MINIMUM DRAG PROFILES
IN INFINITE CAVITY FLOWS

Thesis by

Arthur K. Whitney

In Partial Fulfillment of the
Requirements for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California
1969

(Submitted September 27, 1968)

ACKNOWLEDGMENTS

I wish to thank my research advisor, Professor Theodore Y. Wu, of the California Institute of Technology, for his most generous assistance and encouragement over the last few years. Although previously unpublished, many of the ideas and methods used in this work are his and his alone; specifically, his recognition that non-classical variational methods might profitably be applied to optimization problems involving inviscid free surface flows and the subsequent reduction of the problem to a system of singular integral equations are significant contributions to the field of hydrodynamics.

I am grateful to the National Science Foundation for four years of tuition payments and financial assistance for myself and my family and to the California Institute of Technology for awarding me a Graduate Teaching Assistantship during my last year.

The insight gained in working closely with Dr. Wu over a one-summer period on an optimization problem involving a sprayless planing surface was of invaluable assistance in the preparation of the present work. This research was funded by the Naval Ship Research and Development Center under contract Nonr 220(51).

The excellent typing of Mrs. Barbara Hawk and skillful drafting of Miss Cecilia Lin make any thesis a pleasure to read.

To my friends and relatives, especially my parents, go my thanks for their encouragement.

Finally, the twin virtues of my wife, patience and fortitude, make even a graduate student's life bearable.

ABSTRACT

The problem considered is that of minimizing the drag of a symmetric plate in infinite cavity flow under the constraints of fixed arclength and fixed chord. The flow is assumed to be steady, irrotational, and incompressible. The effects of gravity and viscosity are ignored.

Using complex variables, expressions for the drag, arclength, and chord, are derived in terms of two hodograph variables, Γ (the logarithm of the speed) and β (the flow angle), and two real parameters, a magnification factor and a parameter which determines how much of the plate is a free-streamline.

Two methods are employed for optimization:

(1) The parameter method. Γ and β are expanded in finite orthogonal series of N terms. Optimization is performed with respect to the N coefficients in these series and the magnification and free-streamline parameters. This method is carried out for the case $N = 1$ and minimum drag profiles and drag coefficients are found for all values of the ratio of arclength to chord.

(2) The variational method. A variational calculus method for minimizing integral functionals of a function and its finite Hilbert transform is introduced. This method is applied to functionals of quadratic form and a necessary condition for the existence of a minimum solution is derived. The variational method is applied to the minimum drag problem and a nonlinear integral equation is derived but not solved.

TABLE OF CONTENTS

I. INTRODUCTION

Consider the irrotational, two-dimensional motion of an incompressible fluid past a cylinder of arbitrary shape in the absence of gravity. Let the flow far upstream be uniform with velocity U, pressure p_∞, and density ρ. At sufficiently high speeds, the local pressures in the fluid decrease to the vapor pressure and the liquid "boils," leaving a vapor cavity behind the body. In most cases the density of the cavity vapor is very small compared with that of the liquid so that the kinetic energy of the vapor motion has only a small effect on the pressure within the cavity. Therefore, the cavity pressure can be closely approximated by a constant, say p_c. Bernoulli's equation then prescribes that the velocity at the cavity wall be a constant, say V, where

$$\frac{1}{2}\rho V^2 + p_c = \frac{1}{2}\rho U^2 + p_\infty \quad .$$

This is the "free-streamline model" for cavitating flow.

As the free stream speed increases, the length and width of the cavity grow indefinitely and the flow may be idealized by <u>Helmholtz</u> flow; i.e., the cavity extends to infinity downstream, the fluid motion is steady and irrotational, and the cavity pressure is constant. In this model, the free stream pressure must equal the cavity pressure to maintain the cavity far downstream. Hence, $U = V$ in Helmholtz flow.

It will simplify matters by considering only that portion of the body in contact with the fluid. The lift and drag acting on the body and the cavity shape are determined by only this portion of the cylinder surface which we call P. We may replace the cylindrical body by a plate with the same shape as P without changing the flow.

The specific problem considered here is to find the shape of a symmetric plate (see Fig. 1) of given arclength $2s_o$ and given width $2y_o$, so that the drag of this plate in infinite cavity flow is a minimum. (A precise definition of the class of plates under consideration is given in Chapter II.)

In 1938, M. Lavrentieff (see, e.g., [1], p. 386) published the solution to the following minimum drag problem: Find the shape of a symmetric plate of minimum drag if the plate is confined to a rectangle (see Fig. 2(a)). The nose of the plate must start at $(0, 0)$ and extend to the two corners of the rectangle, (x_o, y_o) and $(x_o, -y_o)$. The solution for the optimum profile consists of a straight section extending from $(0, -h)$ to $(0, h)$ and the free-streamlines which leave the ends of this section and go on to pass through the corners of the rectangle. The length h is uniquely determined by the ratio y_o/x_o.

Lavrentieff's method of solving this problem is completely different from the methods found in this paper. It is based on several comparison and monotonicity theorems ([1], p. 380) which follow from the maximum principle for harmonic functions. The present paper was originally conceived as a confirmation of Lavrentieff's work using the variational calculus technique introduced in Chapter IV; however, no satisfactory method was found to impose the constraint that the plate be confined to lie within the rectangle. If this constraint is dropped, however, one can easily construct a sequence of plate shapes which, in the limit, give zero drag regardless of the ratio y_o/x_o.

Such a sequence is illustrated in Fig. 2(b). A typical plate consists of a rectangular cup with width h_1 and length h_2 and the free-streamlines which leave the edge of the cup and pass through (x_o, y_o) and $(x_o, -y_o)$. We now shrink the width h_1 and increase the length h_2 so that the free-streamlines still pass through (x_o, y_o) and $(x_o, -y_o)$. As $h_1 \to 0$, the region inside the cup becomes essentially a "deadwater" region with stagnation pressure $p = \frac{1}{2}\rho U^2$, so that the drag D of the plate is just

$$D = \frac{1}{2}\rho U^2 h_1$$

which can be made as small as we please by choosing h_1 small enough.

This observation led the author to consider the minimum drag problem described above. It was thought that the addition of a constraint on the arclength of the plate would prevent the possibility of such needle-nosed shapes. The constraint on the length of the plate was dropped for simplification.

The minimization problem considered here is related to a more difficult design problem: For a fixed cavity pressure (not necessarily equal to the free stream pressure), find the shape of a cavitating hydrofoil of given lift, chord, and angle of attack, so that the drag is a minimum. It is not clear if a fixed arclength constraint needs to be imposed in the hydrofoil problem for physically meaningful solutions; however, it is hoped that the treatment of the minimum drag problem found in this paper will clarify this point.

Although we illustrate the minimization procedure with the Helmholtz model, cavities of finite length could be incorporated into the minimum drag problem by the adoption of other flow models. For example, the Riabouchinsky model ([2], p. 335), the re-entrant jet model ([2], p. 332), or the wake model developed by Wu [3] could be used. The difficulties in using these models would be much greater, however, as should become apparent in the work to follow.

II. THE PROBLEM OF THE SYMMETRIC CAVITATING PLATE

1. Physical Description

Consider the infinite cavity flow of an inviscid, incompressible fluid past a two-dimensional plate P. The fluid motion is steady and irrotational with uniform velocity U far upstream parallel to the positive x-axis. The far stream pressure equals the cavity pressure which is assumed to be the vapor pressure of the liquid. By Bernoulli's equation, the flow speed at the cavity wall is constant and equal to U.

We limit the class of flows under consideration to the class of all infinite cavity flows in which the plate P has the following properties:

(a) P starts at the origin O (see Fig. 1) and is symmetric with respect to the x-axis.

(b) The distance between the endpoints A and B of the plate is $2y_o$.

(c) P has arclength $2s_o$.

(d) P has continuous slope except at the origin where the nose angle is 2α, with $0 \leq \alpha \leq \pi$.

(e) Let S be some point on the plate between the nose O and the endpoint A. The pressure p acting on the plate segment OS satisfies $p \geq p_c$, while on SA, $p \equiv p_c$. The same is true for the other half of the plate below the axis; i.e., $p \geq p_c$ on OS' and $p \equiv p_c$ on S'B.

(f) P is coincident with the dividing streamline passing through O.

The condition $p \geqslant p_c$ is an obvious statement of the fact that the vapor pressure of the liquid is the minimum pressure in the flow. The point S is included since we expect an interval on the minimum drag profile with $p \equiv p_c$. (This is motivated by observing that part of the optimum profile for Lavrentieff's problem is a free-streamline.) This feature is included in the class of plates over which we minimize since it is easier to take account of this expectation from the beginning than not. Note that (e) places no undue restrictions on the problem, since the point S for the optimal profile is not known a priori, but must be found as part of the minimization process; it could, in fact, be the same as the endpoint A.

Condition (f) is trivial on the part of the plate S'OS; however, on SA and S'B it is non-trivial since the shape of the plate from S to A (likewise for S'B) may be altered without changing the flow pattern. For example, if the shape of the plate is changed as shown in Fig. 3, vapor would form to fill the cavity so produced since the fluid is at the vapor pressure along the sections of the dividing streamline SA and S'B. The fluid flow, however, is unaffected. In order to avoid an infinity of plate shapes with the same flows, we require that the plate be coincident with the dividing streamline.

2. Mathematical Formulation

By proper choice of origin and magnification, the complex potential plane $f = \varphi + i\psi$ is mapped to the upper half $\zeta = \xi + i\eta$ plane (see Figs. 4) by

$$f = \frac{1}{2} AU\zeta^2 \quad . \tag{2.1}$$

Here, φ is the velocity potential, ψ is the stream function, and A is a real positive constant.

The magnification factor A is chosen so that the section of the plate S'OS maps to the real ζ axis, $-1 \leqslant \xi \leqslant 1$. The sections of the plate which are free-streamlines, SA and S'B, are mapped to the real axis, $1 < \xi \leqslant c$ and $-c \leqslant \xi < -1$, where $c \geqslant 1$ (equality only if S = A, S' = B). The free-streamlines AI and BI map to $c < \xi < \infty$ and $-\infty < \xi < -c$, respectively.

We now introduce the complex velocity

$$w = \frac{df}{dz} = u - iv = qe^{-i\theta} \tag{2.2}$$

and the logarithmic hodograph variable

$$\omega = \log \frac{U}{w} = \log \frac{U}{q} + i\theta \equiv \tau + i\theta \quad , \tag{2.3}$$

where u is the x component of velocity, v is the y component, q is the fluid speed ($q = (u^2 + v^2)^{\frac{1}{2}}$), and θ is the flow angle with respect to the positive x axis measured positive in the counterclockwise direction.

Bernoulli's equation can be written

$$p - p_c = \frac{1}{2} \rho U^2 - \frac{1}{2} \rho q^2 = \frac{1}{2} \rho U^2 (1 - e^{-2\tau}) \quad . \tag{2.4}$$

On the free-streamlines ($\eta = 0$, $|\xi| > 1$), $p \equiv p_c$; therefore

$$\tau(\xi,\ 0+) = 0 \quad \text{for} \quad |\xi| > 1 \quad . \tag{2.5}$$

Let the boundary values of $\omega(\zeta)$ be denoted by

$$\omega(\xi+i0) = \tau(\xi, 0+) + i\theta(\xi, 0+) = \begin{cases} \Gamma(\xi) + i\beta(\xi) & \text{for} \quad |\xi| \leqslant 1 \ . \\ i\beta^*(\xi) & \text{for} \quad |\xi| > 1 \ . \end{cases} \tag{2.6}$$

The condition that p_c is the minimum pressure in the flow means that the inequality

$$\Gamma(\xi) \geqslant 0 \tag{2.7}$$

must hold for $|\xi| \leqslant 1$.

Because of the symmetry of the plate,

$$\Gamma(\xi) = \Gamma(-\xi) \tag{2.8}$$

$$\beta(\xi) = -\beta(-\xi), \qquad \beta^*(\xi) = -\beta^*(-\xi) \quad . \tag{2.9}$$

That is, at corresponding points on the top and bottom of the dividing streamline, the flow speed is the same and the flow angle on the top is minus the flow angle on the bottom.

The condition that the pressure be continuous at S and S' requires that the speed also be continuous; hence,

$$\Gamma(\pm 1) = 0 \quad . \tag{2.10}$$

Since the slope of the plate is continuous, $\beta(\xi)$ and $\beta^*(\xi)$ are continuous and satisfy

$$\beta(\pm 1) = \beta^*(\pm 1) \quad . \tag{2.11}$$

The function $\omega(\zeta)$ is discontinuous at $\zeta = 0$. Since the nose of the plate is a stagnation point of the flow,

$$\tau \to \infty \quad \text{as} \quad |\zeta| \to 0 \quad , \quad \eta \geqslant 0 \quad . \tag{2.12}$$

Since the flow divides at the nose of the plate, there is a jump in the flow angle,

$$\beta(0+) - \beta(0-) = 2\alpha \quad . \tag{2.13}$$

In addition, since the flow approaches that of the free stream at large distances from the plate, we have

$$\omega(\zeta) \to 0 \quad \text{as} \quad |\zeta| \to \infty \quad , \quad \eta \geqslant 0 \quad . \tag{2.14}$$

We now show that $\omega(\zeta)$ can be written in the form

$$\omega(\zeta) = \omega_o(\zeta) + \omega_1(\zeta) \tag{2.15}$$

where $\omega_o(\zeta)$ is the discontinuous part of ω,

$$\omega_o(\zeta) = \frac{2\alpha}{\pi} \log\left(\frac{\sqrt{\zeta^2-1} + i}{\zeta} \right) \quad ; \tag{2.16}$$

and $\omega_1(\zeta)$ is analytic in the upper half plane, continuous on the boundary, and

$$\omega_1(\zeta) \to 0 \quad \text{as} \quad |\zeta| \to \infty \quad , \quad \eta \geqslant 0 \quad .$$

$$\mathrm{Re}\{\omega_1(\xi + i0)\} = 0 \quad \text{for} \quad |\xi| > 1 \quad .$$

We use Re and Im to stand for the real and imaginary parts. In (2.16), and subsequently, we take $\sqrt{\zeta^2-1}$ to be cut along the real axis from -1 to 1 and positive for $\zeta = \xi > 1$. We will use $(\quad)^{\frac{1}{2}}$ to stand for the positive root of the positive quantity inside the parentheses. The logarithm function in (2.16) is defined to be that branch of the function which is real for real, positive argument with a cut along the negative real axis of the argument.

If we let

$$\omega_1(\xi+i0) = \begin{cases} \Gamma_1(\xi) + i\beta_1(\xi) & \text{for} \quad |\xi| \leq 1 \\ i\beta_1^*(\xi) & \text{for} \quad |\xi| > 1 \end{cases} ,$$

then Γ_1, β_1, and β_1^*, must be continuous and satisfy the symmetry conditions

$$\Gamma_1(\xi) = \Gamma_1(-\xi)$$

$$\beta_1(\xi) = -\beta_1(-\xi), \ \beta_1^*(\xi) = -\beta_1^*(-\xi)$$

and the continuity conditions

$$\begin{aligned} \Gamma_1(\pm 1) &= 0 \\ \beta_1(\pm 1) &= \beta_1^*(\pm 1) \quad . \end{aligned} \tag{2.17}$$

Letting $\zeta \to \xi + i0$ in (2.16) for $|\xi| \leq 1$ and comparing the real and imaginary parts of (2.15), we have

$$\begin{aligned} \Gamma(\xi) &= \frac{2\alpha}{\pi} \log \left(\frac{1+(1-\xi^2)^{\frac{1}{2}}}{|\xi|} \right) + \Gamma_1(\xi) \\ \beta(\xi) &= \alpha \operatorname{sgn}\xi + \beta_1(\xi) \quad , \end{aligned} \tag{2.18}$$

where

$$\operatorname{sgn}\xi = \begin{cases} 1 & \text{if} \quad \xi > 0 \\ -1 & \text{if} \quad \xi < 0 \end{cases} .$$

Because of the assumed continuity of Γ_1 and β_1 conditions (2.12) and (2.13) are satisfied. It is obvious that conditions (2.8) through (2.10) are also satisfied by this choice of ω_o and ω_1.

As $\zeta \to \xi + i0$ for $|\xi| > 1$,

$$\omega_o(\zeta) \to \frac{2\alpha}{\pi} \log\left[\frac{(\xi^2-1)^{\frac{1}{2}} \operatorname{sgn}\xi + i}{\xi}\right] = i\,\frac{2\alpha}{\pi}\sin^{-1}\left(\frac{1}{\xi}\right) \quad .$$

This result, together with (2.17) and (2.18), show that (2.11) is satisfied.

As for the boundary values of $\omega_1(\zeta)$ on the real axis, it is easiest to deal with the inverse problem. That is, instead of specifying the shape of the plate and trying to find the corresponding boundary values it is better to let the shape of the plate be determined by giving either $\Gamma_1(\xi)$ or $\beta_1(\xi)$.

We now suppose that $\Gamma_1(\xi)$ is given with the following properties:

(i) $\Gamma_1(\xi)$ is Hölder continuous (see below) on the closed interval $[-1,1]$.

(ii) $\Gamma_1(\pm 1) = 0$.

(iii) $\Gamma_1(\xi) = \Gamma_1(-\xi)$.

A function $f(x)$ is Hölder continuous on some interval if

$$|f(x_1) - f(x_2)| \leqslant M|x_1 - x_2|^{\mu}$$

for real positive constants M and μ with $0 < \mu \leqslant 1$, and for any x_1 and x_2 in the interval.

We have a Dirichlet problem for the determination of $\omega_1(\zeta)$,

$$\operatorname{Re}\{\omega_1(\xi + i0)\} = \begin{cases} \Gamma_1(\xi) & \text{for} \quad |\xi| \leqslant 1 \\ 0 & \text{for} \quad |\xi| > 1 \end{cases} \quad .$$

The further condition $\omega_1 \to 0$ as $|\zeta| \to \infty$, $\eta > 0$, provides the unique solution

$$\omega_1(\zeta) = -\frac{i}{\pi}\int_{-1}^{1}\frac{\Gamma_1(t)dt}{t-\zeta} \quad . \tag{2.19}$$

We now let $\zeta \to \xi + i0$ for $|\xi| \leqslant 1$ in this equation. Using Plemelj's formula (see, e.g., [4], §17) we have an identity in the real parts of the left and right sides. The imaginary parts give

$$\beta_1(\xi) = -\frac{1}{\pi}\fint_{-1}^{1}\frac{\Gamma_1(t)dt}{t-\xi} \quad , \tag{2.20}$$

where $\fint$ denotes the Cauchy principal value. As $\zeta \to \xi + i0$ for $|\xi| > 1$, we have

$$\beta_1^*(\xi) = -\frac{1}{\pi}\int_{-1}^{1}\frac{\Gamma_1(t)dt}{t-\xi} \quad . \tag{2.21}$$

These forms for β_1 and β_1^* satisfy all of the necessary continuity and symmetry conditions. The continuity of $\beta_1(\xi)$ on the closed interval $[-1,1]$ follows from the Hölder continuity of $\Gamma_1(\xi)$ and the fact that $\Gamma_1(\pm 1) = 0$. (See [4], Chapter 4, for the behavior of the Cauchy integral near the endpoints of integration.)

If, on the other hand, $\beta_1(\xi)$ is given, we can consider (2.20) as an integral equation for $\Gamma_1(\xi)$ or solve the Riemann-Hilbert problem

$$\begin{aligned} &\mathrm{Im}\{\omega_1(\xi+i0)\} = \beta_1(\xi) \qquad &&\text{for} \quad |\xi| \leqslant 1 \\ &\mathrm{Re}\{\omega_1(\xi+i0)\} = 0 &&\text{for} \quad |\xi| > 1 \\ &\omega_1(\zeta) \to 0 \quad \text{as} \quad |\zeta| \to \infty, &&\eta \geqslant 0 \quad . \end{aligned}$$

The necessary conditions for the existence of a solution of the integral equation (2.20) such that $\Gamma_1(\pm 1) = 0$ are that $\beta_1(\xi)$ be Hölder continuous on $[-1,1]$ and that it satisfy the orthogonality condition

$$\int_{-1}^{1} \frac{\beta_1(t)dt}{(1-t^2)^{\frac{1}{2}}} = 0 \quad .$$

Under these conditions (the orthogonality condition is trivially satisfied by the symmetry of the problem), the solution for $\Gamma_1(\xi)$ is (see [4], §88)

$$\Gamma_1(\xi) = \frac{1}{\pi}(1-\xi^2)^{\frac{1}{2}} \oint_{-1}^{1} \frac{\beta_1(t)dt}{(1-t^2)^{\frac{1}{2}}(t-\xi)} \quad .$$

The solution of the Riemann-Hilbert problem gives the same results. Equation (2.21) can now be used to find $\beta_1^*(\xi)$.

3. The Physical Plane

From Eqs. (2.2) and (2.3),

$$\frac{df}{dz} = U e^{-\omega(\zeta)} \quad .$$

Using (2.1),

$$dz = \frac{1}{U} e^{\omega(\zeta)} df = A e^{\omega(\zeta)} \zeta \, d\zeta \quad . \tag{2.22}$$

Thus, the physical plane is found by an integration in the ζ-plane:

$$z(\zeta) = A \int_0^{\zeta} e^{\omega(\zeta')} \zeta' d\zeta' \quad . \tag{2.23}$$

The shape of the plate is given parametrically by

$$z(\xi) = x(\xi) + iy(\xi) = A \int_0^{\xi} e^{\omega(\zeta')} \zeta' d\zeta' \quad , \tag{2.24}$$

where $-c \leqslant \xi \leqslant c$. The integration from $\zeta' = 0$ to $\zeta' = \xi$ can be taken on any path in the upper half plane because of the analyticity of the integrand.

Because of the symmetry of the plate, the plate width is given by

$$2iy_o = A\int_{-c}^{c} e^{\omega(\zeta)}\zeta\, d\zeta \quad . \tag{2.25}$$

It will be convenient for subsequent analysis (Chapter V) to convert this expression for y_o to an integral over the real ζ axis from -1 to 1.

Let the function $\omega(\zeta)$ by analytically continued across the real axis into the lower half ζ-plane by

$$\omega(\bar{\zeta}) = -\overline{\omega(\zeta)} \quad . \tag{2.26}$$

By $\omega(\zeta)$, we now mean the function so defined on the entire ζ plane. This function is sectionally holomorphic with a line of discontinuity on the real axis from -1 to 1. Using (2.6) and (2.26),

$$\omega(\xi \pm i0) \equiv \omega_{\pm}(\xi) = \pm\, \Omega(\xi) + i\beta(\xi) \qquad \text{for} \qquad |\xi| \leqslant 1 \tag{2.27}$$

and

$$\omega_{+}(\xi) = \omega_{-}(\xi) = i\beta^{*}(\xi) \qquad \text{for} \qquad |\xi| > 1 \quad .$$

Now consider the function

$$F(\zeta) = \zeta e^{\omega(\zeta)} \quad .$$

This function is uniquely determined on the entire ζ-plane by its jump in value across the line of discontinuity, $-1 \leqslant \xi \leqslant 1$, and by its expansion at infinity (see [4] §78). That is, F can be represented by

$$F(\zeta) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{F_{+}(t)-F_{-}(t)}{t-\zeta}\, dt + \sum_{k=o}^{n} \alpha_k \zeta^k \tag{2.28}$$

if F is of degree n at infinity. The coefficients α_k are found by expanding F as $|\zeta| \to \infty$.

