A MULTIPLE SCATTERING PROBLEM

Thesis by

A. D. Varvatsis

In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1968

(Submitted May 8, 1968)

ACKNOWLEDGMENTS

The author is grateful to Professor C. H. Papas for his continued guidance and encouragement during the course of this work. Special thanks are due to Dr. H. Hodara, Mr. Tom McGill Jr. for many helpful discussions, and Mr. P. Marmarelis for assistance in the programming of the numerical computations.

ABSTRACT

The present work deals with the problem of the interaction of the electromagnetic radiation with a statistical distribution of non-magnetic dielectric particles immersed in an infinite homogeneous isotropic, non-magnetic medium. The wavelength of the incident radiation can be less, equal or greater than the linear dimension of a particle. The distance between any two particles is several wavelengths. A single particle in the absence of the others is assumed to scatter like a Rayleigh-Gans particle, i.e. interaction between the volume elements (self-interaction) is neglected. The interaction of the particles is taken into account (multiple scattering) and conditions are set up for the case of a lossless medium which guarantee that the multiple scattering contribution is more important than the self-interaction one. These conditions relate the wavelength λ and the linear dimensions of a particle a and of the region occupied by the particles D. It is found that for constant λ/a, D is proportional to λ and that $|\Delta\chi|$, where $\Delta\chi$ is the difference in the dielectric susceptibilities between particle and medium, has to lie within a certain range.

The total scattering field is obtained as a series the several terms of which represent the corresponding multiple scattering orders. The first term is a single scattering term. The ensemble average of the total scattering intensity is then obtained as a series which does not involve terms due to products between terms of different orders. Thus the waves corresponding to different orders are independent and

their Stokes parameters add.

The second and third order intensity terms are explicitly computed. The method used suggests a general approach for computing any order. It is found that in general the first order scattering intensity pattern (or phase function) peaks in the forward direction $\theta = 0$. The second order tends to smooth out the pattern giving a maximum in the $\theta = \pi/2$ direction and minima in the $\theta = 0$, $\theta = \pi$ directions. This ceases to be true if ka (where $k = 2\pi/\lambda$) becomes large (> 20). For large ka the forward direction is further enhanced. Similar features are expected from the higher orders even though the critical value of ka may increase with the order.

The first order polarization of the scattered wave is determined. The ensemble average of the Stokes parameters of the scattered wave is explicitly computed for the second order. A similar method can be applied for any order. It is found that the polarization of the scattered wave depends on the polarization of the incident wave. If the latter is elliptically polarized then the first order scattered wave is elliptically polarized, but in the $\theta = \pi/2$ direction is linearly polarized. If the incident wave is circularly polarized the first order scattered wave is elliptically polarized except for the directions $\theta = \pi/2$ (linearly polarized) and $\theta = 0, \pi$ (circularly polarized). The handedness of the $\theta = 0$ wave is the same as that of the incident whereas the handedness of the $\theta = \pi$ wave is opposite. If the incident wave is linearly polarized the first order scattered wave is also linearly polarized. The second order makes the total scattered wave to be elliptically polarized for any θ no matter what the incident wave is. However, the handedness of the total scattered wave is not altered by the second order. Higher orders have similar effects as the second order.

If the medium is lossy the general approach employed for the lossless case is still valid. Only the algebra increases in complexity. It is found that the results of the lossless case are insensitive in the first order of $k_{im}D$ where k_{im} = imaginary part of the wave vector k and D a linear characteristic dimension of the region occupied by the particles. Thus moderately extended regions and small losses make $(k_{im}D)^2 \ll 1$ and the lossy character of the medium does not alter the results of the lossless case. In general the presence of the losses tends to reduce the forward scattering.

TABLE OF CONTENTS

PART	PAGE

I. INTRODUCTION

When one computes the scattered wave, due to the illumination of a collection of particles, ignoring the interaction of the particles one talks about single scattering. Multiple scattering involves the interaction of the particles.

The first sound attempts to attack the multiple scattering problem are due to Arthur Schuster (1905) who formulated a problem in radiative transfer to explain the appearance of absorption and emission lines in stellar spectra, and to Karl Schwarzschild (1906) who introduced and developed the concept of radiative equilibrium in stellar atmospheres. However, a systematic treatment of the multiple scattering problem was first given by W. Hartel (9) in 1941. His method is based on determining successive angular intensity distributions for each successive order of scattering. His theory is applicable to the case of a medium densely packed with scatterers. This approach has been recently followed by D. H. Woodward (11) who has assumed that the scatterers are Mie spheres with a radius large compared to the wavelength. The theory introduced by Hartel, however, does not involve the polarization of the scattered wave. Such a scalar theory is never reliable according to Chandrasekhar (8). But Woodward (11) has extended Hartel's theory to include polarization effects.

Another difficulty which also applies to some other theories is the following: One usually starts with the law of single scattering by individual scatterers. In most of the mathematical theories the derivation of the law of scattering is based upon the concept of the illumination of the scatterers by a plane electromagnetic wave. Thus one talks about the Mie or Rayleigh etc. laws of scattering. Now in a dense medium every point (small region) is a scatterer. The interaction of the scatterers is by far different from the interaction of a plane wave with a single scatterer. We then understand that when multiple scattering is taken into account one cannot assume that every point scatters according to a specified law based on the illumination of a single scatterer by a plane wave. Therefore, the Hartel theory cannot use the single scattering theories mentioned above.

In 1945 S. Chandrasekhar (8) developed in a systematic and mathematically rigorous way the problem of Radiative Transfer. His equation of transfer is a continuity equation for a 4-dimensional vector with components the 4 Stokes parameters of the scattered wave. The radiative transfer theories are best suited to problems such as scattering by planetary atmospheres, radiative equilibrium of a stellar atmosphere and other related problems. Like the Hartel theory, the Radiative Transfer Theories (R.T.T.) assume a medium densely packed with scatterers. Therefore these theories cannot be based upon single scattering theories such as Mie's etc. Another frequent assumption of the R.T.T. is that the scatterers behave like small dipoles. If higher moments are taken into account (10) or one considers particles of a shape other than spherical the computations get pretty complicated.

The theories mentioned above or related ones cannot deal with the problem of the interaction of a plane wave with a collection of particles not densely packed and whose shape might be considerably different from spherical. A rigorous theory for such a situation seems infinitely complicated. If one introduces the element of randomness in the position and orientation of the particles things look brighter. Even so an exact treatment is practically impossible.

The first order scattering or single scattering can be done exactly only when one knows how to find the scattering law for a single particle of a given shape. This is not known in general. A considerable simplification takes place if the single scattering is of the Rayleigh-Gans type, i.e. if the interaction of the volume elements (self-interaction) is neglected. If one wants to find the effect of the *multiple* scattering for such particles one must make sure that the multiple scattering contribution is more important than the self-interaction contribution.

Our theory is an approximate one and deals with the following problem. Consider a collection of non-magnetic dielectric particles of any shape immersed in a homogeneous isotropic non-magnetic medium of infinite extent. We will assume that the particles have random position and orientation. The particles are of the Rayleigh-Gans type and are several wavelengths apart. This last assumption is made to simplify the computations.

Consider now a plane wave illuminating the particles. We want to compute the scattered field. The scattered field will be characterized by the four Stokes parameters. The averaging over the random positions and orientations of the particles will be an ensemble average.

Our aim is to expand the total scattered field into a series the several terms of which represent the corresponding orders of scattering. Thus the first term is a single scattering or first order scattering. The second term is a multiple scattering of the first order or a second order scattering, i.e. the additional current induced within any particle is due to currents inside all other particles induced by the incident field only. The third order involves current due to the second order fields etc.

The essential part then of this thesis is the computation of the several scattering orders.

II. FORMULATION OF THE PROBLEM

2.1. Scattering From a Single Particle

Consider the scattering of a plane wave by a single dielectric particle (see figure 1). The particle has constitutive parameters ϵ_p, which can be complex, and $\mu_p = \mu_o$ = magnetic permeability of vacuum. The surrounding medium is infinite, homogeneous, isotropic with constitutive parameters ϵ_m, complex in general, and $\mu_m = \mu_o$. If we call the incident electric field $\underline{E}_{inc}$ and the scattered one $\underline{E}_{sc}$ then it can be shown (see Appendix A) that:

$$\underline{E}_{sc}(\underline{r}) = \frac{\omega^2}{c^2}\Delta\chi \int_V \underline{\underline{\Gamma}}(\underline{r};\underline{r}') \cdot \underline{E}(\underline{r}')\, dV' \tag{2.1.1}$$

where $\underline{E}(\underline{r})$ = total field at $\underline{r} = \underline{E}_{inc}(\underline{r}) + \underline{E}_{sc}(\underline{r})$, $\Delta\chi = \chi_m - \chi_p = \left(\frac{\epsilon_m - \epsilon_o}{\epsilon_o}\right) - \left(\frac{\epsilon_p - \epsilon_o}{\epsilon_o}\right) = \frac{1}{\epsilon_o}(\epsilon_m - \epsilon_p)$, $\underline{\underline{\Gamma}}(\underline{r};\underline{r}') = (\underline{\underline{u}} + \frac{1}{k^2}\nabla\nabla)\frac{e^{ik|\underline{r}-\underline{r}'|}}{4\pi|\underline{r}-\underline{r}'|}$

where $\underline{\underline{u}}$ = unit dyadic = $\underline{e}_x\underline{e}_x + \underline{e}_y\underline{e}_y + \underline{e}_z\underline{e}_z$, $k = \omega\sqrt{\mu_o\epsilon_m}$.

2.1.1 can be solved by an iteration method. Thus the first order approximation is obtained by replacing $\underline{E}(\underline{r}')$ by $\underline{E}_{inc}(\underline{r}')$:

$$\underline{E}^{[1]}_{sc}(\underline{r}) = \frac{\omega^2}{c^2}\Delta\chi \int_V \underline{\underline{\Gamma}}(\underline{r};\underline{r}') \cdot \underline{E}_{inc}(\underline{r}')\, dV' \tag{2.1.2}$$

The second order approximation replaces $\underline{E}(\underline{r}')$ in (2.1.1) by $\underline{E}(\underline{r}') = \underline{E}_{inc}(\underline{r}') + \underline{E}^{[1]}_{sc}(\underline{r}')$:

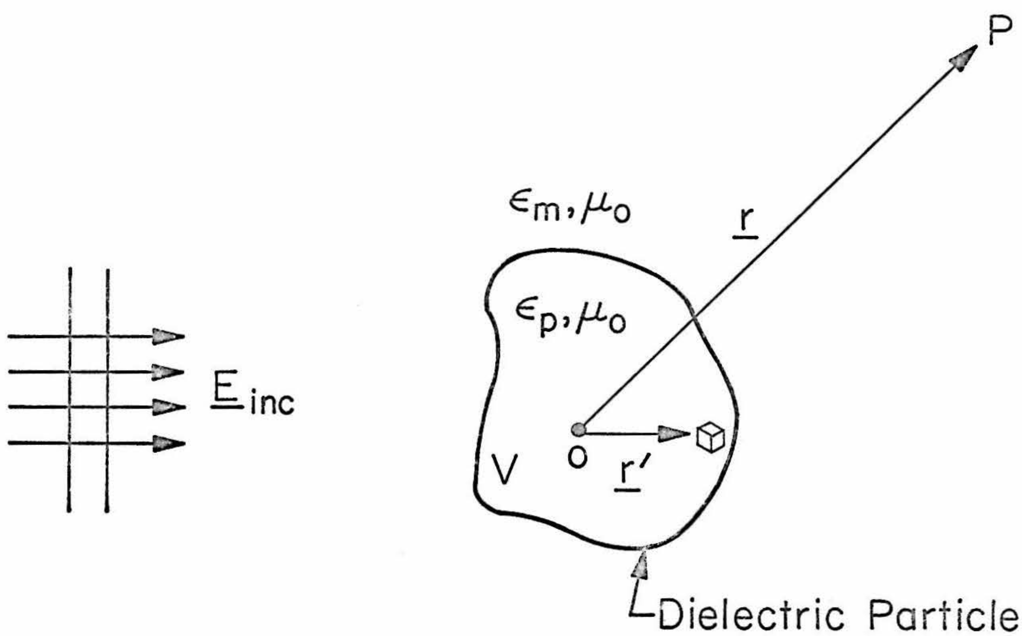


Fig. 1. A dielectric particle illuminated by a plane wave $\underline{E}_{inc}$. P is the observation point.

$$\underline{E}_{sc}^{[2]}(\underline{r}) = \frac{\omega^2}{c^2}\Delta\chi \int_V \underline{\underline{\Gamma}}(\underline{r};\underline{r}') \cdot \underline{E}_{inc}(\underline{r}')\, dV'$$

$$+ \left(\frac{\omega^2}{c^2}\Delta\chi\right)^2 \int_V \underline{\underline{\Gamma}}(\underline{r};\underline{r}') \cdot \left\{ \int_V \underline{\underline{\Gamma}}(\underline{r}';\underline{r}'') \cdot \underline{E}_{inc}(\underline{r}'')\, dV'' \right\} dV'$$

$$= \underline{E}_{sc}^{[1]}(\underline{r}) + \underline{E}_{sc}^{(2)}(\underline{r}) \qquad (2.1.3)$$

It is now easy to see that in general we must have:

$$\underline{E}_{sc}^{[n]}(\underline{r}) = \underline{E}_{sc}^{(1)}(\underline{r}) + \underline{E}_{sc}^{(2)}(\underline{r}) + \dots + \underline{E}_{sc}^{(n-1)}(\underline{r}) + \underline{E}_{sc}^{(n)}(\underline{r}) \qquad (2.1.4)$$

with

$$\underline{E}_{sc}^{[0]} = \underline{E}_{inc}(\underline{r})$$

$$\underline{E}_{sc}^{[1]} = \underline{E}_{sc}^{(1)}(\underline{r})$$

$$\underline{E}_{sc}^{(n)}(\underline{r}) = \left(\frac{\omega^2}{c^2}\Delta\chi\right)^n \int_V \underline{\underline{\Gamma}}(\underline{r};\underline{r}_1) \cdot \left\{ \int_V \underline{\underline{\Gamma}}(\underline{r}_1;\underline{r}_2) \right.$$

$$\cdot \left\{ \int_V \underline{\underline{\Gamma}}(r_2;r_3) \dots \left\{ \int_V \underline{\underline{\Gamma}}(\underline{r}_{n-1};\underline{r}_n) \cdot \underline{E}_{inc}(\underline{r}_n)\, dV_n \right\} \right.$$

$$\dots \Big\} dV_3 \Big\} dV_2 \Big\} dV_1 \Big\}$$

$$= \left(\frac{\omega^2}{c^2}\Delta\chi\right) \int_V \underline{\underline{\Gamma}}(\underline{r};\underline{r}_1) \cdot \underline{E}_{sc}^{(n-1)}(\underline{r}_1)\, dV_1 \qquad (2.1.5)$$

The first order approximation $\underline{E}_{sc}^{[1]}(\underline{r}) = \underline{E}_{sc}^{(1)}(\underline{r})$ is called the Born approximation. From 2.1.5 we see that the n^{th} order in the series expansion 2.1.4 is proportional to $(\Delta\chi)^n$. One then might be led to believe that for sufficiently small $|\Delta\chi|$ the Born approximation is valid. However, this is not true as the following analysis shows: Let's compare $\underline{E}^{(1)}_{sc}$ to $\underline{E}^{(2)}_{sc}$, i.e. let's estimate the absolute value of the fields. We are usually interested in the far zone values (see Appendix A)

$$\underline{E}^{(1)}_{sc}(\underline{r}) = \frac{\omega^2}{c^2}\Delta\chi \int_V \underline{\underline{\Gamma}}(\underline{r};\underline{r}') \cdot \underline{E}_{inc}(\underline{r}')\, dV'$$

$$\approx \frac{\omega^2}{c^2}\Delta\chi \int_V (\underline{\underline{u}} - \underline{e}_r\underline{e}_r)\frac{e^{ikr}}{4\pi r} e^{-ik\underline{e}_r\cdot\underline{r}'} \cdot \underline{E}_o e^{i\underline{k}\cdot\underline{r}'}\, dV'$$

$$= -\frac{\omega^2}{c^2}\Delta\chi\, (\underline{e}_r \times \underline{e}_r \times \underline{E}_o)\frac{e^{ikr}}{4\pi r}\int_V e^{-ik\underline{e}_r\cdot\underline{r}'} e^{i\underline{k}\cdot\underline{r}'}\, dV' \quad (2.1.6)$$

$$\underline{E}^{(2)}_{sc}(\underline{r}) = \left(\frac{\omega^2}{c^2}\Delta\chi\right)^2 \int_V \underline{\underline{\Gamma}}(\underline{r};\underline{r}') \cdot \int_V \underline{\underline{\Gamma}}(\underline{r}';\underline{r}'') \cdot \underline{E}_{inc}(\underline{r}'')\, dV''\, dV'$$

$$= -\left(\frac{\omega^2}{c^2}\Delta\chi\right)^2 (\underline{e}_r \times \underline{e}_r \times \int_V e^{-ik\underline{e}_r\cdot\underline{r}'} (\underline{\underline{u}} + \frac{1}{k^2}\nabla'\nabla')$$

$$\cdot \int_V \frac{e^{ik|\underline{r}'-\underline{r}''|}}{4\pi|\underline{r}'-\underline{r}''|}\underline{E}_o e^{i\underline{k}\cdot\underline{r}''}\, dV''\, dV'$$

$$= -\left(\frac{\omega^2}{c^2}\Delta\chi\right)^2 (\underline{e}_r \times \underline{e}_r \times \int_V e^{-ik\underline{e}_r\cdot\underline{r}'} (\underline{\underline{u}} + \frac{1}{k^2}\nabla'\nabla')$$

$$\cdot \underline{E}_o \int_V \frac{e^{ik|\underline{r}'-\underline{r}''|}}{4\pi|\underline{r}'-\underline{r}''|} e^{i\underline{k}\cdot\underline{r}''}\, dV''\, dV' \quad (2.1.7)$$

Now from 2.1.6 we get

$$|\underline{E}^{(1)}_{sc}(r)| = \frac{\omega^2}{c^2}|\Delta\chi|\,|\underline{e}_r\times\underline{e}_r\times\underline{E}_o|\left|\frac{e^{ikr}}{4\pi r}\right|\left|\int_V e^{-ik\underline{e}_r\cdot\underline{r}'}e^{i\underline{k}\cdot\underline{r}'}dV'\right|$$

If we assume that k is real then $\left|\int_V e^{-ik\underline{e}_r\cdot\underline{r}'}e^{i\underline{k}\cdot\underline{r}'}dV'\right| \leq$ $\int_V |e^{-ik\underline{e}_r\cdot\underline{r}'}e^{i\underline{k}\cdot\underline{r}'}|\,dV' = V_p$ = volume of dielectric particle. If k is not real then by writing $\underline{k} = \underline{k}_r + i\underline{k}_i$ we get $I = \left|\int_V e^{-ik\underline{e}_r\cdot\underline{r}'}e^{i\underline{k}\cdot\underline{r}'}dV'\right|$ $\leq \int_V e^{k_i\underline{e}_r\cdot\underline{r}'}e^{-\underline{k}_i\cdot\underline{r}'}dV'$. Assuming that $\underline{E}_{inc}$ travels in the z direction we have $I \leq \int_V e^{k_i(\underline{e}_r-\underline{e}_z)\cdot\underline{r}'}dV'$.

In the present work we deal with lossy media with k_i of the order of $(1 \div \frac{1}{20})m^{-1}$ or less. Therefore in view of the small dimensions of the particle $(< 10\,\mu)$ we understand that $e^{k_i(\underline{e}_r-\underline{e}_z)\cdot r'} \approx 1$ and $I \leq V_p$. Thus

$$|\underline{E}^{(1)}_{sc}| \leq \frac{\omega^2}{c^2}|\Delta\chi|\,|\underline{e}_r\times\underline{e}_r\times\underline{E}_o|\left|\frac{e^{ikr}}{4\pi r}\right| < \frac{\omega^2}{c^2}|\Delta\chi|\,|E_o|\left|\frac{e^{ikr}}{4\pi r}\right| \tag{2.1.8}$$

Next we estimate $|\underline{E}^{(2)}_{sc}|$:

$$|\underline{E}^{(2)}_{sc}| < \left(\frac{\omega^2}{c^2}|\Delta\chi|\right)^2\left|\left|\frac{e^{ikr}}{4\pi r}\right|\int_V e^{-ik\underline{e}_r\cdot\underline{r}'}(\underline{\underline{u}}+\frac{1}{k^2}\nabla'\nabla')\cdot\underline{E}_o\int_V\frac{e^{ik|\underline{r}'-\underline{r}''|}}{4\pi|\underline{r}'-\underline{r}''|}e^{i\underline{k}\cdot\underline{r}''}dV''\,dV'\right|$$

$$< \left(\frac{\omega^2}{c^2}|\Delta\chi|\right)^2\left|\frac{e^{ikr}}{4\pi r}\right|\int_V\left|(\underline{\underline{u}}+\frac{1}{k^2}\nabla'\nabla')\cdot\underline{E}_o\int_V\frac{e^{ik|\underline{r}'-\underline{r}''|}}{4\pi|\underline{r}'-\underline{r}''|}e^{i\underline{k}\cdot r''}dV''\right|dV'$$

For estimation purposes we may write $\nabla' \sim \frac{1}{k}\underline{e}_r'$, i.e.

$$|\underline{E}_{sc}^{(2)}| < \left(\frac{\omega^2}{c^2}|\Delta\chi|\right)^2 \left|\frac{e^{ikr}}{4\pi r}\right| \left|\int_V 2|\underline{E}_o| \left|\int_V \frac{e^{ik|\underline{r}'-\underline{r}''|}}{4\pi|\underline{r}'-\underline{r}''|} e^{i\underline{k}\cdot\underline{r}''} dV''\right| dV'\right|$$

$$< \left(\frac{\omega^2}{c^2}|\Delta\chi|\right)^2 \left|\frac{e^{ikr}}{4\pi r}\right| \left|\int_V 2|\underline{E}_o| \int_V \frac{1}{4\pi|\underline{r}'-\underline{r}''|} dV'' dV'\right|$$

To estimate the integral $\int_V \frac{1}{|\underline{r}'-\underline{r}''|} dV''$ we choose a spherical particle of radius a and we measure $\underline{r}', \underline{r}''$ from the center. Then the integral is the electric potential of a uniform spherical charge distribution with $\rho = 1$ in electrostatic units, evaluated at $\underline{r}'$ lying within the sphere. The result is well known:

$$\int_V \frac{1}{|\underline{r}'-\underline{r}''|} dV'' = 2\pi a^2 - \frac{2}{3}\pi r'^2$$

and

$$\frac{2}{4\pi}\int_V \left(2\pi a^2 - \frac{2}{3}\pi r'^2\right) dV' = \left(2\pi a^2 V_p - \frac{2}{3}\pi\frac{a^5}{5}\right)\frac{2}{4\pi}$$

$$\approx V_p a^2$$

Finally

$$|\underline{E}_{sc}^{(2)}| < \left(\frac{\omega^2}{c^2}|\Delta\chi|\right)^2 |\underline{E}_o| V_p a^2 \left|\frac{e^{ikr}}{4\pi r}\right| \tag{2.1.9}$$

If we now compare $|\underline{E}_{sc}^{(1)}|$ given by 2.1.8 and $|\underline{E}_{sc}^{(2)}|$ given by 2.1.9 we understand that

$$|\underline{E}_{sc}^{(1)}| >> |\underline{E}_{sc}^{(2)}| \Longrightarrow \frac{\omega^2}{c^2}|\Delta\chi| a^2 << 1 \tag{2.1.10}$$

Condition 2.1.10 further guarantees that $|\underline{E}_{sc}^{(n)}| << |\underline{E}_{sc}^{(n-1)}|$ as we can show, therefore the Born approximation is valid only if 2.1.10 is satisfied. This condition has been discussed and derived in a different way by Van de Hulst (2).

2.2. Scattering From a Collection of Particles

If the scattered electric field by the i^{th} particle (ϵ_i, μ_o) is called $\underline{E}_{sc}^i(\underline{r})$ then the total scattered field is

$$\underline{E}_{sc}(\underline{r}) = \sum_{i=1}^{N} \underline{E}_{sc}^i(\underline{r}) \tag{2.2.1}$$

where N is the number of particles that do the scattering.

If we apply 2.1.1 for the i^{th} particle we have:

$$\underline{E}_{sc}^i(\underline{r}) = \frac{\omega^2}{c^2} \Delta\chi_i \int_{V_i} \underline{\underline{\Gamma}}(\underline{r};\underline{r}_i)\cdot\underline{E}(\underline{r}_i)\, dV_i \tag{2.2.2}$$

where

$$\underline{E}(\underline{r}_i) = \underline{E}_{inc}(\underline{r}_i) + \sum_{j=1}^{N} \underline{E}_{sc}^j(\underline{r}_i)$$

To be able to write down a series expansion for the total scattered field as we did in section 2.1 for a single particle we work as follows: First we write down the formula for the scattered field by the j^{th} particle which induces a current inside the i^{th} particle. According to 2.1.1 we have:

$$\underline{E}_{sc}^j(\underline{r}_i) = \frac{\omega^2}{c^2} \Delta\chi_j \int_{V_j} \underline{\underline{\Gamma}}(\underline{r}_i;\underline{r}_j)\cdot\underline{E}(\underline{r}_j)\, dV_j \tag{2.2.3}$$

where

$$\underline{E}(\underline{r}_j) = \underline{E}_{inc}(\underline{r}_j) + \sum_k \underline{E}_{sc}^k(\underline{r}_j)$$

Next we compute $\underline{E}_{sc}^k(\underline{r}_j)$ using 2.1.1 once more.

$$\underline{E}_{sc}^k(\underline{r}_j) = \frac{\omega^2}{c^2} \Delta\chi_k \int_{V_k} \underline{\underline{\Gamma}}(\underline{r}_j;\underline{r}_k) \cdot \underline{E}(\underline{r}_k)\, dV_k \tag{2.2.4}$$

where

$$\underline{E}(\underline{r}_k) = \underline{E}_{inc}(\underline{r}_k) + \sum_\ell \underline{E}_{sc}^\ell(\underline{r}_k)$$

Again we use 2.1.1 to compute $\underline{E}_{sc}^\ell(\underline{r}_k)$ etc.

We are now in a position to obtain the series expansion for $\underline{E}_{sc}(\underline{r}) = \sum_{i=1}^{N} \underline{E}_{sc}^i(\underline{r})$. The first order term is obtained from 2.2.2 if we replace $\underline{E}(\underline{r}_i)$ by $\underline{E}_{inc}(\underline{r}_i)$, i.e.

$$\underline{E}_{sc}^{[1]i} = \frac{\omega^2}{c^2} \Delta\chi_i \int_{V_i} \underline{\underline{\Gamma}}(\underline{r};\underline{r}_i) \cdot \underline{E}_{inc}(\underline{r}_i)\, dV_i = \underline{E}_{sc}^{(1)i}(\underline{r})$$

and (2.2.5)

$$\underline{E}_{sc}^{[1]}(\underline{r}) = \sum_{i=1}^{N} \frac{\omega^2}{c^2} \Delta\chi_i \int_{V_i} \underline{\underline{\Gamma}}(\underline{r};\underline{r}_i) \cdot \underline{E}_{inc}(\underline{r}_i)\, dV_i$$

The next approximation replaces $\underline{E}(\underline{r}_i)$ by $\underline{E}_{inc}(\underline{r}_i) + \sum_{j=1}^{N} \underline{E}_{sc}^{(1)j}(\underline{r}_j)$, i.e.

$$\underline{E}_{sc}^{[2]\,i}(\underline{r}) = \frac{\omega^2}{c^2}\,\Delta\chi_i \int_{V_i} \underline{\underline{\Gamma}}(\underline{r};\underline{r}_i)\cdot\underline{E}_{inc}(\underline{r}_i)\,dV_i$$

$$+\left(\frac{\omega^2}{c^2}\right)^2 \Delta\chi_i \int_{V_i} \underline{\underline{\Gamma}}(\underline{r};\underline{r}_i)\cdot\left\{\sum_{j=1}^{N}\Delta\chi_j \int_{V_j}\underline{\underline{\Gamma}}(\underline{r}_i;\underline{r}_j)\cdot\underline{E}_{inc}(\underline{r}_j)\,dV_j\right\}dV_i$$

$$= \underline{E}_{sc}^{(1)i}(\underline{r}) + \underline{E}_{sc}^{(2)i}(\underline{r})$$

$$\underline{E}_{sc}^{[2]}(\underline{r}) = \sum_{i=1}^{N} \underline{E}_{sc}^{[2]\,i}(\underline{r}) \tag{2.2.6}$$

To get the third order order $\underline{E}(\underline{r}_i)$ is replaced by $\underline{E}_{inc}(\underline{r}_i) + \sum_{j=1}^{N}\underline{E}_{sc}^{[2]i}(\underline{r}_i)$ i.e.

$$\underline{E}_{sc}^{[3]\,i}(\underline{r}) = \underline{E}_{sc}^{(1)i}(\underline{r}) + \underline{E}_{sc}^{(2)i}(\underline{r}) + \left(\frac{\omega^2}{c^2}\right)^3 \Delta\chi_i \int_{V_j} \underline{\underline{\Gamma}}(\underline{r};\underline{r}_i)$$

$$\cdot\left\{\sum_{j=1}^{N}\Delta\chi_j \int_{V_j} \underline{\underline{\Gamma}}(\underline{r}_i;\underline{r}_j)\cdot\left[\sum_{k=1}^{N}\Delta\chi_k \int_{V_k}\underline{\underline{\Gamma}}(\underline{r}_j;\underline{r}_k)\right.\right.$$

$$\left.\left.\cdot\,\underline{E}_{inc}(\underline{r}_k)\,dV_k\right]dV_j\right\}dV_i$$

$$= \underline{E}_{sc}^{(1)i}(\underline{r}) + \underline{E}_{sc}^{(2)i}(\underline{r}) + \underline{E}_{sc}^{(3)i}(\underline{r})$$

$$\underline{E}_{sc}^{[3]}(\underline{r}) = \sum_{i=1}^{N} \underline{E}_{sc}^{[3]\,i}(\underline{r}) \tag{2.2.7}$$

It is clear now how to obtain the n^{th} order. We have to replace $\underline{E}(\underline{r}_i)$ by $\underline{E}_{inc}(\underline{r}_i) + \sum_{j=1}^{N} \underline{E}_{sc}^{[n-1]j}(\underline{r}_i)$, i.e.

$$\underline{E}_{sc}^{[n]i}(\underline{r}) = \underline{E}_{sc}^{(1)i}(\underline{r}) + \frac{\omega^2}{c^2} \Delta\chi_i \int_{V_i} \underline{\underline{\Gamma}}(\underline{r};\underline{r}_i) \cdot \sum_{j=1}^{N} \underline{E}_{sc}^{[n-1]j}(\underline{r}_i)\, dV_i$$

$$= \underline{E}_{sc}^{(1)i}(\underline{r}) + \underline{E}_{sc}^{(2)i}(\underline{r}) + \ldots + \underline{E}_{sc}^{(n-1)i}(\underline{r}) + \underline{E}_{sc}^{(n)i}(\underline{r}) \quad (2.2.8)$$

The several terms in the series expansion have a simple physical explanation which goes as follows:

The first order scattered field $\underline{E}_{sc}^{(1)}(\underline{r})$ is due to currents induced by the incident field only, i.e. ignoring interaction of the volume elements within a particle or of the particles.

The second order scattered field $\underline{E}_{sc}^{(2)}(\underline{r})$ is due to currents induced by the first order scattered field $\underline{E}_{sc}^{(1)}(\underline{r})$, i.e. a first order interaction between volume elements and particles is taken into account. Therefore, the field $\underline{E}_{sc}^{(2)}(\underline{r})$ is due to a multiple scattering.

The third order $\underline{E}_{sc}^{(3)}(\underline{r})$ is due to currents induced by the second order $\underline{E}_{sc}^{(2)}(\underline{r})$ etc. All the terms $\underline{E}_{sc}^{(n)}(\underline{r})$ with $n > 1$ are multiple scattering terms.

Next the following observation should be made. Consider $\underline{E}_{sc}^{(2)i}$ for example (see 2.2.6)

$$\underline{E}_{sc}^{(2)i} = \left(\frac{\omega^2}{c^2}\right)^2 \Delta\chi_i \int_{V_i} \underline{\underline{\Gamma}}(\underline{r};\underline{r}_i)\cdot \left\{ \sum_{j=1}^{N} \Delta\chi_j \int_{V_j} \underline{\underline{\Gamma}}(\underline{r}_i;\underline{r}_j)\cdot \underline{E}_{inc}(\underline{r}_j)\, dV_j \right\} dV_i$$

$$= \left(\frac{\omega^2}{c^2}\right)^2 (\Delta\chi_i)^2 \int_{V_i} \underline{\underline{\Gamma}}(\underline{r};\underline{r}_i)\cdot \int_{V_i} \underline{\underline{\Gamma}}(\underline{r}_i;\underline{r}_i')\cdot \underline{E}_{inc}(\underline{r}_i')\, dV_i'\, dV_i$$

$$+ \left(\frac{\omega^2}{c^2}\right)^2 (\Delta\chi_i) \int_{V_i} \underline{\underline{\Gamma}}(\underline{r};\underline{r}_i)\cdot \left\{ \sum_{j\neq i} \Delta\chi_j \int_{V_j} \underline{\underline{\Gamma}}(\underline{r}_i;\underline{r}_j) \cdot \underline{E}_{inc}(\underline{r}_j)\, dV_j \right\} dV_i \qquad (2.2.9)$$

The first term involves the interaction of the volume elements of the i^{th} particle which would exist even if all the other particles were absent, whereas the second term describes an interaction between the i^{th} particle and all the others.

If we recall the results of section 2.1 we recognize that the first term is just $\underline{E}_{sc}^{(2)}(\underline{r})$ in 2.1.3. This is a self-field because it involves interaction within the particle itself. As we shall later see the second term in 2.2.9 will depend on the density of the particles whereas the first does not. It is not obvious a priori which term is the most important. Of course they are both of order $(\Delta\chi)^2$ if the χ_i's are comparable but this is not the whole story as we saw in section 2.1. In the present work we will neglect self fields; therefore, we should find out under what conditions the self-field terms are negligible compared to terms due to the interaction of the particles. We should notice here that we also get "mixed" terms which come from interactions within the particles but depending on the presence of the other particles whereas the first term in 2.2.9 does not. Such terms exist in the higher order terms. Consider for example $\underline{E}_{sc}^{(3)}(\underline{r})$. If we assume for simplicity that $N = 2$ we can write 2.2.7 as:

$$
\begin{aligned}
\underline{E}_{sc}^{(3)1}(\underline{r}) = {} & \frac{\omega^2}{c^2}\Delta\chi_1 \int_{V_1} \underline{\underline{\Gamma}}(\underline{r};\underline{r}_1)\cdot \left\{ \frac{\omega^2}{c^2}\Delta\chi_1 \int_{V_1} \underline{\underline{\Gamma}}(\underline{r}_1;\underline{r}_1')\cdot \left[\frac{\omega^2}{c^2}\Delta\chi_1 \int_{V_1} \underline{\underline{\Gamma}}(\underline{r}_1';\underline{r}_1'') \right.\right. \\
& \left.\left. \cdot\, \underline{E}_{inc}(r_1'')\, dV_1'' \right] dV_1' \right\} dV_1 \\
& + \frac{\omega^2}{c^2}\Delta\chi_1 \int_{V_1} \underline{\underline{\Gamma}}(\underline{r};\underline{r}_1)\cdot \left\{ \frac{\omega^2}{c^2}\Delta\chi_1 \int_{V_1} \underline{\underline{\Gamma}}(\underline{r}_1;\underline{r}_1')\cdot \left[\frac{\omega^2}{c^2}\Delta\chi_2 \int_{V_2} \underline{\underline{\Gamma}}(\underline{r}_1';\underline{r}_2) \right.\right. \\
& \left.\left. \cdot\, \underline{E}_{inc}(\underline{r}_2)\, dV_2 \right] dV_1' \right\} dV_1 \\
& + \frac{\omega^2}{c^2}\Delta\chi_1 \int_{V_1} \underline{\underline{\Gamma}}(\underline{r};\underline{r}_1)\cdot \left\{ \frac{\omega^2}{c^2}\Delta\chi_2 \int_{V_2} \underline{\underline{\Gamma}}(\underline{r}_1;\underline{r}_2)\cdot \left[\frac{\omega^2}{c^2}\Delta\chi_1 \int_{V_1} \underline{\underline{\Gamma}}(\underline{r}_2;\underline{r}_1) \right.\right. \\
& \left.\left. \cdot\, \underline{E}_{inc}(\underline{r}_1)\, dV_1 \right] dV_2 \right\} dV_1 \\
& + \frac{\omega^2}{c^2}\Delta\chi_1 \int_{V_1} \underline{\underline{\Gamma}}(\underline{r};\underline{r}_1)\cdot \left\{ \frac{\omega^2}{c^2}\Delta\chi_2 \int_{V_2} \underline{\underline{\Gamma}}(\underline{r}_1;\underline{r}_2)\cdot \left[\frac{\omega^2}{c^2}\Delta\chi_2 \int_{V_2} \underline{\underline{\Gamma}}(\underline{r}_2;\underline{r}_2') \right.\right. \\
& \left.\left. \cdot\, \underline{E}_{inc}(\underline{r}_2')\, dV_2' \right] dV_2 \right\} dV_1
\end{aligned}
\qquad (2.2.10)
$$

The first term in 2.2.10 would exist even in the absence of particle no. 2. It is a pure self-field term belonging to particle no. 1. The second term depends on the presence of particle no. 2. Thus $\frac{\omega^2}{c^2} \Delta\chi_2 \int_{V_2} \underline{\underline{\Gamma}}(\underline{r}_1';\underline{r}_2) \cdot \underline{E}_{inc}(\underline{r}_2)\, dV_2$ is the first order scattered field due to particle no. 2. This field induces a current within no. 1. The current produces a field which in turn induces a current within no. 1 again. This last interaction is a self-interaction depending on the presence of no. 2. The third term is <u>not</u> of similar nature. Thus $\frac{\omega^2}{c^2} \Delta\chi_1 \int_{V_1} \underline{\underline{\Gamma}}(\underline{r}_2;\underline{r}_1) \cdot \underline{E}_{inc}(\underline{r}_1)\, dV_1$ is a first order field due to no. 1. inducing a current within no. 2. This current produces a field which causes a current within no. 1. There is <u>no self-interaction</u> even though no. 1 affects itself through no. 2. The fourth term includes such an interaction, i.e. the field produced by no. 2 induces a current within no. 2 which in turn produces a field acting on no. 1.

