A SCALING TECHNIQUE FOR THE DESIGN OF IDEALIZED ELECTROMAGNETIC LENSES

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ABSTRACT

A technique is developed for the design of lenses for transitioning TEM waves between conical and/or cylindrical transmission lines, ideally with no reflection or distortion of the waves. These lenses utilize isotropic but inhomogeneous media and are based on a solution of Maxwell's equations instead of just geometrical optics. The technique employs the expression of the constitutive parameters, ε and μ, plus Maxwell's equations, in a general orthogonal curvilinear coordinate system in tensor form, giving what we term as formal quantities. Solving the problem for certain types of formal constitutive parameters, these are transformed to give ε and μ as functions of position. Several examples of such lenses are considered in detail.

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I. INTRODUCTION

One of the techniques used in the solution of electromagnetic boundary value problems consists in writing Maxwell's equations in orthogonal curvilinear coordinates and then solving, not for the physical components of the fields, but for quantities which combine the physical components with scale factors of the coordinate transformation. These new quantities are components of tensors and tensor densities referred to the orthogonal curvilinear coordinate system. Similarly the constitutive parameters of the medium are combined with the scale factors in the resulting equations. In making such a transformation one hopes to simplify the equations and/or boundary conditions in some way.

One type of problem on which this technique has been used relates to waveguides (1,2). In this case one takes a waveguide filled with a homogeneous isotropic medium, and transforms to an orthogonal curvilinear coordinate system in which the boundary walls are more conveniently expressed. The resulting transformed constitutive parameters, however, are in general inhomogeneous and anisotropic. Thus while the boundaries have been simplified, the medium has become more complicated.

In this report we consider an extension of this technique. We assume that the formal constitutive parameters, as expressed in some orthogonal curvilinear coordinate system, are of a particularly simple form, i.e., homogeneous, at least as they relate to the allowed field components. Furthermore, we assume that the constitutive parameters, before being transformed to the curvilinear system, are those of an inhomogeneous but isotropic medium. From this we find many cases of isotropic inhomogeneous media for which certain types of electromagnetic wave propagation can be simply expressed.

In this approach the medium is made inhomogeneous and perfectly conducting boundaries are geometrically arranged such that when they are transformed into the appropriate orthogonal curvilinear coordinates a simpler problem results which can be solved by more standard techniques. The present approach can then be used to define geometries for perfectly conducting boundaries and distribution functions for inhomogeneous media such that devices built to such designs will transport electromagnetic waves in certain desirable ways. In particular, we consider cases which in the curvilinear coordinate system corresponds to a problem of a TEM plane wave on a cylindrical transmission line. In the reference cartesian (x,y,z) coordinates the waves are still TEM, but not necessarily plane. For the examples considered the particular conductor geometries and media inhomogeneities can be used to transition waves between two transmission lines, each of which is a conical or cylindrical transmission line. Furthermore, the transition is accomplished with neither reflection nor distortion of the wave. Another application of such examples is for a highly directional high-frequency antenna in which the special geometry and medium inhomogeneity is used to launch an approximate TEM wave over a cross section with dimensions much larger than a wavelength.

The approach followed in this report then represents a design procedure for a certain kind of electromagnetic lens. The properties of such a lens, combined with appropriate perfect conductors, are independent of frequency assuming that the permittivity and permeability of the medium used are real and frequency independent and that its conductivity is zero. This result is in contrast to lenses based on a geometrical optics approximation, such as the well known Luneburg lens (3), which relies on the frequency being sufficiently high. The lenses considered here, used with appropriate transmission lines, can then transmit arbitrary pulse waveforms without distortion.

While the cases considered represent exact solutions to the vector wave equation, there are, of course, approximations involved in the practical realization of such devices. For example, for pulse applications the permittivity and permeability should be frequency independent and have certain prescribed values as functions of position. Such characteristics can only be approximately realized. As another example, it will turn out that the lenses should, in some cases, have infinite extent and so will have to be cut off. If, however, the lens is large enough the relative magnitude of the fields (as compared to the magnitude of the fields near the transmission line passing through the center of the lens) can be small enough that the perturbation is insignificant. The permittivity and permeability will be required to be infinite in some places and less than their free-space values in others, but such positions can be made to be far from any significant fields so that these requirements can be neglected. For certain transmission lines the conductors restrict the fields to a closed region of space so that no lens material at all is needed outside this region. For a particular application of these lens designs, one should consider such things as the range of permittivity and permeability required and the spatial extent of the lens required. In this report we treat the lenses from an idealized viewpoint.

In outline, this report first considers the definition of what we call formal electromagnetic fields, vector operators, and constitutive parameters used with orthogonal curvilinear coordinates. Restricting the forms of the permittivity and permeability the general, but very restrictive, case with field components in all three coordinate directions is briefly considered. This is followed by a consideration of the TEM wave case with electric field components in two coordinate directions. Some general results are obtained for this case and a few lens types are considered. Finally, the simpler case of two-dimensional lenses is considered, together with a few examples.

II. FORMAL VECTORS AND OPERATORS

Let us first consider a cartesian coordinate system (x,y,z) with unit vectors $\vec{e}_x, \vec{e}_y, \vec{e}_z$, and an orthogonal curvilinear coordinate system (u_1,u_2,u_3) with unit vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$. We restrict both coordinate systems to be right handed, i.e.

$$\vec{e}_x \times \vec{e}_y = \vec{e}_z \tag{2.1}$$

and

$$\vec{e}_1 \times \vec{e}_2 = \vec{e}_3 \tag{2.2}$$

The line element is

$$d\vec{r} = \vec{e}_x dx + \vec{e}_y dy + \vec{e}_z dz = h_1 du_1 \vec{e}_1 + h_2 du_2 \vec{e}_2 + h_3 du_3 \vec{e}_3 \tag{2.3}$$

where the scale factors h_i are given for $i=1,2,3$ as

$$h_i^2 = \left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2 = \left[\left(\frac{\partial u_i}{\partial x}\right)^2 + \left(\frac{\partial u_i}{\partial y}\right)^2 + \left(\frac{\partial u_i}{\partial z}\right)^2\right]^{-1} \tag{2.4}$$

The h_i are taken positive and we exclude singular points where $h_i = 0,\infty$ for any $i=1,2,3$ from our consideration. The line element is also often written using the metric tensor (g_{ij}) as

$$(ds)^2 = d\vec{r} \cdot d\vec{r} = \sum_{i=1}^{3} g_{ii}(du_i)^2 \tag{2.5}$$

where for orthogonal curvilinear coordinates the metric tensor has the simple form

$$(g_{ij}) = (g_{ii}\,\delta_{ij}) = \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{pmatrix} = \begin{pmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix} \tag{2.6}$$

For later use we define some combinations of the h_i as

$$H \equiv h_1 h_2 h_3 \tag{2.7}$$

$$(\alpha_{ij}) \equiv (\delta_{ij} h_i) = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \tag{2.8}$$

$$(\beta_{ij}) \equiv (\delta_{ij} \frac{H}{h_i}) = \begin{pmatrix} h_2 h_3 & 0 & 0 \\ 0 & h_3 h_1 & 0 \\ 0 & 0 & h_1 h_2 \end{pmatrix} \tag{2.9}$$

and

$$(\gamma_{ij}) \equiv (\delta_{ij} \frac{H}{h_i^2}) = \begin{pmatrix} \frac{h_2 h_3}{h_1} & 0 & 0 \\ 0 & \frac{h_3 h_1}{h_2} & 0 \\ 0 & 0 & \frac{h_1 h_2}{h_3} \end{pmatrix} \tag{2.10}$$

where δ_{ij} is the Kronecker delta function.

In the u_i coordinate system the standard vector operations are the gradient

$$\nabla \Phi = \frac{1}{h_1} \frac{\partial \Phi}{\partial u_1} \vec{e}_1 + \frac{1}{h_2} \frac{\partial \Phi}{\partial u_2} \vec{e}_2 + \frac{1}{h_3} \frac{\partial \Phi}{\partial u_3} \vec{e}_3 \tag{2.11}$$

curl

$$\begin{aligned} \nabla \times \vec{X} = \frac{1}{H} \{ & h_1 [\frac{\partial}{\partial u_2} (h_3 X_3) - \frac{\partial}{\partial u_3} (h_2 X_2)] \vec{e}_1 \\ & + h_2 [\frac{\partial}{\partial u_3} (h_1 X_1) - \frac{\partial}{\partial u_1} (h_3 X_3)] \vec{e}_2 \\ & + h_3 [\frac{\partial}{\partial u_1} (h_2 X_2) - \frac{\partial}{\partial u_2} (h_1 X_1)] \vec{e}_3 \} \end{aligned} \tag{2.12}$$

and divergence

$$\nabla \cdot \vec{Y} = \frac{1}{H} \{\frac{\partial}{\partial u_1} (h_2 h_3 Y_1) + \frac{\partial}{\partial u_2} (h_3 h_1 Y_2) + \frac{\partial}{\partial u_3} (h_1 h_2 Y_3)\} \tag{2.13}$$

The X_i and Y_i are referred to as the physical components of the vectors $\vec{X}$ and $\vec{Y}$ which have the representations

$$\vec{X} = X_1 \vec{e}_1 + X_2 \vec{e}_2 + X_3 \vec{e}_3$$

and

$$\vec{Y} = Y_1 \vec{e}_1 + Y_2 \vec{e}_2 + Y_3 \vec{e}_3 \tag{2.14}$$

Other common operations such as the scalar and vector Laplacians are formed as combinations of the above operations.

Now we define another set of vectors and operators which we call formal vectors and formal operators and symbolize by the addition of a prime to the standard symbols. Related to $\vec{X}$ we define

$$\vec{X}' = X_1' \vec{e}_1 + X_2' \vec{e}_2 + X_3' \vec{e}_3 \equiv (\alpha_{ij}) \cdot \vec{X} \tag{2.15}$$

Thus the components of $\vec{X}'$ and $\vec{X}$ are related as

$$X_i' = h_i X_i \tag{2.16}$$

In tensor language the X_i' are the covariant components of $\vec{X}$. Related to $\vec{Y}$ we define

$$\vec{Y}' = Y_1' \vec{e}_1 + Y_2' \vec{e}_2 + Y_3' \vec{e}_3 \equiv (\beta_{ij}) \cdot \vec{Y} \tag{2.17}$$

Thus the components of $\vec{Y}'$ and $\vec{Y}$ are related as

$$Y_i' = \frac{H}{h_i} Y_i \tag{2.18}$$

In tensor language the Y_i' are the components of a relative contravariant tensor of weight +1 which can also be called a relative contravariant tensor density (4). Note that we have defined the formal vectors $\vec{X}'$ and $\vec{Y}'$ differently because of the different ways that $\vec{X}$ and $\vec{Y}$ appear in equations 2.12 and 2.13.

Now we define formal vector operators by

$$\nabla'\Phi' \equiv \frac{\partial\Phi'}{\partial u_1}\vec{e}_1 + \frac{\partial\Phi'}{\partial u_2}\vec{e}_2 + \frac{\partial\Phi'}{\partial u_3}\vec{e}_3 \tag{2.19}$$

$$\nabla' \times \vec{X}' \equiv \left[\frac{\partial X_3'}{\partial u_2} - \frac{\partial X_2'}{\partial u_3}\right]\vec{e}_1 + \left[\frac{\partial X_1'}{\partial u_3} - \frac{\partial X_3'}{\partial u_1}\right]\vec{e}_2 + \left[\frac{\partial X_2'}{\partial u_1} - \frac{\partial X_1'}{\partial u_2}\right]\vec{e}_3 \tag{2.20}$$

and

$$\nabla' \cdot \vec{Y}' \equiv \frac{\partial Y_1'}{\partial u_1} + \frac{\partial Y_2'}{\partial u_2} + \frac{\partial Y_3'}{\partial u_3} \tag{2.21}$$

Note that the formal vector operators have precisely the same form in orthogonal curvilinear coordinates as the standard vector operators have in cartesian coordinates. These formal operators are related to the standard ones by

$$\nabla\Phi = (\alpha_{ij})^{-1} \cdot \nabla'\Phi' = \begin{pmatrix} 1/h_1 & 0 & 0 \\ 0 & 1/h_2 & 0 \\ 0 & 0 & 1/h_3 \end{pmatrix} \cdot \nabla'\Phi' \tag{2.22}$$

where the potential function Φ' is related to Φ by

$$\Phi = \Phi' \tag{2.23}$$

and by

$$\nabla \times \vec{X} = \frac{1}{H} (\alpha_{ij}) \cdot (\nabla' \times \vec{X}') = (\beta_{ij})^{-1} \cdot (\nabla' \times \vec{X}') \tag{2.24}$$

and

$$\nabla \cdot \vec{Y} = \frac{1}{H} \nabla' \cdot \vec{Y}' \tag{2.25}$$

Again in tensor language Φ is an invariant scalar, the components of $\nabla'\Phi'$ are the covariant components of $\nabla\Phi$, the components of $\nabla' \times \vec{X}'$ are the components of a relative contravariant tensor of weight +1, and $\nabla \cdot \vec{Y}$ is a relative scalar of weight +1.

Finally we define a formal matrix (ν'_{ij}) related to (ν_{ij}) by

$$(\nu'_{ij}) \equiv (\beta_{ij}) \cdot (\nu_{ij}) \cdot (\alpha_{ij})^{-1} \tag{2.26}$$

The ν'_{ij} are the components of a relative contravariant tensor of weight +1. This transformation will be used later for the constitutive parameters in Maxwell's equations. For the special case that (ν_{ij}) is diagonal we have

$$(\nu'_{ij}) \equiv (\gamma_{ij}) \cdot (\nu_{ij}) \tag{2.27}$$

It is this latter case which will be of concern to us in this report.

III. FORMAL ELECTROMAGNETIC QUANTITIES

Now consider Maxwell's equations

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \tag{3.1}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{3.2}$$

$$\nabla \cdot \vec{D} = \rho \tag{3.3}$$

and

$$\nabla \cdot \vec{B} = 0 \tag{3.4}$$

together with the constitutive relations

$$\vec{D} = (\varepsilon_{ij}) \cdot \vec{E} \tag{3.5}$$

and

$$\vec{B} = (\mu_{ij}) \cdot \vec{H} \tag{3.6}$$

and the equation of continuity

$$\nabla \cdot \vec{J} = - \frac{\partial \rho}{\partial t} \tag{3.7}$$

Note that ρ is the "free" charge density and does not include charge displacement conventionally included in (ε_{ij}). In writing the above equations we have assumed that (ε_{ij}) and (μ_{ij}) are real constant matrices, independent of frequency; they may, however, be functions of position. If we had written the above equations in the frequency domain, then (ε_{ij}) and (μ_{ij}) could easily have been taken as complex functions of frequency.

Equations 3.1 through 3.7 are assumed to be expressed in terms of the u_i coordinates and the $\vec{e}_i$ unit vectors as in Section II. So now we make some appropriate definitions of formal electromagnetic quantities. Since $\vec{E}$ and $\vec{H}$ appear with the curl operator, we define, as in the case of $\vec{X}$,

$$\vec{E}' \equiv (\alpha_{ij}) \cdot \vec{E} \quad , \quad \vec{H}' \equiv (\alpha_{ij}) \cdot \vec{H} \tag{3.8}$$

Since $\vec{B}$, $\vec{D}$, and $\vec{J}$ appear with the divergence operator, we define, as in the case of $\vec{Y}$,

$$\vec{B}' \equiv (\beta_{ij}) \cdot \vec{B} \ , \quad \vec{D}' \equiv (\beta_{ij}) \cdot \vec{D} \ , \quad \vec{J}' \equiv (\beta_{ij}) \cdot \vec{J} \tag{3.9}$$

Now ρ equals a divergence in equation 3.3 so we define

$$\rho' \equiv H\rho \tag{3.10}$$

so that ρ is a relative scalar of weight +1. Substituting for $\vec{E}$, $\vec{H}$, $\vec{B}$, and $\vec{D}$ from equations 3.8 and 3.9 into equations 3.5 and 3.6 and requiring

$$\vec{D}' = (\varepsilon'_{ij}) \cdot \vec{E}' \quad , \quad \vec{B}' = (\mu'_{ij}) \cdot \vec{H}' \tag{3.11}$$

shows that for the formal constitutive parameter matrices we should define

$$(\varepsilon'_{ij}) \equiv (\beta_{ij}) \cdot (\varepsilon_{ij}) \cdot (\alpha_{ij})^{-1}$$

$$(\mu'_{ij}) \equiv (\beta_{ij}) \cdot (\mu_{ij}) \cdot (\alpha_{ij})^{-1} \tag{3.12}$$

For some problems one might include a conductivity matrix (σ_{ij}) so that $\vec{J}$ includes a conduction current density $(\sigma_{ij}) \cdot \vec{E}$. Then we would define

$$(\sigma'_{ij}) \equiv (\beta_{ij}) \cdot (\sigma_{ij}) \cdot (\alpha_{ij})^{-1} \tag{3.13}$$

If (ε_{ij}), (μ_{ij}), and (σ_{ij}) are required to be diagonal, equations 3.12 and 3.13 reduce to

$$(\varepsilon'_{ij}) \equiv (\gamma_{ij}) \cdot (\varepsilon_{ij}) \ , \ (\mu'_{ij}) \equiv (\gamma_{ij}) \cdot (\mu_{ij}) \ ,$$

$$(\sigma'_{ij}) \equiv (\gamma_{ij}) \cdot (\sigma_{ij}) \tag{3.14}$$

The formal electromagnetic quantities defined in equations 3.8 through 3.12 can now be substituted into Maxwell's equations, the constitutive relations and the equation of continuity. The curl and divergence operators can be replaced by the formal operators from equations 2.24 and 2.25. Equations 3.1 through 3.7 can then be rewritten as

$$\nabla' \times \vec{E}' = -\frac{\partial \vec{B}'}{\partial t} \tag{3.15}$$

$$\nabla' \times \vec{H}' = \vec{J}' + \frac{\partial \vec{D}'}{\partial t} \tag{3.16}$$

$$\nabla' \cdot \vec{D}' = \rho' \tag{3.17}$$

$$\nabla' \cdot \vec{B}' = 0 \tag{3.18}$$

$$\vec{D}' = (\varepsilon'_{ij}) \cdot \vec{E}' \tag{3.19}$$

$$\vec{B}' = (\mu'_{ij}) \cdot \vec{H}' \tag{3.20}$$

and

$$\nabla' \cdot \vec{J}' = -\frac{\partial \rho'}{\partial t} \qquad (3.21)$$

Note that equations 3.15 through 3.21 are of the same form as equations 3.1 through 3.7. All electromagnetic quantities and operators are replaced with primed symbols, except for t which has remained unchanged. However, the formal curl and divergence operators, using the u_i coordinates, have the same mathematical forms as have the standard operators, using the x,y,z cartesian coordinates. Suppose that we formally think of the u_i as a cartesian coordinate system and think of the primed quantities as the electromagnetic fields, constitutive parameters, etc. Then we can take a known solution of Maxwell's equations related to cartesian coordinates, directly substitute primed for unprimed quantities and the u_i for the cartesian coordinates, and thereby construct a solution of the above equations. Transforming the formal quantities back to the standard ones by equations 3.8 through 3.13, we then have a solution of Maxwell's equations for which (ε_{ij}), (μ_{ij}), and/or (σ_{ij}) may be anisotropic and/or inhomogeneous. The idea is then to pick (ε'_{ij}), (μ'_{ij}), and (σ'_{ij}) of some particularly convenient form and also to choose any boundary surfaces to have convenient forms in the u_i coordinate system so that we can obtain a solution in terms of the formal electromagnetic quantities. Choosing some particular relationship between the u_i coordinates and x, y, and z, the parameters (ε_{ij}), (μ_{ij}), and (σ_{ij}) as well as the geometry of the boundary surfaces are determined and the solution is applied to the particular case.

