

Applications of the Decoherence Formalism

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Abstract

In this work the decoherence formalism of quantum mechanics is explored and applied to a number of interesting problems in quantum physics. The boundary between quantum and classical physics is examined, and demonstration made that quantum histories corresponding to classical equations of motion become more probable for a broad class of models, including linear and nonlinear models of Brownian motion. The link between noise, dissipation, and decoherence is studied. This work is then applied to systems which classically exhibit dissipative chaotic dynamics. A theory is explicated for treating these systems, and the ideas are applied to a particular model of the forced, damped Duffing oscillator, which is chaotic for certain parameter values. Differences between classical and quantum chaos are examined, particularly differences arising in the structure of fractal strange attractors, and the conceptual difficulties in framing standard notions of chaos in a quantum system. A brief discussion of previous work on quantum chaos is included, and the differences between Hamiltonian and dissipative chaos pointed out; a somewhat different interpretation of quantum chaos from the standard one is suggested. A class of histories for quantum systems, in phase space rather than configuration space, is studied. Different ways of representing projections in phase space are discussed, and expressions for the probability of phase space histories are derived; conditions for such histories to decohere are also estimated in the semiclassical limit.

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Introduction

I. DECOHERENCE

Ever since the discovery of quantum mechanics, questions have arisen about the physical meaning and interpretation of quantum theory. The early schisms over the validity of quantum mechanics and its realm of application are legendary for their passion and brilliant debate. The consensus ultimately arrived at by Bohr, Dirac, and many others was successful in quieting the doubts about the validity of quantum mechanics itself among all but a few; but the resulting theory, enormously successful as it was, nevertheless remained unsatisfactory in many ways.

The Copenhagen interpretation describes quantum phenomena as seen in a laboratory excellently, but has considerable lingering practical and philosophical problems. The rigorous division of the universe into quantum objects and classical measuring devices begs the question of how “classical” behavior arises in a system composed of quantum mechanical constituents. The collapse of the wave function—the magic trick that we all learn in our introductory QM course, which enables us to calculate probabilities for quantum events—has many troubling properties. It is instantaneous and irreversible, introducing a fundamental arrow of time into a system which is otherwise time-symmetric. It is a dramatic departure from the usual form of evolution for quantum systems. It has no convincing physical description or explanation. It places an unhealthy emphasis on the necessity of a human, or at any rate intelligent, observer for wave function collapse. It gives little guidance to studies of the universe as a whole; a system containing everything in the universe must, perforce, have nothing outside to “measure” it.

These problems have been tackled piecemeal over the years, but the first real attempt to alter this picture was Hugh Everett’s “Many-Worlds” interpretation of quantum mechanics [1]. In Everett’s picture, every quantum event causes the universe to split into coexisting but noninteracting universes, representing all possible outcomes of the event. Thus, no collapse of the wavefunction occurs. This is a very

bold notion, avoiding the practical and philosophical problems of the Copenhagen interpretation, but having many difficulties of its own. Why couldn't these different universes interfere with each other, producing measurable results? Indeed, they must do so to be consistent with laboratory experiments on interference. And these infinite coexisting universes are far from a simple picture; such a theory turns the principle of parsimony on its head.

Nevertheless, the germ of this idea has fertilized more recent attempts to explain quantum mechanics in a satisfactory way. Instead of many worlds, one has many *histories* of a single universe, each with some probability, only one of which is assumed actually to occur.

The other histories represent alternative possibilities; and as the progress of time eliminates different possibilities, the histories representing them are also eliminated, just as in the classical view of probability.

A history of the universe consists of sets of values for the variables used to describe the universe at successive points in time. The most fine-grained possible description would be to specify the values of a complete set of variables at every point in time – for example, the trajectories of all particles for all of time (to take a nonrelativistic viewpoint). Such a history is simple conceptually, quite close to a classical view of the world. Since the universe is not classical, however, these fine-grained histories are unsuitable for actual calculations. They cannot be consistently assigned classical probabilities, because the histories can interfere with each other, and therefore do not obey classical probability sum rules.

Instead, we must take a more coarse-grained view. We can coarse-grain our histories by considering only ranges rather than exact values, for only certain variables, and only at certain discrete points in time. If we coarse-grain a set of histories sufficiently that we can assign them classical probabilities which obey classical probability sum rules without interference, we call such a set of histories *decoherent*. If the probabilities also sum to 1, we call this an *exhaustive* set of histories. Given an exact

knowledge of the fundamental laws of nature, plus an exact knowledge of the initial condition of the universe, we could (in principle) enumerate an exhaustive set of decoherent histories which would describe all possible evolutions of the universe, and calculate probabilities for them.

This approach was developed by a number of people, including Griffiths (“Consistent Histories”), Omnès, Zurek (“Environment-Induced Superselection”), and Gell-Mann and Hartle [2–7]. In the theory of Gell-Mann and Hartle, decoherence is determined by a *decoherence functional*. This is a functional on pairs of histories $D[h, h']$. If

$$\text{Re}D[h, h'] = 0, h \neq h', \quad (1)$$

then the set of histories weakly decohere, and their probabilities are given by the diagonal terms

$$p[h] = D[h, h]. \quad (2)$$

Later, in the individual chapters, I give the definition of $D[h, h']$ for standard nonrelativistic quantum mechanics, and examine its behavior for different systems.

Gell-Mann and Hartle have gone on to broaden and explicate this basic concept [7], emphasizing a much stronger view of decoherence than Griffiths and Omnès. They have extended the idea of a history, generalized the decoherence functional, and set up the system as an axiomatic basis for any quantum theory, making an argument that coarse-grained interacting systems tend to decohere strongly ($D[h, h'] \approx 0$) due to the existence of orthogonal generalized records. I, by contrast, have used this formalism to examine several simple but interesting problems. In the chapters which follow I have made extensive use of a class of models first treated by Feynman and Vernon and later by Caldeira and Leggett [8,9]. In these models, a distinguished system, typically a single particle moving in some potential, interacts with a reservoir containing a large number of degrees of freedom whose individual motions can be neglected. For a very

simple reservoir, typically a thermal bath of harmonic oscillators, the average effects of the reservoir on the system can be calculated and summarized in an influence functional. Caldeira and Leggett used this model to examine Brownian motion. I have used it similarly, to calculate the decoherence functional for a system simple enough that the equations remain relatively tractable. All of the chapters are treated nonrelativistically; they largely concern the rise of classical or quasiclassical behavior from a fundamentally quantum-mechanical underlying theory. The chapters were written separately, and are intended to be read independently, as a series of distinct papers treating a common theme.

II. QUASICLASSICAL EQUATIONS OF MOTION FOR NONLINEAR BROWNIAN SYSTEMS

In this chapter I study Brownian motion for systems similar to those described above. The main gist of the paper is the study of systems interacting with thermal reservoirs in both the classical and quantum case. In each I show how the interactions with the bath produces a stochastic force which acts on the system, as well as an averaged force which (in the appropriate limit) serves as a source of dissipation or friction.

This is nothing novel. The rise of such noise and dissipation in models of this sort has been demonstrated for linear systems both classically (by Zwanzig) and quantum mechanically (by Caldeira and Leggett) [10,9]. By using the decoherence formalism, however, I am able to show that noise and dissipation are not the only effects of this type of coarse-grained system. The same interaction which produces them also produces decoherence, i.e., the elimination of interference terms between histories of the system. Noise, dissipation, and decoherence turn out to be intimately linked, in a relationship similar to (and related to) the fluctuation-dissipation theorem.

One can picture the reservoir as continuously “measuring” the system, eliminating

interference by preventing superposition, but at the same time resulting in uncertainties in the system's path. Each tiny "measurement" is a quantum event with a probabilistic outcome.

I am also able to show that in the quasiclassical limit, the most probable histories of the quantum system become centered about the classical path. The quantum mechanical uncertainty of the micro-level shades into the more usual classical uncertainty due to the existence of a stochastic force.

In addition to treating systems interacting linearly with the bath, I also consider weakly nonlinear interactions. While the path integrals in this case can no longer be exactly calculated, they can be approximated in perturbation theory. I use perturbation theory to calculate both the decoherence functional and the classical equation of motion and show that once again the most probable histories center on those that follow the classical trajectories. In this case as well, interactions lead to decoherence, and it is possible to show that this is a very general result.

This chapter has been previously published as a paper in *Physical Review D*, volume 47, pp. 3383-3393. It is entirely my own work, but appeared as a companion to a paper by Gell-Mann and Hartle, p. 3345 in the same issue.

III. QUANTUM DISSIPATIVE CHAOS

Since the formalism developed in the previous paper is so useful for treating quantum mechanical systems with dissipation, I thought it would be interesting to treat a system with inherent interest, dissipative chaos. While "Quantum Chaos" has been around for a few years now, almost all of the work in this field concerns Hamiltonian systems. This is natural enough, in that we have considerable intuitive knowledge of systems with Hamiltonians. Examining the quantum mechanical equivalents to these classically chaotic systems is comparatively straightforward.

