} \left[J^{(1)} \right]^{-4} \frac{\partial J^{(1)}}{\partial x_1} \ , \qquad J^{(1)} = \mu \frac{\partial y_1^{(1)}}{\partial x_1} \ .$$

The boundary conditions on ω are

$$\omega = 0 \quad \text{on} \quad x_2 = \pm \frac{b}{2} \ ,$$

$$\frac{\partial \omega}{\partial x_1} = 0 \quad \text{on} \quad x_1 = \pm \frac{a}{2} \ ,$$

and we see that ω will be zero on $x_1 = 0$.

Now as x_1 goes from 0 to $a/2$, $J^{(1)}$ decreases from μ to $\sqrt{\mu}$ so that $\partial J^{(1)}/\partial x_1$ is negative for $x_1 > 0$. Because $J^{(1)}$ is even in x_1, we see then that ρ is odd in x_1 and negative for $x_1 > 0$. In the terminology of electrostatics, for $x_1 > 0$ the function ω is the potential of a distribtuion of (negative) charge with density ρ in a rectangular sheet which has zero potential at three sides and zero charge line-density at the fourth. Consequently, the potential inside the sheet must be non-positive, $\omega \le 0$ for $x_1 \ge 0$. The shortening S predicted by $y_1^{(2)}$ will then be greater than that calculated from $y_1^{(1)}$. Since (6.11) overestimates the shortening of the middle line the actual value must lie between the two values (6.11), (6.13).

The total "charge" in the right half of the strip is

$$Q = \int_{-b/2}^{b/2}\int_0^{a/2} \rho \, dx_1 \, dx_2 = -\frac{b}{2\pi}\left[\{J^{(1)}\}^{-3}\right]_0^{a/2} = -\frac{b}{2\pi\sqrt{\mu}}\left(1-\mu^{-\frac{3}{2}}\right) .$$

For b/a small, the change in $J^{(1)}$ from μ at $x_1 = 0$ to $\sqrt{\mu}$ at $x_1 = a/2$ occurs mostly in a narrow band near the traction-free edge $x_1 = a/2$. For a good estimate ω^* of ω, therefore, we can assume that all of the charge is concentrated along the line $x_1 = a/2$ with charge density

$$\frac{Q}{b} = -\frac{1}{2\pi\sqrt{\mu}}\left(1-\mu^{-\frac{3}{2}}\right)$$

per unit length. Thus ω^* is the harmonic function which satisfies the boundary conditions

$$\omega^* = 0 \quad \text{on } x_2 = \pm\frac{b}{2}, \quad x_1 = 0 ,$$

$$\frac{\partial\omega^*}{\partial x_1} = -\frac{1}{\sqrt{\mu}}\left(1 - \mu^{-\frac{3}{2}}\right) .$$

It follows that

$$\omega^* = -\frac{1}{\sqrt{\mu}}\left(1 - \mu^{-\frac{3}{2}}\right) F(x_1, x_2)$$

and with this value for ω we have

$$y_1^{(2)} = x_1 - \left(1 - \frac{1}{\mu^2}\right) F(x_1, x_2) \tag{6.14}$$

The second approximation $y_2^{(2)}$ to y_2 differs from $y_2^{(1)}$ by terms which are $O(\mu^{-3})$.

The same expression (6.14) for $y_1^{(2)}$ can be obtained by means of the stress functions φ and ψ. The first approximations $\varphi^{(1)}, \psi^{(1)}$ are harmonic conjugates to $y_\alpha^{(1)}$ in (6.10) and it is easy to see that they are

$$\varphi^{(1)} = G(x_1, x_2), \qquad \psi^{(1)} = -\mu x_1 ,$$

where

$$G(x_1, x_2) = \frac{4a}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \frac{\sinh \frac{2n+1}{a}\pi x_2}{\cosh \frac{2n+1}{2a}\pi b} \cos \frac{2n+1}{a}\pi x_1 .$$