IV. RESTRICTION OF CONSTITUTIVE PARAMETERS TO SCALARS

In this report we are only concerned with problems related to inhomogeneous isotropic media. The later examples of lenses will utilize such media. Thus we restrict the constitutive parameter and conductivity matrices to be of the forms

$$(\epsilon_{ij}) \equiv \epsilon(\delta_{ij}) \ , \quad (\mu_{ij}) \equiv \mu(\delta_{ij}) \ , \quad (\sigma_{ij}) \equiv \sigma(\delta_{ij}) \tag{4.1}$$

where ϵ, μ, and σ are scalar functions of the coordinates. From equations 3.14 the formal constitutive parameters then have the forms

$$(\epsilon'_{ij}) = \epsilon(\gamma_{ij}) \ , \quad (\mu'_{ij}) = \mu(\gamma_{ij}) \ , \quad (\sigma'_{ij}) = \sigma(\gamma_{ij}) \tag{4.2}$$

Also, we restrict $\sigma = 0$ and assume that ϵ and μ are real and frequency independent. However ϵ and μ may, in general, depend on the coordinates. The formal constitutive parameters (ϵ'_{ij}) and (μ'_{ij}) are now diagonal matrices with the three diagonal terms possibly functions of the coordinates.

Thus we are led to consider some possible forms for diagonal (ϵ'_{ij}) and (μ'_{ij}) which are consistent with equations (4.2). We would like (ϵ'_{ij}) and (μ'_{ij}) to have rather simple forms so that electromagnetic waves, as expressed using the formal electromagnetic quantities and u_i coordinates, have desired forms. A first case to consider is defined by requiring (ϵ'_{ij}) and (μ'_{ij}) to be expressible as $\epsilon'(\delta_{ij})$ and $\mu'(\delta_{ij})$ with ϵ' and μ' independent of the coordinates. In terms of the formal quantities, this corresponds to a homogeneous medium problem for which many types of solutions of

Maxwell's equations are available. This first case is considered in Section V and Appendix A.

It is not necessary, however, for (ϵ'_{ij}) and (μ'_{ij}) to each have their three diagonal components equal and independent of the u_i for the problem to correspond to one of a homogeneous medium. In particular, suppose that for each matrix just the first two of the diagonal components are constrained to be equal and independent of the coordinates. An inhomogeneous TEM wave with formal field components with only subscripts 1 and 2 has no interaction with ϵ'_{33} or μ'_{33}, and so ϵ'_{33} and μ'_{33} are unimportant in the case of such a wave. Such TEM solutions are used to define lenses to match waves onto cylindrical and/or conical transmission lines. This second case is considered in Sections VI and VII and Appendix B.

As a further simplification we consider the two-dimensional problem in which $u_3 = z$, one of the formal electromagnetic fields has only a u_3 component, and the other formal electromagnetic field has only a u_2 component. With appropriate restrictions on the components of (ϵ'_{ij}) and (μ'_{ij}) this defines a third case considered in Sections VIII and IX. Solutions for this case are used to define lenses for launching TEM waves on two parallel perfectly conducting plates.

V. GENERAL CASE WITH FIELD COMPONENTS IN ALL THREE COORDINATE DIRECTIONS

Now consider the case in which $\vec{E}'$ and $\vec{H}'$ are both allowed to have all three formal components. For this case we constrain the constitutive parameters to have the forms

$$(\epsilon'_{ij}) \equiv \epsilon'(\delta_{ij}) \quad , \qquad (\mu'_{ij}) \equiv \mu'(\delta_{ij}) \tag{5.1}$$

where $\epsilon' > 0$ and $\mu' > 0$ are both independent of the u_i coordinates. In terms of the formal electromagnetic quantities we have a homogeneous medium problem. One might then apply many known solutions for homogeneous media to this case.

With (ϵ'_{ij}) and (μ'_{ij}) each constrained by both equations 5.1 and 4.2, we have

$$(\gamma_{ij}) \equiv \begin{pmatrix} \frac{h_2 h_3}{h_1} & 0 & 0 \\ 0 & \frac{h_3 h_1}{h_2} & 0 \\ 0 & 0 & \frac{h_1 h_2}{h_3} \end{pmatrix} = \frac{\epsilon'}{\epsilon}(\delta_{ij}) = \frac{\mu'}{\mu}(\delta_{ij}) \tag{5.2}$$

where ϵ and μ are both assumed nonzero at positions of interest. This implies

$$\gamma_{11} = \gamma_{22} = \gamma_{33} = \frac{\epsilon'}{\epsilon} = \frac{\mu'}{\mu} \tag{5.3}$$

From

$$\gamma_{22}\gamma_{33} = \gamma_{33}\gamma_{11} = \gamma_{11}\gamma_{22} \tag{5.4}$$

we obtain

$$h_1^2 = h_2^2 = h_3^2 \tag{5.5}$$

Since the h_i are all taken positive, then we have them all equal which we express as

$$h \equiv h_1 = h_2 = h_3 \tag{5.6}$$

Then from equation 5.3 ε and μ are given by

$$\varepsilon h = \varepsilon' \quad , \quad \mu h = \mu' \tag{5.7}$$

so that εh and μh are both independent of the coordinates.

However, we cannot just choose h to be any function of the coordinates. In Appendix A we show that there are two general forms for h which satisfy the restriction imposed by equation 5.6. The first is given by h equals a constant for which the u_i form a cartesian coordinate system. For this case ε and μ are constant so that the medium is homogeneous.

The second form of h, from equations A.27 and A.31, gives an inhomogeneous medium described by

$$\frac{\varepsilon}{\varepsilon'} = \frac{\mu}{\mu'} = \frac{1}{h} = \frac{a^2}{x^2 + y^2 + z^2} \tag{5.8}$$

where $a \neq 0$ is a real constant. This corresponds to a 6-sphere type of coordinate system. Defining the radius

$$r^2 \equiv x^2 + y^2 + z^2 \tag{5.9}$$

we have

$$\frac{\varepsilon}{\varepsilon'} = \frac{\mu}{\mu'} = \frac{1}{h} = \frac{a^2}{r^2} \tag{5.10}$$

If one were to attempt to construct such a medium for frequency independent ε and μ , then ε and μ would be constrained to be at least as large as their free space values. For fixed ε', μ', and a^2 there is a maximum r for which ε and μ can be realized. Also, a neighborhood of $r = 0$ is excluded because of the singularity in ε and μ there. Thus there are restrictions on realizing such a medium.

With the h_i restricted as in equation 5.6, the associated class of inhomogeneous media is then very restricted, being limited to spherically stratified media of the form given by equation 5.8. In the next section we loosen somewhat this restriction on the h_i .

VI. THREE-DIMENSIONAL TEM WAVES

Now we restrict our attention to waves of a certain form. Consider inhomogeneous TEM plane waves such as propagate on ideal cylindrical transmission lines, including coaxial cables, strip lines, etc. Such a structure supports TEM plane waves which propagate parallel to some fixed direction, say the z axis. It has two or more separate perfect conductors which form a cross section (in a plane perpendicular to the z axis) which is independent of z . Also, let the medium in which the perfect conductors are placed be homogeneous.

Next apply this type of inhomogeneous TEM wave solution to the formal fields discussed in Section III. Let the wave propagate in the $+u_3$ direction and let the formal constitutive parameters have the forms

$$(\epsilon'_{ij}) \equiv \begin{pmatrix} \epsilon' & 0 & 0 \\ 0 & \epsilon' & 0 \\ 0 & 0 & \epsilon'_3 \end{pmatrix} \quad , \quad (\mu'_{ij}) \equiv \begin{pmatrix} \mu' & 0 & 0 \\ 0 & \mu' & 0 \\ 0 & 0 & \mu'_3 \end{pmatrix} \qquad (6.1)$$

where $\epsilon' > 0$ and $\mu' > 0$ are constants but ϵ'_3 and μ'_3 are unspecified. Since we shall only consider waves with no field components parallel to the u_3 direction, then ϵ'_3 and μ'_3 nowhere enter the formal constitutive relations, equations 3.19 and 3.20. Then the dependence of ϵ'_3 and μ'_3 on the coordinates is irrelevant and can be ignored. For this TEM wave the medium can then be formally considered isotropic and homogeneous since only ϵ' and μ' are significant.

Specifically, consider formal fields of the form

$$E'_1 = E'_{1_o}(u_1,u_2)\ f(t - \frac{u_3}{c'})\ ,\ E'_2 = E'_{2_o}(u_1,u_2)\ f(t - \frac{u_3}{c'}),\ E'_3 \equiv 0 \qquad (6.2)$$

and

$$H'_1 = H'_{1_o}(u_1,u_2)\ f(t - \frac{u_3}{c'})\ ,\ H'_2 = H'_{2_o}(u_1,u_2)\ f(t - \frac{u_3}{c'}),\ H'_3 \equiv 0 \qquad (6.3)$$

where we define

$$c' \equiv \frac{1}{\sqrt{\mu'\varepsilon'}}\quad ,\qquad c \equiv \frac{1}{\sqrt{\mu_o\varepsilon_o}} \qquad (6.4)$$

and where we can choose the form of $f(t - \frac{u_3}{c'})$ to specify the waveform. This is the well-known form of TEM waves on cylindrical transmission lines (5). The formal field components are related by

$$E'_1 = Z'_o\ H'_2 \qquad (6.5)$$

and

$$E'_2 = -Z'_o\ H'_1 \qquad (6.6)$$

where Z'_o is the formal wave impedance defined by

$$Z'_o \equiv \sqrt{\frac{\mu'}{\varepsilon'}} \qquad (6.7)$$

Equations 6.5 and 6.6 express the orthogonality of $\vec{E}'$ and $\vec{H}'$, i.e.

$$\vec{E}' \cdot \vec{H}' \equiv 0 \qquad (6.8)$$

Also $\vec{E}$ and $\vec{H}$ can be derived from scalar potential functions as

$$\vec{E}' = f(t - \frac{u_3}{c'})\ \nabla'\ \Phi_e(u_1,u_2)\ ,$$

$$\vec{H}' = f(t - \frac{u_3}{c'})\ \nabla' \Phi_h(u_1,u_2) \tag{6.9}$$

where Φ_e and Φ_h both satisfy the Laplace equation (using the $\nabla'^2 = \nabla' \cdot \nabla'$ operator). These potential functions can be combined to form a complex potential $\Phi_e + i\Phi_h$ which allows one to use conformal transform techniques with the complex variable $u_1 + iu_2$. All these equations, 6.2 through 6.9, are merely the direct application of known results for cylindrical transmission lines to their formal equivalents using formal field components and the u_i coordinates in place of physical field components and cartesian coordinates.

Note, of course, that while the results for cylindrical transmission lines assume constant ε and μ , the present results using the formal quantities assume constant ε' and μ' . Likewise the present results require that the two or more perfect conductors forming the transmission line intersect surfaces of constant u_3 in such a manner that the representation in terms of u_1 and u_2 is independent of u_3 . Put simply, these perfectly conducting boundaries can be represented in terms of only their u_1 and u_2 coordinates.

The important feature of these TEM waves is that we only need restrict the first two diagonal components of the formal constitutive parameter matrices as in equations 6.1. We still assume that (ε_{ij}) and (μ_{ij}) correspond to isotropic but inhomogeneous media having the forms as in equations 4.1

$$(\varepsilon_{ij}) = \varepsilon(\delta_{ij}) \qquad , \qquad (\mu_{ij}) = \mu(\delta_{ij}) \tag{6.10}$$

where $\varepsilon > 0$ and $\mu > 0$ may be functions of the coordinates. Then as in equations 4.2 the formal constitutive parameters have the forms

$$(\varepsilon'_{ij}) = \varepsilon(\gamma_{ij}) \quad , \quad (\mu'_{ij}) = \mu(\gamma_{ij}) \tag{6.11}$$

Combining equations 6.1 and 6.11 then gives

$$(\gamma_{ij}) \equiv \begin{pmatrix} \frac{h_2 h_3}{h_1} & 0 & 0 \\ 0 & \frac{h_3 h_1}{h_2} & 0 \\ 0 & 0 & \frac{h_1 h_2}{h_3} \end{pmatrix} = \frac{1}{\varepsilon} \begin{pmatrix} \varepsilon' & 0 & 0 \\ 0 & \varepsilon' & 0 \\ 0 & 0 & \varepsilon'_3 \end{pmatrix} = \frac{1}{\mu} \begin{pmatrix} \mu' & 0 & 0 \\ 0 & \mu' & 0 \\ 0 & 0 & \mu'_3 \end{pmatrix} \tag{6.12}$$

This implies

$$\frac{h_2 h_3}{h_1} = \frac{h_3 h_1}{h_2} = \frac{\varepsilon'}{\varepsilon} = \frac{\mu'}{\mu} \tag{6.13}$$

and

$$\frac{h_1 h_2}{h_3} = \frac{\varepsilon'_3}{\varepsilon} = \frac{\mu'_3}{\mu} \tag{6.14}$$

From equation 6.13 we find that the first two scale factors are equal which we express as

$$h \equiv h_1 = h_2 \tag{6.15}$$

Note that h_3 is not included in this equation. This will allow us a greater degree of freedom in choosing our u_i coordinate systems.

Now ε and μ are given by

$$\varepsilon h_3 = \varepsilon' \quad , \quad \mu h_3 = \mu' \tag{6.16}$$

so that ϵh_3 and μh_3 are both independent of the coordinates. Then the formal wave impedance from equation 6.7 is the same as the physical wave impedance because

$$Z_o' \equiv \sqrt{\frac{\mu'}{\epsilon'}} = \sqrt{\frac{\mu h_3}{\epsilon h_3}} = \sqrt{\frac{\mu}{\epsilon}} \equiv Z_o \tag{6.17}$$

Since ϵ_3' and μ_3' are arbitrary, then any orthogonal curvilinear coordinate system which satisfies equation 6.15 is acceptable. The h_3 that results defines ϵ and μ by equations 6.16. Not just any orthogonal system, however, satisfies equation 6.15. In Appendix B we show that surfaces of constant u_3 can only be planes or spheres (with respect to an x,y,z cartesian coordinate system). Two examples of such coordinate systems have already appeared in Section V (and Appendix A), namely cartesian coordinates and 6-sphere coordinates. In those examples all three u_i surfaces are planes or spheres, since all three h_i were made equal.

In the next section several examples of orthogonal curvilinear coordinate systems satisfying equation 6.15 are considered. These are used to define types of inhomogeneous lenses which are then combined with conical and/or cylindrical transmission lines. Some of these lenses have rotational symmetry, while the associated u_i coordinate system is not a rotational system. For convenience in such cases we then introduce an additional orthogonal curvilinear coordinate system v_1, v_2, v_3 which is both right handed and rotational. We define the cylindrical coordinates ρ, ϕ, z with

$$\rho \equiv (x^2 + y^2)^{1/2} \tag{6.18}$$

and

$$\tan(\phi) \equiv y/x \tag{6.19}$$

where $\phi = 0$ is taken from the xz plane for x positive. To make the v_i coordinate system a rotational system we define

$$v_2 \equiv \phi \tag{6.20}$$

In order to distinguish the scale factors for the v_i coordinate system we write them as h_{v_1}, h_ϕ, and h_{v_3} where v_1 and v_3 may be replaced by other symbols for a particular rotational coordinate system. There are many well-known rotational coordinate systems for which the h_{v_i} are tabulated (6).

To construct the u_i coordinate systems we consider a transformation from the v_i system of the form

$$u_1 = \lambda(v_1) \cos(\phi) \tag{6.21}$$

$$u_2 = \lambda(v_1) \sin(\phi) \tag{6.22}$$

and

$$u_3 = \xi(v_3) \tag{6.23}$$

with $\lambda(v_1)$ assumed non-negative. There are several reasons for considering this type of transformation. Surfaces of constant u_3 are also surfaces of constant v_3 which must then also be planes or spheres. The functional form $\xi(v_3)$ gives us some flexibility in choosing h_3 which in turn defines ε and μ. The choice for u_1 and u_2 will make $h_1 = h_2$. The functional form $\lambda(v_1)$ is used to gain flexibility in trying to make surfaces of constant u_1 and surfaces of constant u_2 orthogonal. As an illustrative example, let v_1,ϕ,v_2 be cylindrical coordinates ρ,ϕ,z and let $\lambda(\rho) = \rho$, $\xi(z) = z$. Then u_1,u_2,u_3 is just x,y,z . This corresponds to the well-known case of a TEM wave propagating in the +z direction on a cylindrical transmission line in a homogeneous medium.