Letting $|\zeta| \to \infty$ in the equation

$$\omega(\zeta) = -\frac{i}{\pi}\int_{-1}^{1} \frac{\Gamma(t)}{t-\zeta}\, dt$$

we have

$$\omega(\zeta) = \frac{i}{\pi\zeta}\int_{-1}^{1} \Gamma(t)dt + \frac{i}{\pi\zeta^3}\int_{-1}^{1} t^2\Gamma(t)dt + O(\zeta^{-5}) \quad .$$

Thus

$$e^{\omega(\zeta)} = 1 + \frac{i}{\pi\zeta}\int_{-1}^{1} \Gamma(t)dt - \frac{1}{2\pi^2\zeta^2}\left(\int_{-1}^{1} \Gamma(t)dt\right)^2 + O(\zeta^{-3})$$

as $|\zeta| \to \infty$, so that F is of first degree $(n=1)$ at infinity,

$$F(\zeta) \to \frac{i}{\pi}\int_{-1}^{1} \Gamma(t)dt + \zeta \qquad \text{as} \qquad |\zeta| \to \infty \quad . \tag{2.29}$$

From (2.27) we have

$$F_{\pm}(\xi) = \xi e^{\omega_{\pm}(\xi)} = \xi e^{\pm\Gamma(\xi)+i\beta(\xi)} \quad . \tag{2.30}$$

Using Eqs. (2.28) through (2.30),

$$\zeta e^{\omega(\zeta)} = \frac{1}{\pi i}\int_{-1}^{1} \frac{te^{i\beta(t)}\sinh\Gamma(t)}{t-\zeta}\, dt + \frac{i}{\pi}\int_{-1}^{1} \Gamma(t)dt + \zeta \quad . \tag{2.31}$$

If this expression for $\zeta e^{\omega(\zeta)}$ is substituted into (2.25) and the integration in ζ is carried out we obtain

$$2iy_o = \frac{iA}{\pi}\int_{-1}^{1} t \sinh\Gamma(t)\sin\beta(t)dt$$

$$+ \frac{iA}{\pi}\int_{-1}^{1} t \sinh\Gamma(t)\cos\beta(t)\log\frac{c-t}{c+t}\, dt$$

$$+ \frac{2iAc}{\pi}\int_{-1}^{1} \Gamma(t)dt \quad .$$

A somewhat simpler expression for the first integral on the righthand side of this equation is obtained by looking at Eq. (2.31) as $|\zeta| \to \infty$. The lefthand side becomes

$$\zeta + \frac{i}{\pi}\int_{-1}^{1}\Gamma(t)dt - \frac{1}{2\pi^2\zeta}\left(\int_{-1}^{1}\Gamma(t)dt\right)^2 + O\left(\frac{1}{\zeta^2}\right) ,$$

while the righthand side equals

$$\zeta + \frac{i}{\pi}\int_{-1}^{1}\Gamma(t)dt - \frac{1}{\pi\zeta}\int_{-1}^{1} t\,\sin\beta(t)\sinh\Gamma(t)dt + O\left(\frac{1}{\zeta^2}\right) .$$

Comparing the coefficients of $\frac{1}{\zeta}$, we have

$$\int_{-1}^{1} t\,\sinh\Gamma(t)\sin\beta(t)dt = \frac{1}{2\pi}\left(\int_{-1}^{1}\Gamma(t)dt\right)^2 . \qquad (2.32)$$

Thus, y_o takes the form

$$y_o = \frac{A}{4\pi}\left(\int_{-1}^{1}\Gamma(t)dt\right)^2 + \frac{A}{2\pi}\int_{-1}^{1} t\,\sinh\Gamma(t)\cos\beta(t)\log\frac{c-t}{c+t}\,dt$$

$$+ \frac{Ac}{\pi}\int_{-1}^{1}\Gamma(t)dt .$$

Let ds be an element of arclength of the plate, measured positive from B to A. From (2.22),

$$ds = |dz|_{\text{plate}} = \begin{cases} Ae^{\Gamma(\xi)}|\xi|\,d\xi & \text{for} \quad |\xi| \leqslant 1 \\ A|\xi| & \text{for} \quad |\xi| > 1 . \end{cases}$$

The total arclength of the plate is given by

$$2s_o = A\int_{-c}^{-1}|\xi|\,d\xi + A\int_{-1}^{1} e^{\Gamma(\xi)}|\xi|\,d\xi + A\int_{1}^{c}|\xi|\,d\xi$$

or

$$s_o = \frac{A}{2}(c^2-1) + \frac{A}{2}\int_{-1}^{1} e^{\Gamma(\xi)}|\xi|\,d\xi .$$

4. The Drag

The complex force acting on an element of the plate dz is given by

$$dF = (p - p_c)(-i\,dz) \quad .$$

From Bernoulli's equation.

$$p - p_c = \frac{1}{2}\rho U^2 - \frac{1}{2}\rho q^2 = \frac{1}{2}\rho U^2(1 - e^{-2\tau}) \quad .$$

Using this and the expression (2.22) for dz, the total force acting on the plate is just

$$D + iL = -i\int_{BOA}(p-p_c)dz$$

$$= -iA\cdot\frac{1}{2}\rho U^2\int_{-1}^{1}(1-e^{-2\Gamma})e^{\Gamma+i\beta}\xi\, d\xi$$

$$= A\rho U^2\int_{-1}^{1}\xi \sinh\Gamma(\xi)\sin\beta(\xi)d\xi \quad .$$

Therefore, the lift L is zero and using (2.32) the drag D is given by

$$D = \frac{1}{2}\rho U^2\,\frac{A}{\pi}\left(\int_{-1}^{1}\Gamma(t)dt\right)^2 \quad .$$

Since ρ and U are to be kept constant in the optimization problem, it is convenient to give the drag the dimension of length by defining $2D^*$ to be the length over which the hydrodynamic pressure $\frac{1}{2}\rho U^2$ would act to produce the drag D. Thus,

$$D^* = \frac{A}{2\pi}\left(\int_{-1}^{1}\Gamma(t)dt\right)^2 \quad .$$

Note that the drag coefficient of the plate based on the width $2y_o$ is just

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 2y_o} = \frac{D^*}{y_o} \quad .$$

5. Statement of the Minimum Drag Problem

Using the results of the previous sections, the problem of minimizing the drag of a symmetric profile of given width, $2y_o$, and given arclength, $2s_o$, reduces to finding two functions, $\Gamma_1(\xi)$ and $\beta_1(\xi)$, and the constants A, c, and α, such that

$$D^* = \frac{A}{2\pi}\left(\int_{-1}^{1} \Gamma(t)dt\right)^2 \tag{2.33}$$

as a minimum subject to the constraints

$$y_o = \frac{A}{2\pi}\left[\frac{1}{2}\left(\int_{-1}^{1}\Gamma(t)dt\right)^2 + \int_{-1}^{1} t \sinh\Gamma(t)\cos\beta(t)\log\frac{c-t}{c+t}\,dt + 2c\int_{-1}^{1}\Gamma(t)dt\right] \tag{2.34}$$

$$s_o = \frac{A}{2}\left[c^2 - 1 + \int_{-1}^{1} e^{\Gamma(t)}|t|dt\right] , \tag{2.35}$$

where

$$\Gamma(\xi) = \frac{2\alpha}{\pi}\log\left(\frac{1+(1-\xi^2)^{\frac{1}{2}}}{|\xi|}\right) + \Gamma_1(\xi) \geqslant 0 \tag{2.36}$$

$$\beta(\xi) = \alpha \operatorname{sgn}\xi + \beta_1(\xi) \quad . \tag{2.37}$$

Furthermore, Γ_1 and β_1 are related by either of the following sets of equations (one implies the other):

(i) $\Gamma_1(\xi)$ Hölder continuous on $[-1,1]$.

(ii) $\Gamma_1(-\xi) = \Gamma_1(\xi)$.

(iii) $\Gamma_1(\pm 1) = 0$ (2.38)

(iv) $\beta_1(\xi) = -\frac{1}{\pi}\oint_{-1}^{1} \frac{\Gamma_1(t)}{t-\xi}\, dt \qquad \text{for} \qquad |\xi| \leqslant 1$

or

(i) $\beta_1(\xi)$ Hölder continuous on $[-1,1]$.

(ii) $\beta_1(\xi) = -\beta_1(-\xi)$. (2.39)

(iii) $\Gamma_1(\xi) = \frac{1}{\pi}(1-\xi^2)^{\frac{1}{2}} \oint_{-1}^{1} \frac{\beta_1(t)}{(1-t^2)^{\frac{1}{2}}(t-\xi)}\, dt \qquad \text{for} \qquad |\xi| \leqslant 1$.

Finally, α and c are restricted to the intervals

$$0 \leqslant \alpha \leqslant \pi$$

$$1 \leqslant c < \infty \quad .$$

III. OPTIMIZATION OVER A FINITE PARAMETER SPACE

1. Expansion in Orthogonal Series

In this section we investigate the method of expanding $\Gamma_1(\xi)$ and $\beta_1(\xi)$ in a series of known functions with unknown parameter multipliers and minimizing the drag (2.33), subject to the isoperimetric constraints (2.34) and (2.35), by the proper choice of these parameters. This method is similar to that used in the Rayleigh-Ritz method of minimizing integral functionals.

A natural expansion for $\Gamma_1(\xi)$ is afforded by the choice

$$\Gamma_1(\xi) = -\sum_{n=1}^{N} a_n \sin(2n-1)\varphi \tag{3.1}$$

where $\xi = \cos\varphi$, φ running from 0 to π. From (2.38(iv)), $\beta_1(\xi)$ is given by

$$\beta_1(\xi) = -\sum_{n=1}^{N} a_n \cos(2n-1)\varphi \quad , \tag{3.2}$$

where the well-known formula

$$\int_0^{\pi} \frac{\sin m\theta \sin\theta}{\cos\theta - \cos\varphi}\, d\theta = -\pi \cos m\varphi$$

has been used. Notice that this choice of Γ_1 and β_1 satisfy all continuity and symmetry conditions. We now limit the class of plates to those with blunt noses; that is, we set $\alpha = \frac{\pi}{2}$. This choice of α not only simplifies the calculations, but it is also motivated by the results of Chapter V, where it is shown that if a solution to the variational problem exists then this is the only possible value for α. Setting

$\alpha = \frac{\pi}{2}$ in Eq. (2.16),

$$\omega_o(\zeta) = \log\left(\frac{\sqrt{\zeta^2-1}+i}{\zeta}\right) \quad ,$$

so that from Eq. (2.24),

$$z(-c) = x_o - iy_o = A\int_o^{-c} e^{\omega(\zeta)}\zeta\, d\zeta = A\int_o^{-c} e^{\omega_1(\zeta)}(\sqrt{\zeta^2-1}+i)d\zeta \quad . \tag{3.3}$$

This integration is most easily done by a change of variable. Let

$$\zeta = -\frac{1}{2}\left(\nu + \frac{1}{\nu}\right) \quad , \tag{3.4}$$

where $\nu = \sigma + i\tau$. The ζ-plane, $\mathrm{Im}\zeta \geqslant 0$ is mapped into the half circle $|\nu| \leqslant 1$, $\mathrm{Im}\,\nu \geqslant 0$ (see Figs. 4(b) and 5). The inverse transform

$$\nu = -\zeta + \sqrt{\zeta^2 - 1} \tag{3.5}$$

is chosen so that the point at infinity in the ζ-plane maps to $\nu = 0$. The endpoints of the plate, $\zeta = \pm c$, maps to $\nu = \mp \kappa$, where

$$c = \frac{1}{2}\left(\kappa + \frac{1}{\kappa}\right) \tag{3.6}$$

or

$$\kappa = c - (c^2-1)^{\frac{1}{2}} \quad .$$

It is readily verified that ω_1 as a function of ν is given by

$$\omega_1(\nu) = i\sum_{n=1}^{N} a_n \nu^{2n-1} \equiv i\Omega(\nu) \quad . \tag{3.7}$$

On the section of the plate S'OS, corresponding to ζ running from

-1 to 1 on the real axis, ν runs from 1 to -1 on the circle $|\nu| = 1$. That is,

$$\nu = e^{i(\pi - \varphi)} = -e^{-i\varphi} \quad ,$$

where φ goes from π to 0. Therefore, from Eq. (3.7),

$$\omega_1 = -i \sum_{n=1}^{N} a_n e^{-i(2n-1)\varphi}$$

$$= -\sum_{n=1}^{N} a_n \sin(2n-1)\varphi - i \sum_{n=1}^{N} a_n \cos(2n-1)\varphi$$

which agrees with (3.1) and (3.2). Furthermore, on the free streamlines, $\tau = 0$, $|\sigma| \leqslant 1$, ω_1 is purely imaginary and $\omega_1 \to 0$ as $|\nu| \to 0$.

With the use of Eqs. (3.4) and (3.7), Eq. (3.3) becomes

$$x_o - iy_o = -\frac{A}{4} \int_i^{\kappa} e^{i\Omega(\nu)} \left(\nu + 2i - \frac{2}{\nu} - \frac{2i}{\nu^2} + \frac{1}{\nu^3} \right) d\nu \quad . \tag{3.8}$$

In Appendix A, x_o and y_o are evaluated by taking the path of integration along L_ϵ, defined by the imaginary axis from $\nu = i$ to $\nu = i\epsilon$, the circle $|\nu| = \epsilon$ from $\nu = i\epsilon$ to $\nu = \epsilon$, and the real axis from $\nu = \epsilon$ to $\nu = \kappa$. In the limit $\epsilon \to 0$, the expression for y_o is found to be

$$y_o = \frac{1}{4} A \left[\frac{1}{2} \int_o^{\kappa} \sin\Omega(t) \left\{ 2t - \frac{(2 - \Omega'(t))^2}{t} \right\} dt + \int_o^{\kappa} \cos\Omega(t) \left\{ 2 + \frac{\Omega''(t)}{2t} \right\} dt + \frac{\pi}{2} (2 - a_1)^2 + \left(\frac{2}{\kappa} - \frac{\Omega'(\kappa)}{2\kappa} \right) \cos\Omega(\kappa) - \frac{1}{2\kappa^2} \sin\Omega(\kappa) \right] \quad , \tag{3.9}$$

where

$$\Omega(t) = \sum_{n=1}^{N} a_n t^{2n-1} \quad , \quad \Omega'(t) = \frac{d\Omega}{dt} \quad , \quad \text{etc.}$$

Setting $\alpha = \frac{\pi}{2}$ in Eq. (2.36),

$$\Gamma(\xi) = \log\left(\frac{1+(1-\xi^2)^{\frac{1}{2}}}{|\xi|}\right) + \Gamma_1(\xi) \quad .$$

Thus, (2.35) becomes

$$s_o = \frac{A}{2}\left[c^2 - 1 + \int_{-1}^{1} e^{\Gamma_1(t)} \{1+(1-t^2)^{\frac{1}{2}}\}dt\right] \quad .$$

Finally, making the change of variable $t = \cos\varphi$ and using Eqs. (3.1) and (3.6), s_o is given by

$$s_o = \frac{A}{2}\left[\frac{1}{4}\left(\kappa + \frac{1}{\kappa}\right)^2 - 1 + \int_0^{\pi} \exp\left\{-\sum_{n=1}^{N} a_n \sin(2n-1)\varphi\right\}(1+\sin\varphi)\sin\varphi \, d\varphi\right] \tag{3.10}$$

The drag is given by

$$D^* = \frac{A}{2\pi}\left(\int_{-1}^{1} \Gamma(t)dt\right)^2 = \frac{A}{2\pi}\left[\int_{-1}^{1} \log\left(\frac{1+(1-t^2)^{\frac{1}{2}}}{|t|}\right) dt + \int_{-1}^{1} \Gamma_1(t)dt\right]^2$$

$$= \frac{A}{2\pi}\left[\int_0^{\pi} \log\left(\frac{1+\sin\varphi}{|\cos\varphi|}\right) \sin\varphi \, d\varphi - \int_0^{\pi} \sum_{n=1}^{N} a_n \sin(2n-1)\varphi \sin\varphi \, d\varphi\right]^2 \quad .$$

The first term in this expression is easily evaluated by integrating by parts. Since the set $\{\sin(2n-1)\varphi\}$ is orthogonal the second term depends only on a_1. The final expression for D^* is given by

$$D^* = \frac{A\pi}{8}(2-a_1)^2 \quad . \tag{3.11}$$

The minimization problem reduces to minimizing D^* given by (3.1) subject to the constraints (3.9) and (3.10) over the (N+2)-dimensional parameter space $(A, \kappa, a_1, a_2, \ldots, a_N)$.

2. The Case N = 1

For N = 1,

$$\beta(\xi) = \frac{\pi}{2}\,\mathrm{sgn}\,\xi - a_1\xi \quad .$$

Thus, a_1 measures the flow angle from the vertical at the point S on the plate (positive in the clockwise direction). The plate section S'OS is convex or concave when seen from the flow as a_1 is positive or negative (see Figs. 6).

The integrals in Eqs. (3.9) and (3.10) are easily calculated in terms of special functions. With N = 1,

$$\Omega(t) = a_1 t$$

$$\Omega'(t) = a_1$$

$$\Omega''(t) = 0 \quad .$$

Therefore, (3.9) becomes

$$y_o = \frac{1}{4}A\left[\frac{1}{2}\int_o^{\kappa} \sin(a_1 t)\left\{2t - \frac{(2-a_1)^2}{t}\right\}dt + 2\int_o^{\kappa}\cos(a_1 t)dt + \frac{\pi}{2}(2-a_1)^2 + \left(\frac{2}{\kappa} - \frac{a_1}{2\kappa}\right)\cos a_1\kappa - \frac{1}{2}\kappa^2 \sin a_1\kappa\right] \quad ,$$

or

$$y_o = \frac{1}{8} A\left[\left(\frac{2}{a_1^2} + \frac{4}{a_1} - \frac{1}{\kappa^2}\right)\sin a_1\kappa\right.$$

$$+\left(\frac{4}{\kappa} - \frac{a_1}{\kappa} - \frac{2\kappa}{a_1}\right)\cos a_1\kappa$$

$$\left. + (2-a_1)^2\left\{\frac{\pi}{2} - \mathrm{Si}(a_1\kappa)\right\}\right] \quad , \tag{3.12}$$

where Si is the sine integral, $\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt$.

From (3.10), the arclength is given by

$$s_o = \frac{A}{2}\left\{\frac{1}{4}\left(\kappa + \frac{1}{\kappa}\right)^2 - 1 + \int_0^\pi e^{-a_1 \sin\varphi}(1+\sin\varphi)\sin\varphi\, d\varphi\right\}$$

$$= \frac{A}{2}\left\{\frac{1}{4}\left(\kappa^2 + \frac{1}{\kappa^2}\right) - \frac{1}{2} + \left(1 - \frac{d}{da_1}\right)\int_0^\pi e^{-a_1\sin\varphi}\sin\varphi\, d\varphi\right\} \quad .$$

From the integral tables [6],

$$\int_0^\pi e^{-a_1\sin\varphi}\sin\varphi\, d\varphi = 2\int_0^{\pi/2} e^{-a_1\sin\varphi}\sin\varphi\, d\varphi$$

$$= \frac{2}{a_1}\int_0^{a_1}\frac{xe^{-x}dx}{(a_1^2-x^2)^{\frac{1}{2}}} = 2 + \pi[L_1(a_1)-I_1(a_1)] \quad ,$$

where L_1 is the modified Struve function (see e.g., [7], p. 498) and I_1 is the modified Bessel function. Using the expressions for the derivatives,

$$L'_\nu(x) = L_{\nu-1}(x) - \frac{\nu}{x} L_\nu(x)$$

and

$$I'_\nu(x) = I_{\nu-1}(x) - \frac{\nu}{x} I_\nu(x)$$

the final form for s_o is given by

$$s_o = \frac{A}{8}\left\{\kappa^2 + \frac{1}{\kappa^2} + 6 + 4\pi\left(1+\frac{1}{a_1}\right)[L_1(a_1) - I_1(a_1)] - 4\pi[L_o(a_1) - I_o(a_1)]\right\} \quad . \qquad (3.13)$$

The apparent singularity at $a_1 = 0$ in the expressions for y_o and s_o given by (3.12) and (3.13) is removable. For small a_1 the following expansions are valid:

$$y_o = \frac{A}{8}\left\{\left(2\pi + 4\kappa + \frac{4}{\kappa}\right) + \left(\frac{2}{3}\kappa^3 - \frac{2}{\kappa} - 4\kappa - 2\pi\right)a_1 + \left(2\kappa - \frac{2}{3}\kappa^3 + \frac{\pi}{2}\right)a_1^2 + O(a_1^3)\right\}$$

$$s_o = \frac{A}{8}\left\{\left(2\pi + \kappa^2 + \frac{1}{\kappa^2} + 6\right) - \left(\frac{16}{3} + 2\pi\right)a_1 + \left(\frac{8}{3} + \frac{3\pi}{4}\right)a_1^2 + O(a_1^3)\right\} \quad .$$

Setting $a_1 = 0$ and $\kappa = 1$ in these equations gives

$$y_o = s_o = \frac{A}{8}\,(2\pi + 8) \quad ,$$

which checks with the fact that $a_1 = 0$, $\kappa = 1$ corresponds to a flat plate with no free-streamlines.

The condition that the pressure be greater than the vapor pressure of the liquid imposes an upper limit on the value of a_1. The condition $p \geqslant p_c$ imples $\Gamma(\xi) \geqslant 0$ for all $\xi \epsilon [-1, 1]$. That is,

$$\Gamma(\xi) = \log\left(\frac{1+(1-\xi^2)^{\frac{1}{2}}}{|\xi|}\right) + \Gamma_1(\xi)$$

$$= \frac{1}{2}\log\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right) - a_1 \sin\varphi$$

$$= (1-a_1)\sin\varphi + \frac{1}{3}\sin^3\varphi + \frac{1}{5}\sin^5\varphi + \ldots \geqslant 0 \quad .$$

Therefore, $a_1 \leqslant 1$. If this constraint is not imposed, we see from (3.11) that plates with zero drag would be possible by choosing $a_1 = 2$. In general, on plates for which $a_1 > 1$, there is a region near the ends of the segment S'OS where $p < p_c$.

In addition, a_1 cannot become too negatively large if the flow is to remain single sheeted. For large negative values of a_1, the plate curls around on intersects itself near the points S and S'. An arbitrary lower limit $a_1 = -\frac{\pi}{2}$ seems reasonable.

