SMOOTH BANACH SPACES AND APPROXIMATIONS

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ABSTRACT

If E and F are real Banach spaces let $C^{p,q}(E,F)$ $0 \le q \le p \le \infty$, denote those maps from E to F which have p continuous Frechet derivatives of which the first q derivatives are bounded. A Banach space E is defined to be $C^{p,q}$ smooth if $C^{p,q}(E,R)$ contains a nonzero function with bounded support. This generalizes the standard C^p smoothness classification.

If an L^p space, $p \ge 1$, is C^q smooth then it is also $C^{q,q}$ smooth so that in particular L^p for p an even integer is $C^{\infty,\infty}$ smooth and L^p for p an odd integer is $C^{p-1,p-1}$ smooth. In general, however, a C^p smooth B-space need not be $C^{p,p}$ smooth. c_o is shown to be a non-$C^{2,2}$ smooth B-space although it is known to be C^∞ smooth. It is proved that if E is $C^{p,1}$ smooth then $c_o(E)$ is $C^{p,1}$ smooth and if E has an equivalent C^p norm then $c_o(E)$ has an equivalent C^p norm.

Various consequences of $C^{p,q}$ smoothness are studied. If $f \in C^{p,q}(E,F)$, if F is $C^{p,q}$ smooth and if E is non-$C^{p,q}$ smooth, then the image under f of the boundary of any bounded open subset U of E is dense in the image of U. If E is separable then E is $C^{p,q}$ smooth if and only if E admits $C^{p,q}$ partitions of unity; E is $C^{p,p}$ smooth, $p<\infty$, if

and only if every closed subset of E is the zero set of some C^p function.

$f \in C^q(E,F)$, $0 \le q < p \le \infty$, is said to be $C_{p,q}$ approximable on a subset U of E if for any $\epsilon > 0$ there exists a $g \in C^p(E,F)$ satisfying

$$\sup_{x \in U, 0 \le k \le q} \|D^k f(x) - D^k g(x)\| \le \epsilon \quad .$$

It is shown that if E is separable and $C^{p,q}$ smooth and if $f \in C^q(E,F)$ is $C_{p,q}$ approximable on some neighborhood of every point of E, then F is $C_{p,q}$ approximable on all of E.

In general it is unknown whether an arbitrary function in $C^1(\ell^2,R)$ is $C_{2,1}$ approximable and an example of a function in $C^1(\ell^2,R)$ which may not be $C_{2,1}$ approximable is given. A weak form of $C_{\infty,q}$, $q \ge 1$, to functions in $C^q(\ell^2,R)$ is proved: Let $\{U_\alpha\}$ be a locally finite cover of ℓ^2 and let $\{T_\alpha\}$ be a corresponding collection of Hilbert-Schmidt operators on ℓ^2. Then for any $f \in C^q(\ell^2,F)$ with $D^q f$ locally uniformly continuous, there exists a $g \in C^\infty(\ell^2,F)$ such that for all α

$$\sup_{x \in U_\alpha, \|h\| \le 1, 0 \le k \le q} \|D^k(f(x) - g(x))[T_\alpha h]\| \le 1.$$

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INTRODUCTION

The central theme of this dissertation is the study of Frechet differentiable functions on real Banach spaces with bounded derivatives. If E and F are real Banach spaces and $0 \leq q \leq p \leq \infty$, a map f from E to F will be said to be of class $C^{p,q}(E,F)$ if f has p continuous Frechet derivatives, the first q of which are bounded. Contrary to the finite dimensional case, an infinite dimensional Banach space need not have any nontrivial $C^{p,q}$ functions. We define a B-space to be $C^{p,q}$ smooth if there exists a nonzero function in $C^{p,q}(E,R)$ with bounded support. This generalizes the standard concept of C^p smoothness and $C^{p,0}$ smoothness is equivalent to C^p smoothness.

In the first chapter we provide the necessary preliminary material on differential calculus in Banach spaces. Several of the results of Bonic and Frampton [1] concerning C^p smoothness are valid for $C^{p,q}$ smoothness. One property in particular is that a map in $C^{p,q}(E,F)$ has an "analytic" property if F is $C^{p,q}$ smooth and E is non $C^{p,q}$ smooth: the values of F on a bounded open subset U of E are uniquely determined by its values on the boundary of U. This, and other results, including a summary of the

$C^{p,q}$ smoothness of various B-spaces, is contained in Chapter II. An L^p space is shown to be $C^{q,q}$ smooth if it is C^q smooth so that L^p for p an even integer is $C^{\infty,\infty}$ smooth.

In Chapter III we show that a C^p smooth B-space need not be $C^{p,p}$ smooth by proving that c_o , the C^∞ smooth B-space of sequences of real numbers converging to zero, is not $C^{2,2}$ smooth. In addition we show that if E is $C^{p,1}$ smooth, then $c_o(E)$,(the B-space of sequences in E converging to zero), is $C^{p,1}$ smooth. N.H.Kuiper constructed an equivalent C^∞ norm for c_o and we generalize this by proving that $c_o(E)$ has an equivalent C^p norm if E has an equivalent C^p norm.

In Chapter IV the existence of a C^p function with a prescribed zero set is studied. The main result is that a separable B-space E is $C^{p,p}$ smooth, $p<\infty$, if and only if every closed subset of E is the zero set of some C^p function. Secondly, if E is only C^p smooth and A is a closed subset of E, we give a sufficient condition on A to insure that it is the zero set of some C^p function.

If a B-space E admits C^p partitions of unity, then $C^p(E,F)$ is dense in $C^0(E,F)$ for any F, but in general the existence of C^p partitions of unity on E is unknown. Bonic and Frampton in [1] proved that if E is separable then E admits C^p partitions of unity if and only if E is

C^p smooth. We generalize this by proving in Chapter V that a separable $C^{p,q}$ smooth B-space admits $C^{p,q}$ partitions on unity. Along with this we study the $C^{p,q}$-ness of sums of the form $\sum_i a_i \varphi_i(x)$, where $\{\varphi_i\}$ is a $C^{p,q}$ partition of unity.

In Chapter VI we examine the problem of smooth approximation. We say that a map $f \in C^q(E,F)$ is $C_{p,q}$, $0 \leq q < p \leq \infty$, approximable on a subset U of E if for any $\epsilon > 0$ there exists a $g \in C^p(U,F)$ such that

$$\sup_{x \in U, 0 \leq k \leq q} \|D^k f(x) - D^k g(x)\| < \epsilon \quad .$$

We say that f is strongly $C_{p,q}$ approximable on U if it satisfies the above condition with ϵ replaced by an arbitrary positive continuous function e(x). In general the $C_{p,q}$ approximability of an arbitrary C^q function on an infinite dimensional Banach space is unsolved. We prove, however, that if a B-space E is $C^{p,q}$ smooth and separable, and if $f \in C^q(E,F)$ is $C_{p,q}$ approximable in some neighborhood of every point of E, then f is strongly $C_{p,q}$ approximable on all of E. In the last part of the chapter we prove a theorem that suggests that the C^1 function $\sigma(x) = \sum_i x_i |x_i|$ from ℓ^2 to R might not be $C_{2,1}$ approximable on any open subset of ℓ^2.

The last chapter is devoted to a weak form of $C_{\infty,q}$ approximation to functions defined on ℓ^2. Let $\{U_\alpha\}$

be a locally finite cover of ℓ^2 and let $\{T_\alpha\}$ be a collection of Hilbert-Schmidt operators on ℓ^2. Then we show that for any $f \in C^q(\ell^2, F)$, with $D^q f(x)$ locally uniformly continuous, there exists a $g \in C^\infty(\ell^2, F)$ such that for all α

$$\sup_{x \in U_\alpha, \|h\| \leq 1, 0 \leq k \leq q} \|D^k(f(x) - g(x))[T_\alpha h]\| \leq 1.$$

CHAPTER I

DIFFERENTIAL CALCULUS

We will define the two most important types of derivatives on Banach spaces. For the proofs of the theorems of this section, refer to [5],[8],[10], and [17]. From here on all Banach spaces will be assumed to be real.

Definition. If E and F are Banach spaces, a continuous k-multilinear map T from E into F is a continuous map from E x...x E into F satisfying $T[h_1,..,ah_i'+bh_i'',..h_k] = aT[h_1,..h_i',..h_k] + bT[h_1,..h_i'',..h_k]$ for all real a,b and $1 \le i \le k$.

Definition. If E and F are Banach spaces, $L^k(E,F), k \ge 1$, will denote the set of continuous k-multilinear maps from E into F. We will write L(E,F) for $L^1(E,F)$. If T $\in L^k(E,F)$ then the norm, $\|T\|$, is defined as $\sup_{\|h_i\| \le 1, i \le k} \|T[h_1,..h_k]\|$.

$L^k(E,F)$ with the above norm is a Banach space and from here on $L^k(E,F)$ will be assumed to have the topology induced by the above norm. There is a canonical isomorphism, $*$, between $L^k(E,L^p(E,F))$ and $L^{k+p}(E,F)$

given by $(*T)[h_1,..h_{k+p}] = (T[h_1,..h_k])[h_{k+1},.,h_{k+p}]$ and with this isomorphism we will regard $L^k(E,L^p(E,F))$ and $L^{k+p}(E,F)$ as identical.

Remark. If $T \in L^k(E,F)$ then $T[h]$ will be the shorthand notation for $T[h,..h]$.

Definition. $L_s^k(E,F)$ will denote the set of all continuous symmetric k-multilinear maps from $E \times ... \times E$ into F.

Definition. If f is a map from a Banach space E into a Banach space F, f is said to have a Gateaux derivative at x in direction h if $GDf(x)[h] = \lim_{t \to 0} \frac{f(x+th)-f(x)}{t}$ exists.

It is immediate from the definition that $GDf(x)[h]$ is homogeneous in h(i.e. $GDf(x)[ah] = aGDf(x)[h]$) although $GDf(x)[h]$ may be nonlinear in h or may be linear but unbounded.

Proposition 1.1 If $f:E \to F$ and f has a Gateaux derivative at all points on the segment $[x,x+h]$ in direction h, then for any $w \in F^*$(the dual space of F), $\langle f(x+h)-f(x),w \rangle = \langle GDf(x+\tau h)[h],w \rangle$ where $0 < \tau < 1$ and τ depends on w.

Proposition 1.2 If $f:E \to F$ and f has a Gateaux derivative at all points on the segment $[x,x+h]$ in direction h, then $\|f(x+h)-f(x)\| \leq \sup_{0<\tau<1} \|GDf(x+\tau h)[h]\|$.

Proof. Pick w in Prop. 1.1 such that $\|w\|=1$ and $\|f(x+h)-f(x)\| = \langle f(x+h)-f(x),w\rangle$.

Under certain conditions $GDf(x)[h]$ is a bounded linear function in h:

Proposition 1.3 Suppose that $f:E\to F$ and that $GDf(x)[h]$ exists for all h and for all x in a neighborhood of x_o. Suppose that for all fixed h, $GDf(x)[h]$ is continuous at x_o as a function of x and that $GDf(x_o)[h]$ is continuous in h. Then $h\to GDf(x_o)[h]$ is a bounded linear map from E into F.

Definition. If E and F are B-spaces and $f:E\to F$, then f is said to be Frechet differentiable at $x\in E$ if there exists a $Df(x)\in L(E,F)$ such that

$$\lim_{t\to 0}\sup_{\|h\|\le t} \frac{\|f(x+h)-f(x)-Df(x)[h]\|}{\|h\|} = 0.$$

$Df(x)$ is then said to be the Frechet derivative of f at x.

If $Df(x)$ exists at x, then clearly $Df(x)[h] = GDf(x)[h]$ for all h. $Df(x)$ is invariant within the set of equivalent norms.

Proposition 1.4 If f is Frechet differentiable at x then f is continuous at x.

Proposition 1.5 If $f:E\to F$ has a Frechet derivative at all points on the segment $[x,x+h]$, then for any $w\in F^*$

$\langle (f(x+h) - f(x), w\rangle = \langle Df(x+\tau h)[h], w\rangle$, where $0 < \tau < 1$ and τ depends on w.

<u>Proposition 1.6(Mean Value Theorem)</u> If $f: E \to F$ has a Frechet derivative at all x in the segment $[x, x+h]$, then
$$\|f(x+h) - f(x)\| \le \sup_{0<\tau<1} \|Df(x+\tau h)[h]\| \le \sup_{0<\tau<1} \|Df(x+\tau h)\| \cdot \|h\| .$$

<u>Proposition 1.7</u> If f is a map from an open subset U of a B-space E into F, if f has a derivative at x and if $\|f(x) - f(y)\| \le M(\|x-y\|)$, then $\|Df(x)\| \le M$.

<u>Proposition 1.8</u> Suppose that $f: E \to F$ and that $GDf(x)[h]$ exists and is bounded and linear in h for all x in a neighborhood of some x_o. If $GDf(x)[h]$, considered as a map from E into $L(E,F)$, is continuous at x_o, then f has a Frechet derivative at x_o and $Df(x_o)[h] = GDf(x_o)[h]$.

<u>Definition</u>. If $f: E \to F$ and $Df(x)$ exists in a neighborhood of x_o and if the map $Df(x)$ from E into $L(E,F)$ is differentiable at x_o, then we say that f is twice differentiable at x_o and we write $D^2f(x_o) = D(Df(x_o))$. Note that $D^2f(x_o) \in L(E, L(E,F)) \cong L^2(E,F)$. Inductively we say that D^p exists at x_o if $D^{p-1}f(x)$ exists in a neighborhood of x_o and if $D^{p-1}f(x)$ is differentiable at x_o. We write $D^pf(x_o) = D(D^{p-1}f(x_o))$ and again note that $D^pf(x_o) \in L(E, L^{p-1}(E,F)) \cong L^p(E,F)$.