We thus see that $\underline{E}_{sc}^{(3)} = \underline{E}_{sc}^{(3)1} + \underline{E}_{sc}^{(3)2}$ consists of $2 \times 2 \times 2 = 8$ terms with only two terms without self-interaction. For any N the terms without self-interaction are $N(N-1)(N-1)$, i.e. $\sum_i a_i \sum_{j \neq i} b_{ij} \sum_{k \neq j} c_{jk}$ whereas the self-interaction terms are $N^3 - N(N-1)(N-1) = N^3 - N(N^2 - 2N + 1) = 2N^2 - N$

$$\text{Without: } N^3 - 2N^2 + N = N_{out}$$
$$\text{With: } 2N^2 - N = N_w$$

For $N > 3$ $N_{out} > N_w$. Thus as N gets high, whereas the volume within which the particles exist remains constant, we expect the self-interaction contribution to be less important than the contribution from the other terms. Now what we really want is to make the largest self-interaction term, i.e. $\underline{E}_{sc}^{(2)}$ in 2.1.3 smaller than any arbitrary order term $\underline{E}_{sc}^{(n)}$ in 2.2.8 if we exclude self-interaction terms. This seems adequate for a theory which neglects self-interaction but it is not. To see this recall the series expansion 2.1.4 for the scattered field by a single particle in the absence of the others. If we want the intensity pattern of the scattered field we have to compute the far zone Poynting vector

$$\underline{S} = S_r \underline{e}_r \qquad S_r = \frac{1}{2} \operatorname{Re} \underline{E} \times \underline{H}^* \sim |\underline{E}_{sc}|^2$$

Thus

$$S_r \sim |\underline{E}_{sc}|^2 = |\underline{E}_{sc}^{(1)} + \underline{E}_{sc}^{(2)} + \ldots |^2$$

The scattering is not incoherent, i.e.

$$S_r \sim |\underline{E}_{sc}^{(1)}|^2 + |\underline{E}_{sc}^{(2)}|^2 + \ldots + 2 \operatorname{Re} \underline{E}_{sc}^{(1)} \cdot \underline{E}_{sc}^{(2)*} + \ldots$$

$$= |\underline{E}_{sc}^{(1)}|^2 + 2 \operatorname{Re} \underline{E}_{sc}^{(1)} \cdot \underline{E}_{sc}^{(2)*} + |\underline{E}_{sc}^{(2)}|^2 + \ldots \qquad (2.2.11)$$

The terms have been arranged in order of magnitude.

Now if we consider the <u>collection of the particles</u> and neglect the self-interaction terms we can show (see Appendix D) that

$$\langle S_r \rangle \sim \langle |\underline{E}_{sc}|^2 \rangle = \langle |\underline{E}_{sc}^{(1)} + \underline{E}_{sc}^{(2)} + \ldots |^2 \rangle$$

$$= \langle |\underline{E}_{sc}^{(1)}|^2 \rangle + \langle |\underline{E}_{sc}^{(2)}|^2 \rangle + \ldots \qquad (2.2.12)$$

i.e. the several orders add incoherently due to the assumption about randomness in the position and orientation of the particles.

Therefore what we want is to make sure that any term in 2.2.12 is larger than $2N \operatorname{Re} \underline{E}_{sc}^{(1)} \cdot \underline{E}_{sc}^{(2)*}$. The multiplication by N is due to the assumption of randomness which make the intensities from the several particles add.

We thus have to find the conditions under which

$$\left.\langle |\underline{E}_{sc}^{(n)}|^2 \rangle\right|_{\substack{\text{collection}\\ \text{of particles}}} \gg 2N \operatorname{Re}\left(\underline{E}_{sc}^{(1)} \cdot \underline{E}_{sc}^{(2)*}\right)_{\substack{\text{single}\\ \text{particle}}} \tag{2.2.13}$$

We have already estimated $|\underline{E}_{sc}^{(1)}|$ and $|\underline{E}_{sc}^{(2)}|$ for the scattering by a single particle (see 2.1.8 and 2.1.9). Thus we can write

$$2N \operatorname{Re}\left(\underline{E}_{sc}^{(1)} \cdot \underline{E}_{sc}^{(2)}\right)_{\substack{\text{single}\\ \text{particle}}} \lesssim N\left(\frac{\omega^2}{c^2}|\Delta\chi|\right)^3 |E_o|^2 V_p^2 a^2 \frac{1}{(4\pi r)^2} \tag{2.2.14}$$

if losses are neglected. <u>Notice</u> that due to interference the left-hand side is usually much smaller than the right-hand side.

As we will show later in this work we get the following result for the average $|\underline{E}_{sc}^{(n)}|^2$ if losses are neglected:

$$\langle |\underline{E}_{sc}^{(n)}|^2 \rangle = |E_o|^2 \left(\frac{\omega^2}{c^2}|\Delta\chi|\right)^{2n} \left(\frac{1}{4\pi r}\right)^2 N \left(\frac{N}{V}\right)^{n-1} \left(\frac{D}{32\pi^2}\right)^{n-1} F_1^{n-1}$$

$$(K_1 + K_2 \cos^2\theta) \qquad \text{for } n \neq 1 \tag{2.2.15a}$$

and

$$\langle |\underline{E}_{sc}^{(1)}|^2 \rangle = |E_o|^2 \left(\frac{\omega^2}{c^2}|\Delta\chi|\right)^2 \left(\frac{1}{4\pi r}\right)^2 NF(\theta)(1+\cos^2\theta) \tag{2.2.15b}$$

Expressions 2.2.15 have been derived under the assumption of a circularly polarized incident wave. Here V is the volume occupied by the particles, D is a linear dimension of V, F_1 is a function of ka of the order V_p^2 and K_1, K_2 are given by certain integral expressions which will be later derived. One can see almost by inspection but numerical results for the special case of a collection of spheres also confirm that K_1, K_2 are approximately one order greater than V_p^2 if ka is not too large. The maximum value of $F(\theta)$ is V_p^2.

If self-interaction is to be neglected then all multiple scattering terms should be greater than the dominant self-interaction contribution. From 2.2.14 and 2.2.15a we get

$$\left(\frac{\omega^2}{c^2}|\Delta\chi|\right)^{2n-3}\left(\frac{N}{V}\right)^{n-1}\left(\frac{D}{32\pi^2}\right)^{n-1}(10\,V_p^2)^{n-1}a^{-2} >> 1 \qquad (2.2.16)$$

On the other hand the series $\langle |\underline{E}_{sc}|^2\rangle = \sum_n \langle |\underline{E}_{sc}^{(n)}|^2\rangle$ must converge rapidly, i.e. we must have $\langle |\underline{E}_{sc}^{(n-1)}|^2\rangle / \langle |\underline{E}_{sc}^{(n)}|^2\rangle << 1$. To achieve the corresponding conditions we have to notice the following fact. $F(\theta)$ varies significantly with θ only when ka is relatively large. In general $F(\theta)$ is obtained through an averaging procedure and therefore no zeros exist. For the special case of a collection of spheres no averaging takes place and $F(\theta)$ varies significantly even if ka is not large, i.e. ka ≈ 1. On the other hand $F(\theta)$ has a number of zeros provided $2\,\text{ka} > 4.5$. When $F(\theta) = 0$ then $\langle |\underline{E}_{sc}^{(1)}|^2\rangle$ is zero,but the multiple scattering is non-zero, i.e. $\langle |\underline{E}_{sc}^{(1)}|^2\rangle$ is greater than $\langle |\underline{E}_{sc}^{(1)}|^2\rangle$. In such a case a condition of convergence should really involve terms with $n \geq 2$. One should therefore include the condition $\langle |\underline{E}_{sc}^{(2)}|^2\rangle / \langle |\underline{E}_{sc}^{(1)}|^2\rangle \ll 1$ only when it makes sense.

As we said before in general $F(\theta)$ has no zeros and unless we choose ka large $F(\theta)$ does not vary significantly. For estimation purposes we can write $F(\theta) \lesssim V_p^2$ and demand

$$\left(\frac{\omega^2}{c^2}|\Delta\chi|\right)^2 \frac{N}{V} \frac{D}{32\pi^2}(10\, V_p^2) \ll 1 \tag{2.2.17}$$

as we can easily get from 2.2.15a and 2.2.15b.

For the ratios with $n \geq 2$ we can easily find that 2.2.17 must be satisfied.

We can immediately see that 2.2.16 is "hostile" to 2.2.17. Thus 2.2.16 requires high frequencies, high number density whereas 2.2.17 requires exactly the opposite. The anomaly gets worse as n increases. Usually however the rapid convergence of the series $\langle |\underline{E}_{sc}|^2\rangle = \sum_n \langle |\underline{E}_{sc}^{(n)}|^2\rangle$ guarantees that even a few terms will give an accurate scattering intensity.

We can summarize the previous discussion as follows. Condition 2.2.16 requires that

$$\langle |\underline{E}_{sc}^{(n)}|^2\rangle \gg N\, 2\, \mathrm{Re}\left(\underline{E}_{sc}^{(1)}\underline{E}_{sc}^{(2)*}\right)_{\substack{\text{single}\\ \text{particle}}}$$

whereas 2.2.17 requires

$$\langle |\underline{E}_{sc}^{(1)}|^2\rangle \gg \langle |\underline{E}_{sc}^{(2)}|^2\rangle \gg \ldots \gg \langle |\underline{E}_{sc}^{(n)}|^2\rangle$$

We can combine both in one:

$$\langle |\underline{E}_{sc}^{(1)}|^2\rangle >> \langle |\underline{E}_{sc}^{(2)}|^2\rangle >> \ldots >> \langle |\underline{E}_{sc}^{(n)}|^2\rangle >> N\, 2\,\mathrm{Re}\left(\underline{E}_{sc}^{(1)}\underline{E}_{sc}^{(2)*}\right)_{\substack{\text{single}\\ \text{particle}}}$$

We thus see that $\langle |\underline{E}_{sc}^{(1)}|^2\rangle >> N\, 2\,\mathrm{Re}\left(\underline{E}_{sc}^{(1)}\underline{E}_{sc}^{(2)*}\right)_{\substack{\text{single}\\ \text{particle}}}$ or if we recall that $\langle |\underline{E}_{sc}^{(1)}|^2\rangle = N|\underline{E}_{sc}^{(1)}|^2_{\substack{\text{single}\\ \text{particle}}}$ we get $\left(|\underline{E}_{sc}^{(1)}|^2\right)_{s.p.} >> 2\,\mathrm{Re}\left(\underline{E}_{sc}^{(1)}\underline{E}_{sc}^{(2)*}\right)_{s.p.}$

or $|\underline{E}_{sc}^{(1)}|_{s.p.} >> |\underline{E}_{sc}^{(2)}|_{s.p.}$ This is, however, condition 2.1.10. Thus 2.1.10 is compatible with the pair 2.2.16 and 2.2.17. As a matter of fact we can immediately get 2.1.10 if we write 2.2.16 for $n = 2$ and combine it with 2.2.17:

$$\frac{\omega^2}{c^2}\,|\Delta\chi\,|\,\frac{N}{V}\,\frac{D}{32\pi^2}\,10\,V_p^2\,a^{-2} >> 1 >> \left(\frac{\omega^2}{c^2}\,|\Delta\chi\,|\right)^2\frac{N}{V}\,\frac{D}{32\pi^2}\,10\,V_p^2$$

$$\therefore\quad \frac{\omega^2}{c^2}\,|\Delta\chi\,|\,a^2 << 1\;.$$

To see how 2.2.16 and 2.2.17 work we transform them as follows: The number density $\frac{N}{V}$ can be expressed as $\frac{1}{(a+d)^3}$ where d is an average closest distance between two neighboring particles. This is so because N is approximately equal to $\frac{N}{(a+d)^3}$. Here we should notice that 2.2.15 has been derived under the assumption that the particles are of such size and so far apart that they only see the far zone field of any particle. This means that $r >> r'$ and $kr >> 1$ in the expression for $\underline{\underline{\Gamma}}(\underline{r};\underline{r}')$ (see Appendix A). Now $kr = \frac{2\pi r}{\lambda} = \frac{2\pi r}{\lambda_o}\, n_m$ where n_m is the index of refraction of the medium. Therefore we have the following conditions:

$$r >> \frac{\lambda_o}{2\pi n_m} \qquad\qquad r >> r'$$

or

$$a + d >> \frac{\lambda_o}{2\pi n_m} \quad \text{and} \quad a + d >> a$$

If we call $a + d = ma$ then we must require that $m >> 1$ and $m >> \frac{\lambda_o}{2\pi n_m a}$. If $\frac{\lambda_o}{n_m} > 2\pi a$ then $m >> \frac{\lambda_o}{2\pi n_m a}$ implies $m >> 1$, i.e. $m >> \frac{\lambda_o}{2\pi a n_m}$ suffices. If however $\frac{\lambda_o}{n_m} < 2\pi a$ $m >> 1$ does.

Consider now 2.2.16 for $n = 2$. We will therefore assume that the self-interaction contribution is smaller than the 2nd order multiple scattering term which is also greater than the 3rd order term in 2.2.12. Thus in the present case we will neglect multiple scattering terms higher than the second and also self-interaction terms. If we assume the ratio of two successive terms in 2.2.12 equal to 10 and the ratio of the 2nd order multiple scattering intensity to the dominant self-interaction term in 2.2.11 also equal to 10 then we make a mistake of the order of 1%. Under the previous assumptions 2.2.16 and 2.2.17 give:

$$\frac{\omega^2}{c^2} |\Delta\chi| \frac{N}{V} \frac{D}{32\pi^2} 10 V_p^2 a^{-2} > 10 \tag{2.2.16a}$$

$$\left(\frac{\omega^2}{c^2} |\Delta\chi| \right)^2 \frac{N}{V} \frac{D}{32\pi^2} 10 V_p^2 < \frac{1}{10} \tag{2.2.17a}$$

If we take into account that $V_p \approx a^3$, $\frac{N}{V} \approx \frac{1}{m^3 a^3}$ our conditions are transformed into the following:

$$|\Delta\chi| \frac{\omega^2}{c^2} \frac{D}{32\pi^2} \frac{a}{m^3} \geq 1$$

$$|\Delta \chi|^2 \left(\frac{\omega^2}{c^2}\right)^2 \frac{D}{32\pi^2} \frac{a^3}{m^3} \leq \frac{1}{100}$$

or

$$|\Delta \chi| \frac{\omega^2}{c^2} \geq \left(\frac{D}{32\pi^2}\right)^{-1} m^3 a^{-1} \tag{2.2.16b}$$

$$|\Delta \chi| \frac{\omega^2}{c^2} \leq \frac{1}{10} \left(\frac{32\ ^2}{D}\right)^{1/2} a^{-3/2} m^{3/2} \tag{2.2.17b}$$

We now understand that if $A \geq B$ and $A \leq C$ then $C \geq B$, i.e. from 2.2.16b and 2.2.17b:

$$m^3 a \leq 10^{-2} \frac{D}{32\pi^2} = 3 \times 10^{-5} D \tag{2.2.18}$$

i) Assume $2\pi a > \lambda_o/n_m$, i.e. $a > \lambda_o/2\pi n_m$. Then we must choose $m \gg 1$. If we set $m = 15$ then

$$a \leq 9 \times 10^{-9} D \qquad (m = 15) \tag{2.2.19}$$

We usually require λ_o to be in the visible range, i.e.

$$\lambda_o = 4 \times 10^{-5} \text{ cm} \ldots 7 \times 10^{-5} \text{ cm}$$

If we write $a = n\lambda_o/n_m$ where n is some number greater than $1/2\pi$, then D must satisfy the following inequality:

$$D \geq \frac{n}{n_m} \left(\frac{4}{9} \times 10^4 \text{ cm} \ldots \frac{7}{9} \times 10^4 \text{ cm}\right) \quad \left(n > \frac{1}{2\pi},\quad m = 15\right)$$

If for example we choose $n/n_m = 1/4$, i.e. $a/\lambda_o = 1/4$ then

$$D \geq (\tfrac{1}{9} \times 10^4 \text{ cm} \ldots \tfrac{7}{36} \times 10^4 \text{ cm}) \qquad (m = 15)$$

or if $n/n_m = 5$, i.e. $a/\lambda_o = 5$ then

$$D \geq (\tfrac{2}{9} \times 10^5 \text{ cm} \ldots 4 \times 10^5 \text{ cm}) \qquad (m = 15)$$

If we do not specify λ_o we can easily get:

$$D \geq \tfrac{a}{9}\, 10^6 = p(ka)\, \frac{\lambda_o}{18\pi n_m}\, 10^9 \qquad (p \geq 1) \tag{2.2.21a}$$

Thus if λ_o is constant D increases as ka or a/λ increases.

ii) Next assume $2\pi a < \lambda = \lambda_o/n_m$. Then we should have $m >> \lambda_o/2\pi a n_m$. To comply with case i) we choose m such that $ma = 15/k$ or $m = 15/ka$. We can now easily get

$$D \geq m^3 a\, \frac{32\pi^2}{10^{-2}} \quad \text{or} \quad D = p\, \frac{1.7 \times 10^7}{(ka)^2}\, \frac{\lambda_o}{n_m} \tag{2.2.21b}$$

Thus if $a/\lambda_o = 1/10$ and $\lambda_o = 4 \times 10^{-5}$ with $n_m \approx 1$ we get

$$D \geq 1.7 \times 10^3 \text{ cm} = 17 \text{ m}$$

We observe from 2.2.21b that for λ_o constant D increases as ka or a/λ decreases.

This seems paradoxical since everyone knows that as a/λ gets very small the self-interaction contribution becomes negligible and therefore "the multiple scattering should dominate." However, one should be careful enough to notice that the self-interaction goes like

$(a/\lambda)^6$ whereas the 2nd order multiple scattering like $(a/\lambda)^8$, i.e. goes to zero faster. Thus D has to increase to fortify the multiple scattering since N/V is constant for the case $a < \lambda/2\pi$. We thus conclude that the minimum D for constant λ corresponds to $ka = 1$.

Once we specify a and $\lambda_o = 2\pi c/\omega$ or better their ratio, as we show below, we can immediately find the range of $|\Delta\chi|$ for which conditions 2.2.16 and 2.2.17 are satisfied. We start from 2.2.18, i.e.

$$\frac{D}{32\pi^2} = p\, 10^2 m^3 a \qquad (p \geq 1)$$

and conditions 2.2.16b and 2.2.17b give:

$$|\Delta\chi| \geq \left(\frac{\lambda_o}{2\pi}\right)^2 \frac{1}{p}\, 10^{-2} \frac{1}{m^3 a} m^3 a^{-1} = \frac{10^{-2}}{p} \frac{n_m^2}{(ka)^2}$$

$$|\Delta\chi| \leq \left(\frac{\lambda_o}{2\pi}\right)^2 \frac{1}{p^{1/2}} 10^{-2} \left(\frac{1}{m^3 a}\right)^{1/2} a^{-3/2} m^{3/2} = \frac{10^{-2}}{p^{1/2}} \frac{n_m^2}{(ka)^2}$$

i.e.

$$\frac{10^{-2}}{p} \frac{n_m^2}{(ka)^2} \leq |\Delta\chi| \leq \frac{10^{-2}}{p^{1/2}} \frac{n_m^2}{(ka)^2} \tag{2.2.22}$$

If ka is large $|\Delta\chi|$ has to be small whereas a small ka can make $|\Delta\chi|$ of the order unity or larger. Is 2.2.22 the final range of $|\Delta\chi|$? How about condition $\frac{\omega^2}{c^2}|\Delta\chi| a^2 \ll 1$ or $|\Delta\chi| \ll n_m^2/(ka)^2$?

We have already shown that this condition is compatible with 2.2.16 and 2.2.17. As a matter of fact if we rewrite 2.2.16a and

2.2.17a as follows:

$$\frac{\omega^2}{c^2} |\Delta\chi| \frac{N}{V} \frac{D}{32\pi^2} V_p^2 a^{-2} \geq 1 \geq \left(\frac{\omega^2}{c^2} |\Delta\chi|\right)^2 \frac{N}{V} \frac{D}{32\pi^2} 10^2 V_p^2$$

we understand that

$$\frac{\omega^2}{c^2} |\Delta\chi| a^2 \leq \frac{1}{100}$$

or

$$|\Delta\chi| \leq \frac{n_m^2}{(ka)^2} 10^{-2} \qquad (2.2.23)$$

If we recall that $p > 1$ we understand that 2.2.22 is the range for $|\Delta\chi|$ that satisfies all our conditions.

Next we examine the range of applicability of the theory if 2.2.16 with $n = 3$ holds. Then neglecting self-interaction and higher terms than the third in 2.2.12 introduces a mistake of the order of 1‰ or less. Remembering that $N/V \approx (ma)^{-3}$, 2.2.16 with $n = 3$ and 2.2.17 can be written as

$$|\Delta\chi| \frac{\omega^2}{c^2} \geq (10)^{-1/3} \left(\frac{D}{32\pi^2}\right)^{-2/3} \frac{m^2}{a^{4/3}} \qquad (2.2.16c)$$

$$|\Delta\chi| \frac{\omega^2}{c^2} \leq \left(\frac{1}{100}\right)^{1/2} \left(\frac{D}{32\pi^2}\right)^{-1/2} \frac{m^{3/2}}{a^{3/2}} \qquad (2.2.17c)$$

Again we can easily see that

$$\left(\frac{1}{100}\right)^{1/2} \left(\frac{D}{32\pi^2}\right)^{-1/2} \frac{m^{3/2}}{a^{3/2}} \geq (10)^{-1/3} \left(\frac{D}{32\pi^2}\right)^{-2/3} \frac{m^2}{a^{4/3}}$$

or

$$m^3 a \leq 10^{-4} \frac{D}{32\pi^2} \tag{2.2.24}$$

We can immediately see that the choice of a, λ_o will be the same as before (when $n = 2$ in 2.2.16 was chosen) provided D is two orders of magnitude bigger. Therefore, unless very small wavelengths are used, D is too big and the third order is not likely to be practically useful.

The range of $|\Delta\chi|$ is found easily if we set $D/32\pi^2 = pm^3 a \times 10^4$ $(p \geq 1)$, i.e.

$$|\Delta\chi| \geq \frac{10^{-3}}{p^{2/3}} \frac{n_m^2}{(ka)^2}$$

$$|\Delta\chi| \leq \frac{10^{-3}}{p^{1/2}} \frac{n_m^2}{(ka)^2}$$

or

$$\frac{10^{-3}}{p^{2/3}} \frac{n_m^2}{(ka)^2} \leq |\Delta\chi| \leq \frac{10^{-3}}{p^{1/2}} \frac{n_m^2}{(ka)^2} \qquad (p \geq 1) \tag{2.2.25}$$

Thus if the third order is taken into account D has to get larger and $|\Delta\chi|$ has to become smaller than the corresponding quantities in second order.

Do we have to worry about condition $|\Delta\chi| << n_m^2/(ka)^2$? No! because now 2.2.16a should read $\frac{\omega^2}{c^2} |\Delta\chi| \frac{N}{V} \frac{D}{32\pi^2} 10 V_p^2 a^{-2} > 100$ and if it is combined with 2.2.17a we can easily get $|\Delta\chi| \leq 10^{-3} n_m^2/(ka)^2$. Thus 2.2.25 is O.K.

III. FIRST ORDER SCATTERING

3.1. Intensity of the Scattered Wave

We will assume that the particles in general have different shapes, random positions and random orientations. They can have different susceptibilities but of the same order of magnitude. Later for the sake of obtaining a simple form for the intensity of the scattered wave we will assume that our particles have the same shape, dimensions and susceptibilities.

To each particle we attach a triad which will be characterized by three Eulerian angles (see Appendix B) w.r.t. a fixed system of orthogonal cartesian coordinates with the z-axis along the wave vector $\underline{k}$ of the incident wave (see figure 2). The Eulerian angles give the orientation of a particle and will be treated as random variables. We want to find the far zone scattered field at $\underline{r}$ characterized by r, θ, φ w.r.t. the fixed system x, y, z. The expression for the far zone $\underline{\underline{\Gamma}}$ is (see Appendix A)

$$\underline{\underline{\Gamma}} = (\underline{\underline{u}} - \underline{e}_r \underline{e}_r) \frac{e^{ikr}}{4\pi r} e^{-ik\underline{e}_r \cdot \underline{r}'} \quad \text{where } \underline{e}_r = \frac{\underline{r}}{r}$$

Therefore we can write for the far zone field given by 2.2.5

$$\underline{E}_{sc}^{(1)i}(\underline{r}) = \frac{\omega^2 \Delta\chi_i}{c^2} (\underline{\underline{u}} - \underline{e}_r \underline{e}_r) \frac{e^{ikr}}{4\pi r} \int_{V_i} e^{-ik\underline{e}_r \cdot \underline{r}_i} \cdot \underline{E}_{inc}(\underline{r}_i) \, dV_i \qquad (3.1.1)$$

Now $\underline{E}_{inc} = \underline{E}_o e^{ikz}$ where $\underline{E}_o$ has the general form $E_x e^{-i\delta_x} \underline{e}_x +$ $E_y e^{-i\delta_y} \underline{e}_y$, i.e. $\underline{E}_{inc}$ is in general elliptically polarized.

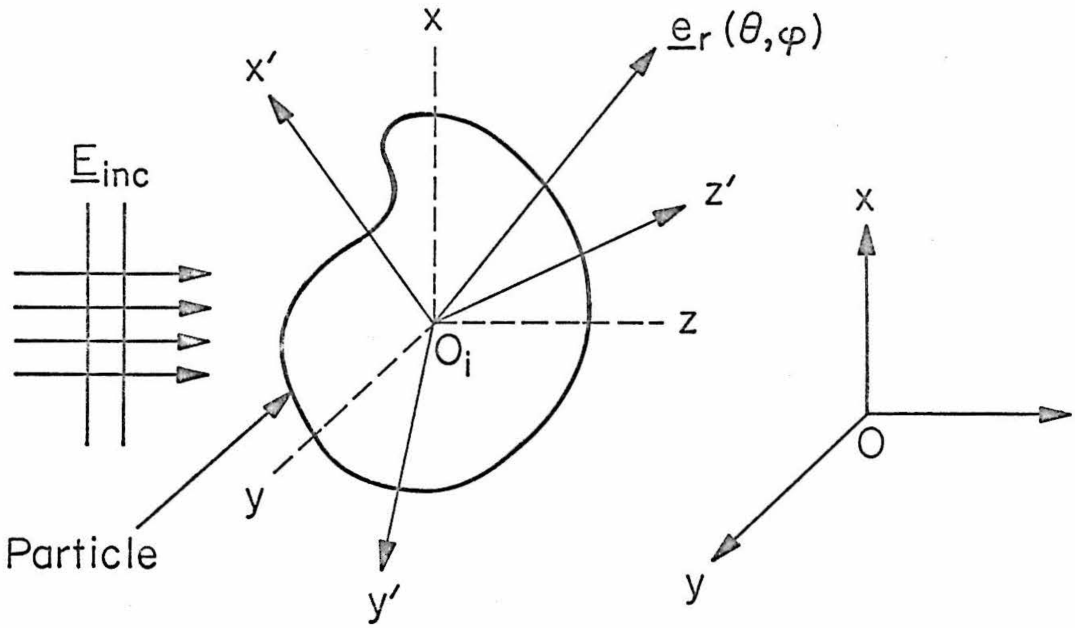


Fig. 2. x'y'z' is a triad attached to the dielectric particle. xyz is the fixed coordinate frame. $\underline{e}_r$ is a unit vector pointing in the direction (θ, φ).

3.1.1 can now be rewritten as:

$$\underline{E}_{sc}^{(1)i}(\underline{r}) = -\frac{\omega^2 \Delta\chi_i}{c^2}(\underline{e}_r \times \underline{e}_r \times \underline{E}_o)\frac{e^{ikr}}{4\pi r}\int_{V_i} e^{-ik\underline{e}_r \cdot \underline{r}_i} e^{ikz_i} \, dV_i \quad (3.1.2)$$

since $\underline{e}_r \times (\underline{e}_r \times \underline{E}_o) = \underline{e}_r \underline{e}_r \cdot \underline{E}_o - (\underline{e}_r \cdot \underline{e}_r)\underline{E}_o = (\underline{e}_r \underline{e}_r - \underline{\underline{u}}) \cdot \underline{E}_o$. Notice that $\underline{e}_r \cdot \underline{E}_{sc}^{(1)i}(\underline{r}) = 0$ as it should. To take into account the randomness in position of the particles we split $\underline{r}_i$ as follows (figure 3)

$$\underline{r}_i = \underline{r}_{io} + \underline{R}_i \quad (3.1.3)$$

Thus we have

$$\underline{r}_i \cdot \underline{e}_r = \underline{r}_{io} \cdot \underline{e}_r + \underline{R}_i \cdot \underline{e}_r \quad (3.1.4)$$

and

$$z_i = \underline{r}_i \cdot \underline{e}_z = z_{io} + Z_i \quad (3.1.5)$$

Substituting 3.1.4 and 3.1.5 into 3.1.2 we get

$$\underline{E}_{sc}^{(1)i}(\underline{r}) = -\frac{\omega^2 \Delta\chi_i}{c^2}(\underline{e}_r \times \underline{e}_r \times \underline{E}_o)\frac{e^{ikr}}{4\pi r} e^{-ik\underline{e}_r \cdot \underline{r}_{io}} e^{ikz_{io}}$$

$$\times \int_{V_i} e^{-ik\underline{e}_r \cdot \underline{R}_i} e^{ikZ_i} \, dV_i \quad (3.1.6)$$

We will temporarily drop the index i. Now we want to evaluate $\underline{e}_r \cdot \underline{R}$ and $\underline{e}_z \cdot \underline{R} = Z$ in terms of the Eulerian angles α, β, γ the polar angles θ, φ and the coordinates characterizing the shape of the body. If we transfer the origin of the fixed system x,y,z and make it coincide

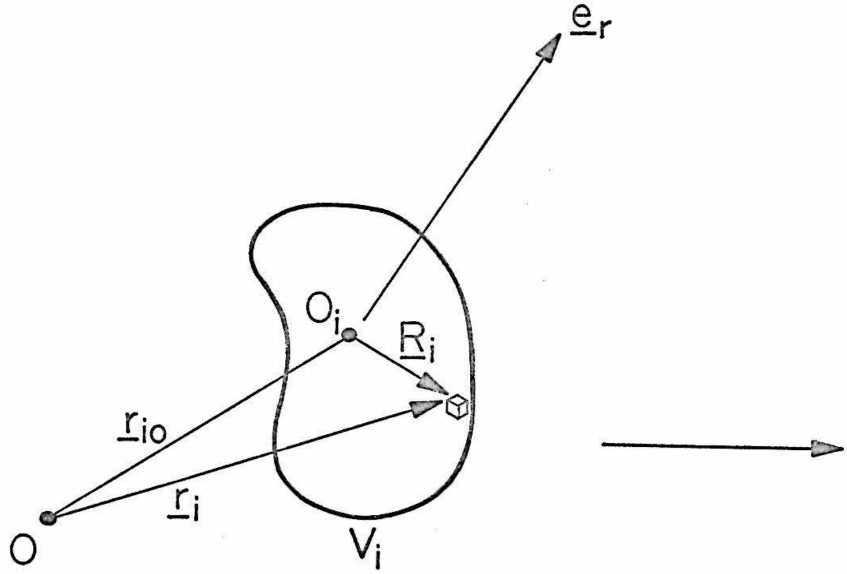


Fig. 3. The splitting of $\underline{r}_i = \underline{r}_{io} + \underline{R}_i$. $\underline{r}_{io}$ characterizes the random position of a dielectric particle, whereas $\underline{R}_i$ characterizes the random orientation.

with O_i the center of the ith particle we can view the $x'y'z'$ system as one obtained from xyz by an appropriate rotation, i.e.

$$R_i = (M^{-1})_{ij} R_j' \qquad (i,j = 1,2,3; \text{ repeated indices are summed})$$

where i,j are indices signifying the cartesian components of $\underline{R} = R_i \underline{e}_i = R_i' \underline{e}_i'$, and M^{-1} is the inverse rotation matrix given by:

$$M^{-1} = \begin{bmatrix} \cos\gamma\cos\beta - \cos\alpha\sin\beta\sin\gamma & -\sin\gamma\cos\beta - \cos\alpha\sin\beta\cos\gamma & \sin\alpha - \sin\beta \\ \cos\gamma\sin\beta + \cos\alpha\cos\beta\sin\gamma & -\sin\gamma\sin\beta + \cos\alpha\cos\beta\cos\gamma & -\sin\alpha\cos\beta \\ \sin\alpha\sin\gamma & \sin\alpha\cos\gamma & \cos\alpha \end{bmatrix}$$

We now write:

$$\underline{e}_r \cdot \underline{R} = (\underline{e}_r)_i R_i = (\underline{e}_r)_i (M^{-1})_{ij} x_j'$$

and

$$\underline{e}_z \cdot \underline{R} = Z = x_3 = (M^{-1})_{3j} x_j'$$

Now

$$(\underline{e}_r)_x = (\underline{e}_r)_1 = \cos\varphi \sin\theta$$

$$(\underline{e}_r)_y = (\underline{e}_r)_2 = \sin\varphi \sin\theta$$

$$(\underline{e}_r)_z = (\underline{e}_r)_3 = \cos\theta$$

Using the above results and 3.1.6 we get the total scattered field:

$$\underline{E}^{(1)}_{sc}(\underline{r}) = \sum_i -\frac{\omega^2}{c^2}\Delta\chi_i(\underline{e}_r\times\underline{e}_r\times\underline{E}_o)\frac{e^{ikr}}{4\pi r}\exp(-ik\underline{e}_r\cdot\underline{r}_{io}+ikz_{io})$$

$$\int_{V_i}\Big[\exp\Big(-ik(\underline{e}_r)_\ell(\theta,\varphi)(M^{-1})_{\ell n}(\alpha_i,\beta_i,\gamma_i)x'_n$$

$$+ ik(M^{-1})_{3n}(\alpha_i,\beta_i,\gamma_i)x'_n\Big)\Big]\,dx'_1\,dx'_2\,dx'_3 \qquad (3.1.7)$$

To simplify 3.1.7 we define

$$\int_{V_i}\Big\{\exp\Big(-ik(M^{-1})_{\ell n}(\alpha_i,\beta_i,\gamma_i)[(\underline{e}_r)_\ell(\theta,\varphi)-\delta_{\ell 3}]x'_n\Big)\Big\}dx'_1\,dx'_2\,dx'_3$$

$$\equiv K_i(\alpha_i,\beta_i,\gamma_i;\theta,\varphi) \qquad (3.1.8)$$

The time-average radiated power per unit solid angle is given by:

$$I = \frac{dP}{d\Omega} = r^2\frac{1}{2}\,\mathrm{Re}\,(\underline{E}\times\underline{H}^*\cdot\underline{e}_r)$$

In the far zone of a localized source we have:

$$E = \sqrt{\frac{\mu_o}{\epsilon_m}}\,(\underline{H}\times\underline{e}_r) \quad \text{and} \quad \underline{H} = \sqrt{\frac{\epsilon_m}{\mu_o}}\,(\underline{e}_r\times\underline{E})$$

therefore

$$\underline{E}\times\underline{H}^* = \underline{E}\times\sqrt{\frac{\epsilon_m^*}{\mu_o}}\,(\underline{e}_r\times\underline{E}^*) = \sqrt{\frac{\epsilon_m^*}{\mu_o}}\Big\{|\underline{E}|^2\underline{e}_r - (\underline{e}_r\cdot\underline{E})\underline{E}^*\Big\}$$

$$= \sqrt{\frac{\epsilon_m^*}{\mu_o}}\,|\underline{E}|^2\underline{e}_r$$

$$I = \frac{dP}{d\Omega} = r^2\frac{1}{2}\,\mathrm{Re}\sqrt{\frac{\epsilon_m^*}{\mu_o}}\,|\underline{E}|^2 \qquad (3.1.9)$$

If we substitute 3.1.7 into 3.1.9 we get

$$I \propto |e^{ikr}|^2 |\underline{e}_r \times \underline{e}_r \times \underline{E}_o|^2$$

$$\times \left| \sum_i \frac{\omega^2}{c^2} \Delta \chi_i \exp(-ik\underline{e}_r \cdot \underline{r}_{io} + ikz_{io}) K_i(\alpha_i, \beta_i, \gamma_i; \theta, \varphi) \right|^2 \qquad (3.1.10)$$

To make the computation of 3.1.10 easy to handle we assume that all the particles have the same shape, same size, and same susceptibility. Then we have:

$$K_i(\alpha_i, \beta_i, \gamma_i; \theta, \varphi) = K(\alpha_i, \beta_i, \gamma_i; \theta, \varphi)$$

i.e. we drop the index i from K because the functional form will be the same for any particle if all have the same shape and size.