Systems with dissipation are much more problematic, and little has been done

in the way of finding quantum equivalents (for some examples, see Chapter II). I showed how the simple Brownian-motion model could be used to represent a chaos, by adding dissipation to a nonlinear system. Outlining the general framework for such a theory, I considered a particular example, and performed explicit calculations with the forced, damped Duffing oscillator. This is a canonical example of the chaotic oscillator, and exhibits all of the typical behavior of such systems.

I once again demonstrated that my quantum system had the same quasiclassical behavior as the classical Duffing oscillator, with the inescapable addition of stochastic noise. In studies of Hamiltonian quantum chaos, workers have observed that the chaotic behavior becomes suppressed as one approaches the quantum limit. Somehow, the uncertainty of chaos becomes less in the quantum case! While this is accurate in one sense, it neglects the need for decoherence in order to say anything meaningful about a quantum system. It is only when a system decoheres, i.e., when we can assign it a probability, when we “measure” it, that we can gain any knowledge of it. When such a measurement is made, the outcome varies probabilistically, as normal. As one goes from the classical to the quantum limit, the uncertainty does not decrease, but rather changes in nature: from the deterministic uncertainty of chaos to the probabilistic uncertainty of quantum mechanics.

In the case of dissipative systems, there is no such suppression of chaos. Instead, to the deterministic chaos is added a stochastic term, resulting from the noise of interaction with a reservoir. While in the classical limit this noise may be tiny, it is characteristic of chaotic systems that even very small noise is amplified until it can affect the macroscopic state of the system. Thus, allowing for quantum effects can only increase uncertainty.

In addition to this, I examined some of the generic traits of dissipative chaos and showed how they were modified by quantum mechanics. The fractal strange attractor can no longer be a true fractal with quantum mechanics; the existence of noise (ultimately arising from the uncertainty principle itself) introduces a fundamental

scale below which the fractal must cease to be self-similar. In a similar way, the usual definition of chaos—sensitive dependence on initial conditions—becomes somewhat ambiguous in the quantum case. Since a completely fine-grained trajectory is not a decoherent history, it makes little sense to speak of starting trajectories arbitrarily close together and following them for arbitrarily long times. I suggest some possible alternatives which could lead to a quantum mechanical definition of chaos.

This paper has not yet been submitted for publication in a journal. It appeared as a Caltech preprint in 1993, and was replaced with a revised version a few months later. It too is entirely my own work.

IV. THE DECOHERENCE OF PHASE SPACE HISTORIES

In looking at the quasiclassical limit of quantum theories, it sometimes becomes desirable to frame problems in a more classical way. For example, the strange attractors of the chaotic systems in the preceding chapter are most suitably described in phase space. Quantum mechanics is somewhat hampered by the fact that there is no completely natural way of framing the theory in phase space.

In an effort to improve this slightly, I looked at possible ways of describing phase space histories—coarse-grained paths in phase space instead of configuration space. This is made difficult by the fact that there are no very good projection operators which project onto local cells of phase space. Thanks to the uncertainty principle, these are not very well-defined. Since the most straightforward definition of the decoherence functional used projection operators to specify alternatives at a given time, this lack is rather important.

Nevertheless, there are a few approximations to such operators. In this paper I consider a couple of such approximations: consecutive projections in x and the p , and coherent states. The former are unsatisfactory because the two projections do not commute, and thus one gets different results depending on the order in which they

are applied. The latter are true projections, but unfortunately are not orthogonal, and are overcomplete. Both of these sets of handicaps diminish as one takes cells of phase space relatively large compared to \hbar .

In this chapter I derive fairly general expressions for the probabilities of a set of phase space histories, and show that highly disjoint or discontinuous histories tend to be suppressed. I then make a number of approximations, and examine the decoherence of phase space histories for the Caldeira and Leggett based models I have been using. In the semiclassical limit I show that the level of coarse-graining required for decoherence is much greater than that which would be naively required by the uncertainty principle alone, i.e., the area $\Delta x \Delta p$ of a cell in phase space must be considerably larger than \hbar . I also elaborate a method for actually calculating the probability and decoherence of such systems numerically.

This paper, too, has not yet been submitted for publication. It is my own work, and will shortly appear as a Caltech preprint.

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Chapter I

Quasiclassical Equations of Motion for Nonlinear Brownian Systems

Quasiclassical Equations of Motion for Nonlinear Brownian Systems

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Abstract

Following the formalism of Gell-Mann and Hartle, phenomenological equations of motion are derived from the decoherence functional of quantum mechanics, using a path-integral description. This is done explicitly for the case of a system interacting with a “bath” of harmonic oscillators whose individual motions are neglected. The results are compared to the equations derived from the purely classical theory. The case of linear interactions is treated exactly, and nonlinear interactions are also compared, using classical and quantum perturbation theory.

I. INTRODUCTION

A. Decoherence and the quasiclassical limit

Two of the most puzzling aspects of the quantum theory have, until recently, remained unclear: the proper interpretation of quantum probabilities, and the mechanism by which deterministic classical “laws” can arise from a probabilistic underlying theory. The idea of wave-function collapse, while providing a useful approximate description of most experimental situations, begs the question of why a system which otherwise undergoes purely unitary evolution should suddenly and dramatically be collapsed upon measurement by a scientist. The procedure is highly asymmetric, instantaneous, and irreversible, and moreover requires the existence of a “classical” measuring device outside the system being measured. When one considers a closed system, the idea of wave-functions collapsing becomes highly ambiguous. There is nothing outside the system to collapse it. The quintessential example of this, of course, is the universe itself. Clearly, if the fundamental laws of the universe are quantum mechanical, there can be no separate “classical domain” to explain our observations. Since the classical realm is itself, presumably, merely a limit of the underlying quantum reality, the probabilities must arise directly from the quantum theory itself, without recourse to the *deus ex machina* of the measurement device. And somehow, the various potential futures of the universe must collapse themselves onto the one possibility which we observe. Quantum cosmology requires a solid formalism for the treatment of closed systems, and work in this field should have that as its goal.

The recent work on the decoherence functional formulation avoids the problems of earlier approaches [1–4]. Physics is described in terms of exhaustive sets of possible histories, coarse-grained, with the restriction that these histories must be decoherent. That is, it must be possible to assign probabilities to these histories such that they

obey the classical laws of probability, with no interference.

Gell-Mann and Hartle have argued that it is possible, in a highly coarse-grained system, to define the classical equation of motion directly from the decoherence functional itself [5]. I will, in this paper, attempt to show that this definition gives the exact classical results, at least for the case of systems interacting with baths of oscillators; and further, that these systems are decoherent in the classical limit. Quantum effects enter as a random, fluctuating force from the effects of neglected degrees of freedom, even in cases where the classical noise would ordinarily be zero. Fluctuations, dissipation, and decoherence turn out to be intimately interlinked.

The linear case has been treated before by a number of people, both classically and quantum mechanically, though not in precisely this same way [6–8]. The correspondence of this quantum system to the classical Langevin equation is thus nothing new. The decoherence of similar systems has also been examined, using a somewhat different definition of decoherence which for these models generally corresponds to my definition [9]. However, to my knowledge, no one has considered the classical correspondence of these sorts of nonlinear systems, nor the relationship between dissipation, noise, and decoherence in these more general cases. Thus, the results herein are of interest in demonstrating that it is possible to define classical equations of motion directly from the quantum theory in a broad range of systems.

B. Path-integral description of the decoherence functional

We will not, for the purposes of this model, be using the decoherence functional in its most general form. Instead, we will consider only one type of history. Suppose that our system is completely described by a set of generalized coordinates q^β (collectively referred to as q). The most fine-grained possible family of histories would be just the set of all possible paths $q(t)$. We can coarse-grain this by dividing the range of the q^β into an exhaustive set of intervals $\Delta_{\alpha_i}^i(t_i)$ at a sequence of times t_1, t_2, \dots , where the

α_i are an index labeling the intervals. We can then specify one particular history by which interval was passed through at each time, labeling it by the sequence of indices $\alpha_1, \alpha_2, \dots$, which I will generally abbreviate as α . Such a history includes all possible paths which pass through the given set of intervals at the given times.

The decoherence functional is a functional on pairs of histories. The value of this functional on a pair of histories α and α' is given by

$$D(\alpha', \alpha) = \int_{\alpha'} \delta q' \int_{\alpha} \delta q \delta(q'_f - q_f) \exp\left\{i(S[q'(t)] - S[q(t)])/ \hbar\right\} \rho(q'_0, q_0). \quad (1.1)$$

Here we are integrating over all paths which pass through the specified sequence of intervals. The functional $S[q(t)]$ is the fundamental action. If $\text{Re}D(\alpha', \alpha) = 0$ for $\alpha' \neq \alpha$, then the system is said to be *decoherent*, and obeys classical laws of probability. The diagonal elements $D(\alpha, \alpha)$ are the probabilities of each history α .

The simplest form of this type of history is that where the intervals are completely fine-grained in certain variables, and completely coarse-grained in others. We divide the coordinates q^β into two groups, x^β (henceforth known as the *system* coordinates), referred to collectively as x , and Q^k (henceforth known as the *reservoir* coordinates), referred to collectively as Q . Our histories will then be complete trajectories $x^\beta(t)$ for the system coordinates, while the reservoir coordinates will be neglected completely.