Some examples of actual plate shapes for various values of a_1 and κ are shown in Figs. 6. The shape of the section OS is found by numerically integrating the differential relations

$$dx = -\frac{1}{\bar{y}} e^{\Gamma_1(\xi)} \sin\beta_1(\xi)[1+(1-\xi^2)^{\frac{1}{2}}]d\xi$$

$$dy = \frac{1}{\bar{y}} e^{\Gamma_1(\xi)} \cos\beta_1(\xi)\,[1+(1-\xi^2)^{\frac{1}{2}}]d\xi$$

from $\xi = 0$ to $\xi = 1$, with $\Gamma_1(\xi) = -a_1(1-\xi^2)^{\frac{1}{2}}$ and $\beta_1(\xi) = -a_1\xi$. The shape of the free-streamline section of the plate SA is given parametrically by the equations

$$x(t) = \frac{1}{8\bar{y}}\left\{(2-a_1)^2\gamma(a_1,t) + e^{-a_1}\left(\frac{2}{a_1^2} + \frac{6}{a_1} + 5 - a_1\right)\right.$$
$$\left. +\left(\frac{4}{t} - \frac{a_1}{t} - \frac{2t}{a_1}\right)\sin a_1 t + \left(\frac{1}{t^2} - \frac{2}{a_1^2} - \frac{4}{a_1}\right)\cos a_1 t\right\}$$

$$y(t) = \frac{1}{8\bar{y}}\left\{\left(\frac{2}{a_1^2} + \frac{4}{a_1} - \frac{1}{t^2}\right)\sin a_1 t + \left(\frac{4}{t} - \frac{a_1}{t} - \frac{2t}{a_1}\right)\cos a_1 t\right.$$
$$\left. +(2-a_1)^2\left[\frac{\pi}{2} - \mathrm{Si}(a_1 t)\right]\right\} \quad ,$$

where t runs from 1 to κ and

$$\gamma(a_1, t) = \begin{cases} Ci(a_1 t) + E_1(a_1) & \text{for} \quad a_1 > 0 \\ \log t & \text{for} \quad a_1 = 0 \\ Ci(-a_1 t) - Ei(-a_1) & \text{for} \quad a_1 < 0 \end{cases} .$$

Here, Ci is the cosine integral and E_1 and Ei are exponential integrals. This expression for x is found by carrying out the integration in the expression for x_o given in Appendix A for the case N = 1. The normalization factor

$$\overline{y} = \frac{1}{8}\left\{\left(\frac{2}{a_1^2} + \frac{4}{a_1} - \frac{1}{\kappa^2}\right) \sin a_1\kappa + \left(\frac{4}{\kappa} - \frac{a_1}{\kappa} - \frac{2\kappa}{a_1}\right) \cos a_1\kappa \right.$$
$$\left. + (2 - a_1)^2\left[\frac{\pi}{2} - \mathrm{Si}(a_1\kappa)\right]\right\}$$

that appears in these equations simply gives all of the plates the same width, $y(\kappa) = 1$.

The problem of finding the optimum profile among the class of plates for which $\omega_1(\nu) = ia_1\nu$ and $\alpha = \frac{\pi}{2}$ reduces to the problem of minimizing D^* given by (3.11) subject to the constraints (3.12) and (3.13). This corresponds to extremizing

$$I = D^*(A, a_1) - \lambda_1 s_o(A, \kappa, a_1) - \lambda_2 y_o(A, \kappa, a_1) ,$$

where λ_1 and λ_2 are unknown constant Lagrange multipliers. For a minimum I, we solve the equations

$$\frac{\partial I}{\partial A} = \frac{\partial D^*}{\partial A} - \lambda_1 \frac{\partial s_o}{\partial A} - \lambda_2 \frac{\partial y_o}{\partial A} = 0$$

$$\frac{\partial I}{\partial a_1} = \frac{\partial D^*}{\partial a_1} - \lambda_1 \frac{\partial s_o}{\partial a_1} - \lambda_2 \frac{\partial y_o}{\partial a_1} = 0 \qquad (3.14)$$

$$\frac{\partial I}{\partial \kappa} = \frac{\partial D^*}{\partial \kappa} - \lambda_1 \frac{\partial s_o}{\partial \kappa} - \lambda_2 \frac{\partial y_o}{\partial \kappa} = 0 \quad .$$

These three equations, together with (3.12) and (3.13) determine the five unknowns A, κ, a_1, λ_1, and λ_2.

A necessary condition for the solution of the three equations (3.14) in two unknowns, λ_1 and λ_2, is that the determinant

$$\Delta = \begin{vmatrix} \frac{\partial D^*}{\partial A} & \frac{\partial s_o}{\partial A} & \frac{\partial y_o}{\partial A} \\ \frac{\partial D^*}{\partial a_1} & \frac{\partial s_o}{\partial a_1} & \frac{\partial y_o}{\partial a_1} \\ \frac{\partial D^*}{\partial \kappa} & \frac{\partial s_o}{\partial \kappa} & \frac{\partial y_o}{\partial \kappa} \end{vmatrix}$$

vanish. Evaluation of these partial derivatives gives

$$\frac{\partial D^*}{\partial A} = \frac{\pi}{8}(2-a_1)^2$$

$$\frac{\partial D^*}{\partial a_1} = -\frac{A\pi}{4}(2-a_1)$$

$$\frac{\partial D^*}{\partial \kappa} = 0$$

$$\frac{\partial s_o}{\partial A} = \frac{1}{8}\left\{\kappa^2 + \frac{1}{\kappa^2} + 6 + 4\pi\left(1 + \frac{1}{a_1}\right)[L_1(a_1) - I_1(a_1)] - 4\pi[L_o(a_1) - I_o(a_1)]\right\}$$

$$\frac{\partial s_o}{\partial a_1} = \frac{A}{8}\left\{4\pi\left(1+\frac{1}{a_1}\right)\left[L_o(a_1) - I_o(a_1)\right] - 4\pi\left(1+\frac{1}{a_1}+\frac{2}{a_1^2}\right)\left[L_1(a_1) - I_1(a_1)\right] - 8\right\}$$

$$\frac{\partial s_o}{\partial \kappa} = \frac{A}{4}\left(1-\frac{1}{\kappa^2}\right)\left(\kappa+\frac{1}{\kappa}\right)$$

$$\frac{\partial y_o}{\partial A} = \frac{1}{8}\left\{\left(\frac{2}{a_1^2}+\frac{4}{a_1}-\frac{1}{\kappa^2}\right)\sin a_1\kappa + \left(\frac{4}{\kappa}-\frac{a_1}{\kappa}-\frac{2\kappa}{a_1}\right)\cos a_1\kappa + (2-a_1)^2\left[\frac{\pi}{2} - \mathrm{Si}(a_1\kappa)\right]\right\}$$

$$\frac{\partial y_o}{\partial a_1} = \frac{A}{8}\left\{2\left(\frac{2\kappa}{a_1^2}+\frac{2\kappa}{a_1}-\frac{1}{\kappa}\right)\cos a_1\kappa + \frac{2}{a_1}\left(\kappa^2-\frac{2}{a_1^2}-\frac{2}{a_1}-2\right)\sin a_1\kappa - 2(2-a_1)\left[\frac{\pi}{2} - \mathrm{Si}(a_1\kappa)\right]\right\}$$

$$\frac{\partial y_o}{\partial \kappa} = \frac{A}{4}\left(1-\frac{1}{\kappa^2}\right)\left\{2\cos a_1\kappa + \left(\kappa-\frac{1}{\kappa}\right)\sin a_1\kappa\right\} .$$

By factoring various rows and columns of Δ, it can be shown that

$$\Delta = \frac{\pi}{4}\left(\frac{A}{8}\right)^2(2-a_1)\left(1-\frac{1}{\kappa^2}\right)\tilde{\Delta}$$

where

$$\tilde{\Delta} = \begin{vmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{vmatrix} \qquad (3.15)$$

and

$$t_{11} = 2 - a_1$$

$$t_{21} = -2$$

$$t_{31} = 0$$

$$t_{12} = \kappa^2 + \frac{1}{\kappa^2} + 6 + 4\pi\left(1 + \frac{1}{a_1}\right)[L_1(a_1) - I_1(a_1)]$$
$$- 4\pi[L_o(a_1) - I_o(a_1)]$$

$$t_{22} = 4\pi\left(1 + \frac{1}{a_1}\right)[L_o(a_1) - I_o(a_1)]$$
$$- 4\pi\left(1 + \frac{1}{a_1} + \frac{2}{a_1^2}\right)[L_1(a_1) - I_1(a_1)] - 8$$

$$t_{32} = \kappa + \frac{1}{\kappa}$$

$$t_{13} = \left(\frac{2}{a_1^2} + \frac{4}{a_1} - \frac{1}{\kappa^2}\right)\sin a_1\kappa + \left(\frac{4}{\kappa} - \frac{a_1}{\kappa} - \frac{2\kappa}{a_1}\right)\cos a_1\kappa$$
$$+ (2 - a_1)^2\left[\frac{\pi}{2} - \mathrm{Si}(a_1\kappa)\right] \qquad (3.16)$$

$$t_{23} = 2\left(\frac{2\kappa}{a_1^2} + \frac{2\kappa}{a_1} - \frac{1}{\kappa}\right)\cos a_1\kappa$$
$$+ \frac{2}{a_1}\left(\kappa^2 - \frac{2}{a_1^2} - \frac{2}{a_1} - 2\right)\sin a_1\kappa - 2(2-a_1)\left[\frac{\pi}{2} - \mathrm{Si}(a_1\kappa)\right]$$

$$t_{33} = 2\cos a_1\kappa - \left(\kappa - \frac{1}{\kappa}\right)\sin a_1\kappa \quad .$$

Therefore, $\Delta = 0$ for the following cases:

(i) $A = 0$

(ii) $a_1 = 2$

(iii) $\kappa = 1$

(iv) $\tilde{\Delta} = 0$.

We can rule out (i) since for $A = 0$ the plate reduces to a point in the z-plane. The case $a_1 = 2$ violates the pressure condition $p \geqslant p_c$. The case $\kappa = 1$ is discussed below. It corresponds to maximum drag profiles if $a_1 < 0$ and plates of stationary drag (neither maximum nor minimum) if $a_1 > 0$.

The fourth possibility, $\tilde{\Delta} = 0$, provides a relation between a_1 and κ. If we are interested only in the drag coefficient, $C_D = D^*/y_o$, and the ratio of arclength to chord, $k = s_o/y_o$, this relation is all that is needed to complete the solution, since the factor A drops out of these quantities.

Let $a_1 = f(\kappa)$ be the curve on which $\tilde{\Delta}(a_1, \kappa) = 0$. In Appendix B, this curve is found for κ near 1 by a Taylor series expansion. It is shown

$$f(\kappa) = \frac{8}{3\pi+16}(1-\kappa)^2 - \frac{24\pi}{(3\pi+16)^2}(1-\kappa)^3 + O(\kappa-1)^4 \quad .$$

The general solution is found by fixing κ at various values between 0 and 1 and numerically solving $\tilde{\Delta}(a_1, \kappa) = 0$ for a_1. This curve is plotted in in Fig. 7. As $\kappa \to 0$ (ratio of arclength to chord goes to infinity), it can be shown that a_1 is the root of the transcendental equation

$$(2+a_1)a_1[L_o(a_1)-I_o(a_1)]-(a_1^2+2a_1+4)[L_1(a_1)-I_1(a_1)] = \frac{1}{\pi}\frac{(3-2a_1)a_1^2}{(2-a_1)} \quad ,$$

which is found to be

$$a_1 \sim 0.1020 \quad .$$

In Fig. 8, $C_D = D^*/y_o$ versus $k = s_o/y_o$ is plotted for the cases for which $\Delta = 0$:

(i) $a_1 = f(\kappa)$, $1 \geqslant \kappa > 0$.

(ii) $\kappa = 1$, $-\frac{\pi}{2} \leqslant a_1 \leqslant 1$.

Case (i) are minimum drag profiles, while case (ii) are maximum profiles if $a_1 < 0$ and neither maximum nor minimum if $a_1 > 0$.

Minimum drag profiles for various values of k are drawn in Fig. 9. These profiles are found to be quite similar to Lavrentieff profiles ($a_1 = 0$); however, from the expansions

$$y_o(a_1,\kappa) = \frac{A}{8}\left\{(8+2\pi) + \left[4(1-\kappa)^2 - \left(2\pi + \frac{16}{3}\right)a_1\right]\right.$$

$$+ 4(1-\kappa)^3 + \left[4(1-\kappa)^4 + \left(\frac{4}{3} + \frac{\pi}{2}\right)a_1^{\,2}\right]$$

$$\left. + O(1-\kappa)^5\right\}$$

$$s_o(a_1,\kappa) = \frac{A}{8}\left\{(8+2\pi) + \left[4(1-\kappa)^2 - \left(2\pi + \frac{16}{3}\right)a_1\right]\right.$$

$$+ 4(1-\kappa)^3 + \left[5(1-\kappa)^4 + \left(\frac{8}{3} + \frac{3\pi}{4}\right)a_1\right]$$

$$\left. + O(1-\kappa)^5\right\} ,$$

which are valid on $a_1 = f(\kappa)$ for $\kappa \sim 1$, it is easily shown that

$$C_D = C_{D_o}\left[1 - 1.16(k-1)^{\frac{1}{2}} + O(k-1)\right]$$

for the minimum drag profiles; while

$$C_D = C_{D_o}\left[1 - 1.06(k-1)^{\frac{1}{2}} + O(k-1)\right]$$

for Lavrentieff profiles. Here, C_{D_o} is the flat plate drag coefficient

$$C_{D_o} = \frac{2\pi}{4+\pi} \quad .$$

These results are not contradictory, since the two families of profiles were found by minimizing the drag under different isoperimetric constraints; however, it is surprising the plate shapes are as close as they are. This indicates the desirability of using a fixed arclength constraint rather than the constraint that the plate be confined to a rectangle, the latter constraint being, in general (for $N > 1$), more difficult to apply than the former.

IV. THE METHOD OF CALCULUS OF VARIATIONS

1. Introduction

In this chapter, calculus of variations is used to minimize an integral functional of a Hölder continuous function $f(x)$ and its finite Hilbert transform,

$$g(x) = -\frac{1}{\pi} \oint_{-1}^{1} \frac{f(t)}{t-x}\, dt \quad . \tag{4.1}$$

Although previously unpublished, this method was developed by Professor Theodore Y. Wu of California Institute of Technology.

By way of introduction, we first consider the case in which the functional to be minimized is an integral of a quadratic in f and g. Consider the problem of minimizing

$$\begin{aligned} J[f(x), g(x), x] = \int_{-1}^{1} \{&a(x)f^2(x) + 2b(x)f(x)g(x) \\ &+ c(x)g^2(x) + 2d(x)f(x) + 2e(x)g(x)\}dx \end{aligned} \tag{4.2}$$

subject to n constraints of similar form

$$\begin{aligned} J_i[f(x), g(x), x] = \int_{-1}^{1} \{&a_i(x)f^2(x) + 2b_i(x)f(x)g(x) \\ &+ c_i(x)g^2(x) + 2d_i(x)f(x) + 2e_i(x)g(x)\}\, dx \\ &= \ell_i \quad , \qquad i = 1, 2, \ldots, n \quad . \end{aligned} \tag{4.3}$$

In the spirit of "classical" calculus of variations (i.e., when the functional involves a function and its derivative), we minimize the new functional

$$I[f(x), g(x), x] = J - \sum_{i=1}^{n} \lambda_i J_i \quad ,$$

where the $\{\lambda_i\}$ are undetermined Lagrange multipliers. Therefore, we need only consider the problem of minimizing

$$\begin{aligned} I[f(x), g(x), x] = \int_{-1}^{1} \{ & A(x)f^2(x) + 2B(x)f(x)g(x) \\ & + C(x)g^2(x) + 2D(x)f(x) + 2E(x)g(x)\} dx \end{aligned} \tag{4.4}$$

with the understanding that the coefficients $A(x), B(x), \ldots, E(x)$ may contain Lagrange multipliers; i.e.,

$$A(x) = a(x) - \sum_{i=1}^{n} \lambda_i a_i(x) \quad , \qquad \text{etc.}$$

Now suppose $f(x)$ and $g(x)$ extremize I and consider the variation of I due to variations $\delta f(x)$ and $\delta g(x)$ away from the extremum solution. The first variation of I, which must vanish if I is extremum, is given by

$$\begin{aligned} \delta I = 2 \int_{-1}^{1} [& \{A(x)f(x) + B(x)g(x) + D(x)\}\delta f(x) \\ & + \{B(x)f(x) + C(x)g(x) + E(x)\}\delta g(x)] dx = 0 \quad . \end{aligned} \tag{4.5}$$

The variations $\delta f(x)$ and $\delta g(x)$ are not independent but are related by

$$\delta g(x) = -\frac{1}{\pi} \int_{-1}^{1} \frac{\delta f(t)}{t-x} dt \quad , \tag{4.6}$$

which follows from (4.1).

We now wish to substitute (4.6) into (4.5) and change the order of integration over t and x. This is permissible if $\delta f(x)$ and the function

$$G(x) = B(x)f(x) + C(x)g(x) + E(x)$$

satisfy the integrability condition

$$\frac{1}{p_1} + \frac{1}{p_2} \leqslant 1 \tag{4.7}$$

where $\delta f(x) \epsilon L_{p_1}$ and $G(x) \epsilon L_{p_2}$ (see [5], §4.2). We assume (4.7) is satisfied with the understanding that this condition should be checked once f(x) and g(x) have been determined. Substituting (4.6) into (4.5), changing the order of integration of t and x, and finally, replacing t by x and x by t gives

$$\delta I = 2 \int_{-1}^{1} \{A(x)f(x) + B(x)g(x) + D(x)$$

$$+ \frac{1}{\pi} \oint_{-1}^{1} \frac{B(t)f(t) + C(t)g(t) + E(t)}{t - x} dt\} \delta f(x) dx = 0 \quad .$$

Since $\delta f(x)$ is arbitrary, the singular integral equation

$$A(x)f(x) + B(x)g(x) + D(x)$$

$$+ \frac{1}{\pi} \oint_{-1}^{1} \frac{B(t)f(t) + C(t)g(t) + E(t)}{t - x} dt = 0 \tag{4.8}$$

must hold for $|x| \leqslant 1$.

If B(x), C(x), and f(x) are Hölder continuous, Eq. (4.8) can be reduced to a Fredholm integral equation with a regular, symmetric kernel. In some cases, however, it is easier to solve Eqs. (4.1) and (4.8) as a system of singular integral equations (see Ex. 2 below). If the notation for the finite Hilbert transform

$$H[f] \equiv \frac{1}{\pi} \oint_{-1}^{1} \frac{f(t)dt}{t-x}$$

is introduced, Eq. (4.8) becomes

$$A(x)f(x) - B(x)H[f] + H[Bf] - H[CH[f]] = -D - H[E] \quad . \qquad (4.9)$$

The second and third terms on the lefthand side combine to give

$$H[Bf] - B(x)H[f] = \frac{1}{\pi} \int_{-1}^{1} \left(\frac{B(t)-B(x)}{t-x} \right) f(t)dt \quad .$$

The Poincaré-Bertrand formula ([4], §23) can be used to rewrite last term of the lefthand side of (4.9) as follows:

$$-H[CH[f]] = -\frac{1}{\pi^2} \oint_{-1}^{1} \frac{C(t)dt}{t-x} \oint_{-1}^{1} \frac{f(u)du}{u-t}$$

$$= C(x)f(x) + \frac{1}{\pi^2} \int_{-1}^{1} \frac{f(t)dt}{t-x} \int_{-1}^{1} \left(\frac{1}{u-t} - \frac{1}{u-x} \right) C(u)du \quad .$$

Thus, Eq. (4.9) reduces to

$$\{A(x)+C(x)\}f(x) + \int_{-1}^{1} K(t,x)f(t)dt = -D(x) - H[E] \qquad (4.10)$$

where

$$K(t,x) = \frac{1}{\pi} \left(\frac{B(t)-B(x)}{t-x} \right) + \frac{1}{\pi^2(t-x)} \int_{-1}^{1} \left(\frac{1}{u-t} - \frac{1}{u-x} \right) C(u)du. \qquad (4.11)$$

In the next section, it is shown that a necessary condition for the existence of a minimizing solution is that $A(x) + C(x) > 0$ for $|x| < 1$. If we let

$$\hat{f}(x) = \{A(x) + C(x)\}^{\frac{1}{2}} f(x) \qquad (4.12)$$

Eq. (4.10) becomes

$$\hat{f}(x) + \int_{-1}^{1} \hat{K}(t,x)\hat{f}(t)dt = -\frac{D(x) + H[E]}{\{A(x)+C(x)\}^{\frac{1}{2}}} \quad , \tag{4.13}$$

where

$$\hat{K}(t,x) = \hat{K}(x,t)$$

$$= \{A(x)+C(x)\}^{-\frac{1}{2}}\{A(t)+C(t)\}^{-\frac{1}{2}}K(t,x) \tag{4.14}$$

and K is given by (4.11).

Equation (4.13) is now subject to the well-developed theory of Fredholm integral equations. The kernel and righthand side of (4.13) may contain unknown Lagrange multipliers. Ideally, the integral equation would be solved for arbitrary values of $\lambda_1, \lambda_2, \ldots, \lambda_n$. Equation (4.1) then gives $g(x)$ and the n constraints (4.3) determine the $\{\lambda_i\}$.

We now consider two examples:

Example 1: Minimize

$$J = \int_{-1}^{1} [f^2(x) + 2\pi x^3 f(x)g(x)]\,dx \tag{4.15}$$

subject to the constraint

$$J_1 = \int_{-1}^{1} f(x)dx = 1 \quad . \tag{4.16}$$

In this case,

$$A(x) = 1$$

$$B(x) = \pi x^3$$

$$D(x) = -\frac{1}{2}\lambda_1$$

$$C(x) = E(x) = 0$$

$$f(x) = \hat{f}(x) \quad .$$

The integral Eq. (4.13) becomes

$$f(x) + \int_{-1}^{1} \hat{K}(t,x)f(t)dt = \frac{1}{2}\lambda_1 \quad ,$$

where

$$\hat{K}(t,x) = K(t,x) = \frac{1}{\pi}\left(\frac{\pi t^3 - \pi x^3}{t - x}\right) = t^2 + xt + x^2 \quad .$$

The solution is easily found to be

$$f_o(x) = \frac{15}{89}\lambda_1\left(\frac{5}{2} - 3x^2\right) \quad .$$

λ_1 is determined from Eq. (4.16).

$$\int_{-1}^{1} f_o(x)dx = \frac{15}{89}\lambda_1(5 - 2) = \frac{45}{89}\lambda_1 = 1 \quad .$$

Thus,

$$f_o(x) = \frac{1}{3}\left(\frac{5}{2} - 3x^2\right) \quad .$$

From Eq. (4.1),

$$g_o(x) = - H[f_o] = - \frac{1}{3\pi}\left\{\left(\frac{5}{2} - 3x^2\right)\log\left(\frac{1-x}{1+x}\right) - 6x\right\} \quad .$$

Using these forms for f and g in Eq. (4.15), the minimum value of J is

$$J_o = \frac{89}{90} = .9888. \ . \ . \ .$$

In some instances f and g may be required to be Hölder continuous on $[-1,1]$. This means, in particular, that $f(\pm 1) = 0$. Only in exceptional cases will the solution to the variational problem have this property. In case f and g are required to be Hölder continuous, a natural approach is to expand f and g in orthogonal series

$$f^{(M)} = \sum_{k=1}^{M} a_k \sin k\varphi$$

$$g^{(M)} = \sum_{k=1}^{M} a_k \cos k\varphi \quad ,$$

where $x = \cos\varphi$, and then minimize J with respect to $a_1, a_2, \ldots, a_N$. Note that the above equations satisfy (4.1) for all $\{a_k\}$.