Next we write $\underline{k} = \underline{k}_r + ik_{im}$ to take into account the losses of the medium. If we now call $-k_r \underline{e}_r \cdot \underline{r}_{io} + k_r z_{io} = \varphi_i$, 3.1.10 becomes

$$I \propto e^{-2k_{im} r} |\underline{e}_r \times \underline{e}_r \times \underline{E}_o|^2 \left(\frac{\omega^2}{c^2} |\Delta\chi| \right)^2$$

$$\times \left| \sum_i e^{i\varphi_i} \exp\left(k_{im}(\underline{e}_r - \underline{e}_z) \cdot \underline{r}_{io}\right) K(\alpha_i, \beta_i, \gamma_i; \theta, \varphi) \right|^2 \qquad (3.1.11)$$

We will now treat φ_i as a random variable (due to randomness of the positions of the particles) which is equally likely to be found everywhere between 0 and 2π. Under this assumption it is shown in Appendix D that the average value of the absolute square of the sum is given by:

$$\left\langle \left| \sum_i e^{i\varphi_i} \exp\left(k_{im}(\underline{e}_r - \underline{e}_z)\cdot \underline{r}_{io}\right) K(\alpha_i,\beta_i,\gamma_i;\theta,\varphi) \right|^2 \right\rangle$$

$$= N \frac{1}{V}\int_V \exp\left(2k_{im}(\underline{e}_r - \underline{e}_z)\cdot \underline{a}\right) dV$$

$$\times \frac{1}{8\pi^2}\int_{\gamma=0}^{2\pi}\int_{\beta=0}^{2\pi}\int_{\alpha=0}^{\pi} |K(\alpha,\beta,\gamma;\theta,\varphi)|^2 \sin\alpha\, d\alpha\, d\beta\, d\gamma \qquad (3.1.12)$$

where $\underline{a}$ is the radius vector from the origin to any point, N the number of particles and V the volume occupied by the particles.

To estimate the importance of the losses we evaluate $J = \frac{1}{V}\int_V \exp\left(k_{im}(\underline{e}_r - \underline{e}_z)\cdot \underline{a}\right) dV$ for an orthogonal parallelopiped L_x, L_y, L_z.

$$J = \frac{1}{V}\int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} e^{2k_{im}e_{rx}x}\,dx \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} e^{2k_{im}e_{ry}y}\,dy \int_{-\frac{L_z}{2}}^{\frac{L_z}{2}} e^{2k_{im}(e_{rz}-1)z}\,dz$$

$$= \frac{1}{V}\,\frac{1}{k_{im}^3 e_{rx} e_{ry}(e_{rz}-1)} \sinh(k_{im}L_x e_{rx}) \sinh(k_{im}L_y e_{ry})$$

$$\times \sinh[k_{im}L_z(e_{rz}-1)] \qquad (3.1.13)$$

Notice that for $\theta = 0$, i.e. forward scattering, $J = 1$. If we expand $\sinh x = x + \frac{x^3}{3!} + \ldots$ we can easily show that 3.1.13 gives:

$$J = 1 + O(Lk_{im})^2 \qquad (3.1.14)$$

where L is a linear dimension of V.

Thus J is independent of the losses, if $(k_{im}L)^2 << 1$. For example k_{im} for water can be as low as $\frac{1}{20}\ m^{-1}$, therefore a region V with $L \leq 2\,m$ will make $(k_{im}L)^2 \doteq 1\%$ and $J = 1$ with an error of 1%. However, our theory does not allow L to get so small if the wavelength is in the visible range. Thus if $\lambda_o = (4\times 10^{-5} \ldots 7\times 10^{-5})$ cm and $n_m = 1.33$ then the min L is obtained for $ka = 1$, i.e. (see 2.2.21a)

$$\min L = \frac{\lambda_o}{18\,\pi n_m} \times 10^9 = 5.3\ m \ldots 9.3\ m$$

If for example $\min L = 5.3$ m then $(k_{im}L)^2 = 7\%$ and we make an error larger than the accuracy of the problem if losses are neglected. The use of smaller wavelengths can reduce L, also $k_{im} = k_{im}(\lambda)$ and then losses can be neglected. If the medium is not too lossy, i.e. $k_{im} \lesssim \frac{1}{100}\ m^{-1}$ then we require $L \gtrsim 10$ m and losses can be neglected for visible wavelengths and ka of the order unity or larger.

We have not worried about the effect of losses on the integral over α, β, γ for the following reason. If we do the integration we will find a function of θ, φ and ka. Now $k_{im}a$ will in general be much smaller than unity since a is about the same order as the wavelength. For example if the medium is water $k_{im} \approx 1 \div \frac{1}{20}\ m^{-1}$ and for the largest k_{im}, $k_{im}a \sim 10^{-7}$ if λ is in the visible range, whereas $k_r a \sim 1$. If losses are taken into account we can easily see that they tend to reduce the forward scattering.

Suppose now that we neglect the losses. Then from 3.1.11 and 3.1.12 we get:

$$\langle I \rangle = \langle I^{(1)} \rangle \propto N\left(\frac{\omega^2}{c^2}\,|\Delta\chi|\right)^2 |\underline{e}_r \times \underline{e}_r \times \underline{E}_o|^2 F(\theta) \tag{3.1.13}$$

where

$$F(\theta) = \frac{1}{8\pi^2}\int_{\gamma=0}^{2\pi}\int_{\beta=0}^{2\pi}\int_{\alpha=0}^{\pi} |K(\alpha,\beta,\gamma;\theta,\varphi)|^2 \sin\alpha \, d\alpha \, d\beta \, d\gamma \tag{3.1.14}$$

We have written $F(\theta)$ and not $F(\theta,\varphi)$ because the averaging procedure will eliminate the φ dependence no matter what the shape of the particles is, provided there are no losses.

Next we compute $|\underline{e}_r \times \underline{e}_r \times \underline{E}_o|^2$ for an elliptically polarized incident wave. We have

$$-\underline{e}_r \times \underline{e}_r \times \underline{E}_o = \text{component of } \underline{E}_o \text{ perpendicular to } \underline{e}_r$$

$$= (\underline{E}_o \cdot \underline{e}_\varphi)\underline{e}_\varphi + (\underline{E}_o \cdot \underline{e}_\theta)\underline{e}_\theta$$

Now $\underline{E}_o = E_x e^{-i\delta_x}\underline{e}_x + E_y e^{-i\delta_y}\underline{e}_y$. We know that

$$\underline{e}_x \cdot \underline{e}_\varphi = -\sin\varphi \qquad \underline{e}_y \cdot \underline{e}_\varphi = \cos\varphi$$

$$\underline{e}_x \cdot \underline{e}_\theta = \cos\theta\,\cos\varphi \qquad \underline{e}_y \cdot \underline{e}_\theta = \cos\theta\,\sin\varphi$$

Thus

$$-\underline{e}_r \times \underline{e}_r \times \underline{E}_o = (-E_x e^{-i\delta_x}\sin\varphi + E_y e^{-i\delta_y}\cos\varphi)\underline{e}_\varphi$$

$$+ (E_x \cos\theta\,\cos\varphi\, e^{-i\delta_x} + E_y e^{-i\delta_y}\cos\theta\,\sin\varphi)\underline{e}_\theta \tag{3.1.15}$$

and

$$|\underline{e}_r \times \underline{e}_r \times \underline{E}_o|^2$$

$$= |-E_x e^{-i\delta_x} \sin\varphi + E_y e^{-i\delta_y} \cos\varphi|^2$$

$$+ |E_x \cos\theta \cos\varphi\, e^{-i\delta_x} + E_y \cos\theta \sin\varphi\, e^{-i\delta_y}|^2$$

$$= |-E_x e^{-i(\delta_x - \delta_y)} \sin\varphi + E_y \cos\varphi|^2$$

$$+ \cos^2\theta\, |E_x e^{-i(\delta_x - \delta_y)} \cos\varphi + E_y \sin\varphi|^2$$

$$= (E_y \cos\varphi - E_x \cos(\delta_x - \delta_y) \sin\varphi)^2 + \cos^2\theta (E_y \sin\varphi + E_x \cos(\delta_x - \delta_y) \cos\varphi)^2$$

$$+ E_x^2 \sin^2(\delta_x - \delta_y) \sin^2\varphi + \cos^2\theta\, E_x^2 \sin^2(\delta_x - \delta_y) \cos^2\varphi$$

Two special cases are of interest

i) $\underline{E}_{inc}$ linearly polarized $\underline{E}_{inc} = E_o e^{-i\delta_x} \underline{e}_x$

ii) $\underline{E}_{inc}$ circularly polarized $\underline{E}_{inc} = E_o' e^{-i\delta_x} (\underline{e}_x \pm i\underline{e}_y)$.

The plus corresponds to a right-handed and the minus to a left-handed wave.

i) $$|\underline{e}_r \times \underline{e}_r \times \underline{E}_o|^2 = E_o^2 (\sin^2\varphi + \cos^2\theta \cos^2\varphi) \tag{3.1.16}$$

The scattered power per unit solid angle is then

$$\langle I^{(1)} \rangle \propto N E_o^2 \left(\frac{\omega^2}{c^2} |\Delta\chi| \right)^2 (\sin^2\varphi + \cos^2\theta \cos^2\varphi) F(\theta) \tag{3.1.17}$$

and the intensity pattern of the scattered wave is

$$\text{I.P.} \propto (\sin^2\varphi + \cos^2\theta \cos^2\varphi) F(\theta)$$

ii) $$|\underline{e}_r \times \underline{e}_r \times \underline{E}_o|^2 = E_o'^2 \, |e^{\pm i\pi/2} \sin\varphi + \cos\varphi|^2$$

$$+ E_o'^2 \cos^2\theta \, |e^{\pm i\pi/2} \cos\varphi + \sin\varphi|^2$$

$$= E_o'^2 \, |e^{\pm i\varphi}|^2 + E_o'^2 \cos^2\theta \, |e^{\pm i\pi/2} e^{\mp i\varphi}|^2$$

$$= (1 + \cos^2\theta) E_o'^2$$

$$\langle I^{(1)} \rangle \propto N E_o'^2 \left(\frac{\omega^2}{c^2} |\Delta\chi| \right)^2 (1 + \cos^2\theta) F(\theta) \qquad (3.1.18)$$

$$\text{I.P.} \propto (1 + \cos^2\theta) F(\theta)$$

Notice, the intensity pattern is independent of φ as it should be since the incident wave is circularly polarized,therefore the time average radiated power per unit solid angle cannot depend on φ. This would not be true if the collection of the particles exhibited a φ-dependence on the average.

3.2. Polarization of the Scattered Wave

Recall equation 3.1.7 for the first order total scattered field:

$$\underline{E}_{sc}^{(1)}(\underline{r}) = -\frac{\omega^2}{c^2} (\underline{e}_r \times \underline{e}_r \times \underline{E}_o) \frac{e^{ikr}}{4\pi r}$$

$$\sum_i \Delta\chi_i \exp(-ik\underline{e}_r \cdot \underline{r}_{io} + ikz_{io}) K_i(\alpha_i, \beta_i, \gamma_i; \theta, \varphi) \qquad (3.2.1)$$

3.2.1 can be rewritten as

$$\underline{E}_{sc}^{(1)}(\underline{r}) = (\underline{e}_r \times \underline{e}_r \times \underline{E}_o)\,|F(\theta,\varphi,r)|\,e^{ig(\theta,\varphi,r)} \qquad (3.2.2)$$

where the meaning of $|F(\theta,\varphi,r)|\,e^{ig(\theta,\varphi,r)}$ is obvious.

The polarization properties of $\underline{E}_{sc}^{(1)}(\underline{r})$ entirely depend upon the vector $\underline{e}_r \times \underline{e}_r \times \underline{E}_o$ which is independent of the material medium, the shape, size, orientation and susceptibility of the particles. This will cease to be true for higher order scattered fields.

If $\underline{E}_o$ has the form: $E_x e^{-i\delta_x}\underline{e}_x + E_y e^{-i\delta_y}\underline{e}_y$ then we saw in section 3.1 that

$$-\underline{e}_r \times \underline{e}_r \times \underline{E}_o = (-E_x e^{-i\delta_x}\sin\varphi + E_y e^{-i\delta_y}\cos\varphi)\underline{e}_\varphi$$

$$+ (E_x\cos\theta\cos\varphi\, e^{-i\delta_x} + E_y e^{-i\delta_y}\cos\theta\sin\varphi)\underline{e}_\theta \qquad (3.2.3)$$

It is obvious from 3.2.3 that the total scattered wave is elliptically polarized. However, for $\theta = \pi/2$ the polarization is linear since $\cos\theta = 0$.

To determine the polarization ellipse it is necessary to cast $\underline{e}_r \times \underline{e}_r \times \underline{E}_o$ into the following form:

$$-\underline{e}_r \times \underline{e}_r \times \underline{E}_o = E_\varphi e^{-i\delta_\varphi}\underline{e}_\varphi + E_\theta e^{-i\delta_\theta}\underline{e}_\theta \qquad (3.2.4)$$

It is shown in Appendix C how one can draw the polarization ellipse if $E_\varphi, E_\theta, \delta_\varphi, \delta_\theta$ are known. The computation of these parameters is easy. For example

$$E_\varphi \cos\delta_\varphi = -E_x\cos\delta_x\sin\varphi + E_y\cos\delta_y\cos\varphi$$

$$E_\varphi \sin\delta_\varphi = -E_x\sin\delta_x\sin\varphi + E_y\sin\delta_y\cos\varphi$$

and

$$E_\varphi^2 = E_x^2\sin^2\varphi + E_y^2\cos^2\varphi \quad \text{etc.}$$

The cases of interest are:

i) $\underline{E}_{inc}$ linearly polarized, i.e. $\underline{E}_{inc} = e^{-i\delta_x}(E_x\underline{e}_x + E_y\underline{e}_y)$

ii) $\underline{E}_{inc}$ circularly polarized, $\underline{E}_{inc} = E_o' e^{-i\delta_x}(\underline{e}_x \pm i\underline{e}_y)$

i) 3.2.3 gives:

$$-\underline{e}_r\times\underline{e}_r\times\underline{E}_o = e^{-i\delta_x}\Big\{(-E_x\sin\varphi + E_y\cos\varphi)\underline{e}_\varphi$$

$$+ (E_x\cos\theta\cos\varphi + E_y\cos\theta\sin\varphi)\underline{e}_\theta\Big\}$$

The scattered wave is obviously linearly polarized.

ii) 3.2.3 gives:

$$-\underline{e}_r\times\underline{e}_r\times\underline{E}_o = E_o' e^{-i\delta_x}(-\sin\varphi \pm i\cos\varphi)\underline{e}_\varphi$$

$$+ \cos\theta\, E_o' e^{-i\delta_x}(\cos\varphi \pm \sin\varphi)\underline{e}_\theta$$

$$= E_o' e^{-i(\delta_x \mp \varphi)}(\cos\theta\,\underline{e}_\theta \pm i\underline{e}_\varphi)$$

The scattered wave is elliptically polarized with an inclination angle $\psi = 0$. However, for $\theta = \pi/2$ is linearly polarized which is true in general as we saw before. Also for $\theta = 0$ (forward scattering) or

$\theta = \pi$ (back scattering) the scattered wave is circularly polarized.

Thus if the incident wave is right-handed circularly polarized then

$$-\underline{e}_r \times \underline{e}_r \times \underline{E}_o = E_o' e^{-i(\delta_x - \varphi)} (\cos\theta\, \underline{e}_\theta + i\, \underline{e}_\varphi)$$

$$= \begin{cases} E_o' e^{-i(\delta_x - \varphi)} (\underline{e}_\theta + i\,\underline{e}_\varphi) & \theta = 0 \\ E_o' e^{-i(\delta_x + \varphi)} e^{i\pi} (\underline{e}_\theta - i\,\underline{e}_\varphi) & \theta = \pi \end{cases}$$

If we take into account the correspondence $(x,y,z) \Longleftrightarrow (\theta,\varphi,r)$ we understand that the back-scattered wave is circularly polarized but of opposite handedness than the incident wave whereas the forward scattered wave is circularly polarized and of the same handedness.

If the incident wave is left-handed circularly polarized then we can easily see that the back-scattered wave is again c.p. but of opposite handedness whereas the forward scattered wave is c.p. and of the same handedness as the incident wave.

We can easily understand the above results if we recall that the observer who decides about the sense of rotation of the electric vector runs always behind the wave front.

IV. SECOND ORDER SCATTERING

4.1. Intensity of the Scattered Wave

Equation 2.2.6 gives if the self-interaction terms are neglected:

$$\underline{E}_{sc}^{(2)i}(\underline{r}) = \left(\frac{\omega^2}{c^2}\right)^2 \Delta\chi_i \int_{V_i} \underline{\underline{\Gamma}}(\underline{r};\underline{r}_i)\cdot \left\{ \sum_{j\neq i} \Delta\chi_j \int_{V_j} \underline{\underline{\Gamma}}(\underline{r}_i;\underline{r}_j)\cdot \underline{E}_{inc}(\underline{r}_j)\, dV_j \right\} dV_i \tag{4.1.1}$$

We have assumed that the interaction between the particles involves far zone fields only. Therefore, we can use the simplified form for $\underline{\underline{\Gamma}}(\underline{r}_i;\underline{r}_j)$

$$\underline{\underline{\Gamma}}(\underline{r}_i;\underline{r}_j) \approx (\underline{\underline{u}} - \underline{e}_{r_i}\underline{e}_{r_i})\, \frac{e^{ikr_i}}{4\pi r_i}\, e^{-ik\underline{e}_{r_i}\cdot \underline{r}_j} \tag{4.1.2}$$

If $\underline{r}_\ell$ for any particle is measured from a common origin, say the center of the volume occupied by the particles, then 4.1.2 is a bad approximation for particles for which $r_i - \underline{e}_{r_i}\cdot\underline{r}_j$ is close to zero. There are, however, two reasons for using 4.1.2. a) The majority of the particles in pairs satisfies 4.1.2 to a good degree of accuracy. b) Any fine details which would result from an accurate form of $\underline{\underline{\Gamma}}(\underline{r}_i;\underline{r}_j)$ will be completely washed out by the final averaging procedure.

The scattered field $\underline{E}_{sc}^{(2)i}(\underline{r})$ is a far zone field, therefore $\underline{\underline{\Gamma}}(\underline{r};\underline{r}_i)$ takes the simplified form 4.1.2. We can now write $\underline{E}_{sc}^{(2)i}(\underline{r})$ in the following form

$$\underline{E}_{sc}^{(2)i}(\underline{r}) = \frac{\omega^4}{c^4}\Delta\chi_i \frac{e^{ikr}}{4\pi r}(\underline{\underline{u}} - \underline{e}_r\underline{e}_r)\cdot \int_{V_i} e^{-ik\underline{e}_r\cdot\underline{r}_i}$$

$$(\underline{\underline{u}} - \underline{e}_{r_i}\underline{e}_{r_i})\cdot\underline{E}_o \frac{e^{ikr_i}}{4\pi r_i}\left\{\sum_{j\neq i}\Delta\chi_j\int_{V_j} e^{-ik\underline{e}_{r_i}\cdot\underline{r}_j + ikz_j}\,dV_j\right\}dV_i \tag{4.1.3}$$

To take account of the randomness in position of the particles we do the same splitting as we did in section 3.1, i.e.

$$\underline{r}_i = \underline{R}_i + \underline{r}_{io}, \quad \underline{r}_j = \underline{R}_j + \underline{r}_{jo}$$

whereupon 4.1.3 becomes

$$\underline{E}_{sc}^{(2)i}(\underline{r}) = \frac{\omega^4}{c^4}\Delta\chi_i \frac{e^{ikr}}{4\pi r}(\underline{\underline{u}} - \underline{e}_r\underline{e}_r)\cdot e^{-ik\underline{e}_r\cdot\underline{r}_{io}}\int_{V_i}(\underline{\underline{u}} - \underline{e}_{r_i}\,\underline{e}_{r_i})$$

$$\cdot\underline{E}_o \frac{e^{ikr_i}}{4\pi r_i} e^{-ik\underline{e}_r\cdot\underline{R}_i}\left\{\sum_{j\neq i}\chi_j e^{-ik\underline{e}_{r_i}\underline{r}_{jo} + ikz_{jo}}\right.$$

$$\left.\int_{V_j} e^{-ik\underline{e}_{r_i}\cdot\underline{R}_j} e^{ikZ_j}\,dV_j\right\} \tag{4.1.4}$$

When we do the integration over the volume of the ith particle we can replace $\underline{e}_{r_i}$ by its average value $\underline{e}_{r_{io}}$. The reason is the following. $\underline{e}_{r_i}$ points along the line joining the origin (lying somewhere in the center of the volume occupied by the particles) with a volume element within the ith particle. Because of the assumption of a far zone field interaction between the particles we understand that if a particle is situated near the origin then any other particle lies at a distance much greater than a wavelength. On the other hand the linear dimensions of the particles are of the order of a wavelength, therefore the change in the direction of $\underline{e}_{r_i}$ over the volume of the ith particle is really negligible for all the particles but the one situated near the origin. However the error we make by ignoring the particle near the origin is really negligible if N is large. Next we will replace e^{ikr_i}/r_i by $e^{ikr_{io}}/r_{io}$ which again is O.K. for all the particles away from the center.

We can now write 4.1.4 as follows:

$$\underline{E}_{sc}^{(2)i}(\underline{r}) = \frac{\omega^4}{c^4}\Delta\chi_i \frac{e^{ikr}}{4\pi r}(\underline{\underline{u}} - \underline{e}_r\underline{e}_r)\cdot \exp(-ik\underline{e}_r\cdot\underline{r}_{io})$$
$$(\underline{\underline{u}} - \underline{e}_{r_{io}}\underline{e}_{r_{io}})\cdot\underline{E}_o \frac{e^{ikr_{io}}}{4\pi r_{io}}\left\{\int_{V_i}\exp(-ik\underline{e}_r\cdot\underline{R}_i)\, dV_i\right\}$$
$$\left\{\sum_{j\neq i}\Delta\chi_j \exp(-ik\underline{e}_{r_{io}}\cdot\underline{r}_{jo} + ikz_{jo})\int_{V_j}\exp(-ik\underline{e}_{r_{io}}\cdot\underline{R}_j + ikZ_j)\, dV_j\right\} \tag{4.1.5}$$

If we now recall definition 3.1.8 and use 4.1.5 we get for the total scattered field:

$$\underline{E}_{sc}^{(2)}(\underline{r}) = \frac{\omega^4}{c^4}\frac{e^{ikr}}{4\pi r}(\underline{\underline{u}} - \underline{e}_r\underline{e}_r)\cdot\sum_i\Delta\chi_i(\underline{\underline{u}} - \underline{e}_{r_{io}}\underline{e}_{r_{io}})$$
$$\cdot\underline{E}_o\exp(-ik\underline{e}_r\cdot\underline{r}_{io})\frac{e^{ikr_{io}}}{4\pi r_{io}}\; L_i(\alpha_i,\beta_i,\gamma_i;\theta,\varphi)\left\{\sum_j\Delta\chi_j\right.$$
$$\left.\exp(-ik\underline{e}_{r_{io}}\cdot\underline{r}_{jo} + ikz_{jo})K_j(\alpha_j,\beta_j,\gamma_j;\theta_{io},\varphi_{io})\right\} \tag{4.1.6}$$

where we have defined

$$L_i(\alpha_i, \beta_i, \gamma_i; \theta, \varphi) \equiv \int_{V_i} \exp(-ik\underline{e}_r \cdot \underline{R}_i)\, dV_i \tag{4.1.6a}$$

To make 4.1.6 look simpler we define

$$\sum_{j \neq i} \Delta\chi_j \exp(-ik\underline{e}_{r_{io}} \cdot \underline{r}_{jo} + ikz_{jo})\, K_j(\alpha_j, \beta_j, \gamma_j; \theta_{io}, \varphi_{io}) \equiv \Lambda_1(\theta_{io}, \varphi_{io}) \tag{4.1.7}$$

Next we drop the index o as redundant and 4.1.6 becomes

$$\underline{E}_{sc}^{(2)}(\underline{r}) = \frac{\omega^4}{c^r} \frac{e^{ikr}}{4\pi r} (\underline{\underline{u}} - \underline{e}_r \underline{e}_r)$$

$$\cdot \Big\{ \sum_i \Delta\chi_i (\underline{\underline{u}} - \underline{e}_{r_i} \underline{e}_{r_i}) \cdot \underline{E}_o \exp(-ik\underline{e}_r \cdot \underline{r}_i) \frac{e^{ikr_i}}{4\pi r_i}$$

$$L_i(\alpha_i, \beta_i, \gamma_i; \theta, \varphi) \Lambda_1(\theta_i, \varphi_i) \Big\} \tag{4.1.8}$$

It is shown in Appendix D that if $\underline{E}_{sc} = \underline{E}_{sc}^{(1)} + \underline{E}_{sc}^{(2)} + \ldots$ then

$$\langle \underline{E}_{sc} \cdot \underline{E}_{sc}^* \rangle = \langle \underline{E}_{sc}^{(1)} \cdot \underline{E}_{sc}^{(1)*} \rangle + \langle \underline{E}_{sc}^{(2)} \cdot \underline{E}_{sc}^{(2)*} \rangle + \ldots$$

i.e.

$$\langle I \rangle = \langle I^{(1)} \rangle + \langle I^{(2)} \rangle + \ldots$$

The above relations say that the fields of the several orders are "orthogonal" to each other when the appropriate averaging is done.

Thus it makes sense to compute $\langle \underline{E}_{sc}^{(2)} \cdot \underline{E}_{sc}^{(2)} \rangle \propto \langle I^{(2)} \rangle$ because in that way we get $\langle I\rangle$ up to the second order by simply adding $\langle I^{(1)}\rangle$ to $\langle I^{(2)}\rangle$. We now multiply 4.1.8 by its complex conjugate to get:

$$\underline{E}^{(2)}_{sc}\cdot\underline{E}^{(2)*}_{sc} = \left(\frac{\omega^4}{c^2}\right)^2 \left|\frac{e^{ikr}}{4\pi r}\right|^2 \Bigg|\sum_i \chi_i(\underline{\underline{u}}\cdot\underline{e}_r\underline{e}_r)$$

$$\cdot(\underline{\underline{u}}-\underline{e}_{r_i}\underline{e}_{r_i})\cdot\underline{E}_o\frac{e^{ikr_i}}{4\ r_i}\exp(-ik\underline{e}_r\cdot\underline{r}_i)L_i(\alpha_i,\beta_i,\gamma_i;\theta,\varphi)$$

$$\Lambda_1(\theta_i,\varphi_i)\Bigg|^2 \qquad (4.1.9)$$

As we did in section 3.1 we will again assume that all the particles have the same shape, size and susceptibility. Thus $\Delta\chi_i = \Delta\chi_j = \Delta\chi$ and we should drop the index from K_i. Then according to the rules set up in Appendix D we have:

$$\langle\underline{E}^{(2)}_{sc}\cdot\underline{E}^{(2)*}_{sc}\rangle = \left(\frac{\omega^4}{c^4}|\Delta\chi|^2\right)^2\left|\frac{e^{ikr}}{4\pi r}\right|^2 N\frac{1}{V}\int_V \exp(2k_{im}\underline{e}_r\cdot\underline{a})\frac{e^{-2k_{im}a}}{(4\ a)^2}$$

$$|(\underline{\underline{u}}-\underline{e}_r\underline{e}_r)\cdot(\underline{\underline{u}}-\underline{e}_{r_i}\underline{e}_{r_i})\cdot\underline{E}_o|^2|\Lambda(\theta_i,\varphi_i)|^2\,dV$$

$$\frac{1}{8\pi^2}\int\!\!\int\!\!\int |L(\alpha,\beta,\gamma;\theta,\varphi)|^2\sin\alpha\,d\alpha\,d\beta\,d\gamma \qquad (4.1.10)$$

where $\underline{e}_{r_i} = \underline{a}/a = (\theta_i,\varphi_i)$ and $\underline{a}$ is the radius vector from the origin to any point within the volume V occupied by the particles. Also $\Lambda(\theta_i,\varphi_i) = \frac{1}{\Delta\chi}\Lambda_1(\theta_i,\varphi_i)$. If we recall definition 4.1.7, we can write

$$|\Lambda(\theta_i,\varphi_i)|^2 = \left|\sum_{j\neq i}\exp(-ik\underline{e}_{r_i}\cdot\underline{r}_j)e^{ikz_j}K(\alpha_i,\beta_i,\gamma_i;\theta_j,\varphi_j)\right|^2$$

To get an approximate expression for $\langle \underline{E}_{sc}^{(2)} \cdot \underline{E}_{sc}^{(2)*} \rangle$ we replace $|\Lambda(\theta_i, \varphi_i)|^2$ by its average, i.e.

$$\langle |\Lambda(\theta_i, \varphi_i)|^2 \rangle = N \frac{1}{V} \int_V \exp\left(2k_{im}(\underline{e}_{r_i} - \underline{e}_z) \cdot \underline{a}'\right) dV'$$

$$\times \frac{1}{8\pi^2} \iiint |K(\alpha, \beta, \gamma; \theta_i, \varphi_i)|^2 \sin\alpha \, d\alpha \, d\beta \, d\gamma \qquad (4.1.11)$$

where $\underline{a}'$ is the radius vector from the origin to any point within the volume V occupied by the particles.

If we now use definition 3.1.14, i.e.

$$F(\theta) = \frac{1}{8\pi^2} \int_{\gamma=0}^{2\pi} \int_{\beta=0}^{2\pi} \int_{\alpha=0}^{\pi} |K(\alpha, \beta, \gamma; \theta, \varphi)|^2 \sin\alpha \, d\alpha \, d\beta \, d\gamma$$

and 4.1.11 we get from 4.1.10

$$\langle \underline{E}_{sc}^{(2)} \cdot \underline{E}_{sc}^{(2)*} \rangle = \left(\frac{\omega^4}{c^2} |\Delta\chi|^2 \right)^2 \frac{e^{-2k_{im} r}}{(4\pi r)^2} \left(\frac{N}{V}\right)^2 F_1(\theta)$$

$$\left\{ \int_V \exp\left(2k_{im}(\underline{e}_r - \underline{e}_z) \cdot \underline{a}\right) \frac{e^{-2k_{im} a}}{(4\pi a)^2} \left| (\underline{\underline{u}} - \underline{e}_r \underline{e}_r) \cdot (\underline{\underline{u}} - \underline{e}_a \underline{e}_a) \cdot \underline{E}_o \right|^2 \right.$$

$$\left. \left[\int_V \exp\left(2k_{im}(\underline{e}_a - \underline{e}_z) \cdot a'\right) dV' \right] F(\theta_i) \, dV \right\} \qquad (4.1.12)$$

Notice that we have defined

$$F_1(\theta) \equiv \frac{1}{8\pi^2} \iiint \left| L(\alpha, \beta, \gamma; \theta, \varphi) \right|^2 \sin\alpha \, d\alpha \, d\beta \, d\gamma \qquad (4.1.12a)$$

We will now show that $F_1(\theta)$ is independent of θ. Recall that

$$L(\alpha,\beta,\gamma;\theta,\varphi) \equiv \int_{V_p} \exp(-ik\underline{e}_r\cdot\underline{R}_i)\, dV_i$$

If we call τ the angle between $\underline{R}_i$ and $\underline{e}_r$ we have $\underline{e}_r\cdot\underline{R}_i = R_i \cos\tau$. Now $\underline{e}_r$ is a fixed direction in space and L will depend on the orientation of the particle w.r.t. the fixed direction. However when we average over α,β,γ all the possible orientations of the particle are included and F_1 cannot possibly depend on the fixed direction $\underline{e}_r(\theta,\varphi)$. The situation with $F(\theta)$ is different because $K = \int_{V_p} \exp\left(-ik(\underline{e}_r-\underline{e}_z)\cdot\underline{R}_i\right)dV_i$ depends on the orientation of the particle w.r.t. the fixed direction $\underline{e}_r-\underline{e}_z$ but also on the magnitude of $|\underline{e}_r-\underline{e}_z|$. This last dependence makes $F(\theta)$ to be θ dependent. (The φ dependence is smeared out because $\underline{e}_r-\underline{e}_z$ rotates about z.)