It is then convenient to break the fundamental action of the system into several parts:

$$S[q(t)] = S_{\text{sys}}[x(t)] + S_{\text{res}}[Q(t)] - \int_{t_0}^{t_f} V(x(t), Q(t)) dt, \quad (1.2)$$

where $S_{\text{sys}}[x(t)]$ is the action of the system, $S_{\text{res}}[Q(t)]$ is the action of the reservoir, and there is an interaction potential $V(x, Q)$ between them. The decoherence functional is then

$$\begin{aligned}
D[x'(t), x(t)] = & \exp\left\{i(S_{\text{sys}}[x'(t)] - S_{\text{sys}}[x(t)])/ \hbar\right\} \int \delta Q' \int \delta Q \delta(Q'(t_f) - Q(t_f)) \\
& \times \exp\left\{i(S_{\text{res}}[Q'(t)] - S_{\text{res}}[Q(t)] \right. \\
& \quad \left. - \int_{t_0}^{t_f} (V(x'(t), Q'(t)) - V(x(t), Q(t))) dt)/ \hbar\right\} \\
& \times \rho(x'_0, Q'_0; x_0, Q_0). \tag{1.3}
\end{aligned}$$

II. LINEAR CASE

The case of a system interacting linearly with a reservoir is a famous one, and has been treated by a number of people; quantum mechanically by Feynman and Vernon, and Caldeira and Leggett, classically by Zwanzig. For convenience, it is customary to make a number of simplifying assumptions:

1. The reservoir variables Q^k are harmonic oscillators, i.e.,

$$S_{\text{res}}[Q(t)] = \sum_k \int_{t_0}^{t_f} \frac{m}{2} (\dot{Q}^k{}^2 - \omega_k^2 Q^k{}^2) dt. \tag{2.1}$$

2. The initial density matrix factors:

$$\rho(x'_0, Q'_0; x_0, Q_0) = \chi(x'_0, x_0) \phi(Q'_0, Q_0). \tag{2.2}$$

A similar assumption classically is to assume that the initial probability distribution of the reservoir coordinates is independent of the initial state of the system coordinates.

3. The interaction $V(x, Q)$ is bilinear:

$$V(x, Q) = - \sum_k \gamma_k x Q^k. \tag{2.3}$$

I will generally assume that x is a single variable; multivariable systems are a trivial generalization, where the γ_k become matrices.

We will relax these assumptions to a certain degree later on, but for now let us consider this case. The classical case is exactly solvable. In this, the equation of motion for the reservoir variable Q^k is

$$\frac{d^2 Q^k}{dt^2}(t) = -\omega_k^2 Q^k(t) + (\gamma_k/m)x(t). \quad (2.4)$$

This has a solution

$$Q^k(t) = Q^k(t_0) \cos(\omega_k(t - t_0)) + \frac{\dot{Q}^k(t_0)}{\omega_k} \sin(\omega_k(t - t_0)) + \frac{\gamma_k}{m\omega_k} \int_{t_0}^t \sin[\omega_k(t - s)]x(s) ds. \quad (2.5)$$

We can then use this in the equation of motion for x :

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) (t) &= \left(\frac{\partial L}{\partial x} \right) (t) + \sum_k \gamma_k Q^k(t) \\ &= \left(\frac{\partial L}{\partial x} \right) (t) + F(t) + \sum_k \frac{\gamma_k^2}{m\omega_k} \int_{t_0}^t \sin[\omega_k(t - s)]x(s) ds, \end{aligned} \quad (2.6)$$

where $F(t)$ is the *Langevin force*. In this case, it is clearly

$$F(t) = \sum_k \gamma_k \left(Q^k(t_0) \cos(\omega_k(t - t_0)) + \frac{\dot{Q}^k(t_0)}{\omega_k} \sin(\omega_k(t - t_0)) \right). \quad (2.7)$$

If we assume that the Q have a thermal probability distribution initially,

$$P(Q) = \prod_k \frac{m\omega_k}{2\pi kT} \exp \left(-\frac{m}{2kT} (\dot{Q}^{k2} + \omega_k^2 Q^{k2}) \right), \quad (2.8)$$

which, when averaged over an ensemble, gives

$$\langle Q^k \rangle = 0, \quad \langle Q^{k2} \rangle = \frac{kT}{m\omega_k^2}, \quad \dots, \quad (2.9a)$$

$$\langle \dot{Q}^k \rangle = 0, \quad \langle \dot{Q}^{k2} \rangle = \frac{kT}{m}, \quad \dots, \quad (2.9b)$$

then

$$\langle F(t) \rangle = 0, \quad (2.10a)$$

$$\langle F(t)F(s) \rangle = \sum_k \gamma_k^2 \left(\frac{kT}{m\omega_k^2} \right) \cos(\omega_k(t - s)). \quad (2.10b)$$

Let us compare this to the quantum results. It is interesting to first consider the system in isolation from the reservoir. In this case we would have

$$D[x'(t), x(t)] = \exp\left\{i(S_{\text{sys}}[x'(t)] - S_{\text{sys}}[x(t)])/ \hbar\right\} \chi(x'_0, x_0). \quad (2.11)$$

If the action $S_{\text{sys}}[x(t)]$ has the usual form

$$S_{\text{sys}}[x(t)] = \int_{t_0}^{t_f} L(x(t), \dot{x}(t)) dt, \quad (2.12)$$

then we can change variables

$$X(t) = \frac{1}{2}[x'(t) + x(t)], \quad (2.13a)$$

$$\xi(t) = x'(t) - x(t), \quad (2.13b)$$

and expand the phase in terms of ξ :

$$\begin{aligned} S_{\text{sys}}[x'(t)] - S_{\text{sys}}[x(t)] &= \int_{t_0}^{t_f} \frac{\partial L}{\partial X}(X(t), \dot{X}(t)) \xi(t) + \frac{\partial L}{\partial \dot{X}}(X(t), \dot{X}(t)) \dot{\xi}(t) + O(\xi^3) dt \\ &= \int_{t_0}^{t_f} \left(-\frac{d}{dt} \frac{\partial L}{\partial \dot{X}}(X(t), \dot{X}(t)) + \frac{\partial L}{\partial X}(X(t), \dot{X}(t)) \right) \xi(t) dt \\ &\quad - \frac{\partial L}{\partial \dot{X}}(X_0, \dot{X}_0) \xi(t_0) + O(\xi^3). \end{aligned} \quad (2.14)$$

So the Euler-Lagrange equation of motion appears in the phase of the decoherence functional!

One should not put too much weight on this occurrence. This system is not decoherent; substantial interference can still occur between different possible trajectories. There is no particular reason to expect $\xi(t)$ to be small, so it is not correct to neglect higher-order terms. This system, on its own, is still essentially quantum-mechanical. It is not even quasiclassical.

This still leaves the effects of the reservoir variables and interaction unaccounted for. Let us turn, then, to this portion of the decoherence functional.

$$\begin{aligned} F[x'(t), x(t)] &= \int \delta Q' \int \delta Q \delta(Q'(t_f) - Q(t_f)) \exp\left\{i\left(S_{\text{res}}[Q'(t)] - S_{\text{res}}[Q(t)]\right.\right. \\ &\quad \left.\left. - \int_{t_0}^{t_f} V(x'(t), Q'(t)) - V(x(t), Q(t)) dt\right) / \hbar\right\} \chi(Q'_0, Q_0) \\ &= \exp\left\{iW[x'(t), x(t)] / \hbar\right\}. \end{aligned} \quad (2.15)$$

$F[x'(t), x(t)]$ is termed the *influence functional* by Feynman and Vernon, and $W[x'(t), x(t)]$ is the *influence phase* [6]. In our simplified model, this is not difficult to evaluate exactly. It is generally assumed that the initial density matrix is in a thermal state. We quote the results of Feynman and Vernon:

$$W[x'(t), x(t)] = \frac{1}{2} \int_{t_0}^{t_f} dt \int_{t_0}^t ds [x'(t) - x(t)] \left(k(t-s)x'(s) + k(t-s)x(s) \right), \quad (2.16)$$

where the real and imaginary parts of $k(t-s)$ are

$$k_R(t-s) = \sum_k \frac{\gamma_k^2}{m\omega_k} \sin(\omega_k(t-s)), \quad (2.17a)$$

$$k_I(t-s) = \sum_k \frac{\gamma_k^2}{m\omega_k} \coth(\hbar\omega_k/kT) \cos(\omega_k(t-s)). \quad (2.17b)$$

Changing to our variables X and ξ , we see that

$$W[X(t), \xi(t)] = \sum_k \frac{\gamma_k^2}{2m\omega_k} \int_{t_0}^{t_f} dt \int_{t_0}^t ds \left\{ 2\xi(t)X(s) \sin(\omega_k(t-s)) + i\xi(t)\xi(s) \coth\left(\frac{\hbar\omega_k}{kT}\right) \cos(\omega_k(t-s)) \right\}. \quad (2.18)$$

Thus, we have a real term which is proportional to $\xi(t)$ and an imaginary term which is proportional to $\xi(t)\xi(s)$. The imaginary term is a double integral over a symmetric kernel whose eigenvalues are strictly non-negative; thus, for large ξ the decoherence functional will be diminished by a decaying exponential

$$\exp\left[-\int_{t_0}^{t_f} dt \int_{t_0}^t ds \xi(t)\xi(s) \cos(\omega_k(t-s))\right].$$

Since ξ essentially measures how far you are from the diagonal of the decoherence functional, the off-diagonal terms tend to vanish and the system becomes decoherent.

