SOME THEOREMS IN CLASSICAL ELASTODYNAMICS

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Abstract

This investigation is concerned with various fundamental aspects of the linearized dynamical theory for mechanically homogeneous and isotropic elastic solids. First, the uniqueness and reciprocal theorems of dynamic elasticity are extended to unbounded domains with the aid of a generalized energy identity and a lemma on the prolonged quiescence of the far field, which are established for this purpose. Next, the basic singular solutions of elastodynamics are studied and used to generate systematically Love's integral identity for the displacement field, as well as an associated identity for the field of stress. These results, in conjunction with suitably defined Green's functions, are applied to the construction of integral representations for the solution of the first and second boundary-initial value problem. Finally, a uniqueness theorem for dynamic concentrated-load problems is obtained.

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Introduction

The linearized dynamical theory of elasticity has long been a highly developed and, in large measure, complete discipline. It is therefore not surprising that most of the recent publications in this area of interest are concerned with the exploration of exact or approximative methods for the solution of relevant problem-classes and with specific wave-propagation problems. The present investigation — although ultimately motivated by physically significant applications — can make no such immediate practical claims. Our main objective is to study certain general consequences of the equations governing classical elastodynamics with limitation to mechanically homogeneous and isotropic solids. Some of the results presented in what follows aim primarily at a clarification, strengthening, and extension of theorems previously available. In contrast, the work on Green's functions, integral representations, and concentrated loads in dynamic elasticity, would appear to fill a gap in the existing literature.

In Section 1 we dispose of required geometric and notational preliminaries. Here we also cite pertinent properties of Riemann convolutions and introduce the notion of an "elastodynamic state", which lends economy to subsequent developments.

In Section 2 we deduce a generalized energy identity and use the latter to extend the conventional uniqueness theorem of elastodynamics to unbounded domains in the absence of artificial restrictions upon the behavior of the velocities or stresses at infinity. Further, we employ the foregoing energy identity to establish sufficient conditions for the prolonged quiescence of the far elastodynamic field belonging to a solution that corresponds to initial quiescence. This result, in turn, supplies the principal tool for a generalization of Graffi's dynamic reciprocal theorem[1] to infinite regions, which concludes Section 2.

Section 3 is partly expository. Here we first cite Stokes' solution for a time-dependent concentrated load at a point of a medium occupying the entire space. We then examine relevant properties of this solution and of the singular solutions appropriate to force-doublets. This material is followed by a systematic derivation, based on the reciprocal theorem given in Section 2, of Love's integral identity for elastodynamic displacement fields. Finally, at the end of the section, we deduce an analogous integral identity for the associated stresses.

In Section 4 we take the integral identities of the preceding section as a point of departure in deriving integral representations for the solution to the first and second boundary-initial value problem of dynamic elasticity. This task is accomplished through the introduction of suitable Green's solutions of the first and second kind. We also include here some supplementary results on properties of

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[1] Detailed references to the literature can be found in the body of this investigation.

the Green's solutions with a view toward facilitating their actual construction.

Finally, Section 5 is devoted to a uniqueness theorem for the second boundary-initial value problem encompassing time-dependent concentrated loads with stationary points of application in the interior or on the surface of an elastic solid. This theorem is proved with the aid of the Green's solutions introduced in Section 4.

1. Notation and mathematical preliminaries.

Throughout this investigation, lower-case Latin or Greek letters, when not underscored, stand for scalars; lower-case Latin letters underscored by a tilde denote vectors, while lower-case Greek letters underscored by a tilde designate second-order tensors. Upper-case letters are ordinarily reserved for sets; in particular, upper-case script letters are used exclusively for sets of functions. The letter E is set aside for the entire three-dimensional euclidean space. If $\underset{\sim}{x}$ is the position vector of a point in E, the symbols $B_\delta(\underset{\sim}{x})$ and $S_\delta(\underset{\sim}{x})$ are employed, respectively, for the open spherical neighborhood (ball) of radius δ about $\underset{\sim}{x}$ and for the spherical surface of radius δ centered at $\underset{\sim}{x}$. Thus,

$$B_\delta(\underset{\sim}{x}) = \{\underset{\sim}{y} \mid \underset{\sim}{y} \in E,\ |\underset{\sim}{y} - \underset{\sim}{x}| < \delta\} \quad (\delta > 0)\,, \tag{1.1}$$

$$S_\delta(\underset{\sim}{x}) = \{\underset{\sim}{y} \mid \underset{\sim}{y} \in E,\ |\underset{\sim}{y} - \underset{\sim}{x}| = \delta\} \quad (\delta > 0)\,. \tag{1.2}$$

Further, we agree to write B_δ and S_δ in place of $B_\delta(\underset{\sim}{0})$ and $S_\delta(\underset{\sim}{0})$.

The symbol R, in the absence of any qualifying restrictions, will always denote an arbitrary region in E, i.e. an open connected set in E together with some, all, or none of its boundary points. The interior, the closure, and the boundary of R — in this order — will be designated by $\mathring{R}$, $\overline{R}$, and ∂R. Further, if $\underset{\sim}{x} \in R$, we agree to let $R_{\underset{\sim}{x}}$ represent the set obtained from R by deletion of the point $\underset{\sim}{x}$ and write $\overline{R}_{\underset{\sim}{x}}$ in place of $(\overline{R})_{\underset{\sim}{x}}$.

In particular, we say that R is a regular region if it is open and there is a $\delta_0 > 0$ such that for every $\delta > \delta_0$ the boundary of $R \cap B_\delta$

consists of a finite number of "closed regular surfaces", the latter term being used in the sense of Kellogg [1] (page 112). Note that a regular region, as defined here[1], need not be bounded and, if unbounded, need not be an exterior region since its boundary may extend to infinity. In addition, the boundary of a regular region may have edges and corners. If R is a regular region, we designate by $\overset{*}{\partial}R$ the subset of ∂R consisting of all "regular boundary points", i.e. the set of all points of ∂R at which its normal is defined.

We will use the symbol $\overset{\infty}{T}$ for the entire real line and T for an arbitrary (open, closed, or half-open) interval of $\overset{\infty}{T}$. The interior and closure of T will be designated by $\overset{\circ}{T}$ and $\overline{T}$, respectively. Finally, we adopt the notation

$$T^{-} = (-\infty, 0], \; T^{+} = [0, \infty) \; . \tag{1.3}$$

If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are vectors, $\underset{\sim}{a} \cdot \underset{\sim}{b}$ and $\underset{\sim}{a} \wedge \underset{\sim}{b}$ are their scalar and vector product, respectively. Standard indicial notation is used in connection with the cartesian components of tensors of any order: Latin subscripts and superscripts — unless otherwise specified — have the range (1, 2, 3), summation over repeated indices being implied. Also, if $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{\sigma}$ are second-order tensors, we write $\underset{\sim}{\varepsilon} \cdot \underset{\sim}{\sigma}$ for the fully contracted outer product $\varepsilon_{ij}\sigma_{ij}$. As usual, δ_{ij} is the Kronecker-delta.

We will frequently need to deal with scalar-valued and tensor-valued functions of position and time, having as their domain of

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[1] Our definition of a regular region differs from, and is considerably broader than, Kellogg's [1] (page 113).

definition the cartesian product of a set P in E and a time-interval T. If φ is a scalar-valued function defined on[1] $P \times T$, we denote its value at $(\underset{\sim}{x}, t) \in P \times T$ by $\varphi(\underset{\sim}{x}, t)$ and mean by $\varphi(\cdot, t)$, the subsidiary mapping of P obtained upon holding t fixed. The analogous interpretation applies to $\varphi(\underset{\sim}{x}, \cdot)$ and to tensor-valued functions. As for time and space-differentiation, we write

$$\varphi^{(m)} = \frac{\partial^m \varphi}{\partial t^m}, \quad \underbrace{\varphi_{,ij\ldots\ldots k}}_{(m \text{ indices})} = \frac{\partial^m \varphi}{\partial x_i \partial x_j \ldots \partial x_k} \quad (m=1, 2, \ldots), \tag{1.4}$$

provided the partial derivatives here involved exist. Ordinarily we shall write $\dot{\varphi}$, $\ddot{\varphi}$ instead of $\varphi^{(1)}$, $\varphi^{(2)}$. Analogous notations will be employed for differentiation of tensor-valued functions of non-zero order.

We turn now to notational agreements related to the smoothness of functions. If P is a set in euclidean n-space, we denote by $C(P)$ the class of all tensor-valued functions of any order that are defined and continuous on P. Next, if α is a positive integer, we say that a function belongs to $C^{\alpha}(P)$ if and only if it is in $C(P)$ and its partial derivatives of order up to and including α exist on the interior of P and there coincide with functions that are continuous on P. If P is a set in E, T is a time-interval, and α a non-negative integer, we let $C^{(\alpha)}(P \times T)$ stand for the set of all functions in $C(P \times T)$ having continuous partial time-derivatives of order up to and including α on

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[1] Here and in the sequel we use the conventional notation for the cartesian product of two sets.

$P\times\mathring{T}$, provided each of these derivatives[1] coincides on $P\times\mathring{T}$ with a function continuous on $P\times T$. Finally, $\mathcal{G}^{\alpha}$ denotes the class of all functions in $C^{\alpha}(\overset{\infty}{T})$ that vanish on T^{-}.

The order-of-magnitude symbols "O" and "o" are used consistently in their standard mathematical connotation. For example, if $\underset{\sim}{y}\in R$, $\underset{\sim}{v}$ is defined on $R_{\underset{\sim}{y}}\times\overset{\infty}{T}$, and n is a real number, we write $\underset{\sim}{v}(\underset{\sim}{x},\cdot)=O(|\underset{\sim}{x}-\underset{\sim}{y}|^{n})$ as $\underset{\sim}{x}\to\underset{\sim}{y}$, uniformly on $(-\infty,t]$, if and only if there exist real numbers $\delta(t)$ and $m(t)$ such that $\underset{\sim}{x}\in R_{\underset{\sim}{y}}\cap B_{\delta}(\underset{\sim}{y})$ implies $|\underset{\sim}{v}(\underset{\sim}{x},\tau)|<m|\underset{\sim}{x}-\underset{\sim}{y}|^{n}$ for every $\tau\in(-\infty,t]$.

For future convenience we now recall a version of the divergence theorem that is adequate for our purposes.

Theorem 1.1. Let R be a regular region and let $\underset{\sim}{n}$ be the outward unit normal of ∂R. Suppose

$$\underset{\sim}{f}\in C^{1}(R)\cap C(\overline{R}),$$

and assume the set

$$\{\underset{\sim}{x}\,|\,\underset{\sim}{x}\in\overline{R},\ \underset{\sim}{f}(\underset{\sim}{x})\neq\underset{\sim}{0}\}$$

has a bounded closure, so that $\underset{\sim}{f}$ is of bounded support. Then,

$$\int_{R}\nabla\cdot\underset{\sim}{f}\,dV=\int_{\partial R}\underset{\sim}{f}\cdot\underset{\sim}{n}\,dA, \qquad (1.5)^{2}$$

provided the volume integral in (1.5) exists.

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[1] Observe that the class of functions $C^{(0)}(P\times T)$ is identical with $C(P\times T)$.

[2] Here ∇ is the usual gradient operator.

The truth of the preceding theorem follows trivially from the strongest form of the divergence theorem considered and proved by Kellogg [1][1] (page 119).

Next, we collect here certain results from the theory of Riemann convolutions that will be needed later on. To this end we first introduce

Definition 1.1. (Convolution). Let P be a set in E and suppose

$$\varphi \in C(P \times T^+), \ \psi \in C(P \times T^+) .$$

We call the function ϑ defined by

$$\vartheta(\underset{\sim}{x}, t) = \begin{cases} 0 \ \text{for all} \ (\underset{\sim}{x}, t) \in P \times T^- \\ \int_0^t \varphi(\underset{\sim}{x}, t-\tau)\psi(\underset{\sim}{x}, \tau)d\tau \ \text{for all} \ (\underset{\sim}{x}, t) \in P \times \mathring{T}^+ \end{cases} \tag{1.6}$$

the convolution of φ and ψ. We also write

$$\vartheta = \varphi * \psi, \quad \vartheta(\underset{\sim}{x}, t) = [\varphi * \psi](\underset{\sim}{x}, t)$$

to denote this function and its values.

Lemma 1.1. (Properties of convolutions). Let P either be an open or closed region in E or a regular surface in E. Let

$$\varphi \in C(P \times T^+), \ \psi \in C(P \times T^+), \ \omega \in C(P \times T^+) .$$

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[1] To avoid confusion we emphasize that Kellogg's "regular region" is a closed region, the boundary of which is a single "closed regular surface" (in Kellogg's sense of the latter term).

Then:

(a) $\varphi * \psi \in C(P \times \overset{\infty}{T})$;

(b) $\varphi * \psi = \psi * \varphi$;

(c) $\varphi * (\psi * \omega) = (\varphi * \psi) * \omega = \varphi * \psi * \omega$;

(d) $\varphi * (\psi + \omega) = \varphi * \psi + \varphi * \omega$;

(e) $\varphi * \psi = 0$ on $P \times T^+$ implies $\varphi = 0$ on $P \times T^+$ or $\psi = 0$ on $P \times T^+$.

Property (a) is an elementary consequence of Definition 1.1. Proofs of (b), (c), (d), and (e) may be found in Mikusinski's [2] [1] book. The following two lemmas are readily inferred from Definition 1.1 and (a) of Lemma 1.1.

Lemma 1.2. (Time-differentiation of convolutions). Let P be as in Lemma 1.1 and let

$$\varphi \in C^{(1)}(P \times T^+),\ \psi \in C(P \times T^+),\ \vartheta = \varphi * \psi .$$

Then:

(a) $\vartheta \in C^{(1)}(P \times T^+)$;

(b) $\dot{\vartheta} = \dot{\varphi} * \psi + \varphi(\cdot, 0)\psi$ on $P \times \mathring{T}^+$;

(c) $\varphi(\cdot, 0) = 0$ on P implies $\vartheta \in C^{(1)}(P \times \overset{\infty}{T})$.

Lemma 1.3. (Space-differentiation of convolutions). Let R be an open or closed region in E and let

$$\varphi \in C^1(R \times T^+),\ \psi \in C^1(R \times T^+),\ \vartheta = \varphi * \psi .$$

Then:

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[1] Properties (b), (c), and (d) are established in Chapter I of [2]. Titchmarsh's theorem (e) is proved in Chapter II.

(a) $\vartheta \in C^1(R \times T^+)$;

(b) $\vartheta_{,i} = \varphi_{,i} * \psi + \varphi * \psi_{,i}$ on $\mathring{R} \times T^+$;

(c) $\varphi(\cdot, 0) = 0$ or $\psi(\cdot, 0) = 0$ on R implies $\vartheta \in C^1(R \times \mathring{T})$.

As for convolutions of a scalar and a vector-valued function or of a scalar and a second-order tensor-valued function, we agree to write

$$\left.\begin{aligned} &\underset{\sim}{v} = \varphi * \underset{\sim}{u} \text{ if and only if } v_i = \varphi * u_i , \\ &\underset{\sim}{\vartheta} = \varphi * \underset{\sim}{\psi} \text{ if and only if } \vartheta_{ij} = \varphi * \psi_{ij} . \end{aligned}\right\} \quad (1.7)$$

Further, we adopt the notation

$$\left.\begin{aligned} &\underset{\sim}{u} * \underset{\sim}{v} = u_i * v_i , \\ &\underset{\sim}{\vartheta} * \underset{\sim}{\psi} = \vartheta_{ij} * \psi_{ij} . \end{aligned}\right\} \quad (1.8)$$

The remainder of this section is devoted to essential preliminaries pertaining to the linearized dynamical theory of homogeneous and isotropic elastic solids. For this purpose we introduce

Definition 1.2. (States. Elastodynamic states). Let R (not necessarily open or closed) be a region in E and let T (open, closed, or half-open) be a time-interval. If $\underset{\sim}{u}$ and $\underset{\sim}{\sigma}$ are, respectively, a vector-valued and a second-order tensor-valued function defined on $R \times T$, we call the ordered pair $S = [\underset{\sim}{u}, \underset{\sim}{\sigma}]$ a state on $R \times T$. We say that $S = [\underset{\sim}{u}, \underset{\sim}{\sigma}]$ is an elastodynamic state with the displacement field $\underset{\sim}{u}$ and the stress field $\underset{\sim}{\sigma}$, corresponding to the body-force density $\underset{\sim}{f}$, the mass density ρ, the dilatational wave speed c_1, and the shear-wave speed c_2, and write

$$S = [\underset{\sim}{u}, \underset{\sim}{\sigma}] \in \mathcal{E}(\underset{\sim}{f}, \rho, c_1, c_2; R \times T) ,$$

provided:

(a) $\underset{\sim}{u} \in C^2(\mathring{R} \times T) \cap C^1(R \times T)$, $\underset{\sim}{\sigma} \in C(R \times T)$, $\underset{\sim}{f} \in C(R \times T)$,

while ρ, c_1, and c_2 are constants subject to

$$\rho > 0, \quad 0 < \frac{2}{\sqrt{3}} c_2 < c_1 ; \tag{1.9}$$

(b) $\underset{\sim}{u}, \underset{\sim}{\sigma}, \underset{\sim}{f}, \rho, c_1$, and c_2 on $\mathring{R} \times \mathring{T}$ satisfy the equations

$$\sigma_{ij,j} + f_i = \rho \ddot{u}_i , \tag{1.10}$$

$$\sigma_{ij} = \rho(c_1^2 - 2c_2^2)\delta_{ij} u_{k,k} + 2\rho c_2^2 u_{(i,j)} . \tag{1.11}$$[1]

If, in particular,

$$T = \overset{\infty}{T}, \ \underset{\sim}{u} = \underset{\sim}{0} \text{ on } R \times T^- , \tag{1.12}$$

we say that S is an elastodynamic state with a quiescent past and write

$$S = [\underset{\sim}{u}, \underset{\sim}{\sigma}] \in \mathcal{E}_o(\underset{\sim}{f}, \rho, c_1, c_2; R) . \tag{1.13}$$

Equations (1.10) represent the stress equations of motion — (1.11) the stress-displacement relations of classical elastodynamics. In view of (1.10) and (1.11), the regularity assumptions under (a), though mutually consistent, are partly redundant. Note that (1.11) implies the symmetry of the stress-tensor field $\underset{\sim}{\sigma}$ on $R \times T$ since $\underset{\sim}{\sigma}$ is continuous on $R \times T$. The wave speeds c_1 and c_2 are expressed by

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[1] If $\underset{\sim}{\psi}$ is a second-order tensor, $\psi_{(ij)}$ and $\psi_{[ij]}$ are the components of the symmetric part and of the skew-symmetric part of $\underset{\sim}{\psi}$, respectively.

$$c_1 = \sqrt{\frac{2(1-\nu)\mu}{(1-2\nu)\rho}}\ , \ c_2 = \sqrt{\frac{\mu}{\rho}} \tag{1.14}$$

in terms of ρ, the shear modulus μ, and Poisson's ratio ν of the elastic solid. Also, the inequalities (1.9) required under (a) of Definition 1.2 are equivalent to

$$\mu > 0, \ -1 < \nu < \frac{1}{2} \ . \tag{1.15}$$

Moreover, (1.9) assure the positive definiteness of the quadratic function e that is defined by

$$e(\underset{\sim}{\varphi}) = \frac{\rho}{2}\left[(c_1^2 - 2c_2^2)\varphi_{ii}\varphi_{jj} + 2c_2^2\varphi_{ij}\varphi_{ij}\right] \tag{1.16}$$

for every symmetric second-order tensor $\underset{\sim}{\varphi}$. If $\underset{\sim}{\epsilon}$ is the infinitesimal strain tensor associated with $\underset{\sim}{u}$, i.e.

$$\epsilon_{ij} = u_{(i,j)} \ , \tag{1.17}$$

then $e(\underset{\sim}{\epsilon})$ represents the strain-energy density appropriate to the elastodynamic state S.

If R is a regular region, $S = [\underset{\sim}{u}, \underset{\sim}{\sigma}]$ is a state on $\overline{R} \times T$, and $\underset{\sim}{n}$ is the unit outward normal vector of ∂R, we call the vector-field $\underset{\sim}{s}$ defined by

$$s_i = \sigma_{ij} n_j \ \text{ on } \overset{*}{\partial} R \times T \ , \tag{1.18}$$[1]

the tractions of S acting on ∂R.

We now define equality and addition of states, as well as multiplication of a state by a scalar constant. To this end let R be an

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[1] Recall that $\overset{*}{\partial} R$ represents the set of all regular boundary points.

arbitrary region, suppose $S = [\underset{\sim}{u}, \underset{\sim}{\sigma}]$ and $S' = [\underset{\sim}{u}', \underset{\sim}{\sigma}']$ are states on $R \times T$ and let λ be a real number. Then,

$$S = S' \Leftrightarrow \underset{\sim}{u} = \underset{\sim}{u}', \ \underset{\sim}{\sigma} = \underset{\sim}{\sigma}' \text{ on } R \times T ,$$

$$S + S' = [\underset{\sim}{u} + \underset{\sim}{u}', \ \underset{\sim}{\sigma} + \underset{\sim}{\sigma}'] \text{ on } R \times T ,$$

$$\lambda S = [\lambda \underset{\sim}{u}, \lambda \underset{\sim}{\sigma}] \text{ on } R \times T .$$

Next, with reference to (1.4), we write

$$S' = S^{(\alpha)} \Leftrightarrow u_i' = u_i^{(\alpha)}, \ \sigma_{ij}' = \sigma_{ij}^{(\alpha)} \text{ on } R \times \mathring{T}, \tag{1.19}$$

and, for fixed k, adopt the notation

$$S' = S_{,k} \Leftrightarrow u_i' = u_{i,k}, \ \sigma_{ij}' = \sigma_{ij,k} \text{ on } \mathring{R} \times T, \tag{1.20}$$

provided the required time and space-derivatives exist. Finally,

$$\varphi * S = [\varphi * \underset{\sim}{u}, \ \varphi * \underset{\sim}{\sigma}] \tag{1.21}$$

whenever the underlying convolutions are meaningful.

2. Extension of the uniqueness and the reciprocal theorem in elastodynamics to unbounded regions.

The current section serves a dual purpose: here we extend Neumann's [3] uniqueness theorem of classical elastodynamics to unbounded domains and subsequently generalize Graffi's [4] reciprocal identity to a pair of elastodynamic states associated with an infinite region. The results thus obtained are essential prerequisites to the determination of integral representations for the two fundamental problems of dynamic elasticity carried out later on; at the same time these results are apt to be of interest in themselves.