The following proposition is a generalization of the formula $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$:

Proposition 1.9 If $D^p f(x_o)$, $p \geq 2$, exists and $D^{p-1}f(x)$ is continuous in a neighborhood of x_o, then $D^p f(x_o) \in L_s^p(E,F)$.

Definition. If $0 \leq p < \infty$ and if U is an open subset of a Banach space E, we will say that $f \in C^p(U,F)$ if $f: U \to F$ and $D^p f(x)$ exists and is continuous in U. We say that $f \in C^\infty(U,F)$ if $f \in C^p(U,F)$ for all p.

Note. From here on the words derivative, differentiation etc, will refer to the Frechet derivative unless otherwise stated. $D^p f$ will always denote an element of $L^p(E,F)$.

Proposition 1.10 Let E,F,G be B-spaces and let U be an open subset of E, V an open subset of F. If $f \in C^p(U,V)$ and $g \in C^p(V,G)$ then $f \circ g \in C^p(U,G)$.

Proposition 1.11 Suppose that $f_n \in C^1(U,F)$ where U is an open subset of a B-space E and that $g: U \to F$, $G: U \to L(E,F)$. Suppose that for every point $x_o \in U$ there is a neighborhood N_{x_o} of x_o contained in U such that $f_n(x)$ and $Df_n(x)$ converge uniformly to $g(x)$ and $G(x)$ in N_{x_o}. Then $g \in C^1(U,F)$ and $Dg(x) = G(x)$ for all $x \in U$.

Proposition 1.12(Taylor's Formula) Let $U \subset E$ be a convex neighborhood of x_o and suppose that $f \in C^p(U,F)$. Then

$$f(x_o+h) = \sum_{k=0}^{p-1} \frac{D^k f(x_o)[h]}{k!} + R_p(x_o,h)$$

where

$$R_p(x_o,h) \quad = \quad \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} D^p f(x_o+th)[h]dt \qquad .$$

For every $\epsilon>0$ there is a $\delta>0$ such that if $\|h\| < \delta$ then $\|R_p(x_o,h)\| \leq \epsilon\|h\|^p$.

Proposition 1.13(Inverse Function Theorem) Let E and F be Banach spaces and suppose that $f \in C^p(U,F)$ where U is a neighborhood of $x_o \in E$. Suppose that $Df(x_o)$ is an isomorphism from E into F. Then there exists a neighborhood V of x_o and a C^p map g from f(V) onto V such that $g \circ f$ and $f \circ g$ are the identity maps on V and f(V).

CHAPTER II

SMOOTHNESS CLASSES

In this chapter we introduce a new smoothness classification for Banach spaces. This generalizes the usual C^p smoothness classes.

Definition. If E and F are Banach spaces, and U is an open subset of E and $0 \le q \le p \le \infty$, then $C^{p,q}(U,F)$ will denote those functions f in $C^p(U,F)$ for which $\sup_{x \in U, 0 \le k \le q} \|D^k f(x)\| < \infty$.

Definition. A Banach space E will be said to be $C^{p,q}$ smooth if $C^{p,q}(E,R)$ contains a non-trivial function with bounded support (sometimes called a $C^{p,q}$ bump function).

The standard concept of C^p smoothness places no boundedness restrictions on f or its derivatives:

Definition. A Banach space is said to be C^p smooth if $C^p(E,R)$ contains a non-trivial function with bounded support.

It is easy to check that any $f \in C^p(E,R)$ can be composed with a suitable function in $C^\infty(R,R)$ to yield a function in $C^{p,0}(E,R)$ which has the same support as f. Hence a Banach space is C^p smooth if and only if it is $C^{p,0}$ smooth.

We will prove in the next chapter that there exists a C^2 smooth B-space which is not $C^{2,2}$ smooth so that $C^{p,q}$ smoothness is more retrictive than C^p smoothness.

If E is $C^{p,q}$ smooth then any B-space equivalent to E and any closed subspace of E is again $C^{p,q}$ smooth. Also if $p \geq p'$ and $q \geq q'$, then E is $C^{p',q'}$ smooth.

Several basic theorems proved by Bonic and Frampton in [1] concerning C^p smoothness are generalized below for $C^{p,q}$ smoothness. The following basic proposition will be essential in manipulating $C^{p,q}$ functions:

Proposition 2.1 If $f \in C^{p,q}(E,F)$ and $g \in C^{p,q}(F,G)$, then $g \circ f \in C^{p,q}(E,G)$.

Proof. The proof can be obtained by induction from the following formula, known as Fas di Bruno's formula:

$$D^k(g\circ f) = \sum_{i=1}^{k}\Big((D^i g)\circ f\Big) \cdot \sum_{\substack{a_1+\ldots+a_k = j \\ a_1+2a_2+\ldots ka_k = k}} \frac{k!}{a_1!\cdots a_k!}(Df)^{a_1}\cdots(D^k f)^{a_k}$$

Proposition 2.2 A Banach space E is $C^{p,q}$ smooth if and only if the norm topology on E is equivalent to the topology induced on E by the functions $C^{p,q}(E,R)$.

Proof. The proof is identical to the proof of Prop. 2 of [1].

Proposition 2.3 Suppose that E is a Banach space with equivalent norm α such that $\alpha \in C^{p,q}(U-\{0\},R)$, where U is an open neighborhood of 0. Then E is $C^{p,q}$ smooth.

Proof. For some r, U contains the ball $\{x \mid \alpha(x)<r\}$. Construct a $g \in C^{\infty,\infty}(R,R)$ with $g(t) = 1$ if $t \leq r/2$ and $g(t) = 0$ if $r \geq 1$. Then by Prop. 2.1, $g(\alpha(x)) \in C^{p,q}(E,R)$. Also, $g(\alpha(0)) = 1$ and $g(\alpha(x))$ has bounded support. Q.E.D.

Remarks. It is not known whether the converse to Prop. 2.3 is true, even for $q = 0$ (i.e. does a C^p smooth space have an equivalent C^p norm?). If $\alpha(x)$ is an equivalent norm for E, then $|\alpha(x+h) - \alpha(x)| \leq \alpha(h) \leq K\|h\|$ for some K, so by Prop. 1.7 if $D\alpha(x)$ exists then $\|D\alpha(x)\| \leq K$. Also $D^k(\alpha(x)) = D^k(\alpha\left(\frac{x}{r}\right))r^{1-k}$ and hence if $D^k\alpha$ is bounded on bounded sets it is bounded everywhere.

Definition. If $f \in C^p(E,F)$ and $q \leq p$, then by $\|f\|_q$, we will denote $\sup_{x\in E, 0\leq k\leq q} \|D^k f(x)\|$.

Proposition 2.4. Suppose that F is $C^{p,q}$ smooth but that E is not $C^{p,q}$ smooth. Suppose that U is a bounded open subset of E and that ∂U and $\overline{U}$ are the boundary and closure of U. Then any $f \in C^{p,q}(U,F)$ and $f \in C^0(\overline{U},F)$ has the property that $f(\partial U)$ is dense in $f(\overline{U})$.

Proof. The proof uses the same argument as the argument in the proof of Prop.4 of [1]. Suppose that $f(x)$ is not contained in the closure of $f(\partial U)$ for some x in U. Then by the hypothesis we can find a $\varphi \in C^{p,q}(F,R)$ with $\varphi(f(x)) = 1$ and $\varphi(x) = 0$ in some neighborhood of $f(\partial U)$.

Let $g(y) = \varphi(f(y))$ if $y \in U$ and $g(y) = 0$ otherwise. Then g is nonzero, has bounded support and by Prop.2.1 $g \in C^{p,q}(E,R)$. This contradicts the non-$C^{p,q}$ smoothness of E. Q.E.D.

Remarks. It follows that if f_1 , f_2 are two functions in $C^{p,q}(E,F)$ which agree on the boundary of U, then they agree on all of U. Thus $C^{p,q}$ functions on a non-$C^{p,q}$ smooth B-space have a type of"analytic" property: the values of the function on a bounded open set are uniquely determined by the values on the boundary. The following two problems were posed by Bonic and Frampton for non-C^p smooth B-spaces and they can also be asked for non-$C^{p,q}$ smooth B-spaces: suppose that E is non-$C^{p,q}$ smooth, that F is $C^{p,q}$ smooth and that U is a bounded open subset of E, then what continuous functions on ∂U are boundary values of functions in $C^{p,q}(E,F)$? Also, given $f \in C^{p,q}(U,F)$ and $f \in C^0(\overline{U},F)$ how can f be determined from $f|\partial U$?

The following is a summary of the $C^{p,q}$ smoothness and related properties of various Banach spaces:

a) All finite dimensional B-spaces are $C^{\infty,\infty}$ smooth.

b) Restrepo([15]) has proved that a separable B-space, E, has an equivalent C^1 norm on E-{0} if and only if E* is separable. By the remarks following Prop. 2.3, all Banach spaces with an equivalent norm in $C^1(E-\{0\},R)$ are $C^{1,1}$ smooth. Hence a separable B-space with a separable dual is $C^{1,1}$ smooth.

c) Bonic and Reis in [3] have shown that if E has a C^2 norm away from zero and the dual norm in E* is also of class C^2 away from zero, then E is a Hilbert space.

d) $L^2(S,\Sigma,\mu)$, where (S,Σ,μ) is a positive measure space, is $C^{\infty,\infty}$ smooth. It is easy to check that $D(\|x\|^2)[h] = 2\langle x,h\rangle$, $D^2(\|x\|^2)[h^1,h^2] = 2\langle h^1,h^2\rangle$ and $D^k(\|x\|^2) = 0$ for $k > 2$. Hence $\|x\| \in C^\infty(L^2(S,\Sigma,\mu)-\{0\},R)$ and all the derivatives of $\|x\|$ are bounded on bounded sets. Hence by Prop. 2.3, $L^2(S,\Sigma,\mu)$ is $C^{\infty,\infty}$ smooth.

e) Bonic and Frampton in [1] have completely classified the C^p smoothness of $L^p(S,\Sigma,\mu)$ for $p \geq 1$. Their results are as follows. L^p is C^∞ smooth if p is an even integer. L^p is D_1^{p-1} smooth if p is an odd integer. This means that there exists a C^{p-1} bump function satisfying $\|D^{p-1}f(x+h)-D^{p-1}f(x)\| \leq O(\|x\|)$ for all x. If p is not an integer let [p] be the greatest integer less than p. Then L^p is $D^{[p]}_{p-[p]}$ smooth. This means that there exists a $C^{[p]}$ bump function satisfying $\|D^{[p]}f(x+h) - D^{[p]}f(x)\| \leq O(\|h^{p-[p]}\|)$. If p is an odd integer, ℓ^p and hence any infinite dimensional L^p space is not D^p smooth. This means there does not exist a C^{p-1} bump function f such that $D^p f(x)$ exists for all x. Finally if p is not an integer then ℓ^p and any infinite dimensional L^p space is not $C^{[p]}_{p-[p]}$ smooth. This means there does not exist a $C^{[p]}$ bump function f satisfying $\|D^{[p]}f(x+h)-D^{[p]}f(x)$ $D^{[p]}f(x)\| \le o(\|h\|^{p-[p]})$. By Prop.2.5 below, L^p is $C^{\infty,\infty}$, $C^{p-1,p-1}$ or $C^{[p],[p]}$ smooth if p is an even integer, odd integer or a non-integer respectively.

f) We show in the next section that $c_o(E)$(i.e. the B-space of sequences in E converging to 0 is $C^{p,1}$ smooth if E is $C^{p,1}$smooth, that $c_o(E)$ has a C^p norm if E has a C^p norm and that c_o(i.e. $c_o(R)$) is not $C^{2,2}$ smooth. This example shows that a C^p smooth B-space need not be $C^{p,p}$ smooth.

Proposition 2.5 $L^p(S,\Sigma,\mu)$ is $C^{\infty,\infty}$ smooth, $C^{p-1,p-1}$ smooth or $C^{[p],[p]}$smooth if p is an even integer, odd integer or non-integer respectively.

Proof. Let $\alpha(f) = (\|f\|)^p = \int |f(x)|^p d\mu(x)$. Then it can be shown(refer to [1]) that

$$D^k(\alpha(f(x))[h_1,..h_k] = \int \frac{p!}{k!} |f(x)|^{p-k}(\text{sgn } f(x))^k \cdot h_1(x)\cdots h_k(x)d\mu(x)$$

for k < [p] and

$$= p! \quad \text{for p an even integer.}$$

By Hölder's inequality, $|D^k(\alpha(f(x))[h_1,..,h_k]|$ is

$$\leq \frac{p!}{k!}\left(\int |f(x)|^{p-k}\right)^{p/(p-k)} \cdot \left(\int |h_1(x)|^p\right)^{1/p} \cdots \left(\int |h_k(x)|^p\right)^{1/p}$$

$$\leq \frac{p!}{k!}\|f\|^{p-k}\|h_1\| \cdots \|h_k\| .$$

Therefore $D^k\alpha$ and hence $D^k\|x\|$ is bounded on bounded sets for $k < [p]$ and if p is an even integer, for all k. Hence by Prop. 2.3 the result is proved.

Definition. A family of functions $\{\varphi_\alpha\} \in C^p(E,R^+)$ will be called a C^p partition of unity if every point of E has a neighborhood on which all but a finite number of φ_α's vanish and $\sum_\alpha \varphi_\alpha = 1$.

Definition. A Banach space E will be said to "admit C^p partitions of unity" if for every open covering $\{U_\beta\}$ of E there is a C^p partition of unity $\{\varphi_\alpha\}$ such that the support of each φ_α is contained in some U_β.

Proposition 2.6.[1] If E is a separable C^p smooth B-space then E admits C^p partitions of unity.