Furthermore, since large ξ is suppressed, it now makes sense to discard terms of $O(\xi^3)$. Thus we can now say

$$S_{\text{sys}}[x'(t)] - S_{\text{sys}}[x(t)] + W[x'(t), x(t)] = \frac{i}{4} \int_{t_0}^{t_f} dt \int_{t_0}^t ds \xi(t)k_I(t-s)\xi(s) + \int_{t_0}^{t_f} dt \xi(t)\epsilon(t) + O(\xi^3), \quad (2.19)$$

where

$$e(t) = -\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{X}}(t) \right) + \frac{\partial L}{\partial X}(t) - \sum_k \frac{\gamma_k^2}{m\omega_k} \int_{t_0}^t ds X(s) \sin(\omega_k(t-s)). \quad (2.20)$$

If we compare this to (2.6), we see that

$$e(t) = 0$$

is identical to the ensemble-averaged classical equation of motion. Note that the bath of harmonic oscillators acts as a retarded force on the system. In the limit as we go to a continuum of oscillator frequencies with a high cut-off, this retarded force becomes a dissipative term, i.e., a frictional force. In this limit, Caldeira and Leggett show that for a Debye distribution of oscillator frequencies, the influence phase becomes [8]

$$W[X, \xi] = \int_{t_0}^{t_f} \left(-2\Gamma \dot{X} \xi(t) + \frac{ikT}{\hbar} \Gamma \xi^2(t) \right) dt. \quad (2.21)$$

where Γ is the usual classical coefficient of friction, defined in terms of γ and the cutoff frequency Ω . See [7,8] for details.

We have seen that the real term of $W[X, \xi]$ corresponds to the classical retarded or (in the limit) dissipative force. The imaginary term also has a classical analog. In the classical case, there is a random stochastic force $F(t)$ given by (2.7), which ensemble-averages to zero $\langle F(t) \rangle = 0$. As we see in (2.10b), however, the two-time correlation function of this force does not vanish. As $\hbar \rightarrow 0$, we get $\coth(\hbar\omega/2kT) \rightarrow 2kT/\hbar\omega$. So the imaginary part of $W[X, \xi]$ has the form

$$\text{Im } W[X(t), \xi(t)] = \int_{t_0}^{t_f} dt \int_{t_0}^t ds \langle F(t)F(s) \rangle \xi(t)\xi(s). \quad (2.22)$$

Here we observe the subtle linkage between noise, dissipation, and decoherence. In interacting with the many degrees of freedom of the reservoir, the system loses energy. It also is subject to random jostlings from the reservoir oscillators. But one last, purely quantum-mechanical effect is that the state of the system is continually

being “measured,” and thus the various possible trajectories tend to decohere, at least on a scale large compared to \hbar . Later we will see that even in situations where the classical noise vanishes, there is still quantum-mechanical noise. This arises essentially from the zero-point energy of the reservoir oscillators.

We can straightforwardly generalize to the case where the potential is nonlinear in x , but still linear in Q . Suppose that

$$V(x, Q) = - \sum_k a_k(x) Q^k. \quad (2.23)$$

Here the influence phase is

$$W[x'(t), x(t)] = \sum_k \int_{t_0}^{t_f} dt \int_{t_0}^t ds \left\{ (a_k(x'(t)) - a_k(x(t))) \right. \\ \left. \times (a_k(x'(s)) k_k(t-s) - a_k(x(s)) k_k^*(t-s)) \right\}, \quad (2.24)$$

where

$$k_k(t-s) = \frac{1}{2m\omega_k} \left[\sin(\omega_k(t-s)) + i \coth(\hbar\omega_k/2kT) \cos(\omega_k(t-s)) \right]. \quad (2.25)$$

We can again separate the real and imaginary parts, and change to variables X and ξ . We then get, to $O(\xi^3)$,

$$W[X(t), \xi(t)] = -\frac{1}{2m} \sum_k \int_{t_0}^{t_f} dt \int_{t_0}^t ds \left\{ a'_k(X(t)) a_k(X(s)) \xi(t) \sin(\omega_k(t-s)) \right. \\ \left. - i \coth\left(\frac{\hbar\omega_k}{2kT}\right) a'_k(X(t)) a'_k(X(s)) \xi(t) \xi(s) \cos(\omega_k(t-s)) \right\}. \quad (2.26)$$

Again, we see that the real term has the same form as the classical retarded force, which becomes dissipative in the limit of continuous frequencies and high cutoff. The imaginary term again corresponds to a double integral over the two-time correlation function of the classical stochastic force. It is strictly non-negative, and exponentially damps the decoherence functional for large ξ .

III. NONLINEAR EXAMPLES

The problem with potentials nonlinear in Q is that the path integral is no longer solvable in closed form. Thus, it is difficult to be certain that this correspondence

with the classical equation of motion which holds in the linear case is truly universal. We can, however, consider weak couplings, and solve for the equation of motion using perturbation theory. We can then compare the classical perturbative equation to that derived from the influence functional.

A. Classical and quantum perturbation theory

Let us consider a system coupled to a bath of harmonic oscillators with a potential of the form

$$V(x, Q) = -\epsilon \sum_k V_k(x, Q^k), \quad (3.1)$$

where $V_k(x, Q^k)$ can be nonlinear in x and Q^k . In general, such a problem cannot be solved exactly. However, if the coupling is weak ($\epsilon \ll 1$), then we can make a perturbation expansion, at least for reasonably well-behaved potentials.

The total Lagrangian is

$$L_{\text{total}}(x, \dot{x}, Q, \dot{Q}) = L(x, \dot{x}) + \sum_k \frac{m}{2} \left((\dot{Q}^k)^2 - \omega_k^2 (Q^k)^2 \right) - \epsilon V_k(x, Q^k). \quad (3.2)$$

Let's suppose that the trajectory $x(t)$ is known. Then the equation of motion for the k th harmonic oscillator is

$$\frac{d^2 Q^k}{dt^2} = -\omega_k^2 Q^k + \frac{\epsilon}{m} \frac{\partial V_k}{\partial Q}(x(t), Q^k). \quad (3.3)$$

If we then write Q^k as an expansion

$$Q^k(t) = Q_0^k(t) + \epsilon Q_1^k(t) + \epsilon^2 Q_2^k(t) + \dots \quad (3.4)$$

and equate equal powers of ϵ , we get a series of equations

$$\frac{dQ_0^k}{dt} = -\omega_k^2 Q_0^k, \quad (3.5a)$$

$$\frac{dQ_1^k}{dt} = -\omega_k^2 Q_1^k + \frac{1}{m} \frac{\partial V_k}{\partial Q}(x(t), Q_0^k(t)), \quad (3.5b)$$

$$\frac{dQ_2^k}{dt} = -\omega_k^2 Q_2^k + \frac{1}{m} Q_1^k(t) \frac{\partial^2 V_k}{\partial Q^2}(x(t), Q_0^k(t)), \quad (3.5c)$$

etc., where we've Taylor-expanded $V_k(x, Q_0^k + \epsilon Q_1^k + \dots)$ in powers of ϵ .

Now we have equations for each $Q_i^k(t)$ in terms of the lower order functions. Notably, the lowest order equation is now a simple harmonic oscillator, and we can solve for it easily in terms of the initial conditions

$$Q_0^k(t) = A_k \cos(\omega_k t) + B_k \sin(\omega_k t), \quad (3.6)$$

where $A_k = Q^k|_{t_0}$ and $B_k = (1/\omega_k)(dQ^k/dt)|_{t_0}$.

The higher-order equations are driven oscillators. We can solve for them exactly, matching initial conditions:

$$Q_1^k(t) = \frac{1}{m\omega_k} \int_{t_0}^t \sin(\omega_k(t-s)) \frac{\partial V_k}{\partial Q}(x(s), Q_0^k(s)) ds, \quad (3.7a)$$

$$Q_2^k(t) = \frac{1}{m\omega_k} \int_{t_0}^t \sin(\omega_k(t-s)) \frac{\partial^2 V_k}{\partial Q^2}(x(s), Q_0^k(s)) Q_1^k(s) ds, \quad (3.7b)$$

and so forth.

Having found the motion of the harmonic oscillators in terms of $x(t)$, we can now turn around and find the equation of motion for x . This is

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}}(t) = \frac{\partial L}{\partial x}(t) + \epsilon \sum_k \frac{\partial V_k}{\partial x}(x(t), Q^k(t)). \quad (3.8)$$

$Q^k(t)$ is the expansion that we solved for above, and it will depend on the earlier behavior of x , in general. Note that causality is strictly obeyed. This classical causality follows as a result of more fundamental quantum causality, as discussed by Gell-Mann and Hartle [5].