The extension ratio $\lambda^{(1)}$ can be obtained from equation (5.13) in terms of $\varphi^{(1)}$ and $\psi^{(1)}$. In view of the fact that a large part of the sheet is under pure shear, we can, for a good estimate of the second approximation $\varphi^{(2)}$ to φ, choose $\lambda^{(1)}$ to be $1/\mu$ throughout the sheet. $\varphi^{(2)}$ is then the harmonic function which satisfies the boundary conditions

$$\frac{\partial \varphi^{(2)}}{\partial x_2} = 1 - \frac{1}{\mu^2} \quad \text{on} \quad x_2 = \pm \frac{b}{2} \, ,$$

and

$$\varphi^{(2)} = 0 \quad \text{on} \quad x_1 = \pm \frac{a}{2} \, .$$

Thus we have

$$\varphi^{(2)} = \left(1 - \frac{1}{\mu^2} \right) G(x_1, x_2) \, .$$

If we now use the inversion relations

$$\frac{\partial y_1^{(2)}}{\partial x_1} = \frac{\partial \varphi^{(2)}}{\partial x_2} + \frac{1}{\mu^2} \, , \quad \frac{\partial y_1^{(2)}}{\partial x_2} = - \frac{\partial \varphi^{(2)}}{\partial x_1} \, ,$$

we obtain the solution given in (6.14).

According to (6.14), the shortening S of the middle line is given by

$$S = 0.742 \frac{b}{a} \left(1 - \frac{1}{\mu^2} \right) \times 100\% \, . \qquad (6.15)$$

When $a/b = 15$, S is 4.8% when μ is 6.2. In an experiment with a strip of rubber having dimensions such that $a/b = 15$, Treloar [6] observed for $\mu = 6.2$ a shortening of the middle line of 12%, which is more than twice the theoretical value $S = 4.8\%$ for a neo-Hookean material. Rivlin and Saunders [7] conducted a similar experiment and they report a shortening of 3% for the extension ratio $\mu = 2.2$.

The ratio a/b for the specimen employed in their experiment is not given in [7] but a figure suggests that the ratio $a/b = 20$ was used. With $a/b = 20$ and $\mu = 2.2$, formula (6.15) predicts a shortening of 2.9%.

The discrepancy between theory and experiment for the large extension ratio $\mu = 6.2$ is due to the fact that the neo-Hookean form is not a good representation for the strain energy function of the rubber for extension ratios greater than 2 or 3. A better strain energy function for rubber is the Mooney form

$$U = h_o C_1 \left[\lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2 \lambda_2^2} + \Gamma \left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \lambda_1^2 \lambda_2^2 \right) \right],$$

where $\Gamma = C_2/C_1$ and C_2, C_1 are the material constants. For a state of pure shear with $\lambda_1 = 1$, $\lambda_2 = \mu$ the stress resultant across the width of the strip is

$$T_1 = \frac{2h_o C_1}{\mu} \left[\left(1 - \frac{1}{\mu^2} \right) + \Gamma(\mu^2 - 1) \right] .$$

Even a small value for Γ increases T_1 significantly at large values of μ. A greater curvature is then required at the traction-free edges in order to provide the resultant T_1 in the middle of the strip, and the shortening of the strip is increased. With the expression (6.14) for y_1 for a neo-Hookean material, numerical results for the case $a/b = 15$ and $\mu = 6.2$ show that straight lines initially vertical on the sheet remain quite straight except in regions very close to the traction-free edges, within a distance of the order of $a/25$. This is in contrast to the experiment of Treloar [6] in which appreciable curvature was observed of vertical lines initially distant $a/5$ from the traction-free edges.

In order to superpose pure shear on simple extension, the strip is stretched in simple extension in the x_2-direction with extension ratio $1/\lambda_2^2$ before the clamps are applied [7]. The clamps are then moved apart so that the extension ratio in the x_2-direction becomes μ while that in the x_1-direction is substantially λ_2 throughout the sheet. In order to ensure tensile stresses everywhere μ must be greater than $1/\lambda_2^2$. If the initial width and height are again a and b, the boundary conditions are

$$\left.\begin{aligned} y_1 &= \lambda_2 x_1 , \\ y_2 &= \mu x_2 , \end{aligned}\right\} \qquad \text{on } x_2 = \pm \frac{b}{2} ,$$

and

$$\left.\begin{aligned} \frac{\partial y_1}{\partial x_1} &= \lambda^3 \frac{\partial y_2}{\partial x_2} , \\ \frac{\partial y_2}{\partial x_1} &= - \lambda^3 \frac{\partial y_1}{\partial x_2} , \end{aligned}\right\} \qquad \text{on } x_1 = \pm \frac{a}{2} .$$