The principal tool used to establish the two theorems alluded to above is supplied by a generalized energy identity, which we state and prove presently. This lemma is an elastodynamic counterpart of a result due to Zaremba [5] for the scalar wave equation[1]. Indeed, our method of proving the generalized uniqueness theorem is suggested by the treatment in [5] of uniqueness issues pertaining to the wave-equation. A lucid account of Zaremba's paper is given by Fritz John in [6].

Lemma 2.1. (Generalized energy identity). *Suppose* R *is a regular region and*

(a) $S = [\underset{\sim}{u}, \underset{\sim}{\sigma}] \in \mathcal{E}_o(\underset{\sim}{f}, \rho, c_1, c_2; \bar{R})$,

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[1] Zaremba's energy scheme was rediscovered independently by Rubinowicz [7] and by Friedrichs and Lewy [8]. See also Courant and Hilbert [9] (pages 659-661) where Zaremba's result is extended and applied to the general second-order hyperbolic equation.

(b) $\tau \in C^1(\bar{R})$ *is a given* (scalar-valued) *function such that the set*

$$\{\underset{\sim}{x} \mid \underset{\sim}{x} \in \bar{R}, \ \tau(\underset{\sim}{x}) > 0\}$$

is bounded. Let $\underset{\sim}{\varphi}$ *be the* (second-order tensor-valued) *function defined by*

$$\varphi_{ij}(\underset{\sim}{x}) = \frac{1}{2}\left[\frac{\partial}{\partial x_j} u_i(\underset{\sim}{x}, \tau(\underset{\sim}{x})) + \frac{\partial}{\partial x_i} u_j(\underset{\sim}{x}, \tau(\underset{\sim}{x}))\right] \text{ for all } \underset{\sim}{x} \in R . \tag{2.1}$$

Then

$$\int_{\partial R} \int_0^{\tau(\underset{\sim}{x})} \dot{\underset{\sim}{u}}(\underset{\sim}{x}, t) \cdot \underset{\sim}{s}(\underset{\sim}{x}, t) dt dA + \int_R \int_0^{\tau(\underset{\sim}{x})} \dot{\underset{\sim}{u}}(\underset{\sim}{x}, t) \cdot \underset{\sim}{f}(\underset{\sim}{x}, t) dt dV =$$

$$\int_R \Big\{ e(\underset{\sim}{\varphi}(\underset{\sim}{x})) + \frac{\rho}{2} \dot{\underset{\sim}{u}}^2(\underset{\sim}{x}, \tau(\underset{\sim}{x}))[1 - c_1^2(\nabla\tau(\underset{\sim}{x}))^2]$$

$$+ \frac{\rho}{2}(c_1^2 - c_2^2)[\dot{\underset{\sim}{u}}(\underset{\sim}{x}, \tau(\underset{\sim}{x})) \wedge \nabla\tau(\underset{\sim}{x})]^2 \Big\} dV , \tag{2.2}$$

where $\underset{\sim}{s}$ *are the tractions of* S *acting on* ∂R *and the function* e *is given by* (1.16).

Proof. For convenience introduce the auxiliary vector-valued functions $\underset{\sim}{p}$ and $\underset{\sim}{v}$ through

$$p_i = \dot{u}_j \sigma_{ij} \text{ on } \bar{R} \times \overset{\infty}{T}, \quad v_i(\underset{\sim}{x}) = \int_0^{\tau(\underset{\sim}{x})} p_i(\underset{\sim}{x}, t) dt \text{ for all } \underset{\sim}{x} \in \bar{R}. \tag{2.3}$$

In view of the smoothness of τ stipulated in (b), and because of the regularity properties implied by (a) and Definition 1.2,

$$\underset{\sim}{v} \in C^1(R) \cap C(\bar{R}) . \tag{2.4}$$

Further, $\underset{\sim}{v}$ has bounded support by virtue of hypotheses (a), (b) and (1.12). From (2.3), (1.10), (1.11), and (1.17) follows

$$\nabla \cdot \underset{\sim}{v}(\underset{\sim}{x})=\int_0^{\tau(\underset{\sim}{x})}\left\{\dot{\underset{\sim}{\varepsilon}}(\underset{\sim}{x},t)\cdot\underset{\sim}{\sigma}(\underset{\sim}{x},t)+\frac{\rho}{2}\frac{\partial}{\partial t}[\dot{\underset{\sim}{u}}(\underset{\sim}{x},t)]^2-\dot{\underset{\sim}{u}}(\underset{\sim}{x},t)\cdot\underset{\sim}{f}(\underset{\sim}{x},t)\right\}dt$$

$$+\underset{\sim}{p}(\underset{\sim}{x},\tau(\underset{\sim}{x}))\cdot\nabla\tau(\underset{\sim}{x}) \text{ for all } \underset{\sim}{x}\in R\,, \tag{2.5}$$[1]

while (1.11), (1.16) yield

$$\dot{\underset{\sim}{\varepsilon}}(\underset{\sim}{x},t)\cdot\underset{\sim}{\sigma}(\underset{\sim}{x},t)=\frac{\partial}{\partial t}e(\underset{\sim}{\varepsilon}(\underset{\sim}{x},t)) \text{ for all } (\underset{\sim}{x},t)\in R\times\overset{\infty}{T}\,. \tag{2.6}$$

Now substitute from (2.6) into (2.5) and use (1.12) to infer

$$\nabla\cdot\underset{\sim}{v}(\underset{\sim}{x})=e(\underset{\sim}{\varepsilon}(\underset{\sim}{x},\tau(\underset{\sim}{x})))+\underset{\sim}{p}(\underset{\sim}{x},\tau(\underset{\sim}{x}))\cdot\nabla\tau(\underset{\sim}{x})$$

$$+\frac{\rho}{2}\dot{\underset{\sim}{u}}^2(\underset{\sim}{x},\tau(\underset{\sim}{x}))-\int_0^{\tau(\underset{\sim}{x})}\dot{\underset{\sim}{u}}(\underset{\sim}{x},t)\cdot\underset{\sim}{f}(\underset{\sim}{x},t)dt\,. \tag{2.7}$$

Next, note from (2.1), (1.17) that

$$\varphi_{ij}(\underset{\sim}{x})=\varepsilon_{ij}(\underset{\sim}{x},\tau(\underset{\sim}{x}))+\frac{1}{2}\left[\dot{u}_i(\underset{\sim}{x},\tau(\underset{\sim}{x}))\tau_{,j}(\underset{\sim}{x})+\dot{u}_j(\underset{\sim}{x},\tau(\underset{\sim}{x}))\tau_{,i}(\underset{\sim}{x})\right],$$

whence (1.16), (1.11), and the first of (2.3) furnish

$$e(\underset{\sim}{\varphi}(\underset{\sim}{x}))=e(\underset{\sim}{\varepsilon}(\underset{\sim}{x},\tau(\underset{\sim}{x})))+\underset{\sim}{p}(\underset{\sim}{x},\tau(\underset{\sim}{x}))\cdot\nabla\tau(\underset{\sim}{x})$$

$$+\frac{\rho}{2}(c_1^2-c_2^2)[\dot{\underset{\sim}{u}}(\underset{\sim}{x},\tau(\underset{\sim}{x}))\cdot\nabla\tau(\underset{\sim}{x})]^2$$

$$+\frac{\rho}{2}c_2^2\dot{\underset{\sim}{u}}^2(\underset{\sim}{x},\tau(\underset{\sim}{x}))(\nabla\tau(\underset{\sim}{x}))^2\,.$$

This equation, because of Lagrange's identity

$$\dot{\underset{\sim}{u}}^2(\nabla\tau)^2=(\dot{\underset{\sim}{u}}\cdot\nabla\tau)^2+(\dot{\underset{\sim}{u}}\wedge\nabla\tau)^2\,,$$

may be written as

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[1] Recall that $\dot{\underset{\sim}{\varepsilon}}\cdot\underset{\sim}{\sigma}=\dot{\varepsilon}_{ij}\sigma_{ij}$.

$$e(\underset{\sim}{\varepsilon}(\underset{\sim}{x},\tau(\underset{\sim}{x})))+\underset{\sim}{p}(\underset{\sim}{x},\tau(\underset{\sim}{x}))\cdot\nabla\tau(\underset{\sim}{x})=e(\underset{\sim}{\varphi}(\underset{\sim}{x}))-\frac{\rho}{2}c_1^2\dot{\underset{\sim}{u}}^2(\underset{\sim}{x},\tau(\underset{\sim}{x}))(\nabla\tau(\underset{\sim}{x}))^2$$

$$+\frac{\rho}{2}(c_1^2-c_2^2)[\dot{\underset{\sim}{u}}(\underset{\sim}{x},\tau(\underset{\sim}{x}))\wedge\nabla\tau(\underset{\sim}{x})]^2 \text{ for all } \underset{\sim}{x}\in R . \qquad (2.8)$$

Combining (2.8) with (2.7) one has

$$\nabla\cdot\underset{\sim}{v}(\underset{\sim}{x})=e(\underset{\sim}{\varphi}(\underset{\sim}{x}))+\frac{\rho}{2}\dot{\underset{\sim}{u}}^2(\underset{\sim}{x},\tau(\underset{\sim}{x}))[1-c_1^2(\nabla\tau(\underset{\sim}{x}))^2]$$

$$-\int_0^{\tau(\underset{\sim}{x})}\dot{\underset{\sim}{u}}(\underset{\sim}{x},t)\cdot\underset{\sim}{f}(\underset{\sim}{x},t)dt+\frac{\rho}{2}(c_1^2-c_2^2)[\dot{\underset{\sim}{u}}(\underset{\sim}{x},\tau(\underset{\sim}{x}))\wedge\nabla\tau(\underset{\sim}{x})]^2 \qquad (2.9)$$

for all $\underset{\sim}{x}\in R$. From (2.9), the regularity assumptions contained in hypotheses (a) and (b), and the boundedness of the support of $\underset{\sim}{v}$, it is clear that $\nabla\cdot\underset{\sim}{v}$ is properly integrable on R. Thus, integrating both members of (2.9) over R, one is entitled subsequently to apply the divergence theorem (Theorem 1.1) to the vector field $\underset{\sim}{v}$ since the latter conforms to (2.4) and is of bounded support. The desired result (2.2) then follows immediately with the aid of (2.3) and (1.18). This completes the proof.

Suppose now in particular R in Lemma 2.1 is bounded and restrict τ to be a positive constant, say $\tau=t$. In these circumstances one recovers from (2.2) the classical energy identity of elastodynamics in the form

$$\int_{\partial R}\int_0^t\dot{\underset{\sim}{u}}(\underset{\sim}{x},\lambda)\cdot\underset{\sim}{s}(\underset{\sim}{x},\lambda)d\lambda dA+\int_R\int_0^t\dot{\underset{\sim}{u}}(\underset{\sim}{x},\lambda)\cdot\underset{\sim}{f}(\underset{\sim}{x},\lambda)d\lambda dV=$$

$$\int_R[e(\underset{\sim}{\varepsilon}(\underset{\sim}{x},t))+\frac{\rho}{2}\dot{\underset{\sim}{u}}^2(\underset{\sim}{x},t)]\,dV . \qquad (2.10)$$

As will become apparent shortly, the role played by the generalized energy identity (2.2) in connection with the extended uniqueness theorem to which we turn now is strictly analogous to that played by (2.10) in Neumann's [3] familiar uniqueness argument for bounded regions.

Theorem 2.1. (Generalized uniqueness theorem). Let R be a regular region and let S', S'' be two states with the following properties:

(a) $S'=[\underset{\sim}{u}',\underset{\sim}{\sigma}']\in\mathcal{E}(\underset{\sim}{f},\rho,c_1,c_2;\overline{R}\times T^+)$,

$S''=[\underset{\sim}{u}'',\underset{\sim}{\sigma}'']\in\mathcal{E}(\underset{\sim}{f},\rho,c_1,c_2;\overline{R}\times T^+)$;

(b) $\underset{\sim}{u}'(\cdot,0)=\underset{\sim}{u}''(\cdot,0)$, $\dot{\underset{\sim}{u}}'(\cdot,0+)=\dot{\underset{\sim}{u}}''(\cdot,0+)$ on R;

further, suppose either

(c) $\underset{\sim}{u}'=\underset{\sim}{u}''$ on $\partial R\times T^+$

or

(d) $\underset{\sim}{s}'=\underset{\sim}{s}''$ on $\overset{*}{\partial}R\times T^+$, where $\underset{\sim}{s}'$ and $\underset{\sim}{s}''$ are the respective tractions of S' and S'' acting on ∂R.

Then $S'=S''$ on $\overline{R}\times T^+$.

Proof. Define the state S on $\overline{R}\times\overset{\infty}{T}$ by

$$S=[\underset{\sim}{u},\underset{\sim}{\sigma}]=S'-S'' \text{ on } \overline{R}\times T^+,\quad \underset{\sim}{u}=\underset{\sim}{\sigma}=\underset{\sim}{0} \text{ on } \overline{R}\times\overset{\circ}{T}^-. \tag{2.11}$$

From (a), (b), (2.11) and Definition 1.2 one finds without difficulty that

$$S \in \mathcal{E}_o(\underset{\sim}{0}, \rho, c_1, c_2; \overline{R}) . \qquad (2.12)^1$$

By (2.11) and (1.18), since either (c) or (d) holds,

$$\dot{\underset{\sim}{u}} \cdot \underset{\sim}{s} = 0 \text{ on } \overset{*}{\partial} R \times \overset{\infty}{T} , \qquad (2.13)$$

where $\underset{\sim}{s}$ are the tractions of S acting on ∂R. Now fix $(\underset{\sim}{x}, t) \in R \times \dot{T}^+$ and define the scalar-valued function τ through

$$\tau(\underset{\sim}{y}) = t - |\underset{\sim}{y} - \underset{\sim}{x}|/2c_1 \text{ for all } \underset{\sim}{y} \in \overline{R} . \qquad (2.14)$$

Then, evidently,

$$\tau \in C^1(\overline{R}_{\underset{\sim}{x}}) \cap C(\overline{R}), \quad [\nabla\tau(\underset{\sim}{y})]^2 = \frac{1}{4c_1^2} \text{ for all } \underset{\sim}{y} \in R_{\underset{\sim}{x}} , \qquad (2.15)^2$$

and because c_1 is positive by (2.12) and Definition 1.2,

$$\{\underset{\sim}{y} \mid \underset{\sim}{y} \in \overline{R}, \; \tau(\underset{\sim}{x}) > 0\} \text{ is bounded.} \qquad (2.16)$$

Choose $\delta_o > 0$ such that $\overline{B}_{\delta_o}(\underset{\sim}{x}) \subset R$ and set

$$R_\delta = R - \overline{B}_\delta(\underset{\sim}{x}) \quad (0 < \delta < \delta_o) . \qquad (2.17)$$

In view of (2.12), (2.15), (2.16) and Lemma 2.1, one concludes that (2.2) holds for each member of the family of regular regions defined in (2.17). Thus, bearing in mind (2.13), the second of (2.15), and the fact that the body-force field of S vanishes identically, one has

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[1] Here and in the sequel, we write $\underset{\sim}{0}$ in place of the body-force argument of the elastodynamic state under consideration if the body forces vanish identically on the appropriate space-time domain.

[2] Note that the gradient of the function τ given by (2.14) has an (irremovable) finite discontinuity at $\underset{\sim}{y}$.

$$\int_{S_\delta(\underset{\sim}{x})}\int_0^{\tau(\underset{\sim}{y})} \dot{\underset{\sim}{u}}(\underset{\sim}{y},\lambda)\cdot \underset{\sim}{s}(\underset{\sim}{y},\lambda)d\lambda dA =$$

$$\int_{R_\delta}\left\{e(\underset{\sim}{\varphi}(\underset{\sim}{y}))+\frac{3\rho}{8}\dot{\underset{\sim}{u}}^2(\underset{\sim}{y},\tau(\underset{\sim}{y}))+\frac{\rho}{2}(c_1^2-c_2^2)[\dot{\underset{\sim}{u}}(\underset{\sim}{y},\tau(\underset{\sim}{y}))\wedge\nabla\tau(\underset{\sim}{y})]^2\right\}dV \qquad (2.18)$$

for every $\delta\in(0,\delta_o)$, where the functions e and $\underset{\sim}{\varphi}$ are given by (1.16) and (2.1), while $\underset{\sim}{s}$ now stands for the tractions of S acting on ∂R_δ. Owing to the continuity of τ on $\overline{R}$ and of $\dot{\underset{\sim}{u}},\underset{\sim}{\sigma}$ on $\overline{R}\times\overset{\infty}{\mathring{T}}$,

$$\lim_{\delta\to 0}\int_{S_\delta(\underset{\sim}{x})}\int_0^{\tau(\underset{\sim}{y})} \dot{\underset{\sim}{u}}(\underset{\sim}{y},\lambda)\cdot \underset{\sim}{s}(\underset{\sim}{y},\lambda)d\lambda dA = 0 ,$$

so that passage to the limit as $\delta\to 0$ in (2.18) gives

$$\int_{R}\left\{e(\underset{\sim}{\varphi}(\underset{\sim}{y}))+\frac{3\rho}{8}\dot{\underset{\sim}{u}}^2(\underset{\sim}{y},\tau(\underset{\sim}{y}))+\frac{\rho}{2}(c_1^2-c_2^2)[\dot{\underset{\sim}{u}}(\underset{\sim}{y},\tau(\underset{\sim}{y}))\wedge\nabla\tau(\underset{\sim}{y})]^2\right\}dV = 0 . \qquad (2.19)$$

Recall next that the inequalities (1.9), which are implied by (2.12), are sufficient for the positive definiteness of e. Moreover, (1.9) assure that all terms in the integrand of (2.19) are non-negative. Therefore, and since the integrand in (2.19) is continuous on $R_{\underset{\sim}{x}}$,

$$\dot{\underset{\sim}{u}}(\underset{\sim}{y},\tau(\underset{\sim}{y}))=\underset{\sim}{0} \text{ for every } \underset{\sim}{y}\in R_{\underset{\sim}{x}} .$$

Finally, invoke the first of (2.15) and the regularity of $\underset{\sim}{u}$ on $R\times\overset{\infty}{\mathring{T}}$ implied by (2.12), and use (2.14) to confirm that

$$\dot{\underset{\sim}{u}}(\underset{\sim}{x},t)=\dot{\underset{\sim}{u}}(\underset{\sim}{x},\tau(\underset{\sim}{x}))=\underset{\sim}{0} .$$

Consequently, $(\underset{\sim}{x},t)$ having been chosen arbitrarily in $R\times\mathring{T}^+$,

$$\dot{\underset{\sim}{u}} = \underset{\sim}{0} \text{ on } R \times \mathring{T}^{+} . \tag{2.20}$$

But (2.20) and (2.12) furnish

$$\underset{\sim}{u} = \underset{\sim}{\sigma} = \underset{\sim}{0} \text{ on } \overline{R} \times \overset{\infty}{T} .$$

The desired conclusion now follows from (2.11).

An extension of Theorem 2.1 to mixed boundary conditions is entirely elementary. Similarly, the generalization of Lemma 2.1 and Theorem 2.1 to anisotropic and nonhomogeneous solids presents no difficulties. Next, in the first boundary-initial value problem (surface displacements prescribed) uniqueness prevails for unbounded domains even if (1.9) is replaced by the weaker requirement that c_1 and c_2 be real, as can be shown by adapting an argument due to Gurtin and Sternberg [10] for bounded isotropic elastic bodies.[1] The relaxation of the rather stringent smoothness hypotheses involved in (a) of Definition 1.2, which render Theorem 2.1 inapplicable to certain physically important problems, is in need of further attention.[2]

It should be pointed out that an elastodynamic uniqueness theorem valid for infinite regions may alternatively be based on the classical energy identity (2.10), following Neumann's procedure, if one introduces suitable restrictions on the orders of magnitude of the velocity and stress field at infinity. The essential advantage of

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[1] See also Gurtin and Toupin [11], where the result of [10] is extended to anisotropic media.

[2] In this connection see a recent paper by Knops and Payne [12], which contains a uniqueness theorem for weak solutions in elastodynamics, with limitation to bounded domains.

Theorem 2.1 stems from the fact that it does not involve such artificial a priori assumptions. In this connection we recall that the analogous uniqueness issue in elastostatics, where the governing equations are elliptic rather than hyperbolic, is considerably more involved. For *exterior* unbounded domains elastostatic uniqueness theorems that avoid extraneous order prescriptions at infinity were established by Fichera [13], as well as by Gurtin and Sternberg [14]. On the other hand, the uniqueness question associated with boundary-value problems in the equilibrium theory for general domains whose boundaries extend to infinity is yet to be disposed of satisfactorily.[1]

In preparation for a generalization of Graffi's [4] dynamic reciprocal identity to unbounded regions we now proceed to

Lemma 2.2. (Sufficient conditions for the prolonged quiescence of the far field). *Let* R *be an unbounded regular region and suppose*:

(a) $S=[\mathbf{u},\boldsymbol{\sigma}]\in\mathcal{E}_0(\mathbf{f},\rho,c_1,c_2;\bar{R})$;

(b) *for every* $t>0$ *there is a bounded set* $\Lambda(t)\subset\bar{R}$ *such that*

$$\mathbf{f}=\mathbf{0} \text{ on } (\bar{R}-\Lambda(t))\times[0,t],$$

and, if ∂R *is unbounded,*

$$\dot{\mathbf{u}}\cdot\mathbf{s}=0 \text{ on } (\overset{*}{\partial}R-\Lambda(t))\times[0,t],$$

where $\mathbf{s}$ *are the tractions of* S *acting on* ∂R.