We treat this same problem quantum-mechanically by trying to find the influence functional $F[x'(t), x(t)]$ as a perturbation expansion. Assume that the reservoir starts in a definite initial state $|a\rangle$, with wave function $\phi_a(Q)$. Then

$$\begin{aligned}
F_a[x', x] &= \int \delta(Q'(t_f) - Q(t_f)) \exp \left\{ \frac{i}{\hbar} (S_{\text{res}}[Q'(t)] - S_{\text{res}}[Q(t)]) \right\} \\
&\quad \times \left[1 + \frac{i\epsilon}{\hbar} \int_{t_0}^{t_f} \left(V(x'(t), Q'(t)) - V(x(t), Q(t)) \right) dt \right. \\
&\quad + \left(\frac{i\epsilon}{\hbar} \right)^2 \int_{t_0}^{t_f} \int_{t_0}^t \left(V(x'(t), Q'(t)) - V(x(t), Q(t)) \right) \\
&\quad \quad \times \left(V(x'(s), Q'(s)) - V(x(s), Q(s)) \right) ds dt + \dots \left. \right] \phi_a(Q'(t_0)) \phi_a^*(Q(t_0)) \delta Q' \delta Q \\
&= 1 + \frac{i\epsilon}{\hbar} \int_{t_0}^{t_f} \left(V_{aa}(x'(t)) - V_{aa}(x(t)) \right) dt \\
&\quad - \left(\frac{\epsilon^2}{\hbar^2} \right) \sum_b \int_{t_0}^{t_f} \int_{t_0}^t \left(V_{ab}(x'(t)) V_{ba}(x'(s)) e^{-i\omega_{ba}(t-s)} - V_{ab}(x(t)) V_{ba}(x'(s)) e^{-i\omega_{ba}(t-s)} \right. \\
&\quad \quad \left. - V_{ba}(x'(t)) V_{ab}(x(s)) e^{i\omega_{ba}(t-s)} + V_{ba}(x(t)) V_{ab}(x(s)) e^{i\omega_{ba}(t-s)} \right) ds dt + \dots \\
&= 1 + \frac{i\epsilon}{\hbar} \int_{t_0}^{t_f} V'_{aa}(X(t)) \xi(t) dt \\
&\quad - \left(\frac{\epsilon^2}{\hbar^2} \right) \sum_b \int_{t_0}^{t_f} \int_{t_0}^t \left(V'_{ba}(X(t)) V'_{ab}(X(s)) \xi(t) \xi(s) \cos(\omega_{ba}(t-s)) \right. \\
&\quad \quad \left. - 2i V'_{ba}(X(t)) V_{ab}(X(s)) \xi(t) \sin(\omega_{ba}(t-s)) \right) ds dt + O(\epsilon^3) + O(\xi^3). \quad (3.9)
\end{aligned}$$

Here we've defined the functions

$$V_{aa}(x) = \langle a | V(x, \hat{Q}) | a \rangle = \int \phi_a(Q) \phi_a^*(Q) V(x, Q) dr, \quad (3.10a)$$

$$V_{ba}(x) = \langle b | V(x, \hat{Q}) | a \rangle = \int \phi_a(Q) \phi_b^*(Q) V(x, Q) dr. \quad (3.10b)$$

In our case, we assume that the reservoir is a collection of harmonic oscillators initially in a thermal state. In this case, the states $|a\rangle$ become the ordinary Fock states $|n\rangle$ and the influence functional is

$$F[x'(t), x(t)] = \sum_n \rho_{nn} F_n[x'(t), x(t)], \quad (3.11)$$

where

$$\rho_{nn} = \prod_k \left(1 - \exp(-\hbar\omega_k/kT) \right) \exp(-n_k \hbar\omega_k/kT). \quad (3.12)$$

B. Polynomial potentials

We will specifically consider a potential of the form (3.1) where the individual potentials are polynomials in Q^k . We will see that it is convenient to separate the even and odd terms:

$$V_k(x, Q^k) = \sum_{l=0}^N a_{kl}(x)(Q^k)^{2l+1} + \sum_{l=1}^N b_{kl}(x)(Q^k)^{2l}. \quad (3.13)$$

The $a_{kl}(x)$ and $b_{kl}(x)$ are arbitrary functions of x , only assuming that the potential as a whole remains relatively well-behaved, integrable, etc. For convenience, I will drop the index k for the rest of this derivation. It should be understood that the final result is to be summed over all the oscillators,

$$W[x'(t), x(t)] = \sum_k W_k[x'(t), x(t)]. \quad (3.14)$$

From the equation (3.5a), we can write down the equations of motion for a classical oscillator $Q(t) = Q_0(t) + \epsilon Q_1(t) + \dots$. We then plug in the solutions (3.6) and (3.7a) to get

$$Q_0(t) = A \cos(\omega t) + B \sin(\omega t) = \beta e^{i\omega t} + \beta^* e^{-i\omega t}, \quad (3.15a)$$

$$Q_1(t) = \frac{1}{m\omega} \int_{t_0}^t \sin[\omega(t-s)] \left(\sum_{j=0}^N (2j+1)a_j(x(s))Q_0^{2j}(s) + \sum_{j=1}^N 2jb_j(x(s))Q_0^{2j-1}(s) \right) ds, \quad (3.15b)$$

etc., where $\beta = (A - iB)/2$. The equation of motion for x is then

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) &= \left(\frac{\partial L}{\partial x} \right) + \epsilon \left(\sum_{j=0}^N a'_j(x(t))Q_0^{2j+1}(t) + \sum_{j=1}^N b'_j(x(t))Q_0^{2j}(t) \right) \\ &\quad + \epsilon^2 \left(\sum_{j=0}^N (2j+1)a'_j(x(t))Q_0^{2j}(t) + \sum_{j=1}^N 2jb'_j(x(t))Q_0^{2j-1}(t) \right) Q_1(t) + O(\epsilon^3) \\ &= \left(\frac{\partial L}{\partial x} \right) + \epsilon \tau_1(t) + \epsilon^2 \tau_2(t) + O(\epsilon^3) \end{aligned} \quad (3.16)$$

We are interested in the ensemble-averaged equation. We can make use of the fact that

$$\langle \beta^m \beta^{*n} \rangle = \delta_{mn} n! \left(\frac{kT}{2m\omega^2} \right)^n. \quad (3.17)$$

So only the even terms contribute to the first-order component of the equation (3.16).

$Q_0^{2j}(t)$ is readily found then with a binomial expansion of

$$(\beta e^{i\omega t} + \beta^* e^{-i\omega t})^{2j} = \sum_{i=0}^{2j} \binom{2j}{i} (\beta e^{i\omega t})^i (\beta^* e^{-i\omega t})^{2j-i}, \quad (3.18)$$

$$\binom{n}{i} = \frac{n!}{i!(n-i)!},$$

yielding

$$\langle \tau_1(t) \rangle = \sum_{j=1}^N \frac{2j!}{j!} b'_j(x(t)) \left(\frac{kT}{2m\omega^2} \right)^j. \quad (3.19)$$

The second order component is more complicated. Plugging expression (3.15b) for $Q_1(t)$ into (3.16), doing a binomial expansion for the powers of $Q_0(t)$ and $Q_0(s)$, pairing $e^{mi\omega t}$ and $e^{-mi\omega t}$ terms, and ensemble-averaging gives us

$$\begin{aligned} \langle \tau_2(t) \rangle = \frac{1}{m\omega} \int_{t_0}^t & \left\{ \sum_{k=0}^N \sum_{i,j=k}^N \sin[(2k+1)\omega(t-s)] C_{ijk}(t,s) \right. \\ & - \sum_{k=1}^N \sum_{i,j=k}^N \sin[(2k-1)\omega(t-s)] C_{ijk}(t,s) \\ & + \sum_{k=1}^N \sum_{i,j=k}^N \sin[2k\omega(t-s)] D_{ijk}(t,s) \\ & \left. - \sum_{k=1}^N \sum_{i,j=k}^N \sin[(2k-2)\omega(t-s)] D_{ijk}(t,s) \right\} ds, \quad (3.20) \end{aligned}$$

where

$$C_{ijk}(t,s) = a'_i(x(t)) a_j(x(s)) (i+j)! (2i+1)(2j+1) \binom{2i}{i-k} \binom{2j}{j-k} \left(\frac{kT}{2m\omega^2} \right)^{i+j}, \quad (3.21a)$$

$$D_{ijk}(t,s) = b'_i(x(t)) b_j(x(s)) (i+j-1)! 4ij \binom{2i-1}{i-k} \binom{2j-1}{j-k} \left(\frac{kT}{2m\omega^2} \right)^{i+j-1}. \quad (3.21b)$$