As in the case $\lambda_2 = 1$, the first approximation for y_2 is

$$y_2^{(1)} = \mu x_2 ,$$

which is correct to $O(\lambda_2^{-2} \mu^{-3})$, while a second approximation for y_1 is found to be

$$y_1^{(2)} = \lambda_2 \left[x_1 - \left(1 - \frac{1}{\lambda_2^4 \mu^2} \right) F(x_1, x_2) \right] .$$

The shortening S of the middle line is then

$$S = 0.742 \frac{b}{a} \left(1 - \frac{1}{\lambda_2^4 \mu^2} \right) \times 100\ \% .$$

Example 4 Boundary with a corner. Concentrated load at a boundary.

We suppose that the undeformed sheet has a sharp corner with straight edges on one of its boundaries. The origin of the coordinate system is taken at the vertex of the corner and the x_1-axis is taken along the bisector of the corner angle. For $r \le a$, say, the boundaries at the corner will be the lines $\theta = \pm \alpha$, where 2α is the angle of the corner.

If the sides of the corner are traction free, the harmonic functions $y_\gamma^{(1)}$ of the first approximation have zero normal derivatives on $\theta = \pm\alpha$. For $r \le a$, the locations $y_\gamma^{(1)}$ will then have the representations

$$\left.\begin{aligned} y_1^{(1)} &= \sum_{n=0}^{\infty} \left[a_n \left(\frac{r}{a}\right)^{n\pi/\alpha} \cos \frac{n\pi}{\alpha}\theta + b_n \left(\frac{r}{a}\right)^{\frac{2n+1}{2\alpha}\pi} \sin \frac{2n+1}{2\alpha}\pi\theta \right], \\ y_2^{(1)} &= \sum_{n=0}^{\infty} \left[c_n \left(\frac{r}{a}\right)^{n\pi/\alpha} \cos \frac{n\pi}{\alpha}\theta + d_n \left(\frac{r}{a}\right)^{\frac{2n+1}{2\alpha}\pi} \sin \frac{2n+1}{2\alpha}\pi\theta \right]. \end{aligned}\right\} \quad (6.16)$$

where the constants a_n, b_n, c_n, d_n will be determined by the deformation elsewhere in the sheet. For example if $y_1 = \mu_1 x_1$, $y_2 = \mu_2 x_2$ on $r = a$ we have

$$a_o = a\mu_1 \sin\alpha/\alpha, \quad a_n = 2a\mu_1(-1)^{n+1}\sin\alpha/\left[\left(\frac{n\pi}{\alpha}\right)^2 - 1\right], \quad n = 1,2,\dots,$$

$$\left.\begin{aligned} b_n &= c_n = 0, \\ d_n &= 2a\mu_2(-1)^n\cos\alpha/\left[\left(\frac{2n+1}{2\alpha}\pi\right)^2 - 1\right], \end{aligned}\right\} \quad n = 0,1,2,\dots .$$

For $\alpha > \pi/2$ the corner is re-entrant and the derivatives $\partial y_\gamma^{(1)}/\partial x_\delta$ in (6.16) become infinite at the vertex. The corner is deformed into a smooth arc with a continuously turning tangent at the boundary point which was initially at the vertex. The radius of curvature of the deformed boundary at this point is

$$\frac{(b_o^2 + d_o^2)^{\frac{3}{2}}}{2\left|d_o a_1 - b_o c_1\right|},$$

where it is supposed that b_o and d_o do not both vanish. An example involving re-entrant corners (with $\alpha = \pi$) has been met in Example 2 of this section where the infinite sheet with a slit was treated. Under deformation the boundary of the crack became a smooth curve.

For $\alpha < \pi/2$ the corner projects and the derivatives $\partial y_\gamma^{(1)}/\partial x_\delta$ vanish at the vertex. For the material with strain energy $U^{(1)}$ this implies that the stress resultants vanish at the corner. A neo-Hookean material would have small strains in the neighborhood of the corner so that the first approximation $y_\gamma^{(1)}$ is not a good approximation near the corner.