In this particular example we can set the coefficients of the even sine and cosine terms to zero; i.e., let

$$f^{(N)} = \sum_{k=1}^{N} \alpha_k \sin(2k-1)\varphi$$

$$g^{(N)} = \sum_{k=1}^{N} \alpha_k \cos(2k-1)\varphi \quad . \tag{4.17}$$

Equation (4.16) requires that $\alpha_1 = \frac{2}{\pi}$ for all N. Let $J^{(N)}$ be the minimum value of J found by substituting (4.17) into (4.15) and minimizing with respect to $\alpha_2, \alpha_3, \ldots, \alpha_N$. We obtain the following results:

N	$J^{(N)}$
1	1.040376
2	0.992061
3	0.989884
4	0.989380
5	0.989185
6	0.989088

N	$J^{(N)}$
7	0.989033
8	0.988998
9	0.988974
10	0.988957

The particular advantage of the result of the variational technique is that it makes it possible to determine just how well the Hölder continuous functions $f^{(N)}$ and $g^{(N)}$ minimize J.

Example 2: Minimize

$$J = \int_{-1}^{1} \{f^2(x) + k\,g^2(x)\}\,dx \tag{4.18}$$

subject to

$$J_1 = \int_{-1}^{1} f(x)dx = 1 \quad .$$

Here k is a constant greater than -1 and

$$A(x) = 1$$

$$C(x) = k$$

$$D(x) = -\frac{1}{2}\lambda_1$$

$$B(x) = E(x) = 0 \quad .$$

In this case it is best to solve the singular integral Eqs. (4.1) and (4.8) directly rather than the Fredholm integral equation.

Equations (4.8) and (4.1) can be written

$$f(x) + kH[g] = \frac{1}{2}\lambda_1 \tag{4.19}$$

$$g(x) + H[f] = 0 \quad , \tag{4.20}$$

respectively.

Case I: $k > 0$. Multiplying (4.20) by $\pm\sqrt{k}$ and adding it to (4.19) gives the separated equations

$$F_+(x) + \sqrt{k}\, H[F_+(t)] = \frac{1}{2}\lambda_1$$

$$F_-(x) - \sqrt{k}\, H[F_-(t)] = \frac{1}{2}\lambda_1 \quad ,$$

where

$$F_\pm(x) = f(x) \pm \sqrt{k}\, g(x) \quad .$$

These singular integral equations are easily solved by the method of Muskhelishvili ([4], Chap. 14) giving

$$F_\pm(x) = \frac{\lambda_1}{2\sqrt{1+k}} \left(\frac{1+x}{1-x}\right)^{\pm\gamma}$$

where

$$\gamma = \frac{1}{\pi}\tan^{-1}\sqrt{k} \quad .$$

Therefore,

$$\begin{aligned} f_o(x) &= \frac{\lambda_1}{4\sqrt{1+k}}\left\{\left(\frac{1+x}{1-x}\right)^{\gamma} + \left(\frac{1-x}{1+x}\right)^{\gamma}\right\} \\ g_o(x) &= \frac{\lambda_1}{4\sqrt{k(1+k)}}\left\{\left(\frac{1+x}{1-x}\right)^{\gamma} - \left(\frac{1-x}{1+x}\right)^{\gamma}\right\} \end{aligned} \tag{4.21}$$

Equations (4.19) and (4.20) are easily checked by the identity

$$H\left[\left(\frac{1+t}{1-t}\right)^{\pm\gamma}\right] = \pm \csc\pi\gamma \mp \cot\pi\gamma \left(\frac{1+x}{1-x}\right)^{\pm\gamma} \quad .$$

where

$$\csc \pi\gamma = \sqrt{\frac{1+k}{k}} \quad , \qquad \cot \pi\gamma = \frac{1}{\sqrt{k}} \ .$$

Using the equation

$$\int_{-1}^{1} \left(\frac{1+x}{1-x}\right)^{\pm\gamma} dx = 2\pi\gamma \csc \pi\gamma = 2\sqrt{\frac{1+k}{k}} \tan^{-1}\sqrt{k} \qquad (4.22)$$

the Lagrange multiplier is found from the constraint equation.

$$\int_{-1}^{1} f_o(x)dx = \frac{\lambda_1}{4\sqrt{1+k}} \cdot 4\sqrt{\frac{1+k}{k}} \tan^{-1}\sqrt{k} = 1 \ .$$

Thus,

$$\lambda_1 = \frac{\sqrt{k}}{\tan^{-1}\sqrt{k}} \ . \qquad (4.23)$$

The minimum value of J is found by substituting (4.21) into (4.18) and using (4.22) and (4.23).

$$\begin{aligned} J_o &= \frac{\lambda_1^2}{16(1+k)} \int_{-1}^{1} \left\{ 2\left(\frac{1+x}{1-x}\right)^{2\gamma} + 2\left(\frac{1-x}{1+x}\right)^{2\gamma} \right\} dx \\ &= \frac{\lambda_1^2}{8(1+k)} \cdot 4\pi(2\gamma) \csc \pi(2\gamma) \\ &= \frac{\sqrt{k}}{2\tan^{-1}\sqrt{k}} \ . \end{aligned}$$

Case II: $k = 0$. In this case it is easily shown that

$$f_o(x) = \text{constant} = \frac{1}{2}$$

$$J_o = \frac{1}{2} \quad ,$$

which agrees with Case I as $k \to 0$.

<u>Case III:</u> $-1 < k < 0$. Using methods similar to those employed in Case I, it is easily shown that

$$f_o(x) = \frac{\lambda_1}{2\sqrt{1+k}} \cos\left(\gamma \log \frac{1+x}{1-x}\right)$$

$$g_o(x) = \frac{\lambda_1}{2\sqrt{-k(1+k)}} \sin\left(\gamma \log \frac{1+x}{1-x}\right) ,$$

where

$$\lambda_1 = \frac{2\sqrt{-k}}{\log\left(\frac{1+\sqrt{-k}}{1-\sqrt{-k}}\right)}$$

$$\gamma = \frac{1}{2\pi} \log\left(\frac{1+\sqrt{-k}}{1-\sqrt{-k}}\right) .$$

The minimum value of J is found to be

$$J_o = \frac{\sqrt{-k}}{\log\left(\frac{1+\sqrt{-k}}{1-\sqrt{-k}}\right)} .$$

To compare the variational method with the Fourier series method, we choose $k = -\frac{1}{4}$, in which case,

$$J_o = \frac{1}{2\log 3} \sim .455119 .$$

The Fourier series method gives the following results:

N	$J^{(N)}$
1	0.472832
2	0.458504
3	0.456182
4	0.455540

N	$J^{(N)}$
5	0.455311
6	0.455216
7	0.455173
8	0.455152
9	0.455142
10	0.455136

Again, $J^{(N)}$ seems to converge to J_o, even though the Fourier expansions, $f^{(N)}$ and $g^{(N)}$, are poor representations of f_o and g_o, particularly near the ends where the extremum solutions oscillate infinitely fast. It should be noted that the higher order coefficients α_k do not seem to converge, but oscillate wildly as N increases.

2. A Necessary Condition for a Minimum Solution

Consider the problem of minimizing a functional of the form

$$I = \int_{-1}^{1} F(f(x), g(x), x)dx \quad ,$$

where f and g are related by

$$g(x) = -\frac{1}{\pi} \oint_{-1}^{1} \frac{f(t)}{t-x}\, dt \quad . \tag{4.24}$$

The first variation of I, which must vanish for arbitrary variations, $\delta f(x)$ and $\delta g(x)$, about the extremal solutions, f(x) and g(x), is given by

$$\delta I = \int_{-1}^{1} [F_f(f,g,x)\delta f(x) + F_g(f,g,x)\delta g(x)]\, dx = 0 \quad . \tag{4.25}$$

The variations $\delta f(x)$ and $\delta g(x)$ are not independent, but are related by

$$\delta g(x) = -\frac{1}{\pi} \oint_{-1}^{1} \frac{\delta f(t)}{t-x}\, dt \qquad (4.26)$$

which follows from (4.24). Now suppose $F_g(f(x), g(x), x) \epsilon L_{p_1}$ and $\delta f(x) \epsilon L_{p_2}$ with

$$\frac{1}{p_1} + \frac{1}{p_2} \leqslant 1 \quad .$$

With this assumption, it is permissible to substitute (4.26) into (4.25) and reverse the order of integration over the t and x variables. This gives

$$\delta I = \int_{-1}^{1} \left[F_f(f, g, x) + \frac{1}{\pi} \oint_{-1}^{1} \frac{F_g(f, g, t)}{t - x}\, dt \right] \delta f(x) dx = 0 \quad .$$

Since $\delta f(x)$ is arbitrary, the following nonlinear singular integral equation must hold for $|x| \leqslant 1$:

$$F_f(f(x), g(x), x) + \frac{1}{\pi} \oint_{-1}^{1} \frac{F_g(f(t), g(t), t)}{t - x}\, dt = 0 \quad . \qquad (4.27)$$

We now suppose that Eqs. (4.27) and (4.24) can be solved for the extremal arcs $f(x)$ and $g(x)$. Under what condition does this solution actually provide a minimum of I?

To study this question, we look at

$$\Delta I = \int_{-1}^{1} F(f+\delta f,\ g+\delta g, x) dx - \int_{-1}^{1} F(f, g, x) dx \quad ,$$

which should be positive for arbitrary δf and δg. Suppose that ΔI can be closely approximated by the first two variations of I,

$$\Delta I \sim \delta I + \frac{1}{2}\, \delta^2 I \quad ,$$

where

$$\delta^2 I = \int_{-1}^{1} [F_{ff}(f,g,x)\{\delta f(x)\}^2 + 2F_{fg}(f,g,x)\delta f(x)\delta g(x)$$

$$+ F_{gg}(f,g,x)\{\delta g(x)\}^2] dx \quad . \tag{4.28}$$

Since $\delta I = 0$, the condition that I be a minimum requires

$$\delta^2 I > 0$$

for all variations $\delta f(x)$ and $\delta g(x)$ consistent with Eq. (4.26).

Consider the case in which $F_{ff}(f(x), g(x), x)$, $F_{fg}(f(x), g(x), x)$, $F_{gg}(f(x), g(x), x)$, and $\delta f(x)$ are all Hölder continuous on $[-1, 1]$. The second term on the righthand side of (4.28) can be rewritten

$$2\int_{-1}^{1} F_{fg}(f,g,x)\delta f(x)\delta g(x)dx$$

$$= -2\int_{-1}^{1} F_{fg}(f,g,x)\delta f(x)H[\delta f(t)]\,dx$$

$$= 2\int_{-1}^{1} H[F_{fg}(f,g,t)\delta f(t)]\,\delta f(x)dx$$

$$= \frac{1}{\pi}\int_{-1}^{1}\int_{-1}^{1}\left(\frac{F_{fg}(f(t),g(t),t)-F_{fg}(f(x),g(x),x)}{t-x}\right)\delta f(t)\delta f(x)dt\,dx \quad .$$

The second line follows by the substitution of Eq. (4.26) and the third line is a result of reversing the order of integration. The last line is just the mean of the two previous lines.

By similar operations and the use of the Poincare-Bertrand formula, the third term on the righthand side of (4.28) can be written

$$\int_{-1}^{1} F_{gg}(f,g,x)\{\delta g(x)\}^2 dx = \int_{-1}^{1} F_{gg}(f,g,x) H[\delta f]\; H[\delta f]\, dx$$

$$= -\int_{-1}^{1} H[\, F_{gg} H[\delta f]\,]\, \delta f(x) dx$$

$$= \int_{-1}^{1} F_{gg}(f,g,x)\{\delta f(x)\}^2 dx$$

$$- \frac{1}{\pi^2} \int_{-1}^{1} \int_{-1}^{1} \frac{1}{t-x} \int_{-1}^{1} \left(\frac{1}{u-t} - \frac{1}{u-x} \right) F_{gg}(f,g,u) du\, \delta f(x) \delta f(t) dt\, dx \quad .$$

Combining these results, Eq. (4.28) becomes

$$\delta^2 I = \int_{-1}^{1} [\, F_{ff}(f(x), g(x), x) + F_{gg}(f(x), g(x), x)]\, \{\delta f(x)\}^2 dx$$

$$+ \frac{1}{\pi} \int_{-1}^{1} \int_{-1}^{1} \left(\frac{\chi(t) - \chi(x)}{t - x} \right) \delta f(x) \delta f(t) dt\, dx \quad , \tag{4.29}$$

where

$$\chi(x) = F_{fg}(f(x), g(x), x) - \frac{1}{\pi} \oint_{-1}^{1} \frac{F_{gg}(f(u), g(u), u)}{u - x}\, du \quad .$$

Since F_{fg} and F_{gg} are assumed to be Hölder continuous on $[-1, 1]$, $\chi(x)$ is also Hölder continuous except possibly at the endpoints where $F_{gg}(f(x), g(x), x) \neq 0$.

Now consider a special choice of $\delta f(x)$: Let

$$\delta f(x) = M\eta\left(\frac{x - x_o}{\epsilon} \right) \quad ,$$

where η is Hölder continuous and

$$0 < \eta(u) \leqslant 1 \qquad \text{for} \quad |u| < 1 \quad ,$$

$$\eta(u) \equiv 0 \qquad |u| \geqslant 1 \quad . \tag{4.30}$$

M is either positive or negative and is chosen so small that $\delta^3 I$, $\delta^4 I$, etc., can be neglected in comparison with $\delta^2 I$. x_0 is any fixed point in the open interval $(-1, 1)$ and, for the time being, ϵ is chosen so that $|x_0 \pm \epsilon| < 1$.

With the choice of δf, the limits of integration in the expression for $\delta^2 I$ given by Eq. (4.29) can be taken from $x_0 - \epsilon$ to $x_0 + \epsilon$. Since $\chi(x)$ and the function

$$\Theta(x) \equiv F_{ff}(f(x), g(x), x) + F_{gg}(f(x), g(x), x)$$

are Hölder continuous on this interval we have

$$\begin{aligned} |\chi(t) - \chi(x)| &\leqslant A_1 \,|t - x)|^{\mu_1} \\ |\Theta(x) - \Theta(x_0)| &\leqslant A_2 \,|x - x_0|^{\mu_2} \quad , \end{aligned} \tag{4.31}$$

with $0 < \mu_1 \leqslant 1$ and $0 < \mu_2 \leqslant 1$, for any x and t in the interval $[x_0 - \epsilon, x_0 + \epsilon]$.

By adding and subtracting a term, Eq. (4.29) can be written

$$\begin{aligned} \delta^2 I &= \Theta(x_0) M^2 \int_{x_0-\epsilon}^{x_0+\epsilon} \eta^2\left(\frac{x - x_0}{\epsilon}\right) dx + R \\ &= \Theta(x_0) M^2 \epsilon \int_{-1}^{1} \eta^2(u) du + R \quad , \end{aligned} \tag{4.32}$$

where

$$R = \int_{x_0-\epsilon}^{x_0+\epsilon} [\Theta(x) - \Theta(x_0)] \{\delta f(x)\}^2 \, dx$$

$$+ \frac{1}{\pi} \int_{x_0-\epsilon}^{x_0+\epsilon} \int_{x_0-t}^{x_0+\epsilon} \left(\frac{\chi(t) - \chi(x)}{t - x}\right) \delta f\,(x) \delta f(t) dt\, dx \quad .$$

Using the inequalities (4.30) and (4.31), it follows that

$$|R| \leqslant A_2 M^2 \epsilon^{1+\mu_2} \int_{-1}^{1} |\mu|^{\mu_2} \eta^2(u)du$$

$$+ A_1 M^2 \epsilon^{1+\mu_1} \int_{-1}^{1}\int_{-1}^{1} |u - v|^{\mu_1 - 1} \eta(u)\eta(v)du\, dv$$

$$\leqslant \left(\frac{2A_2M^2}{1+\mu_2}\right)\epsilon^{1+\mu_2} + \left(\frac{2^{2+\mu_1} A_1 M^2}{\mu_1(1+\mu_1)}\right)\epsilon^{1+\mu_1} \quad .$$

We now shrink the interval over which δf is nonzero by letting ϵ go to zero. As $\epsilon \to 0$, the first term on the righthand side of Eq. (4.32) dominates, so that a necessary condition that $\delta^2 I > 0$ is that

$$\Theta(x_o) = F_{ff}(f(x_o), g(x_o), x_o) + F_{gg}(f(x_o), g(x_o), x_o) > 0 \quad (4.33)$$

for every $x_o \epsilon (-1, 1)$.

If F is quadratic in f and g, as in the previous section, condition (4.33) is independent of the extremum solution; i.e., $\delta^2 I > 0$ requires

$$A(x_o) + C(x_o) > 0 \quad \text{for} \quad |x_o| < 1.$$

This condition is satisfied in the two examples considered in the last section.

V. APPLICATION OF THE VARIATIONAL TECHNIQUE TO THE MINIMUM DRAG PROBLEM

1. Introduction - The Singular Integral Equation

In this chapter we apply some of the theory developed in the last chapter to the minimum drag problem, a concise statement of which appears in Ch. II, §5.

The problem of minimizing the drag (2.33) under the constraints of fixed chord, y_o, and fixed arclength, s_o, is seen to be equivalent to finding the extremal arcs $\Gamma_1(\xi)$ and $\beta_1(\xi)$ and constants A, c, and α, which satisfy the isoperimetric constraints, (2.34) and (2.35), and either of the functional relations, (2.38) or (2.39), and extremizes the functional

$$I[\Gamma_1(\xi),\beta_1(\xi),\xi] = D^* - \lambda_1 s_o - \lambda_2 y_o$$

$$= \int_{-1}^{1} F(\Gamma(\xi),\beta(\xi),\xi;A,c,\lambda_1,\lambda_2)d\xi \quad , \qquad (5.1)$$

where

$$F = \frac{A}{2\pi}\left(\int_{-1}^{1}\Gamma(t)dt\right)\Gamma(\xi) - \lambda_1\left[\frac{A}{4}(c^2-1) + \frac{A}{2}e^{\Gamma(\xi)}|\xi|\right]$$

$$-\lambda_2\left[\frac{A}{4\pi}\left(\int_{-1}^{1}\Gamma(t)dt\right)\Gamma(\xi) + \frac{A}{2\pi}\xi\sinh\Gamma(\xi)\cos\beta(\xi)\log\frac{c-\xi}{c+\xi} + \frac{Ac}{\pi}\Gamma(\xi)\right] ,$$

and $\Gamma(\xi)$ and $\beta(\xi)$ have the form

$$\Gamma(\xi) = \frac{2\alpha}{\pi} \log\left(\frac{1+(1-\xi^2)^{\frac{1}{2}}}{|\xi|}\right) + \Gamma_1(\xi)$$

$$\beta(\xi) = \alpha \operatorname{sgn}\xi + \beta_1(\xi) \quad .$$

λ_1 and λ_2 are undetermined Lagrange multipliers.

The first variation of I is given by

$$\begin{aligned}
\delta I = {} & \frac{\partial I}{\partial A}\,\delta A + \frac{\partial I}{\partial c}\,\delta c \\
& + A\int_{-1}^{1}\left\{\frac{1}{\pi}\int_{-1}^{1}\Gamma(t)dt - \lambda_1\,\frac{1}{2}\,e^{\Gamma(\xi)}|\xi|\right. \\
& \left. - \lambda_2\left(\frac{1}{2\pi}\int_{-1}^{1}\Gamma(t)dt + \frac{1}{2\pi}\,\xi\cosh\Gamma\cos\beta\log\frac{c-\xi}{c+\xi} + \frac{c}{\pi}\right)\right\}\delta\Gamma(\xi)d\xi \\
& + A\int_{-1}^{1}\left(\lambda_2\,\frac{1}{2\pi}\,\xi\sinh\Gamma\sin\beta\log\frac{c-\xi}{c+\xi}\right)\delta\beta(\xi)d\xi \\
= {} & \frac{\partial I}{\partial A}\,\delta A + \frac{\partial I}{\partial c}\,\delta c \\
& + A\int_{-1}^{1}\left\{\frac{1}{\pi}\int_{-1}^{1}\Gamma(t)dt - \lambda_1\,\frac{1}{2}\,e^{\Gamma(\xi)}|\xi|\right. \\
& - \lambda_2\left(\frac{1}{2\pi}\int_{-1}^{1}\Gamma(t)dt + \frac{1}{2\pi}\,\xi\cosh\Gamma\cos\beta\log\frac{c-\xi}{c+\xi} + \frac{c}{\pi}\right. \\
& \left.\left. - \frac{1}{2\pi}H\left[t\sinh\Gamma(t)\sin\beta(t)\log\frac{c-t}{c+t}\right]\right)\right\}\delta\Gamma(\xi)d\xi \quad .
\end{aligned}$$

This last step is possible if

$$\delta\Gamma(\xi) = \frac{2\delta\alpha}{\pi}\log\left(\frac{1+(1-\xi^2)^{\frac{1}{2}}}{|\xi|}\right) + \delta\Gamma_1(\xi)$$

and

$$F_\beta = \lambda_2 \frac{A}{2\pi} \xi \sinh \Gamma(\xi) \sin\beta(\xi) \log \frac{c-\xi}{c+\xi}$$

satisfy the integrability condition

$$\frac{1}{p_1} + \frac{1}{p_2} < 1 \quad ,$$

where $\delta\Gamma(\xi) \epsilon L_{p1}$ and $F_\beta \epsilon L_{p2}$.

This condition can be checked only when Γ and β have been determined; however, if we assume $\delta\Gamma_1(\xi)$ and $\Gamma_1(\xi)$ are bounded, $\delta\Gamma(\xi)$ is finite except at the origin where

$$\delta\Gamma(\xi) \sim \delta\alpha \frac{2}{\pi} \log \frac{2}{|\xi|} \quad .$$

Therefore, p_1 can be chosen as large as we please. In addition, F_β is bounded since as $|\xi| \to 0$,

$$F_\beta \sim -\frac{2\lambda_2 A}{\pi} e^{\Gamma_1(o)} \sin\alpha \left(\frac{|\xi|}{2}\right)^{2\left(1-\frac{\alpha}{\pi}\right)} \quad ,$$

where $0 \leqslant \alpha \leqslant \pi$. Hence, p_2 can also be chosen arbitrarily large.