We now go back to 4.1.12. Notice that we have slightly changed the notation, i.e. we have set $\underline{e}_{r_i} \equiv \underline{e}_a$ = unit vector in the direction θ_i,φ_i. Again the losses have an effect $1 + O(k_{im}L)^2$. If we neglect the losses 4.1.12 becomes

$$\langle \underline{E}_{sc}^{(2)}\cdot\underline{E}_{sc}^{(2)*}\rangle = \left(\frac{\omega^4}{c^2}|\Delta\chi|^2\right)^2\left(\frac{1}{4\pi r}\right)^2 N\frac{N}{V}F_1$$

$$\times\int_V |(\underline{\underline{u}}-\underline{e}_r\underline{e}_r)\cdot(\underline{\underline{u}}-\underline{e}_a\underline{e}_a)\cdot\underline{E}_o|^2 F(\theta_i)\, dV_i \qquad (4.1.13)$$

It is shown in Appendix E-i that for an incident wave circularly polarized (no matter whether right or left-handed) we find

$$\langle \underline{E}_{sc}^{(2)} \cdot \underline{E}_{sc}^{(2)*} \rangle = \left(\frac{\omega^4}{c^2} |\Delta \chi|^2\right)^2 \left(\frac{1}{4\pi r}\right)^2 N \frac{N}{V} \frac{R}{16\pi^2} F_1 \left\{B_1 + B_2 \cos^2\theta\right\}$$

(4.1.14)

where R is the radius of the region occupied by the particles and B_1, B_2 are defined as follows:

$$B_1 = J_1 + J_3 + 4\pi(J_1 - J_3)$$
$$B_2 = J_1 + J_3 - 4\pi(J_1 - J_3) \qquad (4.1.15)$$

and

$$J_1 = \frac{1}{2}\int_0^\pi F(\theta) \sin\theta \, d\theta$$
$$J_3 = \frac{1}{2}\int_0^\pi F(\theta) \cos^4\theta \sin\theta \, d\theta \qquad (4.1.16)$$

If the incident wave is linearly polarized the result is:

$$\langle \underline{E}_{sc}^{(2)} \cdot \underline{E}_{s}^{(2)*} \rangle = \left(\left(\frac{\omega^4}{c^4}\right) |\Delta\chi|^2\right)^2 \left(\frac{1}{4\pi r}\right)^2 N \frac{N}{V} \frac{R}{16\pi^2} F_1$$

$$(A_1 \sin^2\varphi + A_2 \sin^2\varphi \cos^2\theta + A_3 \cos^2\theta + A_4) \qquad (4.1.17)$$

where

$$A_1 = \int_0^\pi \left[\frac{\pi}{2}(1 + \cos^4\theta) - 2\pi\cos^2\theta\right] F(\theta) \sin\theta \, d\theta$$

$$A_2 = -\int_0^\pi \left(\frac{\pi}{2} + \frac{5\pi}{2}\cos^2\theta\right) F(\theta) \sin\theta \, d\theta$$

(4.1.18)

$$A_3 = \int_0^\pi \left(\frac{3\pi}{4} + \frac{7\pi}{4}\cos^2\theta - \pi\cos^2\theta\sin^2\theta\right) F(\theta) \sin\theta \, d\theta$$

$$A_4 = \int_0^{\pi} \left[\frac{\pi}{4}(1 + \cos^4\theta) + \pi \cos^2\theta \right] F(\theta) \sin\theta \, d\theta$$

4.2. Polarization of the Scattered Wave

Recall the expression 4.1.8 for the second order scattered field if the particles have the same shape, size and susceptibility and there are no losses.

$$\underline{E}_{sc}^{(2)}(r) = \frac{\omega^4(\Delta\chi)^2}{c^4} \frac{e^{ikr}}{4\pi r} (\underline{\underline{u}} - \underline{e}_r\underline{e}_r) \cdot \left\{ \sum_i (\underline{\underline{u}} - \underline{e}_{r_i}\underline{e}_{r_i}) \right.$$

$$\left. \cdot \underline{E}_o e^{-ik\underline{e}_r \cdot \underline{r}_i} \frac{e^{ikr_i}}{4\pi r_i} L(\alpha_i, \beta_i, \gamma_i, \theta, \varphi) \Lambda(\theta_i, \varphi_i) \right\} \tag{4.2.1}$$

We want to cast 4.2.1 into the following form

$$\underline{E}_{sc}^{(2)}(\underline{r}) = E_\theta e^{-i\delta_\theta} \underline{e}_\theta + E_\varphi e^{-i\delta_\varphi} \underline{e}_\varphi \tag{4.2.2}$$

and then apply the method developed in Appendix D for the computation of the polarization properties of $\underline{E}_{sc}^{(2)}$. We first compute

$$(\underline{\underline{u}} - \underline{e}_r\underline{e}_r) \cdot (\underline{\underline{u}} - \underline{e}_{r_i}\underline{e}_{r_i}) \cdot \underline{E}_o = \underline{e}_r \times \left\{ \underline{e}_r \times \left[\underline{e}_{r_i} \times (\underline{e}_{r_i} \times \underline{E}_o) \right] \right\} = \underline{e}_r \times (\underline{e}_r \times A)$$

Now assume that $\underline{E}_o = E_o' e^{-i\delta_x}(\underline{e}_x \pm i\underline{e}_y)$. Then as we saw in section 3.2, page 40 we have:

$$-\underline{e}_{r_i} \times \underline{e}_{r_i} \times \underline{E}_o = E_o' e^{-i(\delta_x \mp \varphi_i)} (\cos\theta_i \underline{e}_{\theta_i} \pm i\underline{e}_{\varphi_i})$$

and according to Appendix E-i we get:

$$\underline{e}_r \times \left\{ \underline{e}_r \times \left[\underline{e}_{r_i} \times (\underline{e}_{r_i} \times \underline{E}_o) \right] \right\}$$

$$= E_o' e^{-i(\delta_x \mp \varphi_i)} \left\{ \left[\pm i(\underline{e}_{\varphi_i} \cdot \underline{e}_\theta) + \cos\theta_i (\underline{e}_{\theta_i} \cdot \underline{e}_\theta) \right] \underline{e}_\theta \right.$$

$$\left. + \left[\pm i(\underline{e}_{\varphi_i} \cdot \underline{e}_\varphi) + \cos\theta_i (\underline{e}_{\theta_i} \cdot \underline{e}_\varphi) \right] \underline{e}_\varphi \right\}$$

$$= E_o' e^{-i(\delta_x \mp \varphi_i)} \left\{ \sqrt{(\underline{e}_{\varphi_i} \cdot \underline{e}_\theta)^2 + \cos^2\theta_i (\underline{e}_{\theta_i} \cdot \underline{e}_\theta)^2}\; e^{i\lambda_i} \underline{e}_\theta \right.$$

$$\left. + \sqrt{(\underline{e}_{\varphi_i} \cdot \underline{e}_\varphi)^2 + \cos^2\theta_i (\underline{e}_{\theta_i} \cdot \underline{e}_\varphi)^2}\; e^{i\mu_i} \underline{e}_\varphi \right\}$$

Here the simplifying assumption will be made that λ_i and μ_i which are functions of the random variables φ_i, θ_i represent the same random variable with a constant probability density $1/2\pi$ over the interval $(0, 2\pi)$.

We can also write $L(\alpha_i, \beta_i, \gamma_i; \theta, \varphi) = |L(\alpha_i, \beta_i, \gamma_i; \theta, \varphi)| e^{i\sigma_i}$, $\Lambda(\theta_i, \varphi_i) = |\Lambda(\theta_i, \varphi_i)| e^{i\tau_i}$ with the same assumption about σ_i and τ_i as for λ_i, μ_i.

Finally we get from 4.2.1

$$\underline{E}_{sc}^{(2)}(\underline{r}) = \frac{\omega^4}{c^4} |\Delta\chi|^2 \frac{E_o'}{4\pi r} \left\{ \underline{e}_\theta \left[\sum_i e^{i\kappa_i} \sqrt{(\underline{e}_{\varphi_i} \cdot \underline{e}_\theta)^2 + \cos^2\theta_i (\underline{e}_{\theta_i} \cdot \underline{e}_\theta)^2} \right. \right.$$

$$\left. |\Lambda(\theta_i, \varphi_i)| \frac{1}{4\pi r_i} |L(\alpha_i, \beta_i, \gamma_i; \theta, \varphi)| \right]$$

$$+ \underline{e}_\varphi \left[\sum_i e^{i\kappa_i} \sqrt{(\underline{e}_{\varphi_i} \cdot \underline{e}_\varphi)^2 + \cos^2\theta_i (e_{\theta_i} \cdot \underline{e}_\varphi)^2} \right.$$

$$\left. \left. |\Lambda(\theta_i, \varphi_i) \frac{1}{4\pi r_i} |L(\alpha_i, \beta_i, \gamma_i; \theta, \varphi)| \right] \right\} \quad \text{, i.e.}$$

$$\mathcal{E}_\theta = E_\theta e^{-i\delta_\theta} = \frac{\omega^4}{c^2}|\Delta\chi|^2 \frac{E_o'}{4\pi r}\sum_i e^{-i\kappa_i}\sqrt{(\underline{e}_{\varphi_i}\cdot\underline{e}_\theta)^2+\cos^2\theta_i(\underline{e}_{\varphi_i}\cdot\underline{e}_\theta)^2}$$

$$|\Lambda(\theta_i,\varphi_i)|\frac{1}{4\pi r_i}\;|L(\alpha_i,\beta_i,\gamma_i;\theta,\varphi)| = \sum_i e^{i\kappa_i}A_i \tag{4.2.3}$$

$$\mathcal{E}_\varphi = E_\varphi e^{-i\delta_\varphi} = \frac{\omega^4}{c^4}|\Delta\chi|^2 \frac{E_o'}{4\pi r}\sum_i e^{i\kappa_i}\sqrt{(e_{\varphi_i}\cdot\underline{e}_\varphi)^2+\cos^2\theta_i(\underline{e}_{\theta_i}\cdot\underline{e}_\varphi)^2}$$

$$|\Lambda(\theta_i,\varphi_i)|\frac{1}{4\pi r_i}\;|L(\alpha_i,\beta_i,\gamma_i;\theta,\varphi)| = \sum_i e^{i\kappa_i}B_i \tag{4.2.4}$$

According to Appendix D we have to compute:

$$\langle A_i^2\rangle\;,\;\langle B_i^2\rangle\;,\;\langle A_iB_i\rangle$$

We have

$$\langle A_i^2\rangle = \left(\frac{\omega^4}{c^4}|\Delta\chi|^2\right)^2\frac{E_o'^2}{(4\pi r)^2}\;\frac{1}{V}\int_V\left[(\underline{e}_{\varphi_i}\cdot\underline{e}_\theta)^2+\cos^2\theta_i(\underline{e}_{\varphi_i}\cdot\underline{e}_\theta)^2\right]$$

$$\langle|\Lambda(\theta_i,\varphi_i)|^2\rangle\left(\frac{1}{4\pi r_i}\right)^2 dV\times\frac{1}{8\pi^2}\iiint|L(\alpha,\beta,\gamma;\theta,\varphi)|^2\sin\alpha\;d\alpha\;d\beta\;d\gamma$$

$$\langle B_i^2\rangle = \left(\frac{\omega^4}{c^4}|\Delta\chi|^2\right)^2\frac{E_o'^2}{(4\pi r)^2}\;\frac{1}{V}\int_V\left[(\underline{e}_{\varphi_i}\cdot\underline{e}_\varphi)^2+\cos^2\theta_i(\underline{e}_{\theta_i}\cdot\underline{e}_\varphi)^2\right]$$

$$\langle|\Lambda(\theta_i,\varphi_i)|^2\rangle\left(\frac{1}{4\pi r_i}\right)^2 dV\times\frac{1}{8\pi^2}\iiint|L(\alpha,\beta,\gamma;\theta,\varphi)|^2\sin\alpha\;d\alpha\;d\beta\;d\gamma$$

$$\langle A_i B_i \rangle = \left(\frac{\omega^4}{c^4} |\Delta\chi|^2\right)^2 \frac{E_o'^2}{(4\pi r)^2}$$

$$\times \frac{1}{V} \int_V \left\{ \left[(\underline{e}_{\varphi_i} \cdot \underline{e}_\theta)^2 + \cos^2\theta_i (\underline{e}_{\varphi_i} \cdot \underline{e}_\theta)^2 \right] \left[(\underline{e}_{\varphi_i} \cdot \underline{e}_\varphi)^2 + \cos^2\theta_i (\underline{e}_{\theta_i} \cdot \underline{e}_\varphi)^2 \right] \right\}^{\frac{1}{2}}$$

$$\langle |\Lambda(\theta_i, \varphi_i)|^2 \rangle \left(\frac{1}{4\pi r_i}\right)^2 dV \times \frac{1}{8\pi^2} \int\int\int |L(\alpha, \beta, \gamma; \theta, \varphi)|^2 \sin\alpha \, d\alpha \, d\beta \, d\gamma$$

where $dV = r_i^2 \, dr_i \sin\theta_i \, d\theta_i \, d\varphi_i$ and $\underline{r}_i \, (r_i, \theta_i \varphi_i)$ is the radius vector from the origin to any point within the volume V occupied by the particles.

We can now find the Stokes parameters:

$$S_o = N\left[\langle A_i^2 \rangle + \langle B_i^2 \rangle\right] \qquad S_1 = N\left[\langle A_i^2 \rangle - \langle B_i^2 \rangle\right]$$

$$S_2 = N\langle A_i B_i \rangle \qquad S_3 = N\langle A_i B_i \rangle$$

One can easily see that even if $\theta = 0, \pi$ the second order scattered wave is no larger circularly but eliptically polarized and consequently so is the total field $= \underline{E}_{sc}^{(1)} + \underline{E}_{sc}^{(2)}$. However the handedness is preserved because of the domination of $\underline{E}_{sc}^{(1)}$. If the incident wave is linearly polarized we can make analogous computations to find that the second order scattered wave is elliptically polarized and so is the total scattered field.

V. THIRD ORDER SCATTERING

5.1. Intensity of the Scattered Wave

The third order scattered field is given by 2.2.7 and has the following form if self-interaction terms are neglected

$$\underline{E}_{sc}^{(3)}(\underline{r}) = \sum_i \underline{E}_{sc}^{(3)i}(\underline{r})$$

where

$$\underline{E}_{sc}^{(3)i}(\underline{r}) = \left(\frac{\omega^2}{c^2}\right)^3 \Delta\chi_i \int_{V_i} \underline{\underline{\Gamma}}(\underline{r};\underline{r}_i)\cdot\left\{\sum_{j\neq i} \Delta\chi_j \int_{V_j} \underline{\underline{\Gamma}}(\underline{r}_i;\underline{r}_j)\right.$$

$$\left.\cdot\left[\sum_{k\neq j} \Delta\chi_k \int_{V_k} \underline{\underline{\Gamma}}(\underline{r}_j;\underline{r}_k)\cdot\underline{E}_{inc}(\underline{r}_k)\, dV_k\right] dV_j\right\} dV_i \qquad (5.1.1)$$

Under the assumption of a far zone interaction between the particles and a far zone scattered field $\underline{E}_{sc}^{(3)}(\underline{r})$, all the Γ's will be written in their approximate form. Again the splitting $\underline{r}_\ell = \underline{r}_{\ell o} + \underline{R}_\ell$ will be made to take into account the randomness in the position (through $\underline{r}_{\ell o}$) and the shape, size and orientation (through $\underline{R}_\ell$) of the particles.

We rewrite now 5.1.1 as

$$\underline{E}_{sc}^{(3)i}(\underline{r}) = \left(\frac{\omega^2}{c^2}\right)^3 \Delta\chi_i \frac{e^{ikr}}{4\pi r}(\underline{\underline{u}} - \underline{e}_r\underline{e}_r)\cdot e^{-ik\underline{e}_r\cdot\underline{r}_{io}}$$

$$\int_{V_i}\left\{e^{-ik\underline{e}_r\cdot\underline{R}_i}\frac{e^{ikr_i}}{4\pi r_i}(\underline{\underline{u}} - \underline{e}_{r_i}\underline{e}_{r_i})\cdot\sum_{j\neq i}\Delta\chi_j e^{-ik\underline{e}_{r_i}\cdot\underline{r}_{jo}}\right.$$

$$\int_{V_j}\left[e^{-ik\underline{e}_{r_i}\cdot\underline{R}_j}\frac{e^{ikr_j}}{4\pi r_j}(\underline{\underline{u}} - \underline{e}_{r_j}\underline{e}_{r_j})\cdot\sum_{k\neq j}\underline{E}_o\Delta\chi_k \exp(-ik\underline{e}_{r_j}\cdot\underline{r}_{ko}+ikz_{ko})\right.$$

(relation continued)

$$\int_{V_k} \exp(-ik\underline{e}_{r_j} \cdot \underline{R}_k + ikZ_k)\, dV_k \Big] dV_j \Big\} dV_i \tag{5.1.2}$$

As we did with the second order field again we replace $\underline{e}_{r_\ell}$ by $\underline{e}_{r_{\ell o}}$, $e^{ikr_\ell}/4\pi r_\ell$ by $e^{ikr_{\ell o}}/4\pi r_{\ell o}$, $(\underline{\underline{u}} - \underline{e}_{r_\ell}\underline{e}_{r_\ell})$ by $(\underline{\underline{u}} - \underline{e}_{r_{\ell o}}\underline{e}_{r_{\ell o}})$ (see justification in section 4.1).

Now recall the following definitions:

$$\int_{V_k} \exp(-ik\underline{e}_{r_{jo}} \cdot \underline{R}_k + ikZ_k)\, dV_k \equiv K_k(\alpha_k, \beta_k, \gamma_k; \theta_{jo}, \varphi_{jo})$$

$$\int_{V_j} \exp(-ik\underline{e}_{r_{io}} \cdot \underline{R}_j)\, dV_j \equiv L_j(\alpha_j, \beta_j, \gamma_j; \theta_{io}, \varphi_{io})$$

$$\int_{V_i} \exp(-ik\underline{e}_r \cdot \underline{R}_i)\, dV_i \equiv L_i(\alpha_i, \beta_i, \gamma_i; \theta, \varphi)$$

$$\sum_{k \neq j} \Delta\chi_k \exp(-ik\underline{e}_{r_{jo}} \cdot \underline{r}_{ko} + ikz_{ko}) K(\alpha_k, \beta_k, \gamma_k; \theta_{jo}, \varphi_{jo}) \equiv \Lambda_1(\theta_{jo}, \varphi_{jo})$$

In view of the above definitions 5.1.2 becomes:

$$\begin{aligned}\underline{E}_{sc}^{(3)i}(\underline{r}) = \left(\frac{\omega^2}{c^2}\right)^2 \Delta\chi_i \frac{e^{ikr}}{4\pi r} (\underline{\underline{u}} - \underline{e}_r\underline{e}_r) \cdot e^{-ik\underline{e}_r \cdot \underline{r}_{io}} (\underline{\underline{u}} - \underline{e}_{r_{io}}\underline{e}_{r_{io}}) \\ \cdot \frac{e^{ikr_{io}}}{4\pi r_{io}} L_i(\alpha_i, \beta_i, \gamma_i; \theta, \varphi) \sum_{j \neq i} \Delta\chi_j \exp(-ik\underline{e}_{r_{io}} \cdot \underline{r}_{jo}) (\underline{\underline{u}} - \underline{e}_{r_{jo}}\underline{e}_{r_{jo}}) \\ \cdot \frac{e^{ikr_{jo}}}{4\pi r_{jo}} L(\alpha_j, \beta_j, \gamma_j; \theta_{io}, \varphi_{io}) \Lambda_1(\theta_{jo}, \varphi_{jo}) \underline{E}_o\end{aligned} \tag{5.1.3}$$

In what follows we will drop the index o as redundant. We now define

$$\sum_{j\neq i} \Delta\chi_j e^{-ik\underline{e}_{r_i}\cdot\underline{r}_j}(\underline{\underline{u}}-\underline{e}_{r_j}\underline{e}_{r_j})\cdot\underline{E}_o\frac{e^{ikr_j}}{4\pi r_j}\Lambda_1(\theta_j,\varphi_j)L_j(\alpha_j,\beta_j,\gamma_j;\theta_i,\varphi_i)$$

$$\equiv \underline{T}_1(\theta_i,\varphi_i) \qquad (5.1.4)$$

In view of 5.1.3 and 5.1.4 the total third order scattered field can be written as:

$$\underline{E}_{sc}^{(3)}(\underline{r}) = \left(\frac{\omega^2}{c^2}\right)^3 \frac{e^{ikr}}{4\pi r}\sum_i \Delta\chi_i e^{-ik\underline{e}_r\cdot\underline{r}_i}(\underline{\underline{u}}-\underline{e}_r\underline{e}_r)\cdot(\underline{\underline{u}}-\underline{e}_{r_i}\underline{e}_{r_i})$$

$$\cdot\underline{T}_1(\theta_i,\varphi_i)\frac{e^{ikr_i}}{4\pi r_i}L_i(\alpha_i,\beta_i,\gamma_i;\theta,\varphi) \qquad (5.1.5)$$

To compute the average $\underline{E}_{sc}^{(3)}\cdot\underline{E}_{sc}^{(3)*}$ we again assume that all the particles have the same shape, size and susceptibility. We define $\underline{T}\equiv\frac{1}{(\Delta\chi)^2}\underline{T}_1$ and $\Lambda\equiv\frac{1}{\Delta\chi}\Lambda_1$. Now we compute the average according to the rules of Appendix D to find:

$$\langle\underline{E}_{sc}^{(2)}\cdot\underline{E}_{sc}^{(2)*}\rangle = \left|\left(\frac{\omega^2}{c^2}\right)^3(\Delta\chi)^3\right|^2\frac{e^{-2k_{im}r}}{(4\pi r)^2}$$

$$\frac{N}{V}\left\{\int_V \exp\left(2k_{im}(\underline{e}_r-\underline{e}_z)\cdot\underline{a}'\right)\left|(\underline{\underline{u}}-\underline{e}_r\underline{e}_r)\cdot(\underline{\underline{u}}-\underline{e}_{a'}\underline{e}_{a'})\cdot\underline{T}(\theta',\varphi')\right|^2\right.$$

$$\left.\frac{e^{-2k_{im}a'}}{(4\pi a')^2}dV'\right\}\frac{1}{8\pi^2}\int\int\int|L(\alpha,\beta,\gamma;\theta,\varphi)|^2\sin\alpha\, d\alpha\, d\beta\, d\gamma$$

$$(5.1.6)$$

where $\underline{a}'$ (a',θ',φ') is the radius vector from the origin to any point within the volume V occupied by the particles. As we see from 5.1.6

we have to compute $|(\underline{\underline{u}}-\underline{e}_r\underline{e}_r)\cdot(\underline{\underline{u}}-\underline{e}_{a'}\underline{e}_{a'})\cdot\underline{T}(\theta',\varphi')|^2$. According to previous calculations

$$-(\underline{\underline{u}}-\underline{e}_{a'}\underline{e}_{a'})\cdot\underline{T}(\theta',\varphi') = T_{\varphi'}\underline{e}_{\varphi'} + T_{\theta'}\underline{e}_{\theta'}$$

and

$$(\underline{\underline{u}}-\underline{e}_r\underline{e}_r)\cdot(\underline{\underline{u}}-\underline{e}_{a'}\underline{e}_{a'})\cdot\underline{T}(\theta',\varphi') = \left[T_{\varphi'}(\underline{e}_{\varphi'}\cdot\underline{e}_\varphi) + T_{\theta'}(\underline{e}_{\theta'}\cdot\underline{e}_\varphi)\right]\underline{e}_\varphi$$

$$+ \left[T_{\varphi'}(\underline{e}_{\varphi'}\cdot\underline{e}_\theta) + T_{\theta'}(\underline{e}_{\theta'},\underline{e}_\theta)\right]\underline{e}_\theta$$

Therefore we get:

$$A = |(\underline{\underline{u}}-\underline{e}_r\underline{e}_r)\cdot(\underline{\underline{u}}-\underline{e}_a\underline{e}_a)\cdot\underline{T}(\theta',\varphi')|^2$$

$$= |T_{\varphi'}(\underline{e}_{\varphi'}\cdot\underline{e}_\varphi) + T_{\theta'}(\underline{e}_{\theta'}\cdot\underline{e}_\varphi)|^2 + |T_{\varphi'}(\underline{e}_{\varphi'}\cdot\underline{e}_\theta) + T_{\theta'}(\underline{e}_{\theta'}\cdot\underline{e}_\theta)|^2$$

$$= |T_{\varphi'}|^2\left[(\underline{e}_{\varphi'}\cdot\underline{e}_\varphi)^2 + (\underline{e}_{\varphi'}\cdot\underline{e}_\theta)^2\right] + |T_{\theta'}|^2\left[(e_{\theta'}\cdot\underline{e}_\varphi)^2 + (\underline{e}_{\theta'}\cdot\underline{e}_\theta)^2\right]$$

$$+ (T_{\varphi'}T^*_{\theta'} + T^*_{\varphi'}T_{\theta'})\left[(\underline{e}_{\varphi'}\cdot\underline{e}_\theta)(\underline{e}_{\theta'}\cdot\underline{e}_\theta) + (\underline{e}_{\varphi'}\cdot\underline{e}_\varphi)(\underline{e}_{\theta'}\cdot\underline{e}_\varphi)\right]$$

What we will do next is to replace $|T_{\varphi'}|^2$, $|T_{\theta'}|^2$ and $(T_{\varphi'}T^*_{\theta'} + T^*_{\varphi'}T_\theta)$ by their averages. Recall the definition of $\underline{T}$:

$$\underline{T} = \sum_j \Delta\chi_j \exp(-ik\underline{e}_{r_i}\cdot\underline{r}_j)(\underline{\underline{u}}-\underline{e}_{r_j}\underline{e}_{r_j})\cdot\underline{E}_o \frac{e^{ikr_j}}{4\pi r_j}\Lambda(\theta_j,\varphi_j)$$

Then $(\underline{e}_{r_j}\rightarrow\underline{e}_{a''}, \underline{e}_{r_i}\rightarrow\underline{e}_{a'})$:

$$\langle T_k T_k^* \rangle = \frac{N}{V}\int_V \frac{e^{-2k_{im}a''}}{(4\pi a'')^2} \exp\left(2k_{im}\underline{e}_{a'}\cdot\underline{a}''\right)\langle |\Lambda(\theta'',\varphi'')|^2\rangle$$

$$\left[(\underline{\underline{u}} - \underline{e}_{a''}\underline{e}_{a''})\cdot\underline{E}_o\right]_k \left[(\underline{\underline{u}} - \underline{e}_{a''}\underline{e}_{a''})\cdot\underline{E}_o\right]_k^* dV''$$

$$\frac{1}{8\pi^2}\iiint |L(\alpha,\beta,\gamma;\theta',\varphi')|^2 \sin\alpha \, d\alpha \, d\beta \, d\gamma$$

and

$$\langle T_k T_\ell^* + T_k^* T_\ell \rangle = \frac{N}{V}\int_V \frac{e^{-2k_{im}a''}}{(4\pi a'')^2} \exp\left(2k_{im}\underline{e}_{a'}\cdot\underline{a}''\right)\langle |\Lambda(\theta'',\varphi'')|^2\rangle$$

$$\left\{\left[(\underline{\underline{u}} - \underline{e}_{a''}\underline{e}_{a''})\cdot\underline{E}_o\right]_k \left[(\underline{\underline{u}} - \underline{e}_{a''}\underline{e}_{a''}\cdot\underline{E}_o\right]_\ell^*\right.$$

$$\left. + \left[(\underline{\underline{u}} - \underline{e}_{a''}\underline{e}_{a''})\cdot\underline{E}_o\right]_k^* \left[(\underline{\underline{u}} - \underline{e}_{a''}\underline{e}_{a''})\cdot\underline{E}_o\right]_\ell\right\} dV''$$

$$\frac{1}{8\pi^2}\iiint |L(\alpha,\beta,\gamma;\theta',\varphi')|^2 \sin\alpha \, d\alpha \, d\beta \, d\gamma$$

or in view of 4.1.12a

$$\langle T_k T_k^* \rangle = \frac{N}{V} G_{kk}(\theta,\varphi;\theta',\varphi') F_1 \tag{5.1.7}$$

$$\langle T_k T_\ell^* + T_k^* T_\ell \rangle = \frac{N}{V}\left[G_{k\ell} + G_{\ell k}\right] F_1 \tag{5.1.8}$$

where

$$G_{k\ell} = \int_V \frac{e^{-2k_{im}a''}}{(4\pi a'')^2} e^{2k_{im}\underline{e}_{a'}\cdot\underline{a}''} \langle |\Lambda(\theta'',\varphi'')|^2\rangle$$

$$\left[(\underline{\underline{u}} - \underline{e}_{a''}\underline{e}_{a''})\cdot\underline{E}_o\right]_k \left[(\underline{\underline{u}} - \underline{e}_{a''}\underline{e}_{a''})\right]_\ell^* dV'' \tag{5.1.9}$$

Note also that

$$\langle |\Lambda(\theta'',\varphi'')|^2\rangle = \frac{N}{V}\int_V \exp\left(2k_{im}(\underline{e}_{a''}-\underline{e}_z)\cdot\underline{a}_1\right) dV_1$$

$$\times \frac{1}{8\pi^2}\iiint |K(\alpha,\beta,\gamma;\theta'',\varphi'')|^2 \sin\alpha\, d\alpha\, d\beta\, d\gamma$$

$$= \frac{N}{V}\int_V \exp\left(2k_{im}(\underline{e}_{a''}-\underline{e}_z)\cdot\underline{a}_1\right) dV_1\, F(\theta'') \qquad (5.1.10)$$

We can now write $\langle \underline{E}_{sc}^{(3)}\cdot \underline{E}_{sc}^{(3)*}\rangle$ in the following form

$$\langle \underline{E}_{sc}^{(3)}\cdot \underline{E}_{sc}^{(3)*}\rangle = \left(\frac{\omega^2}{c^2}|\Delta\chi|^3\right)^2 \frac{e^{-2k_{im}r}}{(4\pi r)^2}\, \frac{N}{V}\,\frac{N}{V}\,\frac{N}{V}\, F_1^2$$

$$\int_V \Big\{ G_{\varphi'\varphi'}\left[(\underline{e}_{\varphi'}\cdot\underline{e}_\varphi)^2+(\underline{e}_{\varphi'}\cdot\underline{e}_\theta)^2\right] + G_{\theta'\theta'}\Big[(\underline{e}_{\theta'}\cdot\underline{e}_\varphi)^2$$

$$+ (\underline{e}_{\theta'}\cdot\underline{e}_\theta)^2\Big] + (G_{\varphi'\theta'}+G_{\theta'\varphi'})\Big[(\underline{e}_{\varphi'}\cdot\underline{e}_\varphi)(\underline{e}_{\theta'}\cdot\underline{e}_\varphi)$$

$$+ (\underline{e}_{\varphi'}\cdot\underline{e}_\theta)(\underline{e}_{\theta'}\cdot\underline{e}_\theta)\Big]\Big\} \exp(2k_{im}\underline{e}_r\cdot\underline{a}')\frac{e^{-2k_{im}a'}}{(4\pi a')^2}\, dV'$$

$$= \left(\left(\frac{\omega^2}{c^2}\right)^3 |\Delta\chi|^3\right)^2 \frac{e^{-2k_{im}r}}{(4\pi r)^2}\, \frac{N}{V}\,\frac{N}{V}\,\frac{N}{V}\, F_1^2\, M(\theta,\varphi) \qquad (5.1.11)$$

Now we can easily find that:

$$(\underline{e}_{\varphi'}\cdot\underline{e}_\varphi)^2 = \cos^2(\varphi-\varphi')$$

$$(\underline{e}_{\varphi'}\cdot\underline{e}_\theta)^2 = \cos^2\theta\,\sin^2(\varphi-\varphi')$$

$$(\underline{e}_{\theta'} \cdot \underline{e}_{\varphi})^2 = \cos^2\theta' \sin^2(\varphi - \varphi')$$

$$(\underline{e}_{\theta'} \cdot \underline{e}_{\theta})^2 = [\cos\theta \cos\theta' \cos(\varphi - \varphi') + \sin\theta \sin\theta']^2$$

Next we assume that the incident wave is circularly polarized, i.e. $\underline{E}_o = \underline{E}_o' e^{-i\delta_x}(\underline{e}_x \pm i\underline{e}_y)$. We also assume that losses are negligible, i.e. $(k_{im}L)^2 << 1$. In the absence of losses we have from 5.1.10

$$\langle |\Lambda(\theta'', \varphi'')|^2 \rangle = F(\theta'')$$

and from 5.1.9

$$G_{k\ell} = \int_V \left(\frac{1}{4\pi a''}\right)^2 F(\theta'') \left[(\underline{\underline{u}} - \underline{e}_{a''}\underline{e}_{a''}) \cdot \underline{E}_o\right]_k \left[(\underline{\underline{u}} - \underline{e}_{a''}\underline{e}_{a''}) \cdot \underline{E}_o\right]_\ell^* \, dV''$$

Therefore we have:

$$G_{\varphi'\varphi'} = \int_V \left(\frac{1}{4\pi a''}\right)^2 F(\theta'') E_o'^2 \left\{(\underline{e}_{\varphi''} \cdot \underline{e}_{\varphi'})^2 + \cos^2\theta'' (\underline{e}_{\theta''} \cdot \underline{e}_{\varphi'})^2\right\} \, dV''$$

$$G_{\theta'\theta'} = \int_V \left(\frac{1}{4\pi a''}\right)^2 F(\theta'') E_o'^2 \left\{(e_{\varphi''} \cdot \underline{e}_{\theta'})^2 + \cos^2\theta'' (\underline{e}_{\theta''} \cdot \underline{e}_{\theta'})^2\right\} \, dV''$$

$$G_{\varphi'\theta'} = \int_V \left(\frac{1}{4\pi a''}\right)^2 F(\theta'') E_o'^2 \Big[\pm i(\underline{e}_{\varphi''} \cdot \underline{e}_{\varphi'})$$

$$+ \cos\theta'' (\underline{e}_{\theta''} \cdot \underline{e}_{\varphi'})\Big]\Big[\mp i(\underline{e}_{\varphi''} \cdot \underline{e}_{\varphi'}) + \cos\theta'' (\underline{e}_{\theta''} \cdot \underline{e}_{\varphi'})\Big] \, dV''$$

and

$$G_{\varphi'\theta'}+G_{\theta'\varphi'}= 2\int_V \left(\frac{1}{4\pi a''}\right)^2 F(\theta'')E_o'^2 \Big\{(e_{\varphi''}\cdot \underline{e}_{\varphi'})(\underline{e}_{\varphi''}\cdot \underline{e}_{\theta'})$$

$$+ \cos^2\theta'' \,(\underline{e}_{\theta''}\cdot \underline{e}_{\varphi'})(\underline{e}_{\theta''}\cdot \underline{e}_{\theta'})\Big\} \, dV''$$

Now we can easily find that

$$(\underline{e}_{\varphi''}\cdot \underline{e}_{\varphi'})^2 + \cos^2\theta''(\underline{e}_{\theta''}\cdot \underline{e}_{\varphi'})^2$$

$$= \cos^2(\varphi''-\varphi') + \cos^2\theta''\cos^2\theta''\sin^2(\varphi''-\varphi')(\underline{e}_{\varphi''}\cdot \underline{e}_{\theta'})$$

$$+ \cos^2\theta''(\underline{e}_{\theta''}\cdot \underline{e}_{\theta})^2$$

$$= \cos^2\theta'\sin^2(\varphi''-\varphi') + \cos^2\theta''\cos^2\theta''\cos^2\theta'\cos^2(\varphi''-\varphi')$$

$$+ \cos^2\theta''\sin^2\theta''\sin^2\theta' + 2\cos^3\theta''\sin\theta''\cos\theta'\sin\theta'\cos(\varphi''-\varphi')$$

and

$$(\underline{e}_{\varphi''}\cdot \underline{e}_{\varphi'})(\underline{e}_{\varphi''}\cdot \underline{e}_{\theta'}) + \cos^2\theta''(\underline{e}_{\theta''}\cdot \underline{e}_{\varphi'})(\underline{e}_{\theta''}\cdot \underline{e}_{\theta})$$

$$= \cos(\varphi''-\varphi')\cos\theta'\sin(\varphi''-\varphi')$$

$$+ \cos^2\theta''\cos\theta''\sin(\varphi'-\varphi'')\cos\theta''\cos\theta'\cos(\varphi'-\varphi'')$$

$$+ \cos^2\theta''\cos\theta'\sin(\varphi'-\varphi'')\sin\theta''\sin\theta'$$

Recall now that

$$dV'' = a''^2\, da''\, \sin\theta''\, d\theta''\, d\varphi''$$

We want to compute the $G_{k\ell}$'s. If we now perform the φ''

integration first the only φ'' dependent terms are the products of the unit vectors computed before and we get:

$$\int_0^{2\pi}\left[(e_{\varphi''}\cdot \underline{e}_{\varphi'})^2 + \cos^2\theta''(\underline{e}_{\theta''}\cdot \underline{e}_{\varphi'})^2\right] d\varphi'' = \frac{1}{2} + \frac{1}{2}\cos^2\theta''\cos^2\theta'$$

$$\int_0^{2\pi}\left[(\underline{e}_{\varphi''}\cdot \underline{e}_{\theta'})^2 + \cos^2\theta''(\underline{e}_{\theta''}\cdot \underline{e}_{\theta'})^2\right] d\varphi''$$

$$= \frac{1}{2}\cos^2\theta' + \frac{1}{2}\cos^2\theta'\cos^4\theta'' + 2\pi\cos^2\theta''\sin^2\theta''\sin^2\theta'$$

$$\int_0^{2\pi}\left[(\underline{e}_{\varphi''}\cdot \underline{e}_{\varphi'})(\underline{e}_{\varphi''}\cdot \underline{e}_{\theta'}) + \cos^2\theta''(\underline{e}_{\theta''}\cdot \underline{e}_{\varphi'})(\underline{e}_{\theta''}\cdot \underline{e}_{\theta'})\right] d\varphi'' = 0$$

Therefore, doing the a'' integration also, we find

$$G_{\varphi'\varphi'} = \frac{RE_o'^2}{16\pi^2}\int_0^{\pi}\frac{1}{2}F(\theta'')\left\{1 + \cos^2\theta\cos^2\theta''\right\}\sin\theta''\,d\theta''$$

$$G_{\theta'\theta'} = \frac{RE_o'^2}{16\pi^2}\int_0^{\pi}F(\theta'')\left[\frac{1}{2}(1+\cos^4\theta'')\cos^2\theta'\right.$$

$$\left. + 2\pi\cos^2\theta''\sin^2\theta''\sin^2\theta'\right]\sin\theta''\,d\theta''$$

$$G_{\varphi'\theta'} + G_{\theta'\varphi'} = 0$$

where R is the radius of the region occupied by the particles.