Surfaces of constant v_1, v_2 and v_3 are mutually orthogonal by hypothesis. Then since neither u_1 nor u_2 are functions of v_3 , while u_3 is a function of v_3 only, surfaces of constant u_3 are orthogonal both to surfaces of constant u_1 and to surfaces of constant u_2 . This leaves the question of the mutual orthogonality of surfaces of constant u_1 and surfaces of constant u_2 . For orthogonality of constant u_1 and constant u_2 surfaces we need

$$\frac{\partial\vec{r}}{\partial u_1}\cdot\frac{\partial\vec{r}}{\partial u_2} = 0 \qquad (6.24)$$

or

$$\left[\frac{\partial\vec{r}}{\partial v_1}\frac{\partial v_1}{\partial u_1}+\frac{\partial\vec{r}}{\partial\phi}\frac{\partial\phi}{\partial u_1}\right]\cdot\left[\frac{\partial\vec{r}}{\partial v_1}\frac{\partial v_1}{\partial u_2}+\frac{\partial\vec{r}}{\partial\phi}\frac{\partial\phi}{\partial u_2}\right] = 0 \qquad (6.25)$$

Since the v_i surfaces are orthogonal we have

$$\frac{\partial\vec{r}}{\partial v_1}\cdot\frac{\partial\vec{r}}{\partial\phi} = 0 \qquad (6.26)$$

so that equation 6.25 becomes

$$\frac{\partial\vec{r}}{\partial v_1}\cdot\frac{\partial\vec{r}}{\partial v_1}\frac{\partial v_1}{\partial u_1}\frac{\partial v_1}{\partial u_2}+\frac{\partial\vec{r}}{\partial\phi}\cdot\frac{\partial\vec{r}}{\partial\phi}\frac{\partial\phi}{\partial u_1}\frac{\partial\phi}{\partial u_2} = 0 \qquad (6.27)$$

or

$$h_{v_1}^2 \frac{\partial v_1}{\partial u_1} \frac{\partial v_1}{\partial u_2} + h_\phi^2 \frac{\partial \phi}{\partial u_1} \frac{\partial \phi}{\partial u_2} = 0 \qquad (6.28)$$

Using the relations

$$\phi = \arc \tan(\frac{u_2}{u_1}) \quad , \quad \lambda = (u_1^2 + u_2^2)^{1/2} \qquad (6.29)$$

we find

$$\frac{\partial \phi}{\partial u_1} = - \frac{\sin(\phi)}{\lambda} \quad , \quad \frac{\partial \phi}{\partial u_2} = \frac{\cos(\phi)}{\lambda} \qquad (6.30)$$

and

$$\frac{\partial v_1}{\partial u_1} = \frac{\partial \lambda}{\partial u_1} \frac{dv_1}{d\lambda} = \cos(\phi) \frac{dv_1}{d\lambda}$$

$$\frac{\partial v_1}{\partial u_2} = \frac{\partial \lambda}{\partial u_2} \frac{dv_1}{d\lambda} = \sin \phi \frac{dv_1}{d\lambda} \qquad (6.31)$$

For h_ϕ we have

$$h_\phi = \left[(\frac{\partial x}{\partial \phi})^2 + (\frac{\partial y}{\partial \phi})^2\right]^{1/2} = \rho \qquad (6.32)$$

Substituting these results in equation 6.28 gives

$$h_{v_1} \left|\frac{dv_1}{d\lambda}\right| = \frac{\rho}{\lambda} \qquad (6.33)$$

or

$$\frac{|d\lambda|}{\lambda} = \frac{h_{v_1}}{\rho} |dv_1| = \frac{h_{v_1}}{h_\phi} |dv_1| \qquad (6.34)$$

Now we have

$$\frac{h_{v_1}}{h_\phi} = \frac{1}{\rho}\left[(\frac{\partial x}{\partial v_1})^2 + (\frac{\partial y}{\partial v_1})^2 + (\frac{\partial z}{\partial v_1})^2\right]^{1/2} = \frac{1}{\rho}\left[(\frac{\partial \rho}{\partial v_1})^2 + (\frac{\partial z}{\partial v_1})^2\right]^{1/2} \qquad (6.35)$$

so that h_{v_1}/h_ϕ is independent of ϕ. In Appendix B (equation B.12) we find, from the requirement that surfaces of constant v_3 be spheres or planes, that h_{v_1}/h_ϕ is independent of v_3. Hence h_{v_1}/h_ϕ is only a function of v_1. Then equation 6.34 can be integrated to obtain λ as only a function of v_1. Thus, from an orthogonal system v_1,ϕ,v_3 with surfaces of constant v_3 spheres or planes only, equations 6.21 through 6.23 define an orthogonal u_i system in which surfaces of constant u_3 are spheres or planes only.

Next consider the h_i and relate them to the h_{v_i}. For h_3 we have

$$h_3^2 = \frac{\partial \vec{r}}{\partial u_3} \cdot \frac{\partial \vec{r}}{\partial u_3} = \frac{\partial \vec{r}}{\partial v_3} \cdot \frac{\partial \vec{r}}{\partial v_3} (\frac{dv_3}{du_3})^2 \qquad (6.36)$$

or

$$h_3 = h_{v_3} \left|\frac{dv_3}{du_3}\right| \qquad (6.37)$$

For h_1 we have (using equation 6.26)

$$h_1^2 = \frac{\partial \vec{r}}{\partial u_1} \cdot \frac{\partial \vec{r}}{\partial u_1} = \frac{\partial \vec{r}}{\partial v_1} \cdot \frac{\partial \vec{r}}{\partial v_1} (\frac{\partial v_1}{\partial u_1})^2 + \frac{\partial \vec{r}}{\partial \phi} \cdot \frac{\partial \vec{r}}{\partial \phi} (\frac{\partial \phi}{\partial u_1})^2$$

$$= h_{v_1}^2 (\frac{dv_1}{d\lambda})^2 \cos^2(\phi) + \frac{\rho^2}{\lambda^2} \sin^2(\phi) \qquad (6.38)$$

Substituting for h_{v_1} from equation 6.33 simplifies this last result to

$$h_1 = \frac{\rho}{\lambda} = \frac{h_\phi}{\lambda} \qquad (6.39)$$

For h_2 we have similarly

$$h_2^2 = \frac{\partial\vec{r}}{\partial u_2} \cdot \frac{\partial\vec{r}}{\partial u_2} = \frac{\partial\vec{r}}{\partial v_1} \cdot \frac{\partial\vec{r}}{\partial v_1} \left(\frac{\partial v_1}{\partial u_2}\right)^2 + \frac{\partial\vec{r}}{\partial\phi} \cdot \frac{\partial\vec{r}}{\partial\phi} \left(\frac{\partial\phi}{\partial u_2}\right)^2$$

$$= h_{v_1}^2 \left(\frac{dv_1}{d\lambda}\right)^2 \sin^2(\phi) + \frac{\rho^2}{\lambda^2} \cos^2(\phi) \qquad (6.40)$$

which, using equation 6.33, simplifies to

$$h_2 = \frac{\rho}{\lambda} = \frac{h_\phi}{\lambda} = h_1 = h \qquad (6.41)$$

Thus the form of the u_i given by equations 6.21 through 6.23, with $\lambda(v_1)$ satisfying equation 6.33, also satisfies the requirement of equation 6.15 that $h_1 = h_2$. We then have an acceptable u_i system.

Note that the u_i system defined by equations 6.21 through 6.23 and equation 6.33 is based on a rotational system, v_1,ϕ,v_3, with propagation in the $\pm v_3$ direction where surfaces of constant v_3 are spheres or planes. This is not the only way to define an acceptable u_i system. The last example in the next section will construct the u_i system differently.

VII. THREE-DIMENSIONAL TEM LENSES

In this section we consider some examples of lenses for transporting TEM waves of the form considered in Section VI. These inhomogeneous TEM waves propagate on transmission lines with two or more independent perfectly conducting boundaries described in the form

$$f(u_1, u_2) = 0 \tag{7.1}$$

so that the boundaries are independent of u_3 . The simplest example of this case is given by u_1, u_2, u_3 equal to x, y, z respectively which corresponds to a cylindrical transmission line with a homogeneous medium. We first consider the example of conical transmission lines as a simple illustration of the method developed in the last section. This is followed by two inhomogeneous lenses based on bispherical and toroidal coordinate systems. We also show how these can be used to transition TEM waves between conical and/or cylindrical transmission lines. The bispherical lens can be thought of as a converging lens and the toroidal lens as a diverging lens. Finally we consider a lens, based on cylindrical coordinates, which can be used to transition TEM waves between two different cylindrical transmission lines which have their propagation axes pointing in two different directions.

A. Modified Spherical Coordinates

As a first example start with a rotational orthogonal curvilinear coordinate system v_1, ϕ, v_3 given by the spherical coordinates θ, ϕ, r illustrated in Figure 1 and defined by

$$x \equiv r \sin(\theta) \cos(\phi) \tag{7.2}$$

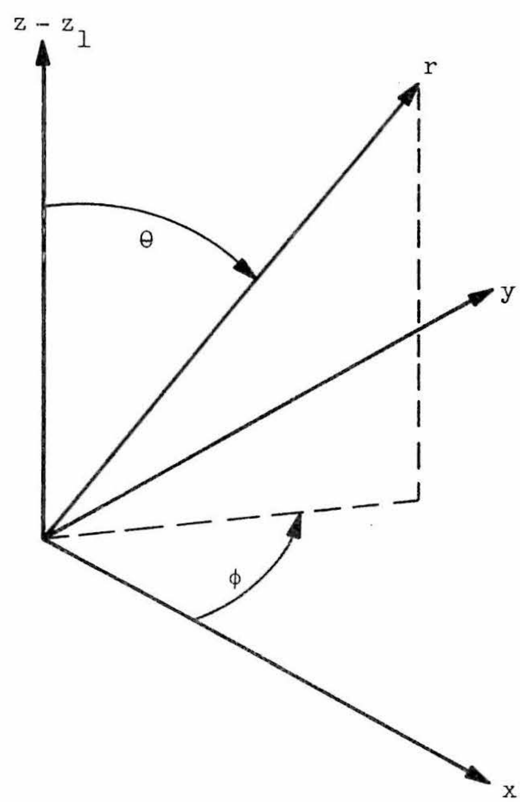


Fig. 1. Spherical Coordinates

$$y \equiv r \sin(\theta) \sin(\phi) \tag{7.3}$$

$$z - z_1 \equiv r \cos(\theta) \tag{7.4}$$

where z_1 is a constant we can choose later. Note that surfaces of constant $v_3 = r$ are spheres and we are considering propagation in the r direction. The scale factors are

$$h_\theta = r \quad , \quad h_\phi = r \sin(\theta) = \rho \quad , \quad h_r = 1 \tag{7.5}$$

Next we construct the u_i system for which we need $\lambda(\theta)$ and $\xi(r)$ for equations 6.21 through 6.23. From equation 6.37 we have

$$h_3 = h_r \left|\frac{dr}{du_3}\right| = \left|\frac{dr}{du_3}\right| \tag{7.6}$$

For convenience we choose $\xi(r) = r + r_o$, where r_o is a constant we can choose later, giving

$$u_3 = r + r_o \quad , \quad h_3 = 1 \tag{7.7}$$

Now we find a λ from integrating equation 6.34 as

$$\int_{2z_o}^{\lambda} \frac{d\lambda'}{\lambda'} = \int_{\pi/2}^{\theta} \frac{d\theta'}{\sin(\theta')} \tag{7.8}$$

where $z_o > 0$ is a constant for later use. This gives

$$\ell n\left(\frac{\lambda}{2z_o}\right) = \ell n\left[\tan\left(\frac{\theta}{2}\right)\right] \tag{7.9}$$

or

$$\lambda = 2z_o \tan\left(\frac{\theta}{2}\right) \tag{7.10}$$

so that

$$u_1 = 2z_o \tan(\frac{\theta}{2}) \cos(\phi) \quad , \quad u_2 = 2z_o \tan(\frac{\theta}{2}) \sin(\phi) \tag{7.11}$$

From equation 6.41 the associated scale factor is

$$h = \frac{h_\phi}{\lambda} = \frac{r \sin(\theta)}{2z_o \tan(\frac{\theta}{2})} = \frac{r}{2z_o} [1 + \cos(\theta)] \tag{7.12}$$

Note that $0 < h < \infty$ on the +z axis ($\theta = 0$, $r > 0$) so that the u_i coordinate system is well behaved there, even though the v_i system is singular there. We call this u_i system modified spherical coordinates.

The required constitutive parameters are given by equations 6.16 as

$$\frac{\varepsilon}{\varepsilon'} = \frac{\mu}{\mu'} = \frac{1}{h_3} = 1 \tag{7.13}$$

Thus for the present choice of u_i coordinates the medium is homogeneous. For convenience we might choose ε', μ' as ε_o, μ_o making ε, μ also ε_o, μ_o so that the medium is free space. The structure defined by the perfect conductors satisfying equation 7.1 is called a conical transmission line. The transformations of equations 7.7 and 7.11, giving the u_i, are the well-known transformation for finding the TEM waves on such a conical structure(7). The present example for the u_i is then a comparatively simple one and the resulting medium is homogeneous. However, this example illustrates how to construct the u_i systems. In addition the conical transmission line is used later in conjunction with inhomogeneous lenses.

B. Modified Bispherical Coordinates

For constructing an example of an inhomogeneous lens, start with the rotational system v_1, ϕ, v_3 given as bispherical coordinates ψ, ϕ, η as illustrated in Figure 2 and defined by (6)

$$x \equiv \frac{a \sin(\psi) \cos(\phi)}{\cosh(\eta) + \cos(\psi)} \tag{7.14}$$

$$y \equiv \frac{a \sin(\psi) \sin(\phi)}{\cosh(\eta) + \cos(\psi)} \tag{7.15}$$

$$z \equiv \frac{a \sinh(\eta)}{\cosh(\eta) + \cos(\psi)} \tag{7.16}$$

with $0 \leq \psi \leq \pi$ and $-\infty < \eta < \infty$. Surfaces of constant $v_3 = \eta$ are spheres; surfaces of constant $v_1 = \psi$ intersect planes of constant ϕ in circles. The scale factors are

$$h_\phi = \frac{a \sin(\psi)}{\cosh(\eta) + \cos(\psi)} = \rho \tag{7.17}$$

$$h_\eta = h_\psi = \frac{a}{\cosh(\eta) + \cos(\psi)} = \frac{\rho}{\sin(\psi)} \tag{7.18}$$

Next construct the u_i system. First we calculate λ from equation 6.34 as

$$\int_{a_o}^{\lambda} \frac{d\lambda'}{\lambda'} = \int_{\pi/2}^{\psi} \frac{d\psi'}{\sin(\psi')} \tag{7.19}$$

which gives

$$\lambda = a_o \tan(\frac{\psi}{2}) \tag{7.20}$$

Rotate ϕ from 0 to 2π to give surfaces of constant η, ψ .

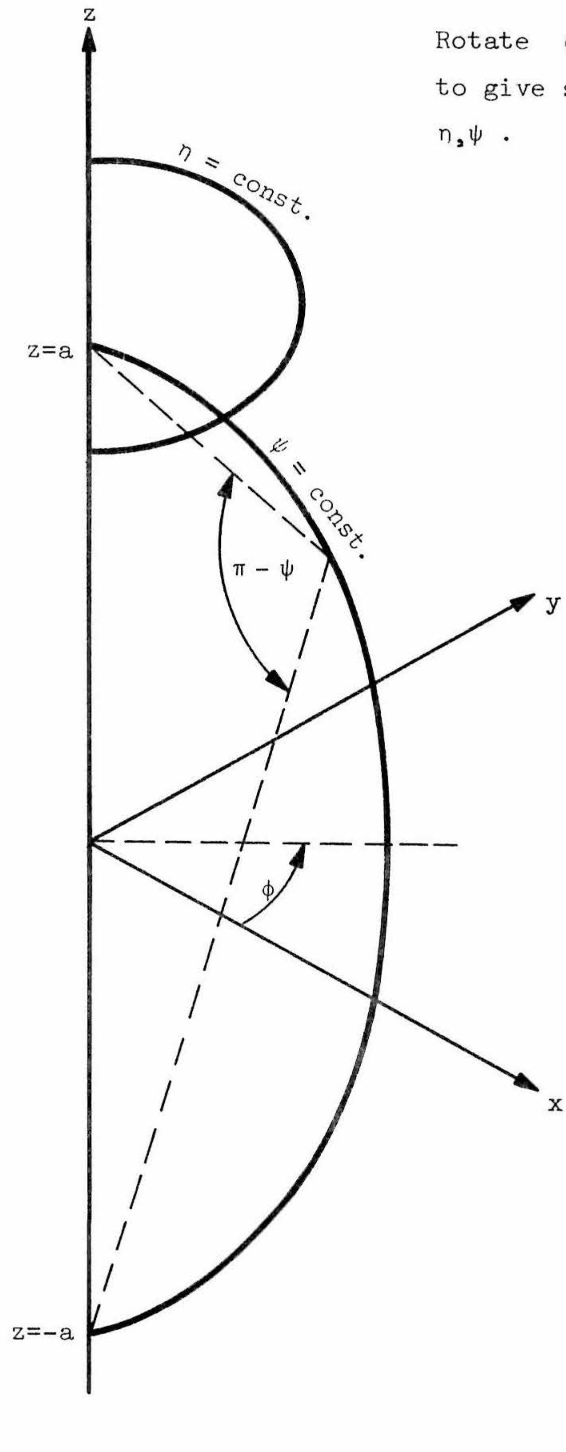


Fig. 2. Bispherical Coordinates

so that

$$u_1 = a_o \tan(\frac{\psi}{2}) \cos(\phi) \quad , \quad u_2 = a_o \tan(\frac{\psi}{2}) \sin(\phi) \tag{7.21}$$

where $a_o > 0$ is a constant we can choose. From equation 6.41 the associated scale factor is

$$h = \frac{h_\phi}{\lambda} = \frac{a}{a_o} \frac{\sin(\psi)}{\tan(\psi/2)} \frac{1}{\cosh(\eta) + \cos(\psi)} = \frac{a}{a_o} \frac{1 + \cos(\psi)}{\cosh(\eta) + \cos(\psi)} \tag{7.22}$$

which has $0 < h < \infty$ for $-a < z < a$ on the z axis so that the u_i system is well behaved there.

From equation 6.37 we have

$$h_3 = h_\eta \left|\frac{d\eta}{du_3}\right| = \frac{a}{\cosh(\eta) + \cos(\psi)} \left|\frac{d\eta}{du_3}\right| \tag{7.23}$$

Now h_3 is related to the constitutive parameters by

$$\frac{\varepsilon}{\varepsilon'} = \frac{\mu}{\mu'} = \frac{1}{h_3} \tag{7.24}$$

For convenience let $\varepsilon' = \varepsilon_o$, $\mu' = \mu_o$ and also restrict $\varepsilon \geqq \varepsilon_o$, $\mu \geqq \mu_o$. This implies the restriction

$$h_3 \leqq 1 \tag{7.25}$$

Next observe for $0 \leqq \psi < \pi$ and for fixed η that h_3 is a monotonically increasing function of ψ. Then consider some maximum ψ of interest and call it ψ_o with $0 < \psi_o < \pi$. Then restrict the space occupied by the inhomogeneous medium to $0 \leqq \psi < \psi_o$. Then to minimize the magnitudes of ϵ and μ required, set $h_3 = 1$ on $\psi = \psi_o$. This gives

$$h_3 = \frac{\cosh(\eta) + \cos(\psi_o)}{\cosh(\eta) + \cos(\psi)} \tag{7.26}$$

and we choose

$$\frac{d\eta}{du_3} = \frac{1}{a}[\cosh(\eta) + \cos(\psi_o)] \tag{7.27}$$

Note that there are many other forms that one could choose for $\frac{d\eta}{du_3}$. The present choice is for the sake of convenience and definiteness. From equation 7.27 we then calculate u_3 as (8)

$$u_3 = a \int_0^\eta \frac{d\eta'}{\cosh(\eta') + \cos(\psi_o)}$$

$$= \frac{2a}{\sin(\psi_o)} \arctan\left[\tanh(\tfrac{\eta}{2}) \tan(\tfrac{\psi_o}{2})\right] \tag{7.28}$$

This last result can be verified by first observing that $u_3 = 0$ for $\eta = 0$ and by second differentiating the result and using the half angle formulas for the trigonometric and hyperbolic functions. We now have all the u_i coordinates which for the present geometry we call modified bispherical coordinates.