Remark. Every metric space is paracompact and hence all B-spaces admit C^0 partitions of unity. It is not known if separability can be dropped from Prop.2.6, in particular, it is not known whether any non-separable Hilbert space admits C^1 partitions of unity.

1) Refer to Bonic and Frampton [1]. A stronger version of this theorem is contained in Chapter V.

<u>Proposition 2.7</u> Suppose that E and F are B-spaces and that E admits C^p partitions of unity. Then given $f \in C^0(E,F)$ and $e(x) \in C^0(E,R)$ with $e(x) > 0$, there exists a $g \in C^p(E,F)$ such that $\|f(x)-g(x)\| < e(x)$ for all x in E.

<u>Proof.</u> For every x in E find neighborhoods N_x^1 and N_x^2 of x such that $\inf_{y \in N_x^1} e(y) \geq e(x)/2$ and if $y \in N_x^2$ then $\|f(y)-f(x)\| < e(x)/4$. Now $\{N_x^1 \cap N_x^2\}$ covers E and by the hypothesis we can find a C^p partition of unity $\{\varphi_\alpha\}$ supported by $\{N_x^1 \cap N_x^2\}$. Pick, for each α, an x_α in the support of φ_α and define $g(x) = \sum_\alpha f(x_\alpha)\varphi_\alpha(x)$. Then $g(x)$ is an element of $C^p(E,F)$ and

$$\|f(x)-g(x)\| = \Big\| \sum_{\{\alpha | x \in \operatorname{supp} \varphi_\alpha\}} (f(x)-f(x_\alpha))\, \varphi_\alpha(x) \Big\|$$

$$\leq \sum_{\{\alpha | x \in \operatorname{supp} \varphi_\alpha\}} \|f(x)-f(x_\alpha)\| \varphi_\alpha(x)$$

$$\leq \sum_{\{\alpha | x \in \operatorname{supp} \varphi_\alpha\}} \Big(\|f(x)-f(x'_\alpha)\| + \|f(x'_\alpha) - f(x_\alpha)\|\Big)\, \varphi_\alpha(x)$$

where x'_α is a point in E such that $\operatorname{supp} \varphi_\alpha \subset \{N^1_{x'_\alpha} \cap N^2_{x'_\alpha}\}$.

The last summation is $\leq \sum_{\{\alpha | x \in \operatorname{supp} \varphi_\alpha\}} e(x'_\alpha)/2 \cdot \varphi_\alpha(x)$

$\leq \sum e(x)\, \varphi_\alpha(x) = e(x).$ Q.E.D.

CHAPTER III

DIFFERENTIABLE FUNCTIONS ON $c_o(E)$

Definition. If E is a Banach space, then $c_o(E)$ denotes the Banach space of all sequences $X = \{x_i\}$ with x_i in E and $\|x_i\| \to 0$. The norm on $c_o(E)$ is defined as

$$\|X\| = \sup_i \|x_i\|$$

We write c_o for $c_o(R)$.

Bonic and Frampton in [1] and [2] proved that if E is C^p smooth then $c_o(E)$ is also C^p smooth. In this chapter we prove several stronger results. We show in Theorem 3.2 that if E is $C^{p,1}$ smooth then $c_o(E)$ is also $C^{p,1}$ smooth and in Theorem 3.1 that if E has a C^p norm then $c_o(E)$ also has a C^p norm. In Theorem 3.3 we show that c_o is not $C^{2,2}$ smooth. This is the first example of a C^p smooth Banach space which is not also $C^{p,p}$ smooth.

There is an important class of spaces equivalent to $c_o(E)$. Suppose that K is a compact subset of R^n and that $0 < \alpha < 1$. If $f \in C^0(K,E)$ define

$$\|f\|_\alpha = \sup_{x \neq y} \|f(x)-f(y)\| / (\|x-y\|)^\alpha .$$

Let $C^\alpha(K,E) = \{f \in C^0(K,E) \mid \|f\|_\alpha < \infty\}$ and let

$\Lambda^\alpha(K,E) = \{f \in C^\alpha(K,E) \mid$ for any $\epsilon > 0$ there is a $\delta > 0$ such that $\|f(x) - f(y)\| \le \epsilon \|x-y\|^\alpha$ whenever $\|x-y\| \le \delta\}$.

Then Bonic, Frampton and Tromba in [4] have proved that $\Lambda^{\alpha}(K,E)$ is equivalent to $c_o(E)$.

N.H.Kuiper has shown that c_o has an equivalent C^{∞} norm(refer to [1]). We give the following generalization of that result:

Theorem 3.1 Suppose that E has a C^p norm, $\|x\|$, away from zero. Then $c_o(E)$ also has a C^p norm away from zero.

Proof. First construct an h in $C^{\infty}(R,R)$ such that h is decreasing, $h(t) = 1$ for $t \leq 1$, $h(3/2) = 1/2$, $h(t) = 0$ for $t \geq 2$ and $h(t)$ is concave downward for $t \leq 3/2$. Now if $X = \{x_1, x_2, .. \}$ is in $c_o(E)$, define

$$\psi(X) = \prod_{i=1}^{\infty} h(\|x_i\|) \quad .$$

ψ locally depends only on a finite number of x_i's and hence $\psi \in C^p(c_o(E),R)$. Now let $G = \{X \mid \psi(X) \geq ½\}$. We show that G is convex. To do this suppose that $\psi(X)$ and $\psi(Y)$ are $\geq ½$ and suppose that $\|x_i\| \leq 1$ and $\|y_i\| \leq 1$ for $i > N$ and that $0 < t < 1$. Then

$$\begin{aligned} \psi(tX+(1-t)Y) &= \prod_{i=1}^{\infty} h(\|tx_i + (1-t)y_i\|) \\ &\geq \prod_{i=1}^{\infty} h(t\|x_i\| + (1-t)\|y_i\|) \\ &= \prod_{i=1}^{N} h(t\|x_i\| + (1-t)\|y_i\|) \quad . \end{aligned}$$

Now $\|x_i\|$ and $\|y_i\|$ are $\leq$ 3/2 for all i, hence by the concavity of h the last product is

$$\geq \prod_{i=1}^{N}\left(th(\|x_i\|) + (1-t)h(\|y_i\|)\right) = \sum_{k=0}^{N} t^{N-k}(1-t)^k \, a_k$$

where $a_k = \sum_{F\in\Xi_k} \left(\prod_{i\in F} h(\|x_i\|) \cdot \prod_{\substack{i\notin F \\ i\leq N}} h(\|y_i\|)\right)$

and where Ξ_k is the set of all subsets of 1,2,..N having k members.

Now if $b_i > 0$ then $\sum_{i=1}^{m} b_i \geq \left(\prod_{i=1}^{m} b_i\right)^{1/m}$. Hence

$$a_i \geq \left(\prod_{F\in\Xi_k}\left(\prod_{i\in F} h(\|x_i\|)\cdot \prod_{i\notin F} h(\|y_i\|)\right)\right)^{1/\binom{N}{k}}$$

$$= \left(\left(\prod_1^N h(\|x_i\|)\right)^{\frac{N-k}{k}\binom{N}{k}} \cdot \left(\prod_1^N h(\|y_i\|)\right)^{\frac{k}{N}\binom{N}{k}}\right)^{1/\binom{N}{k}}$$

$$\geq (1/2)^{(N-k)/N} \cdot (1/2)^{k/N} = 1/2 .$$

Then $\psi(tX + (1-t)Y) \geq \sum_{k=0}^{N} t^{N-k}(1-t)^k \cdot 1/2 = 1/2$. Hence G is convex. Let $\alpha(X)$ be the Minkowski functional of G. Then $\alpha(X)$ is implicitly determined by the equation $\psi\left(\frac{X}{\alpha(X)}\right) = 1/2$. Since $\alpha(X)$ locally depends on only a finite number of variables, we can apply the finite dimensional implicit function theorem to conclude that $\alpha(X) \in C^p(c_o(E),R)$. Then $\alpha(X)$ is an equivalent norm because G is bounded and contains an open neighborhood of 0.

Corollary 3.1 c_o has an equivalent C^∞ norm and therefore by Prop. 2.3, c_o is $C^{\infty,1}$ smooth.

Remark. Although $\alpha(x)$ has a bounded derivative, ψ itself does not. In fact any function, $F(X)$, of the form

$$F(X) = \prod_{1}^{\infty} (h(\|x_i\|))$$

where $h(t) = 1$ for $|t| \le 1$, $h(t) = 0$ for $|t| \ge 2$ is not of class $C^{p,1}(c_o(E),R)$. To show this it suffices to consider $E = R$.

Let a be the largest number between 1 and 2 such that $h(a) = 1$. For any M choose n such that $(h(a+1/2M))^n < ½$ and let $X_o = \{n\ a's,\ 0's\}$ and $X_1 = \{n\ (a+1/2M)'s,\ 0's\}$. Then we have $F(X_o) = 1$, $F(X_1) < ½$ and $\|X_o - X_1\| = 1/2M$. By Prop. 1.7

$$½ \le |F(X_1) - F(X_o)| \le \|X_1 - X_o\| \cdot \sup_X \|DF(X)\|$$

$$= 1/2M \cdot \sup_X \|DF(X)\| \quad .$$

Hence $\sup_X \|DF(X)\| \ge M$, and since M is arbitrary, $\sup_X \|DF(X)\| = \infty$.

It is possible to construct a nontrivial $C^{\infty,1}$ function on c_o without evaluating a Minkowski functional as the following example shows.

Example. Let $h \in C^{\infty,\infty}(R,R)$, $h(t) \ge 0$, $h(t) = 0$ if $|t| \ge ¼$ and $\int_{-¼}^{¼} h(t)dt = 1$. Define

$$\varphi_n(X) = \int_{-¼}^{¼} \cdots \int_{-¼}^{¼} h(y_1)\cdots h(y_n) F(\{x_1+y_1, \ldots x_n+y_n, x_{n+1}, \ldots\}) dy_1 \ldots dy_n$$

where $F(X) = \inf_{\|Y\| \le 1} \|X-Y\|$. F is continuous because $\|F(X)-F(Y)\| \le \|X-Y\|$. Suppose that $|x_m| \le 1/4$ if $m > n(X)$. Now if $\|X'-X\| \le 1/4$, $\|Y\| \le 1/4$ and $x'_m = x'_m$ for $m \le n(X)$, then $F(X'+Y) = F(X+Y)$. Hence when $\|Z-X\| \le 1/4$, $\varphi_{n(X)}(Z)$ depends only on the first n(X) coordinates and therefore is C^∞. Also $\|X'-X\| \le 1/4$, $\|Y\| \le 1/4$ and $y_1 = .. = y_n = 0$ imply that $F(X'+Y) = F(X')$. Hence $\varphi_m(X') = \varphi_{n(X)}(X')$ when $m \ge n(X)$ and $\|X'-X\| \le 1/4$. The above implies that

$$\varphi(X) = \lim_{n\to\infty} \varphi_n(X)$$

exists and is C^∞. Now $|\varphi_n(X) - \varphi_n(Z)|$ is

$$\le \int_{-1/4}^{1/4} \ldots \int_{-1/4}^{1/4} h(y_1)\cdot\cdot h(y_n)\|X-Z\|\, dy_1 \ldots dy_n = \|X-Z\| .$$ Hence

$|\varphi(X) - \varphi(Z)| \le \|X-Z\|$ which gives $\|D\varphi(X)\| \le 1$ for all X.

Finally let $r \in C^\infty(R,R)$, $0 \le r(t) \le 1$, $r(t)=1$ if $t \le 0$ and $r(t) = 0$ if $1/4 \le t$. Then $r(\varphi(X) \in C^{\infty,1}(c_o,R)$, $r(\varphi(0)) = 1$ and the support of $r(\varphi(X)$ is contained in the unit ball.

Theoerm 3.2 If E is $C^{p,1}$ smooth then so is $c_o(E)$.

Proof. First find an f in $C^{p,1}(E,R)$ such that $f(x) = 1$ if $\|x\| \le 1$, $f(x) = 0$ if $2 \le \|x\|$ and $0 \le f(x) \le 1$. Define a map T from $c_o(E)$ into c_o as follows: If $X = \{x_1, x_2, \ldots \} \in c_o(E)$ let $T(X) = \{1-f(x_1), 1-f(x_2), \ldots,\}$. Then since T locally depends on a finite number of coordinates, $T \in C^p(c_o(E), c_o)$. Also $\|T(X) - T(Y)\|$

$$= \sup_i |f(x_i) - f(y_i)|$$

$$\leq \sup_x \|Df(x)\| \cdot \sup_i |x_i - y_i| = \|f\|_1 \cdot \|X-Y\| .$$

Hence $T \in C^{p,1}(c_o(E), c_o)$. By Cor. 3.1 we can find g in $C^{\infty,1}(c_o, R)$ with $g(0) = 1$ and $g(X) = 0$ if $\|X\| \geq 1$. Then $g(T(X)) \in C^{p,1}(c_o(E), R)$ and $g(T(0)) = 1$ and $g(T(X)) = 0$ if $\|X\| \geq 2$. Q.E.D.

The following theorem will imply that a $C^{2,2}$ bump function cannot be found for c_o. We actually prove a slightly stronger result.

<u>Theorem 3.3</u> Let $f \in C^1(c_o, R)$ with $Df(X)$ uniformly continuous. Then the support of f is unbounded.