We now choose λ_1 and λ_2 so that partial derivatives I_A and I_c vanish; i.e.,

$$\frac{\partial I}{\partial A} = \frac{\partial D^*}{\partial A} - \lambda_1 \frac{\partial s_o}{\partial A} - \lambda_2 \frac{\partial y_o}{\partial A} = 0 \tag{5.2}$$

$$\frac{\partial I}{\partial c} = \frac{\partial D^*}{\partial c} - \lambda_1 \frac{\partial s_o}{\partial c} - \lambda_2 \frac{\partial y_o}{\partial c} = 0 \quad . \tag{5.3}$$

Since $\delta\Gamma(\xi)$ is arbitrary, the following nonlinear singular integral equation must hold for $|\xi| < 1$:

$$\frac{1}{\pi}\int_{-1}^{1}\Gamma(t)dt - \lambda_1 \frac{1}{2} e^{\Gamma(\xi)}|\xi|$$

$$- \lambda_2\left(\frac{1}{2\pi}\int_{-1}^{1}\Gamma(t)dt + \frac{1}{2\pi}\xi \cosh\Gamma(\xi)\cos\beta(\xi)\log\frac{c-\xi}{c+\xi}\right.$$

$$\left.+ \frac{c}{\pi} - \frac{1}{2\pi}H\left[t \sinh\Gamma(t)\sin\beta(t)\log\frac{c-t}{c+t}\right]\right) = 0 \quad . \tag{5.4}$$

Since D^*, s_o, and y_o, depend linearly on A, the equation

$$D^* - \lambda_1 s_o - \lambda_2 y_o = 0 \tag{5.5}$$

follows from Eq. (5.2).

From Eqs. (2.33), (2.35), and (2.25), it is easily shown that

$$\frac{\partial D^*}{\partial c} = 0$$

$$\frac{\partial s_o}{\partial c} = Ac$$

$$\frac{\partial y_o}{\partial c} = Ac \sin\beta^*(c) \quad .$$

Here, $\beta^*(c)$ is the flow angle at the endpoint A of the plate (see Fig. 1). Therefore, Eq. (5.3) gives

$$\lambda_1 + \lambda_2 \sin\beta^*(c) = 0 \quad . \tag{5.6}$$

Solving Eqs. (5.5) and (5.6) for λ_1 and λ_2, it is possible to identify the Lagrange multipliers with several parameters of the problem.

$$\lambda_1 = \frac{D^* \sin\beta^*(c)}{s_o \sin\beta^*(c) - y_o} = \frac{C_D \sin\beta^*(c)}{k \sin\beta^*(c) - 1} \tag{5.7}$$

$$\lambda_2 = - \frac{D^*}{s_o \sin\beta^*(c) - y_o} = - \frac{C_D}{k \sin\beta^*(c) - 1} \quad . \quad (5.8)$$

2. Reduction of Eq. (5.4) to an Integral Equation with Regular Kernel

By taking account of the functional relation

$$\beta(\xi) = - \frac{1}{\pi} \oint_{-1}^{1} \frac{\Gamma(t)}{t-\xi} \, dt \quad ,$$

it is possible to reduce the singular integral equation (5.4) to an integral equation with a regular kernel.

If we let $\zeta \to \xi \pm i0$ for $|\xi| < 1$, in Eq. (2.31), we have by Plemelj's formula

$$\xi e^{\pm\Gamma(\xi) + i\beta(\xi)} = \frac{1}{\pi i} \left\{ \pm \pi i \xi e^{i\beta(\xi)} \sinh\Gamma(\xi) + \int_{-1}^{1} \frac{t e^{i\beta(t)} \sinh\Gamma(t)}{t - \xi} \, dt \right\} + \frac{i}{\pi} \int_{-1}^{1} \Gamma(t) dt + \xi \quad .$$

Adding these equations and dividing by two gives

$$\xi \cosh\Gamma(\xi) e^{i\beta(\xi)} = \frac{1}{\pi i} \oint_{-1}^{1} \frac{t e^{i\beta(t)} \sinh\Gamma(t)}{t - \xi} \, dt + \frac{i}{\pi} \int_{-1}^{1} \Gamma(t) dt + \xi \quad .$$

The real part of this equation reads

$$\xi \cosh\Gamma(\xi) \cos\beta(\xi) = H[t \sinh\Gamma(t) \sin\beta(t)] + \xi \quad . \quad (5.9)$$

The substitution of Eq. (5.9) into the integral equation (5.4) gives

$$\frac{1}{\pi}\int_{-1}^{1}\Gamma(t)dt - \frac{\lambda_1}{2}\, e^{\Gamma(\xi)}|\xi| - \lambda_2\Bigg(\frac{1}{2\pi}\int_{-1}^{1}\Gamma(t)dt$$

$$+ \frac{\xi}{2\pi}\log\frac{c-\xi}{c+\xi} + \frac{c}{\pi} + \int_{-1}^{1} t\sinh\Gamma(t)\sin\beta(t)K(t,\xi;c)dt\Bigg) = 0 \quad , (5.10)$$

where

$$K(t,\xi;c) = \frac{1}{2\pi^2}\,\frac{\log\frac{c+t}{c-t} - \log\frac{c+\xi}{c-\xi}}{t-\xi} \qquad (5.11)$$

is a regular kernel

$$\left(K(\xi,\xi;c) = \frac{1}{\pi^2}\,\frac{c}{c^2-\xi^2} \right) \quad .$$

From Eq. (5.10), we see that $e^{\Gamma(\xi)}|\xi|$ possesses the following series expansion about $\xi = 0$:

$$e^{\Gamma(\xi)}|\xi| = \frac{2}{\lambda_1}\Bigg\{\frac{1}{\pi}\left(1-\frac{\lambda_2}{2}\right)\int_{-1}^{1}\Gamma(t)dt - \lambda_2\Bigg(\frac{\xi}{2\pi}\log\frac{c-\xi}{c+\xi}$$

$$+ \frac{c}{\pi} + \int_{-1}^{1} t\sinh\Gamma(t)\sin\beta(t)K(t,\xi;c)dt\Bigg)\Bigg\}$$

$$= C_o + C_1\xi^2 + \dots \quad ,$$

where

$$C_o = \frac{2}{\lambda_1}\Bigg\{\frac{1}{\pi}\left(1-\frac{\lambda_2}{2}\right)\int_{-1}^{1}\Gamma(t)dt$$

$$- \lambda_2\left(\frac{c}{\pi} + \frac{1}{2\pi^2}\int_{-1}^{1}\sinh\Gamma(t)\sin\beta(t)\log\frac{c+t}{c-t}\,dt\right)\Bigg\}$$

$$C_1 = 2\,\frac{\lambda_2}{\lambda_1}\left\{\frac{1}{\pi c} - \frac{1}{2\pi^2}\int_{-1}^{1}\sinh\Gamma\sin\beta\left(\frac{1}{t^2}\log\frac{c+t}{c-t} - \frac{2}{ct}\right)dt\right\} \quad ,$$

etc.

On the hand, from Eq. (2.36),

$$e^{\Gamma(\xi)}|\xi| = \left\{\frac{1+(1-\xi^2)^{\frac{1}{2}}}{|\xi|}\right\}^{\frac{2\alpha}{\pi}} |\xi| e^{\Gamma_1(\xi)}$$

$$\sim |\xi|^{1-\frac{2\alpha}{\pi}} (2)^{\frac{2\alpha}{\pi}} e^{\Gamma_1(o)} \quad \text{as} \quad |\xi| \to 0 \quad .$$

These two results are consistent only if

$$1 - \frac{2\alpha}{\pi} = 0$$

or

$$\alpha = \frac{\pi}{2} \quad .$$

This means that the plate shape which solves the variational problem must have a blunt nose. (Note: We can rule out the case $C_o = 0$, since if $C_o = 0$, the two expansions for $e^{\Gamma}|\xi|$ are consistent only if $1 - \frac{2\alpha}{\pi} = 2$, or $\alpha = -\frac{\pi}{2}$; however, this violates $0 \leq \alpha \leq \frac{\pi}{2}$.)

3. Factorization of the Integral Equation

From Eqs. (5.7) and (5.8), we see that the Lagrange multipliers, λ_1 and λ_2 have the multiplier

$$D^* = \frac{A}{2\pi}\left(\int_{-1}^{1} \Gamma(\xi)d\xi\right)^2$$

in common. Thus, the integral equation (5.10) is satisfied if

$$\int_{-1}^{1} \Gamma(\xi)d\xi = 0 \quad ;$$

i.e., if the plate was zero drag. This case, however, violates the pressure condition $p \geq p_c$. We rule out this case by dividing Eq. (5.10) by the term $\int_{-1}^{1} \Gamma(\xi)d\xi$. This gives

$$\frac{1}{\pi} - \frac{\widetilde{\lambda}_1}{2} e^{\Gamma(\xi)}|\xi| - \widetilde{\lambda}_2 \left(\frac{1}{2\pi} \int_{-1}^{1} \Gamma(t)dt \right.$$

$$\left. + \frac{\xi}{2\pi} \log \frac{c-\xi}{c+\xi} + \frac{c}{\pi} + \int_{-1}^{1} t \sinh \Gamma(t) \sin \beta(t) K(t, \xi; c) dt \right) = 0 \quad , (5.12)$$

where

$$\widetilde{\lambda}_i = \lambda_i \bigg/ \left(\int_{-1}^{1} \Gamma(\xi) d\xi \right) \quad ;$$

i.e., $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$ satisfy the equations

$$\widetilde{\lambda}_1 \left\{ \frac{1}{2} (c^2 - 1) + \frac{1}{2} \int_{-1}^{1} e^{\Gamma(\xi)} |\xi| d\xi \right\}$$

$$+ \widetilde{\lambda}_2 \left\{ \frac{1}{4\pi} \left(\int_{-1}^{1} \Gamma(\xi) d\xi \right)^2 - \frac{1}{2\pi} \int_{-1}^{1} \xi \sinh \Gamma \cos \beta \log \frac{c+\xi}{c-\xi} d\xi \right.$$

$$\left. + \frac{c}{\pi} \int_{-1}^{1} \Gamma(\xi) d\xi \right\} = \frac{1}{2\pi} \int_{-1}^{1} \Gamma(\xi) d\xi \quad (5.13)$$

$$\widetilde{\lambda}_1 + \widetilde{\lambda}_2 \sin \beta^*(c) = 0 \quad . \quad (5.14)$$

These last equations follow from Eqs. (5.5) and (5.6).

The above factorization is equivalent to factoring out the solution $a_1 = 2$ in the parameter problem of Ch. III, §2.

Equations (5.12) through (5.14), in which

$$\Gamma(\xi) = \log \left(\frac{1 + (1 - \xi^2)^{\frac{1}{2}}}{|\xi|} \right) + \Gamma_1(\xi) \quad (5.15)$$

$$\beta(\xi) = \frac{\pi}{2} \operatorname{sgn} \xi + \beta_1(\xi) \quad , \quad (5.16)$$

and the equations

$$\beta_1(\xi) = -\frac{1}{\pi}\oint_{-1}^{1} \frac{\Gamma_1(t)}{t-\xi}\, dt \tag{5.17}$$

$$\beta^*(c) = \sin^{-1}\left(\frac{1}{c}\right) - \frac{1}{\pi}\int_{-1}^{1} \frac{\Gamma_1(t)}{t-c}\, dt \tag{5.18}$$

$$k = \frac{s_o}{y_o} \tag{5.19}$$

$$C_D = \frac{D^*}{y_o} \tag{5.20}$$

complete the system of equations for the variational problem. In Eqs. (5.19) and (5.20), s_o, y_o, and D^*, are given by Eqs. (2.33) through (2.35).

Ideally, the integral equations (5.12) and (5.17) would be solved for arbitrary values of $\tilde{\lambda}_1$, $\tilde{\lambda}_2$, and c, the result of this calculation giving solutions, $\Gamma_1(\xi; \tilde{\lambda}_1, \tilde{\lambda}_2, c)$ and $\beta_1(\xi; \tilde{\lambda}_1, \tilde{\lambda}_2, c)$. Substituting these forms for Γ_1 and β_1 into Eqs. (5.13) and (5.14) gives two relations among the three unknowns, $\tilde{\lambda}_1$, $\tilde{\lambda}_2$, and c. A third relation is given by (5.19), once k has been given a specific value; however, it seems best to leave one parameter free and let it determine k.

The results of the parametric problem for $N = 1$ (Chapter III, §2) would indicate that this free parameter should be c, the parameter which determines how much of the plate is a free-streamline. As $k \to 1+$, we expect the plate shape to approach that of a flat plate with no free-streamline ($c=1$). As $k \to \infty$, a plate which is mostly free-streamline ($c \to \infty$) would seem to be the one of minimum drag.

In summary, the following procedure for solving the variational problem is proposed: For arbitrary values of $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ and for a given fixed value of c, solve Eqs. (5.12) and (5.19) for $\Gamma_1(\xi; \tilde{\lambda}_1, \tilde{\lambda}_2, c)$ and $\beta_1(\xi; \tilde{\lambda}_1, \tilde{\lambda}_2, c)$. Next, $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are determined from Eqs. (5.13) and (5.14). Finally, k and C_D are given by Eqs. (5.19) and (5.20). Changing the value of c should give different values for k. As $c \to 1$, we expect $k \to 1$; as $c \to \infty$, $k \to \infty$.

4. Numerical Methods

An analytic solution of the system of equations in the previous section seems out of the question because of their extreme nonlinearity, our only recourse being a numerical solution. As of now, however, even these attempts have failed.

We mention, briefly, one of the methods which have been tried, an iteration scheme. First, Eq. (5.12) is solved for $\Gamma_1(\xi)$, giving

$$\Gamma_1(\xi) = \log\left\{\frac{2}{\tilde{\lambda}_1(1+(1-\xi^2)^{\frac{1}{2}})}\left[\frac{1}{\pi} - \tilde{\lambda}_2\left(\frac{1}{2\pi}\int_{-1}^{1}\Gamma(t)dt + \frac{\xi}{2\pi}\log\frac{c-\xi}{c+\xi} + \frac{c}{\pi} + \int_{-1}^{1} t \sinh\Gamma(t)\sin\beta(t)K(t,\xi;c)dt\right)\right]\right\}. \qquad (5.21)$$

The integrals in Eqs. (5.21), (5.13), (5.14), and (5.18), are approximated by Gaussian quadratures which involve only the values of Γ_1 and β_1 at the Gaussian abscissas. The Hilbert transform in Eq. (5.17) is accomplished by finding the finite Fourier expansion of Γ_1,

$$\Gamma_1(\xi) = \sum_{n=1}^{M} a_n \sin(2n-1)\varphi \qquad (\xi = \cos\varphi) \quad ,$$

and using the formula

$$\beta_1(\xi) = -\sum_{n=1}^{M} a_n \cos(2n-1)\varphi \quad .$$

c is given a fixed value greater than unity. An initial guess of Γ_1 is made and β_1 is found from Eq. (5.17). Using these values of Γ_1 and β_1, Eqs. (5.13) and (5.14) are solved for $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$. New values of Γ_1 on the Gaussian abscissas are found from Eq. (5.21) by substituting the old Γ_1, β_1, $\tilde{\lambda}_1$, and $\tilde{\lambda}_2$, on the righthand side. The process is then repeated.

This method does not converge. Among its more noticeable defects are (i) the argument of the logarithm in Eq. (5.21) becomes negative, and (ii) the value of Γ_1 as the endpoints are not zero, hence β_1 becomes large as $\xi \rightarrow \pm 1$ and the integration of terms involving $\cos\beta$ and $\sin\beta$ by quadrature fails. Corrective steps such as increasing the order of the Gaussian quadrature and forcing Γ_1 to zero at the endpoints have not helped.

5. Linearized Formulation

In the parametric problem for the case $N = 1$, $|\Gamma_1(\xi)|$ and $|\beta_1(\xi)|$ were found never to exceed 0.102 regardless of the value of k. If this result bears any relation to the solutions of the variation problem, linearization of the system of equations in Section 3 of this chapter seems warranted; i.e., we neglect Γ_1^2, $\Gamma_1\beta_1$, β_1^2, etc., and integrals of these terms.

Equations (5.12), (5.13), (5.14), and (5.18), are linearized, keeping only linear terms in $\Gamma_1(\xi)$ and $\beta_1(\xi)$ and their integrals. The following approximations are used:

$$e^{\Gamma(\xi)}|\xi| = \{1+(1-\xi^2)^{\frac{1}{2}}\}e^{\Gamma_1(\xi)}$$

$$\sim \{1+(1-\xi^2)^{\frac{1}{2}}\}(1+\Gamma_1(\xi))$$

$$\xi \sinh\Gamma(\xi) = \operatorname{sgn}\xi\{\sinh\Gamma_1(\xi) + (1-\xi^2)^{\frac{1}{2}}\cosh\Gamma_1(\xi)\}$$

$$\sim \operatorname{sgn}\xi\{(1-\xi^2)^{\frac{1}{2}}+\Gamma_1(\xi)\}$$

$$\sin\beta(\xi) = \operatorname{sgn}\xi \; \cos\beta_1(\xi) \sim \operatorname{sgn}\xi \quad , \; \cos\beta(\xi) \sim -\operatorname{sgn}\xi\,\beta_1(\xi)$$

$$\sin\beta^*(c) = \frac{1}{c}\cos\left(\frac{1}{\pi}\int_{-1}^{1}\frac{\Gamma_1(t)}{t-c}dt\right) - \frac{(c^2-1)^{\frac{1}{2}}}{c}\sin\left(\frac{1}{\pi}\int_{-1}^{1}\frac{\Gamma_1(t)}{t-c}\,dt\right)$$

$$\sim \frac{1}{c} - \frac{(c^2-1)^{\frac{1}{2}}}{\pi}\int_{-1}^{1}\frac{\Gamma_1(t)}{t^2-c^2}\,dt \quad .$$

In the last equation the symmetry property $\Gamma_1(-\xi) = \Gamma_1(\xi)$ has been used.

With these approximations, Eq. (5.13) can be written as

$$\tilde{\lambda}_1\left[\frac{1}{2}(c^2-1) + \frac{1}{2}\int_{-1}^{1}\{1+(1-t^2)^{\frac{1}{2}}\}(1+\Gamma_1(t))dt\right] \;.$$

$$+\tilde{\lambda}_2\left[\frac{1}{4\pi}\left(\pi^2+2\pi\int_{-1}^{1}\Gamma_1(t)dt\right)+\frac{1}{2\pi}\int_{-1}^{1}\beta_1(t)(1-t^2)^{\frac{1}{2}}\log\frac{c+t}{c-t}\,dt\right.$$

$$\left.+\frac{c}{\pi}\left(\pi+\int_{-1}^{1}\Gamma_1(t)dt\right)\right] = \frac{1}{2\pi}\left(\pi+\int_{-1}^{1}\Gamma_1(t)dt\right) \;.$$

We now substitute $\beta_1(\xi) = -H[\Gamma_1]$ into this equation, change the order of integration, and use the formula

$$H\left[(1-t^2)^{\frac{1}{2}}\log\frac{c+t}{c-t}\right] = -2\{c-(c^2-1)^{\frac{1}{2}}\} + (1-\xi^2)^{\frac{1}{2}}\,\Omega(\xi;c) \quad ,$$

where

$$\Omega(\xi;c) = \sin^{-1}\left(\frac{1+\xi c}{c+\xi}\right) + \sin^{-1}\left(\frac{1-\xi c}{c-\xi}\right) . \qquad (5.22)$$

This results in

$$\tilde{\lambda}_1\left[\frac{1}{2}(c^2-1)+\frac{\pi}{4}+\frac{1}{2}\int_{-1}^{1}\{1+(1-t^2)^{\frac{1}{2}}\}\Gamma_1(t)dt\right]$$
$$+\tilde{\lambda}_2\left[c+\frac{\pi}{4}+\left(\frac{1}{2}+\frac{(c^2-1)^{\frac{1}{2}}}{\pi}\right)\int_{-1}^{1}\Gamma_1(t)dt\right.$$
$$\left.+\frac{1}{2\pi}\int_{-1}^{1}\Gamma_1(t)(1-t^2)^{\frac{1}{2}}\Omega(t;c)dt\right]=\frac{1}{2}+\frac{1}{2\pi}\int_{-1}^{1}\Gamma_1(t)dt \quad . \tag{5.23}$$

Using the above mentioned approximations, Eq. (5.14) becomes

$$\tilde{\lambda}_1+\tilde{\lambda}_2\left[\frac{1}{c}-\frac{(c^2-1)^{\frac{1}{2}}}{\pi}\int_{-1}^{1}\frac{\Gamma_1(t)}{t^2-c^2}dt\right]=0 \quad . \tag{5.24}$$

Solving Eqs. (5.23) and (5.24) for $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ gives the following results, good up to linear order:

$$\tilde{\lambda}_1=\frac{1}{D}\left(\frac{1}{2}+\frac{1}{2\pi}\int_{-1}^{1}\Gamma_1(t)dt+\frac{c(c^2-1)^{\frac{1}{2}}}{2\pi}\int_{-1}^{1}\frac{\Gamma_1(t)}{c^2-t^2}dt\right) \tag{5.25}$$

$$\tilde{\lambda}_2=-\frac{1}{D}\left(\frac{c}{2}+\frac{c}{2\pi}\int_{-1}^{1}\Gamma_1(t)dt\right) , \tag{5.26}$$

where

$$D=\frac{1}{2}(1-c^2)+\frac{\pi}{4}(1-c)+\frac{1}{2}(1-c)\int_{-1}^{1}\Gamma_1(t)dt$$
$$+\frac{1}{2}\int_{-1}^{1}\Gamma_1(t)(1-t^2)\left\{1-\frac{c}{\pi}\Omega(t;c)\right\}dt$$
$$-\frac{c(c^2-1)^{\frac{1}{2}}}{\pi}\int_{-1}^{1}\Gamma_1(t)dt+\frac{c(c^2-1)^{\frac{1}{2}}}{2\pi}\left(c^2+1+\frac{\pi}{2}\right)\int_{-1}^{1}\frac{\Gamma_1(t)}{c^2-t^2}dt \quad . \tag{5.27}$$

The integral equation (5.12) can be written

$$\frac{1}{\pi} - \frac{\widetilde{\lambda}_1}{2}\{1+(1-\xi^2)^{\frac{1}{2}}\}(1+\Gamma_1(\xi))$$

$$- \widetilde{\lambda}_2\left[\frac{1}{2\pi}\left(\pi + \int_{-1}^{1} \Gamma_1(t)dt\right) + \frac{\xi}{2\pi}\log\frac{c-\xi}{c+\xi} + \frac{c}{\pi}\right.$$

$$\left. + \int_{-1}^{1} [\Gamma_1(t) + (1-t^2)^{\frac{1}{2}}]K(t,\xi;c)dt\right] = 0 \quad .$$

Using expression (5.11) for the kernel,

$$\int_{-1}^{1} (1-t^2)^{\frac{1}{2}}K(t,\xi;c)dt = \frac{1}{2\pi^2}\oint_{-1}^{1}\frac{(1-t^2)^{\frac{1}{2}}\log[(c+t)/c-t)]}{t-\xi}\,dt$$

$$-\frac{1}{2\pi^2}\log\frac{c+\xi}{c-\xi}\oint_{-1}^{1}\frac{(1-t^2)^{\frac{1}{2}}}{t-\xi}\,dt$$

$$= -\frac{\xi}{2\pi}\log\frac{c-\xi}{c+\xi} - \frac{\{c-(c^2-1)^{\frac{1}{2}}\}}{\pi} + \frac{(1-\xi^2)^{\frac{1}{2}}}{2\pi}\Omega(\xi;c) \quad ,$$

where Ω is given by (5.22).