Then, for each $t>0$, *there is a bounded set* $\Omega(t)\subset\bar{R}$, *depending only on* $\Lambda(t)$, *such that*

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[1] For the special case of the first and second equilibrium problem appropriate to the half-space this question was settled by Turteltaub and Sternberg [15].

$$\underset{\sim}{u}=\underset{\sim}{\sigma}=\underset{\sim}{0} \text{ on } (\overline{R}-\Omega(t))\times[0,t]\ . \tag{2.21}$$

Proof. Fix $t>0$, let $\delta>0$ be such that

$$\left.\begin{array}{l} \partial R\cup\Lambda(t)\subset\overline{B}_\delta \text{ if } \partial R \text{ is bounded}\ ,\\ \Lambda(t)\subset\overline{B}_\delta \text{ if } \partial R \text{ is unbounded,}\end{array}\right\} \tag{2.22}$$

and consider the set

$$\Omega(t)=\overline{R}\cap\overline{B}_{\delta+2c_1 t}\ . \tag{2.23}$$

Note that $\Omega(t)$, as defined in (2.23), is a bounded subset of $\overline{R}$. With a view toward showing that (2.21) holds, choose

$$(\underset{\sim}{y},\lambda)\in(R-\Omega(t))\times(0,t] \tag{2.24}$$

and regard $(\underset{\sim}{y},\lambda)$ as fixed. Define the function τ by

$$\tau(\underset{\sim}{x})=\lambda-|\underset{\sim}{x}-\underset{\sim}{y}|/2c_1 \text{ for all } \underset{\sim}{x}\in\overline{R}\ . \tag{2.25}$$

Evidently,

$$\tau\in C^1(\overline{R}_{\underset{\sim}{y}})\cap C(\overline{R}),\ [\nabla\tau(\underset{\sim}{x})]^2=\frac{1}{4c_1^2} \text{ for all } \underset{\sim}{x}\in R_{\underset{\sim}{y}}\ , \tag{2.26}$$

and since $c_1>0$,

$$\{\underset{\sim}{x}\,|\,\underset{\sim}{x}\in\overline{R},\ \tau(\underset{\sim}{x})>0\}=\overline{R}\cap B_{2\lambda c_1}(\underset{\sim}{y})\ . \tag{2.27}$$

From (2.23) and (2.24) one draws that $B_{2\lambda c_1}(\underset{\sim}{y})$ does not intersect $\overline{B}_\delta$. Thus, (2.22) and (2.27) imply

$$\left.\begin{array}{l} \{\underset{\sim}{x}\,|\,\underset{\sim}{x}\in\overline{R},\ \tau(\underset{\sim}{x})>0\}\subset\overline{R}-\Lambda(t)\cup\partial R \text{ if } \partial R \text{ is bounded}\ ,\\ \{\underset{\sim}{x}\,|\,\underset{\sim}{x}\in\overline{R},\ \tau(\underset{\sim}{x})>0\}\subset\overline{R}-\Lambda(t) \text{ if } \partial R \text{ is unbounded}\ .\end{array}\right\} \tag{2.28}$$

Now call on (2.24), (2.25) to arrive at

$$\tau(\underset{\sim}{x}) \le t \text{ for all } \underset{\sim}{x} \in \overline{R} \ . \tag{2.29}$$

Hypothesis (a) requires $\underset{\sim}{u}$ to vanish on $\overline{R} \times T^{-}$. This fact, in conjunction with (2.28), (2.29) and hypothesis (b), justifies

$$\left.\begin{aligned} &\int_0^{\tau(\underset{\sim}{x})} \dot{\underset{\sim}{u}}(\underset{\sim}{x},\eta)\cdot\underset{\sim}{f}(\underset{\sim}{x},\eta)d\eta = 0 \text{ for all } \underset{\sim}{x}\in\overline{R} , \\ &\int_0^{\tau(\underset{\sim}{x})} \dot{\underset{\sim}{u}}(\underset{\sim}{x},\eta)\cdot\underset{\sim}{s}(\underset{\sim}{x},\eta)d\eta = 0 \text{ for all } \underset{\sim}{x}\in\overset{*}{\partial}R \ . \end{aligned}\right\} \tag{2.30}$$

Next, let $\xi_o > 0$ be such that $\overline{B}_{\xi_o}(\underset{\sim}{y}) \subset R$ and put

$$R_\xi = R - \overline{B}_\xi(\underset{\sim}{y}) \quad (0 < \xi < \xi_o) \ . \tag{2.31}$$

One concludes from (2.26), (2.27), hypothesis (a), and Lemma 2.1 that (2.2) holds for each R_ξ in (2.31). Thus, (2.30) and the second of (2.26) yield

$$\int_{S_\xi(\underset{\sim}{y})} \int_0^{\tau(\underset{\sim}{x})} \dot{\underset{\sim}{u}}(\underset{\sim}{x},\eta)\cdot\underset{\sim}{s}(\underset{\sim}{x},\eta)d\eta\, dA =$$

$$\int_{R_\xi} \left\{ e(\underset{\sim}{\varphi}(\underset{\sim}{x})) + \frac{3\rho}{8}\dot{\underset{\sim}{u}}^2(\underset{\sim}{x},\tau(\underset{\sim}{x})) + \frac{\rho}{2}(c_1^2 - c_2^2)[\dot{\underset{\sim}{u}}(\underset{\sim}{x},\tau(\underset{\sim}{x}))\wedge\nabla\tau(\underset{\sim}{x})]^2 \right\} dV , \tag{2.32}$$

for every $\xi \in (0,\xi_o)$, where e and $\underset{\sim}{\varphi}$ are given by (1.16) and (2.1), while $\underset{\sim}{s}$ here denotes the tractions of S acting on ∂R_ξ. Since τ is continuous on R and $\dot{\underset{\sim}{u}}$, $\underset{\sim}{\sigma}$ are continuous on $R \times \overset{\infty}{T}$, the left-hand member of (2.32) tends to zero as $\xi \to 0$, whence

$$\int_R \left\{ e(\varphi(x)) + \frac{3\rho}{8}\dot{u}^2(x,\tau(x)) + \frac{\rho}{2}(c_1^2 - c_2^2)[\dot{u}(x,\tau(x))\wedge\nabla\tau(x)]^2 \right\} dV = 0 \ . \qquad (2.33)$$

The inequalities (1.9), which are implied by hypothesis (a), are sufficient for the positive definiteness of e and ensure that each of the three terms of the integrand in (2.33) is non-negative. Accordingly, this integrand being continuous on R_y,

$$\dot{u}(x,\tau(x)) = 0 \ \text{ for all } x \in R_y \ .$$

Invoking once again the continuity of τ on R and of $\dot{u}$ on $R \times \overset{\infty}{T}$, one finds that

$$\dot{u}(y,\lambda) = \dot{u}(y,\tau(y)) = 0 \ .$$

But (y,λ) was selected arbitrarily in $(R-\Omega(t))\times(0,t]$. Hence

$$\dot{u} = 0 \ \text{ on } (R-\Omega(t))\times(0,t] \ ,$$

which, because of the regularity and initial quiescence of u assumed in hypothesis (a), gives

$$u = 0 \ \text{ on } (R-\Omega(t))\times[0,t] \ . \qquad (2.34)$$

By (2.34), and because (1.11) hold on $R \times \overset{\infty}{T}$,

$$\sigma = 0 \ \text{ on } (R-\Omega(t))\times[0,t]. \qquad (2.35)$$

Recalling that $\Omega(t)$ is closed, one shows readily[1] that the closure of $R-\Omega(t)$ contains $\overline{R}-\Omega(t)$. Therefore, appealing to the continuity of u and σ on $\overline{R}\times\overset{\infty}{T}$, one sees that (2.34), (2.35) imply (2.21). Finally,

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[1] Cf. Exercise 1 (page 37) in [16].

note that (2.22) and (2.23) imply that $\Omega(t)$ depends exclusively on $\Lambda(t)$. Since t was chosen arbitrarily, the proof is now complete.

It is essential to recognize that if a state with a quiescent past is characterized as the solution of a standard boundary-initial value problem in elastodynamics, the decision whether or not hypothesis (b) of Lemma 2.2 is met, is immediate from the data.

Theorem 2.2. (Extension of Graffi's reciprocal identity to unbounded regions). *Let R be a regular region and suppose:*

(a) $S=[\underset{\sim}{u},\underset{\sim}{\sigma}]\in\mathcal{E}_o(\underset{\sim}{f},\rho,c_1,c_2;\bar{R})$, $S'=[\underset{\sim}{u}',\underset{\sim}{\sigma}']\in\mathcal{E}_o(\underset{\sim}{f}',\rho,c_1,c_2;\bar{R})$;

(b) S *satisfies hypothesis* (b) *of Lemma* 2.2 *if* R *is unbounded.*

Then, for every $t>0$,

$$\int_{\partial R}[\underset{\sim}{s}*\underset{\sim}{u}'](\underset{\sim}{x},t)dA+\int_{R}[\underset{\sim}{f}*\underset{\sim}{u}'](\underset{\sim}{x},t)dV=\int_{\partial R}[\underset{\sim}{s}'*\underset{\sim}{u}](\underset{\sim}{x},t)dA+\int_{R}[\underset{\sim}{f}'*\underset{\sim}{u}](\underset{\sim}{x},t)dV, \qquad (2.36)^{1}$$

where $\underset{\sim}{s}$ *and* $\underset{\sim}{s}'$ *are the tractions of* S *and* S' *acting on* ∂R.

Proof. It is clear from the present hypotheses and Lemma 2.2 that the integrals in (2.36) are proper even if R is unbounded. Choose $t>0$ and hold t fixed for the remainder of the argument. Define the vector field $\underset{\sim}{v}$ by

$$v_i(\underset{\sim}{x})=[\sigma_{ij}*u_j'](\underset{\sim}{x},t)-[\sigma_{ij}'*u_j](\underset{\sim}{x},t) \text{ for all } \underset{\sim}{x}\in\bar{R}. \qquad (2.37)$$

In view of hypothesis (a), Definition 1.2, Lemma 1.1, and Lemma 1.3,

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[1] Recall the notations adopted in (1.8).

$$\underset{\sim}{v} \in C^1(R) \cap C(\overline{R}) , \tag{2.38}$$

$$v_{i,i}(\underset{\sim}{x}) = [\sigma_{ij,i} * u'_j](\underset{\sim}{x}, t) + [\sigma_{ij} * u'_{j,i}](\underset{\sim}{x}, t)$$

$$- [\sigma'_{ij,i} * u_j](\underset{\sim}{x}, t) - [\sigma'_{ij} * u_{j,i}](\underset{\sim}{x}, t)$$

for all $\underset{\sim}{x} \in R$. Hence hypothesis (a), (1.10), together with symmetry of $\underset{\sim}{\sigma}$ and $\underset{\sim}{\sigma}'$, furnish

$$\nabla \cdot \underset{\sim}{v}(\underset{\sim}{x}) = \rho [\underset{\sim}{\ddot{u}} * \underset{\sim}{u}'](\underset{\sim}{x}, t) - [\underset{\sim}{f} * \underset{\sim}{u}'](\underset{\sim}{x}, t) + [\underset{\sim}{\sigma} * \underset{\sim}{\epsilon}'](\underset{\sim}{x}, t)$$

$$- \rho [\underset{\sim}{\ddot{u}}' * \underset{\sim}{u}](\underset{\sim}{x}, t) + [\underset{\sim}{f}' * \underset{\sim}{u}](\underset{\sim}{x}, t) - [\underset{\sim}{\sigma}' * \underset{\sim}{\epsilon}](\underset{\sim}{x}, t) , \tag{2.39}$$

where

$$\epsilon_{ij} = u_{(i,j)} , \quad \epsilon'_{ij} = u'_{(i,j)} .$$

On the other hand, (1.11), (1.8), and the commutativity of convolutions asserted in (b) of Lemma 1.1, imply

$$\underset{\sim}{\sigma} * \underset{\sim}{\epsilon}' = \underset{\sim}{\sigma}' * \underset{\sim}{\epsilon} \text{ on } R \times \overset{\infty}{T} . \tag{2.40}$$

Now note from hypothesis (a) and Definition 1.2 that

$$\underset{\sim}{u}(\cdot, 0) = \underset{\sim}{u}'(\cdot, 0) = \underset{\sim}{\dot{u}}(\cdot, 0) = \underset{\sim}{\dot{u}}'(\cdot, 0) = \underset{\sim}{0} \text{ on } R .$$

Consequently, two successive applications of (b) in Lemma 1.2 give

$$\underset{\sim}{\dot{u}} * \underset{\sim}{u}' = \underset{\sim}{u} * \underset{\sim}{\dot{u}}', \ \underset{\sim}{\ddot{u}} * \underset{\sim}{u}' = \underset{\sim}{u} * \underset{\sim}{\ddot{u}}' \text{ on } R \times \mathring{T}^{+} . \tag{2.41}$$

Combine (2.39), (2.40), and (2.41) to obtain

$$\nabla \cdot \underset{\sim}{v}(\underset{\sim}{x}) = [\underset{\sim}{f}' * \underset{\sim}{u}](\underset{\sim}{x}, t) - [\underset{\sim}{f} * \underset{\sim}{u}'](\underset{\sim}{x}, t) \text{ for every } \underset{\sim}{x} \in R . \tag{2.42}$$

From hypotheses (a), (b), Lemma 2.2, and (2.37), one infers that $\underset{\sim}{v}$ has bounded support. This being the case, (2.42) and the continuity of $\underset{\sim}{f}' * \underset{\sim}{u}$ and $\underset{\sim}{f} * \underset{\sim}{u}'$ on $\overline{R} \times \overset{\infty}{T}$ assured by Lemma 1.1 imply that $\nabla \cdot \underset{\sim}{v}$ is properly integrable on R. The preceding observations enable one to apply the divergence theorem (Theorem 1.1) to $\underset{\sim}{v}$ on R. In this manner and by recourse to (2.37), (2.42), and (1.18) one confirms that (2.36) holds. This completes the proof since t was chosen arbitrarily.

It is worth mentioning that the foregoing argument, in contrast to Graffi's [4] proof (which is confined to bounded regions), avoids the use of the Laplace transform.

3. Basic singular solutions. Love's integral identity for the displacements and its counterpart for the stresses.

In this section, which is partly expository, we first cite the fundamental singular solution of the field equations in elastodynamics. This solution, due to Stokes [17], corresponds to the problem of a time-dependent concentrated load at a point of a medium occupying the entire space. We then establish certain relevant properties of Stokes' solution and of the associated dynamic doublet solutions. The foregoing singular states are subsequently used to establish in an economical manner Love's [18] integral identity for displacement fields of elastodynamic states with a quiescent past, as well as an analogous identity for the associated fields of stress. The results thus obtained, which are applicable also to unbounded regions, are essential preliminaries to the construction of integral representations for the solutions of the fundamental boundary-initial value problems in dynamic elasticity, carried out in Section 4.

We denote by

$$S^k(\underset{\sim}{x}, t; \underset{\sim}{y} \,|\, g) = [\underset{\sim}{u}^k(\underset{\sim}{x}, t; \underset{\sim}{y} \,|\, g), \underset{\sim}{\sigma}^k(\underset{\sim}{x}, t; \underset{\sim}{y} \,|\, g)] \,, \tag{3.1}$$

for every $(\underset{\sim}{x}, t) \in E_{\underset{\sim}{y}} \times \overset{\infty}{T}$, the values at $(\underset{\sim}{x}, t)$ of the state whose displacement and stress field is given by Stokes' [17] solution[1] appropriate to a concentrated load acting at $\underset{\sim}{y}$ parallel to the x_k-axis. Here $\underset{\sim}{e}^k g(t)$ is the load-vector at the instant t, if $\underset{\sim}{e}^k$ is a unit vector

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[1] See also Love's [19] treatise (page 305).

in the x_k-direction. We assume the "force function" g twice continuously differentiable on $(-\infty,\infty)$. The notation used in (3.1) is to convey that the displacements and stresses, for fixed $\underset{\sim}{x}$, t, and $\underset{\sim}{y}$, are (linear) functionals of g. Since

$$s^k(\underset{\sim}{x},t;\underset{\sim}{y}\,|g)=s^k(\underset{\sim}{x}-\underset{\sim}{y},t;\underset{\sim}{0}|g) \text{ for all } (\underset{\sim}{x},t)\in E_{\underset{\sim}{y}}\times\overset{\infty}{T}, \tag{3.2}$$

it suffices to quote Stokes' solution explicitly merely for the special choice $\underset{\sim}{y}=\underset{\sim}{0}$: for every $(\underset{\sim}{x},t)\in E_{\underset{\sim}{0}}\times\overset{\infty}{T}$ one has

$$4\pi\rho u_i^k(\underset{\sim}{x},t;\underset{\sim}{0}\,|g)=\left[\frac{3x_ix_k}{x^3}-\frac{\delta_{ik}}{x}\right]\int_{1/c_1}^{1/c_2}\lambda g(t-\lambda x)d\lambda$$

$$+\frac{x_ix_k}{x^3}\left[\frac{1}{c_1^2}g(t-x/c_1)-\frac{1}{c_2^2}g(t-x/c_2)\right]+\frac{\delta_{ik}}{xc_2^2}g(t-x/c_2), \tag{3.3}$$

$$4\pi\sigma_{ij}^k(\underset{\sim}{x},t;\underset{\sim}{0}\,|g)=$$

$$-6c_2^2\left[\frac{5x_ix_jx_k}{x^5}-\frac{\delta_{ij}x_k+\delta_{ik}x_j+\delta_{jk}x_i}{x^3}\right]\int_{1/c_1}^{1/c_2}\lambda g(t-\lambda x)d\lambda$$

$$+2\left[\frac{6x_ix_jx_k}{x^5}-\frac{\delta_{ij}x_k+\delta_{ik}x_j+\delta_{jk}x_i}{x^3}\right]\left[g(t-x/c_2)-\left(\frac{c_2}{c_1}\right)^2g(t-x/c_1)\right]$$

$$+\frac{2x_ix_jx_k}{x^4c_2}\left[\dot{g}(t-x/c_2)-\left(\frac{c_2}{c_1}\right)^3\dot{g}(t-x/c_1)\right]$$

$$-\frac{x_k\delta_{ij}}{x^3}\left[1-2\left(\frac{c_2}{c_1}\right)^2\right]\left[g(t-x/c_1)+\frac{x}{c_1}\dot{g}(t-x/c_1)\right]$$

$$-\frac{\delta_{ik}x_j+\delta_{jk}x_i}{x^3}\left[g(t-x/c_2)+\frac{x}{c_2}\dot{g}(t-x/c_2)\right]. \tag{3.4}$$

Here and in the sequel x stands for $|\underset{\sim}{x}|$. The displacements (3.3) are easily seen to agree with the representative displacement field (corresponding to a force parallel to the x_1-axis) appearing in [19] (page 305). The position-dependence of the integration limits in Stokes' original formulas has, for convenience, been eliminated through a change of the integration variable. The stresses (3.4) are readily found from (3.3) by use of (1.11).

Stokes' solution is deduced by Love [18], [19] through a limit process based on a family of time-dependent body-force fields that tends to a concentrated load, in analogy to the limit treatment by Kelvin and Tait [20] (page 279) of the corresponding elastostatic problem.[1] We now adopt

Definition 3.1. (The Stokes-state). *Let* $\underset{\sim}{y} \in E$, $g \in \mathcal{G}^2$, *and let* ρ, c_1, c_2 *satisfy the inequalities* (1.9). *We then call the state* $S^k(\cdot, \cdot; \underset{\sim}{y} | g)$ *defined on* $E_{\underset{\sim}{y}} \times \mathcal{T}^{\infty}$ *by* (3.1) *to* (3.4) *the Stokes-state for a concentrated load at* $\underset{\sim}{y}$ *parallel to the* x_k*-axis, corresponding to the force function* g *and to the material constants* ρ, c_1, c_2.

Theorem 3.1. (Properties of the Stokes-state). *The Stokes-state* $S^k(\cdot, \cdot; \underset{\sim}{y} | g)$ *of Definition* 3.1 *has the properties*:

(a) $S^k(\cdot, \cdot; \underset{\sim}{y} | g) \in \mathcal{E}_o(\underset{\sim}{0}, \rho, c_1, c_2; E_{\underset{\sim}{y}})$;

(b) $\underset{\sim}{u}^k(\underset{\sim}{x}, \cdot; \underset{\sim}{y} | g) = O(|\underset{\sim}{x} - \underset{\sim}{y}|^{-1})$, $\underset{\sim}{\sigma}^k(\underset{\sim}{x}, \cdot; \underset{\sim}{y} | g) = O(|\underset{\sim}{x} - \underset{\sim}{y}|^{-2})$

as $\underset{\sim}{x} \to \underset{\sim}{y}$, *uniformly on* $(-\infty, t]$ *for every* $t \in (-\infty, \infty)$;

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[1] See Sternberg and Eubanks [21] for an explicit version of this limit process. Equations (3.3), (3.4) reduce to the solution of Kelvin's problem if $g(t)=1$ $(-\infty<t<\infty)$.

$$\left.\begin{array}{l} \text{(c)} \quad \lim_{\eta\to 0} \int_{S_\eta(\underset{\sim}{y})} \underset{\sim}{s}^k(\underset{\sim}{x},\cdot;\underset{\sim}{y}|g)\,dA_{\underset{\sim}{x}} = g\underset{\sim}{e}^k \ \underline{\text{on}}\ (-\infty,\infty), \\[2em] \quad\quad \lim_{\eta\to 0} \int_{S_\eta(\underset{\sim}{y})} (\underset{\sim}{x}-\underset{\sim}{y})\wedge \underset{\sim}{s}^k(\underset{\sim}{x},\cdot;\underset{\sim}{y}|g)\,dA_{\underset{\sim}{x}} = \underset{\sim}{0} \ \underline{\text{on}}\ (-\infty,\infty), \end{array}\right\} \quad (3.5)^{1}$$

<u>where</u> $\underset{\sim}{s}^k(\cdot,\cdot;\underset{\sim}{y}|g)$ <u>stands for the traction vector of</u> $S^k(\cdot,\cdot;\underset{\sim}{y}|g)$ <u>acting on the side of</u> $S_\eta(\underset{\sim}{y})$ <u>that faces</u> $\underset{\sim}{y}$, $\underset{\sim}{e}^k$ <u>denotes the unit base-vector in the</u> x_k<u>-direction, and the preceding limits are attained uniformly on</u> $(-\infty, t]$ <u>for every</u> $t\in(-\infty,\infty)$;

(d) <u>if</u> $h\in\mathcal{G}^2$, <u>then</u>

$$h * S^k(\cdot,\cdot;\underset{\sim}{y}|g) = g * S^k(\cdot,\cdot;\underset{\sim}{y}|h) \ \underline{\text{on}}\ E_{\underset{\sim}{y}}\times \overset{\infty}{T}.$$

<u>Proof.</u> In view of the translation identity (3.2) it suffices to take $\underset{\sim}{y}=\underset{\sim}{0}$. To verify (a), note first that (3.3), (3.4), together with the assumed regularity of g, imply that $\underset{\sim}{u}^k(\cdot,\cdot;\underset{\sim}{0}|g)$ and $\underset{\sim}{\sigma}^k(\cdot,\cdot;\underset{\sim}{0}|g)$ satisfy the smoothness requirements in part (a) of Definition 1.2. Moreover, since g vanishes on T^-, one draws from (3.3) that

$$\underset{\sim}{u}^k(\cdot,\cdot;\underset{\sim}{0}|g) = \underset{\sim}{0} \ \text{on}\ E_{\underset{\sim}{0}}\times T^- .$$

To complete the proof of (a) substitute from (3.3), (3.4) into (1.10), (1.11). Property (b) follows at once from (3.3), (3.4) and the hypothesis that $g\in\mathcal{G}^2$.