<u>Proof.</u> If not, then there would exist an f in $C^1(c_o, R)$ such that $f(0) = 1$, $f(X) = 0$ if $\|X\| \geq 1$ and Df is uniformly continuous. Pick N such that $\|H\| \leq 1/N$ implies $\|Df(X+H) - DF(X)\| \leq 1/2$. Now if $\|H\| \leq 1/N$ then by Prop.1.5 there is a τ with $0 < \tau < 1$ such that $f(X+H) - f(X) = Df(X+\tau H)[H]$ so that

$$|f(X+H) - f(X) - Df(X)[H]| = |Df(X+\tau H)[H] - Df(X)[H]|$$
$$\leq \tfrac{1}{2}\|H\| .$$

Let A be the set of all X in c_o such that $2^N - 1$ of the first 2^N components of X have absolute value 1/N, the remaining component of the first 2^N components has absolute value less than or equal to $1/N$ and all the other components are zero. Since A is connected and even, for all X in c_o there exists a Y in A such that $Df(X)[Y] = 0$. Therefore we can pick inductively $H_1, \ldots H_N \in A$ such that $Df(H_1 + \cdots + H_{k-1})[H_k] = 0$ and such that $H_1 + \cdots + H_k$ has at least 2^{N-k} components equal to k/N. But then $\|H_1 + \ldots + H_N\| = 1$ and $|f(H_1 + \cdots + H_N) - f(0)| \le$

$$\sum_{k=1}^{N} |f(H_1 + \cdots + H_k) - f(H_1 + \cdots + H_{k-1}) - Df(H_1 + \cdots + H_{k-1})[H_k]|$$

$\le \sum_{k=1}^{N} \frac{1}{2}\|H_k\| = \frac{1}{2}$ which is a contradiction. Q.E.D.

<u>Corollary 3.2</u> c_o and $c_o(E)$ are not $C^{2,2}$ smooth.

<u>Proof.</u> Any function in $C^{2,2}(c_o, R)$ has a uniformly continuous first derivative.

<u>Corollary 3.3</u> Suppose that U is a bounded open subset of c_o and that $f \in C^0(\overline{U}, R)$, $f \in C^1(U, R)$ and $Df(X)$ is uniformly continuous on U. Then $f(\partial U)$ is dense in $f(\overline{U})$.

CHAPTER IV

ZERO SETS OF C^p FUNCTIONS

In this chapter we consider the problem of finding a C^p function with a prescribed zero set. It will be shown that for separable Banach spaces, $C^{p,p}$ smoothness is a necessary and sufficient condition that every closed set be the locus of zeros of a C^p function. If a B-space is C^p smooth but not $C^{p,p}$ smooth it will be shown that the problem can still be solved for a special class of closed sets.

Theorem 4.1 Let E be a separable $C^{p,p}$ smooth Banach space. Then every closed subset, A, of E is the zero set of some $C^{p,p}$ function.

Proof. First construct an $h \in C^{p,p}(E,R)$ such that $0 \leq h(x) \leq 1$, $h(x) = 1$ if $\|x\| \leq 1$ and $h(x) = 0$ if $2 \leq \|x\|$. Let x_i be a dense countable subset of the complement of A and let $d(x_i,A)$ denote the distance from x_i to A. Define $f_i(x) = h\left(\frac{x-x_i}{2d(x_i,A)}\right)$ and let

$$M_{ik} = \sup_{x \in E}\|D^k f_i(x)\| \quad \text{and} \quad N_p = \max_{i,k \leq p} M_{ik} .$$

Then define $g_n(x) = \sum_{p=1}^{n}\left(f_p(x)/2^p N_p\right)$ and observe that if $n > m > k$ then

$$\sup_{x\in E}\|D^k g_n(x)-D^k g_m(x)\| \le \sum_{p=m+1}^{n}\left(\sup_{x\in E}\frac{D^k f_p(x)}{2^p N_p}\right)$$

$$\le \sum_{p=m+1}^{n}\left(\frac{M_{pk}}{2^p N_p}\right) \le \sum_{p=m+1}^{n}\left(\frac{N_p}{2^p N_p}\right) \le \frac{1}{2^m} .$$

Hence the $D^k g_n(x)$'s converge uniformly to continuous functions $g^{(k)}(x)$. Repeated application of Prop. 1.11 gives $Dg^{(k)}(x) = g^{(k+1)}(x)$. Hence $g^{(0)}(x) \in C^p(E,R)$. That $g^{(0)}$ is also in $C^{p,p}(E,R)$ follows easily. If $x\in A$ then $f_i(x) = 0$ for all i and hence $g^{(0)}(x)=0$. If $x\notin A$ then find an x_i such that $d(x,x_i) < \frac{1}{2}d(x_i,A)$. Then $f_i(x) > 0$ which implies $g^{(0)}(x) > 0$. Q.E.D.

<u>Corollary 4.1</u> Let A and B be disjoint closed subsets of a $C^{p,p}$ smooth separable Banach space. Then there exists a C^p function F such that $0 \le F(x) \le 1$ and $F(x) = 0$ or 1 if and only if $x \in A$ or B.

<u>Proof.</u> By Urysohn's Lemma there is an f in $C^0(E,R)$ satisfying $0 \le f(x) \le 1$ and $f(A) = 0$, $f(B) = 1$. Apply Prop.2.7 to obtain an $f_1 \in C^p(E,R)$ with $|f(x)-f_1(x)| < 1/3$. By the theorem there exists $g_i(x) \in C^p(E,R), i=1,2$, with $g_i(x) \ge 0$ and $g_i(x) = 0$ if and only if $x \in A$ or B for $i = 1$ or 2. Now find a $\varphi \in C^{\infty,\infty}(R,R)$ with $0\le\varphi(t)\le 1$, $\varphi(\{x|x\le 0\}) = 0$ and $\varphi(\{x|x\ge 1\}) = 1$. Then $f_2(x) = \varphi(3(f_1(x)-1/3))$ has values between 0 and 1 and has value 0 on A and 1 on B. We can then take F(x) to equal

$$\left(\frac{1+\varphi(1-g_2(x))}{2}\right)\cdot \varphi\left(f_2(x)+g_1(x)\right) \qquad \text{Q.E.D.}$$

<u>Theorem 4.2</u> Let E be a Banach space which is not $C^{p,p}$, $p<\infty$, smooth. Then there exists a closed subset of E which is not the zero set of any C^p function.

<u>Proof.</u> Let B_i be a sequence of disjoint open balls in E of radii $1/i$ converging to some point x_o in E such that distance$(B_i,B_j) > 0$ if $i \neq j$. Let $A = E - \bigcup_i B_i$ and suppose that $f \in C^p(E,R)$ with $f(x) = 0$ if and only if $x \in A$. Then letting $g_i(x) = f(x)$ when $x \in B_i$ and $g(x) = 0$ when $x \notin B_i$, we have that the $g_i(x)$'s are of class C^p and have bounded supports. By the non $C^{p,p}$ smoothness of E, $\sup_{x\in B_i} \|D^p g_i(x)\| = \infty$. It then follows that $D^p f(x)$ is not continuous at x_o which is a contradiction. Q.E.D.

<u>Corollary 4.2</u> There exists a closed subset of c_o which is not the zero set of any C^2 function.

<u>Proof.</u> c_o is not $C^{2,2}$ smooth by Cor. 3.2 .

By Theorem 4.1 and 4.2 a separable Banach space, E, is $C^{p,p}$ smooth if and only if every closed subset of E is the zero set of a C^p function. Theorem 4.1 may be true for nonseparable B-spaces but this appears to be a difficult problem. Indeed, if every closed subset of a nonseparable Hilbert space H was the zero set of a C^p

function, then H would admit C^p partitions of unity. To see this let $\{U_\alpha\}$ be any locally finite cover of H. By assumption we can find $f_\alpha \in C^p(H,R)$ such that $f_\alpha(x) \geq 0$ and $f_\alpha(x) > 0$ if and only if $x \in U_\alpha$. Then $\varphi_\alpha(x) = f_\alpha(x)/\sum_\alpha f_\alpha(x)$ is a C^p partition of unity refining $\{U_\alpha\}$.

Another question that we can pose is: Given disjoint subsets A and B of a B-space E with distance(A,B) > 0, does there exist a $C^{p,q}$ function f such that f(A)=0 and f(B)=1 ? An equivalent question is: Given a subset A of E and a $\delta > 0$, does there exist a $C^{p,q}$ function f with f(A)=1 and f(x) = 0 if distance(X,A)$\geq \delta$? If A is convex and the space is uniformly convex the answer is yes for p=q=1 as we show in the corollary to the next theorem.

<u>Theorem 4.3</u> Suppose that E is a uniformly convex B-space and that $\|x\| \in C^1(E-\{0\},R)$. Then if A is a closed convex subset of E, $d(x) = \text{distance}(x,A) \in C^1(E-A,R)$.

<u>Proof.</u> A well known consequence of uniform convexity is that there exists a unique closest point p(x) in A to $x \in E$ and that p(x) is continuous. Now suppose that $x \notin A$. Since the norm is C^1, $\|x-(p(x)+h)\| = \|x-p(x)\| - D\|x - p(x)\|[h] + o(\|h\|)$. For all h with $p(x)+h \in A$ we have, by the definition of p(x), that $\|x-(p(x)+h)\| \geq \|x-p(x)\|$, hence $D\|x-p(x)\|[h] \leq 0$. The hyperplane $L = \{y \mid D\|x-p(x)\|(y-p(x))=0\}$ is therefore a supporting hyperplane for A at p(x). Hence for all $\|h\|<p(x)$, $d(x+h,L) \leq d(x+h) \leq d(x+h,p(x))$. This gives

$\|x-p(x)\|+D\|x-p(x)\|[h] \le d(x+h) \le \|x-p(x)\|+D\|x-p(x)\|[h]+o(\|h\|)$. Hence $|d(x+h) - d(x) - D\|x-p(x)\|[h]| \le o(\|h\|)$ so that $Dd(x) = D\|x-p(x)\|$. Since $p(x)$ is continuous, $Dd(x)$ is continuous.

Remark. Uniform convexity implies that E is reflexive and hence by statement b) of Chapter II, if E is separable and uniformly convex then $\|x\| \in C^1(E - \{0\}, R)$.

Corollary 4.3 If A is a convex subset of a uniformly convex B-space E and if $\delta > 0$, there exists an $f \in C^{1,1}(E,R)$ with $f(A) = 1$ and $f(x) = 0$ if distance$(x,A) \ge \delta$.

Proof. Find a $g \in C^{\infty,\infty}(R,R)$ with $g(t) = 1$ if $t \le \delta/3$ and $g(t) = 0$ if $t \ge 2\delta/3$. Then $g \circ d(x,A) \in C^{1,1}(E,R)$ and satisfies the boundary conditions.

If a separable B-space is C^p smooth but not $C^{p,p}$ smooth, then certain closed subsets may still be the zero sets of C^p functions.

Definition. A subset A of a B-space E will be said to have a cylindrical boundary if for all $x \in \partial A$ there exists a neighborhood N_x of x such that $\mathring{A} \cap N_x$ ($\mathring{A}$ is the interior of A) is weakly open in N_x (i.e. $\mathring{A} \cap N_x = N_x \cap W$ for some weakly open W).

Theorem 4.4 Let E be a C^p smooth separable B-space. Then any closed subset A whose complement has a cylindrical boundary is the zero set of some C^p function.

Lemma 4.1 Let A be a weakly open subset of a separable B-space. Then the complement of A is the zero set of a C^∞ function.

Proof. We can write $A = \bigcap_i W_i$ where

$$W_i = \bigcap_{k=1}^{n_i} \{x \mid |y_{i(k)}(x)|<1\}, \quad y_{i(k)} \in E^* .$$

Let $\sigma \in C^\infty(R,R)$ with $\sigma(t) = 0$ if $|t| \geq 1$ and $0<\sigma(t)\leq 1$ if $|t| < 1$. Define $g_i(x) = \prod_{k=1}^{n_i} \sigma\left(y_{i(k)}(x)\right)$. Then $g_i(x) \in C^{\infty,\infty}(E,R)$ and $g_i(x) > 0$ if and only if $x \in W_i$. Let $M_{jk} = \sup_{x\in E}\|D^k g_j(x)\|$ and let $N_p = \sup_{j,k\leq p} M_{jk}$. Define

$f(x) = \sum_{p=1}^{\infty} \frac{1}{2^p N_p} g_p(x)$. We can apply the same method as in the proof of Theorem 4.1 to show that the derivatives of all the partial sums converge uniformly. Prop. 1.11 gives that $f \in C^{\infty,\infty}(E,R)$. Clearly $f(x) > 0$ if and only if $x\in A$.

Q.E.D.

Proof of Theorem 4.4 For each x in E find an N_x such that $N_x \cap C\bar{A}$ is weakly open in N_x. By Prop. 2.6 there exists a C^p partition of unity $\{\varphi_i\}$ refining $\{N_x\}$. Then $CA \cap \operatorname{supp} \varphi_i = \operatorname{supp} \varphi_i \cap W_i$ for some weakly open set W_i. Using Lemma 4.1 we find $f_i(x) \in C^\infty(E,R)$ such that $f_i(x) \geq 0$ and $f_i(x) > 0$ if and only if $x \in W_i$. Defining $F(x) = \sum_i f_i(x)\varphi_i(x)$, we have that $F \in C^p(E,R)$ and $F(x)=0$ if and only if $x \in A$.

Q.E.D.