Now that the u_i coordinates and h_i scale factors are calculated, consider the combination of this bispherical lens with a cylindrical transmission line. On the plane $z = 0$, on which $\eta = 0$ and $u_3 = 0$, we have from equations 7.14, 7.15 and 7.21, and defining $a_o \equiv a$,

$$u_1 = x \quad , \quad u_2 = y \tag{7.29}$$

Let the lens material modifying μ and ε be present only for $u_3 < 0$ (corresponding to $\eta < 0$, $z < 0$). Then for $z > 0$ let the medium be free space with constitutive parameters ε_o, μ_o . Next let there be two or more perfect conductors forming a transmission line described in the form

$$f(x,y)\Big|_{z \geq 0} = 0 \quad , \qquad f(u_1,u_2)\Big|_{z \leq 0} = 0 \tag{7.30}$$

Since on the dividing plane we have $u_1 = x$, $u_2 = y$, then the conductors are continuous through this interface. For $z \geq 0$ these conductors form a cylindrical transmission line on which a TEM mode has the form (from equations 6.9)

$$\vec{E} = f(t - \frac{z}{c})\ \nabla\Phi_e(x,y) \quad , \quad \vec{H} = f(t - \frac{z}{c})\ \nabla\Phi_h(x,y) \tag{7.31}$$

The potential functions solve $\nabla^2\Phi(x,y) = 0$ subject to appropriate boundary conditions from equations 7.30. Similarly for $z \leq 0$, making $u_3 \leq 0$, there is a corresponding TEM mode of the form

$$\vec{E}' = f(t - \frac{u_3}{c'})\ \nabla'\Phi_e(u_1,u_2) \ , \quad \vec{H}' = f(t - \frac{u_3}{c'})\ \nabla'\Phi_h(u_1,u_2) \tag{7.32}$$

We purposely use Φ_e and Φ_h for both $z \geq 0$ and $u_3 \leq 0$ because they solve the same Laplace equation and boundary conditions on both sides of $z = 0$ with u_1,u_2 on one side exchanged for x,y on the other. Note that from equations 6.5 and 6.6 the components of $\vec{E}$ and $\vec{H}$ are related by the wave impedance. Since we want both Φ_e and Φ_h the same on both sides of the boundary then we must have

$$Z_o \equiv \sqrt{\frac{\mu_o}{\epsilon_o}} = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu'}{\epsilon'}} \equiv Z'_o \tag{7.33}$$

which we have already required. Now right at $z = 0$ we have $h = 1$ and $u_1 = x$, $u_2 = y$ so that

$$\vec{E}'\Big|_{z=0^-} = \vec{E}\Big|_{z=0^-} = \vec{E}\Big|_{z=0^+} , \quad \vec{H}'\Big|_{z=0^-} = \vec{H}\Big|_{z=0^-} = \vec{H}\Big|_{z=0^+} \tag{7.34}$$

Thus tangential $\vec{E}$ and $\vec{H}$ are continuous across $z = 0$ and the two TEM waves are exactly matched there. Then a TEM wave as in equations 7.32 in the inhomogeneous lens will propagate into free space in the form of equations 7.31 with no reflection.

An alternative approach to matching the TEM mode through the $z = 0$ interface is to define one u_i coordinate system for both positive and negative z. For $z \geq 0$ let $(u_1,u_2,u_3) \equiv (x,y,z)$ while for $z \leq 0$ let u_1,u_2,u_3 be defined by equations 7.21 and 7.28 with $a_o \equiv a$. Then h is continuous at $z = 0$ while h_3 has a step discontinuity there, since for $z > 0$ we have $h = h_3 = 1$. Note that describing the combination of the lens with free space by a single u_i coordinate system automatically poses the restriction of equation 7.33 in that the ratio μ/ϵ must be the same at all positions of interest in order to satisfy equations 6.16. In terms of this composite u_i coordinate system the TEM wave is then described by equations 7.32. This type of lens-transmission-line combination is illustrated in Figure 3 in which the cylindrical transmission line for $z \geq 0$ is taken as a strip line. The lens is stopped a little before the singularity at $(x,y,z) = (0,0,-a)$ is reached.

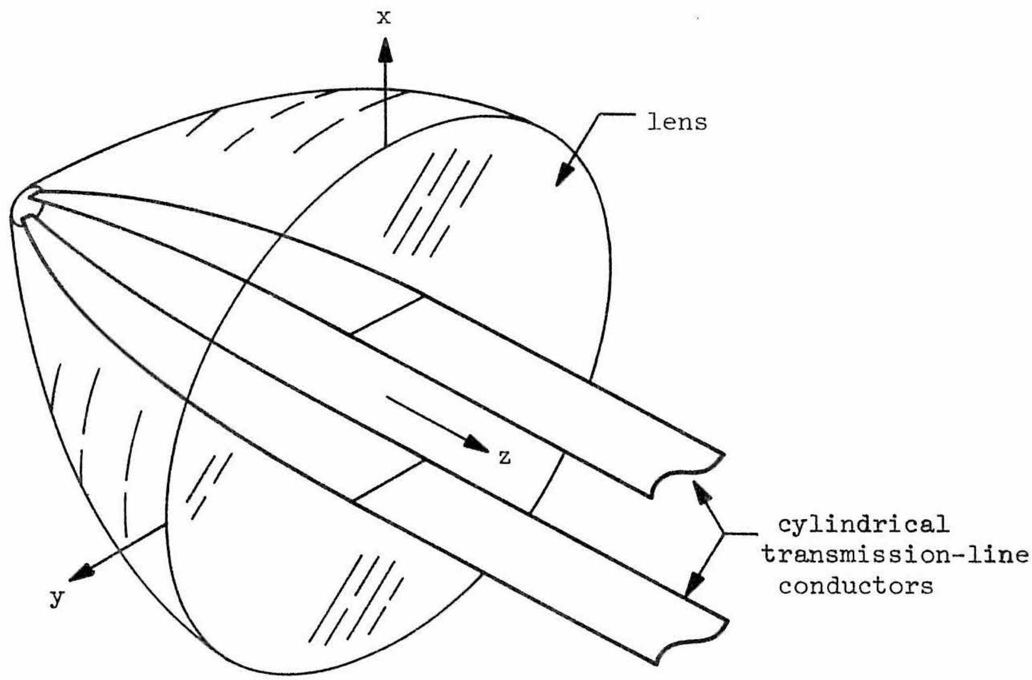


Fig. 3. Bispherical Lens with Cylindrical Transmission Line

Next introduce a second interface at $\eta = \eta_o < 0$. Such a surface is a sphere described by (6)

$$x^2 + y^2 + (z - a\coth(\eta_o))^2 = \frac{a^2}{\sinh^2(\eta_o)} \tag{7.35}$$

This sphere is centered on the z axis at $z = a\coth(\eta_o)$ and has a radius $a|\sinh(\eta_o)|^{-1}$. A cross section of the lens in the zx plane is illustrated in Figure 4 and a perspective view with the transmission line conductors is illustrated in Figure 5. The region inside the sphere $\eta = \eta_o$ is assumed to be free space and in this region we place a conical transmission line with conductors matching to those in the lens.

Recall the conical transmission line discussed in Section VIIA. In order to center the apex of the conical line at the center of the $\eta = \eta_o$ sphere we choose z_1 in equation 7.4 as

$$z_1 \equiv a\coth(\eta_o) \tag{7.36}$$

From equations 7.11 we have for the conical line

$$u_1 = 2z_o \tan(\tfrac{\theta}{2}) \cos(\phi) \quad , \quad u_2 = 2z_o \tan(\tfrac{\theta}{2}) \sin(\phi) \tag{7.37}$$

while from equations 7.21 and $a_o = a$ we have for the lens

$$u_1 = a\tan(\tfrac{\psi}{2}) \cos(\phi) \quad , \quad u_2 = a\tan(\tfrac{\psi}{2}) \sin(\phi) \tag{7.38}$$

We would like u_1 and u_2 to be continuous across the surface $\eta = \eta_o$. Thus we need on $\eta = \eta_o$,

$$a\tan(\tfrac{\psi}{2}) = 2z_o \tan(\tfrac{\theta}{2}) \tag{7.39}$$

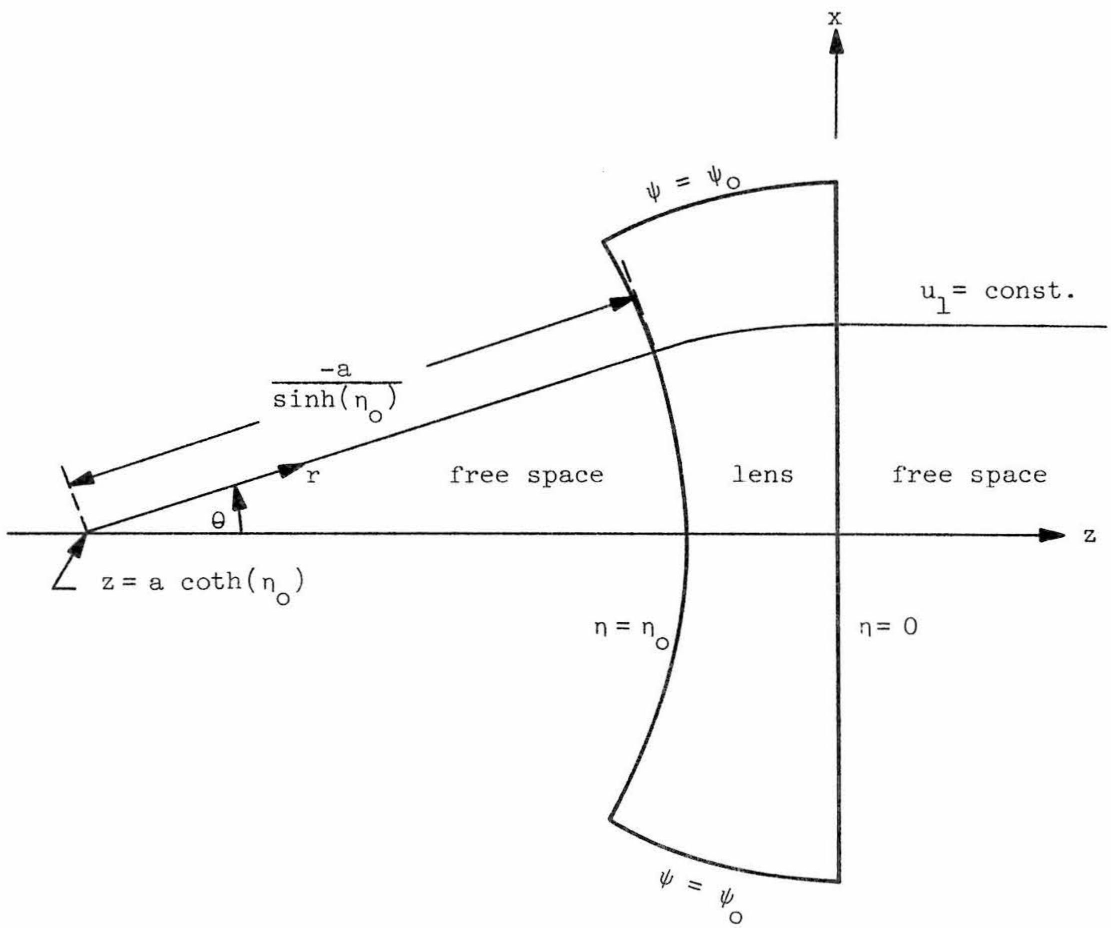


Fig. 4. Bispherical Lens

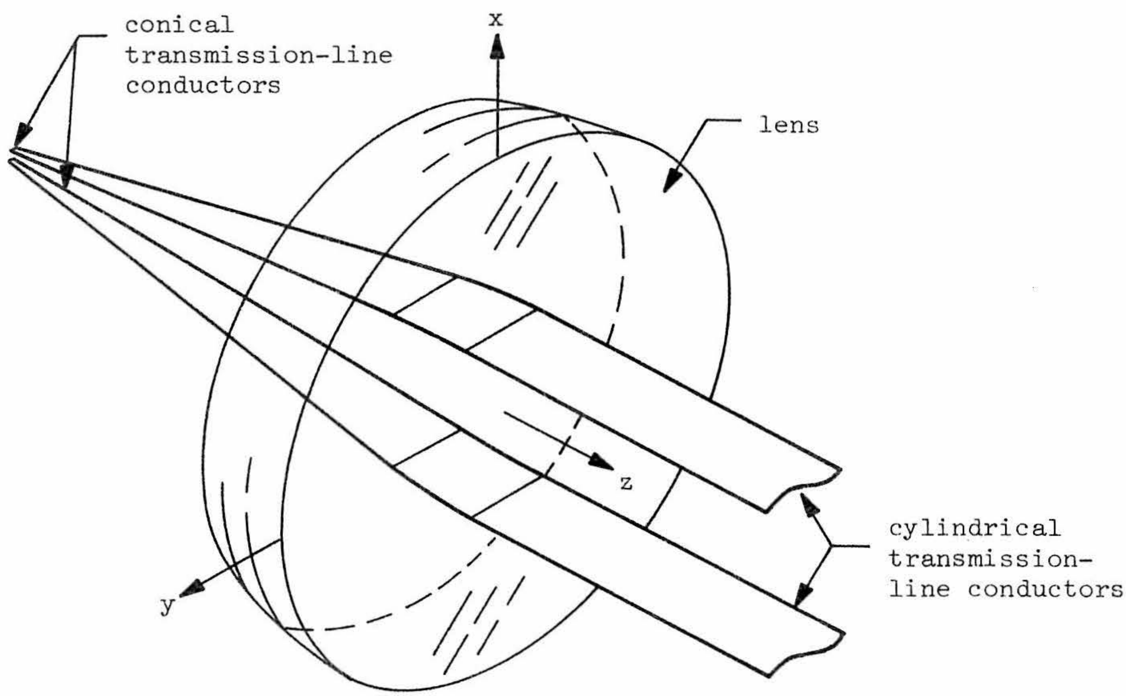


Fig. 5. Bispherical Lens with Cylindrical and Conical Transmission Lines

To do this consider ρ on $\eta = \eta_o$. For the conical line we have

$$\rho = \frac{-a}{\sinh(\eta_o)} \sin(\theta) \tag{7.40}$$

while for the lens we have

$$\rho = \frac{a \sin(\psi)}{\cosh(\eta_o) + \cos(\psi)} \tag{7.41}$$

The θ and ψ coordinates are then related on this surface by

$$\sin(\theta) = \frac{-\sin(\psi)\sinh(\eta_o)}{\cosh(\eta_o) + \cos(\psi)} \equiv A \tag{7.42}$$

Then we have

$$\tan^2(\frac{\theta}{2}) = \frac{1 - \sqrt{1 - A^2}}{1 + \sqrt{1 - A^2}} \tag{7.43}$$

where, after some manipulation, we obtain

$$\sqrt{1 - A^2} = \frac{1 + \cosh(\eta_o)\cos(\psi)}{\cosh(\eta_o) + \cos(\psi)} \tag{7.44}$$

Substituting from equation 7.44 into equation 7.43 and using the half angle formulas for the trigonometric and hyperbolic functions gives

$$\tan^2(\frac{\theta}{2}) = \tanh^2(\frac{\eta_o}{2}) \tan^2(\frac{\psi}{2}) \tag{7.45}$$

or

$$\tan(\frac{\theta}{2}) = -\tanh(\frac{\eta_o}{2}) \tan(\frac{\psi}{2}) \tag{7.46}$$

where the minus sign is used because $\eta_o < 0$. Therefore we define

$$\frac{a}{2z_o} \equiv -\tanh(\frac{\eta_o}{2}) \qquad (7.47)$$

so that equations 7.46 and 7.39 are made equivalent. This makes u_1 and u_2 continuous across the spherical surface $\eta = \eta_o$.

In order to make u_3 continuous at $\eta = \eta_o$ we note from equation 7.7 that on $\eta = \eta_o$ we have

$$u_3 = -\frac{a}{\sinh(\eta_o)} + r_o \qquad (7.48)$$

while from equation 7.28 we have

$$u_3 = \frac{2a}{\sin(\psi_o)} \arctan[\tanh(\frac{\eta_o}{2}) \tan(\frac{\psi_o}{2})] \qquad (7.49)$$

Equating these results gives

$$\frac{r_o}{a} \equiv \frac{1}{\sinh(\eta_o)} + \frac{2}{\sin(\psi_o)} \arctan[\tanh(\frac{\eta_o}{2}) \tan(\frac{\psi_o}{2})] \qquad (7.50)$$

as our definition of r_o .

With u_1 and u_2 continuous across $\eta = \eta_o$, a surface of constant u_3 , h is automatically continuous there. However, h_3 has a step discontinuity at this surface. Then we have the same conditions at $\eta = \eta_o$ as before at $\eta = 0$, namely the TEM wave passes through this surface without reflection and is described by equations 7.32. In summary, inside the $\eta = \eta_o$ sphere the u_i are given by equations 7.7 and 7.11 and the constitutive parameters are just ϵ_o and μ_o . In the lens bounded by $\eta = \eta_o$, $\eta = 0$ and $\psi = \psi_o$, the u_i are given by equations 7.28 and 7.38, and the constitutive parameters are given by equations 7.24 and 7.26 with $\epsilon' = \epsilon_o$, $\mu' = \mu_o$. For $z \geq 0$ u_1, u_2, u_3 are x, y, z and the constitutive parameters are ϵ_o, μ_o . The transmission line conductors in all three regions have exactly the same description as functions of u_1 and u_2 only as in equation 7.1.