Next we define (see also 4.1.16)

$$J_1 = \frac{1}{2}\int_0^{\pi}F(\theta'')\sin\theta''\,d\theta''$$

$$J_2 = \frac{1}{2}\int_0^{\pi}F(\theta'')\cos^2\theta''\sin\theta''\,d\theta'' \qquad (5.1.12)$$

$$J_3 = \frac{1}{2}\int_0^{\pi} F(\theta'') \cos^4\theta'' \sin\theta'' \, d\theta''$$

and we can write the G's in the form

$$G_{\varphi'\varphi'} = \frac{RE_o'^2}{16\pi^2}\left\{J_1 + \cos^2\theta' \, J_2\right\}$$

$$G_{\theta'\theta'} = \frac{RE_o'^2}{16\pi^2}\left\{(J_1+J_3)\cos^2\theta' + 4\pi(J_2 - J_3)\sin^2\theta'\right\}$$

Recall now equation 5.1.11. We have

$$M = \int_V \left\{ G_{\varphi'\varphi'}\left[(\underline{e}_{\varphi'}\cdot\underline{e}_{\varphi})^2 + (\underline{e}_{\varphi'}\cdot\underline{e}_{\theta})^2\right] + G_{\theta'\theta'}\left[(\underline{e}_{\theta'}\cdot\underline{e}_{\varphi})^2 + (\underline{e}_{\theta'}\cdot\underline{e}_{\theta})^2\right]\right\}\left(\frac{1}{4\pi a'}\right)^2 dV'$$

We have

$$(\underline{e}_{\varphi'}\cdot\underline{e}_{\varphi})^2 + (\underline{e}_{\varphi'}\cdot\underline{e}_{\theta})^2 = \cos^2(\varphi-\varphi') + \cos^2\theta \, \sin^2(\varphi-\varphi')$$

$$(\underline{e}_{\theta'}\cdot\underline{e}_{\varphi})^2 + (\underline{e}_{\theta'}\cdot\underline{e}_{\theta})^2 = \cos^2\theta' \sin^2(\varphi-\varphi') + \cos^2\theta \, \cos\theta' \cos^2(\varphi-\varphi')$$
$$+ \sin^2\theta \sin^2\theta' + 2\sin\theta\cos\theta \, \cos\theta' \sin\theta' \cos(\varphi-\varphi')$$

We can then do the φ' and a' integrations to get

$$M = \left(\frac{R}{16\pi^2}\right)^2 E_o'^2 \int_0^{\pi}\left\{(J_1 + \cos^2\theta' J_2)\left(\frac{1}{2} + \frac{1}{2}\cos^2\theta\right)\right.$$
$$+ \left[(J_1+J_3)\cos^2\theta' + 4\pi(J_2 - J_3)\sin^2\theta'\right]\left(\frac{1}{2}\cos^2\theta' + \frac{1}{2}\cos^2\theta \cos^2\theta'\right.$$
$$\left.\left. + 2\pi\sin^2\theta \sin^2\theta'\right)\right\} \sin\theta' \, d\theta'$$

If we finally do the θ' integration we get

$$\langle \underline{E}_{sc}^{(3)} \cdot \underline{E}_{sc}^{(3)*} \rangle = E_o'^2 \left(\frac{\omega^2}{c^2} |\Delta\chi| \right)^6 \left(\frac{1}{4\pi r} \right)^2 N \left(\frac{N}{V} \right)^2 \left(\frac{R}{16\pi^2} \right)^2 F_1^2 (C_1 + C_2 \cos^2\theta) \tag{5.1.14}$$

where

$$C_1 = \left(\frac{7}{10} + \frac{8\pi}{15} \right) J_1 + \left(\frac{1}{3} + \frac{8\pi}{15} + \frac{128\,\pi^2}{15} \right) J_2 + \left(\frac{1}{5} - \frac{128\pi^2}{15} \right) J_3$$

$$C_2 = \left(\frac{5}{6} - \frac{8\pi}{15} \right) J_1 + \left(\frac{1}{3} + \frac{8\pi}{15} - \frac{128\,\pi^2}{15} \right) J_2 + \left(\frac{1}{3} - \frac{16\,\pi}{15} + \frac{128\,\pi^2}{15} \right) J_3$$

or

$$C_1 = 2.3755\ J_1 + 86.2294\ J_2 - 84.0206\ J_3$$

$$C_2 = -0.8422\ J_1 - 82.2118\ J_2 + 81.2029\ J_3$$

If the incident wave is linearly polarized the calculation goes along the same lines. We can now easily predict the form of the nth order if the incident wave is circularly polarized.

$$\langle \underline{E}_{sc}^{(n)} \cdot \underline{E}_{sc}^{(n)*} \rangle = E_o'^2 \left(\frac{\omega^2}{c^2} |\Delta\chi| \right)^{2n} \left(\frac{1}{4\pi r} \right)^2 N \left(\frac{N}{V} \right)^{n-1} \left(\frac{D}{32\pi^2} \right)^{n-1}$$

$$F_1^{n-1} (K_1 + K_2 \cos^2\theta) \tag{5.1.15}$$

where K_1, K_2 will be of the form $\sum_{i=1}^{3} \alpha_i J_i$, $\sum_{i=1}^{3} \beta_i J_i$ respectively, where α_i, β_i numerical constants.

VI. SPECIAL EXAMPLES

All the following examples assume a lossless medium.

6.1. First Order

We have found (equation 3.1.17 and equation 3.1.18) that in first order and in the absence of losses:

$$\langle I^{(1)}\rangle \sim NE_o^2\left(\frac{\omega^2}{c^2}|\Delta\chi|\right)^2(\sin^2\varphi+\cos^2\theta\cos^2\varphi)F(\theta) \tag{6.1.1}$$

if the incident wave is linearly polarized,and

$$\langle I^{(2)}\rangle \sim NE_o'^2\left(\frac{\omega^2}{c^2}|\Delta\chi|\right)^2(1+\cos^2\theta)F(\theta) \tag{6.1.2}$$

if the incident wave is circularly polarized. $F(\theta)$ is defined by

$$\begin{aligned} F(\theta) &= \frac{1}{8\pi^2}\iiint |K(\alpha,\beta,\gamma;\theta,\varphi)|^2 \sin\alpha\, d\alpha\, d\beta\, d\gamma \\ K(\alpha,\beta,\gamma;\theta,\varphi) &= \int_{V_p} \exp(-ik\underline{e}_r\cdot\underline{R}_i + ikZ_i)\, dV_i \end{aligned} \tag{6.1.3}$$

We can immediately see that if $\theta = 0$, i.e. forward scattering, then $\underline{e}_r = \underline{e}_z$ and $-ik\underline{e}_r\cdot\underline{R}_i + ikZ_i = 0$, i.e. $K = V_p$ and $F(0) = V_p^2$. We easily see that

$$\left|\int_{V_p}\exp(-ik\underline{e}_r\cdot\underline{R}_i+ikZ_i)\, dV_i\right| < \int_{V_p}\left|\exp(-ik\underline{e}_r\cdot\underline{R}_i+ikZ_i\right| dV_i$$

$$= V_p = K(\theta = 0)$$

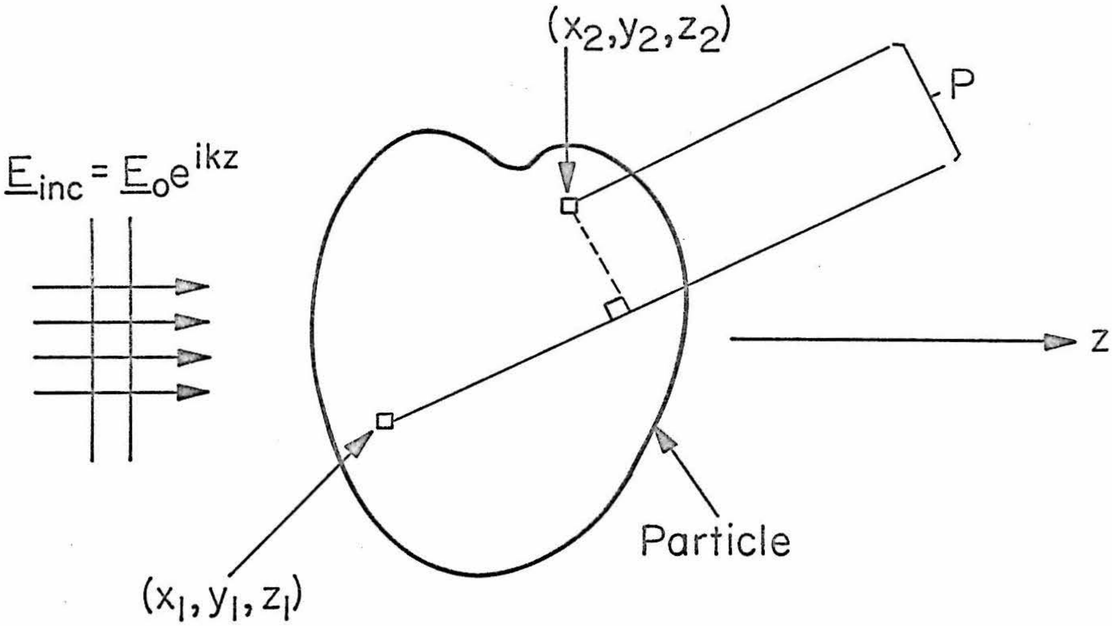


Fig. 4. The rays at P radiated from any two volume elements are not in phase except in the forward direction.

Therefore $K(\theta = 0)$ = maximum and $F(0)$ = maximum.

The physical reason for that is the following. Consider two elements inside a particle (see figure 4). As the plane wave travels inside the volume of the particle the two elements radiate with phases kz_1 and kz_2 respectively. Thus the phase of element 2 is by $k(z_2 - z_1)$ greater than the phase of 1.

If we now consider the rays from the elements in the forward direction toward an observation point P we see that ray 1 travels by $(z_2 - z_1)$ more, i.e. when the rays reach point P they are in phase and the corresponding fields add constructively. For any other direction the phase difference is not zero, therefore the fields add destructively in general. In view of expressions 6.1.1 and 6.1.2 we understand that the intensity pattern will peak in the forward direction.

6.1.1. Our first example is very primitive. However, $F(\theta)$ retains all the essential features of any $F(\theta)$. Consider the arrangements in figure 5a and figure 5b. In figure 5a the two points 1 and 2 get excited by a plane wave passing by. The phase difference between the points is then 2ka. The points radiate and

$$E(\theta) \sim e^{ikb} + e^{ik(2a + b - 2a\cos\theta)}$$

$$= e^{ik(b + a - a\cos\theta)}\left\{e^{-ik(a - a\cos\theta)} + e^{ik(a - a\cos\theta)}\right\}$$

$$= 2e^{ik(b + a - a\cos\theta)}\cos\left[ka(1 - \cos\theta)\right]$$

$$F(\theta) \sim 4\cos^2\left(2ka\sin^2\frac{\theta}{2}\right) \qquad (6.1.4)$$

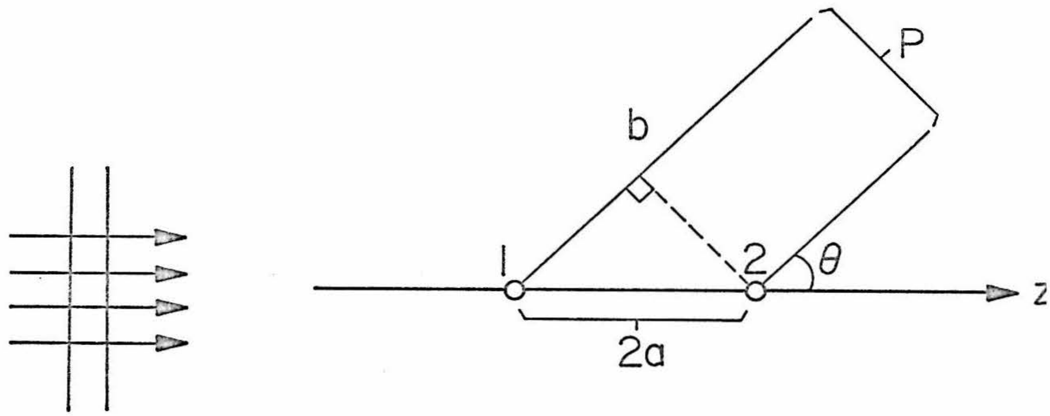


Fig. 5a. Diagram for the two point scatterers 1 and 2 lying in the direction of propagation of the incident wave.

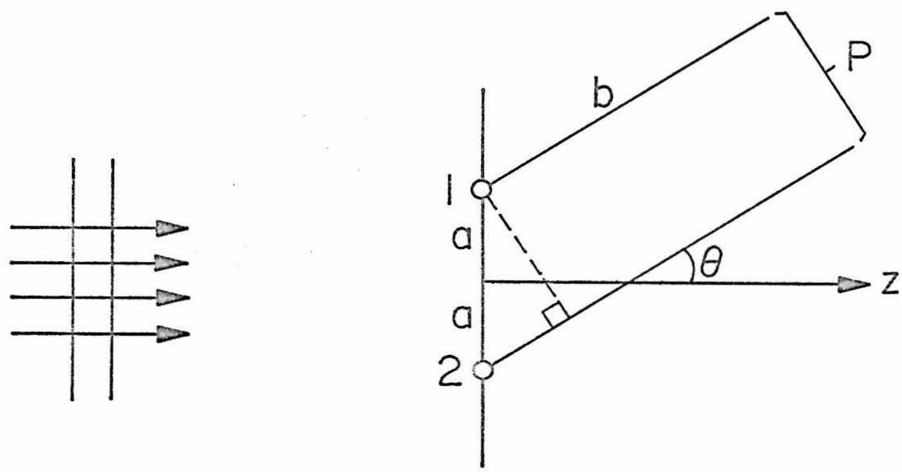


Fig. 5b. Diagram for the two point scatterers with the line joining them normal to the direction of propagation of the incident wave.

Next we consider figure 5b.

$$E(\theta) \sim e^{ikb} + e^{ik(b - 2a \sin\theta)}$$

$$= e^{ikb - ika \sin\theta}(e^{ika \sin\theta} + e^{-ika \sin\theta})$$

$$= 2e^{ik(b - a \sin\theta)} \cos(ka \sin\theta)$$

$$F(\theta) \sim 4 \cos^2(ka \sin\theta) \qquad (6.1.5)$$

We will later see that 6.1.4 and 6.1.5 are pretty close to patterns corresponding to more realistic situations.

6.1.2. Our next example has to do with a collection of spheres of equal radius. This of course is an extreme case but the scattering pattern for particles of any other shape and random orientations will not be too different because of the averaging over orientation. In the present case no averaging w.r.t. the Eulerian angles has to be done because no matter how we rotate a sphere it appears the same.

Let us now compute the K function defined by 6.1.3. It will not of course depend on any Eulerian angles, i.e. $F(\theta) = |K|^2$. We will refer the components of $\underline{e}_r$ and $\underline{R}$ to the fixed x,y,z system, i.e.

$$\underline{e}_r = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta)$$

$$\underline{R} = (R \sin\theta_i \cos\varphi_i, R \sin\theta_i \sin\varphi_i, R \cos\theta_i)$$

and K becomes:

$$K = \int_{\varphi_i=0}^{2\pi}\int_{\theta_i=0}^{\pi}\int_{R=0}^{a} \exp\Big(-ikR\cos(\varphi_i-\varphi)\sin\theta_i\sin\theta + ikR\cos\theta_i(1-\cos\theta)\Big)$$

$$R^2\, dR \sin\theta_i\, d\theta_i\, d\varphi_i$$

As it is shown in Appendix E-ii the integral can be computed in closed form:

$$K(\theta) = \frac{\pi}{2(k\sin\frac{\theta}{2})^3}\left[\sin(2ka\sin\frac{\theta}{2}) - 2ka\sin\frac{\theta}{2}\cos(2ka\sin\frac{\theta}{2})\right]$$

$$F(\theta) = K^2(\theta) \tag{6.1.6}$$

First we check whether $K(0) = V_p = \frac{4}{3}\pi a^3$. We have

$$K\Big|_{\theta\to 0} = \frac{\pi}{2(k\sin\frac{\theta}{2})^3}\left\{\left[2ka\sin\frac{\theta}{2} - \frac{(2ka\sin\frac{\theta}{2})^3}{3!} + \ldots\right]\right.$$

$$\left. - 2ka\sin\frac{\theta}{2}\left[1 - \frac{(2ka\sin\frac{\theta}{2})^2}{2!} + \ldots\right]\right\}$$

$$= \frac{\pi}{2(k\sin\frac{\theta}{2})^3}\left\{\frac{(2ka\sin\frac{\theta}{2})^3}{3} + O(ka\sin\frac{\theta}{2})^5\right\}$$

$$= \frac{4}{3}\pi a^3 = V_p \quad \text{O.K.}$$

If we now recall that the spherical Bessel function of the first kind and of order one is given by:

$$j_1(u) = \frac{1}{u^2}(\sin u - u\cos u)$$

we understand that

$$K(\theta) = \frac{4\pi a^3}{(2\,ka\sin\frac{\theta}{2})}\; j_1(2\,ka\sin\frac{\theta}{2}) \tag{6.1.7}$$

It is easy to show that for θ and k constant the $\max K^2(\theta)$ occurs for

$$\sin(2ka\sin\frac{\theta}{2}) = 0 \quad \text{i.e.}\quad a = \frac{n\pi}{2\,k\sin\frac{\theta}{2}} \qquad n = 1, 2, \ldots$$

If we compute $\max K^2(\theta)$ we find

$$\max K^2(\theta) = (4\pi a^3/(n\pi)^2)^2$$

whereas $K^2(0) = (4\pi a^3/3)^2$

$$\frac{\max K^2(\theta)}{K^2(0)} = \frac{9}{\pi^4 n^4} < 1$$

The zeros of $K^2(\theta)$ are the zeros of $j_1(u)$. From reference 6 we find that

$$j_1(u) = 0 \qquad u = 4.5,\ 7.74,\ \ldots$$

i.e.

$$2\,ka\sin\frac{\theta}{2} = 4.5,\ 7.74,\ \ldots \qquad (\theta \neq 0)$$

If k is constant and a increases then the zero occurs at smaller and smaller θ, whereas $\theta = 0$ always remains a maximum. Thus for a large a, say $ka = 50$, we get a maximum at $\theta = 0$ and many zeros (or minima) close to $\theta = 0$.

Next we draw some diagrams to get some idea about the intensity pattern.

i) $ka = \frac{\pi}{2}$ or $a = \frac{\lambda}{4}$ which corresponds to max $K\ (\theta = \pi)$ with $n = 1$. We can easily find:

$$F(0) = V_p^2, \qquad F(\tfrac{\pi}{4}) = 0.744\ V_p^2\ , \qquad F(\tfrac{\pi}{2}) = 0.34\ V_p^2\ ,$$

$$F(\tfrac{3\pi}{4}) = 0.14\ V_p^2\ , \quad F(\pi) = 0.092\ V_p^2$$

Notice that no zero occurs since $2ka < 4.5$.

If the incident wave is circularly polarized then the actual intensity pattern is given by $\frac{1}{2}(1 + \cos^2\theta)F(\theta)$, i.e.

$$I(0) = V_p^2, \qquad I(\tfrac{\pi}{4}) = 0.56\ V_p^2\ , \qquad I(\tfrac{\pi}{2}) = 0.17\ V_p^2\ ,$$

$$I(\tfrac{3\pi}{4}) = 0.10\ V_p^2, \quad I(\pi) = 0.092\ V_p^2$$

The drawings appear in figure 6.

ii) $ka = \frac{\pi}{\sqrt{2}}$ for which $K^2\ (\theta = \frac{\pi}{2})$ is a maximum. Again $2ka < 4.5$ and there is no zero. We can easily find (set $V_p = 1$):

$$F(0) = 1\ , \qquad F(\tfrac{\pi}{4}) = 0.546\ , \qquad F(\tfrac{\pi}{2}) = 9.2 \times 10^{-2}\ ,$$

$$F(\tfrac{3\pi}{4}) = 4.3 \times 10^{-3}, \qquad F(\pi) = 5.61 \times 10^{-5}$$

and

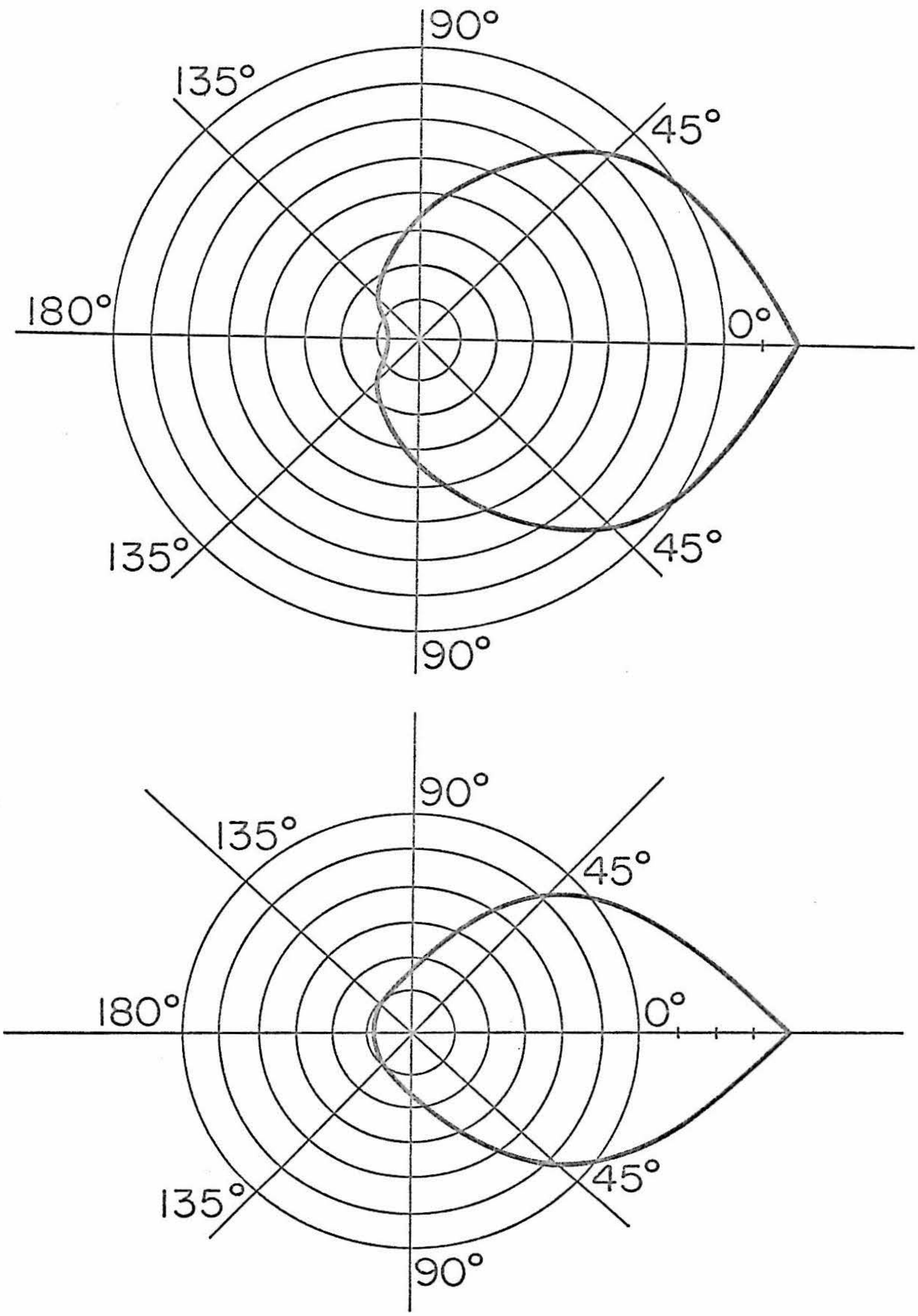


Fig. 6. Drawings of $F(\theta)$ (above) and $I(\theta)$ (below) for spheres with $2ka = \pi$.

$I(0) = 1$, $I(\frac{\pi}{4}) = 0.41$, $I(\frac{\pi}{2}) = 4.6 \times 10^{-2}$,

$I(\frac{3\pi}{4}) = 3.2 \times 10^{-3}$, $I(\pi) = 5.61 \times 10^{-5}$

The drawings appear in figure 7.

Notice that $2ka$ has increased and so has the forward scattering whereas the intensity for all other angles has gone down.

iii) When $2ka \sin\frac{\theta}{2} = 4.5$, i.e. $j_1(u) = 0$, then the smallest $2ka$ corresponds to $\theta = \pi$. Let us choose $2ka \sin\frac{\theta}{2} = 4.5$ with $\theta = \frac{3\pi}{4}$, i.e. $2ka = 4.87$. No other zeros occur since the next zero is 7.74 and $2ka < 7.74$. We have

$F(0) = 1$, $F(\frac{\pi}{4}) = 0.48$, $F(\frac{\pi}{2}) = 4.82 \times 10^{-2}$,

$F(\frac{3\pi}{4}) = 0$, $F(\pi) = 2.07 \times 10^{-3}$

and

$I(0) = 1$, $I(\frac{\pi}{4}) = 0.36$, $I(\frac{\pi}{2}) = 2.41 \times 10^{-2}$,

$I(\frac{3\pi}{4}) = 0$, $I(\pi) = 2.07 \times 10^{-3}$

The drawings appear in figure 8.

Finally we examine the case where we have more zeros. For example if we choose $2ka \sin\frac{\theta}{2} = 4.5$ with $\theta = \frac{\pi}{4}$ then $2ka = 11.7$ and we get two more zeros:

$$11.7 \sin\frac{\theta}{2} = 7.74$$

$$11.7 \sin\frac{\theta}{2} = 10.95$$

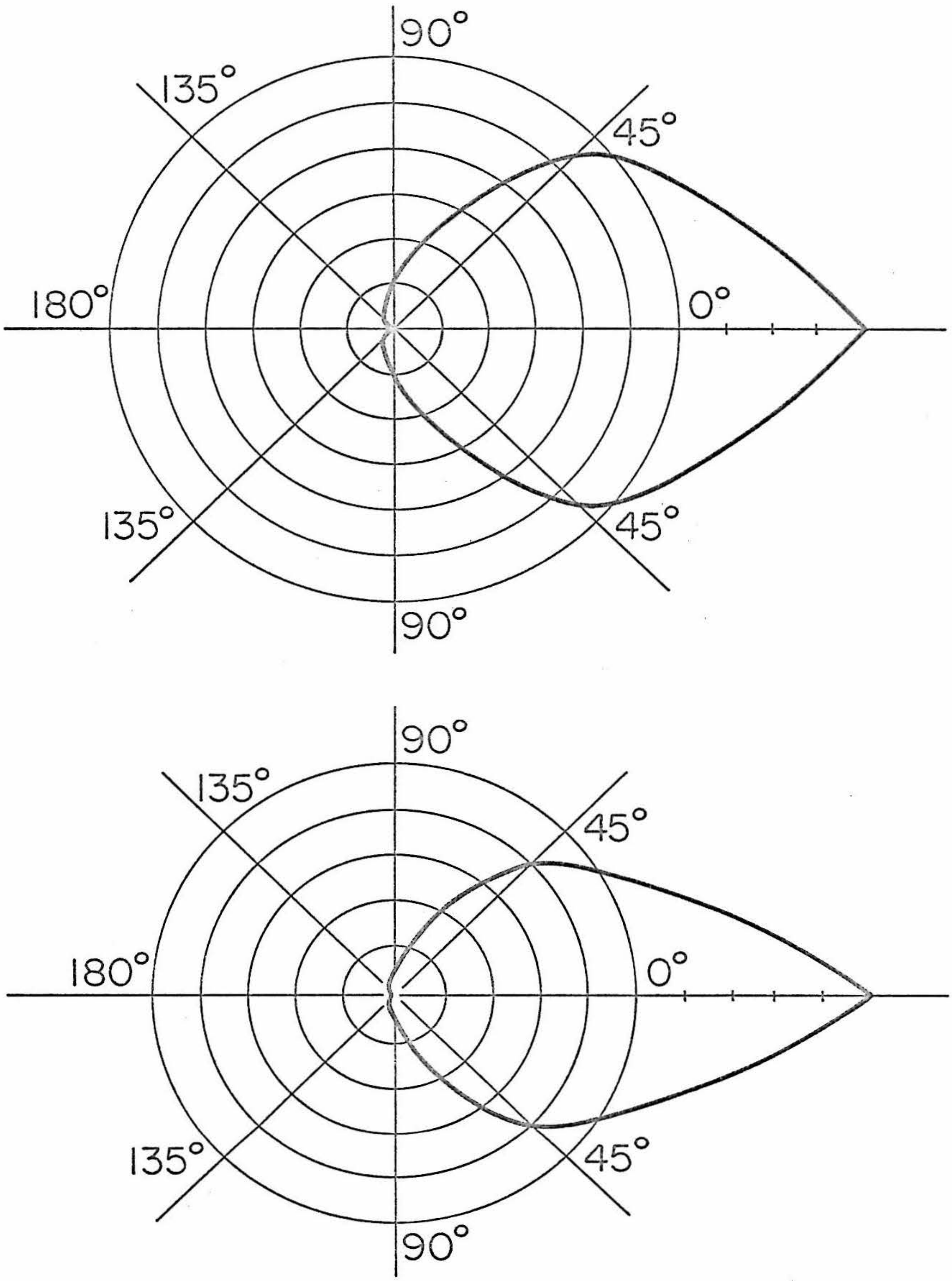


Fig. 7. Drawings of $F(\theta)$ (above) and $I(\theta)$ (below) for spheres with $2ka = \pi\sqrt{2}$.

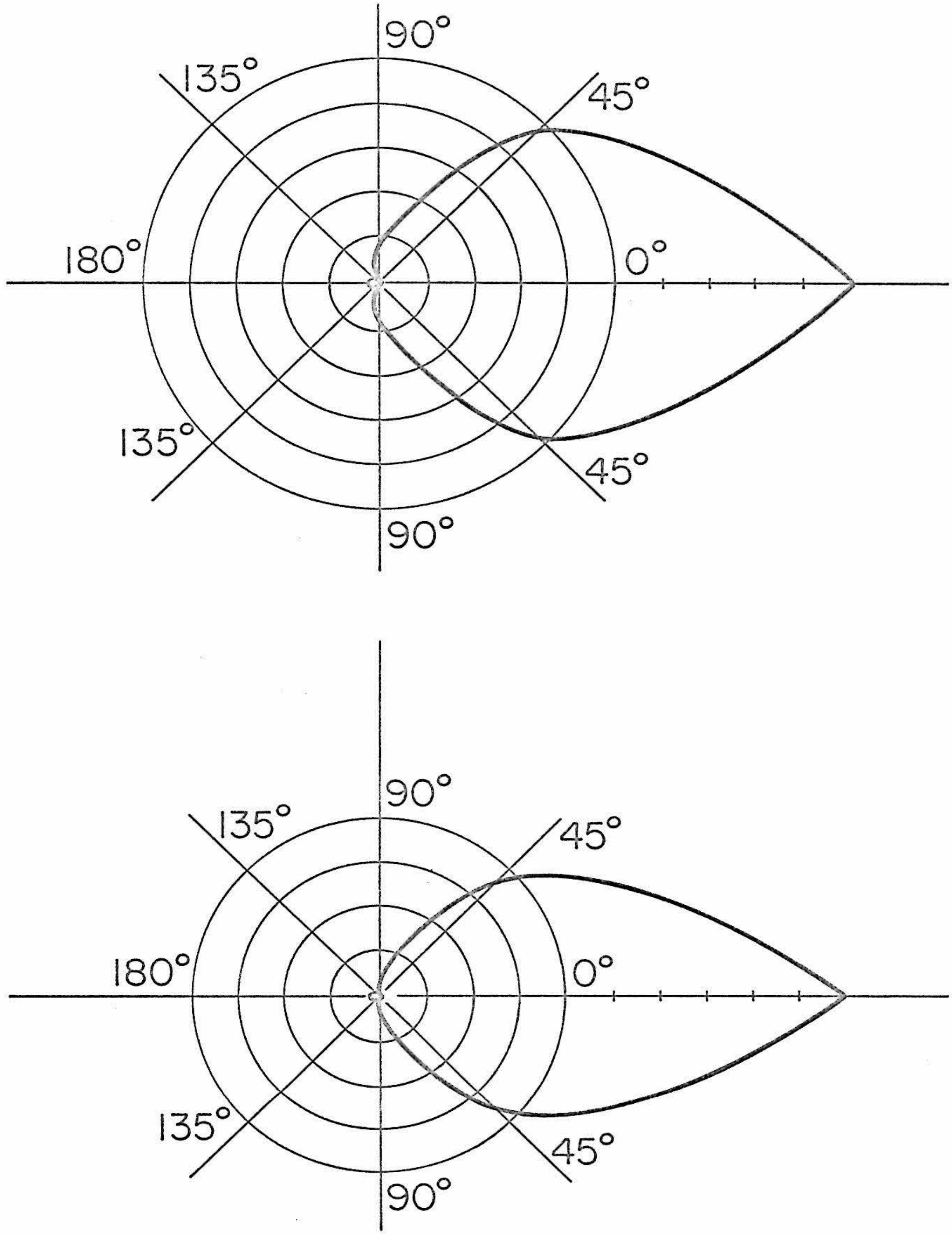


Fig. 8. Drawings of $F(\theta)$ (above) and $I(\theta)$ (below) for spheres with $2ka = 4.87$ for which $I(135^{\circ}) = 0$.

We thus have three zeros at $\theta = \frac{\pi}{4}$, $\theta = 82^\circ 36'$, $\theta = 138^\circ 30'$. We also have:

$$F(0) = 1, \quad F(\tfrac{\pi}{4}) = 0, \quad F(\tfrac{\pi}{8}) = 0.322, \quad F(66^\circ) = 4.9 \times 10^{-3},$$

$$F(\tfrac{\pi}{2}) = 5.14 \times 10^{-4}, \quad F(116^\circ) = 7.9 \times 10^{-4},$$

$$F(\tfrac{3\pi}{4}) = 5.8 \times 10^{-6}, \quad F(\pi) = 2.44 \times 10^{-4}$$

and

$$I(0) = 1, \quad I(\tfrac{\pi}{4}) = 0, \quad I(\tfrac{\pi}{8}) = 2.98 \times 10^{-1}, \quad I(66^\circ) = 2.9 \times 10^{-3},$$

$$I(\tfrac{\pi}{2}) = 2.57 \times 10^{-4}, \quad I(116^\circ) = 7 \times 10^{-4},$$

$$I(\tfrac{3\pi}{4}) = 4.39 \times 10^{-6}, \quad I(\pi) = 2.44 \times 10^{-4}$$

The intensity pattern is shown in figure 9.

6.1.3. We will now consider a collection of needle-like particles. The axis of a needle will be characterized by the two polar angles θ_i, φ_i w.r.t. the fixed system xyz. The averaging then will be done over them. We have to compute $K(\theta_i, \varphi_i, \theta, \varphi)$ where

$$K(\theta_i, \varphi_i; \theta, \varphi) = \int_{V_p} \exp(-ik\underline{e}_r \cdot \underline{R}_i + ikZ_i)\, dV_i$$

Notice that $\underline{R}_i = (R, \theta_i, \varphi_i)$ for half of the needle and $\underline{R}_i = (R, \pi - \theta_i, \pi + \varphi_i)$ for the rest of the needle (see figure 10).