Remarks. a) An example in ℓ^2 of an open set with cylindrical boundary which is not itself weakly open is $C = \{x \mid |x_i| < 1, \text{ for all } i\}$. For any $y \in \ell^2$ suppose that $|y_i| < 1/4$ for $i > n(y)$. Then if B is the open ball of radius $1/4$ about y, $B \cap C = B \cap \{x \mid |x_i| < 1, i \leq n(y)\}$. Hence C has a cylindrical boundary. Since C contains no linear subspace it is not weakly open.

b) Open sets with cylindrical boundaries are closed under finite intersections and finite unions but not under countable unions. Again consider ℓ^2 and let $C_n = \{x \mid |x_1 - 1/n| < 1/3n, |x_i| < 1/3n \text{ for } i \geq 2\}$. By a) each C_n has a cylindrical boundary but we show that $\bigcup C_n$ does not. Let U be any neighborhood of zero and suppose that U contains the open ball of radius r. Find n such that $1/n < r/2$ and suppose that W is any weak neighborhood of e_1/n, ($\{e_i\}$ is the orthonornal basis). Then W contains a set of the form

$$N = \{x \mid \langle y^j, (x - e_1/n) \rangle < 1\}, \quad j = 1, \ldots m .$$

Find k such that $|y^j_k| < 2n/3$. Then the point $e_1/n + 3e_k/2n$ lies in both N and U but is not contained in $\bigcup C_n$. Hence $\bigcup C_n$ does not have a cylindrical boundary at 0.

c) We pose the following question: Suppose that E is C^p smooth but not $C^{p,p}$ smooth. Then if A is the zero set of some C^p function, is $\partial(CA)$ cylindrical?

CHAPTER V

$C^{p,q}$ PARTITIONS OF UNITY

Definition. $\{\varphi_\alpha\}$ will be called a $C^{p,q}$ partition of unity on a Banach space E if $\{\varphi_\alpha\}$ is a partition of unity and $\varphi_\alpha \in C^{p,q}(E,R)$ for each α.

Definition. A B-space E will be said to "admit $C^{p,q}$ partitions of unity" if for every open cover $\{U_\beta\}$ of E there exists a $C^{p,q}$ partition of unity $\{\varphi_\alpha\}$ supported by $\{U_\beta\}$.

We show in this chapter that a separable $C^{p,q}$ smooth B-space admits $C^{p,q}$ partitions of unity. We also show that while $\{\varphi_i\}$ may be a C^2 partition of unity for ℓ^2, there exists a bounded sequence of real numbers $\{a_i\}$ such that $\sum_i a_i\varphi_i(x) \notin C^{2,2}(\ell^2,R)$. We begin with a lemma.

Lemma 5.1 Let E be a separable $C^{p,q}$ smooth B-space and suppose that $\{U_\alpha\}$ is an open cover of E. Then there exists four countable locally finite open covers $\{V_i^1\}$, $\{V_i^2\}$, $\{V_i^3\}$ and $\{V_i^4\}$ of E and maps $g_i \in C^{p,q}(E,R)$ such that:

1) $\bar{V}_i^1 \subset V_i^2$, $\bar{V}_i^2 \subset V_i^3$, $\bar{V}_i^3 \subset V_i^4$, $i \geq 1$.
2) $\{V_i^4\}$ refines $\{U_\alpha\}$ and is locally finite.
3) $0 \leq g_i(x) \leq 1$, $g_i(\bar{V}_i^2) = 1$ and $g_i(CV_i^3) = 0$.
4) If $q \geq 1$ then $\text{dist}(V_i^j, CV_i^{j+1}) > 0$ for $j = 1,2,3$.

Proof. Using the $C^{p,q}$ smoothness, find for every $x \in E$ a $\varphi_x \in C^{p,q}(E,R)$ such that $0 \le \varphi_x \le 1$, $\varphi_x(x)=1$ and support φ_x is contained in some U_α. Let $A_x = \{y | \varphi_x(y) > ½\}$. Then $\{A_x\}$ covers E and since E is Lindelof, we can extract a countable subset of $\{A_x\}$ which also covers E. Denote the elements of this subset by $A_i = \{y | \varphi_i(y) > ½\}$. Now we can find $f_j \in C^{\infty,\infty}(R,R)$, $j \ge 2$, such that

$$f_j(t_1,\ldots,t_j) = 1 \text{ if } t_j \ge ½ \text{ and } t_i \le \frac{1}{2} + \frac{1}{j}, i < j,$$
$$= 0 \text{ if } t_j \le \frac{1}{2} - \frac{1}{j}, \text{and } t_i \ge \frac{1}{2} + \frac{2}{j}, i < j.$$

Now let $\Psi_1(x) = \varphi_1(x)$ and $\Psi_j(x) = f_j(\varphi_1(x),\ldots,\varphi_j(x))$ for $j \ge 2$. Define

$$V_i^1 = \{x | \Psi_i(x) > ¾\}$$
$$V_i^2 = \{x | \Psi_i(x) > ½\}$$
$$V_i^3 = \{x | \Psi_i(x) > ¼\}$$
$$V_i^4 = \{x | \Psi_i(x) > 0\} \quad .$$

Property 1) follows from the definition. Since $V_i^4 \subset \operatorname{supp} \varphi_i$, $\{V_i^4\}$ refines $\{U_\alpha\}$. To show that $\{V_i^1\}$ covers E suppose that $x \in E$ and that $i(x)$ is the first integer for which $\varphi_i(x) \ge ½$. Such an integer exists because the A_i's cover E. Then $\Psi_{i(x)} = 1$ and hence $x \in V_{i(x)}^1$ so $\{V_i^1\}$ covers E. Now again suppose that $x \in E$ and find an integer $n(x)$ such that $\varphi_{n(x)}(x) > ½$. Then there exists, by the continuity of $\varphi_{n(x)}$, a neighbohood N_x of x and an $a_x > ½$ such that

$\inf_{y \in N_x} \varphi_{n(x)}(y) \geq a_x$. Pick k large enough so that $2/k < a_x - ½$. Then $\varphi_{n(x)}(y) > \frac{1}{2} + \frac{2}{k}$ for $y \in N_x$ and hence $\Psi_j(y) = 0$ for $y \in N_x$ and $j \geq k$. Therefore $N_x \cap V_j^4 = \emptyset$ for $j \geq k$ so that $\{V_i^4\}$ is locally finite. Finally take some $h \in C^\infty(R,R)$ with $h(t) = 0$ if $t \leq ¼$, $h(t) = 1$ if $t \geq ¾$ and $0 \leq h(t) \leq 1$. Defining $g_i(x)$ $h(\Psi_i(x))$ we have that $g_i \in C^{p,q}(E,R)$ and that property 3) holds. Property 4) follows from Prop. 1.5. Q.E.D.

<u>Theorem 5.1</u> A separable $C^{p,q}$ smooth Banach space admits $C^{p,q}$ partitions of unity.

<u>Proof.</u> Let $\{U_\alpha\}$ be any open cover of E and use Lemma 5.1 to get four locally finite covers $\{V_i^j\}$, j =1,2,3,4 refining $\{U_\alpha\}$ and maps $g_i \in C^{p,q}(E,R)$ satisfying the conditions of the lemma. Let $f_1(x) = g_1(x)$ and $f_i(x) = g_i(x)(1-g_1(x))\cdots(1-g_{i-1}(x))$ for $i > 1$. Then $f_i \in C^{p,q}(E,R)$ and $\operatorname{supp} f_i(x) \subset \operatorname{supp} g_i(x) \subset V_i^3$, hence every point of E has a neighborhood on which all but a finite number of f_i's vanish. Now since $\{x \mid g_i(x) = 1\} \supset V_i^2$, and $\{V_i^2\}$ covers E, for every x,

$$\prod_{i=1}^{n}(1-g_i(x)) = 0, \text{ for some } n.$$

Hence $\sum_{i=1}^{\infty} f_i(x) = 1 - \lim_{n\to\infty} \prod_{i=1}^{n} (1 - g_i(x)) = 1$ and $\{f_i(x)\}$ is a $C^{p,q}$ partition of unity refining $\{U_\alpha\}$ Q.E.D.

If E is $C^{p,q}$ smooth and separable then by the above theorem we can find a $C^{p,q}$ partition of unity $\{\varphi_i\}$ such that diam.(supp φ_i) are uniformly bounded. We ask the question: Does there exist a $C^{p,q}$ partition of unity $\{\varphi_i\}$ such that

$$(5.1) \qquad \sum_i a_i\varphi_i(x) \in C^{p,q}(E,R) \text{ for all bounded real } a_i?$$

If $E = R^n$ the answer is yes:

<u>Theorem 5.2</u> Suppose that $d > 0$. Then there exists a $C^{\infty,\infty}$ partition of unity $\{\varphi_i\}$ on R^n such that diam(supp φ_i) $< d$ and for every bounded sequence $a_i \in R$, $\sum_i a_i\varphi_i(x) \in C^{\infty,\infty}(R^n,R)$.

<u>Proof.</u> Find $h \in C^\infty(R,R)$ with $h(t) > 0$ if $|t| < 1$ and $h(t) = 0$ if $|t| \geq 1$. Write $x = \{x_1,\ldots,x_n\}$ and let L be the lattice of points $\{dk_1/2n,\ldots,dk_n/2n\}$ where $k_1,\ldots k_n$ are integers. Label these points by x^i, $i = 1,2,\ldots$ and define

$$f_i(x) = \prod_{j=1}^{n} h(x_j - x_j^i) .$$

Then the supports of the f_i's cover R^n, $f_i \in C^{\infty,\infty}(R^n,R)$ and diam(supp f_i) $= d$ for all i. Finally let $\varphi_i(x) = f_i(x)/\sum_1^\infty f_i(x)$. Then $\varphi_i(x) = \varphi_j(x+x^j-x^i)$ so that there exists an M_k such that $\sup_x\|D^k\varphi_i(x)\| < M_k < \infty$ for all i. Now any point of R^n is covered by the supports of at most $(2n+1)$ φ_i's. Hence if $|a_i| \leq M$, $\|D^k\sum a_i\varphi_i(x)\| \leq (2n+1)M_k \cdot M < \infty$. Q.E.D.

When E is infinite dimensional then no such canonical construction is possible. In fact if the supports of the partition functions have uniformly bounded diameters then by Lebesque's Covering Theorem for any N there are points of E at which more than N of the partition functions are non-zero. This seems to suggest that the answer to the question is false for $q \geq 1$. We show below that this is the case for $E = \ell^2$ and $q = 2$. The first theorem is of interest in itself. We consider the continuous map $\sigma: \ell^2 \to \ell^2$ defined by $\sigma(x) = \sum_i |x_i| e_i$, where e_i is the orthonormal basis, and show that $\sigma(x)$ is not uniformly approximable by a $C^{2,2}$ function. We will look at $\sigma(x)$ again in the next chapter when we study C^p approximations.

<u>Theorem 5.3</u> Suppose that B is a ball of radius d and center z, that $f \in C^2(\ell^2, \ell^2)$ and that $\sup_{x \in B} \|f(x) - \sigma(x)\| \leq a < d\sqrt{3}/6$. Then $\sup_{x \in B} \|D^2 f(x)\| = \infty$.

<u>Proof.</u> Since a C^2 function with $\sup_{x \in B} \|D^2 f(x)\| < \infty$ has a uniformly continuous derivative on B, the theorem will follow from the following lemma:

<u>Lemma 5.2</u> Let B be the closed ball in ℓ^2 of radius d and center z. Suppose that $f \in C^1(\ell^2, \ell^2)$ and that Df is uniformly continuous on B. Then $\sup_{x \in B} \|f(x) - \sigma(x)\| \geq d\sqrt{3}/6$.

Proof. Let x_i and $f_i(x)$ denote $\langle x, e_i\rangle$ and $\langle f(x), e_i\rangle$ where $\{e_i\}$ is an orthonormal basis. For an arbitrary $1 > \epsilon > 0$ choose k such that $\sum_{k+1}^{\infty} z_i^2 < \epsilon$ and pick $\delta < \epsilon$ such that $x, y \in B$ and $\|x - y\| < \delta$ implies $\|Df(x) - Df(y)\| < \epsilon$. Let N be the greatest integer less than or equal to $(d^2 - \epsilon^2)/\delta^2$ so that $d^2 - 2\epsilon^2 < \delta^2 N < d^2 - \epsilon^2$. Denote $\sum_{i=1}^{k} z_i e_i$ by z'. Then if we let F be the N dimensional box $\{y \mid |y_i - z_i'| \leq \delta$ for $i = k+1, .., k+N$ and $|y_i - z_i'| = 0$ otherwise$\}$ we will have $F \subset B$. By the mean value theorem(Prop.1.5) if $y \in F$ and $y_{k+1}, .., \hat{y}_i, .., y_{k+N}$ are fixed then

$$f_i(y) = a + by_i + e(y_i) \quad ,$$

where $a = f_i(y - y_i e_i)$, $b = \langle Df(y - y_i e_i)[e_i], e_i\rangle$ and $e(y_i) = \langle (Df(y - y_i e_i + \theta_i(y_i)y_i) - Df(y - y_i e_i))[y_i], e_i\rangle$ for some $0 < \theta_i(y_i) < 1$.

Since $|y_i| < \delta$, $|e(y_i)| \leq \epsilon |y_i|$. Now a simple calculation shows that

$$\int_{-\delta}^{\delta} (a + by_i + e(y_i) - |y_i|)^2 dy_i > \delta^3\left(\frac{1}{6} - 4\epsilon\right) .$$

Hence $\int_F \|f(y) - \sigma(y)\|^2 dy_{k+1}, \ldots, dy_{k+N}$

$$= \sum_{i=k+1}^{k+N} \int_F (f_i(y) - |y_i|)^2 dy_{k+1}, \ldots, dy_{k+N}$$

$> \quad N(2\delta)^{N-1}\delta^3\left(\frac{1}{6} - 4\epsilon\right)$. Therefore $\sup_{x \in F} \|f(x) - \sigma(x)\|^2$

$> N\delta^2\left(\frac{1}{12} - 2\epsilon\right) > \left(d^2 - 2\epsilon^2\right)\left(\frac{1}{12} - 2\epsilon\right)$. Hence $\sup_{x\in B}\|f(x) - \sigma(x)\|$ $\geq \sup_{x\in F}\|f(x) - \sigma(x)\| > \sqrt{\left(d^2 - 2\epsilon^2\right)\left(\frac{1}{12} - 2\epsilon\right)}$. Since ϵ is arbitrary, $\sup_{x\in B}\|f(x) - \sigma(x)\| \geq d/2\sqrt{3}$. Q.E.D.