If a concentrated load L^* acts at the vertex of the corner in a direction at an angle β counterclockwise from the negative x_1-direction, the derivatives $\partial y_\gamma/\partial x_\delta$ will be $O(r^{-1})$ as $r \to 0$, and we require

$$\lim_{r\to 0}\int_{-\alpha}^{\alpha} \frac{\partial y_1}{\partial r} r\, d\theta = \frac{L^*\cos\beta}{2h_o C_1}, \quad \lim_{r\to 0}\int_{-\alpha}^{\alpha} \frac{\partial y_2}{\partial r} r\, d\theta = \frac{L^*\sin\beta}{2h_o C_1}.$$

For $r \leq a$ the first approximation will be

$$y_1^{(1)} = \frac{L^*\cos\beta}{h_o C_1 \alpha} \log\frac{r}{a} + u(r,\theta) ,$$

$$y_2^{(1)} = \frac{L^*\sin\beta}{h_o C_1 \alpha} \log\frac{r}{a} + v(r,\theta) ,$$

where the harmonic functions u, v are finite at $r = 0$ and have representations of the form (6.16). The angle β must be such that the force L^* is directed away from the sheet, $|\beta| < \pi-\alpha$, but β may have to be restricted further, depending on conditions elsewhere in the sheet, in order to ensure tensile stresses everywhere.

If the load L^* acts in the negative x_1-direction, the first approximation will be

$$\left.\begin{aligned} y_1^{(1)} &= \frac{L^*}{h_o C_1 \alpha}\log\frac{r}{a} + \frac{a\sin\alpha}{\alpha} \\ &\quad + 2a\sin\alpha \sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{r}{a}\right)^{n\pi/\alpha} \cos\frac{n\pi}{\alpha}\theta \Big/ \left[\left(\frac{n\pi}{\alpha}\right)^2 - 1\right] , \\ y_2^{(1)} &= 2a\cos\alpha \sum_{n=0}^{\infty} (-1)^n \left(\frac{r}{a}\right)^{\frac{2n+1}{2\alpha}\pi} \sin\frac{2n+1}{2\alpha}\pi\theta \Big/ \left[\left(\frac{2n+1}{2\alpha}\pi\right)^2 - 1\right] , \end{aligned}\right\} \quad (6.17)$$

when the edge $r = a$ of the membrane is held fixed. This solution is valid for all values of α less than π other than $\pi/2$. For r small, we have

$$\left.\begin{aligned} y_1^{(1)} &= \frac{1}{h_o C_1 \alpha} \log \frac{r}{a} + O(1) \,, \\ y_2^{(1)} &= O(r^{\pi/2\alpha}) \,, \end{aligned}\right\} \quad \text{as } r \to 0 \,.$$

Hence we see that the principal extension ratio in the x_2-direction on the line $\theta = 0$ has the limit zero as r approaches zero if $\alpha < \pi/2$, but for $\alpha > \pi/2$, the limit is infinite.

When $\alpha = \pi/2$, we have the case of a concentrated load acting normal to the straight edge of a semi-circular membrane of radius a whose curved boundary is held fixed. Expression (6.17) can then be written as

$$\left.\begin{aligned} y_1^{(1)} &= A \log \frac{r}{a} + u(r,\theta) \,, \\ y_2^{(1)} &= x_2 \,, \end{aligned}\right\} \tag{6.18}$$

where

$$A = \frac{2L^*}{h_o C_1 \pi} \,, \quad u = \frac{2a}{\pi} + 2a \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{r}{a}\right)^{2n} \cos 2n\theta / \; (4n^2-1) \,,$$

so that $u = O(1)$ as $r \to 0$. Along the traction-free edge and at a distance A from the load, the square of the principal extension ratio in the direction tangential to the edge is 2. According to (6.18), for small r straight lines θ = constant of the undeformed sheet become logarithmic curves while the circular lines r = constant become straight vertical lines in the deformed membrane.