If A is the cross section of the needle then we can write

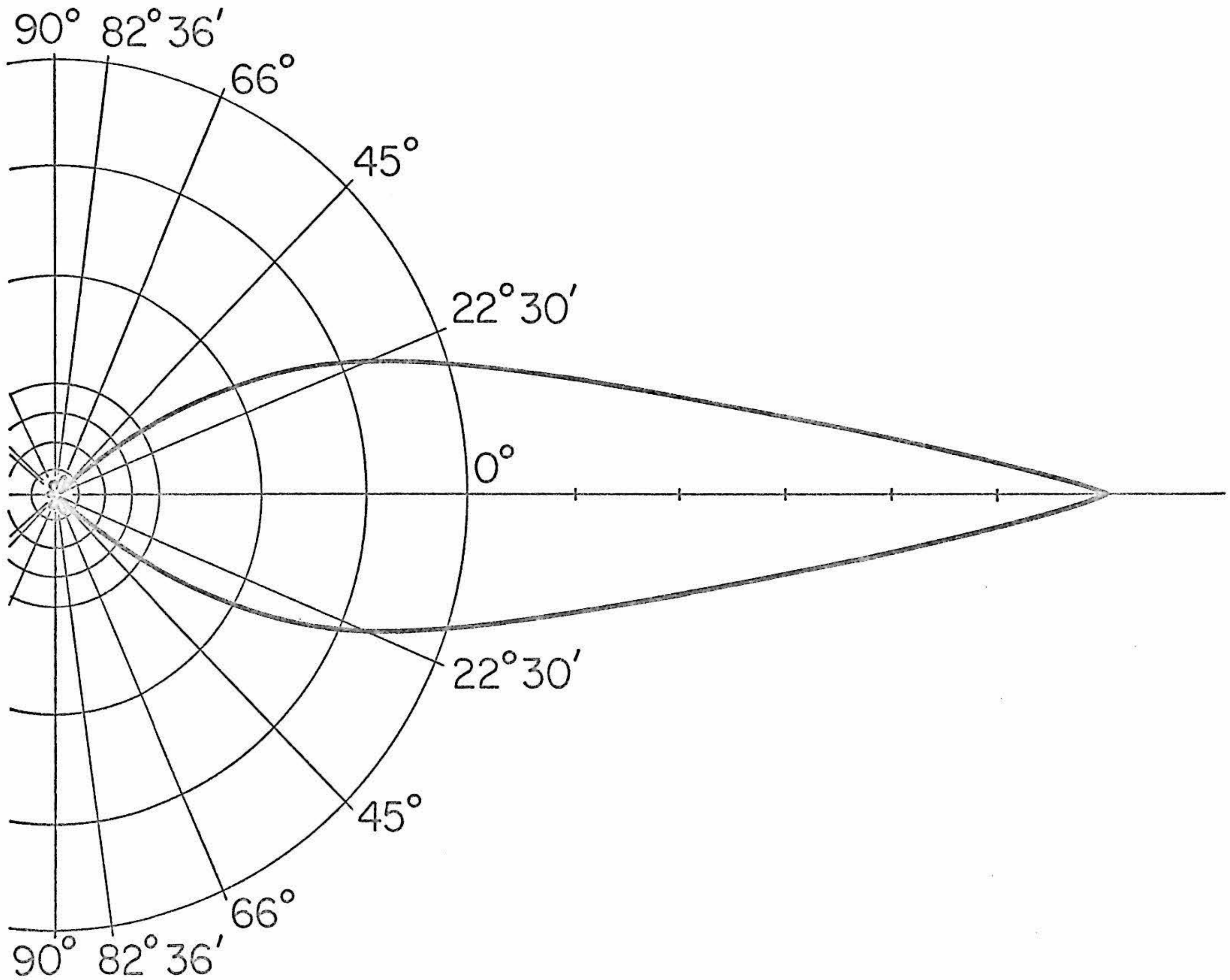


Fig. 9. Drawing of $I(\theta)$ for spheres with $2ka = 11.70$ for which $I(45^\circ) = 0$, $I(82^\circ 36') = 0$, $I(138^\circ 30') = 0$.

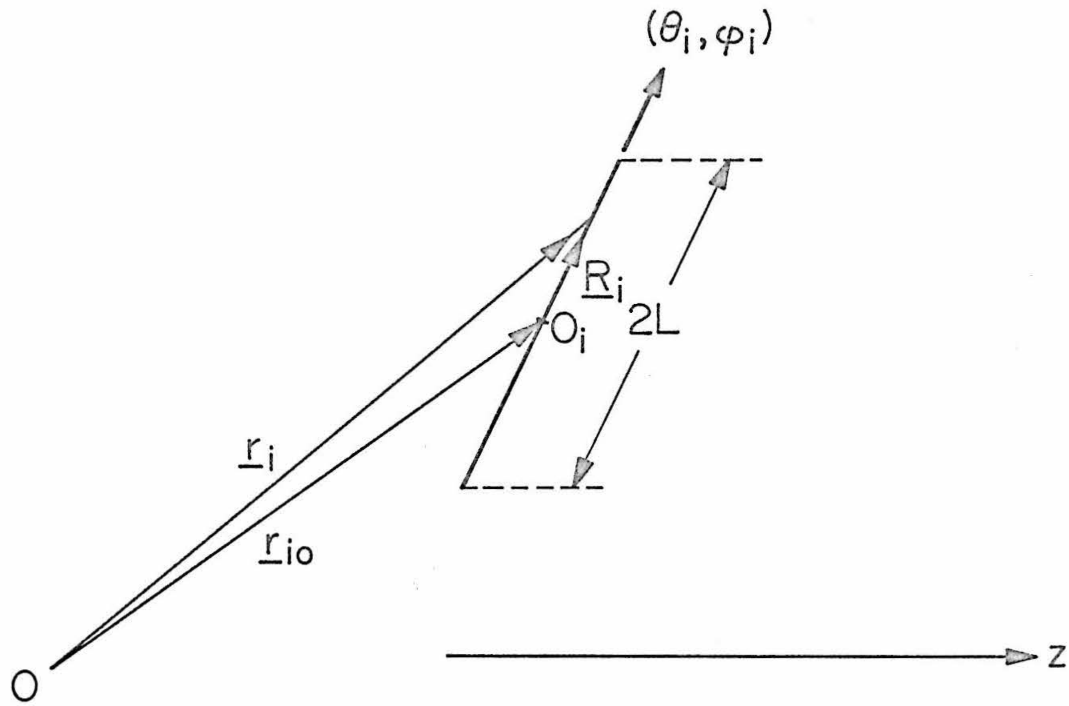


Fig. 10. Needle-like particle. The axis of the needle points in the (random) direction θ_i, φ_i .

$$K(\theta_i, \varphi_i; \theta, \varphi) \approx A\int_o^L \exp\{-ik[\cos(\varphi-\varphi_i)\sin\theta_i\sin\theta + \cos\theta_i(\cos\theta - 1)]R\}\, dR$$

$$+ A\int_o^L \exp\{ik[\cos(\varphi-\varphi_i)\sin\theta_i\sin\theta + \cos\theta_i(\cos\theta - 1)]R\}\, dR$$

$$= A\int_{-L}^L \exp\{-ik[\cos(\varphi-\varphi_i)\sin\theta_i\sin\theta + \cos\theta_i(\cos\theta - 1)]x\}\, dx$$

$$= \frac{2A}{k\Gamma_i}\sin k\Gamma_i L \tag{6.1.8}$$

where $\Gamma_i = \cos(\varphi-\varphi_i)\sin\theta_i\sin\theta + \cos\theta_i(\cos\theta - 1)$. We can easily see that as $\theta \to 0$, $\sin k\Gamma_i L/k\Gamma_i \to L$ and $K \to 2AL = V_p$ as it should. Next we must compute $F(\theta)$ given by:

$$F(\theta) = \frac{1}{4\pi}\int\int |K(\theta_i, \varphi_i; \theta, \varphi)|^2\, d\Omega_i$$

$$= \frac{1}{4\pi}\frac{4A^2}{k^2}\int_o^{2\pi}\int_o^{\pi}\left\{\frac{\sin\{kL[\cos(\varphi-\varphi_i)\sin\theta_i\sin\theta + \cos\theta_i(\cos\theta-1)]\}}{\cos(\varphi-\varphi_i)\sin\theta_i\sin\theta + \cos\theta_i(\cos\theta - 1)}\right\}^2$$

$$\sin\theta_i\, d\theta_i\, d\varphi_i\ .$$

It is shown in Appendix E-iii that

$$F(\theta) = \frac{2V_p^2}{2kL_o\sin\frac{\theta}{2}}\left\{\frac{\cos(2kL_o\sin\frac{\theta}{2}) - 1}{2kL_o\sin\frac{\theta}{2}} + s_i(2kL_o\sin\frac{\theta}{2})\right\} \tag{6.1.9}$$

where V_p is the volume of a particle, $L_o = 2L$ = total length of the needle-like particle and $s_i(x)$ is the sine integral defined as

$$s_i(x) = \int_o^x \frac{\sin u}{u}\, du$$

Again we can check whether $F(\theta=0) = V_p^2$. We have

$$F(\theta)\Big|_{\theta\to 0} = \frac{2V_p^2}{2kL_o \sin\frac{\theta}{2}} \left\{ \frac{1 - \frac{1}{2}(2kL_o \sin\frac{\theta}{2})^2 - 1 + \ldots}{2kL_o \sin\frac{\theta}{2}} \right.$$

$$\left. + \frac{2kL_o \sin\frac{\theta}{2}}{2kL_o \sin\frac{\theta}{2}} 2kL_o \sin\frac{\theta}{2} + \ldots \right\}\Big|_{\theta\to 0} = V_p^2 \quad \text{O.K.}$$

One can easily see that $F(\theta)$ has no zeros, i.e. the averaging procedure has eliminated the sharp behavior of just one of the needles.

Next we draw some diagrams for the intensity pattern <u>if the incident wave is circularly polarized.</u>

i) $L_o = \frac{\lambda}{4}$, i.e. $2kL_o = \pi$

One can find in reference 6 the values of $s_i(x)$ for given x. Thus we get $(V_p^2 = 1)$

$F(0) = 1$ $\quad F(\frac{\pi}{4}) = 0.96$ $\quad F(\frac{\pi}{2}) = 0.865$

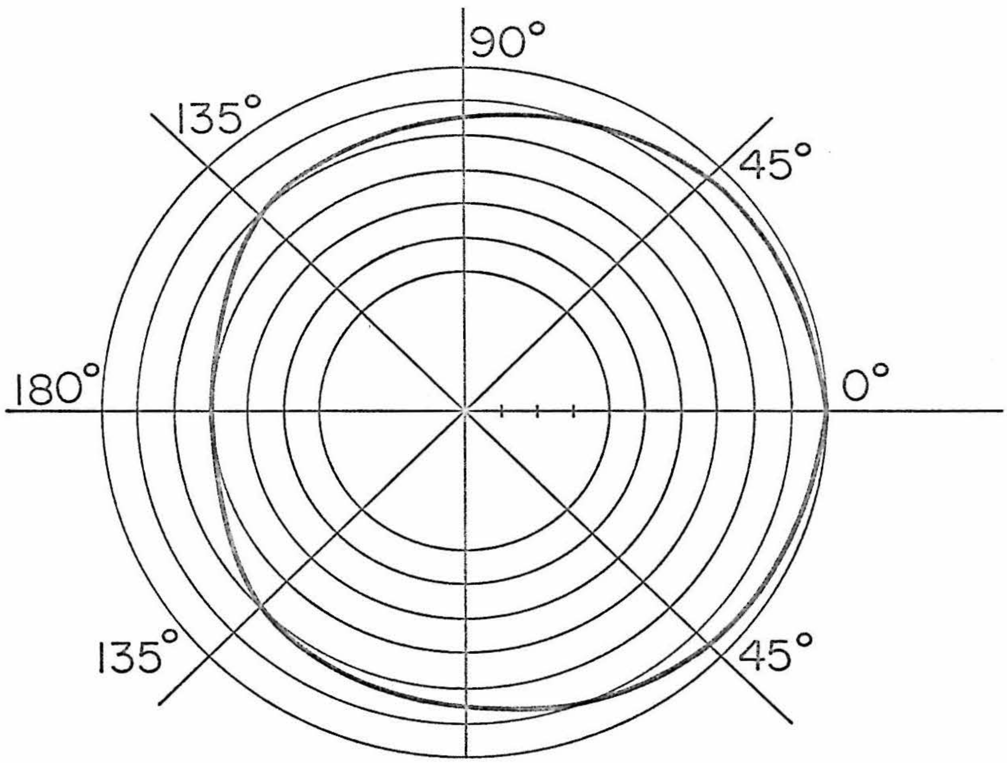
$F(\frac{3\pi}{4}) = 0.795$ $\quad F(\pi) = 0.70$

and

$I(0) = 1$ $\quad I(\frac{\pi}{4}) = 0.72$ $\quad I(\frac{\pi}{2}) = 0.43$

$I(\frac{3\pi}{4}) = 0.60$ $\quad I(\pi) = 0.70$

The diagrams appear in figure 11.



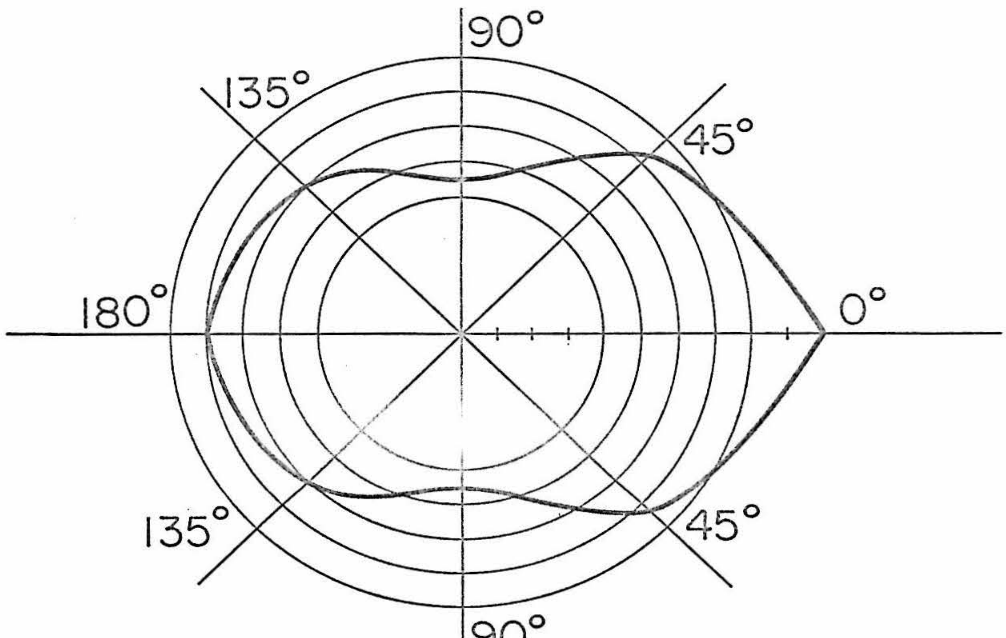


Fig. 11. Drawing of $F(\theta)$ (above) and $I(\theta)$ (below) for needles with $2kL_o = \pi$.

ii) $L_o = 2\lambda$, $2kL_o = 4\pi$

$$F(0) = 1 \qquad F(\tfrac{\pi}{4}) = 0.31 \qquad F(\tfrac{\pi}{2}) = 0.16$$

$$F(\tfrac{3\pi}{4}) = 0.13 \qquad F(\pi) = 0.12$$

and

$$I(0) = 1 \qquad I(\tfrac{\pi}{4}) = 0.23 \qquad I(\tfrac{\pi}{2}) = 0.08$$

$$I(\tfrac{3\pi}{4}) = 0.10 \qquad I(\pi) = 0.24$$

The diagrams appear in figure 12.

As L_o gets bigger the forward scattering gets more pronounced; a tendency which has been observed for the spheres too. However, we should notice that the peaking in the forward direction was much more dramatic for the spheres rather than the needles. This is not surprising because the spheres scatter the same way no matter how we rotate them, but the needles do not. Thus for a needle perpendicular to the axis of propagation the forward scattering is equal to the back scattering whereas for a needle parallel to the axis of propagation the maximum occurs in the forward direction. When we average over θ_i, φ_i we get a pattern which lies in between the extreme cases considered above. Of course the forward scattering is still a maximum.

6.1.4. As our final example we consider a collection of particles possessing an azimuthal symmetry about a certain axis, say z'. Here we need the full formalism of the Eulerian angles because $\underline{R}_i$ does not coincide with the z' axis. However because of the

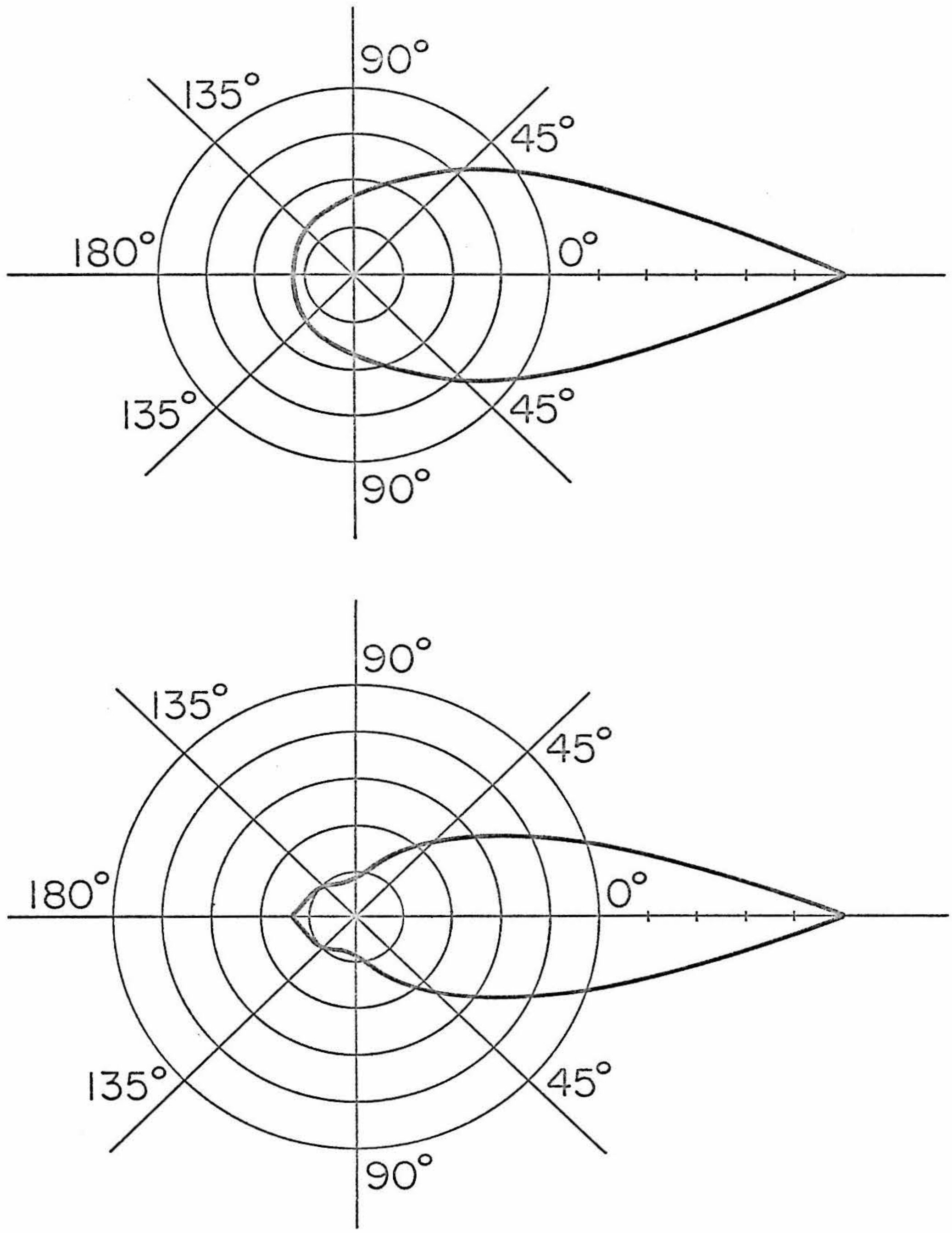


Fig. 12. Drawing of $F(\theta)$ (above) and $I(\theta)$ (below) for needles with $2kL_o = 4\pi$.

azimuthal symmetry we can get rid of one of the Eulerian angles. This is evident because all we have to care for is the direction of z' which can be characterized by the two polar angles θ', φ'. Thus if we choose the x' axis in the fixed xy plane we can get $\gamma = 0$. (As it is shown in Appendix B we have $\theta' = \alpha$ and $\varphi' = -\frac{\pi}{2} + \beta$.) The matrix M^{-1} becomes

$$M^{-1} = \begin{bmatrix} \cos\beta & -\cos\alpha\sin\beta & \sin\alpha\sin\beta \\ \sin\beta & \cos\alpha\cos\beta & -\sin\alpha\cos\beta \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

We then find that

$$K(\alpha, \beta; \theta, \varphi) = \iiint \exp[-ik(A\rho\cos\tau + B\rho\sin\tau + Cz')]\ \rho\ d\rho\ d\tau\ dz' \quad (6.1.10)$$

where ρ, τ, z' are cylindrical coordinates in the attached to the particle system $x'y'z'$. The surface of the particle is given by $\rho = \rho(z')$. One can easily find that

$$A = \sin\theta\cos(\varphi-\beta)$$

$$B = \sin\theta\cos\alpha\sin(\varphi-\beta) + \cos\theta\sin\alpha - \sin\alpha$$

$$C = \sin\theta\sin\alpha\sin(\beta-\varphi) + (\cos\theta - 1)\cos\alpha$$

We can immediately do the τ integration. As it is shown in Appendix E-iv we find that:

$$\int_0^{2\pi} \exp\left(-ik(A\cos\tau + B\sin\tau)\rho\right)\ d\tau = 2\pi J_0(k\rho\sqrt{A^2 + B^2})$$

i.e.

$$K(\alpha,\beta;\theta,\varphi) = \int_{z_1}^{z_2} e^{-ikCz'}\, dz' \int_{\rho=0}^{\rho(z')} 2\pi J_o(k\rho(z')\ \sqrt{A^2+B^2})\ \rho\ d\rho$$

We can also do the ρ integration (see Appendix E-iv) to find:

$$K(\alpha,\beta;\theta,\varphi) = \int_{z_1}^{z_2} 2\pi\rho(z')\ \frac{J_1(k\rho(z')\ \sqrt{A^2+B^2})}{k\ \sqrt{A^2+B^2}}\ e^{-ikCz'}\ dz'$$

$$= \frac{2\pi}{k}\ \frac{1}{\sqrt{A^2+B^2}} \int_{z_1}^{z_2} \rho(z')J_1(k\rho(z')\ \sqrt{A^2+B^2})e^{-ikCz'}\ dz' \qquad (6.1.11)$$

Next we check whether 6.1.11 gives the right answer for needle-like particles, i.e. equation 6.1.8. We have $\rho = 0...a$, i.e.

$$K(\alpha,\beta;\theta,\varphi) = \frac{2\pi}{k}\ \frac{1}{\sqrt{A^2+B^2}} \int_{-L}^{L} aJ_1(ka\ \sqrt{A^2+B^2})e^{-ikCz'}\ dz'$$

$$= \frac{2\pi a}{k\ \sqrt{A^2+B^2}}\ 2J_1(ka\ \sqrt{A^2+B^2})\ \sin kLC \qquad (6.1.12)$$

Now we write

$$J_1(x) = \frac{x}{2}\left[1 - \frac{1}{2}\left(\frac{x}{2}\right)^2 + \ldots\right]$$

i.e.

$$\left.\frac{J_1(ka\ \sqrt{A^2+B^2})}{\sqrt{A^2+B^2}}\right|_{ka\to 0} = \frac{ka}{2} + \ldots$$

and

$$K(\alpha,\beta;\theta,\varphi) = \frac{2\pi a^2}{kC}\sin kLC = \frac{2A}{kC}\sin kLC$$

Now

$$C = \sin\theta\sin\alpha\sin(\beta-\varphi) + (\cos\theta - 1)\cos\alpha$$

or in view of the equality $\theta_i = \alpha$, $\varphi_i = -\frac{\pi}{2} + \beta$ (see Appendix B) we get

$$C = \sin\theta\sin\theta_i\cos(\varphi_i - \varphi) + (\cos\theta - 1)\cos\theta_i$$

which is identical with the Γ_i of 6.1.8. Therefore, 6.1.11 gives the right answer for the needles as it should. Now we compute $F(\theta)$ from

$$F(\theta) = \frac{1}{4\pi}\int\int |K(\alpha,\beta;\theta,\varphi)|^2 \sin\alpha\, d\alpha\, d\beta$$

or

$$F(\theta) = \left(\frac{2\pi}{k}\right)^2\int_0^{2\pi}\int_0^{\pi}\frac{\sin\alpha\, d\alpha\, d\beta}{A^2+B^2}\int_{z_1}^{z_2}\int_{z_1}^{z_2}\rho(z')\rho(z'')J_1(k\beta(z')\sqrt{A^2+B^2})$$

$$J_1(k\rho(z'')\sqrt{A^2+B^2})\, e^{ikC(z'-z'')}dz'\, dz''$$

$$= \left(\frac{2\pi}{k}\right)^2\int_{z_1}^{z_2}\int_{z_1}^{z_2}\rho(z')\rho(z'')$$

$$\left\{\int_0^{2\pi}\int_0^{\pi}\frac{J_1(k\rho' D)J_1(k\rho'' D)e^{ik(z'-z'')C}}{D}\sin\alpha\, d\alpha\, d\beta\right\} dz'\, dz''$$

where

$$D = \sqrt{A^2+B^2} = D(\alpha,\beta;\theta,\varphi) \quad C = (\alpha,\beta;\theta,\varphi) \quad \rho' = \rho(z') \qquad \rho'' = \rho(z'')$$

I have not been able to compute the integral w.r.t. α and β in closed form. Of course $F(\theta)$ is independent of φ because D and C are functions of $\cos(\varphi-\beta)$ and a change $\varphi-\beta=\beta'$ eliminates φ since β runs from 0 to 2π.

6.2. Second Order

We have found that in the absence of losses equation 4.1.14 holds:

$$\langle \underline{E}_{sc}^{(2)} \cdot \underline{E}_{sc}^{(2)} \rangle = E_o'^2 \left(\frac{\omega^4}{c^4} |\Delta\chi|^2\right)^2 \left(\frac{1}{4\pi r}\right)^2 N\frac{N}{V} \frac{R}{16\pi^2} F_1\{B_1+B_2\cos^2\theta\}$$

where

$$F_1 = \frac{1}{8\pi^2}\int\int\int |L(\alpha,\beta,\gamma;\theta,\varphi)|^2 \sin\alpha \, d\alpha \, d\beta \, d\gamma$$

and

$$L(\alpha_i,\beta_i,\gamma_i;\theta,\varphi) = \int_{V_i} e^{-ik\underline{e}_r\cdot\underline{R}_i} dV_i$$

Next we consider the collection of the spheres as our example. Then L is independent of $\alpha_i, \beta_i, \gamma_i$ and $F_1(\theta) = |L|^2$. Now

$$L = \int_{V_p} e^{-ik\underline{e}_r\cdot\underline{R}_i} dV_i$$

If we pick $\underline{e}_r$ as the polar axis then $\underline{e}_r\cdot\underline{R}_i = R_i\cos\theta_i$ and

$$L = 2\pi\int_o^a R_i^2 \, dR_i \int_o^\pi e^{-ikR_i\cos\theta'} \sin\theta' \, d\theta' = 2\pi\int_o^a R_i^2 \, dR_i \frac{2\sin kR_i}{kR_i}$$

$$= \frac{4\pi}{k}(\sin ka - ka\cos ka) \qquad \text{independent of } \theta.$$

Of course we get the same result if we choose z as the polar axis and write

$$\underline{e}_r = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta)$$

$$\underline{R}_i = (\sin\theta_i \cos\varphi_i, \sin\theta_i \sin\varphi_i, \cos\theta_i)$$

It is not of course true that L is independent of θ for any particle. It is only the spherical symmetry of a sphere that renders L independent of θ. However F_1 is always independent of θ. We thus have

$$F_1(\theta) = \left(\frac{4\pi}{k^3}\right)^2 (\sin ka - ka \cos ka)^2 = \left[\frac{4\pi a^3}{ka} j_1(ka)\right]^2$$

We now understand that the zeros of F_1, i.e. the zeros of $j_1(ka)$, except the first one $ka = 0$, make the second order contribution vanish. This is of course only approximately true and it is due to the approximations made when the second order correction was derived. As a matter of fact the method of deriving the several orders is such that for a collection of spheres all the orders higher than the first vanish for $j_1(ka) = 0$ with $ka \neq 0$. Our theory does not give good results for the non-zero values of ka that make $j_1(ka) = 0$ for a collection of spheres of equal radius. <u>In general, however, the averaging procedure will wash out the zeros and F_1 will be non-zero</u>.

Next we have to compute the coefficients B_1 and B_2. We have the following definitions:

$$B_1 = J_1 + J_3 + 4\pi(J_1 - J_3)$$

$$B_2 = J_1 + J_3 - 4\pi(J_1 - J_3)$$

where

$$J_1 = \frac{1}{2}\int_o^\pi F(\theta) \sin\theta \, d\theta$$

$$J_3 = \frac{1}{2}\int_o^\pi F(\theta) \cos^4\theta \sin\theta \, d\theta$$

Recall now that $F(\theta) > 0$ always and $\sin\theta > 0$ in the region $(0,\pi)$. We thus understand that J_1 and J_3 are positive numbers. We also observe that:

$$J_3 = \frac{1}{2}\int_o^\pi F(\theta) \cos^4\theta \sin\theta \, d\theta < \frac{1}{2}\int_o^\pi F(\theta) \sin\theta \, d\theta = J_1$$

since $0 \lesssim \cos^4 \theta \lesssim 1$.

Thus $J_1 > J_3$ and $B_1 > 0$. What about B_2? B_2 is negative if $J_1 + J_3 - 4\pi(J_1 - J_3) < 0$ or $J_3 < \frac{4\pi - 1}{4\pi + 1} J_1 \approx 0.85 \, J_1$. We know that $\max F(\theta) = F(0) = V_p^2$. Therefore:

$$\max J_1 = \frac{V_p^2}{2}\int_o^\pi \sin\theta \, d\theta = V_p^2$$

$$\max J_3 = \frac{V_p^2}{2}\int_o^\pi \cos^4\theta \sin\theta \, d\theta = \frac{V_p^2}{5}$$

and

$$\text{Max } J_3 = 0.2 \max J_1$$

Now in general $F(\theta)$ has no zeros and does not vary significantly with θ as long as ka is small. It is understood that the maximum variation of $F(\theta)$ occurs for a collection of spheres. In this case B_2

is negative up to $2ka \approx 17$ where $2a$ is the diameter of the spheres. In general we expect B_2 to stay negative for ka larger than 17 where a is the linear characteristic dimension of the particle. We thus conclude that if ka is not too large $F(\theta)$ varies smoothly and $J_3 < \frac{4\pi-1}{4\pi+1} J_1$, i.e. $B_2 < 0$. The consequence of $B_1 > 0$, $B_2 < 0$ is that the second order intensity pattern has its maximum at $\theta = \frac{\pi}{2}$ and its minima at $\theta = 0, \pi$. This means that the multiple scattering tends to smooth out the forward peaking of the first order intensity pattern. This ceases to be true if ka becomes very large, in which case $B_1 + B_2 \cos^2\theta$ with $B_1, B_2 > 0$ has its maximum at $\theta = 0, \pi$ and its minimum at $\theta = \frac{\pi}{2}$. Thus for very large ka the multiple scattering makes the forward scattering even more pronounced. This is expected to be true for all the orders. However the critical value of ka may increase with the order.

The ratio of the intensity in the θ direction to the forward direction is $R(\theta) = (B_1 + B_2\cos^2\theta)/(B_1 + B_2)$. We can easily see that $B_1 \sim \frac{1}{k^6} B_1'(ka)$ and $B_2 \sim \frac{1}{k^6} B_2'(ka)$; we thus understand that $R(\theta)$ will only depend on ka but not on k (or λ) alone. This is also true for the first order scattering intensity pattern and it is true for all the orders.

We now return to the example of the collection of the spheres. Recall that:

$$F(\theta) = \left(\frac{4\pi a^3}{2ka \sin\frac{\theta}{2}} \right)^2 j_1^2 (2ka \sin\frac{\theta}{2})$$

If we call $2\, ka \sin\frac{\theta}{2} = x$ then we have to evaluate the *following* integrals:

$$J_1 = 8 \left(\frac{\pi a^2}{k}\right)^2 \int_o^{2ka} \frac{1}{x} j_1^2(x)\, dx$$

$$J_3 = \frac{\pi^2}{32k^6} \frac{1}{(ka)^4} \int_o^{2ka} \frac{1}{x} j_1^2(x) \left[(2ka)^2 - 2x^2 \right]^4 dx$$

One can easily find:

$$J_1 = \frac{\pi^2}{2k^6} (2ka)^4 I(5)$$

$$J_3 = \frac{\pi^2}{2k^6} (2ka)^{-4} \{(2ka)^8 I(5) - 8(2ka)^6 I(3) + 24(2ka)^4 I(1)$$

$$- 32(2ka)^2 I(-1) + 16\, I(-3)\}$$

where

$$I(m) = \int_o^{2ka} \frac{(\sin x - x \cos x)^2}{x^m} dx$$

One finds (see Appendix E-v) ($u = 2ka$)

$$I(5) = -\frac{1}{4u^4} (\sin u - u \cos u)^2 - \frac{1}{4} \frac{\sin^2 u}{u^2} + \frac{1}{4}$$

$$I(3) = -\frac{1}{2} \frac{1}{u^2} (\sin u - u \cos u)^2 - \frac{1}{2} \sin^2 u + \frac{1}{2}\{C + \ln 2u - ci(2u)\}$$

where

$$C = 0.577215\ldots$$

$$ci(u) = \int_\infty^u \frac{\cos x}{x} dx$$

$$I(1) = \frac{1}{2}[\,C + \ln 2u - ci(2u)\,] + \frac{u^2}{4} + \frac{u\sin 2u}{4} + \frac{5}{8}(\cos 2u - 1)$$

$$I(-1) = \frac{u^2}{2}(\sin u - u\cos u)^2 + \frac{1}{4}u^4 \sin^2 u$$

$$- 3\left\{ \frac{u^4}{8} - \left(\frac{u^3}{4} - \frac{3u}{8}\right)\sin 2u - \left(\frac{3u^2}{8} - \frac{3}{16}\right)\cos 2u - \frac{3}{16} \right\}$$

$$I(-3) = \frac{1}{4}u^4(\sin u - u\cos u)^2 + \frac{1}{4}u^6 \sin^2 u - \frac{u^6}{6}$$

$$+ \frac{1}{8}(10\,u^4 - 30\,u^2 + 15)\cos 2u + \frac{1}{4}(2u^5 - 10u^3 + 15)\sin 2u - \frac{15}{8}$$

If we compute the ratio $R(\theta) = (B_1 + B_2\cos^2\theta)/(B_1 + B_2)$ we find that for a fixed θ, $R(\theta)$ decreases as $2ka$ increases with $R(\theta) \geq 1$ for $2ka < 17$ and $R(\theta) < 1$ for $2ka > 17$. For a fixed $2ka$ $R(\theta) = R(\pi - \theta)$ and $R(\theta)$ increases as θ increases from 0° to 90°. Below we give $R(\theta)$ for the values of $2ka$ which we have used in the first order, i.e. $2ka = \pi, \pi\sqrt{2}, 4.87, 11.70$. The drawings appear on pages 97, 98 .