Now suppose that $\{\varphi_i\}$ is a C^2 partition of unity for ℓ^2 and that diam(supp φ_i)< d for all i. If we pick points $x^i \in \operatorname{supp} \varphi_i$ and let $b_i = \sigma(x^i)$ then

$$\|\sum_i b_i\varphi_i(x) - \sigma(x)\| = \|\sum_i\left(\sigma(x^i)\varphi_i(x) - \sigma(x)\varphi_i(x)\right)\|$$

$\leq \sum_i\|x^i - x\|\varphi_i(x) \leq d$. Hence by Theorem 5.3 if B is a ball of radius $r > 2\sqrt{3}\, d$, then

$$\sup_{x\in B} \|D^2\sum_i b_i\varphi_i(x)\| = \infty. \tag{5.2}$$

Let $a_i = b_i$ if $\operatorname{supp} \varphi_i \cap B \neq \emptyset$ and $a_i = 0$ otherwise, Then the a_i's are bounded and $\sum_i a_i\varphi_i(x) = \sum_i b_i\varphi_i(x)$ when $x \in B$ and therefore

$$\sup_{x\in B}\|D^2 \sum_i a_i\varphi_i(x)\| = \infty . \tag{5.3}$$

The next theorem will show that (5.3) also holds when the a_i's are a suitable bounded real sequence.

<u>Theorem 5.4</u> Let $\{\varphi_i\}$ be a $C^p, p \geq 2$, partition of unity on ℓ^2 and suppose that diam (supp φ_i) < d for all i. Then there exists a sequence $\{a_i\}$ of bounded real numbers such that $\sup_x \|D^k \sum_i a_i\varphi_i(x)\| = \infty$ for $2 \leq k \leq p$.

<u>Proof.</u> Choose $b_i = \sigma(x^j)$ and $r > 2\sqrt{3}\, d$ as above and let $B_j = \{x \mid \|x - 2re_j\| \le r\}$. Then $\sup_{x \in B_j} \|D^2 \sum_i b_i \varphi_i(x)\| = \infty$ so we can pick $y^j \in B_j$ and $h^j \in \ell^2$ with $\|h^j\| = 1$ such that $j < |\sum_i b_i D^2 \varphi_i(y^j)[h^j, h^j]|$. Let $F_j = \{i \mid \varphi_i(y^j) > 0\}$. Then if $j \ne j'$, $F_j \cap F_{j'} = \emptyset$. Now define

$$
\begin{aligned}
a_i &= \operatorname{sign}(D^2\varphi_i(y^j)[h^j, h^j]) \text{ if } i \in F_j \\
&= 0 \quad \text{if } i \notin F_j \text{ for any } j \quad .
\end{aligned}
$$

Then $\sup_{x \le 3r} \|D^2 \sum_i a_i \varphi_i(x)\| \ge \sup_{x \in B_j} \|\sum_i a_i D^2 \varphi_i(x)\|$

$\ge \|\sum_i a_i D^2 \varphi_i(y^j)\| \ge |\sum_{i \in F_j} a_i D^2 \varphi_i(y^j)[h^j, h^j]|$

$= \sum_{i \in F_j} |D^2 \varphi_i(y^j)[h^j, h^j]|$. Since $\|b_i\| < 3r$ the last expression is $\ge \frac{1}{3r} \sum_{i \in F_j} \|b_i\| \, |D^2 \varphi_i(y^j)[h^j, h^j]|$

$\ge \frac{1}{3r} \sum_{i \in F_j} \|b_i D^2 \varphi_i(y^j)[h^j, h^j]\| > j/3r$.

Since j is arbitrary, $\sup_{\|x\| \le 3r} \|D^2 \sum_i a_i \varphi_i(x)\| = \infty$ and by Prop. 1.6 the Theorem follows. Q.E.D.

CHAPTER VI

SMOOTH APPROXIMATION

<u>Definition</u>. Suppose that E and F are Banach spaces, that U is an open subset of E and that $f \in C^q(U,F)$. Then if $0 \le q < p \le \infty$ we will say that f is $C_{p,q}$ approximable on U if given $\epsilon > 0$ there exists a $g \in C^p(U,F)$ such that $\sup_{x \in U, 0 \le k \le q} \|D^k f(x) - D^k g(x)\| < \epsilon$. We will say that f is strongly $C_{p,q}$ approximable on U if given any $e(x) \in C^0(U,R^+)$ there exists a $g \in C^p(U,F)$ such that for x in U, $\sup_{0 \le k \le q} \|D^k f(x) - D^k g(x)\| < e(x)$. In both cases the functions g will be called $C_{p,q}$ approximations.

It is well known that if E is finite dimensional then every $f \in C^q(E,F)$,($q \ge 1$), is strongly $C_{p,q}$ approximable on E. When E is infinite dimensional but separable, Prop. 2.7 implies that every $f \in C^0(E,F)$ is strongly $C_{p,0}$ approximable if and only if E is C^p smooth. However when $q > 0$ it is not known whether there exist any infinite dimensional Banach spaces such that every C^q function is $C_{p,q}$ approximable. In particular, it is not known whether every C^1 function on separable Hilbert space is $C_{2,1}$ approximable.

The theorem below will show that if a C^q function on a separable $C^{p,q}$ smooth B-space is locally $C_{p,q}$ approximable, then it is strongly approximable on the whole space. This would be an essential theorem in constructing $C_{p,q}$ approximations on manifolds modeled on $C^{p,q}$ smooth Banach spaces.

Theorem 6.1 Let E be a separable $C^{p,q}$ smooth B-space and let F be another B-space. Let $f \in C^q(E,F)$ and suppose that for every x in E there is a neighborhood N_x of x such that f is $C_{p,q}$ approximable on N_x. Then f is strongly $C_{p,q}$ approximable.

Proof. If $e(x) > 0$, let $\{U_\alpha\}$ be an open cover of E refining $\{N_x\}$ and such that $\inf_{x \in U_\alpha} e(x) > 0$. Apply Lemma 5.1 to get four locally finite subcovers $\{V_i^j\}$ refining $\{U_\alpha\}$ and functions $g_i \in C^{p,q}(E,R)$ satisfying the conditions of the lemma. Let $\epsilon_i = \inf_{x \in V_i^4} e(x)$ and let $M_i = \|(1-g_1(x))\cdots(1-g_{i-1}(x))g_i(x)\|_q$.

By the hypothesis, there exists an $h_i(x) \in C^{p,q}(V_i^4,F)$ with

$$\sup_{x \in V_i^4, 0 \le k \le q} \|D^k f(x) - D^k h_i(x)\| < \epsilon_i/(2^{q+i}M_i) \quad (6.1)$$

Now define $f_0(x) = f(x)$ and $f_i(x) = f(x)(1-g_1(x))\cdots(1-g_i(x)) + h_1(x)g_1(x) + h_2(x)g_2(x)(1-g_1(x)) + \cdots + h_i(x)g_i(x)(1-g_1(x))\cdots(1-g_{i-1}(x))$ for $i > 1$.

If $x \in V_1^2 \cup \cdots \cup V_i^2$ then $(1-g_1(x))\cdots(1-g_i(x)) = 0$, hence

(6.2) $\quad f_i(x) \in C^{p,q}(V_1^1 \cup \ldots \cup V_i^1, F)$, $i \geq 1$.

Also if $x \notin V_i^4$ then $g_i(x) = 0$ so that

(6.3) $\quad f_i(x) = f_{i-1}(x)$ when $x \notin V_i^4$.

Now using (6.2) and (6.3) and the fact that $\{V_i^1\}$ and $\{V_i^4\}$ cover E, for every x there is a neighborhood U_x of x and an integer k_x such that $f_{i+1}(y) = f_i(y)$ for $y \in U_x$ and $i > k_x$ and $f_i(y) \in C^p(U_x, F)$. Hence

$h(x) = \lim_{i\to\infty} f_i(x)$ exists and $h(x) \in C^p(E,F)$. Now

$f_i(x) - f_{i-1}(x) = (h_i(x) - f(x))\cdot(1 - g_1(x))\cdots(1-g_{i-1}(x))g_i(x)$ and hence $\sup_{x\in V_i^4} \|D^k(f_i(x) - f_{i-1}(x))\| \leq \sum_{j=0}^{k} \binom{k}{j} \sup_{x\in V_i^4} \|D^k(h_i(x) - f(x))\| \cdot \sup_{x\in V_i^4} \|D^k\big((1-g_1(x))\cdot\cdot(1-g_{i-1}(x))g_i(x)\big)\| \leq$

$\sum_{j=0}^{k} \binom{k}{j} \epsilon_i/2^{q+i}M_i \cdot M_i \leq \epsilon_i/2^i$ for $k \leq q$. Using this and (6.3) we have for $0\leq k\leq q$, $\|D^k f(x) - D^k h(x)\| = \|D^k f(x) - D^k f_N(x)\|$ for some N, and this is $\leq \sum_{\{j | x\in V_j^4, j\leq N\}} \|D^k\big(f_j(x) - f_{j-1}(x)\big)\|$ $< e(x) \cdot \sum_{j=1}^{N} 1/2^j < e(x)$. Hence f(x) is strongly $C_{p,q}$ approximable. Q.E.D.

Consider separable Hilbert space, ℓ^2 , with orthonormal basis $\{e_i\}$. Write $x = \sum_i x_i e_i$ and define $\sigma(x) = \sum_i |x_i| e_i$ as in Chapter V and $\Sigma(x) = \sum_i x_i |x_i|$. Then

$\|\sigma(x) - \sigma(y)\| \le \|x - y\|$ and $\left|\sum_i (x_i+y_i)|x_i+y_i| - \sum_i x_i|x_i|\right.$ $\left. - \langle 2\sigma(x), y\rangle\right| \le |\sum_i y_i^2| = \|y\|^2$. Hence $D\Sigma(x) = 2\sigma(x)$ and $\Sigma(x) \in C^1(\ell^2, R)$. We observe that $\sigma(x)$ is nowhere differentiable. To show this let $x \in \ell^2$ and suppose that σ is differentiable at x. Then there exists a δ such that when $\|y\| < \delta$, $\|\sigma(x+y) - \sigma(x) - D\sigma(x)[y]\| < \|y\|/8$. Choose n such that $|x_n| < \delta/4$ and let $y = \delta e_n$. Then

$\|\sigma(x+y) + \sigma(x-y) - 2\sigma(x)\| = \left|\, |x_n+y_n| + |x_n-y_n| - 2|x_n|\,\right|$ $\ge 3\delta/4 + 3\delta/4 - 2\delta/4 = \delta/2$. On the other hand

$\|\sigma(x+y) + \sigma(x-y) - 2\sigma(x)\| = \|\sigma(x+y) - \sigma(x) - D\sigma(x)[y] + \sigma(x-y)$ $- \sigma(x) - D\sigma(x)[-y]\| \le \|\sigma(x+y) - \sigma(x) - D\sigma(x)[y]\| + \|\sigma(x-y)$ $- \sigma(x) - D\sigma(x)[-y]\| \le \delta/4$, contradiction.

We pose the question: Is there any better $C_{2,1}$ approximation to $\Sigma(x)$ on the unit ball than a constant function? From Theorem 5.2 it follows that $\Sigma(x)$ is not $C_{2,1}$ approximable by $C^{2,2}$ functions on any ball. The following theorem shows that if $\|\Sigma(x) - g(x)\|_1 < R/2$ on a ball of radius R, where $g \in C^2(\ell^2, R)$, then g can not have a decomposition of the form $g(x) = \sum_i g_i(x_i)$.

<u>Theorem 6.2</u> Suppose that $G(x) \in C^1(\ell^2, \ell^2)$ and that $G(x) = \sum_i h_i(x_i)e_i$. Then if B is a ball of radius R,

$$\sup_{x\in B} \|G(x) - \sigma(x)\| \ge R/2.$$

<u>Proof.</u> Let B have center a and suppose that $R > \epsilon > 0$. Pick n such that $\sum_{j=n+1}^{\infty} a_j^2 < \epsilon$ and let $b = \sum_1^n a_j e_j$. Now find δ such that $\|x - b\| < \delta$ implies $\|G(x) - G(b) - DG(b)[x-b]\| \leq \epsilon\|x-b\|/R$. Thus when $|x_i - b_i| < \delta$,

$$(6.4) \quad |h_i(x_i) - h_i(b_i) - \frac{dh_i}{dx_i}(b_i)(x_i - b_i)| < \frac{\epsilon}{R}|x_i - b_i| .$$

Choose N large enough so that $\frac{R-\epsilon}{N} < \delta$ and let $z = \sum_{j=n+1}^{n+N} \frac{R-\epsilon}{N} e_j$. Then $\|z\| = R - \epsilon$ so that $(b \pm z) \in B$.

By applying (6.4) with $i = n+1, .., n+N$ we obtain

$$(6.5) \quad \begin{aligned} \|G(b+z) + G(b-z) - 2G(b)\| &\leq \|G(b+z) - G(b) - DG(b)[z]\| \\ &+ \|G(b-z) - G(b) - DG(b)[-z]\| \leq 2\epsilon\|z\|/R < 2\epsilon . \end{aligned}$$

Since $\sigma(b+z) = \sigma(b-z)$ we have $\|G(b+z) - \sigma(b+z)\| + \|G(b-z) - \sigma(b-z)\| + 2\|G(b) - \sigma(b)\| \geq \|G(b+z) + G(b-z) - 2\sigma(b+z)\| + 2\|G(b) - \sigma(b)\|$, which by (6.5) is $\geq \|2G(b) - 2\sigma(b+z)\| - 2\epsilon + 2\|G(b) - \sigma(b)\| \geq 2\|\sigma(b+z) - \sigma(b)\| - 2\epsilon = 2\|z\| - 2\epsilon = 2R - 4\epsilon$. Therefore either $\|G(b+z) - \sigma(b+z)\|$, $\|G(b-z) - \sigma(b-z)\|$ or $\|G(b) - \sigma(b)\|$ is $\geq \frac{R}{2} - \epsilon$. Hence $\sup_{x \in B} \|G(x) - \sigma(x)\| \geq \frac{R}{2} - \epsilon$. Since ϵ is arbitrary, the theorem is proved.