We note that since the outer edges $\theta = \pm\pi/2$ of the membrane are free of traction, the material there is in a state of simple extension and for r small the extension ratio is found to be

$$\lambda_s = (1 + A^2/r^2)^{\frac{1}{2}} \sim A/r \ .$$

Hence on the boundary near the load,

$$\lambda \sim (r/A)^{\frac{1}{2}} \ .$$

On the line $\theta = 0$ and for r small, $J \sim A/r$ and $\lambda \sim r/A$. Since the principal extension ratio in the x_2-direction is unity in this case, we see that the central line $\theta = 0$ is in pure shear in a plane perpendicular to the (r,θ) plane. Thus, for a neo-Hookean material, the curvature of the boundary is sufficient to build up enough tensile transverse stress so that the material in the center of the band is in pure shear even though the membrane in the neighborhood of the load is stretched out into a narrow band.

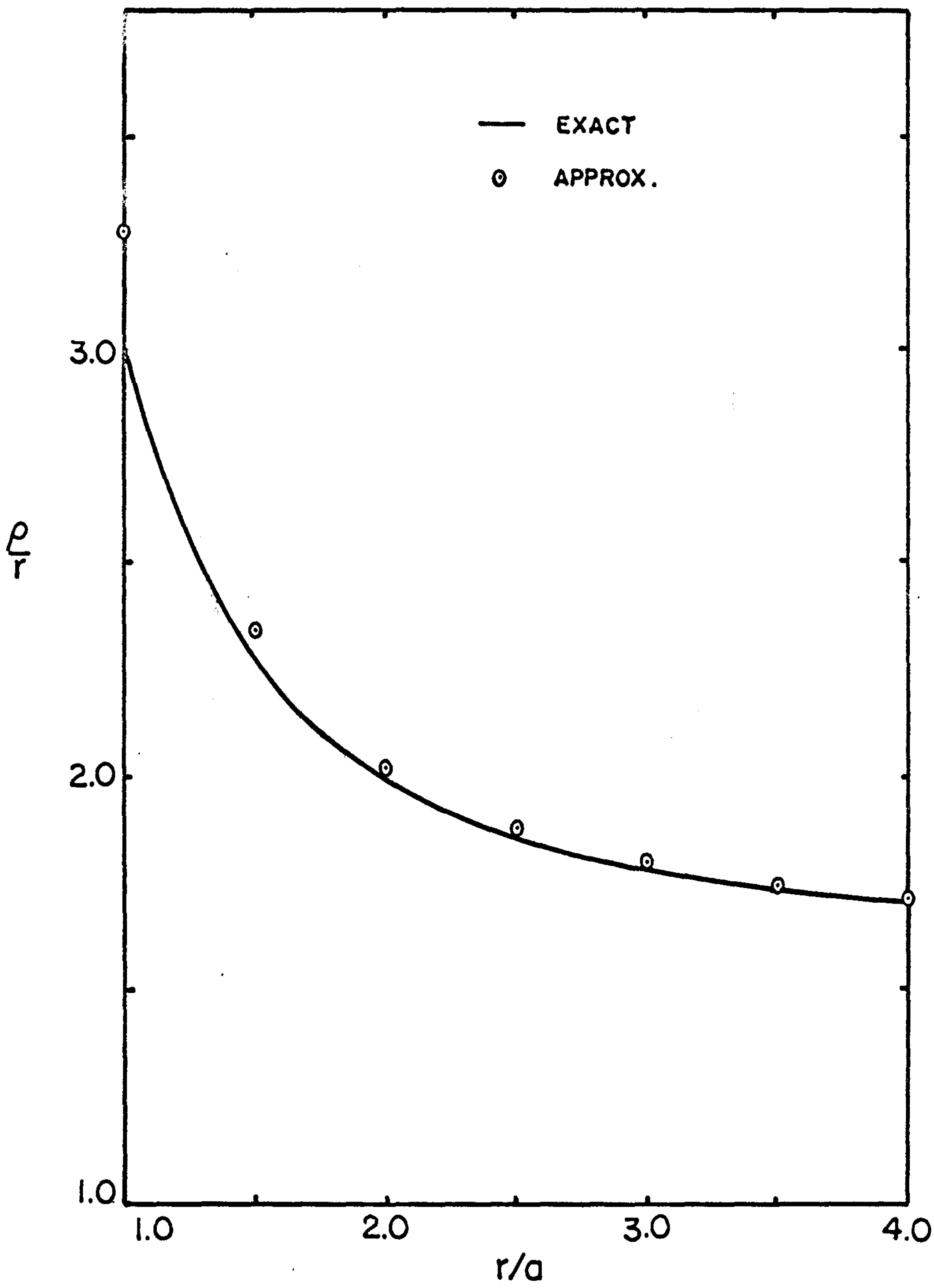

Fig. 1. Variation of ρ/r with r for sheet with circular hole under extension ratio $\mu = 1.62$ at infinity; exact and first approximation values.