TABLE 1

2ka	θ (rad)	R(θ)
π	$\pi/8$	1.49
$\pi\sqrt{2}$	$\pi/8$	1.40
4.87	$\pi/8$	1.33
11.70	$\pi/8$	1.04
π	$\pi/4$	2.69
$\pi\sqrt{2}$	$\pi/4$	2.29
4.87	$\pi/4$	2.14
11.70	$\pi/4$	1.13
π	$3\pi/8$	3.88
$\pi\sqrt{2}$	$3\pi/8$	3.21
4.87	$3\pi/8$	2.95
11.70	$3\pi/8$	1.23
π	$\pi/2$	4.37
$\pi\sqrt{2}$	$\pi/2$	3.59
4.87	$\pi/2$	3.28
11.70	$\pi/2$	1.27

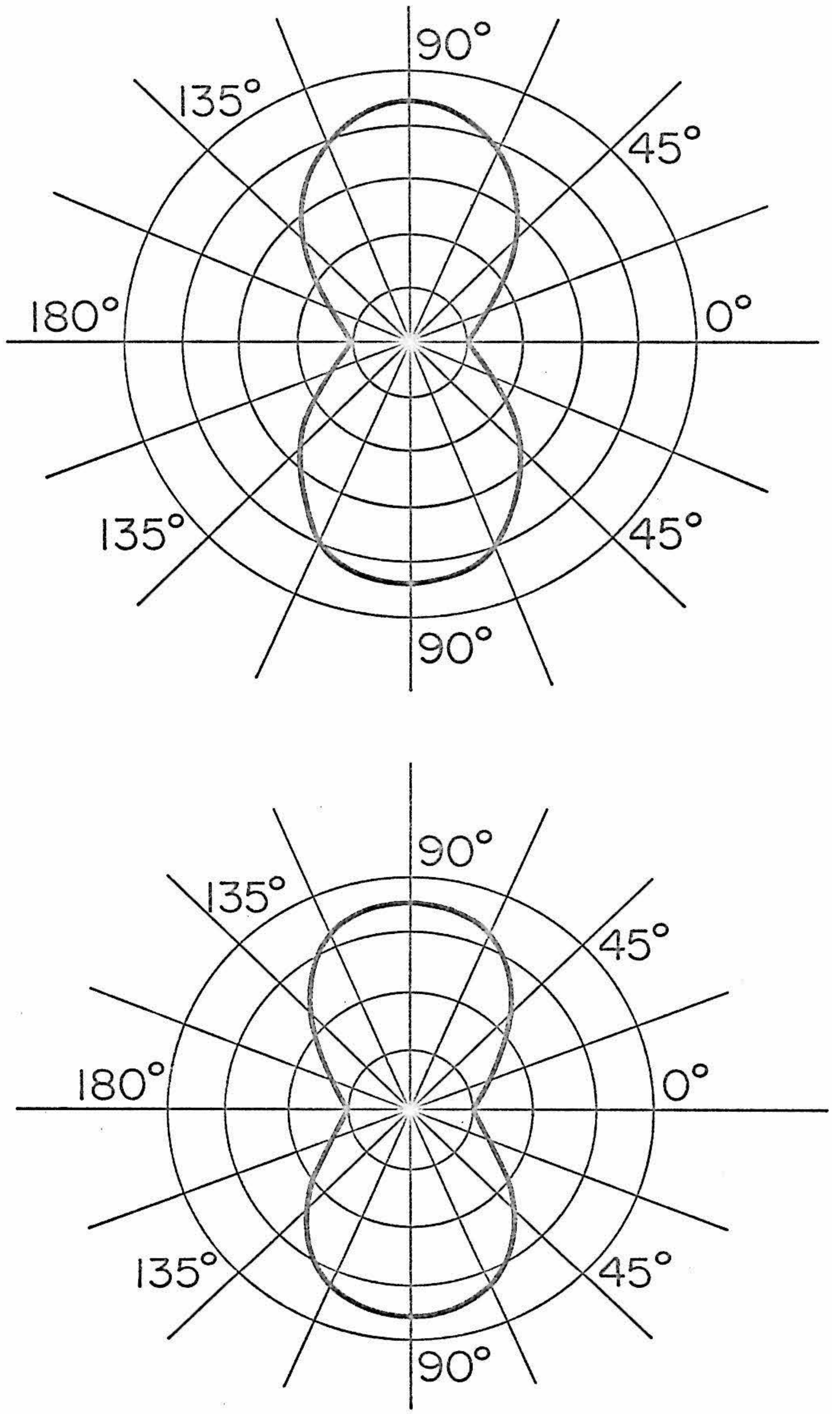


Fig. 13. Drawing of $R(\theta)$ for spheres with $2ka = \pi$ (above) and $2ka = \pi\sqrt{2}$ (below).

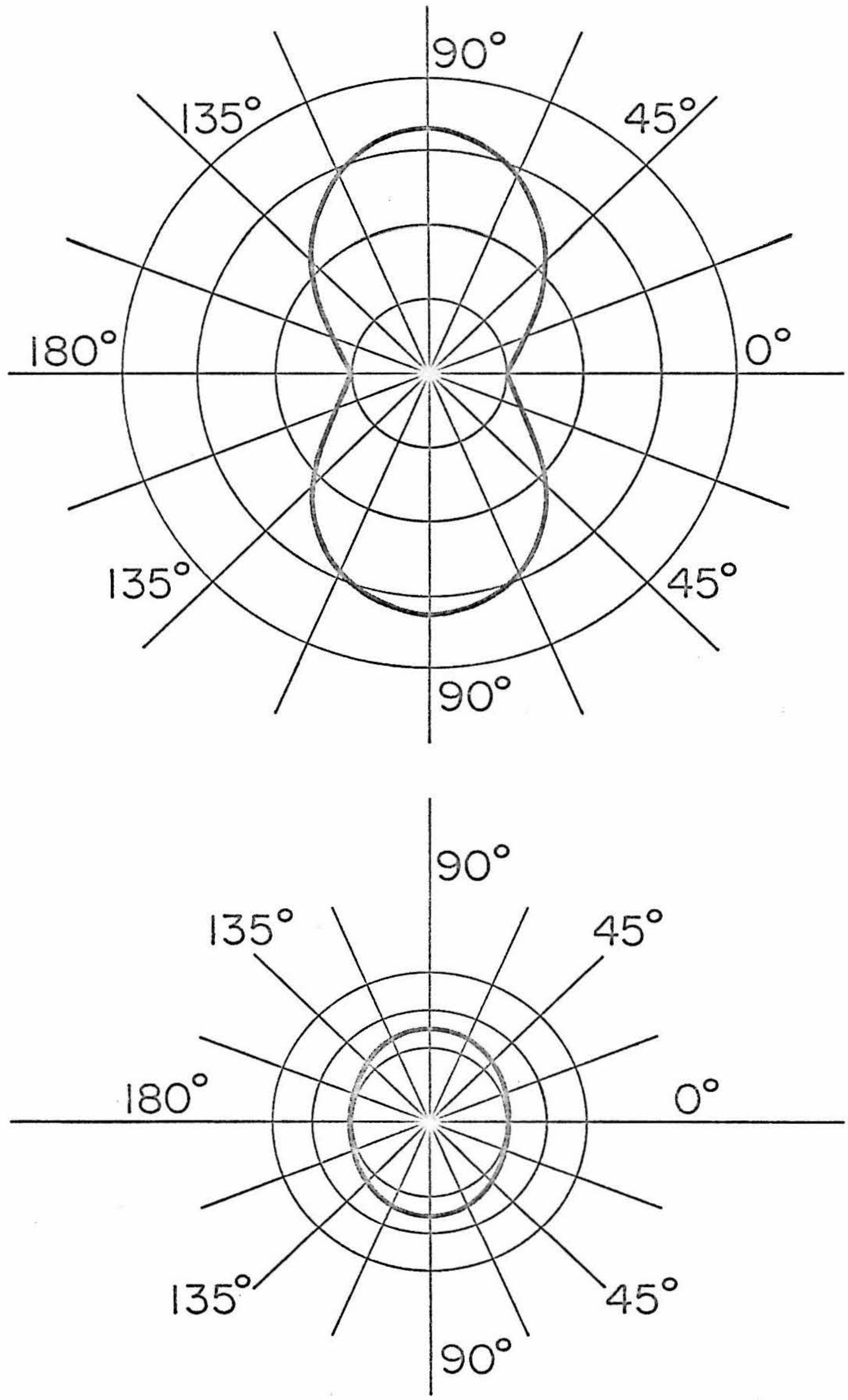


Fig. 14. Drawing of $R(\theta)$ for spheres with $2ka = 4.87$ (above) and $2ka = 11.70$ (below).

Next we want to compare the first order to the second order intensity using data which satisfy our conditions developed in section II. In this way we can check the validity of the conditions with the results of our specific example. Recall that

$$I^{(1)} \propto \langle \underline{E}_{sc}^{(1)} \cdot \underline{E}_{sc}^{(1)*} \rangle = \left(\frac{\omega^2}{c^2} |\Delta\chi| \right)^2 N \left(\frac{1}{4\pi r} \right)^2 (1 + \cos^2\theta) F(\theta) E_o'^2$$

$$I^{(2)} \propto \langle \underline{E}_{sc}^{(2)} \cdot \underline{E}_{sc}^{(2)*} \rangle = \left(\frac{\omega^2}{c^2} |\Delta\chi| \right)^4 N \left(\frac{1}{4\pi r} \right)^2 \frac{N}{V} \frac{D}{32\pi^2} F_1 (B_1 + B_2 \cos^2\theta)$$

Thus

$$\frac{I^{(2)}}{I^{(1)}} = \left(\frac{\omega^2}{c^2} |\Delta\chi| \right)^2 \frac{N}{V} \frac{D}{32\pi^2} \frac{F_1 (B_1 + B_2 \cos^2\theta)}{F(\theta)(1 + \cos^2\theta)}$$

Whenever $F(\theta) = 0$ the ratio blows up. In such a case $I^{(2)} > I^{(1)}$. However, away from the zeros of $F(\theta)$ we expect $I^{(2)}$ to be one to two orders smaller than $I^{(1)}$. Our theory is not good wherever $j(ka) = 0$, i.e. $ka = 4.5,\ 7.74,\ 10.95, \ldots$

Recall now that according to 2.2.21a

$$D \geq (2ka) \frac{\lambda_o}{18\pi n_m} \times 10^9 \qquad (a \rightarrow 2a \text{ for a sphere})$$

or

$$D = p(2ka) \frac{\lambda}{18\pi} 10^9 \quad \text{where } \lambda = \frac{\lambda_o}{n_m} \qquad p \geq 1$$

Also from 2.2.22 we get

$$\frac{1}{p} \frac{n_m^2}{(2ka)^2} 10^{-2} \leq |\Delta\chi| \leq \frac{1}{p^{\frac{1}{2}}} \frac{n_m^2}{(2ka)^2} 10^{-2}$$

We may choose $p = 1$, i.e.

$$D = (2ka)\frac{\lambda}{18\pi}\, 10^{9}$$

$$|\Delta\chi| = \frac{n_m^2}{(2ka)^2}\, 10^{-2}$$

We can now rewrite the ratio $I^{(2)}/I^{(1)}$ as

$$\frac{I^{(2)}}{I^{(1)}} = \left(\frac{2\pi}{\lambda_o}\right)^4 \frac{n_m^4}{(2ka)^4}\, 10^{-4}\, \frac{1}{m^3(2a)^3}\, \frac{1}{32\pi^2}\, (2ka)\, \frac{\lambda}{18\pi}\, 10^{9}\, \frac{F_1(B_1 + B_2\cos^2\theta)}{F(\theta)(1+\cos^2\theta)}$$

$$= (2\pi)^4 \frac{1}{\lambda^4}\frac{1}{(2ka)^4}\, 10^{-4}\, \frac{1}{15^3}\, \frac{1}{(2a)^3}\, \frac{1}{32\pi^2}\, (2ka)\, \frac{\lambda}{8\pi} \times 10^{-9}\, \frac{F_1(B_1 + B_2\cos^2\theta)}{F(\theta)(1+\cos^2\theta)}$$

Recall now that $F_1 \sim \lambda^6 F_1'(ka)$, $B_1 \sim \lambda^6 B_1'(2ka)$, $B_2 \sim \lambda^6 B_2'(2ka)$. Also notice that $2a = (2ak)\lambda/2\pi$. Therefore,the ratio $I^{(2)}/I^{(1)}$ only depends on ka. The same is true for $I^{(n)}/I^{(n-1)}$.

We have computed the ratio $I^{(2)}/I^{(1)}$ for several values of 2ka. For 2ka such that $j_1(ka) = 0$ the ratio is zero and the theory fails to describe the second order multiple scattering. For 2ka and θ such that $F(\theta) = 0$ the intensity pattern is solely given by the second order. One can see from table 2 that $I^{(2)}/I^{(1)}$ is smallest for $\theta = 0$ and has an average value of 1% for θ and 2ka different from those making $F(\theta)$ vanish. According to our conditions set up in section II one would expect a ratio $I^{(2)}/I^{(1)}$ larger than 1%. This discrepancy along with other peculiarities should be attributed to the specific example of the spheres. Thus $F(\theta)$ varies rather wildly even for small 2ka and B_1, B_2 are of the order V_p^2 instead of $10\, V_p^2$ that was assumed in section II. In general, however, $F(\theta)$ has no zeros, does not vary significantly for $2ka$ not large and B_1, B_2 are indeed approximately equal to $10\ V_p^2$.

Some of the computed values of the ratio $I^{(2)}/I^{(1)}$ are given in Table 2. As we can see in table 2 the value $2ka = 9$ gives $I^{(2)}/I^{(1)} = 0$ since $j_1(ka) = 0$. Thus our theory is not good near $2ak = 9$. Notice also that $I^{(2)}/I^{(1)}$ for $\theta = \pi/2$, $2ka = 6$ is large due to the fact that $F(\pi/2) = 0$ for $2ka = 6.38$. The same occurs for $\theta = \pi$, $2ka = 4$ or $2ka = 5$ since $F(\pi) = 0$ for $2ka = 4.5$.

TABLE 2

2ka	θ (rad)	$I^{(2)}/I^{(1)}$
1	0	5.90×10^{-3}
2	0	3.59×10^{-3}
3	0	1.87×10^{-3}
4	0	7.6×10^{-4}
5	0	3.17×10^{-4}
6	0	1.14×10^{-4}
7	0	3.12×10^{-5}
8	0	4.5×10^{-6}
9	0	0
10	0	1.32×10^{-6}
1	$\pi/2$	5.63×10^{-2}
2	$\pi/2$	4.93×10^{-2}
3	$\pi/2$	4.16×10^{-2}
4	$\pi/2$	3.84×10^{-2}
5	$\pi/2$	5.27×10^{-2}
6	$\pi/2$	3.65×10^{-1}
7	$\pi/2$	4.96×10^{-2}
8	$\pi/2$	2.34×10^{-3}
9	$\pi/2$	0
10	$\pi/2$	2.98×10^{-3}
1	π	6.66×10^{-3}
2	π	8.24×10^{-3}
3	π	1.50×10^{-2}
4	π	1.03×10^{-1}
5	π	9.70×10^{-2}
6	π	1.62×10^{-2}
7	π	1.91×10^{-2}
8	π	2.83×10^{-2}
9	π	0
10	π	2.38×10^{-3}

VII. CONCLUSIONS

The main conclusions of this work are the following:

a) The linear dimension D of the region occupied by the particles is related to the wavelength λ and the linear dimension of a particle a. It is found that D is minimum when $ka = 1$. The minimum D is proportional to the wavelength λ. For a constant ka, D is proportional to λ. When $ka \geq 1$ and λ is constant D is proportional to ka but if $ka \lesssim 1$ and λ is constant D is proportional to $1/(ka)^2$. Thus by making ka very small, i.e. a much smaller than λ, we can not get rid of the self-interaction contribution unless D gets large to make the multiple scattering more important than the self-interaction.

b) $|\Delta\chi|$ should not satisfy the inequality $|\Delta\chi| \ll 1$ as one intuitively expects but it has to lie within a certain range. The end limits of this range depend on ka, the index of refraction n_m of the surrounding medium, and the size of the region occupied by the particles. The range narrows down when ka increases but it gets wider as ka decreases.

c) The ratio of the multiple scattering intensity to the single scattering intensity depends on ka but not on the wavelenth λ. Thus no matter what the wavelength is, the relative effect of the multiple scattering to the single scattering will be the same provided ka and D stay constant. Also the ratio of the forward scattering intensity to the scattering intensity at any θ does not depend on λ.

d) The effect of the multiple scattering on the single scattering intensity and polarization has been explained in the abstract. Also, the effect of the losses on the intensity pattern is discussed in the abstract.

APPENDIX A

THE INTEGRAL EQUATION OF THE SCATTERING PROBLEM

We assume that the constitutive parameters of the medium are ϵ_m, μ_o. We can consider the system of the medium plus the particle as a new medium with constitutive parameters ϵ, μ_o. The dielectric permittivity ϵ is equal to ϵ_p within the region occupied by the particle and ϵ_m outside. If a wave is generated in the medium a scattered wave will be produced due to the dielectric discontinuity.

Consider now the Maxwell equations:

$$\begin{aligned} \nabla \times \underline{E} &= i\omega\mu_o \underline{H} \\ \nabla \times \underline{H} &= -i\omega\epsilon \underline{E} \end{aligned} \qquad \text{(A-1)}$$

where the time dependence $e^{-i\omega t}$ has been assumed. From A-1 one can easily get:

$$\nabla \times \nabla \times \underline{E} - \omega^2 \mu_o \epsilon \underline{E} = 0 \qquad \text{(A-2)}$$

A-2 can be rewritten as:

$$\nabla \times \nabla \times \underline{E}(\underline{r}) - \omega^2 \mu_o \epsilon_m \underline{E}(\underline{r}) = 0, \quad \underline{r} \text{ outside } V_p \qquad \text{(A-3)}$$

$$\nabla \times \nabla \times \underline{E}(\underline{r}) - \omega^2 \mu_o \epsilon_p \underline{E}(\underline{r}) = 0, \quad \underline{r} \text{ inside } V_p \qquad \text{(A-4)}$$

The second equation can be rewritten as:

$$\nabla \times \nabla \times \underline{E}(\underline{r}) - \omega^2 \mu_o \epsilon_m \underline{E}(\underline{r}) = \omega^2 \mu_o (\epsilon_p - \epsilon_m (\underline{E}) \underline{r})$$

which can be combined with A-3 to give:

$$\nabla \times \nabla \times \underline{E}(\underline{r}) - \omega^2 \mu_o \epsilon_m \underline{E}(\underline{r}) = i\omega\mu_o \left\{ -i\omega \Delta\epsilon \underline{E}(\underline{r}) \right\}$$

$$= i\omega\mu_o \underline{J}(\underline{r}) \tag{A-5}$$

where $\underline{J}(\underline{r}) = 0$ outside V_p.

If a plane wave falls upon the particle it produces a scattered field $\underline{E}_{sc}(\underline{r})$. Now the homogeneous solution of A-5 is just the incident plane wave $\underline{E}_{inc}(\underline{r})$ whereas the particular solution is the scattered field $\underline{E}_{sc}(\underline{r})$. The total field is $\underline{E} = \underline{E}_{inc} + \underline{E}_{sc}$ satisfying equation A-5. It can now be shown (1) that:

$$\underline{E}(\underline{r}) = \underline{E}_{inc} + i\omega\mu_o \int_{V_p} \underline{\underline{\Gamma}}(\underline{r};\underline{r}') \cdot \underline{J}(r')\, dV'$$

i.e.

$$\underline{E}_{sc}(\underline{r}) = i\omega\mu_o \int_{V_p} \underline{\underline{\Gamma}}(\underline{r};\underline{r}') \cdot \underline{J}(\underline{r}')\, dV' \tag{A-6}$$

where

$$\underline{\underline{\Gamma}}(\underline{r};\underline{r}') = (\underline{\underline{u}} + \frac{1}{k^2} \nabla\nabla) \frac{e^{ik|\underline{r}-\underline{r}'|}}{4\pi|\underline{r}-\underline{r}'|}$$

Now $\underline{\underline{u}}$ is the unit dyadic $= \sum_{ij} \delta_{ij} \underline{e}_i \underline{e}_j$ and $k^2 = \omega^2 \mu_o \epsilon_m$.

If we recall that $\underline{J} = -i\omega \Delta\epsilon \underline{E}(\underline{r}) = -i\omega\epsilon_o \Delta\chi\, \underline{E}(\underline{r})$ then

$$\underline{E}_{sc}(\underline{r}) = \frac{\omega^2}{c^2} \Delta\chi \int_V \underline{\underline{\Gamma}}(\underline{r};\underline{r}') \cdot (\underline{E}_{inc} + \underline{E}_{sc})\, dV' \tag{A-7}$$

where $\Delta = \frac{1}{\epsilon_o}(\epsilon_p - \epsilon_m)$ is assumed to be independent of $\underline{r}$.

We could have derived A-7 by using less algebra but more physical reasoning. The incident wave induces a current inside the volume V_p of the particle which is given by $\underline{J} = -i\omega\underline{P}$ where $\underline{P}$ is the relative (to the surrounding medium) polarization given by $\epsilon_o \Delta\chi \underline{E}(\underline{r})$. Therefore:

$$\underline{J} = -i\omega\underline{P} = -i\omega\epsilon_o \Delta\chi \underline{E}(\underline{r}).$$

Now the scattered wave is entirely due to $\underline{J}$, therefore A-6 is true and A-7 follows.

Suppose now we are interested in the far zone scattered field, i.e. at $\underline{r}$ such that $kr >> 1$ and $r >> r'$. Under these conditions one can show (1) that

$$\underline{\underline{\Gamma}}(\underline{r};\underline{r}') \approx (\underline{\underline{u}} - \underline{e}_r\underline{e}_r) \frac{e^{ikr}}{4\pi r} e^{-ik\underline{e}_r \cdot \underline{r}'}$$

and A-6 becomes

$$\underline{E}_{sc}(\underline{r}) = i\omega\mu_o(\underline{\underline{u}} - \underline{e}_r\underline{e}_r)\cdot \frac{e^{ikr}}{4\pi r}\int_{V_p} e^{-ik\underline{e}_r \cdot \underline{r}'} \underline{J}(\underline{r}')\, dV' \qquad \text{(A-8)}$$

APPENDIX B

THE EULER ANGLES

Consider two cartesian orthogonal systems xyz and x'y'z' (see figure B-1). We bring xyz into coincidence with x'y'z' by three successive rotations. The angles about the corresponding axes are the Euler angles. This is accomplished as follows. First rotate xyz about z counterclockwise by an angle β. The new system is labelled $x_1y_1z_1$ (figure B-2). Next rotate $x_1y_1z_1$ about x_1 counterclockwise by an angle α. The new system is called $x_2y_2z_2$ (figure B-3). Finally we rotate system $x_2y_2z_2$ about z_2 by an angle γ again counterclockwise to get x'y'z'.

If M is the rotation matrix then the components of a true vector $\underline{A}$ w.r.t. xyz and x'y'z' are related in the following way:

$$A_i' = M_{ij}A_j \qquad \text{(repeated indices must be summed)} \quad (i,j=1,2,3)$$

and $A_i = (M^{-1})_{ij}A_j'$. One can show (4) that $M^{-1} = \tilde{M} = M$ transposed. We can also find that

$$M^{-1} = \begin{bmatrix} \cos\gamma\cos\beta - \cos\alpha\sin\beta\sin\gamma & -\sin\gamma\cos\beta - \cos\alpha\sin\beta\cos\gamma & \sin\alpha\sin\beta \\ \cos\gamma\sin\beta + \cos\alpha\cos\beta\sin\gamma & -\sin\gamma\sin\beta + \cos\alpha\cos\beta\cos\gamma & -\sin\alpha\cos\beta \\ \sin\alpha\sin\gamma & \sin\alpha\cos\gamma & \cos\alpha \end{bmatrix}$$

The relation between the polar angles characterizing z' in the system xyz and the Eulerian angles can be found as follows: We have $\underline{A} = A_i\underline{e}_i = \underline{A}_j\underline{e}_j'$ or $(M^{-1})_{ij}A_j'\underline{e}_i = A_j'\underline{e}_j'$, i.e. $(M^{-1})_{ij}\underline{e}_i = \underline{e}_j'$

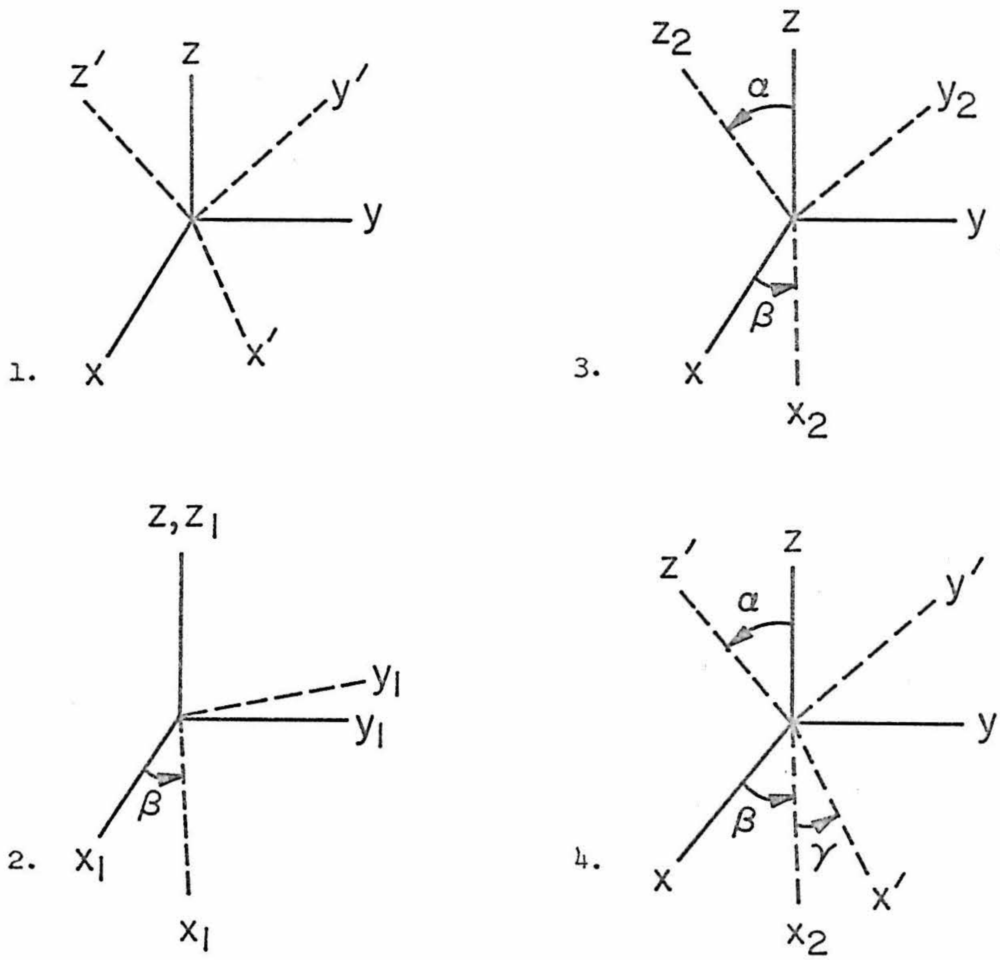


Fig. B. Successive rotations through the Euler angles bring xyz into coincidence with x'y'z' .

since $\underline{A}$ can be any vector. We thus understand that $\underline{e}_i \cdot \underline{e}'_j = (M^{-1})_{ij}$. We now have:

$$\underline{e}'_z \cdot \underline{e}_x = (M^{-1})_{xz} = \sin\theta' \cos\varphi'$$

$$\underline{e}'_z \cdot \underline{e}_y = (M^{-1})_{yz} = \sin\theta' \sin\varphi'$$

$$\underline{e}'_z \cdot \underline{e}_z = (M^{-1})_{zz} = \cos\theta'$$

i.e.

$$\sin\alpha \sin\beta = \sin\theta' \cos\varphi'$$

$$-\sin\alpha \cos\beta = \sin\theta' \cos\varphi'$$

$$\cos\alpha = \cos\theta'$$

We can easily show that $\theta' = \alpha$, $\varphi' = \beta - \frac{\pi}{2}$ $(0 \leq \theta', \alpha \leq \pi$, $0 \leq \varphi', \beta \leq 2\pi)$.

Assume that $A = f(\alpha,\beta,\gamma)$. If we want the average of A over α,β,γ we write

$$\langle A\rangle = \frac{1}{8\pi^2}\int_{\gamma=0}^{2\pi}\int_{\beta=0}^{2\pi}\int_{\alpha=0}^{\pi} f(\alpha,\beta,\gamma)\sin\alpha \, d\alpha \, d\beta \, d\gamma$$

The relation above becomes clear if we recall that if we keep γ constant and average over α,β is like averaging over θ,φ, i.e. over all the possible directions of an axis:

$$\frac{1}{4\pi}\int f \, d\Omega = \frac{1}{4\pi}\iint f \sin\alpha \, d\alpha \, d\beta$$

If we finally vary γ, i.e. take into account rotations about our z' axis $(0 \leq \gamma \leq 2\pi)$ we obtain

$$\frac{1}{4\pi}\,\frac{1}{2\pi}\iiint f \sin\alpha \, d\alpha \, d\beta \, d\gamma = \langle A \rangle$$

APPENDIX C

POLARIZATION ELLIPSE. STOKES PARAMETERS

Consider the scattered field in the far zone

$$\underline{E} = E_\theta e^{-i\delta_\theta} \underline{e}_\theta + E_\varphi e^{-i\delta_\varphi} \underline{e}_\varphi = \mathcal{E}_\theta \underline{e}_\theta + \mathcal{E}_\varphi \underline{e}_\varphi \tag{C-1}$$

The far zone field is a TEM wave and behaves like a plane wave in the vicinity of a given direction.

When the phase difference $\delta_\theta - \delta_\varphi$ is zero or a multiple of π the wave is linearly polarized. In general C-1 represents an elliptically polarized wave. One is interested in the size, orientation and handedness of the polarization ellipse.

One can show (1) that the ellipse is specified as follows: The semiaxes a and b can be found from

$$a^2 + b^2 = E_\theta^2 + E_\varphi^2, \qquad \pm \frac{b}{a} = \tan \chi$$

where χ is given by

$$\sin 2\chi = \frac{2E_\theta E_\varphi}{E_\theta^2 + E_\varphi^2} \sin(\delta_\varphi - \delta_\theta) \qquad \left(-\frac{\pi}{4} < \chi < \frac{\pi}{4}\right)$$

The (+) sign corresponds to left-handed polarization and the (-) sign to right-handed polarization. The sense of rotation is to be specified by an observer who sees the wave receding from him.

The inclination angle ψ (see figure C) is given by

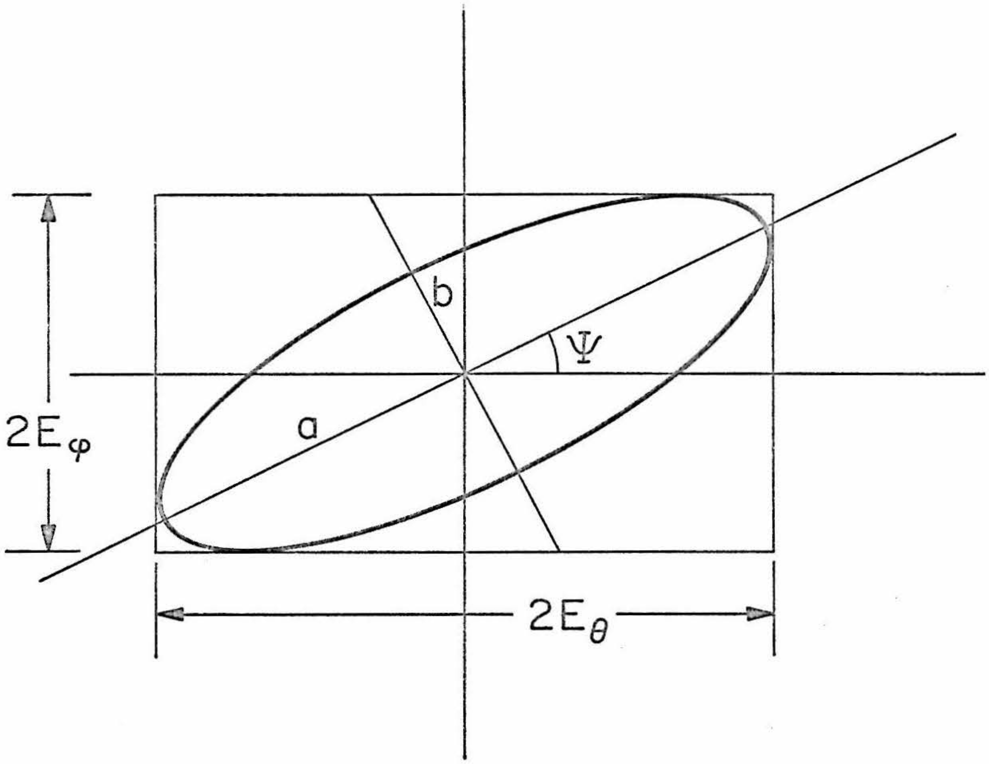


Fig. C. The Polarization Ellipse

$$\tan 2\psi = \frac{2E_\theta E_\varphi}{E_\theta^2 - E_\varphi^2} \cos (\delta_\theta - \delta_\varphi)$$

A circularly polarized wave is such that

$$E_\theta = E_\varphi \text{ and } \delta_\theta - \delta_\varphi = \pm \frac{\pi}{2}$$

The (+) is for left-handed and (-) for right-handed polarization.

Another way to specify the polarization is to give the Stokes parameters:

$$S_o = E_\theta^2 + E_\varphi^2, \quad S_1 = E_\theta^2 - E_\varphi^2, \quad S_2 = 2E_\theta E_\varphi \cos (\delta_\varphi - \delta_\theta)$$

$$S_3 = 2E_\theta E_\varphi \sin (\delta_\varphi - \delta_\theta)$$

or

$$S_1 = S_o \cos 2\chi \cos 2\psi, \quad S_2 = S_o \cos 2\chi \sin 2\psi, \quad S_3 = S_o \sin 2\chi$$

The properties of the Stokes parameters are described in reference (1).