We can collect together and combine those terms with the same sine factor, to get

$$\begin{aligned} \langle \tau_2(t) \rangle = & \frac{1}{m\omega} \int_{t_0}^t \left\{ \sum_{k=0}^N \sum_{i,j=k}^N \sin[(2k+1)\omega(t-s)] E_{ijk}(t,s) \right. \\ & \left. + \sum_{k=0}^N \sum_{i,j=k}^N \sin[2k\omega(t-s)] F_{ijk}(t,s) \right\} ds, \end{aligned} \quad (3.22)$$

where

$$E_{ijk}(t,s) = a'_i(x(t))a_j(x(s))(i+j)! \left(\frac{kT}{2m\omega^2} \right)^{i+j} (2k+1)(i+j+1) \binom{2i+1}{i-k} \binom{2j+1}{j-k}, \quad (3.23a)$$

$$F_{ijk}(t,s) = b'_i(x(t))b_j(x(s))(i+j-1)! \left(\frac{kT}{2m\omega^2} \right)^{i+j-1} 2k(i+j) \binom{2i}{i-k} \binom{2j}{j-k}. \quad (3.23b)$$

We are also interested in the correlation function $\langle F(t)F(s) \rangle$, where $F(t)$ is the force due to the interaction with the reservoir. To second order this is

$$\begin{aligned} \langle F(t)F(s) \rangle = & \epsilon^2 \left\{ \sum_{k=0}^N \sum_{i,j=k}^N 2 \cos[(2k+1)\omega(t-s)] G_{ijk}(t,s) \right. \\ & + \sum_{k=1}^N \sum_{i,j=k}^N 2 \cos[2k\omega(t-s)] H_{ijk}(t,s) \\ & \left. + \sum_{i,j=1}^N H_{ij0}(t,s) \right\} + O(\epsilon^3), \end{aligned} \quad (3.24)$$

$$G_{ijk}(t,s) = a'_i(x(t))a'_j(x(s))(i+j+1)! \left(\frac{kT}{2m\omega^2} \right)^{i+j+1} \binom{2i+1}{i-k} \binom{2j+1}{j-k}, \quad (3.25a)$$

$$H_{ijk}(t,s) = b'_i(x(t))b'_j(x(s))(i+j)! \left(\frac{kT}{2m\omega^2} \right)^{i+j} \binom{2i}{i-k} \binom{2j}{j-k}. \quad (3.25b)$$

We can subtract off the average values to get

$$\langle F(t), F(s) \rangle = \langle F(t)F(s) \rangle - \langle F(t) \rangle \langle F(s) \rangle, \quad (3.26)$$

where $\langle F(t) \rangle$ is the first order ensemble averaged force from (3.19).

We can compare this result to that obtained from our quantum mechanical procedure. Suppose that the reservoir begins in a definite state $|n\rangle$. Then the influence functional is given by (3.9),

$$F_n[X(t), \xi(t)] = 1 + \epsilon \alpha_{n1}[X(t), \xi(t)] + \epsilon^2 \alpha_{n2}[X(t), \xi(t)] + \dots, \quad (3.27)$$

and in the thermal case by (3.11),

$$F[X(t), \xi(t)] = 1 + \epsilon \alpha_1[X(t), \xi(t)] + \epsilon^2 \alpha_2[X(t), \xi(t)] + \dots = \sum_n \rho_{nn} F_n[X(t), \xi(t)], \quad (3.28)$$

where

$$\alpha_i[X(t), \xi(t)] = \sum_n \rho_{nn} \alpha_{ni}[X(t), \xi(t)]. \quad (3.29)$$

The influence phase is then

$$\begin{aligned} W[X(t), \xi(t)] &= -i\hbar \ln F[X(t), \xi(t)] \\ &= -i\hbar \epsilon \alpha_1[X(t), \xi(t)] \\ &\quad -i\hbar \epsilon^2 \left(\alpha_2[X(t), \xi(t)] - \frac{1}{2} \alpha_1^2[X(t), \xi(t)] \right) + \dots \end{aligned} \quad (3.30)$$

From (3.9), then, we see that we must find an expression for $\langle m|r^l|n\rangle$. This will, in general, be a polynomial in n , for certain values of m , and zero for the rest. In comparing to the classical result, we need keep only the highest power of n , since the lower powers will be higher-order in $\hbar\omega/kT$ as we let $\hbar \rightarrow 0$. This will be

$$\langle m|r^l|n\rangle = \binom{l}{k} \left(\frac{\hbar}{2m\omega} \right)^{l/2} n^{l/2} + \dots, m = n + l - 2k, 2k \leq l, \quad (3.31)$$

$$\langle m|r^l|n\rangle = \binom{l}{k} \left(\frac{\hbar}{2m\omega} \right)^{l/2} m^{l/2} + \dots, m = n - l + 2k, 2k \leq l, \quad (3.32)$$

and zero otherwise.

We can then use the fact that as $\hbar \rightarrow 0$,

$$\sum_n \rho_{nn} n^l \approx l! \left(\frac{kT}{\hbar\omega} \right)^l. \quad (3.33)$$

Thus, from equation (3.9) we get

$$\begin{aligned} \alpha_1[X(t), \xi(t)] &= \sum_n \rho_{nn} \frac{i}{\hbar} \int_{t_0}^{t_f} \sum_j b'_j(X(t)) \langle n | r^{2n} | n \rangle \xi(t) dt \\ &= \frac{i}{\hbar} \int_{t_0}^{t_f} \sum_j b'_j(X(t)) \xi(t) \frac{2j!}{j!} \left(\frac{kT}{2m\omega^2} \right)^j dt, \end{aligned} \quad (3.34)$$

which agrees exactly with the first order term in the classical equation of motion (3.19).

Similarly, we can calculate the second order term to get

$$\begin{aligned} \alpha_2[X(t), \xi(t)] &= -\frac{1}{\hbar^2} \left\{ \sum_{k=0}^N \sum_{i,j=k}^N \int_{t_0}^{t_f} \int_{t_0}^t 2 \cos[(2k+1)\omega(t-s)] G_{ijk}(t,s) \xi(t) \xi(s) ds dt \right. \\ &\quad + \sum_{k=1}^N \sum_{i,j=k}^N \int_{t_0}^{t_f} \int_{t_0}^t 2 \cos[2k\omega(t-s)] H_{ijk}(t,s) \xi(t) \xi(s) ds dt \\ &\quad + i \sum_{k=1}^N \sum_{i,j=k}^N \int_{t_0}^{t_f} \int_{t_0}^t \sin[(2k+1)\omega(t-s)] E_{ijk}(t,s) \xi(t) ds dt \\ &\quad + i \sum_{k=1}^N \sum_{i,j=k}^N \int_{t_0}^{t_f} \int_{t_0}^t \sin[2k\omega(t-s)] F_{ijk}(t,s) \xi(t) ds dt \\ &\quad \left. + \sum_{i,j=1}^N \int_{t_0}^{t_f} \int_{t_0}^t H_{ij0}(t,s) \xi(t) \xi(s) ds dt \right\}. \end{aligned} \quad (3.35)$$

Here we've used the same definitions of E_{ijk} , etc., where the classical system variable x has become the quantum variable X .

We can clearly see from this the exact correspondence with the classical equation of motion, at least to second order in ϵ . The real part of $W[X(t), \xi(t)]$ is just an integral of the classical retarded force, just as in the linear case, and the imaginary part consists of a double integral

$$\int_{t_0}^{t_f} \int_{t_0}^t \left[\langle F(t)F(s) \rangle - \langle F(t) \rangle \langle F(s) \rangle \right] \xi(t) \xi(s) ds dt; \quad (3.36)$$

note that the $-\langle F(t) \rangle \langle F(s) \rangle$ comes from subtracting $\alpha_1^2/2$ from the second order term. Again, we note the non-negativity of this imaginary part; the presence of noise

both makes the behavior unpredictable, and causes different trajectories to decohere. So we see that in perturbation theory, the nonlinear problem has exactly the same classical correspondence as the linear problem.

IV. MORE GENERAL CASES

Though the above discussion is fairly general, it leaves unexamined the far broader range of possible strong, nonlinear interactions, as well as the possibilities of non-oscillator reservoirs. This is, of course, a product of computational convenience, as it is very difficult to get analytical answers in other cases. Are there any arguments that can be made for more general systems?