From equations 7.24 and 7.26 the constitutive parameters for the lens are given by

$$\frac{\epsilon}{\epsilon_o} = \frac{\mu}{\mu_o} = \frac{1}{h_3} = \frac{\cosh(\eta) + \cos(\psi)}{\cosh(\eta) + \cos(\psi_o)} \tag{7.51}$$

For convenience one might prefer to have this relation expressed in terms of ρ and z . To do this we form complex variables from equations 7.16 and 7.17 as

$$\frac{\rho + iz}{a} = \frac{\sin(\psi) + i\sinh(\eta)}{\cos(\psi) + \cosh(\eta)} = \tan\left(\frac{\psi + i\eta}{2}\right) \tag{7.52}$$

so that

$$\frac{\psi + i\eta}{2} = \arctan\left(\frac{\rho + iz}{a}\right)$$

$$= \frac{k\pi}{2} + \frac{1}{2}\arctan\left[\frac{2a\rho}{a^2 - \rho^2 - z^2}\right] + \frac{i}{4}\ln\left[\frac{\rho^2 + (z+a)^2}{\rho^2 + (z-a)^2}\right] \tag{7.53}$$

where k is an integer or zero (9). Separately equating real and imaginary parts gives ψ and η as functions of ρ and z . Then we have

$$\cosh(\eta) = \frac{1}{2}\left\{\left[\frac{\rho^2 + (z+a)^2}{\rho^2 + (z-a)^2}\right]^{1/2} + \left[\frac{\rho^2 + (z+a)^2}{\rho^2 + (z-a)^2}\right]^{-1/2}\right\} \tag{7.54}$$

and

$$\cos(\psi) = \pm[1 + \tan^2(\psi)]^{-1/2} = \frac{a^2 - \rho^2 - z^2}{[(a^2 - \rho^2 - z^2)^2 + 4a^2\rho^2]^{1/2}} \tag{7.55}$$

These can be substituted in equation 7.51 to find ϵ and μ as functions of ρ and z for a given value of ψ_o . Note from equation 7.51 that since $0 \leq \psi \leq \psi_o < \pi$ the maximum ϵ and μ for any fixed η occur at $\psi = 0$. Since $\cos(\psi_o) < 1$ then varying η for $\psi = 0$ we see that the maximum ϵ and μ occur at the minimum of $\cosh(\eta)$. Assuming $\eta = 0$ is in the region of interest, the minimum occurs there and we have

$$\left.\frac{\epsilon}{\epsilon_o}\right|_{max} = \left.\frac{\mu}{\mu_o}\right|_{max} = \frac{2}{1 + \cos(\psi_o)} = \frac{1}{\cos^2(\frac{\psi_o}{2})} \tag{7.56}$$

The minimum ϵ and μ are, by previous choice, ϵ_o and μ_o which occur on $\psi = \psi_o$, the maximum ψ for the region of interest.

Referring to Figures 3 through 5 one can better appreciate the approximation involved in placing a boundary on the lens at $\psi = \psi_o$. In these figures we have used a strip line to illustrate a typical cylindrical transmission line. For such a transmission line the fields for the TEM mode extend over the entire cross-section surface, a plane of constant z, or more generally a surface of constant u_3 . However, these fields fall off in amplitude with distance from the conductors, for large distances. Thus we require that ψ_o be chosen large enough that the fields in the TEM mode for $\psi \geq \psi_o$ are insignificant compared to the fields near the conductors. For certain types of cylindrical transmission lines, such as coaxial lines, the fields are zero outside some closed outer perfectly conducting boundary. For such cases the lens material is not needed outside the outer conducting boundary and stopping the lens at some external $\psi \doteq \psi_o$ creates no disturbance in the fields.

This lens, based on a bispherical coordinate system, can be classified as a converging lens. Referring to Figure 5, a spherical TEM wave launched near the apex of the conical transmission line is converted into a plane TEM wave on the cylindrical transmission line.

C. Modified Toroidal Coordinates

For an example of an inhomogeneous diverging lens define the rotational system v_1,ϕ,v_3 as toroidal coordinates ν,ϕ,ζ as illustrated in Figure 6 and defined by (6)

$$x \equiv \frac{a \sinh(\nu) \cos(\phi)}{\cosh(\nu) + \cos(\zeta)} \tag{7.57}$$

$$y \equiv \frac{a \sinh(\nu) \sin(\phi)}{\cosh(\nu) + \cos(\zeta)} \tag{7.58}$$

$$z \equiv \frac{a \sin(\zeta)}{\cosh(\nu) + \cos(\zeta)} \tag{7.59}$$

with $-\pi < \zeta \leq \pi$ and $0 \leq \nu < \infty$. Surfaces of constant $v_3 = \zeta$ are spheres; surfaces of constant $v_1 = \nu$ are toroids. The scale factors are

$$h_\phi = \frac{a \sinh(\nu)}{\cosh(\nu) + \cos(\zeta)} = \rho \tag{7.60}$$

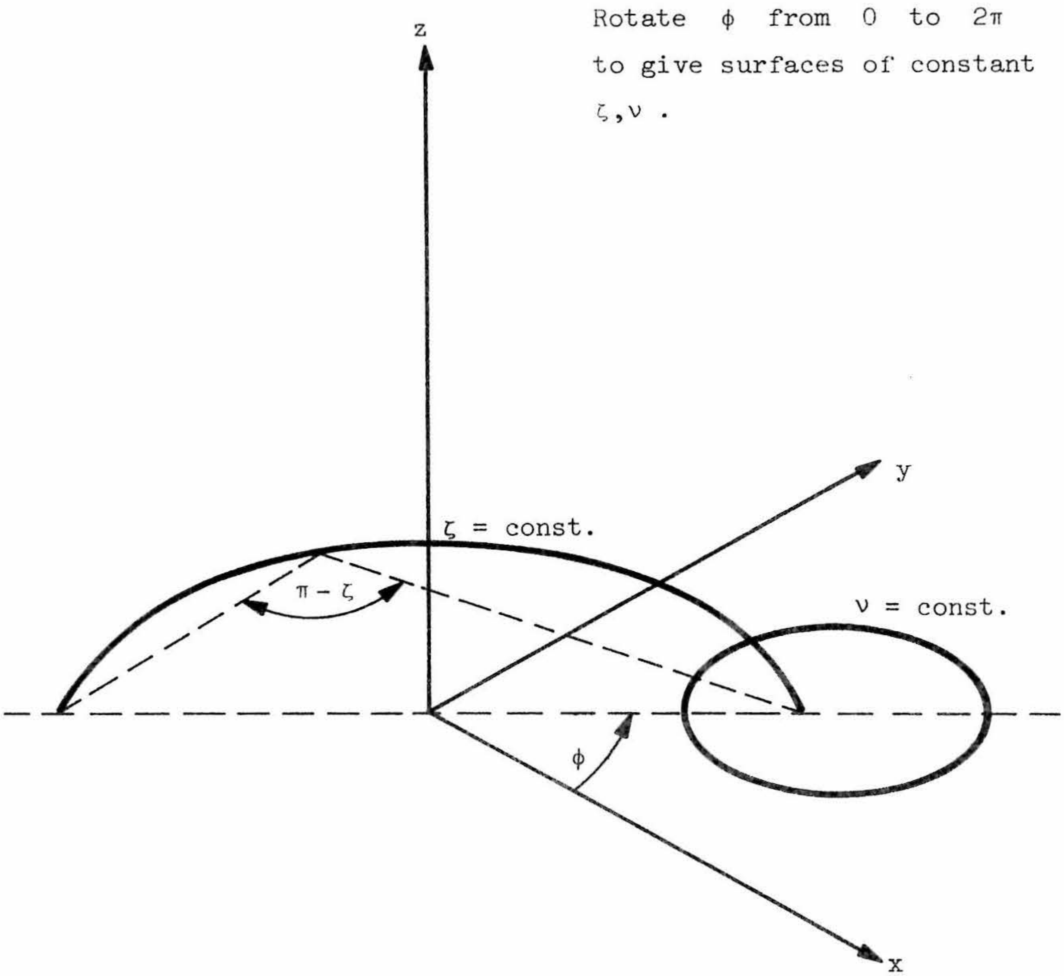


Fig. 6. Toroidal Coordinates

$$h_\nu = h_\zeta = \frac{a}{\cosh(\nu) + \cos(\zeta)} = \frac{\rho}{\sinh(\nu)} \qquad (7.61)$$

To construct the u_i system first calculate λ from equation 6.34 as

$$\int_{a_o}^{\lambda} \frac{d\lambda'}{\lambda'} = \int_{\infty}^{\nu} \frac{d\nu'}{\sinh(\nu')} \qquad (7.62)$$

which gives

$$\ell n(\frac{\lambda}{a_o}) = \ell n[\tanh(\frac{\nu}{2})] \qquad (7.63)$$

or

$$\lambda = a_o \tanh(\frac{\nu}{2}) \qquad (7.64)$$

so that

$$u_1 = a_o \tanh(\frac{\nu}{2}) \cos(\phi) \quad , \quad u_2 = a_o \tanh(\frac{\nu}{2}) \sin(\phi) \qquad (7.65)$$

where $a_o > 0$ is a constant which is chosen later. We obtain h from equation 6.41 as

$$h = \frac{h_\phi}{\lambda} = \frac{a}{a_o} \frac{\sinh(\nu)}{\tanh(\frac{\nu}{2})} \frac{1}{\cosh(\nu) + \cos(\zeta)} = \frac{a}{a_o} \frac{\cosh(\nu) + 1}{\cosh(\nu) + \cos(\zeta)} \qquad (7.66)$$

From equation 6.37 we have

$$h_3 = h_\zeta \left|\frac{d\zeta}{du_3}\right| = \frac{a}{\cosh(\nu) + \cos(\zeta)} \left|\frac{d\zeta}{du_3}\right| \qquad (7.67)$$

As before we set $\varepsilon' = \varepsilon_o$, $\mu' = \mu_o$ and restrict $\varepsilon \geqq \varepsilon_o$, $\mu \geqq \mu_o$ which together require $h_3 \leqq 1$. Observe that for $\nu \geqq 0$ and for fixed ζ with $-\pi < \zeta < \pi$, h_3 is a monotonically decreasing function of ν . Thus for fixed ζ , h_3 is a maximum for $\nu = 0$, the z axis. Then to minimize the required ε and μ set $h_3 = 1$ on $\nu = 0$. This gives

$$h_3 = \frac{1 + \cos(\zeta)}{\cosh(\nu) + \cos(\zeta)} \tag{7.68}$$

and we choose

$$\frac{d\zeta}{du_3} = \frac{1}{a} [1 + \cos(\zeta)] \tag{7.69}$$

Again $\frac{d\zeta}{du_3}$ could have many other forms. Then u_3 is calculated as

$$u_3 = a \int_0^{\zeta} \frac{d\zeta'}{1 + \cos(\zeta')} = a \tan(\frac{\zeta}{2}) \tag{7.70}$$

We now have all the u_i coordinates and call them modified toroidal coordinates.

Having the u_i and h_i for this toroidal lens we now join cylindrical and conical transmission lines to the lens. One boundary surface for the lens is taken as the plane $z = 0$ on which $\zeta = 0$, $u_3 = 0$. Combining equations 7.57, 7.58, and 7.65 and defining $a_o \equiv a$ gives for $z = 0$,

$$u_1 = x \quad , \quad u_2 = y \tag{7.71}$$

Let the lens material be present only for $u_3 > 0$ (corresponding to $\zeta > 0$, $z > 0$). Let the medium for $z < 0$ be free space with constitutive parameters ε_o, μ_o and let u_1, u_2, u_3 for $z \leq 0$ be simply x,y,z. The transmission line conductors are constrained by equation

7.1 for all u_3 considered. Thus, for $z \leqq 0$ we have a cylindrical transmission line, while in the lens the conductors are curved to satisfy equations 7.1 and 7.65 with $a_o = a$. Note that there is a singularity in the u_i at $\rho = a$, $z = 0$ corresponding to $\nu = +\infty$. Thus for the toroidal coordinates we confine our interest to ν satisfying $0 \leqq \nu \leqq \nu_o < +\infty$. We let $\nu = \nu_o$ be a boundary for the lens material.

The u_i, h, and the transmission line conductors are continuous through the plane $z = 0$. We have a TEM wave, as before, of the form

$$\vec{E}' = f(t - \frac{u_3}{c'}) \nabla' \Phi_e(u_1,u_2)$$

$$\vec{H}' = f(t - \frac{u_3}{c'}) \nabla' \Phi_h(u_1,u_2) \tag{7.72}$$

Since h is continuous through $z = 0$, tangential $\vec{E}$ and $\vec{H}$ are continuous through $z = 0$ as required. Note, however, that $h_3 = h = 1$ for $z < 0$ and that h_3 has a step discontinuity at $z = 0$.

Introduce another lens surface at $\zeta = \zeta_o$ with $0 < \zeta_o < \pi$. This surface is a sphere described by

$$x^2 + y^2 + (z + a \cot(\zeta_o))^2 = \frac{a^2}{\sin^2(\zeta_o)} \tag{7.73}$$

This sphere is centered on the z axis at $z = -a \cot(\zeta_o)$ and has a radius $a|\sin(\zeta_o)|^{-1}$. Figure 7 illustrates a lens cross section in the zx plane and Figure 8 gives a perspective view with the transmission line conductors. The region outside the sphere described above is assumed to be free space and contains a conical transmission line with

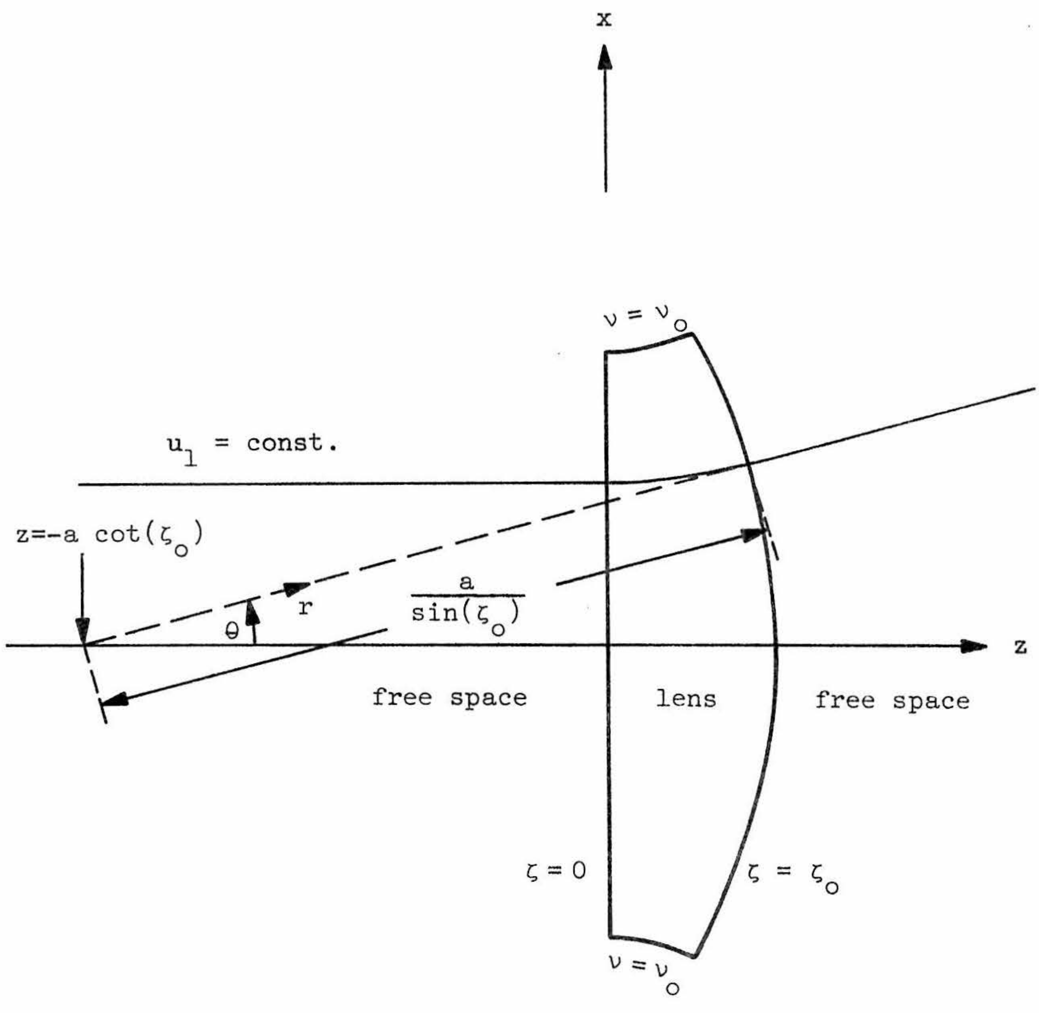


Fig. 7. Toroidal Lens

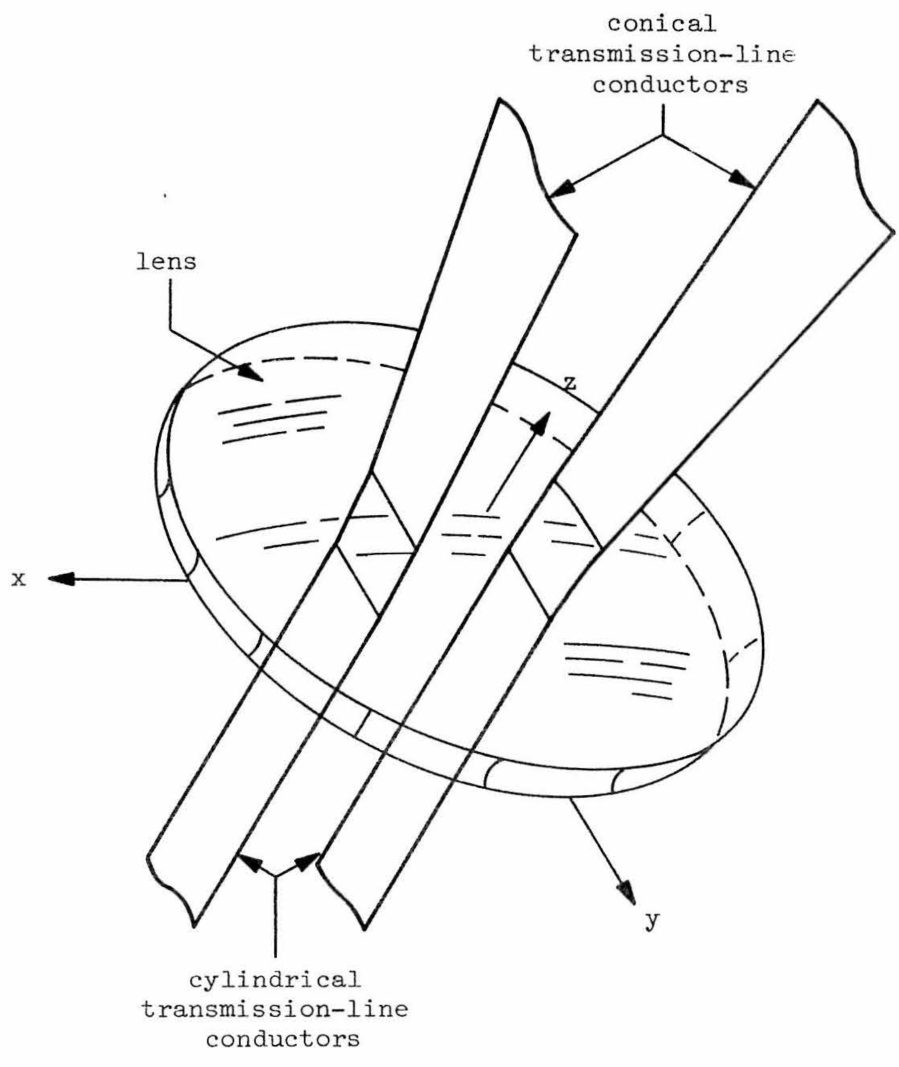


Fig. 8. Toroidal Lens with Cylindrical and Conical Transmission Lines

conductors matched to those in the lens.