APPENDIX D

RANDOM SUMS

We are given the sum:

$$S = \sum_{i=1}^{N} a_i e^{i\varphi_i}$$

where φ_i, a_i are independent random variables. We assume that the φ_i's are equally likely to be found anywhere between 0 and 2π, i.e. the probability density is $P(\varphi) = \frac{1}{2\pi}$ such that $\int_0^{2\pi} P(\varphi)\, d\varphi = 1$. We also assume that all the a_i's have the same probability distribution. We write

$$\begin{aligned} S &= \sum a_i \cos\varphi_i + i \sum a_i \sin\varphi_i \\ &= S_r + i\, S_i \end{aligned} \tag{D-1}$$

The average of S is by definition:

$$\langle S \rangle = \iint P(S_r, S_i)(S_r + iS_i)\, dS_r\, dS_i$$

or if we treat S_r and S_i as independent random variables

$$\begin{aligned} \langle S \rangle &= \iint P_r(S_r) P_i(S_i)(S_r + iS_i)\, dS_r\, dS_i \\ &= \int P_r(S_r) S_r\, dS_r + i \int P_i(S_i) S_i\, dS_i \\ &= \langle S_r \rangle + i \langle S_i \rangle \end{aligned} \tag{D-2}$$

We know that if we have $A = \sum x_i$ and x_i are independent random variables with the same probability distribution, then

$$\langle A \rangle = N \langle x_i \rangle \tag{D-3}$$

$$\sigma_A^2 = \text{var } A = \langle (A - \langle A \rangle)^2 \rangle = N \text{ var } x_i = N\sigma_{x_i}^2 \tag{D-4}$$

Using D-3 in D-2 we get

$$\langle S \rangle = N\langle a_i \cos \varphi_i \rangle + i\langle a_i \sin \varphi_i \rangle N$$

$$= N\langle a_i \rangle \langle \cos \varphi_i \rangle + iN\langle a_i \rangle \langle \sin \varphi_i \rangle$$

$$= 0 + 0 = 0 \quad \text{since } \langle \cos \varphi_i \rangle = \langle \sin \varphi_i \rangle = 0$$

Next we compute $\langle SS^* \rangle$

$$\langle SS^* \rangle = \int\int P_r(S_r)P_i(S_i)(S_r^2 + S_i^2)\, dS_r\, dS_i$$

$$= \langle S_r^2 \rangle + \langle S_i^2 \rangle = \text{var } S_r + \text{var } S_i \quad \text{since } \langle S_r \rangle = \langle S_i \rangle = 0$$

If we apply D-4 we find:

$$\text{var } S_r = N\sigma_r^2 , \qquad \text{var } S_i = N\sigma_i^2$$

where

$$\sigma_r^2 = \langle (a_i \cos \varphi_i - \langle a_i \cos \varphi_i \rangle)^2 \rangle$$

$$= \langle a_i^2 \cos^2 \varphi_i \rangle = \langle a_i^2 \rangle \langle \cos^2 \varphi_i \rangle$$

$$= \frac{1}{2} \langle a_i^2 \rangle$$

and

$$\sigma_i^2 = \langle (a_i \sin \varphi_i - \langle a_i \sin \varphi_i \rangle)^2 \rangle$$

$$= \langle a_i^2 \sin^2 \varphi_i \rangle = \frac{1}{2} \langle a_i^2 \rangle$$

Thus

$$\text{var } S_r = N \frac{1}{2} \langle a_i^2 \rangle$$

$$\text{var } S_i = N \frac{1}{2} \langle a_i^2 \rangle$$

and

$$\langle SS^* \rangle = \text{var } S_r + \text{var } S_i = N \langle a_i^2 \rangle \tag{D-5}$$

Assume now that a_i is a function of the independent random variables u, w, v, i.e.

$$a_i = f(u, w, v)$$

Then

$$\langle a_i^2 \rangle = \int P_a(a_i) a_i^2 \, da_i$$

$$= \int\int\int f^2(u, w, v) P_w(w) P_u(u) P_v(v) \, du \, dw \, dv$$

The last result can be shown as follows. We know that if $f = f(u, w, v)$ then

$$P(a) = \frac{1}{2\pi} \int e^{-ika} \phi(k) \, dk \tag{D-6}$$

where $\phi(k)$ is the characteristic function defined by

$$\phi(k) = \int e^{ikf(u,w,v)} \underbrace{P_u(u)P_w(w)P_v(v)}_{P(u,w,v)} \, du\, dw\, dv \qquad \text{(D-7)}$$

Thus we can write D-6 as

$$P(a) = \frac{1}{2\pi}\int\int e^{-ika} e^{ikf(u,w,v)} P(u,w,v)\, du\, dw\, dv\, dk$$

$$= \frac{1}{2\pi}\int P(u,w,v)\, du\, dw\, dv \int_{-\infty}^{\infty} d^{ik(f-a)}\, dk$$

$$= \int P(u,w,v)\, du\, dw\, dv\, \delta(a-f)$$

Thus

$$\langle a_i^2 \rangle = \int P_a(a) a^2\, da = \int f^2(u,w,v) P(u,w,v)\, du\, dw\, dv \qquad \text{Q.E.D.}$$

If for example the u, w, v are the cartesian coordinates xyz and x_i $(i = 1,2,3)$ has a probability density $P_{x_i}(x_i) = \frac{1}{2L_{x_i}}$ $(-L_{x_i} < x_i < L_{x_i})$ then

$$\langle a^2 \rangle = \int f^2 \frac{1}{2L_x} \frac{1}{2L_y} \frac{1}{2L_z}\, dx\, dy\, dz$$

$$= \frac{1}{V}\int f^2(x,y,z)\, dV$$

a result which agrees with common intuition. If the independent variables are the Euler angles then

$$P(\alpha,\beta) = \frac{\sin\alpha}{4\pi} \qquad \text{since} \qquad \int\int P(\alpha,\beta)\, d\alpha\, d\beta = 1$$

and $P(\gamma) = \frac{1}{2\pi}$, i.e.

$$\langle a^2 \rangle = \iiint f^2(\alpha,\beta,\gamma) \frac{\sin\alpha}{4\pi} \frac{1}{2\pi} d\alpha\, d\beta\, d\gamma$$

$$= \frac{1}{8\pi^2} \iiint f^2(\alpha,\beta,\gamma) \sin\alpha\, d\alpha\, d\beta\, d\gamma$$

which of course agrees with the result obtained in Appendix B.

Suppose now that we have two random sums:

$$S_1 = \sum_{i=1}^{N} a_i e^{i\varphi_i}, \quad S_2 = \sum_{i=1}^{N} b_i e^{i\theta_i}$$

We assume that φ_i and θ_i have probability densities

$$P(\varphi) = \frac{1}{2\pi}, \quad P(\theta) = \frac{1}{2\pi} \quad (0 \leq \varphi \leq 2\pi,\ 0 \leq \theta \leq 2\pi) .$$

We want to compute $\langle S_1 S_2^* \rangle$.

Now

$$S_1 S_2^* = \sum_{ij} a_i b_j e^{i(\varphi_i - \theta_j)}$$

$$= \sum_{ij} a_i b_j \cos(\varphi_i - \theta_j) + i \sum_{ij} a_i b_j \sin(\varphi_i - \theta_j)$$

$$= \sum_{ij} a_i b_j \cos\varphi_i \sin\theta_j + \sum_{ij} a_i b_j \sin\varphi_i \sin\theta_j$$

$$+ i\left\{\sum_{ij} a_i b_j \sin\varphi_i \cos\theta_j - \sum_{ij} a_i b_j \cos\varphi_i \sin\theta_j\right\}$$

$$= A_1 + A_2 + i(A_3 - A_4)$$

Thus

$$\langle S_1 S_2^* \rangle = \langle A_1 \rangle + \langle A_2 \rangle + i\langle A_3 \rangle - i\langle A_4 \rangle$$

We have

$$\begin{aligned} \langle A_1 \rangle &= \langle \sum_{ij} a_i \sin \varphi_i b_j \cos \theta_j \rangle \\ &= \langle \sum_i a_i \cos \varphi_i \sum_j b_j \cos \theta_j \rangle \\ &= \langle \sum_i a_i \cos \varphi_i \rangle \langle \sum_j b_j \cos \theta_j \rangle \\ &= 0 \end{aligned}$$

Similarly $\langle A_2 \rangle = \langle A_3 \rangle = \langle A_4 \rangle = 0.$

For three sums $S_1 S_2 S_3$ we have $\langle S_1 S_2^* S_3^* \rangle = 0$ and $\langle S_1 S_2 S_3^* \rangle = 0.$ Same for more sums.

Now if we recall that $\underline{E}_{sc} = \underline{E}_{sc}^{(1)} + \underline{E}_{sc}^{(2)} + \ldots$ and that each field can be expressed as a random sum $\underline{E}_{sc}^{(n)} = \sum_{i=1}^{N} \underline{a}_{in} e^{i\varphi_{in}}$ we understand that in computing $\langle \underline{E}_{sc} \cdot \underline{E}_{sc}^* \rangle$ all the cross terms will give zero and $\langle \underline{E}_{sc} \cdot \underline{E}_{sc}^* \rangle = \sum_{n=1}^{\infty} \langle \underline{E}_{sc}^{(n)} \cdot \underline{E}_{sc}^{(n)*} \rangle$. Thus the fields corresponding to the several orders are orthogonal upon averaging.

Next we consider the polarization properties of

$$\underline{E} = \underline{e}_x \sum_i A_i e^{i\varphi_i} + \underline{e}_y \sum_i B_i e^{i\varphi_i} \qquad \text{(D-8)}$$

If we consider $\underline{E}$ as an electric field then its polarization properties are completely determined if we know the corresponding Stokes parameters (see Appendix C). We have

$$\underline{E} = E_x e^{-i\delta_x} \underline{e}_x + E_y e^{-i\delta_y} \underline{e}_y \tag{D-9}$$

and

$$\begin{aligned} S_o &= E_x^2 + E_y^2 , \qquad S_1 = E_x^2 - E_y^2 \\ S_2 &= 2E_x E_y \cos(\delta_y - \delta_x) \\ S_3 &= 2E_x E_y \sin(\delta_y - \delta_x) \end{aligned} \tag{D-10}$$

From D-8 and D-9 we understand that

$$\mathcal{E}_x = E_x e^{-i\delta_x} = \sum_i A_i e^{i\varphi_i}, \quad \mathcal{E}_y = E_y e^{-i\delta_y} = \sum_i B_i e^{i\varphi_i} \tag{D-11}$$

We can now write the Stokes parameters as follows:

$$\begin{aligned} S_o &= E_x^2 + E_y^2 , \qquad S_1 = E_x^2 - E_y^2 \\ S_2 &= 2E_x \cos\delta_x E_y \cos\delta_y + 2E_x \sin\delta_x E_y \sin\delta_y \\ S_3 &= 2E_x \cos\delta_x E_y \sin\delta_y - 2E_x \sin\delta_x E_y \cos\delta_y \end{aligned} \tag{D-12}$$

The sums in D-11 are random sums. We therefore want to find the average values of the Stokes parameters:

$$\langle S_o \rangle = \langle E_x^2 + E_y^2 \rangle = \langle \mathcal{E}_x \mathcal{E}_x^* \rangle + \langle \mathcal{E}_y \mathcal{E}_y^* \rangle$$

$$\langle S_1 \rangle = \langle E_x^2 - E_y^2 \rangle = \langle \mathcal{E}_x \mathcal{E}_x^* \rangle - \langle \mathcal{E}_y \mathcal{E}_y^* \rangle$$

$$\langle S_2 \rangle = 2\langle E_x \cos\delta_x E_y \cos\delta_y \rangle + 2\langle E_x \sin\delta_x E_y \sin\delta_y \rangle$$

$$\langle S_3 \rangle = 2\langle E_x \cos\delta_x E_y \sin\delta_y \rangle - 2\langle E_x \sin\delta_x E_y \cos\delta_y \rangle$$

We have already computed averages like $\langle \mathcal{E}_x \mathcal{E}_x^* \rangle$ and $\langle \mathcal{E}_y \mathcal{E}_y^* \rangle$.

We now compute $\langle S_2 \rangle$ and $\langle S_3 \rangle$. We have

$$\begin{aligned}
\langle E_x \cos\delta_x E_y \cos\delta_y \rangle &= \langle \sum_i A_i \cos\varphi_i \sum_j B_j \cos\varphi_j \rangle \\
&= \langle \sum_{ij} A_i B_j \cos\varphi_i \cos\varphi_j \rangle \\
&= \langle \sum_{i=j} A_i B_i \cos^2\varphi_i + \sum_{i\neq j} A_i B_j \cos\varphi_i \cos\varphi_j \rangle \\
&= \langle A_i B_i \rangle \frac{N}{2} + 0 = \frac{N}{2} \langle A_i B_i \rangle
\end{aligned}$$

Similarly

$$\langle E_x \sin\delta_x E_y \sin\delta_y \rangle = \frac{1}{2} \langle A_i B_i \rangle N$$

$$\begin{aligned}
\langle E_x \cos\delta_x E_y \sin\delta_y \rangle &= \langle \sum_{i=j} A_i B_i \cos\varphi_i \sin\varphi_i \rangle + \langle \sum_{i\neq j} A_i B_j \cos\varphi_i \sin\varphi_j \rangle \\
&= 0 + 0 = 0
\end{aligned}$$

$$\langle E_x \sin\delta_x E_y \cos\delta_y \rangle = 0$$

In view of the above partial results we find:

$$\begin{aligned}
\langle S_o \rangle &= N\langle A_i^2 \rangle + N\langle B_i^2 \rangle \\
\langle S_1 \rangle &= N\langle A_i^2 \rangle - N\langle B_i^2 \rangle \\
\langle S_2 \rangle &= N\langle A_i B_i \rangle \\
\langle S_3 \rangle &= N\langle A_i B_i \rangle
\end{aligned} \tag{D-13}$$

If D-8 represents the nth order scattered field, i.e. $\underline{E}_{sc}^{(n)}$, then it does make sense to compute the Stokes parameters D-13 for this order. This is so because the several orders are independent waves, therefore the average Stokes parameter of the composite waves say S_i $(i = 0,1,2,3)$ is just the sum of the Stokes parameters S_i for the several orders, i.e. $\langle S_i \rangle = \sum_j \langle S_i^{(j)} \rangle$. This is a well known theorem which can be easily shown in our case. We have already shown the above property for S_o and S_1 (pp. 119 - 120).

APPENDIX E

ALGEBRAIC AND INTEGRAL COMPUTATIONS

i) Computation of

$$I = \int_0^{2\pi} \left| (\underline{\underline{u}} - \underline{e}_r\underline{e}_r)\cdot(\underline{u} - \underline{e}_a\underline{e}_a)\cdot\underline{E}_o \right|^2 d\varphi_i \tag{E-1}$$

where $\underline{e}_r = (\theta, \varphi)$, $\underline{e}_a = (\theta_i, \varphi_i)$.

We have

$$-\underline{e}_a \times (\underline{e}_a \times \underline{E}_o) = \text{component of } \underline{E}_o \text{ perpendicular to } \underline{e}_a$$

$$= (\underline{E}_o \cdot \underline{e}_{\varphi_i})\underline{e}_{\varphi_i} + (\underline{E}_o \cdot \underline{e}_{\theta_i})\underline{e}_{\theta_i}$$

If $\underline{E}_o$ is circularly polarized then

$$\underline{E}_o = E_o' e^{-i\delta_x}(\underline{e}_x \pm i\underline{e}_y)$$

and

$$-\underline{e}_a \times (\underline{e}_a \times \underline{E}_o) = E_o' e^{-i\delta_x}\left[(\underline{e}_x \pm i\underline{e}_y)\cdot\underline{e}_{\varphi_i}\underline{e}_{\varphi_i} + (\underline{e}_x \pm i\underline{e}_y)\cdot\underline{e}_{\theta_i}\underline{e}_{\theta_i}\right]$$

$$= E_o' e^{-i\delta_x}\left[\pm i\underline{e}_{\varphi_i} + \cos\theta_i \underline{e}_{\theta_i}\right]e^{\pm i\varphi_i} \equiv \underline{A}$$

Next

$$-\underline{e}_r \times (\underline{e}_r \times A) = (\underline{A}\cdot\underline{e}_\varphi)\underline{e}_\varphi + (\underline{A}\cdot\underline{e}_\theta)\underline{e}_\theta$$

$$= \left[A_{\varphi_i}(\underline{e}_{\varphi_i}\cdot\underline{e}_\varphi) + A_{\theta_i}(\underline{e}_{\theta_i}\cdot\underline{e}_\varphi)\right]\underline{e}_\varphi$$

$$+ \left[A_{\varphi_i}(\underline{e}_{\varphi_i}\cdot\underline{e}_\theta) + A_{\theta_i}(\underline{e}_{\theta_i}\cdot\underline{e}_\theta)\right]\underline{e}_\theta$$

$$= E_o' e^{i(-\delta_x \pm i\varphi_i)} \left\{ \left[\pm i(\underline{e}_{\varphi_i} \cdot \underline{e}_\varphi) + \cos\theta_i (\underline{e}_{\theta_i} \cdot \underline{e}_\varphi) \right] \underline{e}_\varphi \right.$$

$$\left. + \left[\pm i(\underline{e}_{\varphi_i} \cdot \underline{e}_\theta) + \cos\theta_i (\underline{e}_{\theta_i} \cdot \underline{e}_\theta) \right] \underline{e}_\theta \right\}$$

Now

$$S \equiv \left| (\underline{\underline{u}} - \underline{e}_r \underline{e}_r) \cdot (\underline{\underline{u}} - \underline{e}_a \underline{e}_a) \cdot \underline{E}_o \right|^2 \tag{E-2}$$

or

$$S = E_o'^2 \left\{ \left| i(\underline{e}_{\varphi_i} \cdot \underline{e}_\varphi) + \cos\theta_i (\underline{e}_{\theta_i} \cdot \underline{e}_\varphi) \right|^2 \right.$$

$$\left. + \left| i(\underline{e}_{\varphi_i} \cdot \underline{e}_\theta) + \cos\theta_i (\underline{e}_{\theta_i} \cdot \underline{e}_\theta) \right|^2 \right\}$$

$$= E_o'^2 \left\{ (\underline{e}_{\varphi_i} \cdot \underline{e}_\varphi)^2 + \cos^2\theta_i (\underline{e}_{\theta_i} \cdot \underline{e}_\varphi)^2 \right.$$

$$\left. + (\underline{e}_{\varphi_i} \cdot \underline{e}_\theta)^2 + \cos^2\theta_i (\underline{e}_{\theta_i} \cdot \underline{e}_\theta)^2 \right\}$$

We can easily show that

$$\underline{e}_{\varphi_i} \cdot \underline{e}_\varphi = \cos(\varphi - \varphi_i)$$

$$\underline{e}_{\varphi_i} \cdot \underline{e}_\theta = \cos\theta \sin(\varphi - \varphi_i)$$

$$\underline{e}_{\theta_i} \cdot \underline{e}_\theta = \cos\theta \cos\theta_i \cos(\varphi - \varphi_i) + \sin\theta \sin\theta_i$$

$$\underline{e}_{\theta_i} \cdot \underline{e}_\varphi = \cos\theta_i \sin(\varphi - \varphi_i)$$

and from E-2

$$S = E_o'^2 \left\{ \cos^2(\varphi - \varphi_i) + \cos^2\theta_i \cos^2\theta_i \sin^2(\varphi_i - \varphi) + \cos^2\theta \sin^2(\varphi - \varphi_i) + \cos^2\theta_i \left[\cos\theta \cos\theta_i \cos(\varphi - \varphi_i) + \sin\theta \sin\theta_i \right]^2 \right\}$$

If we now do the φ_i integration E-1 gives

$$I = \int_o^{2\pi} S \, d\varphi_i = \left\{ \frac{1}{2}(1 + \cos^2\theta)(1 + \cos^4\theta_i) + 2\pi \sin^2\theta \sin^2\theta_i \cos^2\theta_i \right\} E_o'^2$$

If the incident wave is linearly polarized the computation goes along the same lines.

ii) Next we have to compute the following integral:

$$I = \int_{V_p} \exp\{-ikR[\cos(\varphi-\varphi_i)\sin\theta \sin\theta_i + \cos\theta_i(\cos\theta - 1)]\} \, dV_i \tag{E-3}$$

where $dV_i = R^2 dR \sin\theta_i \, d\theta_i \, d\varphi_i$.

First we do the φ_i integration:

$$I_1 = \int_o^{2\pi} \exp[-ikR\cos(\varphi-\varphi_i)\sin\theta \sin\theta_i] \, d\varphi_i = 2\pi J_o(kR \sin\theta_i \sin\theta)$$

Next we do the θ_i integration

$$I_2 = 2\pi \int_o^{\pi} J_o(kR\sin\theta_i \sin\theta) \exp[ikR \operatorname{sos}\theta_i(1 - \cos\theta)] \sin\theta_i \, d\theta_i$$

$$= 2\pi \int_{-1}^{+1} J_o(kR\sin\theta \sqrt{1 - \mu^2}) \exp[ik(1 - \cos\theta)R\mu] \, d\mu$$

Now

$$\int_{-1}^{+1} J_o(\alpha\sqrt{1-\mu^2})e^{i\beta\mu}\,d\mu = \int_{-1}^{+1} J_o(\alpha\sqrt{1-\mu^2})\cos\beta\mu\,d\mu$$

$$+\, i\int_{-1}^{+1} J_o(\alpha\sqrt{1-\mu^2})\sin\beta\mu\,d\mu$$

The imaginary part is zero because the integrand is an odd function of μ. Finally

$$\int_{-1}^{+1} J_o(\alpha\sqrt{1-\mu^2})d^{i\beta\mu}\,d\mu = 2\int_{o}^{1} J_o(\alpha\sqrt{1-\mu^2})\cos\beta\mu\,d\mu$$

It can be shown (5) that

$$\int_{o}^{1} J_o(\alpha\sqrt{1-\mu^2})\cos\beta\mu\,d\mu = \frac{\sin(\sqrt{\alpha^2+\beta^2})}{\sqrt{\alpha^2+\beta^2}} \tag{E-4}$$

Now $\alpha^2+\beta^2 = k^2R^2\sin^2\theta + k^2R^2(1-\cos\theta)^2 = (2kR\sin\frac{\theta}{2})^2$. Finally we get

$$I_2 = 2\pi\,\frac{2\sin(2kR\sin\frac{\theta}{2})}{2kR\sin\frac{\theta}{2}}$$

and from E-3

$$I = 4\pi\int_{o}^{a}\frac{\sin(2kR\sin\frac{\theta}{2})}{2kR\sin\frac{\theta}{2}}R^2\,dR$$

$$= \frac{2\pi}{4(k\sin\frac{\theta}{2})^3}\int_{o}^{2ka\sin\frac{\theta}{2}} x\sin x\,dx$$

$$I = \frac{\pi}{2(k\sin\frac{\theta}{2})^3}\left[\sin(2ka\sin\frac{\theta}{2}) - 2ka\sin\frac{\theta}{2}\cos(2ka\sin\frac{\theta}{2})\right]$$

iii)

$$I=\int_{\alpha=0}^{2\pi}\int_{\beta=0}^{\pi}\left\{\frac{\sin\{kL[\cos(\varphi-\alpha)\sin\beta\sin\theta+\cos\beta(\cos\theta-1)]\}}{\cos(\varphi-\alpha)\sin\beta\cos\theta+\cos\beta(\cos\theta-1)}\right\}^2\sin\beta\,d\beta\,d\alpha$$

First we observe that if we make the substitution $\alpha-\varphi=\alpha'$ then α' varies from $-\varphi$ to $-\varphi+2\pi$. Because of the periodicity of the integrand (= $f(\cos\alpha')$) we can replace the limits by 0 and 2π. Therefore I becomes independent of the azimuthal angle φ. We now call $\alpha'=\alpha$ and we first do the α integration

$$I_1=\int_0^{2\pi}\left[\frac{\sin(\alpha_1\cos\alpha+\alpha_2)x}{\alpha_1\cos\alpha+\alpha_2}\right]^2 d\alpha$$

where $\alpha_1=\sin\beta\sin\theta$, $\alpha_2=\cos\beta(\cos\theta-1)$, $x=kL$. We observe that

$$I_2=\frac{dI_1}{dx}=\int_0^{2\pi}\frac{2\sin[(\alpha_1\cos\alpha+\alpha_2)x]\cos[(\alpha_1\cos\alpha+\alpha_2)x]}{\alpha_1\cos\alpha+\alpha_2}\,d\alpha$$

and

$$I_3=\frac{dI_2}{dx}=\frac{d^2I_1}{dx^2}=2\int_0^{2\pi}\cos[2x(\alpha_1\cos\alpha+\alpha_2)]\,d\alpha$$

$$=2\cos(2x\alpha_2)\int_0^{2\pi}\cos[(2x\cos\alpha)\alpha_1)]\,d\alpha$$

$$-2\sin(2x\alpha_2)\int_0^{2\pi}\sin[(2x\cos\alpha)\alpha_1]\,d\alpha$$

$$=2\cos(2x\alpha_2)\int_0^{2\pi}\cos[2x\alpha_1\cos\alpha]\,d\alpha$$

$$=4\pi\cos(2x\alpha_2)J_o(2x\alpha_1)$$

Now observe that $I_2(0) = 0$, i.e.

$$I_2(x) = 4\pi \int_o^x J_o(2\alpha_1 \lambda_1) \cos (2\lambda_1 \alpha_2)\, d\lambda_1$$

Next $I_1(0) = 0$, i.e.

$$I_1(x) = \int_o^x I_2(\lambda_2)\, d\lambda_2$$

$$= 4\pi \int_o^x \int_o^{\lambda_2} J_o(2\alpha_1 \lambda_1) \cos (2\lambda_1 \alpha_2)\, d\lambda_1\, d\lambda_2$$

and

$$I = 4\pi \int_{-1}^{+1} \int_o^x \int_o^{\lambda_2} J_o(2\lambda_1 \sin\theta \sqrt{1-\mu^2}) \cos [\, 2\lambda_1 (1-\cos\theta)\mu]\, d\lambda_1\, d\lambda_2\, d\mu$$

$$= 4\pi \int_o^x d\lambda_2 \int_o^{\lambda_2} d\lambda_1 \int_{-1}^{+1} J_o(b_1 \sqrt{1-\mu^2}) \cos (b_2 \mu)\, d\mu$$

If E-4 is recalled:

$$I = 4\pi \int_o^x d\lambda_2 \int_o^{\lambda_2} d\lambda_1\, 2\, \frac{\sin \sqrt{b_1^2 + b_2^2}}{\sqrt{b_1^2 + b_2^2}}$$

where

$$b_1^2 + b_2^2 = (2\lambda_1 \sin\theta)^2 + (2\lambda_1 (1 - \cos\theta))^2$$

$$= (4\lambda_1 \sin \frac{\theta}{2})^2$$

and

$$I = 8\pi \int_o^x d\lambda_2 \int_o^{4\lambda_2 \sin\frac{\theta}{2}} \frac{1}{4\sin\frac{\theta}{2}} \frac{\sin z}{z}\, dz$$

$$= \frac{2\pi}{\sin\frac{\theta}{2}} \int_o^x d\lambda_2 \int_o^{4\lambda_2 \sin\frac{\theta}{2}} \frac{\sin z}{z}\, dz$$

Call now $\int_o^{\ell} \frac{\sin z}{z}\, dz = s_i(\ell)$ = sine integral

$$I = \frac{2\pi}{\sin\frac{\theta}{2}} \int_o^x d\lambda_2 \; {}_i(4\lambda_2 \sin\frac{\theta}{2})$$

If we integrate by parts we obtain:

$$I = \frac{2\pi}{\sin\frac{\theta}{2}} \left\{ s_i(4\lambda_2 \sin\frac{\theta}{2})\lambda_2 \Big|_o^x - \int_o^x \lambda_2 \frac{\sin(4\lambda_2 \sin\frac{\theta}{2})}{4\lambda_2 \sin\frac{\theta}{2}} 4\sin\frac{\theta}{2}\, d\lambda_2 \right\}$$

$$= \frac{2\pi}{\sin\frac{\theta}{2}} \left\{ x s_i(4x\sin\frac{\theta}{2}) + \frac{1}{4\sin\frac{\theta}{2}} (\cos(4x\sin\frac{\theta}{2}) - 1 \right\}$$

$$= \frac{2\pi}{\sin\frac{\theta}{2}} \left\{ kL \; {}_i(4kL\sin\frac{\theta}{2}) + \frac{1}{4\sin\frac{\theta}{2}} \left[\cos(4kL\sin\frac{\theta}{2}) - 1\right] \right\}$$

$$I = \frac{2\pi kL}{\sin\frac{\theta}{2}} \left\{ \frac{\cos(4kL\sin\frac{\theta}{2} - 1)}{4kL\sin\frac{\theta}{2}} + s_i(4\,kL\sin\frac{\theta}{2}) \right\}$$

Thus if we call $2L = L_o$ and $V_p^2 = 2LA$ we get

$$K = I\frac{A^2}{\pi k^2} = \frac{2V_p^2}{(2kL_o \sin\frac{\theta}{2})} \left\{ \frac{\cos(2kL_o\sin\frac{\theta}{2}) - 1}{2kL_o\sin\frac{\theta}{2}} + s_i(2kL_o\sin\frac{\theta}{2}) \right\}$$

where $s_i(x) = \int_o^x \frac{\sin z}{z}\, dz$.

iv) Consider

$$K = \int_{\tau=0}^{2\pi} \int_{z=z_1}^{z_2} \int_{\rho=0}^{\rho(z')} \exp[-ik(A\rho\cos\tau + B\rho\sin\tau + Cz'] \; dz'\rho\, d\rho\, d\tau \tag{E-6}$$

We can first do the τ integration:

$$I = \int_0^{2\pi} \exp[-ik(A\rho\cos\tau + B\rho\sin\tau)] \; d\tau$$

We cast $A\rho\cos\tau + B\rho\sin\tau$ into the form:

$$A\rho\cos\tau + B\rho\sin\tau = \rho\sqrt{A^2+B^2}\,\sin(\tau - \tau_o)$$

where $\tan\tau_o = \frac{A}{B}$. Thus

$$I = \int_0^{2\pi} \exp[-ik\rho\sqrt{A^2+B^2}\,\sin(\tau-\tau_o)] \; d\tau$$

$$= \int_0^{2\pi} \exp[+ik\rho\sqrt{A^2+B^2}\,\sin\tau] \; d\tau = 2\pi J_o(k\rho\sqrt{A^2+B^2})$$

Next we do the ρ integration

$$I_1 = 2\pi\int_0^{\rho(z')} J_o(k\rho\sqrt{A^2+B^2})\,\rho\, d\rho$$

$$= \frac{2\pi}{k^2(A^2+B^2)} \int_0^{k\rho(z')\sqrt{A^2+B^2}} J_o(x)x\,dx$$

$$= \frac{2\pi}{k^2(A^2+B^2)}\; xJ_1(x)\Big|_0^{k\rho(z')\sqrt{A^2+B^2}} = 2\pi\rho(z')\,\frac{J_1(k\rho(z')\sqrt{A^2+B^2})}{k\sqrt{A^2+B^2}}$$

Finally

$$K = \frac{2\pi}{k\sqrt{A^2+B^2}} \int_{z_1}^{z_2} dz' \, \rho(z')e^{-ikCz'} J_1(k\rho(z')\sqrt{A^2+B^2})$$

v) We must compute the integrals

$$I(n) = \int_0^a \frac{(\sin x - x\cos x)^2}{x^n}\, dx \quad \text{for } n = 5,3,1,-1,-3$$

The integrand is finite at $x = 0$ since

$$\left.\frac{(\sin x - x\cos x)^2}{x^5}\right|_{x\to 0} = \left.\left(\frac{\sin x - x\cos x}{x^3}\right)^2 x\right|_{x\to 0}$$

$$= \left.\left(\frac{x - \frac{x^3}{3!} + \ldots - x(1 - \frac{x^2}{2} + \ldots)}{x^3}\right)^2 x\,\right|_{x\to 0} = 0$$

We start with $n = 5$ as an indefinite integral

$$I'(5) = \int \frac{(\sin x - x\cos x)^2}{x^5}\, dx = -\frac{1}{4x^4}(\sin x - x\cos x)^2$$

$$+ \frac{1}{4}\int \frac{1}{x^4}\, 2(\sin x - x\cos x)(\cos x - \cos x + x\sin x)\, dx$$

$$= -\frac{1}{4x^4}(\sin x - x\cos x)^2 + \frac{1}{2}\int \frac{(\sin x - x\cos x)\sin x}{x^3}\, dx$$

Now

$$\frac{1}{2}\int \frac{(\sin x - x\cos x)\sin x}{x^3}\, dx = \frac{1}{2}\int \frac{\sin^2 x}{x^3}\, dx - \frac{1}{2}\int \frac{\cos x \sin x}{x^2}\, dx$$

$$= \frac{1}{2}\int \frac{\sin^2 x}{x^3}\, dx - \frac{1}{4}\int \frac{d \sin^2 x}{x^2}$$

$$= \frac{1}{2}\int \frac{\sin^2 x}{x^3} dx - \frac{1}{4}\, \frac{\sin^2 x}{x^2} - \frac{1}{2}\int \frac{\sin^2 x}{x^3}\, dx$$

$$= - \frac{1}{4}\, \frac{\sin^2 x}{x^2}$$

and

$$I(5) = \int_o^a \frac{(\sin x - x \cos x)^2}{x^5}\, dx = - \frac{1}{4a^4}(\sin a - a \cos a)^2$$

$$- \frac{1}{4} \frac{\sin^2 a}{a^2} + \frac{1}{4}$$

$$I(3) = \int_o^a \frac{(\sin x - x \cos x)^2}{x^3} dx = \left[- \frac{1}{2}\, \frac{1}{x^2} (\sin x - x \cos x)^2 - \frac{1}{2} \sin^2 x \right]_o^a$$

$$+ \int_o^a \frac{\sin^2 x}{x}\, dx \qquad \text{(E-7)}$$

Now

$$\int \frac{\sin^2 x}{x} dx = \int \frac{\sin^2 x}{x/a}\, d\, \frac{x}{a} = \int \sin^2 x\, d \ln \frac{x}{a}$$

$$= \sin^2 x \ln \frac{x}{a} - \int \ln \frac{x}{a} \sin 2x\, dx$$

and

$$\int_o^a \frac{\sin^2 x}{x} dx = \sin^2 x \ln \frac{x}{a} \Big|_o^a - \int_o^a \ln \frac{x}{a} \sin 2x\, dx$$

$$= - \int_o^a \ln \frac{x}{a} \sin 2x\, dx = -a \int_o^1 \ln u \sin 2ua\, du$$

It can be shown (5) that

$$\int_0^1 \ln u \sin 2au \, du = -\frac{1}{2a}\left[C + \ln 2a - ci(2a)\right] \quad a > 0$$

where $C = 0.577215...$

$$ci(x) = \int_\infty^x \frac{\cos u}{u} \, du$$

Thus

$$\int_0^a \frac{\sin^2 x}{x} \, dx = \frac{1}{2}\left[C + \ln 2a - ci(2a)\right]$$

and from E-7

$$I(3) = -\frac{1}{2}\frac{1}{a^2}(\sin a - a\cos a)^2 - \frac{1}{2}\sin^2 a$$

$$+\frac{1}{2}\left[C + \ln 2a - ci(2a)\right]$$

$$I(1) = \int_0^a \frac{(\sin x - x\cos x)^2}{x} dx$$

$$= \int_0^a \frac{\sin^2 x}{x} \, dx + \int_0^a x\cos^2 x \, dx - 2\int_0^a \sin x \cos x \, dx$$

$$= \frac{1}{2}\left[C + \ln 2a - ci(2a)\right] + \frac{a^2}{4} + \frac{a \sin 2a}{4} + \frac{5}{8}(\cos 2a - 1)$$

$$I(-1) = \int_0^a (\sin x - x\cos x)^2 x \, dx$$

$$= \frac{1}{2}x^2(\sin x - x\cos x)^2 \Big|_0^a - \int_0^a x^2(\sin x - x\cos x)x\sin x \, dx$$

$$= (\sin x - x\cos x)^2 \frac{x^2}{2}\Big|_0^a + \frac{1}{2}x^4\sin^2 x\Big|_0^a - 3\int_0^a x^3\sin^2 x \, dx$$

$$= \frac{a^2}{2}(\sin a - a\cos a)^2 + \frac{1}{2}a^4\sin^2 a - 3\left\{\frac{a^4}{8} - \left(\frac{a^3}{4} - \frac{3a}{8}\right)\sin 2a\right.$$

$$\left. - \left(\frac{3a^2}{8} - \frac{3}{16}\right)\cos 2a - \frac{3}{16}\right\}$$

$$I(-3) = \int_o^a x^3(\sin x - x\cos x)^2 dx$$

$$= \frac{1}{4}x^4(\sin x - x\cos x)^2\Big|_o^a - \int_o^a \frac{1}{2}x^4(\sin x - x\cos x)x\sin x\, dx$$

$$= \frac{1}{4}a^4(\sin a - a\cos a)^2 - \frac{1}{2}\int_o^a x^5\sin^2 x\, dx + \frac{1}{2}\int_o^a x^6 \sin x\cos x\, dx$$

Now

$$\frac{1}{2}\int_o^a x^6\sin x\cos x\, dx = \frac{1}{4}x^6\sin^2 x\Big|_o^a - \frac{3}{2}\int_o^a x^5\sin^2 x\, dx$$

and

$$I(-3) = \frac{1}{4}a^4(\sin a - a\cos a)^2 + \frac{1}{4}a^6\sin^2 a - 2\int_o^a x^5\sin^2 x\, dx$$

$$= \frac{1}{4}a^4(\sin a - a\cos a)^2 + \frac{1}{4}a^6\sin^2 a - \frac{a^6}{6}$$

$$+ \frac{1}{8}(10a^4 - 30a^2 + 15)\cos 2a$$

$$+ \frac{1}{4}(2a^5 - 10a^3 + 15)\sin 2a - \frac{15}{8}$$

REFERENCES

(1) Papas, C. H., Theory of Electromagnetic Wave Propagation, McGraw-Hill Book Company, New York (1965), 26-28, 31-33, 119-123, 126-134.

(2) Van de Hulst, H. C., Light Scattering by Small Particles, John Wiley and Sons, Inc., New York (1957), 172-173.

(3) Papoulis, A., Probability, Random Variables, and Stochastic Processes, McGraw Hill Book Company, New York (1965).

(4) Goldstein, H., Classical Mechanics, Addison-Wesley Publishing Company, Inc., Readings, Mass. (1950), 107-109.

(5) Gradshteyn, I. S., and Ryzhik, I. M., Table of Integrals, Series and Products, Academic Press, New York (1965), 737 No. 6.677-6, 583 No. 4.381-1.

(6) Abramowitz, M., and Stegun, I. A. (Editors), Handbook of Mathematical Functions, National Bureau of Standards (1964), 438, 469-470, 238-242.

(7) Rowell, R. L., and Stein, R. S. (Editors), Electromagnetic Scattering, Gordon and Breach Science Publishers, New York (1967).

(8) Chandrasekhar, S., Radiative Transfer, Dover Publications, Inc., New York (1960).

(9) Hartel, N., Licht 10, 141 (1940).

(10) Chu, C. M., Leacock, J. A., Chen, J. C., and Churchill, S. W., in Electromagnetic Scattering, edited by Kerker, M., Pergamon Press, Inc., New York (1963), 567.

(11) Woodward, D. M., Journal of the Optical Society of America, 54, 1325 (1964).