In any case where the action can be decomposed

$$S[x(t), Q(t)] = S_{\text{sys}}[x(t)] + S_{\text{res}}[Q(t)] + S_{\text{int}}[x(t), Q(t)], \quad (4.1)$$

it is possible formally to write the decoherence functional in the form

$$D[x(t), x'(t)] = \exp \frac{i}{\hbar} \left\{ S_{\text{sys}}[x(t)] - S_{\text{sys}}[x'(t)] + W[x(t), x'(t)] \right\}. \quad (4.2)$$

If we restrict ourselves, for the moment, to systems in a factorizable pure state,

$$\rho(x', Q'; x, Q) = \Psi^*(x')\Psi(x)\Phi^*(Q')\Phi(Q), \quad (4.3)$$

then this influence phase is defined simply by (2.15)

$$\begin{aligned} \exp \{iW[x'(t), x(t)]/\hbar\} &= \int \delta Q' \int \delta Q \delta(Q'(t_f) - Q(t_f)) \exp \frac{i}{\hbar} \left\{ S_{\text{res}}[Q'(t)] - S_{\text{res}}[Q(t)] \right. \\ &\quad \left. + S_{\text{int}}[x'(t), Q'(t)] - S_{\text{int}}[x(t), Q(t)] \right\} \Phi^*(Q'_0)\Phi(Q_0). \end{aligned} \quad (4.4)$$

By bringing the integral over the final condition Q_f, Q'_f to the front, we can rewrite this as a product of two path integrals:

$$\begin{aligned} \exp \{iW[x'(t), x(t)]/\hbar\} &= \int \int dQ_f dQ'_f \delta(Q'(t_f) - Q(t_f)) \\ &\quad \times \left[\int \delta Q' \exp \frac{i}{\hbar} \left\{ S_{\text{res}}[Q'(t)] + S_{\text{int}}[x'(t), Q'(t)] \right\} \Phi^*(Q'_0) \right] \end{aligned}$$

$$\begin{aligned}
& \times \left[\int \delta Q \exp \frac{i}{\hbar} \left\{ -S_{\text{res}}[Q(t)] - S_{\text{int}}[x(t), Q(t)] \right\} \Phi(Q_0) \right] \\
& = \int \int dQ_f dQ'_f \delta(Q'(t_f) - Q(t_f)) \Phi_{x'(t)}^*(Q'_f) \Phi_{x(t)}(Q_f) \\
& = \int \Phi_{x'(t)}^*(Q_f) \Phi_{x(t)}(Q_f) dQ_f = \langle \Phi_{x'(t)} | \Phi_{x(t)} \rangle, \tag{4.5}
\end{aligned}$$

where $|\Phi_{x(t)}\rangle$ and $|\Phi_{x'(t)}\rangle$ are the states that $|\Phi\rangle$ will evolve into under the influence of the interaction, given the trajectories $x(t)$ and $x'(t)$, respectively.

Clearly, $\langle \Phi_{x'(t)} | \Phi_{x(t)} \rangle \leq 1$, which implies equally clearly that $\text{Im}W[x'(t), x(t)] \geq 0$. So the non-negativity that we saw in the cases I and II above is generally true. This is also clearly the case for mixed states, since we can represent any mixed state as

$$\rho(x', Q'; x, Q) = \sum_i p_i \Psi_i^*(x') \Psi_i(x) \Phi_i^*(Q') \Phi_i(Q), \tag{4.6}$$

where

$$\sum_i p_i = 1, \quad p_i \geq 0; \tag{4.7}$$

so if the $F_i[x'(t), x(t)] < 1$, then clearly

$$\exp \{iW[x'(t), x(t)]/\hbar\} = \sum_i p_i F_i[x'(t), x(t)] \leq 1 \tag{4.8}$$

and $\text{Im}W[x'(t), x(t)] \geq 0$ still holds. Also, $\text{Im}W[x'(t), x(t)] = 0$ for $x'(t) = x(t)$. Thus, without assuming *anything* about the interaction *or* the reservoir, we see that there will be a maximum at $\xi(t) = 0$, and that the off-diagonal $\xi(t) \neq 0$ terms will tend to be suppressed. This is not surprising, as one expects almost any sort of interaction with neglected degrees of freedom to result in the loss of phase coherence. However, it does show how these highly simplified models might actually demonstrate behavior important to the emergence of classical physics from quantum mechanics in physical systems.

For example, in considering quantum gravity, decoherence might arise from neglected gravitational degrees of freedom. The usual semi-classical treatment of quantum gravity, which omits the “back action” of mass-energy on the curvature of space-time, cannot exhibit this effect. The weakness of the gravitational interaction would

in general make it less important in causing decoherence than stronger forces, such as electromagnetism; but it might well become important in quantum cosmology.

There are, of course, still questions. All that has been demonstrated is the non-negativity of $\text{Im}W[x'(t), x(t)]$. Can there not be zeroes for some choices of $\xi(t) \neq 0$? And *how strongly*, in general, are the off-diagonal terms suppressed?

There can certainly be zeroes for nonzero $\xi(t)$ in some cases. Indeed, if we consider the form of $\text{Im}W[x'(t), x(t)]$ for the linear case

$$\text{Im}W[x'(t), x(t)] \sim \int_{t_0}^{t_f} \int_{t_0}^{t_f} \xi(t)\xi(s) \cos[\omega(t-s)] ds dt \quad (4.9)$$

(for a one-oscillator “reservoir” of frequency ω), there are an infinite number of choices of $\xi(t)$ which make this zero. Thus, one cannot call this system truly decoherent. However, as the number of oscillator frequencies is increased, the number of possible choices of $\xi(t)$ is further and further restricted, so that as the reservoir becomes infinite only $\xi(t) = 0$ remains. One would expect similar behavior in the more general case. While it is certainly possible to construct cases where $\text{Im}W[x'(t), x(t)]$ has many zeroes even for a very large reservoir, in practice one expects $\text{Im}W[x'(t), x(t)] > 0$ for $x(t) \neq x'(t)$, as the degrees of freedom of the reservoir are increased.

Similarly, the strength with which off-diagonal terms will be suppressed depends on the details of the system. However, one would expect that $|\Phi_{x(t)}\rangle$ and $|\Phi_{x'(t)}\rangle$ differ more in the case of strong interactions than small, and hence that $\langle \Phi_{x'(t)} | \Phi_{x(t)} \rangle$ would be more strongly suppressed, in general.

V. CONCLUSIONS

It is clear that it is possible to define a “classical” equation of motion directly from the underlying quantum theory, and that, at least in many cases, this corresponds closely to the equation obtained from the classical theory. While correspondences of this sort have often been demonstrated in the past, never before has there been a rigorous, *a priori* technique for deriving them.

Using the formalism of Gell-Mann and Hartle, we can now see classical physics as, very simply, a limit of the underlying quantum theory; and we can systematically determine, at least in principle, the deviations from strict classical equations due to quantum effects. Using the decoherence functional as a criterion for determining if an effect is experimentally observable, we can once and for all avoid the problem of collapsing the wave function; there is no longer any need for an independent “classical realm” of measurement.

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After the completion of this research, I learned that Bei Lok Hu, Juan Pablo Paz, and Yuhong Zhang had studied a very similar class of nonlinear brownian systems more or less simultaneously with me [10]. While their study is from a considerably different point of view, with very different goals, being chiefly concerned with deriving master equations and treating the thermodynamics of these generalized systems, their results overlap mine to a certain extent.

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Chapter II

Quantum Dissipative Chaos

QUANTUM DISSIPATIVE CHAOS

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Abstract

Using the decoherence formalism of Gell-Mann and Hartle, a quantum system is found which is the equivalent of the classical dissipative chaotic Duffing oscillator. The similarities and differences from the classical oscillator are examined; in particular, a new concept of quantum maps is introduced, and alterations in the classical strange attractor due to the presence of scale-dependent quantum effects are studied. Classical quantities such as the Lyapunov exponents and fractal dimension are examined, and quantum analogs are suggested. These results are generalized into a framework for quantum dissipative chaos, and there is a brief discussion of other work in this area.

I. INTRODUCTION

Since classical chaos first began to be studied, a conspicuous puzzle has been how to reconcile this purely classical phenomenon with an underlying quantum theory. If we believe, as we must to be consistent, that all of physics is fundamentally quantum mechanical in nature, then we must further believe that true chaotic systems do not exist. At some point, at length scales determined by Planck's constant, the deterministic uncertainties of classical chaos must give way to the probabilistic uncertainties of quantum mechanics.

But tackling these problems is not simple. The nonlinear equations of chaos are, in general, only solvable with modern high-speed computers, and their quantum analogs share this limitation. Also, chaos itself encompasses two major types of behavior: Hamiltonian chaos in which energy is conserved, and dissipative chaos. It is in principle straightforward to find and solve quantum equivalents to Hamiltonian systems, if difficult in practice, and considerable progress has been made in recent years in understanding these systems. Dissipative systems are much more foreign to quantum mechanics as it is usually studied.

Recently, Murray Gell-Mann and James Hartle have used the decoherence functional formalism of quantum mechanics to show how quasiclassical laws can arise from an underlying quantum theory [1-3]. I applied this approach to the problem of Brownian motion, demonstrating how their scheme reproduces exactly the classical Langevin equation in a fairly broad class of systems [4].

A natural next step is to apply this to systems with interesting classical behavior. Since dissipation is easily and indeed naturally included in such systems, an obvious candidate for study is dissipative chaos. Once a quantum system is found whose limiting behavior is equivalent to a classical chaotic system, we can study how the residual quantum mechanical effects alter the system, and what difference this makes to the classical behavior.

In section II we briefly examine the family of quantum systems from which we will draw our model and derive the quasiclassical equations of motion for them. We then go to the limit of an infinite reservoir of oscillators with a continuum of frequencies, and specialize to the case of a forced, damped nonlinear oscillator.