---

[1] A subscript attached to an "element of area" or an "element of volume" in a surface or volume integral indicates the appropriate space variable of integration.

Consider now part (c). After a brief computation based on (3.4) and (1.18) one finds that

$$\int_{S_\eta} \underset{\sim}{s}^k(\underset{\sim}{x},\tau;\underset{\sim}{0}|g)dA_{\underset{\sim}{x}} = \frac{1}{3}\Big[g(\tau-\eta/c_1)+2g(\tau-\eta/c_2) + \frac{\eta}{c_1}\dot{g}(\tau-\eta/c_1)+\frac{2\eta}{c_2}\dot{g}(\tau-\eta/c_2)\Big]\underset{\sim}{e}^k \tag{3.6}$$

for every $\tau\in(-\infty,\infty)$ and every $\eta>0$, so that

$$\lim_{\eta\to 0}\int_{S_\eta} \underset{\sim}{s}^k(\underset{\sim}{x},\tau;\underset{\sim}{0}|g)dA_{\underset{\sim}{x}}=g(\tau)\underset{\sim}{e}^k \text{ for every } \tau\in(-\infty,\infty) .$$

The uniformity of this limit follows from the inequality

$$\Big|\int_{S_\eta} \underset{\sim}{s}^k(\underset{\sim}{x},\tau;\underset{\sim}{0}|g)dA_{\underset{\sim}{x}}-g(\tau)\underset{\sim}{e}^k\Big| \le \frac{2\eta}{c_2}\max_{(-\infty,t]}|\dot{g}| ,$$

which holds for every $t\in(-\infty,\infty)$ and every $\eta>0$, provided $\tau\in(-\infty,t]$, by virtue of (3.6) and since g vanishes on $(-\infty,0]$ and is continuously differentiable on $(-\infty,\infty)$. The second of (3.5), for $\underset{\sim}{y}=\underset{\sim}{0}$, is immediate from (3.4) and (1.18).

Finally, property (d) is readily inferred from (3.2), (3.3), (3.4), Definition 1.1, (1.21), and the assumption that g and h are both in $\mathcal{C}^2$. This completes the proof in its entirety.

Definition 3.2. (Dynamic doublet-states). Let $\underset{\sim}{y}\in E$, $g\in\mathcal{C}^3$, and let $S^k(\cdot,\cdot;\underset{\sim}{y}|g)$ be the Stokes-state of Definition 3.1. We call the state defined on $E_{\underset{\sim}{y}}\times\mathcal{T}^\infty$ by

$$S^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|g)=[\underset{\sim}{u}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|g),\ \underset{\sim}{\sigma}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|g)]=S^{k}_{,\ell}(\cdot,\cdot;\underset{\sim}{y}|g) \qquad (3.7)^{1}$$

<u>the dynamic doublet-state for the pole</u> $\underset{\sim}{y}$, <u>corresponding to the</u> x_k-<u>axis and the</u> x_ℓ-<u>axis, the force function</u> g, <u>as well as to the material constants</u> ρ, c_1, c_2.

From (3.7), (3.2) follows

$$S^{k\ell}(\underset{\sim}{x},t;\underset{\sim}{y}|g)=S^{k\ell}(\underset{\sim}{x}-\underset{\sim}{y},t;\underset{\sim}{0}|g) \text{ for every } (\underset{\sim}{x},t)\in E_{\underset{\sim}{y}}\times \overset{\infty}{T}. \qquad (3.8)$$

We list next the cartesian components of displacement and stress belonging to $S^{k\ell}(\cdot,\cdot;\underset{\sim}{0}|g)$, which may be computed from (3.3), (3.4) by means of (3.7). For every $(\underset{\sim}{x},t)\in E_{\underset{\sim}{0}}\times\overset{\infty}{T}$ one thus obtains

$$4\pi\rho u_i^{k\ell}(\underset{\sim}{x},t;\underset{\sim}{0}|g)=$$

$$-3\left[\frac{5x_ix_kx_\ell}{x^5}-\frac{\delta_{ik}x_\ell+\delta_{i\ell}x_k+\delta_{k\ell}x_i}{x^3}\right]\int_{1/c_1}^{1/c_2}\lambda g(t-\lambda x)\,d\lambda$$

$$-\left[\frac{6x_ix_kx_\ell}{x^5}-\frac{\delta_{ik}x_\ell+\delta_{i\ell}x_k+\delta_{k\ell}x_i}{x^3}\right]\left[\frac{1}{c_1^2}g(t-x/c_1)-\frac{1}{c_2^2}g(t-x/c_2)\right]$$

$$-\frac{\delta_{ik}x_\ell}{x^3c_2^2}\left[g(t-x/c_2)+\frac{x}{c_2}\dot{g}(t-x/c_2)\right]$$

$$-\frac{x_ix_kx_\ell}{x^4}\left[\frac{1}{c_1^3}\dot{g}(t-x/c_1)-\frac{1}{c_2^3}\dot{g}(t-x/c_2)\right], \qquad (3.9)$$

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[1] Recall the differentiation convention (1.20). For functions of more than one position vector, the space differentiation so indicated is always understood to be performed with respect to the coordinates of the first position vector.

$$
\begin{aligned}
4\pi\,\sigma_{ij}^{k\ell}(\underset{\sim}{x},t;\underset{\sim}{0}\,|\,g) = 6c_2^2\Big[& \frac{35x_ix_jx_kx_\ell}{x^7} \\
& - \frac{5(\delta_{ij}x_kx_\ell+\delta_{i\ell}x_jx_k+\delta_{k\ell}x_ix_j+\delta_{jk}x_ix_\ell+\delta_{j\ell}x_ix_k+\delta_{ik}x_jx_\ell)}{x^5} \\
& + \frac{\delta_{ij}\delta_{k\ell}+\delta_{ik}\delta_{j\ell}+\delta_{jk}\delta_{i\ell}}{x^3}\Big]\int_{1/c_1}^{1/c_2}\lambda g(t-\lambda x)\,d\lambda - 2c_2^2\Big[\frac{45x_ix_jx_kx_\ell}{x^7} \\
& - \frac{6(\delta_{ij}x_kx_\ell+\delta_{i\ell}x_jx_k+\delta_{k\ell}x_ix_j+\delta_{jk}x_ix_\ell+\delta_{j\ell}x_ix_k+\delta_{ik}x_jx_\ell)}{x^5} \\
& + \frac{\delta_{ij}\delta_{k\ell}+\delta_{ik}\delta_{j\ell}+\delta_{jk}\delta_{i\ell}}{x^3}\Big]\Big[\frac{1}{c_2^2}g(t-x/c_2)-\frac{1}{c_1^2}g(t-x/c_1)\Big] \\
& + \frac{2}{c_2}\Big[\frac{\delta_{ij}x_kx_\ell+\delta_{i\ell}x_jx_k+\delta_{k\ell}x_ix_j+\delta_{jk}x_ix_\ell+\delta_{j\ell}x_ix_k+\delta_{ik}x_jx_\ell}{x^4} \\
& \qquad - \frac{10x_ix_jx_kx_\ell}{x^6}\Big]\Big[\dot g(t-x/c_2)-\Big(\frac{c_2}{c_1}\Big)^3\dot g(t-x/c_1)\Big] \\
& + \Big[\frac{3\delta_{ij}x_kx_\ell}{x^5}-\frac{\delta_{ij}\delta_{k\ell}}{x^3}\Big]\Big[1-2\Big(\frac{c_2}{c_1}\Big)^2\Big]\Big[g(t-x/c_1)+\frac{x}{c_1}\dot g(t-x/c_1)\Big] \\
& + \Big[\frac{3(\delta_{ik}x_jx_\ell+\delta_{jk}x_ix_\ell)}{x^5}-\frac{\delta_{ik}\delta_{j\ell}+\delta_{jk}\delta_{i\ell}}{x^3}\Big]\Big[g(t-x/c_2)+\frac{x}{c_2}\dot g(t-x/c_2)\Big] \\
& + \frac{\delta_{ik}x_jx_\ell+\delta_{jk}x_ix_\ell}{c_2^2x^3}\ddot g(t-x/c_2)+\frac{\delta_{ij}x_kx_\ell}{c_1^2x^3}\Big[1-2\Big(\frac{c_2}{c_1}\Big)^2\Big]\ddot g(t-x/c_1) \\
& - \frac{2x_ix_jx_kx_\ell}{c_2^2x^5}\Big[\ddot g(t-x/c_2)-\Big(\frac{c_2}{c_1}\Big)^4\ddot g(t-x/c_1)\Big]\,. \qquad (3.10)
\end{aligned}
$$

We observe that if $g(t)=1$ $(t_o \le t < \infty)$, (3.9), (3.10) reduce to the corresponding elastostatic doublet-states[1] for $t \geq t_0 + x/c_2$. In analogy to Theorem 3.1 one has

Theorem 3.2. (Properties of the dynamic doublet-states). *The dynamic doublet-state* $S^{k\ell}(\cdot,\cdot;\mathbf{y}|g)$ *of Definition 3.2 has the properties:*

(a) $S^{k\ell}(\cdot,\cdot;\mathbf{y}|g)\in\mathcal{E}_o(\mathbf{0},\rho,c_1,c_2;E_{\mathbf{y}})$;

(b) $\mathbf{u}^{k\ell}(\mathbf{x},\cdot;\mathbf{y}|g)=O(|\mathbf{x}-\mathbf{y}|^{-2})$, $\boldsymbol{\sigma}^{k\ell}(\mathbf{x},\cdot;\mathbf{y}|g)=O(|\mathbf{x}-\mathbf{y}|^{-3})$ *as* $\mathbf{x}\to\mathbf{y}$, *uniformly on* $(-\infty,t]$ *for every* $t\in(-\infty,\infty)$;

(c) $$\lim_{\eta\to 0}\int_{S_\eta(\mathbf{y})}\mathbf{s}^{k\ell}(\mathbf{x},\cdot;\mathbf{y}|g)\,dA_{\mathbf{x}}=\mathbf{0} \text{ on } (-\infty,\infty),$$

$$\lim_{\eta\to 0}\int_{S_\eta(\mathbf{y})}(\mathbf{x}-\mathbf{y})\wedge\mathbf{s}^{k\ell}(\mathbf{x},\cdot;\mathbf{y}|g)\,dA_{\mathbf{x}}=g\,\epsilon_{jk\ell}\mathbf{e}^j \text{ on } (-\infty,\infty),$$

where $\mathbf{s}^{k\ell}(\cdot,\cdot;\mathbf{y}|g)$ *are the tractions of* $S^{k\ell}(\cdot,\cdot;\mathbf{y}|g)$ *acting on the side of* $S_\eta(\mathbf{y})$ *that faces* $\mathbf{y}$, *while* $\mathbf{e}^j$ *is the unit base-vector in the* x_j*-direction,* $\epsilon_{jk\ell}$ *denotes the usual alternating symbol, and the preceding limits are attained uniformly on* $(-\infty,t]$ *for every* $t\in(-\infty,\infty)$;

(d) *if* $h\in\mathcal{C}^3$, *then*

$$h*S^{k\ell}(\cdot,\cdot;\mathbf{y}|g)=g*S^{k\ell}(\cdot,\cdot;\mathbf{y}|h) \text{ on } E_{\mathbf{y}}\times\overset{\infty}{T}.$$

Proof. Property (a) is a direct consequence of Definition 3.2, Theorem 3.1, and Definition 1.2. Properties (b) and (c) may be

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[1] Cf. [21] (page 150).

established by the same procedures used to verify their counterparts in Theorem 3.1. Finally, (d) may be confirmed directly with the aid of Definition 3.2, Lemma 1.3 and part (d) of Theorem 3.1.

A physical interpretation of the dynamic doublet-states is easily arrived at on the basis of (3.7) and (3.2). In this connection we refer also to Love's [18] discussion of the singular solutions under consideration. In preparation for a proof of Love's integral identity, we introduce next

Lemma 3.1. Let $\underset{\sim}{y} \in E$, $\alpha > 0$,

$$S = [\underset{\sim}{u}, \underset{\sim}{\sigma}] \in \mathcal{E}_o(\underset{\sim}{f}, \rho, c_1, c_2; B_\alpha(\underset{\sim}{y})) ,$$

and suppose $S^k(\cdot, \cdot; \underset{\sim}{y}|g)$ is the Stokes-state of Definition 3.1.

Then, for each $t \in (-\infty, \infty)$,

$$\text{(a)} \quad \lim_{\eta \to 0} \int_{S_\eta(\underset{\sim}{y})} [\underset{\sim}{s} * \underset{\sim}{u}^k(\cdot, \cdot; \underset{\sim}{y}|g)](\underset{\sim}{x}, t)\, dA_{\underset{\sim}{x}} = 0 ,$$

$$\text{(b)} \quad \lim_{\eta \to 0} \int_{S_\eta(\underset{\sim}{y})} [\underset{\sim}{s}^k(\cdot, \cdot; \underset{\sim}{y}|g) * \underset{\sim}{u}](\underset{\sim}{x}, t)\, dA_{\underset{\sim}{x}} = [g * u_k](\underset{\sim}{y}, t) ,$$

where $\underset{\sim}{s}$ and $\underset{\sim}{s}^k(\cdot, \cdot; \underset{\sim}{y}|g)$ are the tractions of S and $S^k(\cdot, \cdot; \underset{\sim}{y}|g)$ acting on the side of $S_\eta(\underset{\sim}{y})$ that faces $\underset{\sim}{y}$.

Proof. The truth of (a) and (b) for $t \in (-\infty, 0]$ is at once apparent from Definition 1.1 and (1.8). Thus choose $t > 0$, hold t fixed for the remainder of the argument, and let $\beta \in (0, \alpha)$. With a view toward proving (a) for the present choice of t, set

$$I_1^k(\eta)=\int_{S_\eta(\underset{\sim}{y})}[\underset{\sim}{s}*\underset{\sim}{u}^k(\cdot,\cdot;\underset{\sim}{y}|g)](\underset{\sim}{x},t)dA_{\underset{\sim}{x}} \text{ for every } \eta\in(0,\beta]$$

and appeal again to Definition 1.1 and (1.8) to see that

$$I_1^k(\eta)=\int_{S_\eta(\underset{\sim}{y})}\int_0^t \underset{\sim}{s}(\underset{\sim}{x},t-\tau)\cdot\underset{\sim}{u}^k(\underset{\sim}{x},\tau;\underset{\sim}{y}|g)d\tau dA_{\underset{\sim}{x}}.$$

Therefore, bearing in mind the present hypotheses, one has the estimate

$$|I_1^k(\eta)|\leq 4\pi\eta^2 tM_1(\eta)M_1^k(\eta) \text{ for every } \eta\in(0,\beta], \tag{3.11}$$

where

$$\left.\begin{aligned} M_1(\eta)&=\max|\underset{\sim}{s}(\underset{\sim}{x},\tau)|,\quad (\underset{\sim}{x},\tau)\in S_\eta(\underset{\sim}{y})\times[0,t],\\ M_1^k(\eta)&=\max|\underset{\sim}{u}^k(\underset{\sim}{x},\tau;\underset{\sim}{y}|g)|,\quad (\underset{\sim}{x},\tau)\in S_\eta(\underset{\sim}{y})\times[0,t]. \end{aligned}\right\} \tag{3.12}$$

The function M_1 is bounded on $[0,\beta]$ by virtue of (1.18) and the continuity of $\underset{\sim}{\sigma}$ on $\overline{B}_\beta(\underset{\sim}{y})\times[0,t]$, whereas

$$M_1^k(\eta)=O(\eta^{-1}) \text{ as } \eta\to 0$$

because of (b) in Theorem 3.1. Hence (3.11), (3.12) imply conclusion (a).

Next, set

$$I_2^k(\eta)=\int_{S_\eta(\underset{\sim}{y})}[\underset{\sim}{s}^k(\cdot,\cdot;\underset{\sim}{y}|g)*\underset{\sim}{u}](\underset{\sim}{x},t)dA_{\underset{\sim}{x}} \text{ for every } \eta\in(0,\beta]$$

and define an auxiliary function $\underset{\sim}{v}$ through

$$\underset{\sim}{v}(\underset{\sim}{x},\tau)=\underset{\sim}{u}(\underset{\sim}{x},\tau)-\underset{\sim}{u}(\underset{\sim}{y},\tau) \text{ for all } (\underset{\sim}{x},\tau)\in\overline{B}_\beta(\underset{\sim}{y})\times[0,\infty). \tag{3.13}$$

Accordingly,

$$|I_2^k(\eta) - [g * u_k](\underset{\sim}{y}, t)| \le \left| \int_{S_\eta(\underset{\sim}{y})} \int_0^t \underset{\sim}{s}^k(\underset{\sim}{x}, t-\tau; \underset{\sim}{y}|g) \cdot \underset{\sim}{v}(\underset{\sim}{x}, \tau) d\tau dA_{\underset{\sim}{x}} \right|$$

$$+ \left| \int_0^t \underset{\sim}{u}(\underset{\sim}{y}, t-\tau) \cdot \left[\int_{S_\eta(\underset{\sim}{y})} \underset{\sim}{s}^k(\underset{\sim}{x}, \tau; \underset{\sim}{y}|g) dA_{\underset{\sim}{x}} - g(\tau) \underset{\sim}{e}^k \right] d\tau \right| . \tag{3.14}$$

The second term in the right-hand member of (3.14) tends to zero with η since this limit may be taken under the time-integral[1] and because of (c) in Theorem 3.1. Consequently,

$$|I_2^k(\eta) - [g * u_k](\underset{\sim}{y}, t)| \le 4\pi \eta^2 t M_2(\eta) M_2^k(\eta) + o(1) \text{ as } \eta \to 0 , \tag{3.15}$$

where

$$\left.\begin{aligned} M_2(\eta) &= \max |\underset{\sim}{v}(\underset{\sim}{x}, \tau)|, \quad (\underset{\sim}{x}, \tau) \in S_\eta(\underset{\sim}{y}) \times [0, t] , \\ M_2^k(\eta) &= \max |\underset{\sim}{s}^k(\underset{\sim}{x}, \tau; \underset{\sim}{y}|g)|, \quad (\underset{\sim}{x}, \tau) \in S_\eta(\underset{\sim}{y}) \times [0, t] \end{aligned}\right\} \tag{3.16}$$

for every $\eta \in (0, \beta]$. From (3.13) and the continuity of $\underset{\sim}{u}$ on $\overline{B}_\beta(\underset{\sim}{y}) \times [0, t]$ follows

$$M_2(\eta) = o(1) \text{ as } \eta \to 0 .$$

On the other hand, (1.18) and (b) of Theorem 3.1 imply

$$M_2^k(\eta) = O(\eta^{-2}) \text{ as } \eta \to 0 .$$

Thus (b) follows from (3.15), (3.16). The proof is now complete.

Theorem 3.3. (Love's integral identity for the displacement field). *Let* R *be a regular region. Suppose:*

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[1] See Mikusinski [2] (page 143).

(a) $S=[\underset{\sim}{u},\underset{\sim}{\sigma}]\in\mathcal{E}_o(\underset{\sim}{f},\rho,c_1,c_2;\bar{R})$;

(b) $\underset{\sim}{u}\in C^{(2)}(\partial R\times\overset{\infty}{T})$, $\underset{\sim}{\sigma}\in C^{(2)}(\partial R\times\overset{\infty}{T})$, $\underset{\sim}{f}\in C^{(2)}(\bar{R}\times\overset{\infty}{T})$.

Further, let $S^k(\cdot,\cdot;\underset{\sim}{y}|g)$ *be the Stokes-state of Definition* 3.1 *for a concentrated load at* $\underset{\sim}{y}$ *parallel to the* x_k-*axis, corresponding to the force function* g *and to the material constants* ρ, c_1, c_2.

Then, for every $(\underset{\sim}{y},t)\in R\times(-\infty,\infty)$,

$$u_k(\underset{\sim}{y},t)=\sum_{i=1}^{3}\int_{\partial R}[u_i^k(\underset{\sim}{x},t;\underset{\sim}{y}|s_i(\underset{\sim}{x},\cdot))-s_i^k(\underset{\sim}{x},t;\underset{\sim}{y}|u_i(\underset{\sim}{x},\cdot))]\,dA_{\underset{\sim}{x}}$$

$$+\sum_{i=1}^{3}\int_{R}u_i^k(\underset{\sim}{x},t;\underset{\sim}{y}|f_i(\underset{\sim}{x},\cdot))\,dV_{\underset{\sim}{x}}\,, \tag{3.17}$$

where $\underset{\sim}{s}$ *and* $\underset{\sim}{s}^k(\cdot,\cdot;\underset{\sim}{y}|g)$ *are the tractions of* S *and* $S^k(\cdot,\cdot;\underset{\sim}{y}|g)$ *acting on* ∂R.

Proof. Note that the integrands in (3.17) involve Stokes-states with the respective force functions $s_i(\underset{\sim}{x},\cdot)$, $u_i(\underset{\sim}{x},\cdot)$, $f_i(\underset{\sim}{x},\cdot)$ and that these integrands may be written in fully explicit form by making the appropriate substitutions for g in (3.3), (3.4) and by recourse to (3.2), (1.18).