CHAPTER VII

WEAK $C_{p,q}$ APPROXIMATION ON ℓ^2

As stated in Chapter VI, it is unknown whether every C^1 function on ℓ^2 is $C_{2,1}$ approximable. In this chapter we show that $C_{p,q}$ approximation can be performed on ℓ^2 provided we use a weaker approximation condition on the derivatives. The approximation is first done locally and then the $C^{\infty,\infty}$ smoothness of ℓ^2 is used to build up a global approximation.

We first point out that the usual finite dimensional technique of convoluting a C^p function with a C^∞ function having a small bounded support(i.e. letting $\tilde{f}(x) = \int_{R^n} f(x+y)\varphi(y)d\mu(y)$) to obtain a $C_{\infty,p}$ approximation, fails on ℓ^2. There is of course no translation invariant borel measure on ℓ^2 but we might hope that given $f \in C^q(\ell^2,F)$ there would exists a probability measure μ on ℓ^2 such that $\tilde{f}(x) = \int f(x+y)d\mu(y)$ is of class C^p, $p > q$. This , however,is not the case and we sketch a proof for $q = 1$. Define

$$F_1(x) = \sum_{n=1}^{\infty} \frac{(1 - \cos\sqrt{n}\, x_n)}{n}$$

where $x = \sum_n x_n e_n$. Then it is not hard to show that

$F_1(x) \in C^1(\ell^2,R)$ and that $F_1(x)$ is nowhere second differentiable. Suppose now that μ is a probabilty measure on ℓ^2 with bounded support and define $\tilde{F}_1(x) = \int F_1(x+y)d\mu(y)$. Then

$$\tilde{F}_1(x) = c + \sum_n a_n(\cos\sqrt{n}\,\phi_n - \cos\sqrt{n}(x_n+\phi_n))$$

where $c = \int \sum_n (1-\cos\sqrt{n}\,y_n)/n \, d\mu(y) < \infty$

$$a_n^2 = \left(\int \cos\sqrt{n}\,y_n d\mu(y)\right)^2 + \left(\int \sin\sqrt{n}\,y_n d\mu(y)\right)^2$$

and

$$\phi_n = \tan^{-1}\left(\frac{\int \sin\sqrt{n}\,y_n d\mu(y)}{\int \cos\sqrt{n}\,y_n d\mu(y)}\right) .$$

Now $0 \le a_n \le 2$ and $a_n \ge \int \cos\sqrt{n}\,y_n d\mu(y) \ge \int (1-ny_n^2/2)d\mu(y)$ $\ge 1 - n\gamma_n/2$ where $\gamma_n = \int y_n^2 d\mu(y)$. From $\sum_n^\infty \gamma_n =$ $\int \|y\|^2 d\mu(y) < \infty$ follows $\liminf n\gamma_n = 0$ which gives $\limsup a_n = 1$. Since the a_n's do not approach 0, the same method of proving $F_1(x)$ is nowhere second differentiable can be used to show that $\tilde{F}_1(x)$ is nowhere second differentiable.

This can be generalized. Define

$$F_p(x) = \sum_{n=1}^\infty (1-\cos\sqrt{n}\,x_n)/\,n^{(p+1)/2}.$$

Then $F_p \in C^p(\ell^2,R)$ and $F_p(x)$ is nowhere p+1 differentiable. If μ is any probability measure on ℓ^2 and if we define $\tilde{F}_p(x) = \int F_p(x+y)d\mu(y)$, then $\tilde{F}_p$ is nowhere p+1 differentiable.

In the constructions to follow we will need two propositions about measures on Banach spaces. The first proposition is well known. We recall that a probability measure μ on E is a positive regular Borel measure satisfying $\mu(E) = 1$.

Proposition 7.1 Let μ be a probability measure on a complete metric space Ω. Then for every $\epsilon > 0$ there exists a compact subset K_ϵ of Ω such that $\mu(K_\epsilon) \geq 1 - \epsilon$.

Lemma 7.1 Let $f \in C^0(E,F)$ where E and F are Banach spaces and let K be a compact subset of E. Then

$$\lim_{t \to 0} \sup_{h \in K, \|y\| \leq t} \|f(y+h) - f(h)\| = 0 .$$

Proof. Suppose $\epsilon > 0$. For every $h \in K$ find R_h such that $\|y-h\| < R_h$ implies $\|f(y)-f(h)\| < \epsilon/2$. Let $\{B(h_i, R_{h_i})\}$ be a finite subcover of the cover $\{B(h,R_h)\}$, where $B(h,R_h)$ is the ball with center h and radius R_h. Let δ be the Lebesque number of $\{B(h_i,R_{h_i})\}$. Then for every $h \in K$ and $y \in E$ with $\|y\| \leq \delta$ we have $h, y+h \in B(h_i, R_{h_i})$ for some i. Hence $\|f(h+y) - f(h)\| \leq \|f(h+y) - f(h_i)\| + \|f(h) - f(h_i)\| \leq \epsilon/2 + \epsilon/2 = \epsilon$. Q.E.D.

Proposition 7.2 Suppose that μ is a probability measure on a B-space E with compact support K and suppose that $f \in C^p(U,F)$, $p \geq 0$, where U is an open subset of E. Then if V is an open subset of U such that the algebraic sum $V+K$ is contained in U, $g(x) = \int f(x+y)d\mu(y) \in C^p(V,F)$ and $D^k g(x) = \int D^k f(x+y)d\mu(y)$ for $0 \leq k \leq p$.

Proof. Suppose $x \in V$ and $\epsilon > 0$. By Lemma 7.1 there is a $\delta > 0$ such that $\|z\| < \delta$ implies

$$\sup_{y \in K} \|f(x+y+z) - f(x+y)\| < \epsilon .$$

But then $\|z\| < \delta$ implies $\|g(x+z) - g(x)\|$

$$\leq \int \|f(x+y+z) - f(x+y)\| d\mu(y) \leq \epsilon .$$ Hence $g \in C^0(V,F)$.

Assume that $g(x) \in C^q(V,F)$ for $q < p$ and $D^k g(x) = \int D^k f(x+y) d\mu(y)$ for $0 \leq k \leq q$. We show that $g(x) \in C^{q+1}(V,F)$ and $D^{q+1}g(x) = \int D^{q+1} f(x+y) d\mu(y)$. For any x in V,

$$\lim_{t \to 0} \sup_{\|y\|=1, z \in K} \|(D^q f(x+ty+z) - D^q f(x+z) - D^{q+1} f(x+z)[ty])/t\|$$

$$= \lim_{t \to 0} \sup_{\substack{\|y\|=1, z \in K \\ w \in F^*, \|w\| \leq 1}} \langle (D^q f(x+ty+z) - D^q f(x+z) - D^{q+1} f(x+z)[ty])/t, w \rangle .$$

Now by Prop. 1.5, $\langle D^q f(x+ty+z) - D^q f(x+z), w \rangle = \langle D^{q+1} f(x+z+\tau y)[ty], w \rangle$ for some $0 < \tau < t$ so the last limit is

$$\leq \lim_{t \to 0} \sup_{\substack{\|y\|=1, z \in K, w \in F^*, \|w\| \leq 1 \\ 0 < \tau < t}} \langle (D^{q+1} f(x+z+\tau y) - D^{q+1} f(x+z))[y], w \rangle$$

$$= \lim_{t \to 0} \sup_{\|y\|=1, z \in K, 0 < \tau < t} \|(D^{q+1} f(x+z+\tau y) - D^{q+1} f(x+z))[y]\|$$

$= 0$, by Lemma 7.1. Hence

$$\lim_{t \to 0} \sup_{\|y\|=1} \left\| \frac{D^q g(x+ty) - D^q g(x)}{t} - \left(\int D^{q+1} f(x+z) d\mu(z) \right)[y] \right\| = 0,$$

so that $D^{q+1} g(x)$ exists and equals $\int D^{q+1} f(x+z) d\mu(z)$. Q.E.D.

<u>Corollary 7.2</u> Let μ be any probability measure on a B-space E and suppose that $f(x) \in C^{p,p}(E,F)$. Then $g(x) = \int f(x+y)d\mu(y) \in C^{p,p}(E,F)$ and $D^k g(x) = \int D^k f(x+y)d\mu(y)$ for $k \leq p$.

<u>Proof.</u> By Prop. 7.1 there exists compact sets K_ε with $\mu(K_\varepsilon) \geq 1-\varepsilon$. Define $g_\varepsilon(x) = \int_{K_\varepsilon} f(x+y)d\mu(y)$. Then by Prop.7.1, for $k \leq p$, $D^k g_\varepsilon(x) = \int_{K_\varepsilon} D^k f(x+y)d\mu(y)$ and this implies that $D^k g_\varepsilon(x)$ converges uniformly as $\varepsilon \to 0$ to $\int D^k f(x+y)d\mu(y)$ for $k \leq p$. So by Prop. 1.11, $D^k g(x)$ exists and $D^k g(x) = \int D^k f(x+y)d\mu(y)$. $g(x)$ is in $C^{p,p}(E,F)$ because $\|D^k g(x)\| \leq \int \|D^k f(x+y)\| d\mu(y)$. Q.E.D.

Consider now separable Hilbert space ℓ^2 and let $\{e_i\}$ be an orthonormal basis. We will define for each nonnegative sequence $\{a_i\}$, with $\sum_i a_i^2 < \infty$, a probability measure μ^A, $A = \{a_i\}$. Let $\eta(t)$ be a fixed function in $C^\infty(R,R)$ satisfying $\eta(t) = 0$ if $|t| \geq 1$ and $\int_{-\infty}^{\infty} \eta(t)dt = 1$. Define for each positive integer n an integral on $C^{0,0}(\ell^2,R)$ as follows:

$$\Lambda_n^A(f(x)) = \prod_{i=1}^{n}{}' \left(\frac{1}{a_i}\right) \int_{H_n'} \prod_{i=1}^{n}{}' \eta\left(\frac{y_i}{a_i}\right) f\left(\sum_{i=1}^{n}{}' y_i e_i\right) d\mu_n'(y)$$

where H_n' is the space spanned by $e_{\{i \mid 1 \leq i \leq n, a_i > 0\}}$, μ_n' is the standard Lebesque measure on H_n' and $\prod'$ and $\sum'$ denote the product and summation over only those i's for which

$a_i > 0$. Let K denote the compact Hilbert cube K = $\{x \mid |x_i| \le a_i\}$. For any $f \in C^{0,0}(\ell^2, R)$ find δ such that $z, y \in K$ and $\|z-y\| < \delta$ implies $|f(z) - f(y)| < \epsilon$. Then if we take N such that $\sum_{i=N+1}^{\infty} a_i^2 < \delta^2$ we have for $m \ge n \ge N$,

$$| \Lambda_m^A(f(x)) - \Lambda_n^A(f(x))| \le \left| \prod_{i=1}^{m}{}' \left(\frac{1}{a_i}\right) \int_{H_m'} \prod^{m}{}' \eta\left(\frac{y_i}{a_i}\right) f\left(\sum^{m}{}' y_i e_i\right) d\mu_m'(y) \right.$$

$$\left. - \prod_{i=1}^{n}{}' \left(\frac{1}{a_i}\right) \int_{H_n'} \prod^{n}{}' \eta\left(\frac{y_i}{a_i}\right) f\left(\sum^{n}{}' y_i e_i\right) d\mu_n'(y) \right|$$

$$\le \prod_{i=1}^{m}{}' \left(\frac{1}{a_i}\right) \int_{H_m'} \prod^{m}{}' \eta\left(\frac{y_i}{a_i}\right) \left| f\left(\sum^{m}{}' y_i e_i\right) - f\left(\sum^{n}{}' y_i e_i\right) \right| d\mu_m'(y)$$

$$\le \prod_{i=1}^{m}{}' \left(\frac{1}{a_i}\right) \int_{H_m'} \prod^{m}{}' \eta\left(\frac{y_i}{a_i}\right) \cdot \epsilon \, d\mu_m'(y) \quad = \quad \epsilon \, .$$

Hence $\lim_{n \to \infty} \Lambda_n^A(f(x))$ exists and we define this limit to be $\Lambda^A(f(x))$. The functional Λ^A is clearly linear, bounded positive and satisfies $\Lambda^A(1) = 1$. Since supp $\Lambda^A \subset K$ and K is compact, Λ^A is an integral. By the Riesz Representation Theorem there is a unique probability measure μ^A on ℓ^2 such that $\int f(x) d\mu^A(x) = \Lambda^A(f(x))$ for all $f \in C^{0,0}(\ell^2, R)$.

In the proof of the next theorem we will use the measures μ^A to mollify C^p functions on ℓ^2. We recall that a Hilbert-Schmidt operator T on ℓ^2 is an element of $L(\ell^2, \ell^2)$ satisfying $\sum_{i,j=1}^{\infty} \langle Te_i, Te_j \rangle < \infty$.