We now substitute the expressions for $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$ given by (5.25) and (5.26) into the above equation and multiply through by D as given by Eq. (5.27). Again, keeping only linear terms, this finally gives

$$\{1 + (1-\xi^2)^{\frac{1}{2}}\}\Gamma_1(\xi) - \int_{-1}^{1}\overline{K}(t,\xi)\Gamma_1(t)dt$$

$$= \frac{2}{\pi}(c^2-1)^{\frac{1}{2}}\{c-(c^2-1)^{\frac{1}{2}}\} + (1-\xi^2)^{\frac{1}{2}}\left[\frac{c}{\pi}\Omega(\xi;c)-1\right] \tag{5.28}$$

with

$$\begin{aligned}\overline{K}(t,\xi) &= 2\,cK(t,\xi)\\ &+ \frac{1}{\pi}\left\{1 - \frac{2c}{\pi}(c^2-1)^{\frac{1}{2}} + (1-\xi^2)^{\frac{1}{2}}\left[\frac{c}{\pi}\,\Omega(\xi;c) - 1\right]\right\}\\ &+ \frac{2c}{\pi}(c^2-1)^{\frac{1}{2}}\left[\frac{c^2+1}{\pi} - \frac{1}{2}(1-\xi^2)^{\frac{1}{2}}\right]\frac{1}{c^2-t^2}\\ &- \frac{2}{\pi}(1-t^2)^{\frac{1}{2}}\left[\frac{c}{\pi}\,\Omega(t;c) - 1\right] .\end{aligned}$$

Equation (5.28) was solved numerically for various value of c. These results are plotted in Fig. 10. Since $\Gamma_1(\pm 1) \neq 0$, $\beta_1(\xi)$ is logarithmically singular as $\xi \to \pm 1$ so that linearization is dubious at least near the endpoints. This situation is partially remedied by approximating Γ_1 by a function which does vanish at the endpoints (for instance, by taking the first few Fourier sine components of Γ_1); however, this gives values of C_D which, for a given value of k, are consistently greater than those given by the minimum drag profiles of Ch. III, §2. We conclude from this that the linearization of the nonlinear equations of the variational problem is not justified.

VI. DISCUSSION AND CONCLUSION

We have studied the problem of minimizing the drag D^* of a symmetric plate in infinite cavity flow, under the constraints of fixed arclength s_o and fixed chord y_o, by two methods - - the parameter method and the variational technique.

Both approaches depend on the fact that D^*, s_o, and y_o, can be expressed as integral functionals involving two functions, $\Gamma(\xi)$ (essentially the logarithm of the velocity) and $\beta(\xi)$ (the flow angle), and two parameters, A (a magnification factor) and c (or $\kappa = c-(c^2-1)^{\frac{1}{2}}$ (a free-streamline parameter). It was shown that Γ and β have the form

$$\Gamma(\xi) = \frac{2\alpha}{\pi} \log\left(\frac{1+(1-\xi^2)^{\frac{1}{2}}}{|\xi|}\right) + \Gamma_1(\xi)$$

$$\beta(\xi) = \alpha \operatorname{sgn}\xi + \beta_1(\xi) \quad ,$$

where 2α is the nose angle of the plate and

$$\beta_1(\xi) = -\frac{1}{\pi} \oint_{-1}^{1} \frac{\Gamma_1(t)}{t-\xi}\, dt \quad .$$

In the variational problem, we minimize the functional $I = D^* - \lambda_1 s_o - \lambda_2 y_o$, where λ_1 and λ_2 are unknown Lagrange multipliers. This corresponds to solving $\partial I/\partial A = 0$, $\partial I/\partial c = 0$, and a nonlinear, singular integral equation (5.4). This system of equations is subsequently factored and the integral equation is reduced to a non-singular integral equation (5.12) the solution of which requires that $\alpha = \pi/2$.

Since no solution of this integral equation has been found we can only speculate as to the outcome of the variational method. To sound a pessimistic note, it may turn out that the solution fails to have direct physical application; it might, for instance, fail to satisfy the pressure condition $\Gamma(\xi) \geqslant 0$ or it could be that $\Gamma_1(\pm 1) \neq 0$ in which case β_1 would become logarithmically infinite as $\xi \rightarrow \pm 1$ and the flow would be many sheeted.

Nonetheless, such a solution (if it actually does minimize I) would provide an absolute lower bound for the drag which could then be used as a basis for comparison with other minimum drag profiles which are physically relevant (such as those found by the parameter method). Final judgment of the usefulness of the variational technique as a design tool should await a more thorough investigation (most likely numerical) of the equations of Ch. V, §3.

In the parameter method, expressions for D^*, s_o, and y_o, were found in terms of (N+2) parameters - - A, κ, a_1, a_2, . . . , a_N - - by setting $\alpha = \frac{\pi}{2}$ and expanding Γ_1 and β_1 in finite Fourier series with N terms. For a minimum I, we solve the (N+2) equations $\partial I/\partial A = 0$, $\partial I/\partial \kappa = 0$, and $\partial I/\partial a_n = 0$, $n = 1, 2, \ldots, N$. These equations plus the equation $s_o/y_o = k$ (k, a given number greater than unity) are all that are needed to solve for the (N+3) unknowns - - $\lambda_1, \lambda_2, \kappa, a_1, \ldots, a_N$ - - since A drops out of these equations. Finally, the drag coefficient is given by $C_D = D^*/y_o$.

This procedure was carried out for the case N = 1 and found to give reasonable results, although it is difficult to tell just how good they are. To do this, one should solve the variational problem (which essentially corresponds to $N = \infty$) or do the parameter method for $N = 2, 3$, etc., as was done in the two examples considered in Ch. IV, §1.

The parameter problem for $N > 1$ would most likely require use of the computer, although it may be possible to find analytic solutions by series expansion for κ near unity.

Aside from investigation of the variational equations and extension of the parameter method to $N > 1$, several areas for further study include the following: (1) An extension of the variational calculus method of Ch. IV to handle inequality constraints and constraints on the values of f (or g) at the endpoints, the case $f(\pm 1) = 0$ being particularly important; (2) Application of the methods of this paper to the minimum drag problem and the hydrofoil problem with the possible use of more complicated finite cavity flow models, and; (3) Application of the variational technique of Ch. IV to other fields. The linearity of the integral equations for the case of functionals of quadratic form makes this method particularly well suited for application to problems involving energy constraints.

REFERENCES

1. Gilbarg, D. (1960) Jets and Cavities. Handbuch der Physik, Vol. IX, p. 311 - 438. Berlin: Springer-Verlag.

2. Milne-Thomson, L. M. (1960), Theoretical Hydrodynamics, 4th edition. Macmillan Company.

3. Wu, T. Y. (1962), A Wake Model for Free-Streamline Flow Theory. J. Fluid Mech. 13, p. 161 - 181.

4. Muskhelishvili, N. I. (1946), Singular Integral Equations. Groningen, Holland: P. Noordhoff Ltd.

5. Tricomi, F. G. (1957), Integral Equations. Interscience Publishers Inc.

6. Gradshteyn, I. S. and Ryshik, I. M. (1965), Table of Integrals, Series, and Products. p. 316, formula 3.365. Academic Press.

7. Abramowitz, M. and Stegun, I.A. (1964), Handbook of Mathematical Functions. National Bureau of Standards.

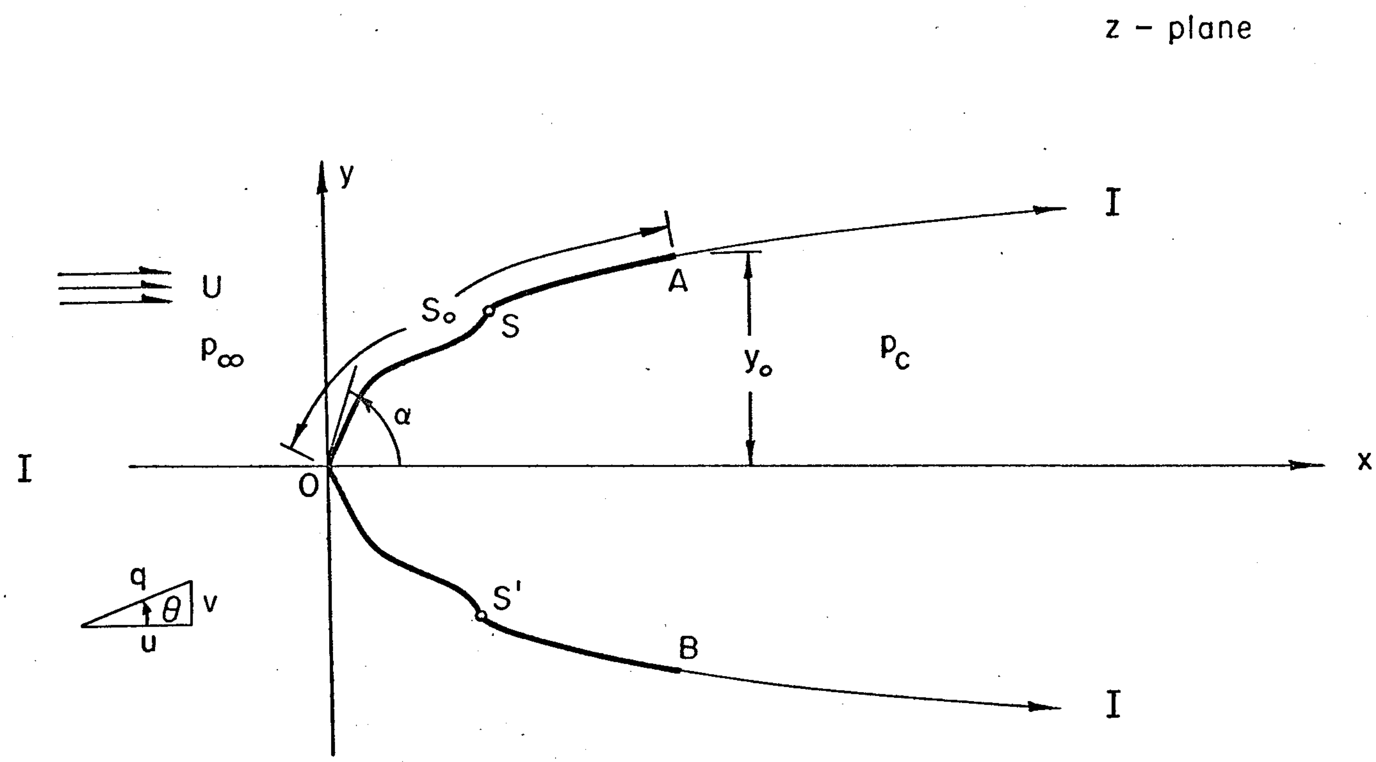


Fig. 1 - The Physical Plane

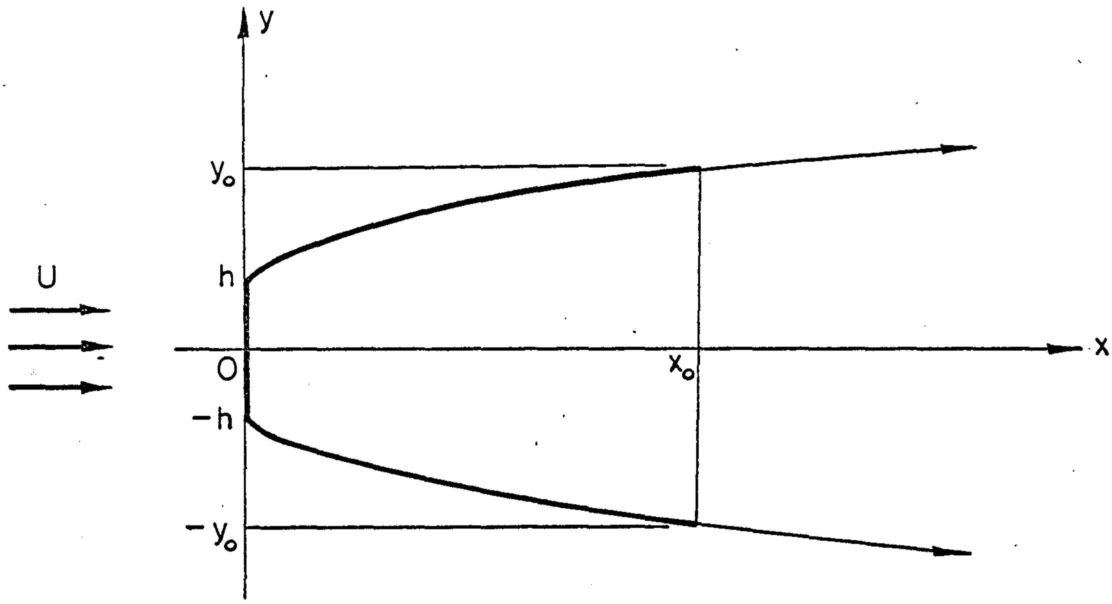


Fig. 2(a) - Lavrentieff's Solution

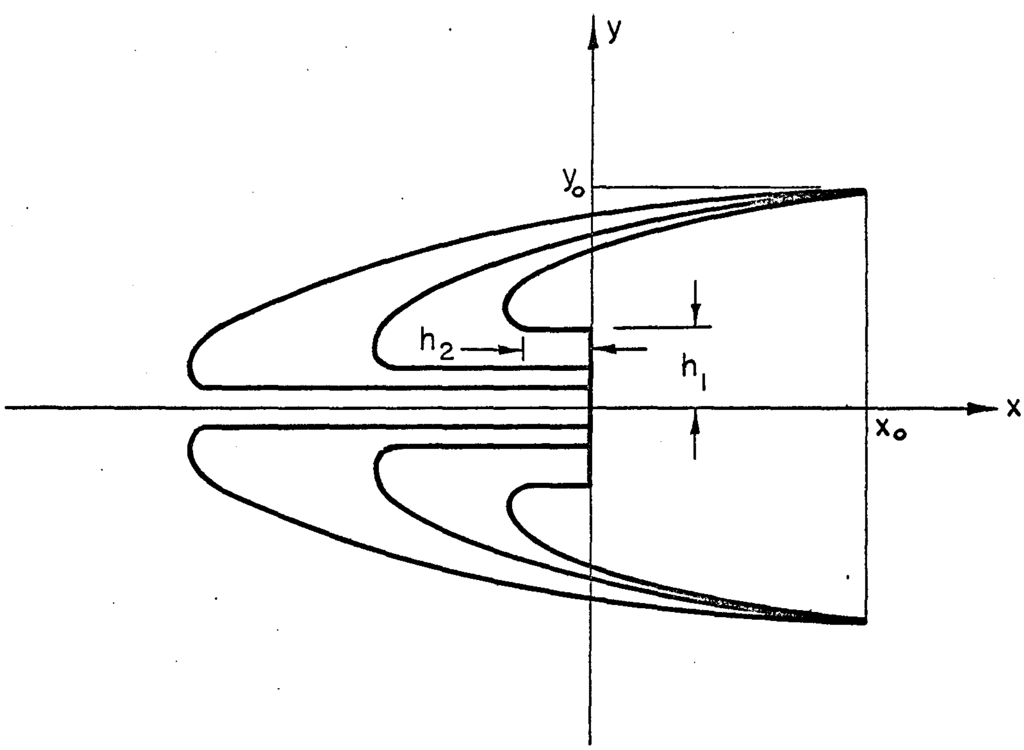


Fig. 2(b) - A Sequence of Plate Shapes with Zero Drag in the Limit as $h_1 \to 0$.

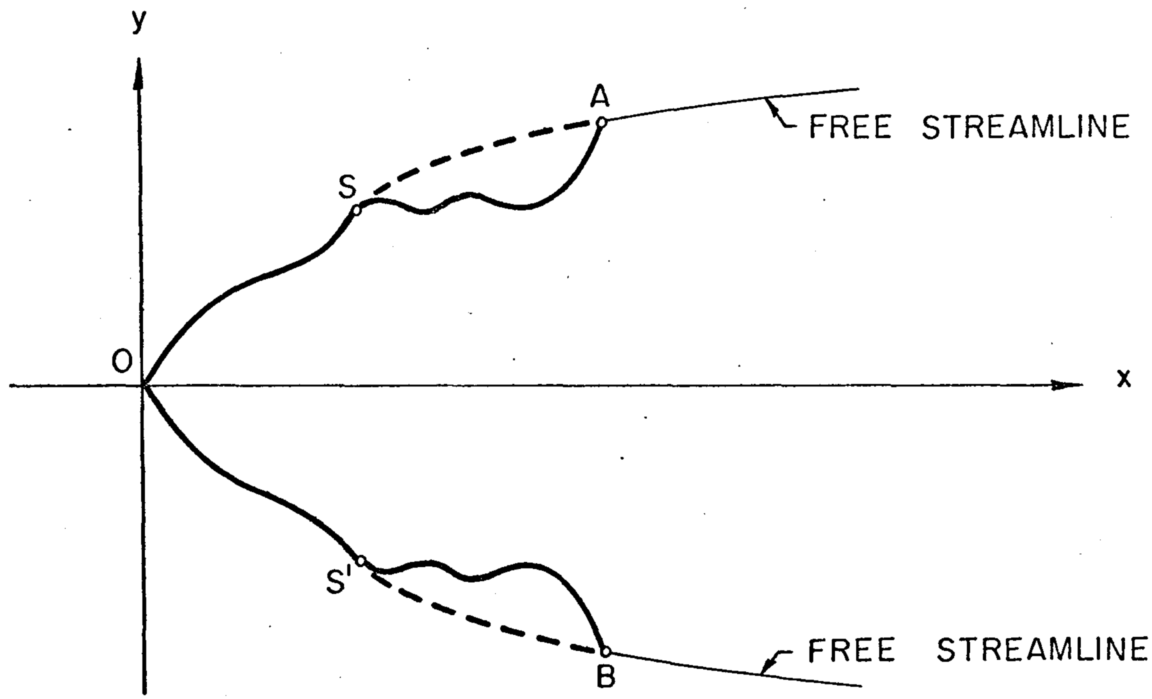


Fig. 3 - Example of Changing the Plate Shape without Changing the Flow.

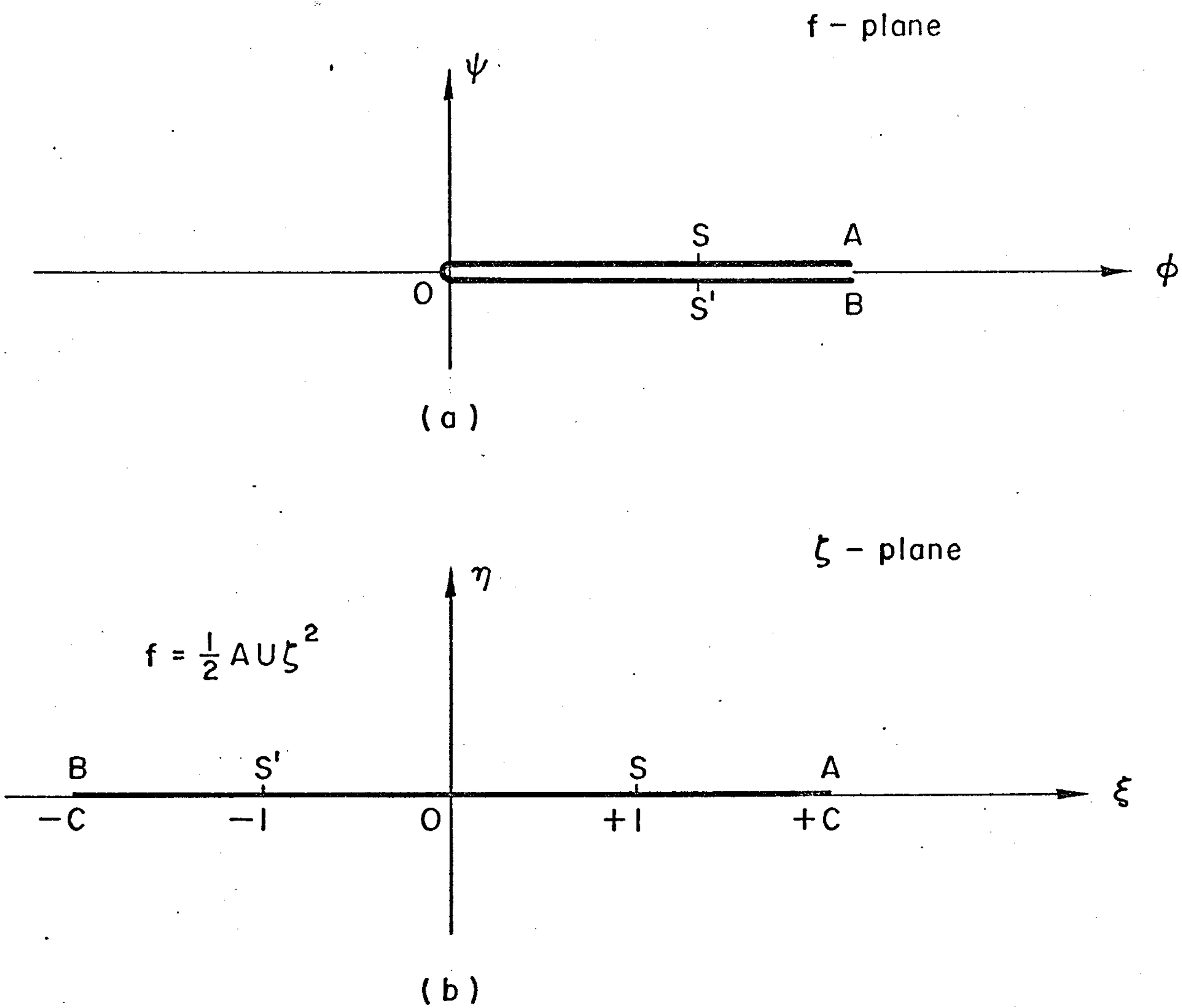


Fig. 4 - Transformation from Complex Potential Plane to ζ-plane.