The validity of (3.17) for $(\underset{\sim}{y},t)\in R\times(-\infty,0]$ is evident from the fact that both S and $S^k(\cdot,\cdot;\underset{\sim}{y}|g)$ have quiescent pasts. Choose $(\underset{\sim}{y},t)\in R\times(0,\infty)$, hold $(\underset{\sim}{y},t)$ fixed until further notice, take $\alpha>0$ such that $\bar{B}_\alpha(\underset{\sim}{y})\subset R$, and set

$$R_\eta=R-\bar{B}_\eta(\underset{\sim}{y}) \text{ for every } \eta\in(0,\alpha)\,.$$

Let $h \in \mathcal{C}^2$ and assume h does not vanish identically on $[0, \infty)$. From (3.2), (3.3), (3.4) one then infers

$$\underset{\sim}{u}^k(\cdot,\cdot;\underset{\sim}{y}|h)=\underset{\sim}{\sigma}^k(\cdot,\cdot;\underset{\sim}{y}|h)=\underset{\sim}{0} \text{ on } (E-B_{c_1\tau}(\underset{\sim}{y}))\times[0,\tau] \text{ for every } \tau>0,$$

while (a) of Theorem 3.1 ensures that the body-force field of $S^k(\cdot,\cdot;\underset{\sim}{y}|h)$ vanishes on $E_{\underset{\sim}{y}}\times \overset{\infty}{T}$. In view of the preceding observations and hypothesis (a), one is entitled to apply the reciprocal theorem (Theorem 2.2) to the pair of states $\mathcal{S}$ and $S^k(\cdot,\cdot;\underset{\sim}{y}|h)$ on R_η. Thus

$$\int_{\partial R_\eta}[\underset{\sim}{s}*\underset{\sim}{u}^k(\cdot,\cdot;\underset{\sim}{y}|h)](\underset{\sim}{x},t)dA_{\underset{\sim}{x}}+\int_{R_\eta}[\underset{\sim}{f}*\underset{\sim}{u}^k(\cdot,\cdot;\underset{\sim}{y}|h)](\underset{\sim}{x},t)dV_{\underset{\sim}{x}}=$$

$$\int_{\partial R_\eta}[\underset{\sim}{s}^k(\cdot,\cdot;\underset{\sim}{y}|h)*\underset{\sim}{u}](\underset{\sim}{x},t)dA_{\underset{\sim}{x}} \text{ for every } \eta\in(0,\alpha)\,, \tag{3.18}$$

where $\underset{\sim}{s}$ and $\underset{\sim}{s}^k(\cdot,\cdot;\underset{\sim}{y}|h)$ are the respective tractions acting on ∂R_η.

Next, pass to the limit as $\eta\to 0$ in (3.18) and use Lemma 3.1 to conclude that

$$[h*u_k](\underset{\sim}{y},t)=\int_R[\underset{\sim}{f}*\underset{\sim}{u}^k(\cdot,\cdot;\underset{\sim}{y}|h)](\underset{\sim}{x},t)dV_{\underset{\sim}{x}}$$

$$+\int_{\partial R}\left\{[\underset{\sim}{s}*\underset{\sim}{u}^k(\cdot,\cdot;\underset{\sim}{y}|h)](\underset{\sim}{x},t)-[\underset{\sim}{s}^k(\cdot,\cdot;\underset{\sim}{y}|h)*\underset{\sim}{u}](\underset{\sim}{x},t)\right\}dA_{\underset{\sim}{x}}\,. \tag{3.19}$$

From (3.19), hypotheses (a), (b), conclusion (d) in Theorem 3.1, as well as (1.21), (1.8) and (b), (d) in Lemma 1.1, one now draws

$$[h*u_k](\underset{\sim}{y},t)=\sum_{i=1}^{3}\int_R [h*u_i^k(\circ,\cdot;\underset{\sim}{y}|f_i(\underset{\sim}{x},\cdot))](\underset{\sim}{x},t)dV_{\underset{\sim}{x}}$$

$$+\sum_{i=1}^{3}\int_{\partial R}[h*\{u_i^k(\cdot,\cdot;\underset{\sim}{y}|s_i(\underset{\sim}{x},\circ))-s_i^k(\cdot,\cdot;\underset{\sim}{y}|u_i(\underset{\sim}{x},\cdot))\}](\underset{\sim}{x},t)dA_{\underset{\sim}{x}}. \qquad (3.20)$$

If R is unbounded, it follows from (3.2), (3.3), and the fact that S is a state with a quiescent past, that

$$\sum_{i=1}^{3}u_i^k(\underset{\sim}{x},\tau;\underset{\sim}{y}|f_i(\underset{\sim}{x},\cdot))=0 \qquad (3.21)$$

for every $(\underset{\sim}{x},\tau)\in(\bar{R}-B_{c_1t}(\underset{\sim}{y}))\times[0,t]$. Similarly, if in addition ∂R is unbounded,

$$\sum_{i=1}^{3}u_i^k(\underset{\sim}{x},\tau;\underset{\sim}{y}|s_i(\underset{\sim}{x},\cdot))=\sum_{i=1}^{3}s_i^k(\underset{\sim}{x},\tau;\underset{\sim}{y}|u_i(\underset{\sim}{x},\circ))=0 \qquad (3.22)$$

for every $(\underset{\sim}{x},\tau)\in(\overset{*}{\partial}R-B_{c_1t}(\underset{\sim}{y}))\times[0,t]$. Because of (3.21), (3.22), the integrands in (3.20) are of bounded support. Interchanging the orders of the space-integrations and convolutions in (3.20), as is permissible in the present circumstances[1], and using again the distributivity of the convolution ((d) in Lemma 1.1), one arrives at

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[1] This reversal is trivially justified for the surface-integrals in (3.20) because of the regularity of the integrands; in the case of the improper volume integrals, whose integrands are singular at $\underset{\sim}{y}$, the reversal is easily legitimized by an elementary limit process.

$$\left[h*\left\{u_k(\underset{\sim}{y},\cdot)-\sum_{i=1}^{3}\int_{\partial R}[u_i^k(\underset{\sim}{x},\cdot;\underset{\sim}{y}|s_i(\underset{\sim}{x},\cdot))-s_i^k(\underset{\sim}{x},\cdot;\underset{\sim}{y}|u_i(\underset{\sim}{x},\cdot))]\,dA_{\underset{\sim}{x}}\right.\right.$$

$$\left.\left.-\sum_{i=1}^{3}\int_{R}u_i^k(\underset{\sim}{x},\cdot;\underset{\sim}{y}|f_i(\underset{\sim}{x},\cdot))dV_{\underset{\sim}{x}}\right\}\right](t)=0\ . \tag{3.23}$$

Since $(\underset{\sim}{y},t)$ was chosen arbitrarily in $R\times(0,\infty)$, (3.23) holds for all $(\underset{\sim}{y},t)\in R\times(0,\infty)$. The term within braces in (3.23) is readily shown to be continuous on $R\times[0,\infty)$, whereas h, by assumption, is continuous on $[0,\infty)$ and does not vanish identically. Thus, the desired conclusion now follows from (e) in Lemma 1.1. This completes the proof.

The integral identity (3.17) represents an extension to elastodynamics of the corresponding formula due to Kirchhoff [22] (1882) for the scalar wave equation. At the same time (3.17) is a dynamic counterpart of Somigliana's [23] (1889) integral identity in the equilibrium theory.[1] A result similar to (3.17), but confined to two-dimensional elastodynamics, was deduced by Volterra [24] (1894). Love [18] (1904) sketched a proof of (3.17), applicable to bounded regions, with the aid of Betti's elastostatic reciprocal theorem, treating the inertia forces as body forces. A somewhat more detailed derivation of (3.17) along these lines may be found in a recent dissertation by DeHoop [25] (1958). Somigliana [26] (1906) arrived at a closely related integral identity by different means, taking Kirchhoff's formula as his point of departure.

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[1] See also Love [19] (page 245).

A precise statement of Theorem 3.3, which also covers unbounded domains, is not available in the previous literature, so far as we are aware. Further, the present proof, which rests on the dynamic reciprocal theorem, would appear to be more direct and more explicit than the proofs referred to above. Our next objective consists in establishing an identity analogous to (3.17), for the stresses of an elastodynamic state with a quiescent past. To this end we require

Lemma 3.2. Let $\underset{\sim}{y} \in E$, $\alpha > 0$,

$$S = [\underset{\sim}{u}, \underset{\sim}{\sigma}] \in \mathcal{E}_o(\underset{\sim}{f}, \rho, c_1, c_2; B_\alpha(\underset{\sim}{y})),$$

and suppose $S^{k\ell}(\cdot, \cdot; \underset{\sim}{y}|g)$ is the dynamic doublet-state of Definition 3.2 for the pole $\underset{\sim}{y}$, corresponding to the x_k-axis and the x_ℓ-axis, the force function g, as well as to the material constants ρ, c_1, c_2.

Then, for each $t \in (-\infty, \infty)$,

(a) $$\lim_{\eta \to 0} \int_{S_\eta(\underset{\sim}{y})} [\underset{\sim}{s} * \underset{\sim}{u}^{k\ell}(\cdot, \cdot; \underset{\sim}{y}|g)](\underset{\sim}{x}, t) dA_{\underset{\sim}{x}} =$$

$$\frac{1}{15}\left[g * \{(3 - 8c^2) u_{i,i} \delta_{k\ell} + 2(3 + 2c^2) u_{(k,\ell)}\}\right](\underset{\sim}{y}, t),$$

(b) $$\lim_{\eta \to 0} \int_{S_\eta(\underset{\sim}{y})} [\underset{\sim}{s}^{k\ell}(\cdot, \cdot; \underset{\sim}{y}|g) * \underset{\sim}{u}](\underset{\sim}{x}, t) dA_{\underset{\sim}{x}} =$$

$$\frac{1}{15}\left[g * \{(3 - 8c^2) u_{i,i} \delta_{k\ell} - (9 - 4c^2) u_{(k,\ell)}\}\right](\underset{\sim}{y}, t) - \left[g * u_{[k,\ell]}\right](\underset{\sim}{y}, t),$$

where $c=c_2/c_1$, while $\underset{\sim}{s}$ and $\underset{\sim}{s}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|g)$ are the tractions of S and $S^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|g)$ acting on the side of $S_\eta(\underset{\sim}{y})$ that faces $\underset{\sim}{y}$.

Proof. If $t\in(-\infty, 0]$, conclusions (a) and (b) follow at once from Definition 1.1 and (1.8). Thus fix $t\in(0,\infty)$ for the remainder of the argument. Bearing in mind that $g\in\mathcal{G}^3$, one infers from (3.8), (3.9), (3.10), (1.18), after a tedious computation, that

$$\lim_{\eta\to 0}\int_{S_\eta(\underset{\sim}{y})} u_i^{k\ell}(\underset{\sim}{x},\cdot;\underset{\sim}{y}|g)n_j(\underset{\sim}{x})dA_{\underset{\sim}{x}} =$$

$$\frac{1}{15\rho c_2^2}\left[(c^2-1)(\delta_{ij}\delta_{k\ell}+\delta_{i\ell}\delta_{jk})+(c^2+4)\delta_{ik}\delta_{j\ell}\right]g \text{ on } [0,t], \tag{3.24}$$

$$\lim_{\eta\to 0}\int_{S_\eta(\underset{\sim}{y})} (x_j-y_j)s_i^{k\ell}(\underset{\sim}{x},\cdot;\underset{\sim}{y}|g)dA_{\underset{\sim}{x}} =$$

$$\frac{1}{15}\left[(3-8c^2)\delta_{ij}\delta_{k\ell}-2(6-c^2)\delta_{ik}\delta_{j\ell}+(3+2c^2)\delta_{i\ell}\delta_{jk}\right]g \text{ on } [0,t], \tag{3.25}$$

where n_j are the components of the inner unit normal of $S_\eta(\underset{\sim}{y})$, and the limits in (3.24), (3.25) are attained uniformly on $[0,t]$.

Next, let $\beta\in(0,\alpha)$, set

$$I_1^{k\ell}(\eta)=\int_{S_\eta(\underset{\sim}{y})}[\underset{\sim}{s}*\underset{\sim}{u}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|g)](\underset{\sim}{x},t)dA_{\underset{\sim}{x}} \text{ for every } \eta\in(0,\beta],$$

and define $\underset{\sim}{\varphi}$ through

$$\underset{\sim}{\varphi}(\underset{\sim}{x},\tau)=\underset{\sim}{\sigma}(\underset{\sim}{x},\tau)-\underset{\sim}{\sigma}(\underset{\sim}{y},\tau) \text{ for all } (\underset{\sim}{x},\tau)\in\overline{B}_\beta(\underset{\sim}{y})\times[0,t]. \tag{3.26}$$

Then Definition 1.1, (1.8), (1.18), and (1.11) enable one to conclude that

$$I_1^{k\ell}(\eta) - \frac{1}{15}\left[g * \left\{(3-8c^2)u_{i,i}\delta_{k\ell} + 2(3+2c^2)u_{(k,\ell)}\right\}\right](\underset{\sim}{y}, t) =$$

$$\int\limits_{S_\eta(\underset{\sim}{y})} \int\limits_0^t u_i^{k\ell}(\underset{\sim}{x}, \tau; \underset{\sim}{y}|g)\varphi_{ij}(\underset{\sim}{x}, t-\tau)n_j(\underset{\sim}{x})d\tau dA_{\underset{\sim}{x}}$$

$$+ \int\limits_0^t \sigma_{ij}(\underset{\sim}{y}, t-\tau)\left\{\int\limits_{S_\eta(\underset{\sim}{y})} u_i^{k\ell}(\underset{\sim}{x}, \tau; \underset{\sim}{y}|g)n_j(\underset{\sim}{x})dA_{\underset{\sim}{x}}\right.$$

$$\left. - \frac{1}{15\rho c_2^2}\left[(c^2-1)(\delta_{ij}\delta_{k\ell} + \delta_{i\ell}\delta_{jk}) + (c^2+4)\delta_{ik}\delta_{j\ell}\right]g(\tau)\right\} d\tau .$$

Equation (3.24) consequently furnishes the estimate

$$\left|I_1^{k\ell}(\eta) - \frac{1}{15}\left[g * \left\{(3-8c^2)u_{i,i}\delta_{k\ell} + 2(3+2c^2)u_{(k,\ell)}\right\}\right](\underset{\sim}{y}, t)\right| \le$$

$$4\pi\eta^2 t M_1(\eta) M_1^{k\ell}(\eta) + o(1) \text{ as } \eta \to 0 , \tag{3.27}$$

where

$$\left.\begin{aligned} M_1(\eta) &= \max\sqrt{\underset{\sim}{\underset{\sim}{\varphi}}(\underset{\sim}{x}, \tau)\cdot\underset{\sim}{\underset{\sim}{\varphi}}(\underset{\sim}{x}, \tau)}, \quad (\underset{\sim}{x}, \tau) \in S_\eta(\underset{\sim}{y}) \times [0, t], \\ M_1^{k\ell}(\eta) &= \max|\underset{\sim}{u}^{k\ell}(\underset{\sim}{x}, \tau; \underset{\sim}{y}|g)|, \quad (\underset{\sim}{x}, \tau) \in S_\eta(\underset{\sim}{y}) \times [0, t] \end{aligned}\right\} \tag{3.28}$$

for every $\eta \in (0, \beta]$. From (3.26), (3.28), and the continuity of $\underset{\sim}{\sigma}$ on $\overline{B}_\beta(\underset{\sim}{y}) \times [0, t]$ follows

$$M_1(\eta) = o(1) \text{ as } \eta \to 0 ,$$

whereas (3.28) and (b) of Theorem 3.2 imply

$$M_1^{k\ell}(\eta) = O(\eta^{-2}) \text{ as } \eta \to 0.$$

Thus, combining (3.27) and (3.28), one confirms (a).

To verify (b), let

$$I_2^{k\ell}(\eta)=\int_{S_\eta(\mathbf{y})}[\mathbf{s}^{k\ell}(\cdot,\cdot;\mathbf{y}|g)*\mathbf{u}](\mathbf{x},t)dA_{\mathbf{x}} \text{ for every } \eta\in(0,\beta]. \tag{3.29}$$

From the assumed regularity of $\mathbf{u}$ on $B_\alpha(\mathbf{y})\times(-\infty,\infty)$ one draws, for every $(\mathbf{x},\tau)\in B_\alpha(\mathbf{y})\times(-\infty,\infty)$,

$$\mathbf{u}(\mathbf{x},\tau)=\mathbf{u}(\mathbf{y},\tau)+\mathbf{u}_{,i}(\mathbf{y},\tau)(x_i-y_i)+\mathbf{v}(\mathbf{x},\tau), \tag{3.30}$$

where

$$\mathbf{v}\in C^2(B_\alpha(\mathbf{y})\times\overset{\infty}{T}),\ \mathbf{v}(\mathbf{x},\cdot)=O(|\mathbf{x}-\mathbf{y}|^2) \text{ as } \mathbf{x}\to\mathbf{y}, \tag{3.31}$$

uniformly on $[0,t]$. On the basis of (3.29), (3.30), the first of (3.31), Definition 1.1, (1.8), and (b) in Lemma 1.1, one arrives at

$$I_2^{k\ell}(\eta)-\frac{1}{15}\Big[g*\Big\{(3-8c^2)u_{i,i}\delta_{k\ell}-(9-4c^2)u_{(k,\ell)}\Big\}\Big](\mathbf{y},t)+\Big[g*u_{[k,\ell]}\Big](\mathbf{y},t)=$$

$$\int_0^t \mathbf{u}(\mathbf{y},t-\tau)\cdot\int_{S_\eta(\mathbf{y})}\mathbf{s}^{k\ell}(\mathbf{x},\tau;\mathbf{y}|g)dA_{\mathbf{x}}d\tau+\int_{S_\eta(\mathbf{y})}\int_0^t \mathbf{v}(\mathbf{x},t-\tau)\cdot\mathbf{s}^{k\ell}(\mathbf{x},\tau;\mathbf{y}|g)d\tau dA_{\mathbf{x}}$$

$$+\int_0^t u_{i,j}(\mathbf{y},t-\tau)\Big\{\int_{S_\eta(\mathbf{y})}(x_j-y_j)s_i^{k\ell}(\mathbf{x},\tau;\mathbf{y}|g)dA_{\mathbf{x}}$$

$$-\frac{1}{15}\Big[(3-8c^2)\delta_{ij}\delta_{k\ell}-2(6-c^2)\delta_{ik}\delta_{j\ell}+(3+2c^2)\delta_{i\ell}\delta_{jk}\Big]g(\tau)\Big\}d\tau. \tag{3.32}$$

The first and third terms in the right-hand member of (3.32) tend to zero with η because of (c) in Theorem 3.2 and (3.25), respectively. Hence

$$|I_2^{k\ell}(\eta) - \frac{1}{15}\Big[g*\Big\{(3-8c^2)u_{i,i}\delta_{k\ell} - (9-4c^2)u_{(k,\ell)}\Big\}\Big](\underset{\sim}{y},t) + \Big[g*u_{[k,\ell]}\Big](\underset{\sim}{y},t)|$$
$$\leq 4\pi\eta^2 t M_2(\eta) M_2^{k\ell}(\eta) + o(1) \text{ as } \eta\to 0 , \tag{3.33}$$

where

$$\left.\begin{aligned} M_2(\eta) &= \max|\underset{\sim}{v}(\underset{\sim}{x},\tau)|, \ (\underset{\sim}{x},\tau)\in S_\eta(\underset{\sim}{y})\times[0,t] , \\ M_2^{k\ell}(\eta) &= \max|\underset{\sim}{s}^{k\ell}(\underset{\sim}{x},\tau;\underset{\sim}{y}\,|g)|, \ (\underset{\sim}{x},\tau)\in S_\eta(\underset{\sim}{y})\times[0,t], \end{aligned}\right\} \tag{3.34}$$

for every $\eta\in(0,\beta]$. Now invoke (3.31) to see that

$$M_2(\eta) = O(\eta^2) \text{ as } \eta\to 0 ,$$

and call on (1.18), as well as (b) in Theorem 3.2 to justify that

$$M_2^{k\ell}(\eta) = O(\eta^{-3}) \text{ as } \eta\to 0 .$$

Conclusion (b) thus follows from (3.33), (3.34). This completes the proof.

Theorem 3.4. (Integral identity for the stress field). Let R be a regular region. Suppose:

(a) $S = [\underset{\sim}{u},\underset{\sim}{\sigma}] \in \mathcal{E}_o(\underset{\sim}{f}, \rho, c_1, c_2; \overline{R})$;

(b) $\underset{\sim}{u}\in C^{(3)}(\partial R\times\overset{\infty}{T})$, $\underset{\sim}{\sigma}\in C^{(3)}(\partial R\times\overset{\infty}{T})$, $\underset{\sim}{f}\in C^{(3)}(\overline{R}\times\overset{\infty}{T})$.

Further, let $S^{k\ell}(\cdot,\cdot;\underset{\sim}{y}\,|g)$ be the dynamic doublet-state of Definition 3.2 for the pole $\underset{\sim}{y}$, corresponding to the x_k-axis and the x_ℓ-axis, the force function g, as well as to the material constants ρ, c_1, c_2. Define the state

$$\overline{S}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|g) = [\overline{\underset{\sim}{u}}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}\,|g), \overline{\underset{\sim}{\sigma}}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|g)]$$

through

$$\bar{S}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|g)=\rho(c_1^2-2c_2^2)\,S^{ii}(\cdot,\cdot;\underset{\sim}{y}|g)\,\delta_{k\ell}$$

$$+2\rho c_2^2 S^{(k\ell)}(\cdot,\cdot;\underset{\sim}{y}|g) \text{ on } E_{\underset{\sim}{y}}\times \overset{\infty}{T} . \tag{3.35}$$

Then, for every $(\underset{\sim}{y},t)\in R\times(-\infty,\infty)$,

$$\sigma_{k\ell}(\underset{\sim}{y},t)=\sum_{i=1}^{3}\int_{\partial R}[\bar{s}_i^{k\ell}(\underset{\sim}{x},t;\underset{\sim}{y}|u_i(\underset{\sim}{x},\cdot))-\bar{u}_i^{k\ell}(\underset{\sim}{x},t;\underset{\sim}{y}|s_i(\underset{\sim}{x},\cdot))]\,dA_{\underset{\sim}{x}}$$

$$-\sum_{i=1}^{3}\int_{R}\bar{u}_i^{k\ell}(\underset{\sim}{x},t;\underset{\sim}{y}|f_i(\underset{\sim}{x},\cdot))\,dV_{\underset{\sim}{x}} , \tag{3.36}$$

where $\underset{\sim}{s}$ *and* $\bar{\underset{\sim}{s}}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|g)$ *are the tractions of* S *and* $\bar{S}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|g)$ *acting on* ∂R.