Now we match the u_i coordinates at $\zeta = \zeta_o$ using the modified spherical coordinates of Section VIIA to describe the continuation of the u_i coordinates past $\zeta = \zeta_o$. Center the apex of the conical line at the center of the sphere corresponding to $\zeta = \zeta_o$ by choosing z_1 in equation 7.4 as

$$z_1 \equiv -a \cot(\zeta_o) \tag{7.74}$$

For the conical line we have, from equations 7.11

$$u_1 = 2z_o \tan(\frac{\theta}{2}) \cos(\phi) \quad , \qquad u_2 = 2z_o \tan(\frac{\theta}{2}) \sin(\phi) \tag{7.75}$$

and for the lens

$$u_1 = a \tanh(\frac{\nu}{2}) \cos(\phi) \quad , \qquad u_2 = a \tanh(\frac{\nu}{2}) \sin(\phi) \tag{7.76}$$

Thus on $\zeta = \zeta_o$ we need

$$a \tanh(\frac{\nu}{2}) = 2z_o \tan(\frac{\theta}{2}) \tag{7.77}$$

Then considering ρ on $\zeta = \zeta_o$ we have for the conical line

$$\rho = \frac{a}{\sin(\zeta_o)} \sin(\theta) \tag{7.78}$$

and for the lens

$$\rho = \frac{a \sinh(\nu)}{\cosh(\nu) + \cos(\zeta_o)} \tag{7.79}$$

The θ and ν coordinates are then related on this surface by

$$\sin(\theta) = \frac{\sinh(\nu) \sin(\zeta_o)}{\cosh(\nu) + \cos(\zeta_o)} \tag{7.80}$$

This has the same form as equation 7.42 if we replace η_o by ν and ψ by $-\zeta_o$. Then from equation 7.46 we have the result

$$\tan(\frac{\theta}{2}) = \tanh(\frac{\nu}{2}) \tan(\frac{\zeta_o}{2}) \tag{7.81}$$

Therefore we define for this case

$$\frac{a}{2z_o} \equiv \tan(\frac{\zeta_o}{2}) \tag{7.82}$$

making equations 7.81 and 7.77 equivalent. Then u_1 and u_2 are continuous across the spherical surface $\zeta = \zeta_o$.

From equation 7.7 we have, on $\zeta = \zeta_o$,

$$u_3 = \frac{a}{\sin(\zeta_o)} + r_o \tag{7.83}$$

while from equation 7.70 we have

$$u_3 = a \tan(\frac{\zeta_o}{2}) \tag{7.84}$$

These results give, as a definition of r_o for this case,

$$\frac{r_o}{a} \equiv \tan(\frac{\zeta_o}{2}) - \frac{1}{\sin(\zeta_o)} = - \cot(\zeta_o) \tag{7.85}$$

Now u_1 and u_2 are continuous across $\zeta = \zeta_o$ so that h is also continuous there. However, h_3 has a step discontinuity there. Then the TEM wave described by equations 7.72 passes through this surface without reflection. In summary, for $z \leq 0$ we have u_1, u_2, u_3 equal to x,y,z and the constitutive parameters are ϵ_o and μ_o. In the lens, bounded by $\zeta = 0$, $\zeta = \zeta_o$, and $\nu = \nu_o$, the u_i are given by equations 7.70 and 7.76; the constitutive parameters are given by equations 7.24 and 7.68 with $\epsilon' = \epsilon_o$, $\mu' = \mu_o$. For $u_3 \doteq a \tan(\frac{\zeta_o}{2})$ the u_i are given by equations 7.7 and 7.11 with constitutive parameters ϵ_o, μ_o . The transmission line conductors are described by equation 7.1 for all values of u_3 of interest.

The constitutive parameters for the lens are given by

$$\frac{\epsilon}{\epsilon_o} = \frac{\mu}{\mu_o} = \frac{1}{h_3} = \frac{\cosh(\nu) + \cos(\zeta)}{1 + \cos(\zeta)} \tag{7.86}$$

To express this result in terms of ρ and z , as in Section VIIB, we form a complex variable

$$\frac{z + i\rho}{a} = \frac{\sin(\zeta) + i \sinh(\nu)}{\cos(\zeta) + \cosh(\nu)} = \tan(\frac{\zeta + i\nu}{2}) \tag{7.87}$$

so that, just as in equations 7.52 and 7.53

$$\frac{\zeta + i\nu}{2} = \arctan(\frac{z + i\rho}{a})$$

$$= \frac{k\pi}{2} + \frac{1}{2} \arctan[\frac{2\,az}{a^2 - z^2 - \rho^2}] + \frac{i}{4} \ell n\left[\frac{z^2+(\rho+z)^2}{z^2 + (\rho-z)^2}\right] \tag{7.88}$$

with k an integer or zero. From this we obtain

$$\cosh(\nu) = \frac{1}{2}\left\{ \left[\frac{z^2 + (\rho+a)^2}{z^2 + (\rho-a)^2}\right]^{1/2} + \left[\frac{z^2+(\rho+a)^2}{z^2+(\rho-a)^2}\right]^{-1/2} \right\} \tag{7.89}$$

and

$$\cos(\zeta) = \pm [1 + \tan^2(\zeta)]^{-1/2} = \frac{a^2 - z^2 - \rho^2}{[(a^2 - z^2 - \rho^2)^2 + 4a^2 z^2]^{1/2}} \tag{7.90}$$

These can be substituted in equation 7.86 to find ε and μ in terms of ρ and z. From equation 7.86 the maximum ε and μ for fixed ζ occur at $\nu = \nu_o$, the maximum ν considered for the lens. Then varying ζ for $0 \leq \zeta \leq \zeta_o < \pi$ for $\nu = \nu_o$, we find that the maximum ε and μ occur at $\zeta = \zeta_o$, $\nu = \nu_o$, for which we have

$$\left.\frac{\varepsilon}{\varepsilon_o}\right|_{max} = \left.\frac{\mu}{\mu_o}\right|_{max} = \frac{\cosh(\nu_o) + \cos(\zeta_o)}{1 + \cos(\zeta_o)} \tag{7.91}$$

The minimum ε and μ were made ε_o and μ_o on the z axis, or equivalently $\nu = 0$.

Referring to Figures 7 and 8, note that for this toroidal lens, as in the previous case, we require that the fields in the TEM mode for $\nu \geq \nu_o$ be negligible compared to the fields near the conductors so that the TEM mode is not significantly disturbed. The present lens, based on toroidal coordinates, can be classified as a diverging lens. Referring to Figure 8, a plane wave on the cylindrical transmission line is converted on passing through the lens, into an expanding spherical wave on the conical transmission line.

D. Modified Cylindrical Coordinates

As a last example we consider a lens which can be used to transition TEM waves between two cylindrical transmission lines with different propagation directions. Specifically, choose the u_i based on cylindrical coordinates as

$$u_1 \equiv z \tag{7.92}$$

$$u_2 \equiv \rho \equiv (x^2 + y^2)^{1/2} \tag{7.93}$$

$$u_3 \equiv \rho_o \phi \quad , \quad \tan \phi \equiv y/x \tag{7.94}$$

with $0 \leq \rho < \infty$, $0 \leq \phi < 2\pi$ where $\rho_o > 0$ is a constant and $\phi = 0$ corresponds to $y = 0, x > 0$. Note that the intermediate rotational v_i coordinate system is not used for this example. Surfaces of constant u_3 are planes. The scale factors are

$$h \equiv h_1 = h_2 = 1 \tag{7.95}$$

$$h_3 = h_\phi \frac{d\phi}{du_3} = \frac{\rho}{\rho_o} \tag{7.96}$$

Since h_1 and h_2 are equal we have an acceptable coordinate system satisfying equation 6.15. The resulting lens and transmission lines are illustrated in Figure 9. We call the u_i modified cylindrical coordinates.

The constitutive parameters are given from equations 6.16 with $\epsilon' = \epsilon_o$, $\mu' = \mu_o$ as

$$\frac{\epsilon}{\epsilon_o} = \frac{\mu}{\mu_o} = \frac{1}{h_3} = \frac{\rho_o}{\rho} \tag{7.97}$$

Consider $\rho = \rho_o$ one surface of the lens and constrain $\rho \leq \rho_o$ for the lens to make $\epsilon \geq \epsilon_o$, $\mu \geq \mu_o$. Fix other lens surfaces as $\rho = \rho_1$ with $0 < \rho_1 < \rho_o$, $\phi = 0$, $\phi = \phi_o$ with $0 < \phi_o < 2\pi$, $z = \pm z_2$ with $z_2 > 0$. The maximum ϵ and μ then occur for $\rho = \rho_1$ giving

$$\left.\frac{\epsilon}{\epsilon_o}\right|_{max} = \left.\frac{\mu}{\mu_o}\right|_{max} = \frac{\rho_o}{\rho_1} \tag{7.98}$$

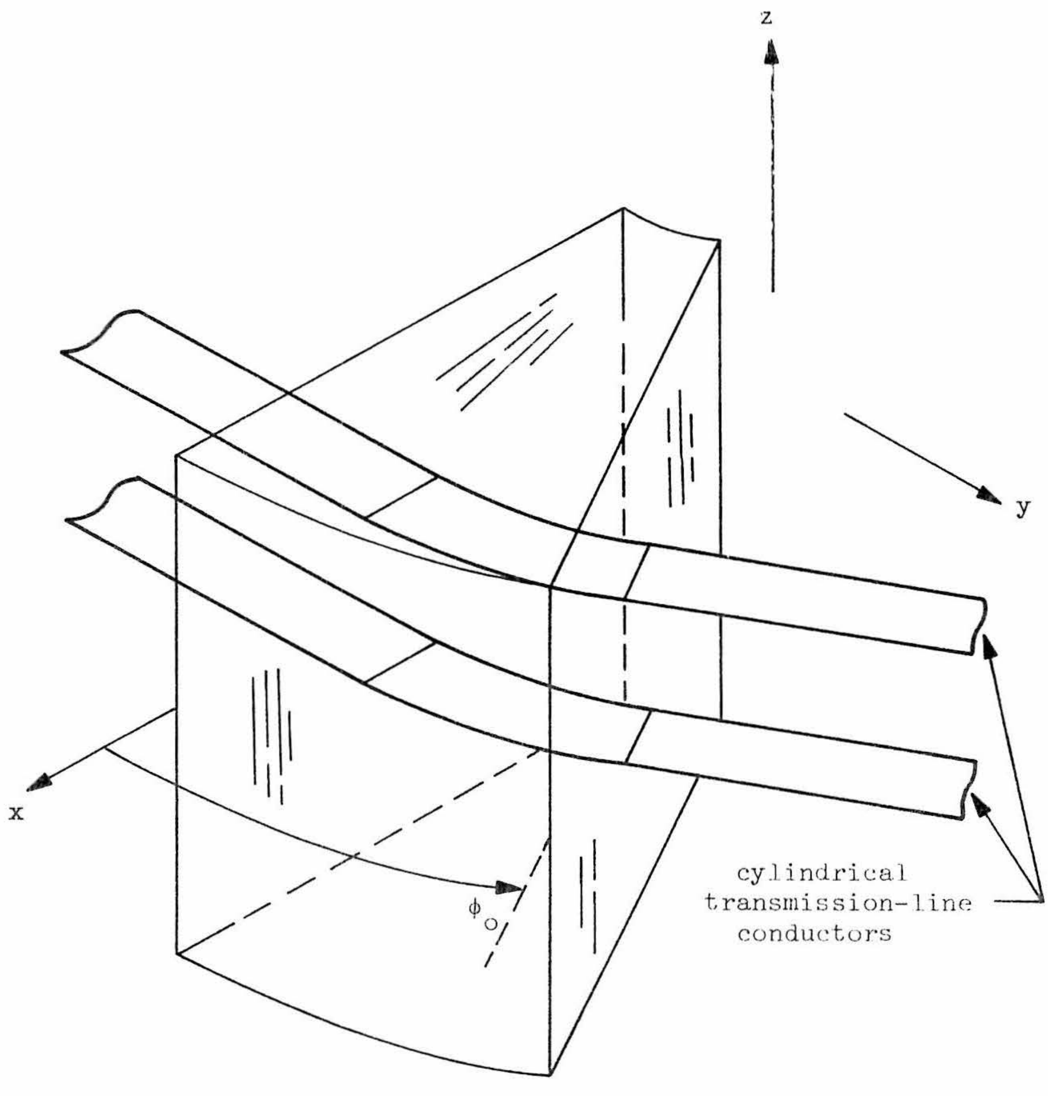


Fig. 9. Cylindrical Lens with Cylindrical Transmission Lines

Including the cylindrical transmission lines in the u_i coordinate system, we define, for $u_3 \leq 0$, u_1, u_2, u_3 as z, x, y. For $0 \leq u_3 \leq \rho_o \phi_o$ the u_i are defined by equations 7.92 through 7.94. For $u_3 \geq \rho_o \phi_o$ the u_i are defined (from a rotation in the xy plane) by

$$u_1 \equiv z \ , \ u_2 \equiv x \cos(\phi_o) + y \sin(\phi_o),$$

$$u_3 \equiv -x \sin(\phi_o) + y \cos(\phi_o) + \rho_o \phi_o \qquad (7.99)$$

With these definitions the u_i are all continuous across the surfaces $\phi = 0$ and $\phi = \phi_o$. For u_3 outside the lens we have $h = h_3 = 1$ so that h is continuous across these latter two lens surfaces while h_3 has step discontinuities there. The transmission line conductors are described by equation 7.1 for all u_3 of interest. The TEM wave in all three regions of u_3 is described by equations 6.9. Note that if $\phi_o > \pi$ then one or both cylindrical transmission lines may need to be cut short to prevent their intersecting each other.

Referring to Figure 9 we require for this cylindrical lens that the fields in the TEM mode for $\rho \leq \rho_1$, for $\rho \geq \rho_o$, and for $|z| \geq z_2$ (separately) be negligible compared to the fields near the transmission line conductors. This lens is neither a converging nor a diverging lens but might be better termed a prism or a redirecting lens.

VIII. TWO-DIMENSIONAL TEM WAVES

Now consider a restricted form of the u_i coordinates by defining

$$u_3 \equiv z \tag{8.1}$$

which implies

$$h_3 = 1 \tag{8.2}$$

while u_1 and u_2 are taken independent of z. Also let either the formal electric field or formal magnetic field have only a u_3 component and let the remaining formal field have only a u_2 component. Let the formal field components be only functions of u_1 and let the wave propagate in the $+u_1$ direction. In terms of the u_i and the formal field components this represents a uniform TEM wave.

Again we assume, for the constitutive parameters, that

$$(\varepsilon_{ij}) = \varepsilon(\delta_{ij}) \quad , \quad (\mu_{ij}) = \mu(\delta_{ij}) \tag{8.3}$$

with the conductivity zero. Thus the medium is isotropic but, in general, inhomogeneous. The formal constitutive parameters are assumed to have the forms

$$(\varepsilon'_{ij}) = \begin{pmatrix} \varepsilon'_1 & 0 & 0 \\ 0 & \varepsilon'_2 & 0 \\ 0 & 0 & \varepsilon'_3 \end{pmatrix} \quad , \quad (\mu'_{ij}) = \begin{pmatrix} \mu'_1 & 0 & 0 \\ 0 & \mu'_2 & 0 \\ 0 & 0 & \mu'_3 \end{pmatrix} \tag{8.4}$$

We also have

$$(\varepsilon'_{ij}) = \varepsilon(\gamma_{ij}) \quad , \quad (\mu'_{ij}) = \mu(\gamma_{ij}) \tag{8.5}$$

where, because of equation 8.2,

$$(\gamma_{ij}) = \begin{pmatrix} h_2/h_1 & 0 & 0 \\ 0 & h_1/h_2 & 0 \\ 0 & 0 & h_1 h_2 \end{pmatrix} \tag{8.6}$$

Note in equations 8.4 that the diagonal components of (ε'_{ij}) and (μ'_{ij}) may be all unequal. However, since the formal electric and magnetic fields are each assumed to have only one component, then only one of the ε'_i and one of the μ'_i will be significant. These significant ε'_i and μ'_i will be assumed independent of the coordinates so that in terms of the u_i coordinates and formal fields the medium is effectively homogeneous.

We have two cases to consider. Call the case with the electric field parallel to the z axis Case 1; call the case with the magnetic field parallel to the z axis Case 2.

For Case 1 we assume a wave of the form

$$\vec{E}' = \vec{e}_3 \, E'_{3_o} \, f(t - \frac{u_1}{c'}) \quad , \qquad \vec{H}' = \vec{e}_2 \, H'_{2_o} \, f(t - \frac{u_1}{c'}) \tag{8.7}$$

with

$$E'_{3_o} = -\sqrt{\frac{\mu'_2}{\varepsilon'_3}} \, H'_{2_o} \quad , \qquad c' = \frac{1}{\sqrt{\mu'_2 \, \varepsilon'_3}} \tag{8.8}$$

where E'_{3_o} and H'_{2_o} are independent of the coordinates. Then for Case 1 we assume that $\mu'_2 > 0$ and $\varepsilon'_3 > 0$ are independent of the u_i. Then from equations 8.4 through 8.6 we have

$$\epsilon = \frac{\epsilon'_3}{h_1 h_2} \quad , \quad \mu = \mu'_2 \frac{h_2}{h_1} \tag{8.9}$$

Note for Case 1 that since $\vec{E}$ is parallel to the z axis, perfectly conducting planar sheets can be placed perpendicular to the z axis and used as boundaries for this TEM wave.