Theorem 7.1 Suppose that $f \in C^{p,p}(U,F), 1 \leq p < \infty$, where U is an open subset of ℓ^2, that $D^p f(x)$ is uniformly continuous on U, that V is an open subset of U with dist(V,CU)>0, and that T is a Hilbert-Schmidt operator on ℓ^2. Then there exists a $g(x) \in C^\infty(V,F)$ satisfying

$$\sup_{x\in V, \|h\|\leq 1, 0\leq k\leq p} \|D^k(f(x) - g(x))[Th]\| \leq 1.$$

Proof. Let T = SW be a polar decomposition for T, where $S = \sqrt{TT^*}$ and W is a partial isometry. Then S is positive definite self-adjoint Hilbert-Schmidt and if we denote the unit ball by B, then

$$(7.1) \qquad T(B) \subset SW(B) \subset S(B) \quad .$$

Assume that the orthonormal basis $\{e_i\}$ is a set of eigenvectors for S and that $Se_i = \alpha_i e_i$. Then $\alpha_i \geq 0$ and $\sum_i \alpha_i^2 < \infty$. Now $D^k f(x)$ is uniformily continuous on U for $k \leq p$ so we can find $\delta > 0$ such that $\delta <$ dist(V,CU) and

$$(7.2) \qquad \sup_{x,y\in U, \|x-y\|<\delta, 0\leq k\leq p} \|D^k f(x) - D^k f(y)\| \leq 1/2\|T\|.$$

Let $t = \min(1, \delta/2\sum_i \alpha_i^2)$, $a_i = t\alpha_i$, $A = \{a_i\}$ and define μ^A as above. Letting K be the compact set $\{x \mid |x_i| \leq a_i\}$, we have diam $K < \delta$, supp $\mu^A \subset K$ and $V + K \subset U$. Now let $M = \sup_{k\leq p} \int_{-1}^{1} |\frac{d^k}{dt^k}\eta(t)|dt$ and use Prop. 2.7 to obtain $g(x) \in C^\infty(\ell^2,F)$ such that

$$(7.3) \qquad \sup_x \|f(x) - g(x)\| \le t^q/2M^q \quad .$$

Let $\tilde{f}(x) = \int f(x+y)d\mu^A(y)$ and $\tilde{g}(x) = \int g(x+y)d\mu^A(y)$, then by Prop.7.2, $\tilde{f} \in C^p(V,F)$ and $\tilde{g} \in C^\infty(V,F)$. (7.2) gives

$$(7.4) \quad \|D^k f(x) - D^k\tilde{f}(x)\| \le \int \|f(x) - f(x+y)\| d\mu^A(y) \le 1/2\|T\| .$$

Suppose now that $x \in V$, that $i_1, \ldots i_k, k \le p$, are integers with $a_{i_j} > 0$ for $j \le k$ and $N = \max(i_1, \ldots i_k)$. Then

$$\left\| \frac{\partial^k}{\partial x_{i_1} \cdots \partial x_{i_k}} (\tilde{f}(x) - \tilde{g}(x)) \right\|$$

$$= \left\| \frac{\partial^k}{\partial x_{i_1} \cdots \partial x_{i_k}} \lim_{n\to\infty} \prod_{j=1}^{n}{}' \left(\frac{1}{a_j}\right) \int_{H_n'} \prod^{n}{}' \eta\left(\frac{y_j}{a_j}\right) \left[f(x + \sum^{n}{}' y_j e_j) \right.\right.$$

$$(7.5) \qquad \left.\left. - g(x + \sum^{n}{}' y_j e_j) \right] d\mu_n'(y) \right\|$$

$$\le \lim_{n\to\infty} \prod_{j=1}^{n}{}' \left(\frac{1}{a_j}\right) \int_{H_n'} \frac{\partial^k}{\partial y_{i_1} \cdots \partial y_{i_k}} \left(\prod_{j=1}^{n}{}' \eta\left(\frac{y_j}{a_j}\right)\right) \| f(x + \sum^{n}{}' y_j e_j)$$

$$- g(x + \sum^{n}{}' y_j e_j)\| \, d\mu_n'(y)$$

(which by (7.3) is)

$$\le \prod_{j=1}^{N}{}' \left(\frac{1}{a_j}\right) \int_{H_n'} \frac{\partial^k}{\partial y_{i_1} \cdots \partial y_{i_k}} \left(\prod^{N}{}' \eta\left(\frac{y_j}{a_j}\right)\right) \cdot t^q/2M^q \, d\mu_N'(y)$$

$$\le \frac{1}{a_{i_1}} \cdots \frac{1}{a_{i_k}} \cdot M^k \cdot t^q/2M^q \le \frac{1}{2} \frac{1}{\alpha_{i_1}} \cdots \frac{1}{\alpha_{i_k}} \quad .$$

It follows now from (7.1) and (7.5) that

$$\sup_{x\in V,\|h\|\le 1} \|D^k(\tilde{f}(x) - \tilde{g}(x))[Th]\| \le \sup_{x\in V,\|h\|\le 1} \|D^k(\tilde{f}(x)-\tilde{g}(x))[Sh]\|$$

$$= \sup_{x\in V,\|h\|\le 1} \Big\| \sum_{i_1,\dots i_k=1}^{\infty} \frac{\partial^k}{\partial x_{i_1}\dots\partial x_{i_k}} (\tilde{f}(x)-\tilde{g}(x))\alpha_{i_1}h_{i_1}\dots\alpha_{i_k}h_{i_k} \Big\|$$

$$\le \sup_{\|h\|\le 1} \Big\| \sum_{i_1\dots i_k=1}^{\infty}{}' \tfrac{1}{2} \frac{1}{\alpha_{i_1}}\dots\frac{1}{\alpha_{i_k}} \alpha_{i_1}h_{i_1}\dots\alpha_{i_k}h_{i_k} \Big\|$$

$$\le \sup_{\|h\|\le 1} \tfrac{1}{2}\|h\|^k = \tfrac{1}{2}.$$

Combining this with (7.4), we obtain for $0\le k\le p$,

$$\sup_{x\in V,\|h\|\le 1} \|D^k(f(x) - \tilde{g}(x))[Th]\| \le \sup_{x\in V,\|h\|\le 1} \|D^k(f(x) - \tilde{f}(x))[Th]\|$$

$$+ \sup_{x\in V,\|h\|\le 1} \|D^k(\tilde{f}(x) - \tilde{g}(x))[Th]\| \le \sup_{x\in V}\|D^k(f(x)-\tilde{f}(x))\|\cdot\|T\| + \tfrac{1}{2}$$

$$\le \tfrac{1}{2} + \tfrac{1}{2} = 1 .$$

Q.E.D.

Remark. Suppose that the f in Theorem 7.1 has the property that for any $\epsilon > 0$ there exists a $g_\epsilon \in C^{\infty,p}(V,F)$ such that $\|f(x) - g_\epsilon(x)\|_0 \le \epsilon$ and $\|g_\epsilon\|_p \le M$, where M is independent of ϵ. Then the conclusion of the theorem would be true if the operator T were only assumed to be compact. To show this assume T compact and find P in $L(\ell^2,\ell^2)$ with finite dimensional range and such that

$\|T - P\| < 1/2(\|f\|_p + M)$. Apply the theorem to get a $\tilde{g} \in C^\infty(\ell^2, F)$ with $\sup_{x\in V,\|h\|\le 1,k\le p}\|D^k(f(x) - \tilde{g}(x))[2Ph]\| < 1$. Since

$\tilde{g}(x) = \int g(x+y)d\mu^A(y)$ where g is a $C_{\infty,0}$ approximation to f and since by assumption we can take $\|g\|_p \le M$, it follows that we can assume $\|\tilde{g}\|_p \le M$. Therefore

$$\sup_{x\in V,\|h\|\le 1,k\le p}\|D^k(f(x) - \tilde{g}(x))[Th]\| \le \sup_{x\in V,\|h\|\le 1,k\le p}\|D^k(f(x) - \tilde{g}(x))[(T-P)h]\|$$

$$+ \sup_{x\in V,\|h\|\le 1,k\le p}\|D^k(f(x) - \tilde{g}(x))[Ph]\| \le (\|f\|_p + M)\|T-P\| + ½ \le 1.$$

We now give a global formulation of Theorem 7.1. The proof is similar to the proof of Theorem 6.1 in which Lemma 5.1 played a key role.

Theorem 7.2 Let $f \in C^p(\ell^2, F), 1\le p<\infty$, and suppose that $D^p f(x)$ is uniformly continuous in some neighborhood of every point of ℓ^2. Then for any locally finite cover $\{U_\alpha\}$ of ℓ^2 and collection $\{T_\alpha\}$ of Hilbert-Schmidt operators on ℓ^2 there exists a $g(x) \in C^\infty(\ell^2, F)$ such that

$$\sup_\alpha \sup_{x\in U_\alpha,\|h\|\le 1,0\le k\le p}\|D^k(f(x) - g(x))[T_\alpha h]\| \le 1 .$$

Proof. As in the proof of Theorem 7.1, let $S_\alpha = \sqrt{T_\alpha T_\alpha^*}$ so that S_α is self-adjoint positive definite Hilbert-Schmidt and $T_\alpha(B) \subset S_\alpha(B)$, where B is the unit ball.

For every x in ℓ^2 find a ball $B(x,R_x)$ of radius R_x about x such that $B(x,R_x)$ intersects only a finite number of U_α's and $D^p f(x)$ is uniformly continuous on $B(x,R_x)$. Now since ℓ^2 is $C^{\infty,\infty}$ smooth, we can apply Lemma 5.1 to the cover $\{B(x,R_x/2)\}$ to obtain covers $\{V_i^j\}$, $j=1,2,3,4$, and functions $g_i(x) \in C^{\infty,\infty}(\ell^2,R)$ such that

1) $\mathrm{dist}(V_i^j, CV_i^{j+1}) > 0$, $j = 1,2,3$
2) $\{V_i^1\}$ covers ℓ^2
3) $\{V_i^4\}$ is locally finite and refines $\{B(x,R_x/2)\}$
4) $0 \le g_i(x) \le 1$, $g_i(x)(V_i^2) = 1$ and $g_i(x)(CV_i^3) = 0$.

Now define $\varphi\ (x) = g_1(x)$, $\varphi_i(x) = (1-g_1(x))\cdots(1-g_{i-1}(x))g_i(x)$ if $i>1$ and $M_i = \|\varphi_i\|_p$. If we let

$$S_i = \sum_{\{\alpha \mid U_\alpha \cap V_i^4 \ne \emptyset\}} S_\alpha$$

(note that the sum is over a finite number of α's) then S_i is positive definite self-adjoint Hilbert-Schmidt and $S_\alpha(B) \subset S_i(B)$. Set

$$S_i' = 2^{p+i} M_i (\max(1,\|S_i\|))^p S_i$$

and use Theorem 7.1, observing that $f(x) \in C^{p,p}(B(x,R_x),F)$ and $\mathrm{dist}(V_i^4, B(x,R_x)) > 0$, to obtain functions $h_i \in C^\infty(V_i^4,F)$ satisfying

$$(7.6) \qquad \sup_{x \in V_i^4, \|h\| \le 1, k \le p} \|D^k(f(x) - h_i(x))[S_i' h]\| \le 1.$$

Define $f_0(x) = f(x), \ldots, f_i(x) = f(x)(1-g_1(x))\cdots(1-g_i(x)) +$

$h_1(x)\varphi_1(x) + \cdots + h_i(x)\varphi_i(x)$. When $x \in V_1^2 \cup \ldots \cup V_i^2$, $(1-g_1(x))\cdots(1-g_i(x)) = 0$, hence

$$(7.7) \qquad f_i(x) \in C^\infty(V_1^1 \cup \ldots \cup V_i^1, F) .$$

Also

$$(7.8) \qquad f_i(x) = f_{i-1}(x) \text{ when } x \notin V_i^4 .$$

For every $x \in \ell^2$ there is a neighborhood N_x of x and an integer n such that $N_x \subset V_1^1 \cup \ldots \cup V_n^1$ and $N_x \cap V_i^4 = \emptyset$ for $i>n$. Hence by (7.7) and (7.8) we can define

$$g(x) = \lim_{i\to\infty} f_i(x) \text{ and } g(x) \in C^\infty(\ell^2, F).$$

Now $f_i(x) - f_{i-1}(x) = (h_i(x) - f(x))\varphi_i(x)$, hence

$$(7.9) \qquad \sup_{x,\|h\|\le 1,k\le p} \|D^k(f_i(x) - f_{i-1}(x))[S_i h]\|$$

$$\le \sum_{n=0}^{k} \binom{k}{n} \sup_{x,\|h\|\le 1,k\le p} \|D^n(h_i(x) - f(x))[S_i h]\| \cdot \sup_{x,\|h\|\le 1,k\le p} \|D^{k-n}\varphi_i(x)[S_i h]\|$$

$$\le \sum_{n=0}^{k} \binom{k}{n} 1/(2^{p+i} M_i \|S_i\|^p) \cdot M_i \|S_i\|^{k-n} \le 1/2^{p+i} \le 1/2^i$$

by (7.6) and (7.8). Therefore if $x \in U_\alpha$

$$\sup_{\|h\|\le 1,k\le p} \|D^k(f(x) - g(x))[T_\alpha h]\| \le \sup_{\|h\|\le 1,k\le p} \|D^k(f(x) - g(x))[S_\alpha h]\|$$

$$\le \sum_{\{j|U_\alpha \cap V_j^4 \ne \emptyset\}} \sup_{\|h\|\le 1,k\le p} \|D^k(f_j(x)-f_{j-1}(x))[S_\alpha h]\| \le \sum_{\{j|U_\alpha \cap V_j^4 \ne \emptyset\}}$$

$$\sup_{\|h\|\le 1,k\le p} \|D^k(f_j(x) - f_{j-1}(x))[S_j h]\| \le \sum_{\{j|U_\alpha \cap V_j^4 \ne \emptyset\}} 1/2^j \le 1. \text{ Q.E.D.}$$

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