Proof. Since the following argument is quite similar to the one used in proving Theorem 3.3, it may be summarized in condensed form. If $(\underset{\sim}{y},t)\in R\times(-\infty,0]$, (3.36) is a consequence of the fact that S and $\bar{S}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|g)$ have quiescent pasts. Hence choose $(\underset{\sim}{y},t)\in R\times(0,\infty)$ and hold $(\underset{\sim}{y},t)$ fixed until further notice. Take $\alpha>0$ such that $\bar{B}_\alpha(\underset{\sim}{y})\subset R$ and set

$$R_\eta = R-\bar{B}_\eta(\underset{\sim}{y}) \text{ for every } \eta\in(0,\alpha) .$$

Let $h\in\mathcal{G}^3$ and assume h does not vanish identically on $[0,\infty)$. Observe from (3.8), (3.9), (3.10), and (a) of Theorem 3.2 that, for every $\eta\in(0,\alpha)$, $S^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|h)$ qualifies as a candidate for the state S of Theorem 2.2 on R_η. Thus, in view of the present hypothesis (a), the reciprocal theorem (Theorem 2.2) is applicable to the present pair of states S and $\underset{\sim}{S}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|h)$ on R_η. On passing to the limit as $\eta \to 0$ in the resulting identity, and using Lemma 3.2, one arrives at

$$[h*u_{k,\ell}](\underset{\sim}{y},t) = -\int_R [\underset{\sim}{f}*\underset{\sim}{u}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|h)](\underset{\sim}{x},t)\,dV_{\underset{\sim}{x}}$$

$$+\int_{\partial R}\left\{[\underset{\sim}{s}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|h)*\underset{\sim}{u}](\underset{\sim}{x},t)-[\underset{\sim}{s}*\underset{\sim}{u}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|h)](\underset{\sim}{x},t)\right\}dA_{\underset{\sim}{x}}. \qquad (3.37)$$

From (3.37), hypotheses (a) and (b), conclusion (d) in Theorem 3.2, and (b), (d) in Lemma 1.1, one now draws

$$[h*u_{k,\ell}](\underset{\sim}{y},t) = -\sum_{i=1}^{3}\int_R [h*u_i^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|f_i(\underset{\sim}{x},\cdot))](\underset{\sim}{x},t)\,dV_{\underset{\sim}{x}}$$

$$+\sum_{i=1}^{3}\int_{\partial R}[h*\{s_i^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|u_i(\underset{\sim}{x},\cdot))-u_i^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|s_i(\underset{\sim}{x},\cdot))\}](\underset{\sim}{x},t)\,dA_{\underset{\sim}{x}}. \qquad (3.38)$$

After permissible reversals of the space-integrations and convolutions involved in (3.38), one finds that

$$\Big[h*\Big\{u_{k,\ell}(\underset{\sim}{y},\cdot)-\sum_{i=1}^{3}\int_{\partial R}[s_i^{k\ell}(\underset{\sim}{x},\cdot;\underset{\sim}{y}|u_i(\underset{\sim}{x},\cdot))-u_i^{k\ell}(\underset{\sim}{x},\cdot;\underset{\sim}{y}|s_i(\underset{\sim}{x},\cdot))]\,dA_{\underset{\sim}{x}}$$

$$+\sum_{i=1}^{3}\int_R u_i^{k\ell}(\underset{\sim}{x},\cdot;\underset{\sim}{y}|f_i(\underset{\sim}{x},\cdot))\,dV_{\underset{\sim}{x}}\Big\}\Big](t)=0\,. \qquad (3.39)$$

Since $(\underset{\sim}{y},t)$ was chosen arbitrarily in $R\times(0,\infty)$, equation (3.39) holds for all $(\underset{\sim}{y},t)\in R\times(0,\infty)$. But the term within braces in (3.39) is continuous on $R\times[0,\infty)$, while h is continuous on $[0,\infty)$ and does not vanish identically, so that (3.39) and (e) of Lemma 1.1 furnish

$$u_{k,\ell}(\underset{\sim}{y}, t) = \sum_{i=1}^{3} \int_{\partial R} [s_i^{k\ell}(\underset{\sim}{x}, t; \underset{\sim}{y} | u_i(\underset{\sim}{x}, \cdot)) - u_i^{k\ell}(\underset{\sim}{x}, t; \underset{\sim}{y} | s_i(\underset{\sim}{x}, \cdot))] dA_{\underset{\sim}{x}}$$

$$- \sum_{i=1}^{3} \int_{R} u_i^{k\ell}(\underset{\sim}{x}, t; \underset{\sim}{y} | f_i(\underset{\sim}{x}, \cdot)) dV_{\underset{\sim}{x}} \tag{3.40}$$

for every $(\underset{\sim}{y}, t) \in R \times (0, \infty)$. The desired conclusion now follows from (3.35), (3.40) and (1.11). This completes the proof.

It is clear that (3.40) may be obtained formally from Love's identity (3.17) by differentiating the latter under the integral signs and by making use of the relations

$$\frac{\partial}{\partial y_\ell} u_i^k(\underset{\sim}{x}, t; \underset{\sim}{y} | g) = -u_i^{k\ell}(\underset{\sim}{x}, t; \underset{\sim}{y} | g) ,$$

$$\frac{\partial}{\partial y_\ell} \sigma_{ij}^k(\underset{\sim}{x}, t; \underset{\sim}{y} | g) = -\sigma_{ij}^{k\ell}(\underset{\sim}{x}, t; \underset{\sim}{y} | g) ,$$

which hold for every $(\underset{\sim}{x}, t) \in E_{\underset{\sim}{y}} \times \overset{\infty}{T}$ because of (3.2) and (3.7). A rigorous proof of Theorem 3.4 based on this alternative procedure is, however, quite cumbersome.

Finally, we remark that (3.40) enables one to write down immediately formulas analogous to (3.17) and (3.36) for the dilatation and rotation fields of an elastodynamic state with a quiescent past. The linear combinations of doublet-states entering the formulas just alluded to are those characteristic of a dynamic center of dilatation and a dynamic center of rotation. Closely related integral identities for the dilatation and rotation were obtained by Tedone [27].

4. Green's states. Integral representations for the solutions to the fundamental boundary-initial value problems of elastodynamics.

In the present section we aim at integral representations for the displacements and stresses of the solutions to the first and second fundamental boundary-initial value problems in classical elastodynamics. The appropriate boundary data consist of the surface displacements in the first problem and of the surface tractions in the second problem. Further, we confine our attention at present to elastodynamic states with a quiescent past.[1]

The integral identities (3.17) and (3.36) involve both the surface displacements and the surface tractions on the boundary of the region at hand. In order to arrive at the desired representations, we need to eliminate from the integrands in (3.17), (3.36) the surface tractions in connection with the first problem and the surface displacements in connection with the second problem. This purpose may be accomplished by means of suitable elastodynamic Green's states. With a view toward the first boundary-initial value problem we introduce

Definition 4.1. (Green's states of the first kind). Let R be a regular region, $\underset{\sim}{y} \in R$, and let $g \in \mathcal{G}^3$. We call

$$\hat{S}^k(\cdot,\cdot;\underset{\sim}{y}|g) = [\hat{\underset{\sim}{u}}^k(\cdot,\cdot;\underset{\sim}{y}|g), \hat{\underset{\sim}{\sigma}}^k(\cdot,\cdot;\underset{\sim}{y}|g)]$$

the displacement Green's states of the first kind and

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[1] See the end of Section 4 for a relaxation of this restriction upon the initial conditions.

$$\hat{S}^{k\ell}(\cdot,\cdot;\mathbf{y}|g)=[\hat{\mathbf{u}}^{k\ell}(\cdot,\cdot;\mathbf{y}|g),\hat{\boldsymbol{\sigma}}^{k\ell}(\cdot,\cdot;\mathbf{y}|g)]$$

<u>the stress Green's states of the first kind for the region</u> R <u>and the pole</u> $\mathbf{y}$, <u>corresponding to the force function</u> g <u>and to the material constants</u> ρ, c_1, c_2, <u>provided</u>:

(a) $\hat{S}^{k}(\cdot,\cdot;\mathbf{y}|g)=S^{k}(\cdot,\cdot;\mathbf{y}|g)+\tilde{S}^{k}(\cdot,\cdot;\mathbf{y}|g)$ <u>on</u> $\overline{R}_{\mathbf{y}}\times\overset{\infty}{T}$,

$\hat{S}^{k\ell}(\cdot,\cdot;\mathbf{y}|g)=\overline{S}^{k\ell}(\cdot,\cdot;\mathbf{y}|g)+\tilde{S}^{k\ell}(\cdot,\cdot;\mathbf{y}|g)$ <u>on</u> $\overline{R}_{\mathbf{y}}\times\overset{\infty}{T}$,

<u>where</u> $S^{k}(\cdot,\cdot;\mathbf{y}|g)$ <u>and</u> $\overline{S}^{k\ell}(\cdot,\cdot;\mathbf{y}|g)$ <u>respectively denote the Stokes-state of Definition</u> 3.1 <u>and the linear combination of doublet-states</u> (3.35);

(b) $\tilde{S}^{k}(\cdot,\cdot;\mathbf{y}|g)=[\tilde{\mathbf{u}}^{k}(\cdot,\cdot;\mathbf{y}|g),\tilde{\boldsymbol{\sigma}}^{k}(\cdot,\cdot;\mathbf{y}|g)]\in\mathcal{E}_o(\mathbf{0},\rho,c_1,c_2;\overline{R})$,

$\tilde{S}^{k\ell}(\cdot,\cdot;\mathbf{y}|g)=[\tilde{\mathbf{u}}^{k\ell}(\cdot,\cdot;\mathbf{y}|g),\tilde{\boldsymbol{\sigma}}^{k\ell}(\cdot,\cdot;\mathbf{y}|g)]\in\mathcal{E}_o(\mathbf{0},\rho,c_1,c_2;\overline{R})$,

$\tilde{\boldsymbol{\sigma}}^{k}(\cdot,\cdot;\mathbf{y}|g)\in C^{(2)}(\partial R\times\overset{\infty}{T})$, $\tilde{\boldsymbol{\sigma}}^{k\ell}(\cdot,\cdot;\mathbf{y}|g)\in C^{(1)}(\partial R\times\overset{\infty}{T})$;

(c) $\tilde{\mathbf{u}}^{k}(\cdot,\cdot;\mathbf{y}|g)=-\mathbf{u}^{k}(\cdot,\cdot;\mathbf{y}|g)$ <u>on</u> $\partial R\times\overset{\infty}{T}$,

$\tilde{\mathbf{u}}^{k\ell}(\cdot,\cdot;\mathbf{y}|g)=-\overline{\mathbf{u}}^{k\ell}(\cdot,\cdot;\mathbf{y}|g)$ <u>on</u> $\partial R\times\overset{\infty}{T}$.

The regular parts $\tilde{S}^{k}(\cdot,\cdot;\mathbf{y}|g)$ and $\tilde{S}^{k\ell}(\cdot,\cdot;\mathbf{y}|g)$ of the displacement and stress Green's states of the first kind are each evidently defined through requirements (b), (c) as the solution to a first boundary-initial value problem for R. Moreover, they are <u>uniquely</u> determined by these conditions because of Theorem 2.1. In contrast, the <u>existence</u> of these regular states, and hence of the corresponding Green's states, is contingent upon the existence of a solution to the first problem for the region under consideration in the presence of sufficiently smooth boundary data. Before proceeding with our immediate task it is convenient to have available

Lemma 4.1. Let R be a regular region, $\underset{\sim}{y} \in R$, $g \in \mathcal{G}^3$, $h \in \mathcal{G}^3$, and let $\tilde{S}^k(\cdot,\cdot;\underset{\sim}{y}|g)$, $\tilde{S}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|g)$ be the regular parts of the Green's states of the first kind introduced in Definition 4.1.

Then:

(a) $h * \tilde{S}^k(\cdot,\cdot;\underset{\sim}{y}|g) = g * \tilde{S}^k(\cdot,\cdot;\underset{\sim}{y}|h)$ on $\overline{R} \times \overset{\infty}{T}$;

(b) $h * \tilde{S}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|g) = g * \tilde{S}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|h)$ on $\overline{R} \times \overset{\infty}{T}$.

Proof. Consider first (a). From Definition 1.2 and (b) in Definition 4.1 one obtains after two successive applications of Lemma 1.2 and Lemma 1.3,

$$\left.\begin{aligned} &h * \tilde{S}^k(\cdot,\cdot;\underset{\sim}{y}|g) \in \mathcal{E}_o(\underset{\sim}{0}, \rho, c_1, c_2; \overline{R}), \\ &g * \tilde{S}^k(\cdot,\cdot;\underset{\sim}{y}|h) \in \mathcal{E}_o(\underset{\sim}{0}, \rho, c_1, c_2; \overline{R}) . \end{aligned}\right\} \quad (4.1)$$

Next, call on (c) in Definition 4.1 and (d) of Theorem 3.1 to see that

$$\begin{aligned} h * \tilde{\underset{\sim}{u}}^k(\cdot,\cdot;\underset{\sim}{y}|g) &= -h * \underset{\sim}{u}^k(\cdot,\cdot;\underset{\sim}{y}|g) \\ &= -g * \underset{\sim}{u}^k(\cdot,\cdot;\underset{\sim}{y}|h) = g * \tilde{\underset{\sim}{u}}^k(\cdot,\cdot;\underset{\sim}{y}|h) \text{ on } \partial R \times \overset{\infty}{T} . \end{aligned} \quad (4.2)$$

Conclusion (a) now follows from (4.1), (4.2) and the uniqueness theorem (Theorem 2.1). The proof of (b) is strictly analogous.

We are now in a position to turn to

Theorem 4.1. (Integral representation for the solution of the first boundary-initial value problem). Let R be a regular region.

Suppose:

(a) $S = [\underset{\sim}{u}, \underset{\sim}{\sigma}] \in \mathcal{E}_o(\underset{\sim}{f}, \rho, c_1, c_2; \bar{R})$;

(b) $\underset{\sim}{u} \in C^{(4)}(\partial R \times \overset{\infty}{T})$, $\underset{\sim}{\sigma} \in C^{(4)}(\partial R \times \overset{\infty}{T})$, $\underset{\sim}{f} \in C^{(4)}(\bar{R} \times \overset{\infty}{T})$.

Further, let $\hat{S}^k(\cdot, \cdot; \underset{\sim}{y}|g)$ and $\hat{S}^{k\ell}(\cdot, \cdot; \underset{\sim}{y}|g)$ be the Green's states of the first kind of Definition 4.1 for the region R and the pole $\underset{\sim}{y}$ corresponding to the force function g and to the material constants ρ, c_1, c_2. If these Green's states exist for all $\underset{\sim}{y} \in R$ and all $g \in \mathcal{G}^3$, then for every $(\underset{\sim}{y}, t) \in R \times \overset{\infty}{T}$,

$$u_k(\underset{\sim}{y}, t) = \sum_{i=1}^{3} \left[\int_R \hat{u}_i^k(\underset{\sim}{x}, t; \underset{\sim}{y} | f_i(\underset{\sim}{x}, \cdot)) dV_{\underset{\sim}{x}} - \int_{\partial R} \hat{s}_i^k(\underset{\sim}{x}, t; \underset{\sim}{y} | u_i(\underset{\sim}{x}, \cdot)) dA_{\underset{\sim}{x}} \right], \tag{4.3}$$

$$\sigma_{k\ell}(\underset{\sim}{y}, t) = -\sum_{i=1}^{3} \left[\int_R \hat{u}_i^{k\ell}(\underset{\sim}{x}, t; \underset{\sim}{y} | f_i(\underset{\sim}{x}, \cdot)) dV_{\underset{\sim}{x}} - \int_{\partial R} \hat{s}_i^{k\ell}(\underset{\sim}{x}, t; \underset{\sim}{y} | u_i(\underset{\sim}{x}, \cdot)) dA_{\underset{\sim}{x}} \right], \tag{4.4}$$

where $\hat{\underset{\sim}{s}}^k(\cdot, \cdot; \underset{\sim}{y}|g)$ and $\hat{\underset{\sim}{s}}^{k\ell}(\cdot, \cdot; \underset{\sim}{y}|g)$ are the tractions of $\hat{S}^k(\cdot, \cdot; \underset{\sim}{y}|g)$ and $\hat{S}^{k\ell}(\cdot, \cdot; \underset{\sim}{y}|g)$ acting on ∂R.

Proof. If $(\underset{\sim}{y}, t) \in R \times T^-$, (4.3) and (4.4) follow trivially from (1.18), Definition 4.1, and hypotheses (a) and (b). Define a function h by setting

$$h(t) = \begin{cases} 0 \text{ for every } t \in (-\infty, 0] \\ t^4/4! \text{ for every } t \in (0, \infty) \end{cases} \tag{4.5}$$

and observe that $h \in \mathcal{G}^3$. Choose $\underset{\sim}{y} \in R$ and note from (4.5), (c) in Definition 4.1, (3.2), (3.3) that if ∂R is unbounded,

$$\tilde{\underset{\sim}{u}}^k(\cdot, \cdot; \underset{\sim}{y}|h) = \underset{\sim}{0} \text{ on } (\partial R - E_{c_1 t}(\underset{\sim}{y})) \times [0, t] \text{ for every } t > 0 . \tag{4.6}$$

From (4.6) and (b) in Definition 4.1 one concludes that $\tilde{S}^k(\cdot,\cdot;\underset{\sim}{y}|h)$ satisfies the conditions imposed on the state S of Theorem 2.2 in hypotheses (a) and (b) of that theorem. Thus, and because of the present hypothesis (a), one may apply the reciprocal theorem (Theorem 2.2) to the pair of states S, $\tilde{S}^k(\cdot,\cdot;\underset{\sim}{y}|h)$ on R. Accordingly, and by virtue of (c) in Definition 4.1,

$$\int_{\partial R} [\underset{\sim}{s} * \underset{\sim}{u}^k(\cdot,\cdot;\underset{\sim}{y}|h)](\underset{\sim}{x},t)\,dA_{\underset{\sim}{x}} = \int_R [\underset{\sim}{f} * \tilde{\underset{\sim}{u}}^k(\cdot,\cdot;\underset{\sim}{y}|h)](\underset{\sim}{x},t)\,dV_{\underset{\sim}{x}}$$

$$- \int_{\partial R} [\tilde{\underset{\sim}{s}}^k(\cdot,\cdot;\underset{\sim}{y}|h) * \underset{\sim}{u}](\underset{\sim}{x},t)\,dA_{\underset{\sim}{x}} \text{ for every } t\in(0,\infty)\,. \tag{4.7}$$

From hypotheses (a) and (b), (3.2), (3.3), (4.5), (b) of Lemma 1.1, and Lemma 1.2 there follows

$$\sigma_{ij} * u_i^k(\cdot,\cdot;\underset{\sim}{y}|h) \in C^{(5)}(\partial R \times \overset{\infty}{T})\,. \tag{4.8}$$

Furthermore, hypotheses (a) and (b), Lemma 1.2, (b) of Lemma 1.1, and (b) in Definition 4.1 imply

$$\left.\begin{aligned} &\underset{\sim}{f} * \tilde{\underset{\sim}{u}}^k(\cdot,\cdot;\underset{\sim}{y}|h) \in C^{(5)}(\bar{R} \times \overset{\infty}{T})\,, \\ &\tilde{\sigma}_{ij}^k(\cdot,\cdot;\underset{\sim}{y}|h) * u_j \in C^{(5)}(\partial R \times \overset{\infty}{T})\,. \end{aligned}\right\} \tag{4.9}$$

Let $t_o > 0$. If R is unbounded, then (4.6), (b) of Definition 4.1, and Lemma 2.2 ensure that there is a bounded set $\Omega(t_o) \subset \bar{R}$ such that

$$\tilde{\underset{\sim}{u}}^k(\cdot,\cdot;\underset{\sim}{y}|h) = \tilde{\underset{\sim}{\sigma}}^k(\cdot,\cdot;\underset{\sim}{y}|h) = \underset{\sim}{0} \text{ on } (\bar{R} - \Omega(t_o)) \times [0, t_o]\,. \tag{4.10}$$

On the other hand, (3.2), (3.3), and (4.5) furnish

$$\underset{\sim}{u}^k(\cdot,\cdot;\underset{\sim}{y}|h)=\underset{\sim}{0} \text{ on } (E-B_{c_1 t_o}(\underset{\sim}{y}))\times[0,t_o]. \tag{4.11}$$

Assertions (4.8) to (4.11), together with (1.18), justify five successive time-differentiations of (4.7) under the integral signs on the interval $(0,t_o)$. Since t_o was chosen arbitrarily in $(0,\infty)$, one thus has

$$\int_{\partial R}[\underset{\sim}{s}*\underset{\sim}{u}^k(\cdot,\cdot;\underset{\sim}{y}|h)]^{(5)}(\underset{\sim}{x},t)dA_{\underset{\sim}{x}}=\int_R[\underset{\sim}{f}*\tilde{\underset{\sim}{u}}^k(\cdot,\cdot;\underset{\sim}{y}|h)]^{(5)}(\underset{\sim}{x},t)dV_{\underset{\sim}{x}}$$

$$-\int_{\partial R}[\tilde{\underset{\sim}{s}}^k(\cdot,\cdot;\underset{\sim}{y}|h)*\underset{\sim}{u}]^{(5)}(\underset{\sim}{x},t)dA_{\underset{\sim}{x}} \text{ for every } t\in(0,\infty). \tag{4.12}$$

Next, appeal to hypotheses (a) and (b), (1.18), (d) in Theorem 3.1, and (a) in Lemma 4.1 to see that

$$\left.\begin{aligned}
&[\underset{\sim}{s}*\underset{\sim}{u}^k(\cdot,\cdot;\underset{\sim}{y}|h)](\underset{\sim}{x},t)=\sum_{i=1}^{3}[h*u_i^k(\cdot,\cdot;\underset{\sim}{y}|s_i(\underset{\sim}{x},\cdot))](\underset{\sim}{x},t)\\
&\qquad\qquad\text{for every } (\underset{\sim}{x},t)\in\overset{*}{\partial}R\times\overset{\infty}{T},\\
&[\underset{\sim}{f}*\tilde{\underset{\sim}{u}}^k(\cdot,\cdot;\underset{\sim}{y}|h)](\underset{\sim}{x},t)=\sum_{i=1}^{3}[h*\tilde{u}_i^k(\cdot,\cdot;\underset{\sim}{y}|f_i(\underset{\sim}{x},\cdot))](\underset{\sim}{x},t)\\
&\qquad\qquad\text{for every } (\underset{\sim}{x},t)\in\overline{R}\times\overset{\infty}{T},\\
&[\tilde{\underset{\sim}{s}}^k(\cdot,\cdot;\underset{\sim}{y}|h)*\underset{\sim}{u}](\underset{\sim}{x},t)=\sum_{i=1}^{3}[h*\tilde{s}_i^k(\cdot,\cdot;\underset{\sim}{y}|u_i(\underset{\sim}{x},\cdot))](\underset{\sim}{x},t)\\
&\qquad\qquad\text{for every } (\underset{\sim}{x},t)\in\overset{*}{\partial}R\times\overset{\infty}{T}.
\end{aligned}\right\} \tag{4.13}$$

Now note that for $\psi\in\mathcal{C}(T^+)$ equation (4.5) and Lemma 1.2 imply

$$[h*\psi]^{(5)}=\psi \text{ on } (0,\infty) .$$

Therefore, and by (4.12), (4.13),

$$\sum_{i=1}^{3}\int_{\partial R} u_i^k(\underset{\sim}{x},t;\underset{\sim}{y}|s_i(\underset{\sim}{x},\cdot))dA_{\underset{\sim}{x}}=\sum_{i=1}^{3}\int_{R}\tilde{u}_i^k(\underset{\sim}{x},t;\underset{\sim}{y}|f_i(\underset{\sim}{x},\cdot))dV_{\underset{\sim}{x}}$$

$$-\sum_{i=1}^{3}\int_{\partial R}\tilde{s}_i^k(\underset{\sim}{x},t;\underset{\sim}{y}|u_i(\underset{\sim}{x},\cdot))dA_{\underset{\sim}{x}} \text{ for every } t\in(0,\infty) . \tag{4.14}$$

Finally, combine (4.14) with (3.17) and use (a) of Definition 4.1 to conclude that (4.3) holds for every $(\underset{\sim}{y},t)\in R\times\overset{\infty}{T}$. The verification of (4.4) is easily carried out in a strictly analogous manner with the aid of the reciprocal theorem (Theorem 2.2) and the integral identity (3.36).