For Case 2 we assume a wave of the form

$$\vec{E}' = \vec{e}_2 E'_{2_o} f(t - \frac{u_1}{c'}) \quad , \quad \vec{H}' = \vec{e}_3 H'_{3_o} f(t - \frac{u_1}{c'}) \tag{8.10}$$

with

$$E'_{2_o} = \sqrt{\frac{\mu'_3}{\epsilon'_2}} H'_{3_o} \quad , \quad c' = \frac{1}{\sqrt{\mu'_3 \epsilon'_2}} \tag{8.11}$$

where E'_{2_o} and H'_{3_o} are independent of the u_i . For this case we then assume that $\mu'_3 > 0$ and $\epsilon'_2 > 0$ are independent of the u_i . We also have

$$\epsilon = \epsilon'_2 \frac{h_2}{h_1} \quad , \quad \mu = \frac{\mu'_3}{h_1 h_2} \tag{8.12}$$

For Case 2 since $\vec{E}$ is perpendicular to surfaces of constant u_2 , perfectly conducting sheets can be placed along these generally curved surfaces and used as boundaries for this TEM wave.

There are many possible ways to choose $u_1(x,y)$ and $u_2(x,y)$ and form an orthogonal curvilinear coordinate system. Then calculating h_1 and h_2 one can find ϵ and μ from equations 8.9 or 8.12. For the examples in the next section we consider coordinate systems with

$$h \equiv h_1 = h_2 \tag{8.13}$$

Define the complex variables

$$p \equiv x + iy \qquad , \qquad q \equiv u_1 + iu_2 \tag{8.14}$$

Then we have for the line element

$$|dp|^2 = (dx)^2 + (dy)^2 = h_1^2(du_1)^2 + h_2^2(du_2)^2$$

$$= h^2[(du_1)^2 + (du_2)^2] = h^2|dq|^2 \tag{8.15}$$

Thus if we are given a conformal transformation of the form $q(p)$ or its inverse, we can calculate an h as

$$h = \left|\frac{dp}{dq}\right| = \left|\frac{dq}{dp}\right|^{-1} \tag{8.16}$$

Then from $q(p)$ we can also obtain u_1 and u_2 .

With the restriction of equation 8.13, look again at Case 1. Equations 8.9 become

$$\varepsilon = \frac{\varepsilon_3'}{h^2} \qquad , \qquad \mu = \mu_2' \tag{8.17}$$

so that μ is homogeneous for Case 1. Similarly for Case 2, equations 8.12 become

$$\varepsilon = \varepsilon_2' \qquad , \qquad \mu = \frac{\mu_3'}{h^2} \tag{8.18}$$

so that ε is homogeneous for Case 2.

For convenience we choose $\varepsilon_3' = \varepsilon_o$, $\mu_2' = \mu_o$ for Case 1, and $\varepsilon_2' = \varepsilon_o$, $\mu_3' = \mu_o$ for Case 2. Then for each case one of the constitutive parameters is the same as for free space. Requiring $\varepsilon \doteq \varepsilon_o$,

$\mu \doteq \mu_o$, then for both cases we require $h \leq 1$. In the next section we choose examples of two-dimensional lenses which might be appropriate for launching TEM waves between wide perfectly conducting parallel sheets. After defining the conformal transformation, giving u_1 and u_2, regions with $h > 1$ are excluded from consideration.

IX. TWO-DIMENSIONAL TEM LENSES

As a first example of a coordinate system for a two-dimensional lens consider the conformal transformation defined by

$$\frac{q}{a} \equiv \frac{1}{\pi} \ln[e^{\pi\frac{p}{a}} - 1] \quad , \qquad \frac{p}{a} \equiv \frac{1}{\pi} \ln[e^{\pi\frac{q}{a}} + 1] \tag{9.1}$$

This is illustrated in Figure 10. This transformation also describes the potential distribution around a uniformly charged wire grid (in a homogeneous medium) terminating a uniform electric field for $x >> 0$. From equations 9.1 we have, for u_1 and u_2

$$u_1 = \frac{a}{2\pi} \ln[e^{\frac{2\pi x}{a}} - 2e^{\frac{\pi x}{a}} \cos(\frac{\pi y}{a}) + 1] \tag{9.2}$$

$$u_2 = \frac{a}{\pi} \arctan\left[\frac{e^{\frac{\pi x}{a}} \sin(\frac{\pi y}{a})}{e^{\pi x/a} \cos(\frac{\pi y}{a}) - 1}\right] + ak \tag{9.3}$$

where $k = 0, \pm 1$. Note that a is just a parameter which can be used to scale the dimensions.

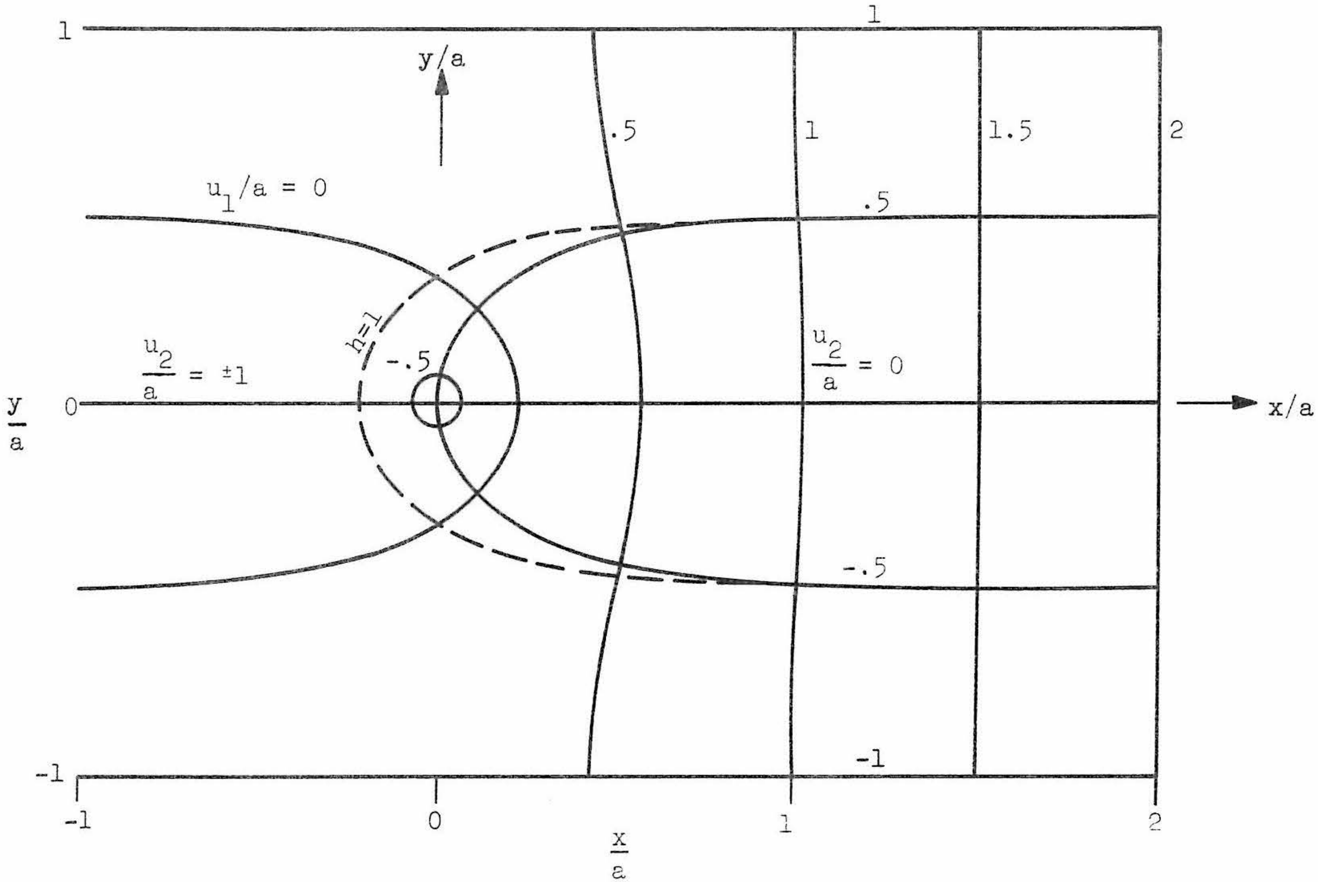
The scale factor h given by equation 8.16 is

$$h = \left|1 + e^{-\pi q/a}\right|^{-1} = \left|1 - e^{-\pi p/a}\right| \tag{9.4}$$

In terms of x and y this is

$$h^2 = 1 - 2e^{-\pi x/a} \cos(\frac{\pi y}{a}) + e^{-2\pi x/a} \tag{9.5}$$

Since only regions with $h \leq 1$ are of interest we find the contour for $h = 1$ given by



$$\frac{u_1 + iu_2}{a} = \frac{1}{\pi}\,\ell n\left[e^{\frac{\pi}{a}(x+iy)} - 1\right]$$

Fig. 10. Coordinates for First Example

$$2\cos\left(\frac{\pi y}{a}\right) = e^{-\pi x/a} \tag{9.6}$$

which is indicated in Figure 10. In terms of u_1 and u_2 the scale factor has the form

$$\frac{1}{h^2} = 1 + 2\,e^{-\pi u_1/a}\cos\left(\frac{\pi u_2}{a}\right) + e^{-2\pi u_1/a} \tag{9.7}$$

Concentrating our attention on the region near the positive x axis, we see from this last equation that if we restrict $|u_2/a| \leq 1/2$, then this will assure having $h \leq 1$. For Case 2 (magnetic field parallel to the z axis) one might place perfectly conducting boundaries on surfaces of constant u_2 within this restriction. From equations 8.17 or 8.18 the maximum ε or μ , as appropriate, is related to the maximum of h^{-2} in the region of interest. Consider some $u_1 = u_{1_o}$ as the minimum u_1 of interest. Note that the maximum of h^{-2} for fixed u_1 occurs on $u_2 = 0$ for which $y = 0$. Then varying u_1 we find that the maximum of h^{-2} occurs at u_{1_o} so that

$$\left.\frac{1}{h^2}\right|_{max} = \left[1 + e^{-\pi u_{1_o}/a}\right]^2 \tag{9.8}$$

Also note that as $x \to \infty$ we have

$$h \to 1\,, \quad u_1 \to x\,, \quad u_2 \to y \tag{9.9}$$

One of the constitutive parameters of the lens is the same as free space; the other tends to the free-space value as $x \to \infty$. Then for sufficiently large x the lens material can be stopped without significantly distorting the TEM wave.

As a second example, consider the conformal transformation defined by

$$\frac{q}{a} \equiv \frac{2}{\pi} \ln[\sinh(\frac{\pi}{2}\frac{p}{a})] \quad , \qquad \frac{p}{a} \equiv \frac{2}{\pi}\text{arcsinh}[e^{\pi q/2a}] \tag{9.10}$$

This is illustrated in Figure 11. This transformation also describes the potential distribution around a uniformly charged wire grid (in a homogeneous medium) terminating uniform, equal but opposite, electric fields for $x >> 0$ and $x << 0$. Obtaining u_1 and u_2 from equations 9.10, we have

$$u_1 = \frac{a}{\pi} \ln[\sinh^2(\frac{\pi}{2}\frac{x}{a}) \cos^2(\frac{\pi}{2}\frac{y}{a}) + \cosh^2(\frac{\pi}{2}\frac{x}{a}) \sin^2(\frac{\pi}{2}\frac{y}{a})] \tag{9.11}$$

$$u_2 = \frac{2a}{\pi} \arctan[\coth(\frac{\pi}{2}\frac{x}{a}) \tan(\frac{\pi}{2}\frac{y}{a})] + 2ak \tag{9.12}$$

where $k = 0, \pm 1$.

The scale factor h is given by

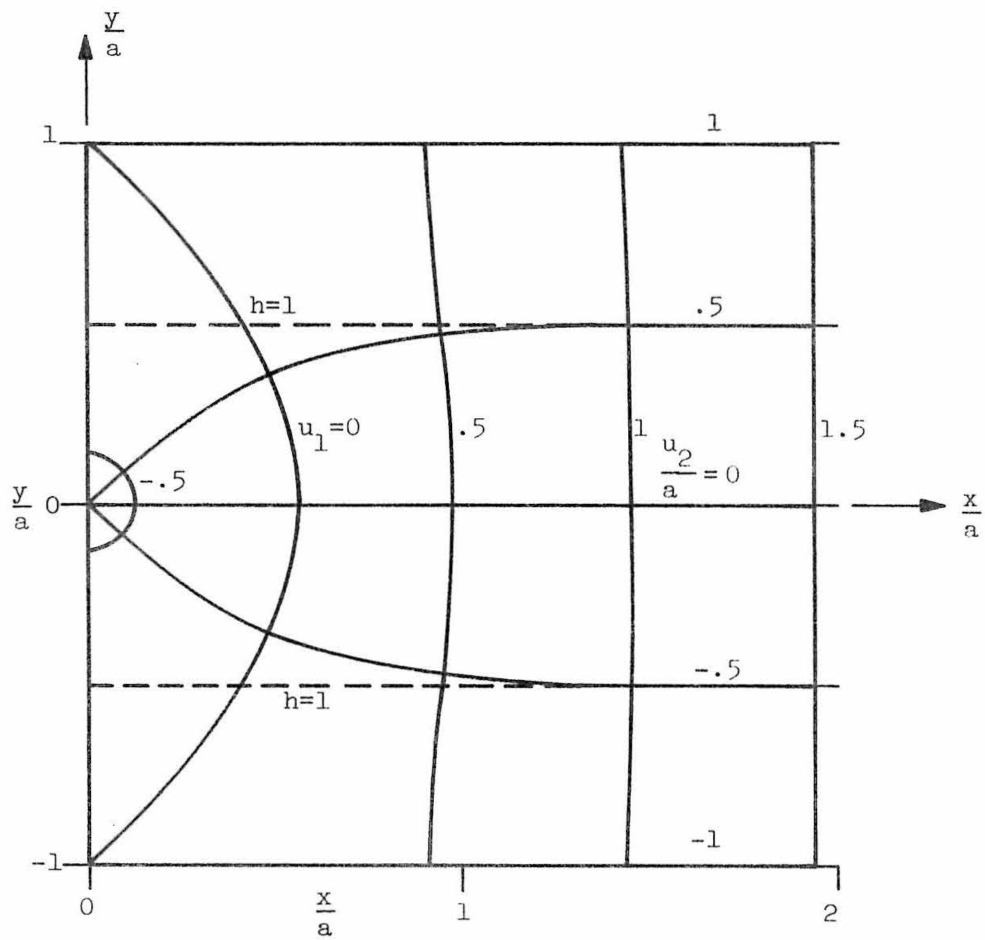
$$h = |1 + e^{-\pi q/a}|^{-1/2} = |\tanh(\frac{\pi}{2}\frac{p}{a})| \tag{9.13}$$

In terms of x and y this is

$$h^2 = \frac{1 - 2e^{-\pi x/a} \cos(\frac{\pi y}{a}) + e^{-2\pi x/a}}{1 + 2e^{-\pi x/a} \cos(\frac{\pi y}{a}) + e^{-2\pi x/a}} \tag{9.14}$$

The contour for $h = 1$ is

$$\frac{y}{a} = \pm \frac{1}{2} \tag{9.15}$$



$$\frac{u_1 + iu_2}{a} = \frac{2}{\pi} \ln \left[\sinh\left(\frac{\pi}{2} \frac{x+iy}{a}\right) \right]$$

Fig. 11. Coordinates for Second Example

which is indicated in Figure 11. In terms of u_1 and u_2 we have

$$\frac{1}{h^4} = 1 + 2e^{-\pi u_1/a} \cos(\frac{\pi u_2}{a}) + e^{-2\pi u_1/a} \qquad (9.16)$$

Considering the region near the positive x axis, note that by restricting $|u_2/a| \leq 1/2$ this will assure having $h \leq 1$. For this second example consider u_{1_o} as the minimum u_1 of interest and note that the minimum h occurs on $u_2 = 0$ (for which $y = 0$) and at $u_1 = u_{1_o}$, so that

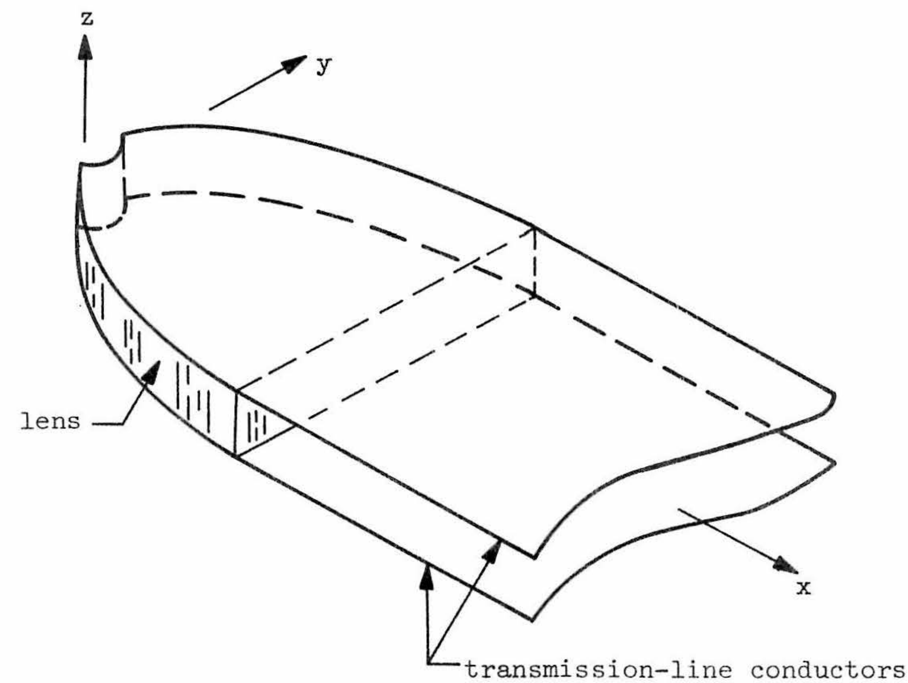
$$\left.\frac{1}{h^2}\right|_{max} = 1 + e^{-\pi u_{1_o}/a} \qquad (9.17)$$

The maximum ϵ or μ can then be found from equations 8.17 or 8.18. Also as $x \to \infty$ we have

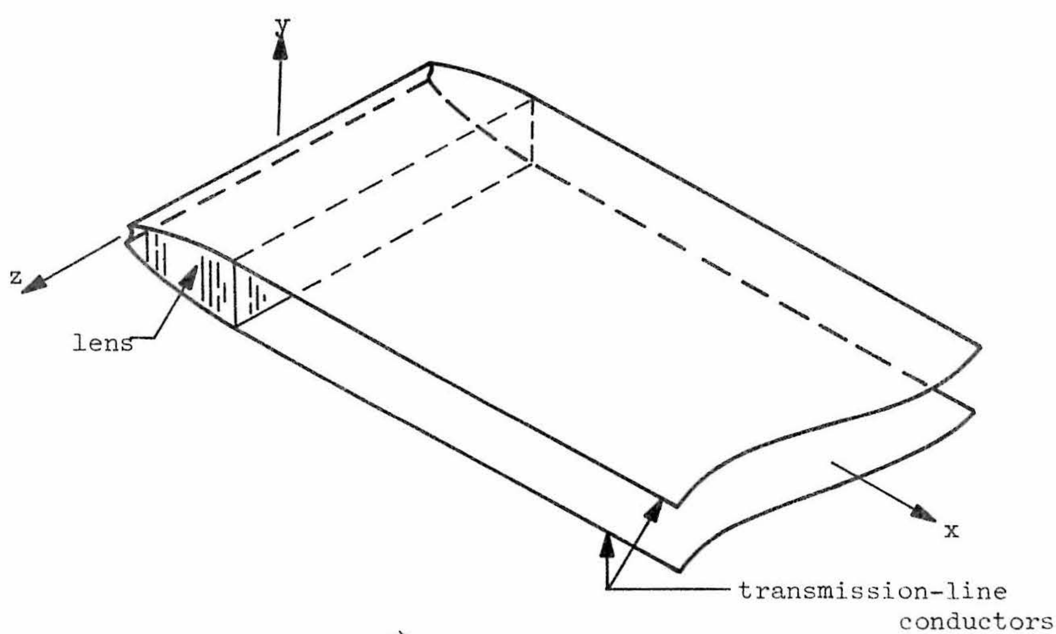
$$h \to 1 , \quad u_1 \to x - \frac{2a}{\pi} \ln(2) , \quad u_2 \to y \qquad (9.18)$$

Then, as in the first example, the lens material can be stopped at sufficiently large x without significantly distorting the TEM wave.