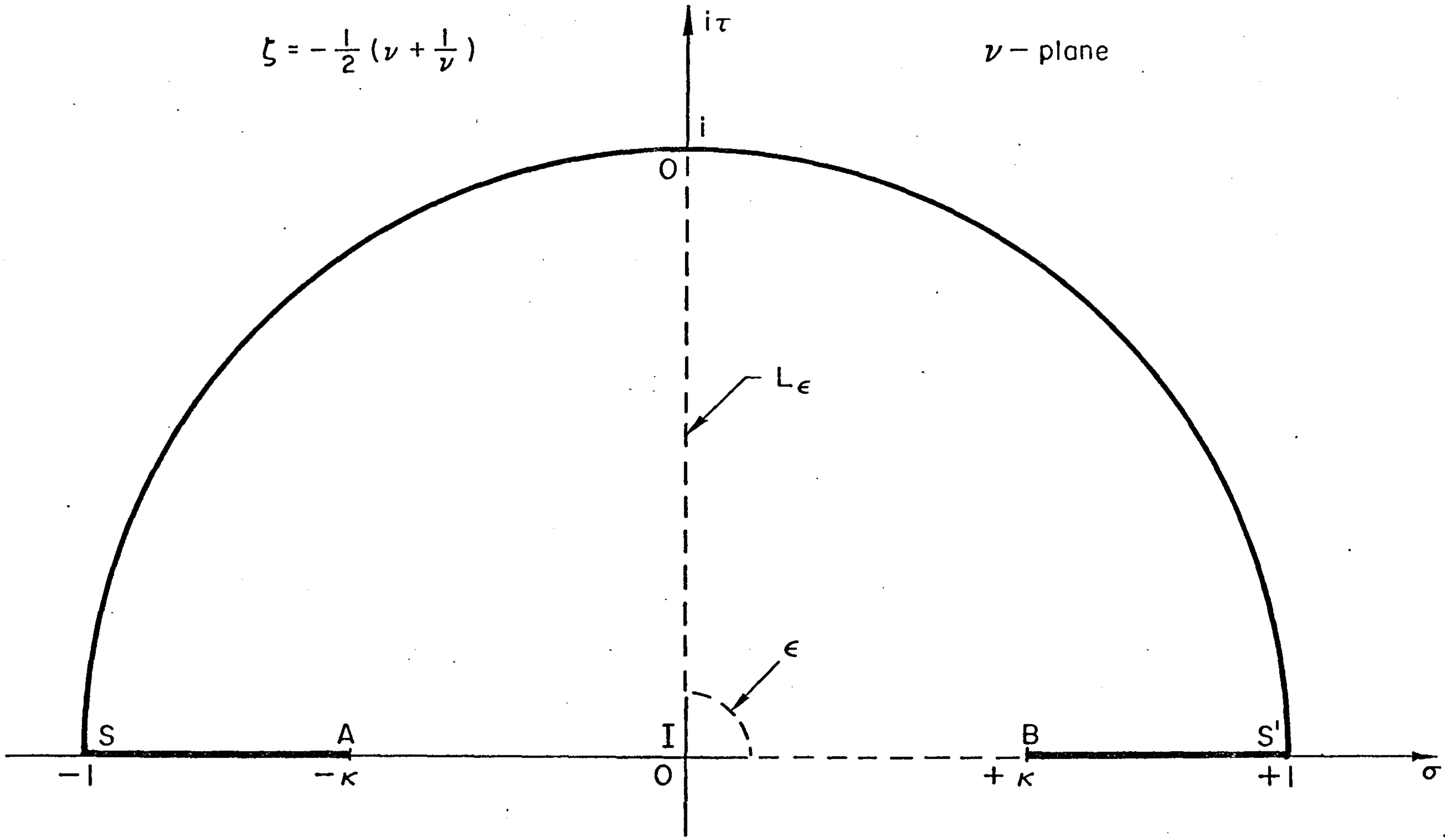


Fig. 5 - The Path of Integration in the ν-plane.

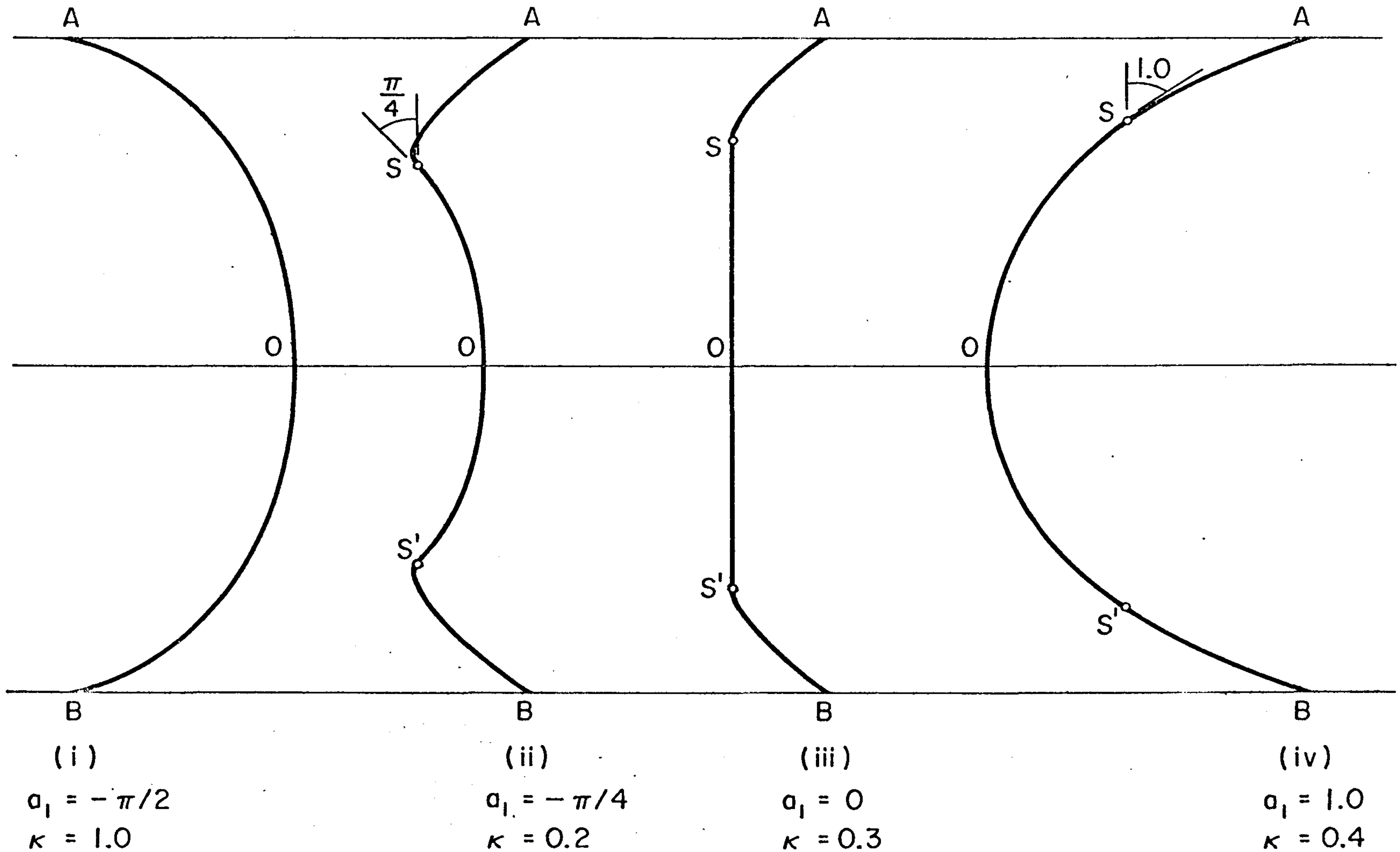


Fig. 6 - Various Plate Shapes for the Case N = 1.

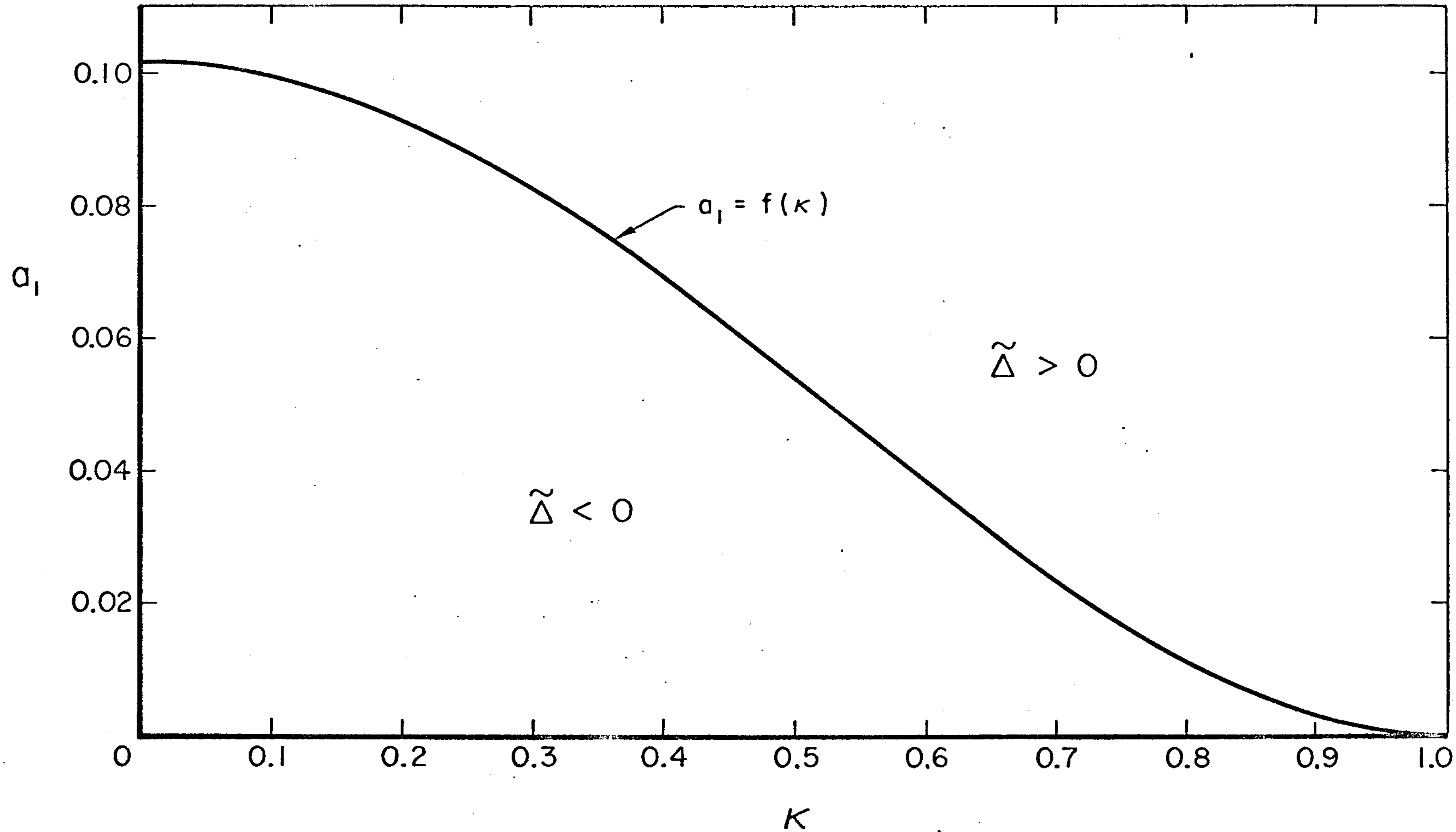
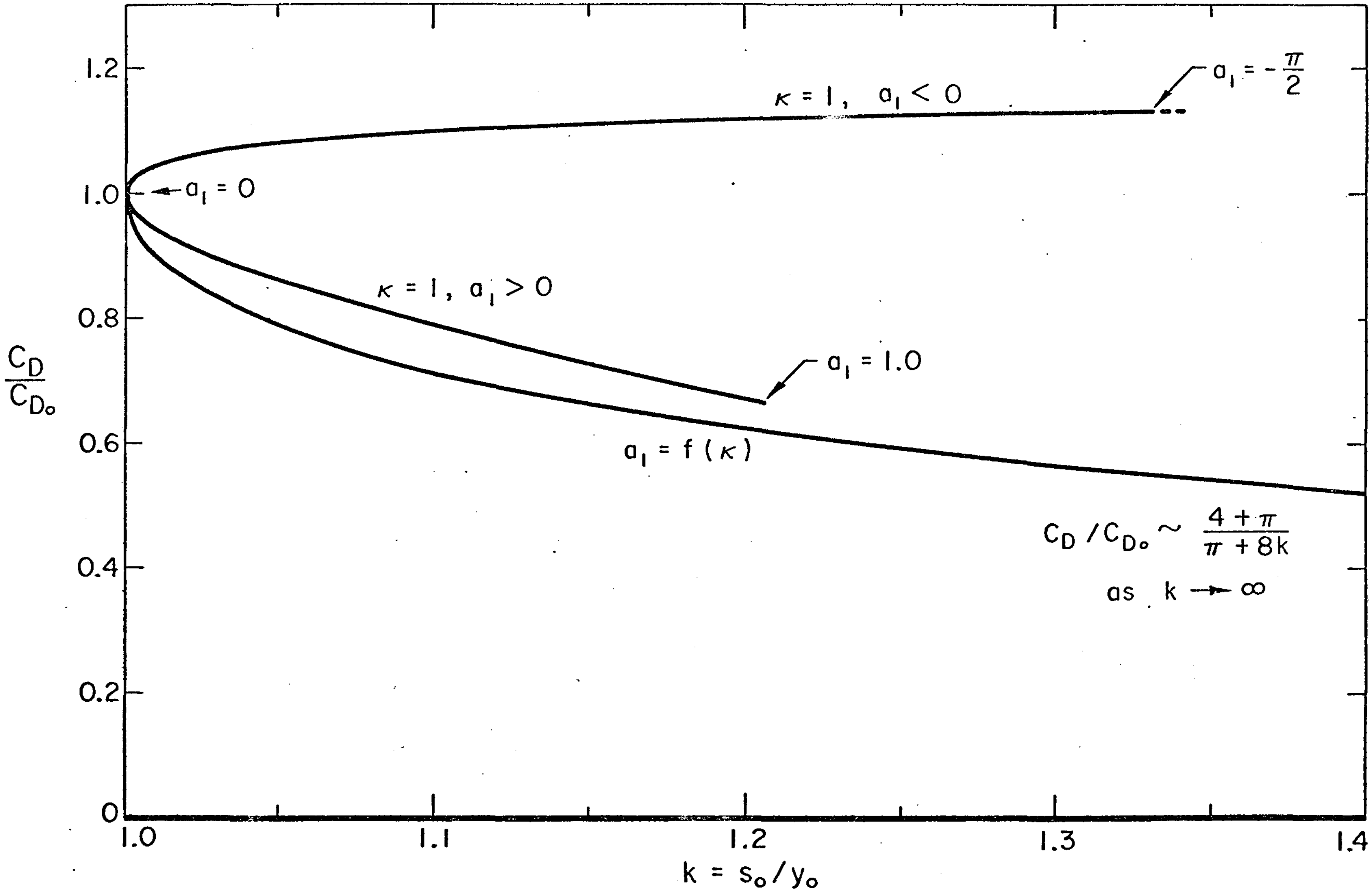


Fig. 7 - The Curve $a_1 = f(\kappa)$ on which $\tilde{\Delta}(a_1, \kappa) = 0$.



Fig, 8 - C_D vs k Plots for the Two Cases for which $\Delta = 0$.
$C_{Do} = 2\pi/(4+\pi)$ is the Drag Coefficient of a Flat Plate.

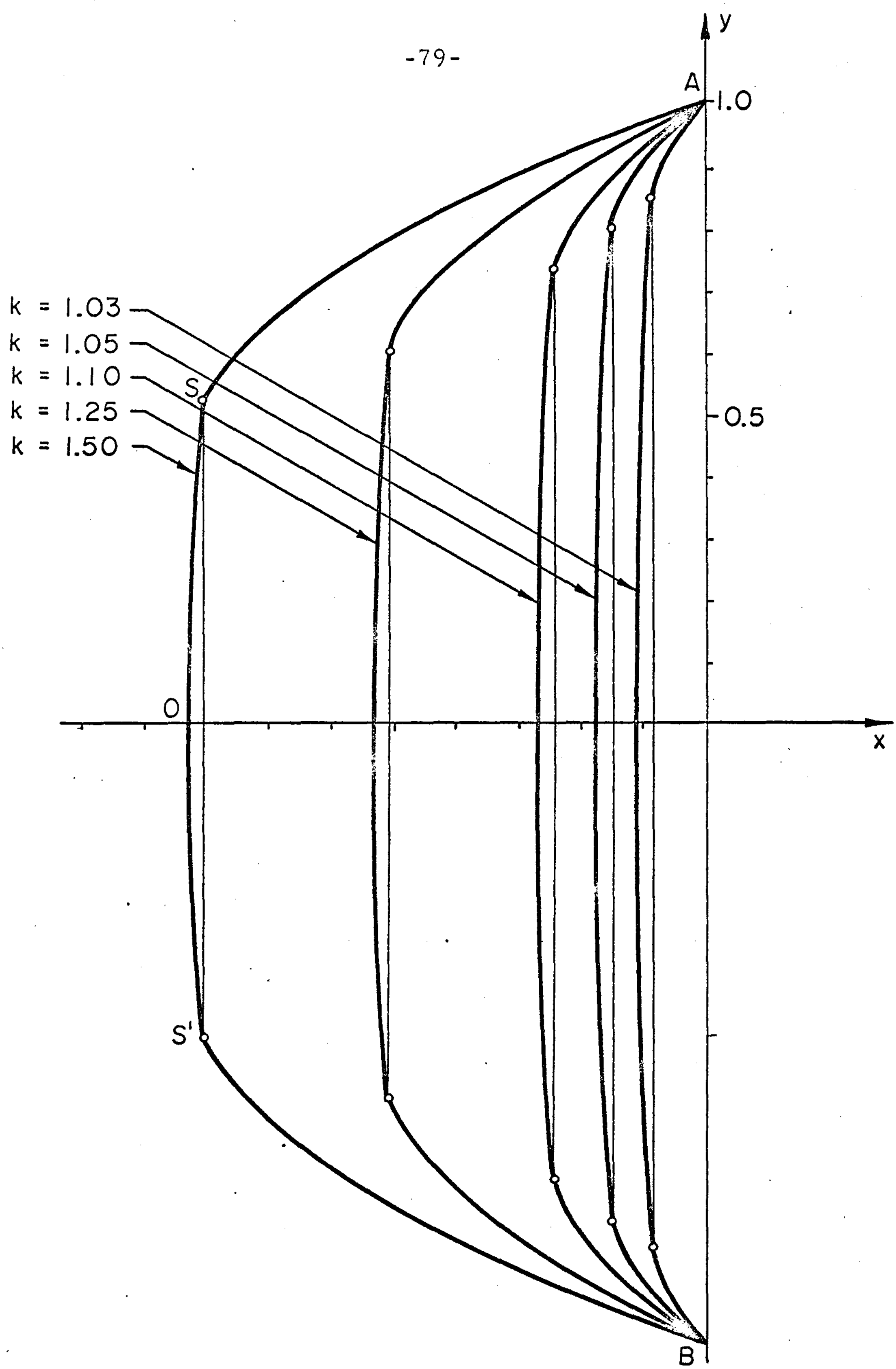


Fig. 9 - Optimum Plate Profiles for the Case N = 1.

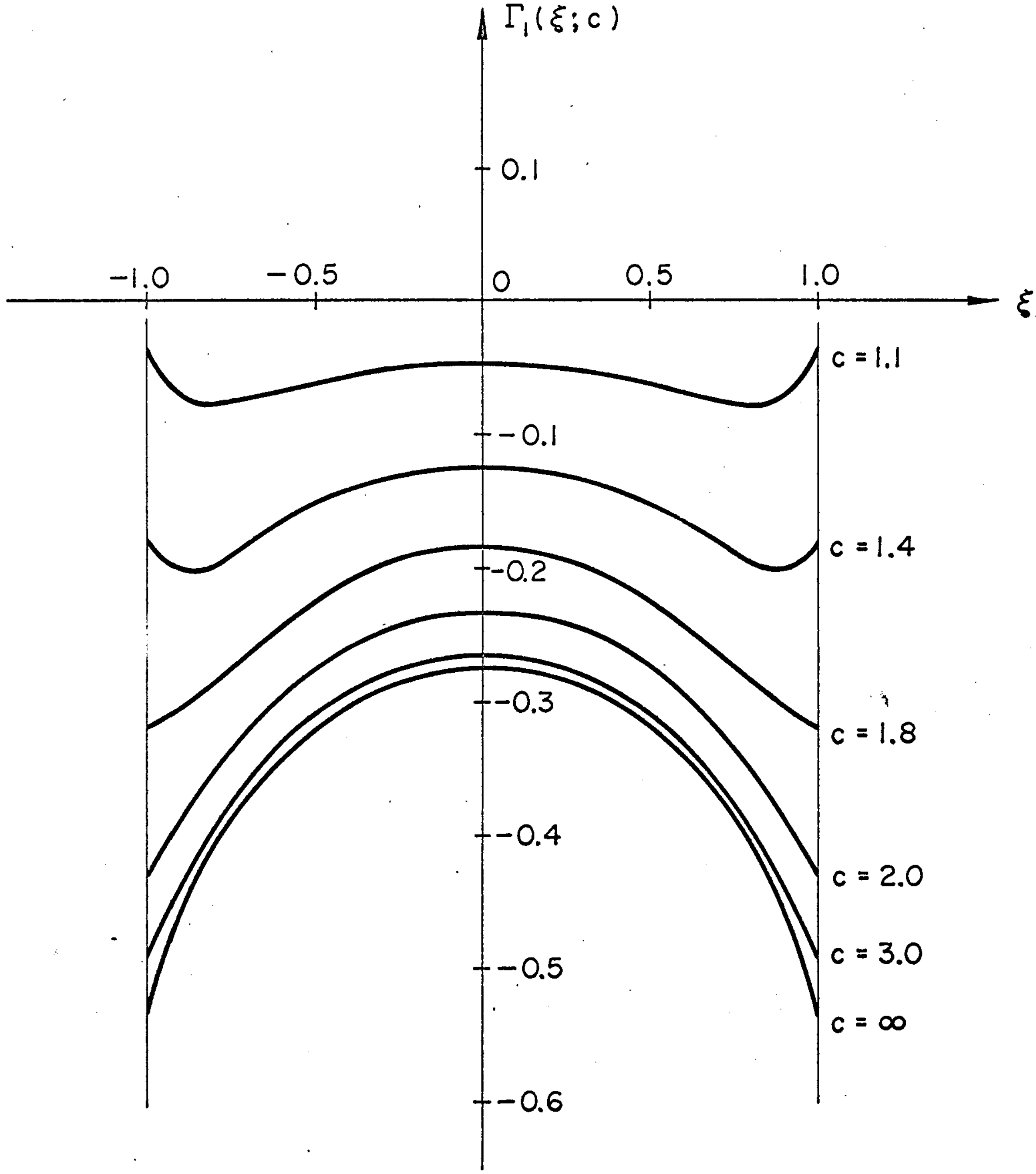


Fig. 10 - Linearized Solution.