Turning to the second boundary-initial value problem, we adopt

Definition 4.2. (Green's states of the second kind). *Let* R *be a regular region,* $\underset{\sim}{y}\in R$, *and let* $g\in \mathcal{C}^3$. *We call*

$$\hat{S}^k(\cdot,\cdot;\underset{\sim}{y}|g)=[\hat{\underset{\sim}{u}}^k(\cdot,\cdot;\underset{\sim}{y}|g),\hat{\underset{\sim}{\sigma}}^k(\cdot,\cdot;\underset{\sim}{y}|g)]$$

the displacement Green's states of the second kind and

$$\hat{S}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|g)=[\hat{\underset{\sim}{u}}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|g),\hat{\underset{\sim}{\sigma}}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|g)]$$

the stress Green's states of the second kind for the region R *and the pole* $\underset{\sim}{y}$, *corresponding to the force function* g *and to the material constants* ρ, c_1, c_2, *provided:*

(a) $\hat{S}^k(\cdot,\cdot;\mathbf{y}|g) = S^k(\cdot,\cdot;\mathbf{y}|g) + \tilde{S}^k(\cdot,\cdot;\mathbf{y}|g)$ *on* $\overline{R}_{\mathbf{y}} \times \overset{\infty}{T}$,

$\hat{S}^{k\ell}(\cdot,\cdot;\mathbf{y}|g) = \overline{S}^{k\ell}(\cdot,\cdot;\mathbf{y}|g) + \tilde{S}^{k\ell}(\cdot,\cdot;\mathbf{y}|g)$ *on* $\overline{R}_{\mathbf{y}} \times \overset{\infty}{T}$,

where $S^k(\cdot,\cdot;\mathbf{y}|g)$ *and* $\overline{S}^{k\ell}(\cdot,\cdot;\mathbf{y}|g)$ *respectively denote the Stokes-state of Definition* 3.1 *and the linear combination of doublet-states* (3.35);

(b) $\tilde{S}^k(\cdot,\cdot;\mathbf{y}|g) = [\tilde{\mathbf{u}}^k(\cdot,\cdot;\mathbf{y}|g), \tilde{\boldsymbol{\sigma}}^k(\cdot,\cdot;\mathbf{y}|g)] \in \mathcal{E}_o(\mathbf{0}, \rho, c_1, c_2; \overline{R})$,

$\tilde{S}^{k\ell}(\cdot,\cdot;\mathbf{y}|g) = [\tilde{\mathbf{u}}^{k\ell}(\cdot,\cdot;\mathbf{y}|g), \tilde{\boldsymbol{\sigma}}^{k\ell}(\cdot,\cdot;\mathbf{y}|g)] \in \mathcal{E}_o(\mathbf{0}, \rho, c_1, c_2; \overline{R})$;

(c) $\tilde{\mathbf{s}}^k(\cdot,\cdot;\mathbf{y}|g) = -\mathbf{s}^k(\cdot,\cdot;\mathbf{y}|g)$ *on* $\overset{*}{\partial}R \times \overset{\infty}{T}$,

$\tilde{\mathbf{s}}^{k\ell}(\cdot,\cdot;\mathbf{y}|g) = -\overline{\mathbf{s}}^{k\ell}(\cdot,\cdot;\mathbf{y}|g)$ *on* $\overset{*}{\partial}R \times \overset{\infty}{T}$,

where $\tilde{\mathbf{s}}^k(\cdot,\cdot;\mathbf{y}|g)$ *etc. denote the tractions of* $\tilde{S}^k(\cdot,\cdot;\mathbf{y}|g)$ *etc. acting on* ∂R.

The regular parts of the displacement and stress Green's states of the second kind are uniquely characterized, in view of (b), (c), and the uniqueness theorem (Theorem 2.1), as solutions to second boundary-initial value problems for R. The existence of the Green's states of the second kind evidently depends on the solvability of the second dynamic problem on R for sufficiently regular surface tractions. The following lemma is a counterpart of, and may be proved in the same way as, Lemma 4.1.

Lemma 4.2. *Let* R *be a regular region,* $\mathbf{y} \in R$, $g \in \mathcal{G}^3$, $h \in \mathcal{G}^3$, *and let* $\tilde{S}^k(\cdot,\cdot;\mathbf{y}|g)$, $\tilde{S}^{k\ell}(\cdot,\cdot;\mathbf{y}|g)$ *be the regular parts of the Green's states of the second kind introduced in Definition* 4.2.

Then:

(a) $h*\tilde{S}^{k}(\cdot,\cdot;\underset{\sim}{y}|g)=g*\tilde{S}^{k}(\cdot,\cdot;\underset{\sim}{y}|h)$ on $\bar{R}\times\overset{\infty}{T}$;

(b) $h*\tilde{S}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|g)=g*\tilde{S}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|h)$ on $\bar{R}\times\overset{\infty}{T}$.

Theorem 4.2. (Integral representation for the solution of the second boundary-initial value problem). Let R be a regular region.

Suppose:

(a) $S=[\underset{\sim}{u},\underset{\sim}{\sigma}]\in\mathcal{E}_o(\underset{\sim}{f},\rho,c_1,c_2;\bar{R})$;

(b) $\underset{\sim}{u}\in C^{(4)}(\partial R\times\overset{\infty}{T})$, $\underset{\sim}{\sigma}\in C^{(4)}(\partial R\times\overset{\infty}{T})$, $\underset{\sim}{f}\in C^{(4)}(\bar{R}\times\overset{\infty}{T})$.

Further, let $\hat{S}^{k}(\cdot,\cdot;\underset{\sim}{y}|g)$ and $\hat{S}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|g)$ be the Green's states of the second kind of Definition 4.2 for the region R and the pole $\underset{\sim}{y}$, corresponding to the force function g and to the material constants ρ, c_1, c_2. If these Green's states exist for all $\underset{\sim}{y}\in R$ and all $g\in\mathcal{G}^3$, then for every $(\underset{\sim}{y},t)\in R\times\overset{\infty}{T}$,

$$u_k(\underset{\sim}{y},t)=\sum_{i=1}^{3}\Big[\int_R \hat{u}_i^{k}(\underset{\sim}{x},t;\underset{\sim}{y}|f_i(\underset{\sim}{x},\cdot))dV_{\underset{\sim}{x}}+\int_{\partial R}\hat{u}_i^{k}(\underset{\sim}{x},t;\underset{\sim}{y}|s_i(\underset{\sim}{x},\cdot))dA_{\underset{\sim}{x}}\Big], \tag{4.15}$$

$$\sigma_{k\ell}(\underset{\sim}{y},t)=\sum_{i=1}^{3}\Big[\int_R \hat{u}_i^{k\ell}(\underset{\sim}{x},t;\underset{\sim}{y}|f_i(\underset{\sim}{x},\cdot))dV_{\underset{\sim}{x}}+\int_{\partial R}\hat{u}_i^{k\ell}(\underset{\sim}{x},t;\underset{\sim}{y}|s_i(\underset{\sim}{x},\cdot))dA_{\underset{\sim}{x}}\Big], \tag{4.16}$$

where $\underset{\sim}{s}$ are the tractions of S acting on ∂R.

The truth of this theorem may be confirmed with the aid of Lemma 4.2 by an argument parallel to that employed in the proof of Theorem 4.1. The smoothness restrictions imposed under (b) of Theorem 4.1 and Theorem 4.2 may be relaxed somewhat at the expense of more elaborate regularity hypotheses. As will become clear at the end of this section, the foregoing two theorems may be used to generate representations of the solution to the first and second elastodynamic problem in the absence of a quiescent past. Finally, integral representations for the solution of mixed boundary-initial value problems in elastodynamics, similar to those contained in Theorem 4.1 and Theorem 4.2, are easily established by means of suitable generalizations of the Green's states of the first and second kind.

Equations (4.3), (4.4) in Theorem 4.1 and (4.15), (4.16) in Theorem 4.2, for a fixed choice of the pole $\underset{\sim}{y}$, involve elements of the relevant Green's states corresponding to an infinite family of force functions (depending on the position parameter $\underset{\sim}{x}$). Accordingly, the representation at a single point of the given region of the solution to either fundamental problem of elastodynamics would seem to require that one solve an infinity of boundary-initial value problems in order to determine the requisite families of displacement and stress Green's states. We show next that this apparent difficulty is easily overcome, and in this connection consider first the representation of states whose body forces and surface displacements or surface tractions are separable functions of position and time.

Thus, suppose the state S in Theorem 4.1 is such that

$$\left.\begin{aligned} &\underset{\sim}{u}(\underset{\sim}{x}, t)=\overset{\circ}{\underset{\sim}{u}}(\underset{\sim}{x})p(t) \text{ for every } (\underset{\sim}{x}, t)\in \partial R\times \overset{\infty}{T}, \\ &\underset{\sim}{f}(\underset{\sim}{x}, t)=\overset{\circ}{\underset{\sim}{f}}(\underset{\sim}{x})q(t) \text{ for every } (\underset{\sim}{x}, t)\in \overline{R}\times \overset{\infty}{T}. \end{aligned}\right\} \quad (4.17)$$

Then, as is clear from (3.3), (3.4), Definition 4.1, and Theorem 2.1, Equations (4.3), (4.4) give way to

$$\left.\begin{aligned} u_k(\underset{\sim}{y}, t) &= \int_R \overset{\circ}{\underset{\sim}{f}}(\underset{\sim}{x}) \cdot \underset{\sim}{\hat{u}}^k(\underset{\sim}{x}, t; \underset{\sim}{y} | q) dV_{\underset{\sim}{x}} - \int_{\partial R} \overset{\circ}{\underset{\sim}{u}}(\underset{\sim}{x}) \cdot \underset{\sim}{\hat{s}}^k(\underset{\sim}{x}, t; \underset{\sim}{y} | p) dA_{\underset{\sim}{x}} , \\ \sigma_{k\ell}(\underset{\sim}{y}, t) &= -\int_R \overset{\circ}{\underset{\sim}{f}}(\underset{\sim}{x}) \cdot \underset{\sim}{\hat{u}}^{k\ell}(\underset{\sim}{x}, t; \underset{\sim}{y} | q) dV_{\underset{\sim}{x}} + \int_{\partial R} \overset{\circ}{\underset{\sim}{u}}(\underset{\sim}{x}) \cdot \underset{\sim}{\hat{s}}^{k\ell}(\underset{\sim}{x}, t; \underset{\sim}{y} | p) dA_{\underset{\sim}{x}} . \end{aligned}\right\} \quad (4.18)$$

Similarly, if the state S in Theorem 4.2 has the separable data

$$\left.\begin{aligned} \underset{\sim}{s}(\underset{\sim}{x}, t) &= \overset{\circ}{\underset{\sim}{s}}(\underset{\sim}{x}) p(t) \text{ for every } (\underset{\sim}{x}, t) \in \overset{*}{\partial} R \times \overset{\infty}{T} , \\ \underset{\sim}{f}(\underset{\sim}{x}, t) &= \overset{\circ}{\underset{\sim}{f}}(\underset{\sim}{x}) q(t) \text{ for every } (\underset{\sim}{x}, t) \in \overline{R} \times \overset{\infty}{T} , \end{aligned}\right\} \quad (4.19)$$

then (4.15), (4.16) may be replaced by

$$\left.\begin{aligned} u_k(\underset{\sim}{y}, t) &= \int_R \overset{\circ}{\underset{\sim}{f}}(\underset{\sim}{x}) \cdot \underset{\sim}{\hat{u}}^k(\underset{\sim}{x}, t; \underset{\sim}{y} | q) dV_{\underset{\sim}{x}} + \int_{\partial R} \overset{\circ}{\underset{\sim}{s}}(\underset{\sim}{x}) \cdot \underset{\sim}{\hat{u}}^k(\underset{\sim}{x}, t; \underset{\sim}{y} | p) dA_{\underset{\sim}{x}} , \\ \sigma_{k\ell}(\underset{\sim}{y}, t) &= -\int_R \overset{\circ}{\underset{\sim}{f}}(\underset{\sim}{x}) \cdot \underset{\sim}{\hat{u}}^{k\ell}(\underset{\sim}{x}, t; \underset{\sim}{y} | q) dV_{\underset{\sim}{x}} - \int_{\partial R} \overset{\circ}{\underset{\sim}{s}}(\underset{\sim}{x}) \cdot \underset{\sim}{\hat{u}}^{k\ell}(\underset{\sim}{x}, t; \underset{\sim}{y} | p) dA_{\underset{\sim}{x}} . \end{aligned}\right\} \quad (4.20)$$

In order to facilitate the construction of integral representations for states whose data are not necessarily separable we insert here

Theorem 4.3. (Standardization of the force function in the construction of Green's states). *Let R be a regular region, and let $\underset{\sim}{y} \in R$. Further, let $\hat{S}^k(\cdot, \cdot; \underset{\sim}{y} | g)$ and $\hat{S}^{k\ell}(\cdot, \cdot; \underset{\sim}{y} | g)$ be the Green's states of the first kind of Definition 4.1 or the Green's states of the second kind of Definition 4.2, and let h be the function defined by*

$$h(t) = \begin{cases} 0 \text{ for every } t \in (-\infty, 0] \\ t^4/4! \text{ for every } t \in (0, \infty) . \end{cases} \quad (4.21)$$

Then:

(a) $\hat{S}^k(\cdot,\cdot;\underset{\sim}{y}|g)=[g*\hat{S}^k(\cdot,\cdot;\underset{\sim}{y}|h)]^{(5)}$ on $\overline{R}_{\underset{\sim}{y}}\times(0,\infty)$;

(b) $\hat{S}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|g)=[g*\hat{S}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|h)]^{(5)}$ on $\overline{R}_{\underset{\sim}{y}}\times(0,\infty)$.

Proof. If $\hat{S}^k(\cdot,\cdot;\underset{\sim}{y}|g)$ and $\hat{S}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|g)$ are Green's states of the first kind, then (1.21), (d) in Theorem 3.1, (d) in Lemma 1.1, Lemma 4.1, and Definition 4.1 yield

$$\left.\begin{aligned} h*\hat{S}^k(\cdot,\cdot;\underset{\sim}{y}|g)&=g*\hat{S}^k(\cdot,\cdot;\underset{\sim}{y}|h) \text{ on } \overline{R}_{\underset{\sim}{y}}\times\overset{\infty}{T},\\ h*\hat{S}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|g)&=g*\hat{S}^{k\ell}(\cdot,\cdot;\underset{\sim}{y}|h) \text{ on } \overline{R}_{\underset{\sim}{y}}\times\overset{\infty}{T}\,. \end{aligned}\right\} \quad (4.22)$$

On the other hand, (4.22) hold true also for Green's states of the second kind by virtue of (1.21), (d) in Theorem 3.2, (3.35), (d) in Lemma 1.1, Lemma 4.2, and Definition 4.2. Further, note from Lemma 1.2 that for the present choice of h, every function $\psi\in C(T^+)$ obeys the identity

$$\psi=[h*\psi]^{(5)} \text{ on } (0,\infty).$$

Thus, conclusions (a) and (b) follow from (1.21), (4.22), (1.19), and the regularity properties of the Green's states of the first and second kind implied by Definition 4.1 and Definition 4.2. This completes the proof.

Theorem 4.3 enables one to generate directly the Green's states of the first and second kind for a given region and a fixed pole, corresponding to an arbitrary (sufficiently smooth) force function from those corresponding to the standard force function h given by (4.21). For example, (4.3) may now be written as

$$u_k(\underset{\sim}{y}, t) = \int_R [\underset{\sim}{f} * \underset{\sim}{\hat{u}}^k(\cdot, \cdot; \underset{\sim}{y}|h)]^{(5)}(\underset{\sim}{x}, t)dV_{\underset{\sim}{x}} - \int_{\partial R} [\underset{\sim}{u} * \underset{\sim}{\hat{s}}^k(\cdot, \cdot; \underset{\sim}{y}|h)]^{(5)}(\underset{\sim}{x}, t)dA_{\underset{\sim}{x}}.$$

Additional properties of the Green's states are supplied by

Theorem 4.4. (Symmetry of the Green's states). Let R be a regular region, let $\underset{\sim}{x}$ and $\underset{\sim}{y}$ be distinct points in R, and let $g \in \mathcal{G}^3$. Further, let $\hat{\mathcal{S}}^k(\cdot, \cdot; \underset{\sim}{y}|g)$, $\hat{\mathcal{S}}^{k\ell}(\cdot, \cdot; \underset{\sim}{y}|g)$ be the Green's states of the first kind of Definition 4.1 or the Green's states of the second kind of Definition 4.2.

Then, for every $t \in \overset{\infty}{T}$,

$$\text{(a)} \quad \hat{u}_i^k(\underset{\sim}{x}, t; \underset{\sim}{y}|g) = \hat{u}_k^i(\underset{\sim}{y}, t; \underset{\sim}{x}|g), \quad \text{(b)} \quad \hat{\sigma}_{ij}^{k\ell}(\underset{\sim}{x}, t; \underset{\sim}{y}|g) = \hat{\sigma}_{k\ell}^{ij}(\underset{\sim}{y}, t; \underset{\sim}{x}|g).$$

Proof. It will be sufficient to illustrate the proof of this theorem by demonstrating merely (a) for the case in which $\hat{\mathcal{S}}^k(\cdot, \cdot; \underset{\sim}{y}|g)$ is a Green's state of the first kind for the region R and the pole $\underset{\sim}{y}$. If $t \in T^-$, (a) is immediate from Definition 4.1. Also, (a) holds trivially for every $t \in \overset{\infty}{T}$ if $g=0$ on $[0, \infty)$. Hence, assume that g fails to vanish identically on $[0, \infty)$. Now choose $\alpha > 0$ such that $\overline{B}_\alpha(\underset{\sim}{x}) \subset R$, $\overline{B}_\alpha(\underset{\sim}{y}) \subset R$, while $\overline{B}_\alpha(\underset{\sim}{x}) \cap \overline{B}_\alpha(\underset{\sim}{y})$ is empty. Then, for each $\eta \in (0, \alpha)$, the region

$$R_\eta = R - \overline{B}_\eta(\underset{\sim}{x}) - \overline{B}_\eta(\underset{\sim}{y})$$

is regular and, by hypothesis and Definition 4.1,

$$\left.\begin{aligned} &\hat{\mathcal{S}}^k(\cdot, \cdot; \underset{\sim}{y}|g) \in \mathcal{E}_o(\underset{\sim}{0}, \rho, c_1, c_2; \overline{R}_\eta), \\ &\hat{\mathcal{S}}^i(\cdot, \cdot; \underset{\sim}{x}|g) \in \mathcal{E}_o(\underset{\sim}{0}, \rho, c_1, c_2; \overline{R}_\eta). \end{aligned}\right\} \quad (4.23)$$

Further, (a) and (c) in Definition 4.1 imply

$$\hat{\underset{\sim}{u}}^{k}(\cdot,\cdot;\underset{\sim}{y}|g)=\underset{\sim}{0} \text{ on } \partial R\times \overset{\infty}{T},$$

so that the state $\hat{\underset{\sim}{S}}^{k}(\cdot,\cdot;y|g)$ conforms to condition (b) imposed on S in Lemma 2.2. Accordingly, Theorem 2.2 may be applied to the pair of states in (4.23), whence

$$\int_{S_\eta(\underset{\sim}{y})}[\hat{\underset{\sim}{s}}^{k}(\cdot,\cdot;\underset{\sim}{y}|g)*\hat{\underset{\sim}{u}}^{i}(\cdot,\cdot;\underset{\sim}{x}|g)](\underset{\sim}{z},t)dA_{\underset{\sim}{z}}+\int_{S_\eta(\underset{\sim}{x})}[\hat{\underset{\sim}{s}}^{k}(\cdot,\cdot;\underset{\sim}{y}|g)*\hat{\underset{\sim}{u}}^{i}(\cdot,\cdot;\underset{\sim}{x}|g)](\underset{\sim}{z},t)dA_{\underset{\sim}{z}}$$

$$=\int_{S_\eta(\underset{\sim}{y})}[\hat{\underset{\sim}{s}}^{i}(\cdot,\cdot;\underset{\sim}{x}|g)*\hat{\underset{\sim}{u}}^{k}(\cdot,\cdot;\underset{\sim}{y}|g)](\underset{\sim}{z},t)dA_{\underset{\sim}{z}}+\int_{S_\eta(\underset{\sim}{x})}[\hat{\underset{\sim}{s}}^{i}(\cdot,\cdot;\underset{\sim}{x}|g)*\hat{\underset{\sim}{u}}^{k}(\cdot,\cdot;\underset{\sim}{y}|g)](\underset{\sim}{z},t)dA_{\underset{\sim}{z}}$$

for every $t>0$. Next, pass to the limit as $\eta\to 0$ in this equation, bearing in mind Lemma 3.1 and Definition 4.1, to arrive at

$$[g*\hat{u}_{k}^{i}(\cdot,\cdot;\underset{\sim}{x}|g)](\underset{\sim}{y},t)=[g*\hat{u}_{i}^{k}(\cdot,\cdot;\underset{\sim}{y}|g)](\underset{\sim}{x},t)$$

for every $t>0$. Conclusion (a), for the displacement Green's states of the first kind now follows from (e) in Lemma 1.1. Conclusion (b) for the stress Green's states of the first kind, as well as both conclusions for the Green's states of the second kind, may be reached in a strictly analogous manner.