Figure 12 illustrates the present types of two-dimensional lenses together with appropriate parallel-plate transmission lines for both cases of field polarization discussed in Section VIII. Note that the conductors and the inhomogeneous medium are stopped before reaching the singularity on the z axis; sources to launch the TEM wave might be placed here. The perfectly conducting sheets and the inhomogeneous medium are also stopped on surfaces of constant u_2. This distorts the TEM wave, particularly near the edges of the sheets. However, the sheets are



A. Case 1: $\vec{E}$ parallel to z axis



B. Case 2: $\vec{H}$ parallel to z axis

Fig. 12. Two-Dimensional Lenses with Transmission Lines

assumed to be much wider than the sheet separation to minimize the influence of this distortion.

X. CONCLUSION

In summary there appear to be many ways of specifying inhomogeneous media such that simple electromagnetic waves, such as the TEM waves used here, can propagate in the medium. These types of inhomogeneous media can be used to define lenses for transitioning TEM waves, without reflection or distortion, between conical and/or cylindrical transmission lines. Of course, there are practical limitations in the realization of such lenses. For example, in some cases the lens should ideally be infinite in extent; limiting the extent of the lens can introduce perturbations into the desired pure TEM wave, and care will have to be taken to insure that these perturbations are small. Another limitation lies in the characteristics of practical materials used to realize the desired permittivity and permeability of the inhomogeneous medium. The available range of these parameters will be limited and their frequency dependence imperfect. Of course, perfect characteristics are not really necessary. Elsewhere we have proposed a lens based on geometrical optics for transitioning TEM waves between conical and cylindrical transmission lines (10). The lenses discussed in the present report, however, have the advantage that, within the limitations mentioned above, the TEM wave passes through the lens undistorted, based on a solution of Maxwell's equations.

In this report we only consider isotropic inhomogeneous media for the lenses. Within this area we consider a few examples each of three-dimensional and two-dimensional lenses. To extend the present work one might consider several other such examples in order to have available other types of geometries and inhomogeneities. For a wider extension one might allow the medium to be anisotropic as well as inhomogeneous. This would remove some of the restrictions on the coordinate systems which could be used. However, such an anisotropic inhomogeneous medium might be more difficult to realize.

APPENDIX A: CHARACTERISTICS OF COORDINATE SYSTEMS FOR FIELD COMPONENTS IN ALL THREE COORDINATE DIRECTIONS

In Section V, when considering the case of formal field components in all three coordinate directions, we require that (ϵ'_{ij}) and (μ'_{ij}) reduce to constant scalars times the identity matrix. Combining this with the earlier requirement that (ϵ_{ij}) and (μ_{ij}) reduce to scalars times the identity matrix leads to the result of equation 5.6, namely

$$h \equiv h_1 = h_2 = h_3 \tag{A.1}$$

This very restrictive form of the h_i leads to the natural question of what forms of $h(x,y,z)$ or $h(u_1,u_2,u_3)$ are possible.

Now the u_i form an orthogonal curvilinear coordinate system by hypothesis. It is then necessary and sufficient that the h_i satisfy the Lamé equations, namely

$$\frac{\partial^2 h_i}{\partial u_j\, \partial u_k} - \frac{1}{h_j}\frac{\partial h_j}{\partial u_k}\frac{\partial h_i}{\partial u_j} - \frac{1}{h_k}\frac{\partial h_k}{\partial u_j}\frac{\partial h_i}{\partial u_k} = 0 \tag{A.2}$$

and

$$\frac{\partial}{\partial u_i}\left(\frac{1}{h_i}\frac{\partial h_j}{\partial u_i}\right) + \frac{\partial}{\partial u_j}\left(\frac{1}{h_j}\frac{\partial h_i}{\partial u_j}\right) + \frac{1}{h_k^2}\frac{\partial h_i}{\partial u_k}\frac{\partial h_j}{\partial u_k} = 0 \tag{A.3}$$

where i,j,k is a permutation of 1,2,3 yielding six independent equations (11). Substituting from equation A.1 gives

$$\frac{\partial^2 h}{\partial u_j \partial u_k} - \frac{2}{h}\frac{\partial h}{\partial u_j}\frac{\partial h}{\partial u_k} = 0 \tag{A.4}$$

and

$$\frac{\partial}{\partial u_i}\left(\frac{1}{h}\frac{\partial h}{\partial u_i}\right) + \frac{\partial}{\partial u_j}\left(\frac{1}{h}\frac{\partial h}{\partial u_j}\right) + \frac{1}{h^2}\left(\frac{\partial h}{\partial u_k}\right)^2 = 0 \tag{A.5}$$

Introduce a change of variable defined by

$$v \equiv \frac{1}{h} \tag{A.6}$$

where any points with $v = 0,\infty$ are excluded from our consideration. Equations A.4 and A.5 then become, respectively

$$\frac{\partial^2 v}{\partial u_j \partial u_k} = 0 \tag{A.7}$$

and

$$-\frac{1}{v}\frac{\partial^2 v}{\partial u_i^2} + \frac{1}{v^2}\left(\frac{\partial v}{\partial u_i}\right)^2 - \frac{1}{v}\frac{\partial^2 v}{\partial u_j^2} + \frac{1}{v^2}\left(\frac{\partial v}{\partial u_j}\right)^2 + \frac{1}{v^2}\left(\frac{\partial v}{\partial u_k}\right)^2 = 0 \tag{A.8}$$

Rewrite equation A.8 as

$$\frac{\partial^2 v}{\partial u_i^2} = \sum_{\ell=1}^{3} \frac{\partial^2 v}{\partial u_\ell^2} - \frac{1}{v}\sum_{\ell=1}^{3}\left(\frac{\partial v}{\partial u_\ell}\right)^2 \tag{A.9}$$

Since this holds for $i=1,2,3$ we deduce

$$\frac{\partial^2 v}{\partial u_i^2} = \frac{\partial^2 v}{\partial u_j^2} \tag{A.10}$$

Also, from equation A.7 we have, for $i \neq j$

$$\frac{\partial^2 v}{\partial u_i \partial u_j} = 0 \tag{A.11}$$

Now from equation A.11 $\partial v/\partial u_i$ is at most a function of u_i. But then $\partial^2 v/\partial u_i^2$ is at most a function of u_i. From equation A.10 we have a function of u_i equal to a function of u_j and thus both are a real constant, say c_1, i.e.

$$\frac{\partial^2 v}{\partial u_i^2} = c_1 = \frac{\partial^2 v}{\partial u_j^2} \tag{A.12}$$

Integrating we obtain

$$\frac{\partial v}{\partial u_i} = c_1 u_i + d_i \tag{A.13}$$

where d_i is a real constant, since $\partial v/\partial u_i$ is at most a function of u_i. Integrating again gives

$$v = \frac{c_1 u_i^2}{2} + d_i u_i + e_i(u_j, u_k) \tag{A.14}$$

where, as indicated, e_i is at most a function of u_j and u_k for i,j,k distinct. Summing over i (from 1 to 3) on both sides of equation A.9 gives

$$2 \sum_{\ell=1}^{3} \frac{\partial^2 v}{\partial u_\ell^2} = \frac{3}{v} \sum_{\ell=1}^{3} \left(\frac{\partial v}{\partial u_\ell}\right)^2 \tag{A.15}$$

Substituting in this last equation from equations A.12 and A.13 gives

$$2c_1 v = \sum_{\ell=1}^{3} (c_1 u_\ell + d_\ell)^2 \tag{A.16}$$

We now consider two cases.

For the first case assume that $c_1 = 0$. Then from equation A.16 we have

$$0 = \sum_{\ell=1}^{3} d_\ell^2 \tag{A.17}$$

which implies for i=1,2,3,

$$d_i = 0 \tag{A.18}$$

Thus equation A.14 becomes

$$v = e_i(u_j, u_k) \tag{A.19}$$

so that v is independent of u_i for i=1,2,3. This implies that v is a constant, and thus h as well, i.e.

$$h = \frac{1}{v} = c_2^2 \tag{A.20}$$

where $c_2 \neq 0$ is a real constant. Then for this first case the u_i are just a cartesian coordinate system. This is the trivial case of a homogeneous medium in which ε and μ are independent of the coordinates.

For the second case assume that $c_1 \neq 0$. Then from equation A.16 we have

$$v = \sum_{\ell=1}^{3} \frac{(c_1 u_\ell + d_\ell)^2}{2c_1} \tag{A.21}$$

Note that $c_1 > 0$ since $v > 0$. Defining new real constants a and b_ℓ, the general form for h is

$$h = \frac{1}{v} = \left[\sum_{\ell=1}^{3} \left(\frac{u_\ell}{a} + b_\ell\right)^2 \right]^{-1} \qquad (A.22)$$

where $a \neq 0$ since $c_1 \neq 0$. Now by a simple linear shift of the u_ℓ coordinates we make $b_\ell = 0$ giving the simple symmetrical result

$$h = \frac{a^2}{u_1^2 + u_2^2 + u_3^2} \qquad (A.23)$$

From equation 2.5 we can write the line element as

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = h^2[(du_1)^2 + (du_2)^2 + (du_3)^2] \qquad (A.24)$$

Rewrite this as

$$(du_1)^2 + (du_2)^2 + (du_3)^2 = \frac{1}{h^2}[(dx)^2 + (dy)^2 + (dz)^2] \qquad (A.25)$$

So momentarily regarding the u_i as cartesian coordinates, x,y,z as the orthogonal curvilinear coordinates, and $1/h$ as the scale factor, we can repeat the foregoing derivation from the Lamé equations and, by interchanging the quantities in equation A.22, obtain the result

$$\frac{1}{h} = \left[\left(\frac{x}{a'} + b_1'\right)^2 + \left(\frac{y}{a'} + b_2'\right)^2 + \left(\frac{z}{a'} + b_3'\right)^2\right]^{-1} \qquad (A.26)$$

where a' and b_ℓ' are real constants and $a' \neq 0$. Make a linear shift in the x,y,z coordinates so that $b_\ell' = 0$ giving

$$h = \frac{1}{a'^2}[x^2 + y^2 + z^2] \qquad (A.27)$$

This type of h in equations A.23 and A.27 corresponds to 6-sphere coordinates or the inversion of cartesian coordinates, given in one of its forms by (6)

$$x = -\frac{a^2 u_1}{u_1^2 + u_2^2 + u_3^2} \tag{A.28}$$

$$y = -\frac{a^2 u_2}{u_1^2 + u_2^2 + u_3^2} \tag{A.29}$$

$$z = -\frac{a^2 u_3}{u_1^2 + u_2^2 + u_3^2} \tag{A.30}$$

We have included minus signs in these equations to make the u_i coordinate system right handed. The scaling constant a^2 is required for these equations to be consistent with equations A.23 and 2.4. We also have

$$a^2 = a'^2 \tag{A.31}$$

which comes from equating the right sides of equations A.23 and A.27 and using equations A.28 through A.30 to relate x, y, z and u_1, u_2, u_3 .

Thus for this special h given by equation A.1 there are two types of solutions. The u_i form either cartesian or 6-sphere type of coordinate systems.

APPENDIX B: CHARACTERISTICS OF COORDINATE SYSTEMS FOR FIELD COMPONENTS IN TWO COORDINATE DIRECTIONS

In Section VI we consider the case of formal field components in only the u_1 and u_2 coordinate directions. Imposing appropriate requirements on the constitutive parameters and on the formal constitutive parameters (equations 6.10 and 6.1, respectively) leads to the result of equation 6.15, namely

$$h \equiv h_1 = h_2 \tag{B.1}$$

In this appendix we consider a restriction which this imposes on the orthogonal curvilinear coordinate systems.

Eisenhart (12) defines the second fundamental form of a surface (which we take as defined by any particular u_3) as the quadratic differential form

$$\Phi = D_3(du_1)^2 + 2D_3'du_1du_2 + D_3''(du_2)^2 \tag{B.2}$$

which for an orthogonal curvilinear system reduces to

$$\Phi = D_3(du_1)^2 + D_3''(du_2)^2 \tag{B.3}$$

with

$$D_3 = -\frac{h_1}{h_3}\frac{\partial h_1}{\partial u_3} \quad , \qquad D_3'' = -\frac{h_2}{h_3}\frac{\partial h_2}{\partial u_3} \tag{B.4}$$

The first fundamental form of a u_3 surface is just the line element given by

$$(ds)^2 = h_1^2(du_1)^2 + h_2^2(du_2)^2 \tag{B.5}$$

However, substituting $h \equiv h_1 = h_2$ into equations B.4 gives

$$D_3 = D_3'' = -\frac{h}{h_3}\frac{\partial h}{\partial u_3} \tag{B.6}$$

The first and second fundamental coefficients are then in proportion, i.e.

$$\frac{D_3}{h_1^2} = \frac{D_3''}{h_2^2} = -\frac{1}{hh_3}\frac{\partial h}{\partial u_3} \tag{B.7}$$

We then apply a result of Eisenhart (12) that these coefficients are in proportion if and only if the surface (given by constant u_3 in this case) is a plane or a sphere. Thus for this restriction on the coordinate system, given by equation B.1, surfaces of constant u_3 can only be planes or spheres (with respect to an x,y,z cartesian coordinate system).

Also in Section VI another orthogonal curvilinear coordinate system with coordinates v_1,ϕ,v_3 is introduced. This system is rotational with

$$v_2 \equiv \phi \;, \quad \tan(\phi) = \frac{y}{x} \;, \quad h_\phi = \rho \equiv (x^2 + y^2)^{1/2} \tag{B.8}$$

from equations 6.18 through 6.20 and equation 6.32. Note that the scale factors for the v_i system are designated by h_{v_i}. Because of equation 6.23 relating u_3 and v_3, surfaces of constant v_3 are also planes or spheres. Since surfaces of constant v_3 are planes or spheres, we again invoke the result of Eisenhart that the first and second fundamental coefficients for a v_3 surface must be in proportion, which we express as

$$\frac{D_{v_3}}{h^2_{v_1}} = \frac{D''_{v_3}}{h^2_{\phi}} \tag{B.9}$$

where

$$D_{v_3} = -\frac{h_{v_1}}{h_{v_3}}\frac{\partial h_{v_1}}{\partial v_3} \quad , \quad D''_{v_3} = -\frac{h_{\phi}}{h_{v_3}}\frac{\partial h_{\phi}}{\partial v_3} \tag{B.10}$$

Combining equations B.9 and B.10 gives

$$\frac{\partial \, \ell n(h_{v_1})}{\partial v_3} = \frac{\partial \, \ell n(h_{\phi})}{\partial v_3} \tag{B.11}$$

But this implies

$$\frac{\partial \, \ell n(\frac{h_{v_1}}{h_{\phi}})}{\partial v_3} = \frac{\partial \, \ell n(h_{v_1})}{\partial v_3} - \frac{\partial \, \ell n(h_{\phi})}{\partial v_3} = 0 \tag{B.12}$$

or, in other words, h_{v_1}/h_{ϕ} is independent of v_3 . This result is used in constructing the u_i from the v_i .

REFERENCES

1. Borgnis, F. E. and Papas, C. H., "Electromagnetic Waveguides and Resonators", in Handbuch der Physik, Vol. XVI, Springer-Verlag, Berlin, 1958, pp. 358-364.

2. Tischer, F. J., "Conformal Mapping Applied to Three-Dimensional Wave Problems", Proc. IEEE 53, Feb. 1965, p. 168.

3. Luneburg, R. K., Mathematical Theory of Optics, U. Calif. Press, Berkeley, 1966, pp. 182-188.

4. Sokolnikoff, I. S., Tensor Analysis Theory and Applications to Geometry and Mechanics of Continua, Second Edition, John Wiley & Sons, New York, 1964, pp. 69-71.

5. Collin, R. E., Field Theory of Guided Waves, McGraw-Hill, New York, 1960, Chapter 4.

6. Moon, P. and Spencer, D. E., Field Theory Handbook, Springer-Verlag, Berlin, 1961, Sections I and IV.

7. Smythe, W.R., Static and Dynamic Electricity, Second Edition, McGraw-Hill, New York, 1950, p. 479.

8. Gradshteyn, I. S. and Ryzhik, I. M., Table of Integrals, Series, and Products, Fourth Edition, Academic Press, New York, 1965, 2.444-2.

9. Abramowitz, M. and Stegun, I. A., ed., Handbook of Mathematical Functions, U.S. Govt. Printing Office, Washington, D.C., 1964, 4.3.57 and 4.4.39.

10. Baum, Capt. C.E., A Lens Technique for Transitioning Waves between Conical and Cylindrical Transmission Lines, Electromagnetic Pulse Sensor and Simulation Notes, Note 32, DASA 1880-2, Jan. 1967.

11. Eisenhart, L.P., A Treatise on the Differential Geometry of Curves and Surfaces, Dover, New York, 1960, p. 449.

12. Eisenhart, op. cit., pp. 114-117, 446-449.