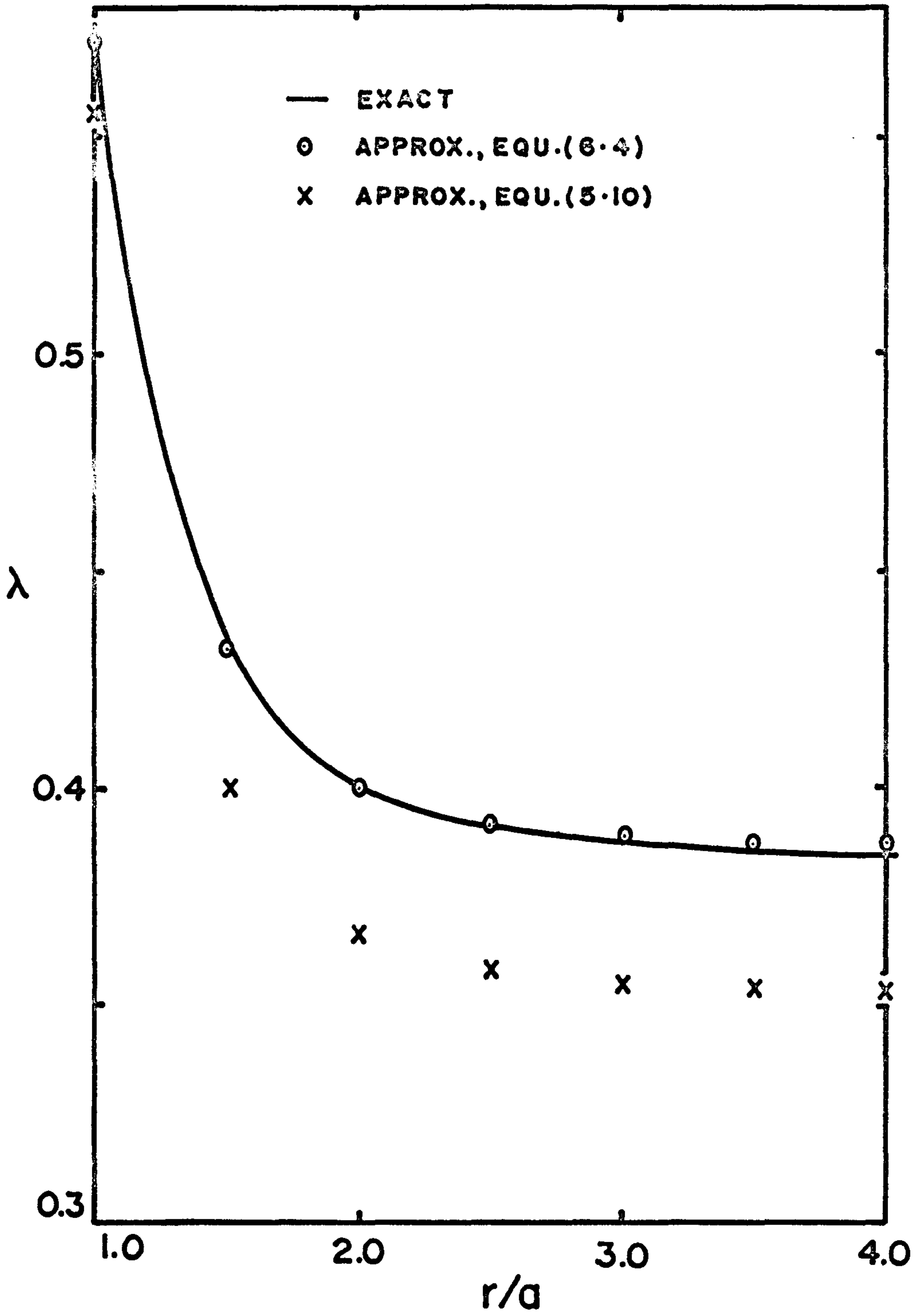

Fig. 2. Variation of thickness ratio λ for μ = 1.62; exact and first approximation values.