Theorem 4.1 and Theorem 4.2 presuppose that the state to be represented has a quiescent past and possesses regularity properties beyond those introduced in the definition of an elastodynamic state with a quiescent past (see Definition 1.2). We conclude this section with a theorem permitting one to obtain from the results established already representations of states that are free of the restrictions just mentioned.

Theorem 4.5. (Regularization of elastodynamic states). Let R be a regular region and let

$$S=[\underset{\sim}{u}, \underset{\sim}{\sigma}]\in \mathcal{E}(\underset{\sim}{f}, \rho, c_1, c_2; \bar{R}\times T^+) .$$

Let $n\geq 2$ be an integer and let φ be the function defined by

$$\varphi(t)= \begin{cases} 0 \text{ for every } t\in(-\infty, 0] \\ t^n/n! \text{ for every } t\in(0,\infty). \end{cases}$$

Suppose further

$$S'=[\underset{\sim}{u}', \underset{\sim}{\sigma}']=\varphi * S \text{ on } \bar{R}\times \overset{\infty}{T} .$$

Then:

(a) $S'\in \mathcal{E}_0(\underset{\sim}{f}', \rho, c_1, c_2; \bar{R})$, where, for every $(\underset{\sim}{x},t)\in\bar{R}\times\overset{\infty}{T}$,

$\underset{\sim}{f}'(\underset{\sim}{x}, t)=[\varphi * \underset{\sim}{f}](\underset{\sim}{x}, t)+\rho\varphi(t)\dot{\underset{\sim}{u}}(\underset{\sim}{x}, 0+)+\rho\dot{\varphi}(t)\underset{\sim}{u}(\underset{\sim}{x}, 0)$;

(b) $\underset{\sim}{u}'\in C^{(n-1)}(\partial R\times\overset{\infty}{T})$, $\underset{\sim}{\sigma}'\in C^{(n-1)}(\partial R\times\overset{\infty}{T})$, $\underset{\sim}{f}'\in C^{(n-2)}(\bar{R}\times\overset{\infty}{T})$;

(c) $\underset{\sim}{u}'\in C^{(n+1)}(\bar{R}\times T^+)$, $\underset{\sim}{\sigma}'\in C^{(n+1)}(\bar{R}\times T^+)$,

$$S=S'^{(n+1)} \text{ on } \bar{R}\times(0,\infty) . \tag{4.24}$$

Proof. Observe that the function φ has the properties

$$\varphi\in C^{n-1}\cap C^{\infty}(T^+),\ \varphi^{(n)}(0+)=1,\ \varphi^{(n+1)}=0 \text{ on } (0,\infty) . \tag{4.25}$$ [1]

The first of (4.25), in conjunction with the conditions imposed on S

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[1] We write $\psi\in C^{\infty}(T)$ if $\psi\in C^m(T)$ for every positive integer m.

in Definition 1.2 and the properties of convolutions given in Lemmas 1.1, 1.2, 1.3, enable one to reach conclusion (a) without difficulty.

Next, appeal to the first of (4.25) and the continuous differentiability of $\underset{\sim}{u}$ on $\overline{R} \times T^+$ to see that

$$\varphi\dot{\underset{\sim}{u}}(\cdot, 0+) + \dot{\varphi}\underset{\sim}{u}(\cdot, 0) \in C^{(n-2)}(\overline{R} \times \overset{\infty}{T}) .$$

Thus (b) follows from the first of (4.25), the above definition of $\underset{\sim}{f}'$, the continuity of $\underset{\sim}{u}$, $\underset{\sim}{\sigma}$, and $\underset{\sim}{f}$ on $\overline{R} \times T^+$, and Lemma 1.2. Finally, (c) is a consequence of (4.25), the continuity of $\underset{\sim}{u}$ and $\underset{\sim}{\sigma}$ on $\overline{R} \times T^+$, as well as Lemma 1.2. This completes the proof.

The preceding theorem owes its usefulness to the fact that, while the state S is not assumed to conform to hypotheses (a) and (b) in Theorems 4.1, 4.2, it is conveniently recoverable in the manner of (4.24) from a state that does meet these hypotheses, provided $n \geq 6$.

5. A uniqueness theorem for concentrated-load problems in elastodynamics.

As a further application of the Green's states introduced in Section 4 we treat in this section a uniqueness issue associated with the second boundary-initial value problem of elastodynamics in the presence of concentrated loads acting at fixed material (interior or boundary) points of the body. The uniqueness theorem arrived at here asserts the completeness of a direct formulation of concentrated-load problems that rests on prescribing — in addition to the body forces, regular surface tractions, and initial conditions — the orders of the displacement and stress singularities at the load points, as well as the stress resultants of the latter singularities. This formulation of the singular class of problems with which we are concerned clearly lies beyond the scope of ordinary uniqueness theorems in dynamic elasticity, such as Neumann's theorem or Theorem 2.1 in the current investigation. The uniqueness theorem constituting our present objective is a dynamic analogue of a recent elastostatic result due to Turteltaub and Sternberg [28] (see Theorem 5.2 of [28]) and will be proved by parallel means.

With a view toward clarifying the relevance of the theorem presented in what follows, we emphasize that the idealization of a "concentrated load" in elasticity theory derives its physical significance from a limit definition of the solution to problems involving such loads. Accordingly, the solution to the <u>singular</u> problem under consideration would have to be defined as the limit of a sequence of regular solutions, corresponding to distributed body forces and surface tractions that tend to the given concentrated loads.

A program aimed at confirming the equivalence of the direct and the limit-formulation of concentrated-load problems may be pursued in three stages. First, one would seek to demonstrate the existence of the limit solution by proving the appropriate convergence of the sequence of approximating regular solutions. Next one would examine the limit solution and attempt to verify that it possesses the properties underlying the direct formulation of the problem; in particular, one would have to determine the orders and stress resultants of the singularities inherent in the limit solution at points of application of concentrated loads. Finally, one would aim at showing that these properties suffice to characterize the limit solution uniquely. The direct formulation of the singular problem at hand has the advantage of obviating the need for a limit process that is apt to be highly cumbersome in actual applications.

The program outlined above was proposed in [21] for the equilibrium theory and was carried out rigorously in [28] with limitation of the first two stages to concentrated surface loads acting on finite bodies with sufficiently smooth boundaries. The limit treatment of internal concentrated loads in elastostatics is in essence disposed of by the derivation due to Kelvin and Tait [20] (page 279) of the solution to Kelvin's problem[1]. Further, the requisite

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[1] For an explicit version of the underlying limit process see [21].

properties of Kelvin's solution, which is in elementary form, are trivially inferred. Similarly, Love's [19] (page 304) derivation through a limit process of the Stokes-state verifies its physical significance. Moreover, conclusions (a), (b), (c) in our Theorem 3.1 furnish the pertinent properties of Stokes' solution. In contrast, a limit treatment of concentrated surface loads in dynamic elasticity — even under very stringent restrictions upon the body geometry — represents an extremely difficult task with which we do not propose to cope at present. Thus, we rely solely on Stokes' solution as a motivation for the a priori assumptions regarding the order of the singularities at the points of application of concentrated loads introduced in

Theorem 5.1. (A uniqueness theorem for elastodynamic problems involving concentrated internal and surface loads). Let R be a regular region and assume that for each $\underset{\sim}{y} \in R$ there is at least one $g \in \mathcal{C}^3$, not identically zero on $(-\infty, \infty)$, such that the displacement Green's states of the second kind exist for the region R and the pole $\underset{\sim}{y}$, corresponding to the force function g and to given material constants ρ, c_1, c_2. Let

$$P = \{\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n\}$$

be a set consisting of n distinct points in $\overline{R}$. Further, let S', S'' be two states with the following properties:

(a) $S' = [\underset{\sim}{u}', \underset{\sim}{\sigma}'] \in \mathcal{E}(\underset{\sim}{f}, \rho, c_1, c_2; (\overline{R}-P)\times T^+)$,

$S'' = [\underset{\sim}{u}'', \underset{\sim}{\sigma}''] \in \mathcal{E}(\underset{\sim}{f}, \rho, c_1, c_2; (\overline{R}-P)\times T^+)$;

(b) as $\underset{\sim}{x} \to \underset{\sim}{a}_k$ (k=1, ..., n),

$$\underset{\sim}{u}'(\underset{\sim}{x}, \cdot) = O(|\underset{\sim}{x} - \underset{\sim}{a}_k|^{-1}), \quad \underset{\sim}{\sigma}'(\underset{\sim}{x}, \cdot) = O(|\underset{\sim}{x} - \underset{\sim}{a}_k|^{-2}),$$

$$\underset{\sim}{u}''(\underset{\sim}{x}, \cdot) = O(|\underset{\sim}{x} - \underset{\sim}{a}_k|^{-1}), \quad \underset{\sim}{\sigma}''(\underset{\sim}{x}, \cdot) = O(|\underset{\sim}{x} - \underset{\sim}{a}_k|^{-2}),$$

uniformly on $[0, t]$ for every $t > 0$;

(c) $$\lim_{\eta \to 0} \int_{\Lambda_k(\eta)} \underset{\sim}{s}'(\underset{\sim}{x}, \cdot) dA = \underset{\sim}{\ell}_k, \quad \lim_{\eta \to 0} \int_{\Lambda_k(\eta)} \underset{\sim}{s}''(\underset{\sim}{x}, \cdot) dA = \underset{\sim}{\ell}_k \text{ on } [0, \infty) \, (k=1, \ldots, n),$$

where $\underset{\sim}{\ell}_k$ (k=1, ..., n) are given vector-valued functions of the time,

$$\Lambda_k(\eta) = R \cap S_\eta(\underset{\sim}{a}_k) \quad (k=1, \ldots, n),$$

while $\underset{\sim}{s}'$, $\underset{\sim}{s}''$ are the tractions of S', S'' acting on the side of $\Lambda_k(\eta)$ that faces the point $\underset{\sim}{a}_k$, and the preceding limits are attained uniformly on $[0, t]$ for every $t > 0$;

(d) $$\underset{\sim}{u}'(\cdot, 0) = \mathring{\underset{\sim}{u}}, \; \dot{\underset{\sim}{u}}'(\cdot, 0+) = \mathring{\underset{\sim}{v}}, \; \underset{\sim}{u}''(\cdot, 0) = \mathring{\underset{\sim}{u}}, \; \dot{\underset{\sim}{u}}''(\cdot, 0+) = \mathring{\underset{\sim}{v}} \text{ on } \overline{R} - P,$$

$$\underset{\sim}{s}' = \underset{\sim}{p}, \quad \underset{\sim}{s}'' = \underset{\sim}{p} \text{ on } (\overset{*}{\partial} R - P) \times [0, \infty),$$

provided $\underset{\sim}{s}'$, $\underset{\sim}{s}''$ here denote the surface tractions of S', S'' whereas $\mathring{\underset{\sim}{u}}$, $\mathring{\underset{\sim}{v}}$, and $\underset{\sim}{p}$ are functions prescribed on their respective domains of definition.

Then,

$$S' = S'' \text{ on } (\overline{R} - P) \times [0, \infty).$$

Proof. Choose $\underset{\sim}{y} \in R$ and hold $\underset{\sim}{y}$ fixed. Let

$$\hat{S}^i(\cdot, \cdot; \underset{\sim}{y} | g) = [\hat{\underset{\sim}{u}}^i(\cdot, \cdot; \underset{\sim}{y} | g), \; \hat{\underset{\sim}{\sigma}}^i(\cdot, \cdot; \underset{\sim}{y} | g)]$$

be the displacement Green's states of the second kind for R and $\underset{\sim}{y}$, corresponding to g, ρ, c_1, and c_2, where $g \in \mathcal{G}^3$ and fails to vanish identically. It is clear from Definition 4.2 and conclusion (a) in Theorem 3.1 that

$$\left.\begin{aligned} &\hat{S}^i(\cdot,\cdot;\underset{\sim}{y}|g) \in \mathcal{E}_o(\underset{\sim}{0},\rho,c_1,c_2;\overline{R}_{\underset{\sim}{y}}), \\ &\hat{\underset{\sim}{s}}^i(\cdot,\cdot;\underset{\sim}{y}|g)=\underset{\sim}{0} \text{ on } \overset{*}{\partial}R\times\overset{\infty}{T}, \end{aligned}\right\} \quad (5.1)$$

if $\hat{\underset{\sim}{s}}^i(\cdot,\cdot;\underset{\sim}{y}|g)$ are the tractions of $\hat{S}^i(\cdot,\cdot;\underset{\sim}{y}|g)$ acting on ∂R.

Next, define the state $S=[\underset{\sim}{u},\underset{\sim}{\sigma}]$ on $(\overline{R}-P)\times\overset{\infty}{T}$ through

$$S=S'-S'' \text{ on } (\overline{R}-P)\times(0,\infty),\ \underset{\sim}{u}=\underset{\sim}{\sigma}=\underset{\sim}{0} \text{ on } (\overline{R}-P)\times(-\infty,0]. \quad (5.2)$$

Then, by hypotheses (a), (b), (c), (d), and Definition 1.2,

$$S\in\mathcal{E}_o(\underset{\sim}{0},\rho,c_1,c_2;\overline{R}-P), \quad (5.3)$$

$$\underset{\sim}{u}(\underset{\sim}{x},\cdot)=O(|\underset{\sim}{x}-\underset{\sim}{a}_k|^{-1}),\ \underset{\sim}{\sigma}(\underset{\sim}{x},\cdot)=O(|\underset{\sim}{x}-\underset{\sim}{a}_k|^{-2}) \text{ as } \underset{\sim}{x}\to\underset{\sim}{a}_k\ (k=1,\ldots,n), \quad (5.4)$$

uniformly on $[0,t]$ for every $t>0$,

$$\lim_{\eta\to 0}\int_{\Lambda_k(\eta)}\underset{\sim}{s}(\underset{\sim}{x},\cdot)\,dA=\underset{\sim}{0} \text{ on } [0,\infty)\ (k=1,\ldots,n), \quad (5.5)$$

this limit being attained uniformly on $[0,t]$ for every $t>0$, and

$$\underset{\sim}{s}=\underset{\sim}{0} \text{ on } (\overset{*}{\partial}R-P)\times\overset{\infty}{T}, \quad (5.6)$$

where $\underset{\sim}{s}$ are the appropriate surface tractions of S.

Take $\eta_o>0$ such that any two spheres (balls) of radius η_o centered at distinct points of P are disjoint and do not intersect $B_{\eta_o}(\underset{\sim}{y})$, while, for every $\eta\in(0,\eta_o)$, $B_\eta(\underset{\sim}{y})\subset R$ and the region

$$R_\eta = R - \bigcup_{k=1}^{n} \overline{B}_\eta(\underset{\sim}{a}_k) - \overline{B}_\eta(\underset{\sim}{y})$$

is a regular region. Evidently, (5.1) and (5.3) now permit an application of the reciprocal theorem (Theorem 2.2) to the pair of states $\hat{S}^i(\cdot,\cdot;\underset{\sim}{y}|g)$, S on R_η. Because of (5.6), the second of (5.1), and the vanishing of the body forces of $\hat{S}^i(\cdot,\cdot;\underset{\sim}{y}|g)$ and S, one finds in this manner that

$$\sum_{k=1}^{n} \int_{\Lambda_k(\eta)} [\underset{\sim}{s}*\hat{\underset{\sim}{u}}^i(\cdot,\cdot;\underset{\sim}{y}|g)](\underset{\sim}{x},t)dA_{\underset{\sim}{x}} + \int_{S_\eta(\underset{\sim}{y})} [\underset{\sim}{s}*\hat{\underset{\sim}{u}}^i(\cdot,\cdot;\underset{\sim}{y}|g)](\underset{\sim}{x},t)dA_{\underset{\sim}{x}} =$$

$$\sum_{k=1}^{n} \int_{\Lambda_k(\eta)} [\hat{\underset{\sim}{s}}^i(\cdot,\cdot;\underset{\sim}{y}|g)*\underset{\sim}{u}](\underset{\sim}{x},t)dA_{\underset{\sim}{x}} + \int_{S_\eta(\underset{\sim}{y})} [\hat{\underset{\sim}{s}}^i(\cdot,\cdot;\underset{\sim}{y}|g)*\underset{\sim}{u}](\underset{\sim}{x},t)dA_{\underset{\sim}{x}} \qquad (5.7)$$

for every $\eta \in (0, \eta_o)$ and for all $t \in (0, \infty)$.

At this stage hold $t>0$ fixed and invoke (5.5), bearing in mind the uniformity on $[0, t]$ of the limit in (5.5), to see that for $k=1,\ldots,n$,

$$\int_{\Lambda_k(\eta)} [\underset{\sim}{s}*\hat{\underset{\sim}{u}}^i(\cdot,\cdot;\underset{\sim}{y}|g)](\underset{\sim}{x},t)dA_{\underset{\sim}{x}} =$$

$$\int_{\Lambda_k(\eta)} \int_0^t \underset{\sim}{s}(\underset{\sim}{x},t-\tau)\cdot[\hat{\underset{\sim}{u}}^i(\underset{\sim}{x},\tau;\underset{\sim}{y}|g)-\hat{\underset{\sim}{u}}^i(\underset{\sim}{a}_k,\tau;\underset{\sim}{y}|g)]d\tau dA_{\underset{\sim}{x}} + o(1) \text{ as } \eta\to 0 .$$

Hence (5.4) and the continuity of $\hat{\underset{\sim}{u}}^i(\cdot,\cdot;\underset{\sim}{y}|g)$ on $\overline{R}_{\underset{\sim}{y}} \times \overset{\infty}{T}$ yield

$$\lim_{\eta\to 0} \int_{\Lambda_k(\eta)} [\underset{\sim}{s}*\hat{\underset{\sim}{u}}^i(\cdot,\cdot;\underset{\sim}{y}|g)](\underset{\sim}{x},t)dA_{\underset{\sim}{x}} = 0 \quad (k=1,\ldots,n). \qquad (5.8)$$

On the other hand, (5.4) and the continuity of $\hat{\underset{\sim}{\sigma}}^{i}(\cdot, \cdot; \underset{\sim}{y}|g)$ on $\overline{R}_{\underset{\sim}{y}} \times \overset{\infty}{T}$ furnish

$$\lim_{\eta \to 0} \int_{\Lambda_k(\eta)} [\hat{\underset{\sim}{s}}^{i}(\cdot, \cdot; \underset{\sim}{y}|g) * \underset{\sim}{u}](\underset{\sim}{x}, t) dA_{\underset{\sim}{x}} = 0 \quad (k=1, \ldots, n). \tag{5.9}$$

Now pass to the limit as $\eta \to 0$ in (5.7) and appeal to (5.8), (5.9), together with Lemma 3.1 and Definition 4.2, to arrive at

$$[g * \underset{\sim}{u}](\underset{\sim}{y}, t) = \underset{\sim}{0} \quad .$$

But $t > 0$ was chosen arbitrarily, so that

$$[g * \underset{\sim}{u}](\underset{\sim}{y}, \cdot) = \underset{\sim}{0} \text{ on } (0, \infty) . \tag{5.10}$$

Since, by hypothesis, g does not vanish identically on $[0, \infty)$, one infers from (5.10) and (e) in Lemma 1.1 that

$$\underset{\sim}{u}(\underset{\sim}{y}, \cdot) = \underset{\sim}{0} \text{ on } [0, \infty) .$$

Recalling that $\underset{\sim}{y}$ was chosen arbitrarily in R-P, one draws

$$\underset{\sim}{u} = \underset{\sim}{0} \text{ on } (R-P) \times [0, \infty) . \tag{5.11}$$

Moreover, (5.11), (1.11) imply that $\underset{\sim}{\sigma}$ vanishes on $(R-P)\times[0, \infty)$. The desired conclusion now follows from the continuity of $\underset{\sim}{u}$, $\underset{\sim}{\sigma}$ on $(\overline{R}-P)\times[0, \infty)$ assured by (5.3) and from (5.2).

The preceding theorem is at once broader and more restrictive than Theorem 2.1. While Theorem 5.1 encompasses a class of _singular_ elastodynamic states, not covered by Theorem 2.1, it presupposes the existence of the displacement Green's states of the second kind — and hence the solvability of a class of _regular_ second boundary-initial value problems, for the region at hand. No such existence hypothesis is involved in Theorem 2.1.

It follows from Theorem 5.1, in particular, that the Stokes-state is uniquely characterized by (a), (b), and the first of (c) in Theorem 3.1. On the other hand, (a) together with both of (c) in Theorem 3.1 fail to characterize the Stokes-state uniquely. To see this, consider the state

$$S^k(\cdot,\cdot;\underset{\sim}{y}|g)+S^{ii}(\cdot,\cdot;\underset{\sim}{y}|h)\ , \tag{5.12}$$

where $S^k(\cdot,\cdot;\underset{\sim}{y}|g)$ is the Stokes-state of Definition 3.1, $h\in\mathcal{G}^3$ and is not identically zero, while $S^{ii}(\cdot,\cdot;\underset{\sim}{y}|g)$ is the linear combination of doublet-states (appropriate to a dynamic center of dilatation) accounted for through Definition 3.2. The state defined by (5.12), in view of Theorem 3.2, evidently conforms to (a) and (c) in Theorem 3.1 but is distinct from the Stokes-state; it possesses, however, displacement and stress singularities at $\underset{\sim}{y}$ of a higher order than those inherent in $S^k(\cdot,\cdot;\underset{\sim}{y}|g)$. This example makes clear that hypothesis (b) in Theorem 5.1 cannot be omitted; nor can it be relinquished in favor of the weaker requirement that, uniformly on $[0,t]$ for every $t>0$,

$$\lim_{\eta\to 0}\int_{\Lambda_k(\eta)}(\underset{\sim}{x}-\underset{\sim}{a}_k)\wedge\underset{\sim}{s}'(\underset{\sim}{x},\cdot)dA=\underset{\sim}{0}\ \text{ on } [0,\infty)\ \ (k=1,\ldots,n)\ ,$$

$$\lim_{\eta\to 0}\int_{\Lambda_k(\eta)}(\underset{\sim}{x}-\underset{\sim}{a}_k)\wedge\underset{\sim}{s}''(\underset{\sim}{x},\cdot)dA=\underset{\sim}{0}\ \text{ on } [0,\infty)\ \ (k=1,\ldots,n)\ ,$$

without invalidating the conclusion. An analogous counter-example related to a concentrated surface load on the boundary of an elastic half-space is easily constructed.

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