APPENDIX A

CALCULATION OF PLATE LENGTH AND CHORD

From Eq. (3.8),

$$x_o - iy_o = -\frac{1}{4} A \int_i^{\kappa} e^{i\Omega(\nu)}\left(\nu + 2i - \frac{2}{\nu} - \frac{2i}{\nu^2} + \frac{1}{\nu^3}\right) d\nu$$

$$= -\frac{1}{4} A(I_1 + 2iI_2 - 2I_3 - 2iI_4 + I_5) \quad , \qquad (A.1)$$

where $\Omega(\nu) = \sum_{n=1}^{N} a_n \nu^{2n-1}$. This expression is evaluated by taking the path of integration along L_ϵ defined as follows: The imaginary axis, from $\nu = i$ to $\nu = i\epsilon$; the arc $|\nu| = \epsilon$, from $\nu = i\epsilon$ to $\nu = \epsilon$; and the real axis, from $\nu = \epsilon$ to $\nu = \kappa$ (see Fig. 5). We then let the radius of the circular arc go to zero. Integrals I_4 and I_5 are integrated by parts before integrating along L_ϵ.

The following notation for $\Omega(\nu)$ on $\nu = i\tau$ is used:

$$-i\Omega(i\tau) \equiv \psi(\tau) = \sum_{n=1}^{N} (-1)^{n+1} a_n \tau^{2n-1} \quad .$$

Integrals I_1 and I_2 can be evaluated directly on L_o (i.e., $\epsilon = 0$).

$$I_1 = \int_i^{\kappa} e^{i\Omega(\nu)} \nu d\nu = -\int_1^{o} e^{-\psi(\tau)} \tau d\tau + \int_o^{\kappa} e^{i\Omega(\sigma)} \sigma d\sigma$$

$$= \left(\int_o^{1} e^{-\psi(\tau)} t\, dt + \int_o^{\kappa} \cos\Omega(t) t\, dt\right) + i \int_o^{\kappa} \sin\Omega(t) t\, dt \qquad (A.2)$$

$$I_2 = \int_i^{\kappa} e^{i\Omega(\nu)} d\nu = \int_1^{o} e^{-\psi(\tau)} i\, d\tau + \int_o^{\kappa} e^{i\Omega(\sigma)} d\sigma$$

$$= \int_o^{\kappa} \cos\Omega(t)dt + i\left(\int_o^{\kappa} \sin\Omega(t)dt - \int_o^{1} e^{-\psi(t)}dt\right) \quad . \qquad (A.3)$$

I_3 is evaluated on L_ϵ:

$$I_3 = \int_i^{\kappa} e^{i\Omega(\nu)} \frac{d\nu}{\nu} = \int_1^{\epsilon} e^{-\psi(\tau)} \frac{d\tau}{\tau} + \int_{\frac{\pi}{2}}^{o} e^{i\Omega(\epsilon e^{i\theta})} i\, d\theta$$

$$+ \int_\epsilon^{\kappa} e^{i\Omega(\sigma)} \frac{d\sigma}{\sigma} \quad .$$

In the limit $\epsilon \rightarrow 0$,

$$I_3 = \left(\overset{*}{\int_o^{\kappa}} \cos\Omega(t)\frac{dt}{t} - \overset{*}{\int_o^{1}} e^{-\psi(t)} \frac{dt}{t}\right)$$

$$+ i\left(\int_o^{\kappa} \sin\Omega(t)\frac{dt}{t} - \frac{\pi}{2}\right) \quad . \qquad (A.4)$$

The asterisks on the integral signs mean that the singular parts of the integrands should be combined; i.e.,

$$\overset{*}{\int_o^{\kappa}} \cos\Omega(t)\frac{dt}{t} - \overset{*}{\int_o^{1}} e^{-\psi(t)} \frac{dt}{t} \equiv \int_o^{\kappa}\left\{\frac{\cos\Omega(t) - e^{-\psi(t)}}{t}\right\}dt$$

$$+ \int_\kappa^{1} e^{-\psi(t)} \frac{dt}{t} \quad .$$

The integral I_4 is first integrated by parts and then evaluated on L_ϵ.

$$I_4 = \int_i^{\kappa} e^{i\Omega(\nu)} \frac{d\nu}{\nu} = - \frac{e^{-i\Omega(\nu)}}{\nu}\bigg|_i^{\kappa} + \int_i^{\kappa} \frac{e^{i\Omega(\nu)} i\Omega'(\nu)}{\nu} d\nu$$

$$= - \frac{e^{-i\Omega(\kappa)}}{\kappa} - ie^{-\psi(1)} + i \int_1^{\epsilon} e^{-\psi(\tau)} \psi'(\tau) \frac{d\tau}{\tau}$$

$$+ \int_{\pi/2}^{0} e^{i\Omega(\epsilon e^{i\theta})} i\Omega'(\epsilon e^{i\theta}) i d\theta + i \int_{\epsilon}^{\kappa} e^{i\Omega(\sigma)} \Omega'(\sigma) \frac{d\sigma}{\sigma} .$$

In the limit $\epsilon \to 0$,

$$I_4 = \left(- \frac{\cos \Omega(\kappa)}{\kappa} + a_1 \frac{\pi}{2} - \int_0^{\kappa} \Omega'(t) \sin \Omega(t) \frac{dt}{t} \right)$$

$$+ i \left(- \frac{\sin \Omega(\kappa)}{\kappa} - e^{-\psi(1)} - {}^{*}\!\!\int_0^{1} e^{-\psi(t)} \psi'(t) \frac{dt}{t} \right.$$

$$\left. + {}^{*}\!\!\int_0^{\kappa} \Omega'(t) \cos \Omega(t) \frac{dt}{t} \right) . \qquad \text{(A.5)}$$

The integral I_5 must first be integrated by parts twice:

$$I_5 = \int_i^{\kappa} e^{i\Omega(\nu)} \frac{d\nu}{\nu^3} = - \frac{1}{2} \frac{e^{i\Omega(\nu)}}{\nu^2}\bigg|_i^{\kappa} + \frac{1}{2} \int_i^{\kappa} e^{i\Omega(\nu)} \frac{i\Omega'(\nu)}{\nu^2} d\nu$$

$$= \left(\frac{1}{2} \frac{e^{i\Omega(\nu)}}{\nu^2} + \frac{1}{2} e^{i\Omega(\nu)} \frac{i\Omega'(\nu)}{\nu} \right)\bigg|_i^{\kappa} + I_{51} + I_{52} ,$$

where

$$I_{51} = \frac{1}{2} \int_i^{\kappa} e^{i\Omega(\nu)} [i\Omega'(\nu)]^2 \frac{d\nu}{\nu}$$

$$I_{52} = \frac{1}{2} \int_i^{\kappa} e^{i\Omega(\nu)} i\Omega''(\nu) \frac{d\nu}{\nu} .$$

Now,

$$I_{51} = -\frac{1}{2}\int_1^{\epsilon} e^{-\psi(\tau)}[\psi'(\tau)]^2 \frac{d\tau}{\tau}$$

$$+\frac{1}{2}\int_{\pi/2}^{0} e^{i\Omega(\epsilon e^{i\theta})}[i\Omega'(\epsilon e^{i\theta})]^2 \, id\theta$$

$$-\frac{1}{2}\int_{\epsilon}^{\kappa} e^{i\Omega(\sigma)}[\Omega'(\sigma)]^2 \frac{d\sigma}{\sigma} \quad .$$

Letting $\epsilon \rightarrow 0$, gives

$$I_{51} = \frac{1}{2}\overset{*}{\int_0^1} e^{-\psi(t)}[\psi'(t)]^2 \frac{dt}{t} - \frac{1}{2}\overset{*}{\int_0^{\kappa}} [\Omega'(t)]^2 \cos\Omega(t) \frac{dt}{t}$$

$$+ i\frac{\pi}{4} a_1^2 - \frac{i}{2}\int_0^{\kappa} [\Omega'(t)]^2 \sin\Omega(t) \frac{dt}{t} \quad .$$

I_{52} can be evaluated directly on L_o since the integrand is regular at $\nu = 0$.

$$I_{52} = \int_1^{o} e^{-\psi(t)}\psi''(t) \frac{dt}{t} + i\int_o^{\kappa} e^{i\Omega(\sigma)}\Omega''(\sigma) \frac{d\sigma}{\sigma}$$

$$= -\int_o^1 e^{-\psi(t)}\psi''(t) \frac{dt}{t} - \int_o^{\kappa} \Omega''(t)\sin\Omega(t) \frac{dt}{t}$$

$$+ i\int_o^{\kappa} \Omega''(t)\cos\Omega(t) \frac{dt}{t} \quad .$$

Combining these results gives

$$I_5 = -\frac{1}{2}e^{-\psi(1)} - \frac{1}{2\kappa^2}\cos\Omega(\kappa) + \frac{1}{2}\psi'(1)e^{-\psi(1)}$$

$$+ \frac{1}{2\kappa}\Omega'(\kappa)\sin\Omega(\kappa) + \frac{1}{2}\,{}^{*}\!\!\int_0^1 e^{-\psi(t)}[\psi'(t)]^2\,\frac{dt}{t}$$

$$- \frac{1}{2}\,{}^{*}\!\!\int_0^{\kappa}[\Omega'(t)]^2\cos\Omega(t)\,\frac{dt}{t} - \frac{1}{2}\int_0^1 e^{-\psi(t)}\psi''(t)\,\frac{dt}{t}$$

$$- \frac{1}{2}\int_0^{\kappa}\Omega''(t)\sin\Omega(t)\,\frac{dt}{t} + i\left(-\frac{1}{2\kappa^2}\sin\Omega(\kappa)\right. \tag{A.6}$$

$$- \frac{1}{2\kappa}\Omega'(\kappa)\cos\Omega(\kappa) + a_1^2\frac{\pi}{4} - \frac{1}{2}\int_0^{\kappa}[\Omega'(t)]^2\sin\Omega(t)\,\frac{dt}{t}$$

$$\left. + \frac{1}{2}\int_0^{\kappa}\Omega''(t)\cos\Omega(t)\,\frac{dt}{t}\right) .$$

Finally, substituting Eqs. (A.2) - (A.6) in (A.1) and separating real and imaginary parts; we have

$$x_o = \frac{1}{4}A\left\{\frac{1}{2}\,{}^{*}\!\!\int_0^1 e^{-\psi(t)}\left[\frac{\psi''(t)}{t} - 4 - 2t - \frac{(2-\psi'(t))^2}{t}\right]dt\right.$$

$$+ \frac{1}{2}\,{}^{*}\!\!\int_0^{\kappa}\cos\Omega(t)\left[\frac{(2-\Omega'(t))^2}{t} - 2t\right]dt$$

$$+ \int_0^{\kappa}\sin\Omega(t)\left[2 + \frac{\Omega''(t))^2}{2t}\right]dt$$

$$+\left(\frac{2}{\kappa} - \frac{\Omega'(\kappa)}{2\kappa}\right)\sin\Omega(\kappa) + \frac{1}{2\kappa^2}\cos\Omega(\kappa)$$

$$\left. +\left(\frac{5}{2} - \frac{1}{2}\psi'(1)\right)e^{-\psi(1)}\right\} .$$

$$y_o = \frac{1}{4} A\left\{\frac{1}{2}\int_o^{\kappa} \sin\Omega(t)\left[2t - \frac{(2-\Omega'(t))^2}{t}\right]dt\right.$$

$$+ \int_o^{\kappa} \cos\Omega(t)\left[2 + \frac{\Omega''(t)}{2t}\right]dt + \frac{\pi}{4}(2-a_1)^2$$

$$\left. + \left(\frac{2}{\kappa} - \frac{\Omega'(\kappa)}{2\kappa}\right)\cos\Omega(\kappa) - \frac{1}{2\kappa^2}\sin\Omega(\kappa)\right\} \quad .$$

APPENDIX B

THE SOLUTION OF $\widetilde{\Delta}(a_1, \kappa) = 0$ FOR $\kappa \sim 1$

Let $a_1 = f(\kappa)$ be the curve on which $\widetilde{\Delta}(a_1, \kappa) = 0$, where $\widetilde{\Delta}$ is given by (3.15) and (3.16); that is,

$$\widetilde{\Delta}(f(\kappa), \kappa) = 0 \quad .$$

Differentiating this expression with respect to κ gives

$$\widetilde{\Delta}_{a_1} \frac{df}{d\kappa} + \widetilde{\Delta}_\kappa = 0 \quad ,$$

where the subscripts denote partial differentiation. Thus,

$$\frac{df(\kappa)}{d\kappa} = -\frac{\widetilde{\Delta}_\kappa(f(\kappa), \kappa)}{\widetilde{\Delta}_{a_1}(f(\kappa), \kappa)} \quad . \tag{B.1}$$

We now propose to solve this differential equation for κ near 1 by using the Taylor series expansion

$$f(\kappa) = f(1) + (\kappa - 1) \frac{df(1)}{d\kappa} + \frac{1}{2} (\kappa - 1)^2 \frac{d^2 f(1)}{d\kappa^2}$$

$$+ \frac{1}{6} (\kappa - 1)^3 \frac{d^3 f(1)}{d\kappa^3} + O(\kappa - 1)^4 \quad . \tag{B.2}$$

We first show that $f(1) = 0$; i.e., $\widetilde{\Delta}(0, 1) = 0$. From (3.16), the elements of the determinant $\widetilde{\Delta}$ are given by

$$\begin{aligned} t_{11} &= 2 - a_1 \\ t_{21} &= -2 \\ t_{31} &= 0 \end{aligned} \tag{B.3}$$

$$t_{12} = \left(\kappa^2 + \frac{1}{\kappa^2} + 6 + 2\pi\right) - \left(\frac{16}{3} + 2\pi\right) a_1 + \left(\frac{8}{3} + \frac{3\pi}{4}\right) a_1^2 + O(a_1^2)$$

$$t_{22} = -\left(\frac{16}{3} + 2\pi\right) + \left(\frac{16}{3} + \frac{3\pi}{2}\right) a_1 - \left(\frac{32}{15} + \frac{3\pi}{4}\right) a_1^2 + O(a_1^3)$$

$$t_{32} = \kappa + \frac{1}{\kappa}$$

$$t_{13} = \left(2\pi + 4\kappa + \frac{4}{\kappa}\right) + \left(\frac{2\kappa^3}{3} - \frac{2}{\kappa} - 4\kappa - 2\pi\right) a_1$$
$$+ \left(2\kappa - \frac{2\kappa^3}{3} + \frac{\pi}{2}\right) a_1^2 + O(a_1^3)$$

$$t_{23} = \left(\frac{2\kappa^3}{3} - \frac{2}{\kappa} - 4\kappa - 2\pi\right) + \left(4\kappa - \frac{4\kappa^3}{3} + \pi\right) a_1$$
$$- \left(\kappa - 2\kappa^3 + \frac{\kappa^5}{5}\right) a_1^2 + O(a_1^3).$$

$$t_{33} = 2 - (1 - \kappa^2) a_1 - \kappa^2 a_1^2 + O(a_1^3) \quad .$$

Thus,

$$\tilde{\Delta}(0,1) = \begin{vmatrix} 2 & 8 + 2\pi & 8 + 2\pi \\ -2 & -\left(\frac{16}{3} + 2\pi\right) & -\left(\frac{16}{3} + 2\pi\right) \\ 0 & 2 & 2 \end{vmatrix} = 0 \quad .$$

Using the well-known rule for the differentiation of determinants,

$$\tilde{\Delta}_{a_1} = \begin{vmatrix} \frac{\partial t_{11}}{\partial a_1} & t_{12} & t_{13} \\ \frac{\partial t_{21}}{\partial a_1} & t_{22} & t_{23} \\ 0 & t_{32} & t_{33} \end{vmatrix} + \begin{vmatrix} t_{11} & \frac{\partial t_{12}}{\partial a_1} & t_{13} \\ t_{21} & \frac{\partial t_{22}}{\partial a_1} & t_{23} \\ 0 & \frac{\partial t_{32}}{\partial a_1} & t_{33} \end{vmatrix} + \begin{vmatrix} t_{11} & t_{12} & \frac{\partial t_{13}}{\partial a_1} \\ t_{21} & t_{22} & \frac{\partial t_{23}}{\partial a_1} \\ 0 & t_{32} & \frac{\partial t_{33}}{\partial a_1} \end{vmatrix} .$$

These partial derivatives can be read directly from Eqs. (B.3). Setting $a_1 = 0$, $\kappa = 1$, we have

$$\tilde{\Delta}a_1(0,1) = \begin{vmatrix} -1 & 8+2\pi & 8+2\pi \\ 0 & -\left(\frac{16}{3}+2\pi\right) & -\left(\frac{16}{3}+2\pi\right) \\ 0 & 2 & 2 \end{vmatrix}$$

$$+ \begin{vmatrix} 2 & -\left(\frac{16}{3}+2\pi\right) & 8+2\pi \\ -2 & \frac{16}{3}+2\pi & -\left(\frac{16}{3}+2\pi\right) \\ 0 & 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 8+2\pi & -\left(\frac{16}{3}+2\pi\right) \\ -2 & -\left(\frac{16}{3}+2\pi\right) & \frac{8}{3}+\pi \\ 0 & 2 & 0 \end{vmatrix}$$

$$= 2\pi + \frac{32}{3} \quad .$$

The evaluation of $\tilde{\Delta}_\kappa$ is done similarly:

$$\tilde{\Delta}_\kappa = \begin{vmatrix} \frac{\partial t_{11}}{\partial\kappa} & t_{12} & t_{13} \\ \frac{\partial t_{21}}{\partial\kappa} & t_{22} & t_{23} \\ 0 & t_{32} & t_{33} \end{vmatrix} + \begin{vmatrix} t_{11} & \frac{\partial t_{12}}{\partial\kappa} & t_{13} \\ t_{21} & \frac{\partial t_{22}}{\partial\kappa} & t_{23} \\ 0 & \frac{\partial t_{32}}{\partial\kappa} & t_{33} \end{vmatrix}$$

$$+ \begin{vmatrix} t_{11} & t_{12} & \frac{\partial t_{13}}{\partial\kappa} \\ t_{21} & t_{22} & \frac{\partial t_{23}}{\partial\kappa} \\ 0 & t_{32} & \frac{\partial t_{33}}{\partial\kappa} \end{vmatrix} \quad . \tag{B.4}$$

Using (B.3), it is easily shown that at $a_1 = 0, \kappa = 1$,

$$\frac{\partial t_{ij}}{\partial\kappa} = 0 \tag{B.5}$$

for all i and j. Therefore,

$$\tilde{\Delta}_\kappa(0,1) = 0$$

and from (B.1),

$$\frac{df(1)}{d\kappa} = -\frac{\widetilde{\Delta}_{\kappa}(0,1)}{\widetilde{\Delta}_{a_1}(0,1)} = 0 \quad . \tag{B.6}$$

The second derivative of f is found from (B.1):

$$\frac{d^2 f}{d\kappa^2} = -\frac{\widetilde{\Delta}_{\kappa\kappa}}{\widetilde{\Delta}_{a_1}} - \frac{2\widetilde{\Delta}_{\kappa a_1}}{\widetilde{\Delta}_{a_1}} f'(\kappa) - \frac{\widetilde{\Delta}_{a_1 a_1}}{(\widetilde{\Delta}_{a_1})^2} (f'(\kappa))^2 \quad .$$

Since $f'(\kappa)$ vanishes at $\kappa = 1$,

$$\frac{d^2 f(1)}{d\kappa^2} = -\frac{\widetilde{\Delta}_{\kappa\kappa}(0,1)}{\widetilde{\Delta}_{a_1}(0,1)} \quad . \tag{B.7}$$

From Eqs. (B.4) and (B.5), we see that the only terms contributing to $\widetilde{\Delta}_{\kappa\kappa}(0,1)$ are those that involve the second derivatives of t_{ij} with with respect to κ; i.e.,

$$\widetilde{\Delta}_{\kappa\kappa}(0,1) = \begin{vmatrix} \frac{\partial^2 t_{11}}{\partial\kappa^2} & t_{12} & t_{13} \\ \frac{\partial^2 t_{21}}{\partial\kappa^2} & t_{22} & t_{23} \\ 0 & t_{32} & t_{33} \end{vmatrix}$$

$$+ \begin{vmatrix} t_{11} & \frac{\partial^2 t_{12}}{\partial\kappa^2} & t_{13} \\ t_{21} & \frac{\partial^2 t_{22}}{\partial\kappa^2} & t_{23} \\ 0 & \frac{\partial^2 t_{32}}{\partial\kappa^2} & t_{33} \end{vmatrix} + \begin{vmatrix} t_{11} & t_{12} & \frac{\partial^2 t_{13}}{\partial\kappa^2} \\ t_{21} & t_{22} & \frac{\partial^2 t_{23}}{\partial\kappa^2} \\ 0 & t_{32} & \frac{\partial^2 t_{33}}{\partial\kappa^2} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 8+2\pi & 8+2\pi \\ 0 & -\left(\frac{16}{3}+2\pi\right) & -\left(\frac{16}{3}+2\pi\right) \\ 0 & 2 & 2 \end{vmatrix}$$

$$+ \begin{vmatrix} 2 & 8 & 8+2\pi \\ -2 & 0 & -\left(\frac{16}{3}+2\pi\right) \\ 0 & 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 8+2\pi & 8 \\ -2 & -\left(\frac{16}{3}+2\pi\right) & 0 \\ 0 & 2 & 0 \end{vmatrix} = -\frac{32}{3} .$$

Therefore, from (B.7),

$$\frac{d^2 f(1)}{d\kappa^2} = -\frac{\left(-\frac{32}{3}\right)}{2\pi + \frac{32}{3}} = \frac{16}{16+3\pi} . \qquad \text{(B.8)}$$

Similar calculations for the third derivative of f give

$$\frac{d^3 f(1)}{d\kappa^3} = -\frac{\widetilde{\Delta}_{\kappa\kappa\kappa}(0,1)}{\widetilde{\Delta}_{a_1}(0,1)} - \frac{3\widetilde{\Delta}_{a_1\kappa}(0,1)}{\widetilde{\Delta}_{a_1}(0,1)} f''(1) ,$$

$$\widetilde{\Delta}_{\kappa\kappa\kappa}(0,1) = -32 \quad ,$$

$$\widetilde{\Delta}_{a_1\kappa}(0,1) = \frac{32}{3} \quad .$$

Thus,

$$\frac{d^3 f(1)}{d\kappa^3} = \frac{144\pi}{(16+3\pi)^2} \quad . \tag{B.9}$$

Finally, substituting (B.6), (B.8), and (B.9), into (B.2), we have

$$f(\kappa) = \frac{8}{16+3\pi}(1-\kappa)^2 - \frac{24\pi}{(16+3\pi)^2}(1-\kappa)^3 + O(1-\kappa)^4 \quad ,$$

which is in good agreement with the numerical solution of $\widetilde{\Delta}(a_1, \kappa) = 0$ for $0.7 < \kappa < 1$.