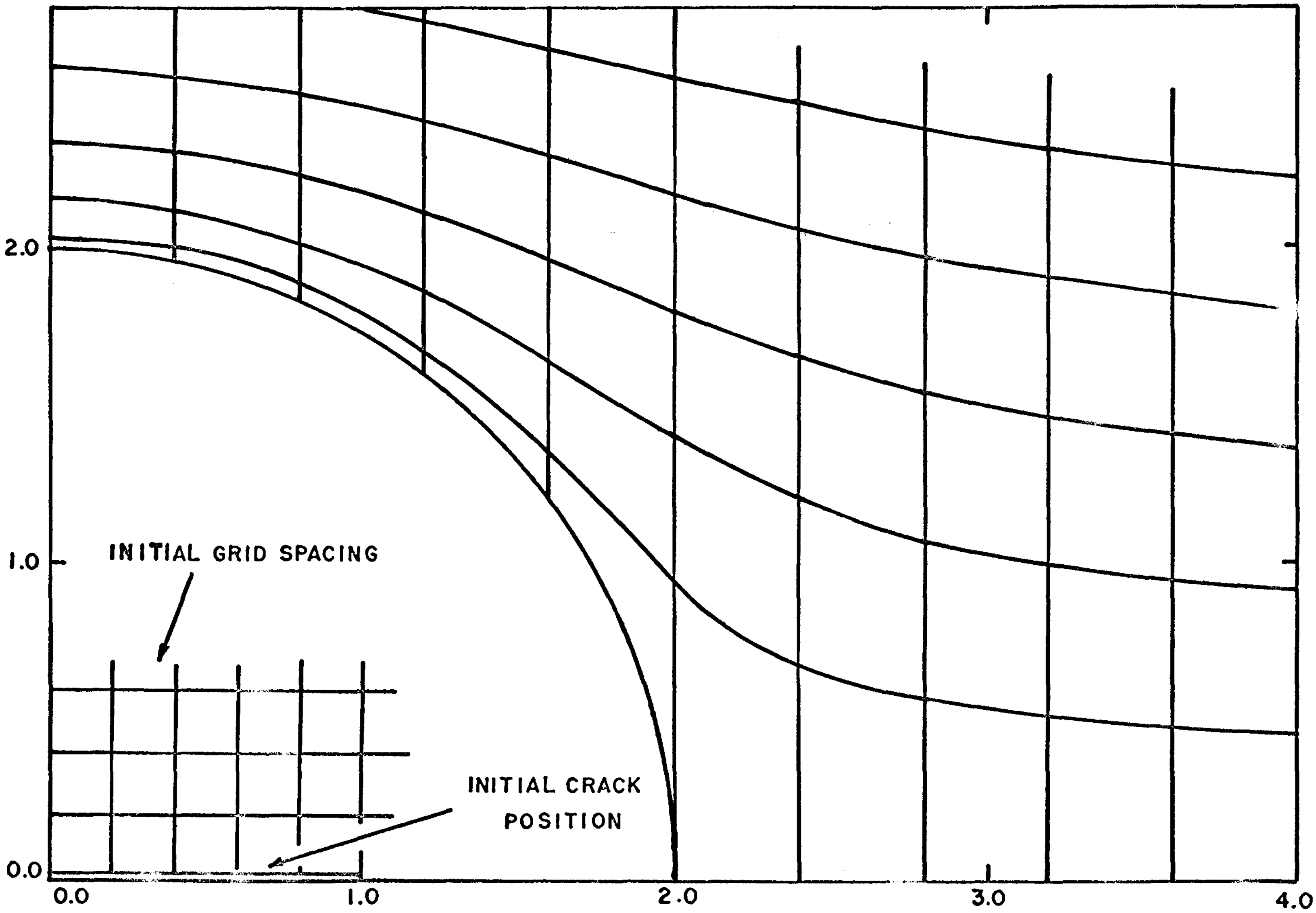

Fig. 3. Deformation of square grid near a crack; $\mu_1 = \mu_2 = 2.0$ at infinity.

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