In section III we examine the classical behavior of one such system, the damped, driven Duffing oscillator. There is a brief discussion of dissipative chaos, the structure of the strange attractor, and the bifurcations leading to chaos. Several quantities useful for characterizing the chaotic behavior are defined: the fractal dimension and Lyapunov exponents, and their relationships are examined.

In section IV we look at the decoherence functional and define the idea of a quantum map. The system is examined as a Wigner distribution, and we see how the invariant measure of the strange attractor goes over to the quantum case. Problems of coarse-graining and decoherence are discussed. Then we look at the system from a master equation point of view and compare this description to the decoherence functional approach. In section V we see how the various classical quantities used to characterize chaotic behavior can be reinterpreted for our quantum system, by treating it as a classical system with noise for sufficiently coarse length scales.

A few other treatments of quantum dissipative chaos are mentioned in section VI, and the differences between Hamiltonian and dissipative chaos are pointed out. Finally, in section VII a case is made for a general theory of quantum dissipative chaos.

II. DAMPED DRIVEN QUANTUM SYSTEMS

A. The quantum systems

Picking a good set of candidate systems requires some thought. Many widely studied sets of chaotic equations have only a loose connection to actual physical sys-

tems; many others are extreme coarse grainings of very complicated systems with many degrees of freedom, e.g., fluid dynamics. It is much better to deal with comparatively simple systems, whose decoherence functionals can be calculated easily. For this reason, I have elected to study damped driven nonlinear oscillators, which can be easily modelled as particles moving in a potential well, interacting with a reservoir of simple linear oscillators. In particular, I will concentrate on one such system, the damped, driven Duffing oscillator.

Earlier work has chiefly considered systems with reservoirs in a thermal state. For the purpose of this model, I wish to consider instead a system whose reservoir is initially in a coherent state.

Consider a system of N harmonic oscillators. We assume them to be in a state $|\{\nu\}\rangle$, where $\{\nu\}$ represents a set of N complex numbers ν_j . If \hat{a}_i is the annihilation operator for the i th oscillator, then $\hat{a}_i|\{\nu\}\rangle = \nu_i|\{\nu\}\rangle$.

As shown in earlier papers [3,4], the decoherence functional for a system interacting with a reservoir is

$$\begin{aligned}
D[x'(t), x(t)] &= \exp\left\{i(S_{\text{sys}}[x'(t)] - S_{\text{sys}}[x(t)])/ \hbar\right\} \int \delta Q' \delta Q \delta(Q'(t_f) - Q(t_f)) \\
&\quad \times \exp\left\{i(S_{\text{res}}[Q'(t)] - S_{\text{res}}[Q(t)] \right. \\
&\quad \left. - \int_{t_0}^{t_f} (V(x'(t), Q'(t)) - V(x(t), Q(t))) dt / \hbar\right\} \\
&\quad \times \rho(x'_0, Q'_0; x_0, Q_0) \\
&= \exp\left\{i(S_{\text{sys}}[x'(t)] - S_{\text{sys}}[x(t)] + W[x'(t), x(t)]) / \hbar\right\} \tilde{\rho}(x'_0; x_0). \quad (2.1)
\end{aligned}$$

Here $S_{\text{sys}}[x(t)]$ is the action of the system for a given trajectory $x(t)$, $S_{\text{res}}[Q(t)]$ is the action of the reservoir for a given trajectory $Q(t)$, and $V(x, Q)$ is the interaction potential between the system and reservoir variables. We will assume this to be a bilinear potential of the form

$$V(x, Q) = - \sum_k \gamma_k x Q^k, \quad (2.2)$$

where Q^k is the coordinate of the k th oscillator in the reservoir. We will shortly allow the number of oscillators to go to infinity, and assume a continuum of oscillator frequencies, but for now let us deal with the discrete case.

We will also assume that the density matrix factors:

$$\rho(x'_0, Q'_0; x_0, Q_0) = \chi(x'_0; x_0)\phi(Q'_0; Q_0), \quad (2.3)$$

where $\phi(Q'_0; Q_0) = \langle Q'_0 | \{\nu\} \rangle \langle \{\nu\} | Q_0 \rangle$ is the pure coherent state described above.

We can readily calculate the influence functional for this system. It is just

$$\begin{aligned} \exp\left\{iW[x'(t), x(t)]/\hbar\right\} &= \int \int \int dQ_f dQ_0 dQ'_0 K_{x'(t)}^*(Q_f; Q'_0) K_{x(t)}(Q_f; Q_0) \phi(Q'_0; Q_0) \\ &= \langle \{\nu\} | \hat{S}_{x'(t)}^\dagger \hat{S}_{x(t)} | \{\nu\} \rangle, \end{aligned} \quad (2.4)$$

where $K_{x(t)}(Q_f; Q_0)$ is the transition amplitude from Q_0 to Q_f of the reservoir and $\hat{S}_{x(t)}$ is the time evolution operator of a forced harmonic oscillator driven by the time-dependant interaction $V(x(t), Q)$ given in (2.2). This is a well-known problem [5].

For a single oscillator of frequency ω , the operator is

$$\hat{S}_{x(t)} = \exp\left[\alpha \hat{a}^\dagger - \alpha^* \hat{a}\right] = \hat{D}(\alpha), \quad (2.5)$$

where

$$\alpha = \frac{i\gamma}{\sqrt{2m\omega\hbar}} \int_{t_0}^{t_f} e^{i\omega s} x(s) ds, \quad (2.6)$$

and

$$\hat{D}(\alpha)|\nu\rangle = |\nu + \alpha\rangle e^{i\text{Im}(\alpha\nu^*)}. \quad (2.7)$$

For $\nu = 0$ this just reduces to the usual form of the influence functional for an oscillator initially in the ground state:

$$\begin{aligned} \exp\left\{iW[x'(t), x(t)]/\hbar\right\} &= \exp\left\{\frac{i}{\hbar}\left\{+\frac{i\gamma^2}{4m\omega} \int_{t_0}^{t_f} dt \int_{t_0}^{t_f} ds \cos(\omega(t-s))(x'(t)-x(t))(x'(s)-x(s))\right.\right. \\ &\quad \left.\left.-\frac{\gamma^2}{2m\omega} \int_{t_0}^{t_f} dt \int_{t_0}^t ds \sin(\omega(t-s))(x'(t)-x(t))(x'(s)+x(s))\right\}\right\}. \end{aligned} \quad (2.8)$$

For non-zero ν we get an additional exponent of the form

$$\text{Im}(2\nu\alpha^* + 2\nu^*\alpha') = \gamma\sqrt{\frac{2}{m\omega\hbar}} \int_{t_0}^{t_f} dt (\text{Re}\nu \cos(\omega t) + \text{Im}\nu \sin(\omega t))(x(t) - x'(t)). \quad (2.9)$$

Generalizing this to many oscillators, we get the influence phase

$$\begin{aligned} W[x'(t), x(t)] &= \sum_k \frac{i\gamma_k^2}{4m\omega_k} \int_{t_0}^{t_f} dt \int_{t_0}^{t_f} ds \cos(\omega_k(t-s))(x'(t) - x(t))(x'(s) - x(s)) \\ &\quad - \frac{\gamma_k^2}{2m\omega_k} \int_{t_0}^{t_f} dt \int_{t_0}^t ds \sin(\omega_k(t-s))(x'(t) - x(t))(x'(s) + x(s)) \\ &\quad + \gamma_k \sqrt{\frac{2\hbar}{m\omega_k}} \int_{t_0}^{t_f} dt (\text{Re}\nu_k \cos(\omega_k t) + \text{Im}\nu_k \sin(\omega_k t))(x'(t) - x(t)). \end{aligned} \quad (2.10)$$

For practical purposes, we generally assume that the interaction began at $t_0 = 0$ and continued up until some final time t_f , so as to avoid having infinite limits in the integrals.

We will now assume that the action of the system variables is of the usual form

$$S_{\text{sys}}[x(t)] = \int_{t_0}^{t_f} L(x(t), \dot{x}(t)) dt. \quad (2.11)$$

We can then change variables to

$$X = \frac{1}{2}(x + x'), \quad (2.12a)$$

$$\xi = x - x', \quad (2.12b)$$

and write the decoherence functional in terms of the new variables. This is easily shown to be

$$\begin{aligned} D[X(t), \xi(t)] &= \exp \frac{i}{\hbar} \left\{ \sum_k \int_{t_0}^{t_f} dt \xi(t) \left[-\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{X}} \right) (X(t), \dot{X}(t)) + \frac{\partial L}{\partial X} (X(t), \dot{X}(t)) \right. \right. \\ &\quad \left. \left. - \gamma_k \sqrt{\frac{2\hbar}{m\omega_k}} (\text{Re}\nu_k \cos(\omega_k(t)) + \text{Im}\nu_k \sin(\omega_k(t))) + \frac{\gamma_k^2}{m\omega_k} \int_{t_0}^t ds \sin(\omega_k(t-s)) X(s) \right] \right. \\ &\quad \left. + \frac{i\gamma_k^2}{4m\omega_k} \int_{t_0}^{t_f} dt \int_{t_0}^t ds \cos(\omega_k(t-s)) \xi(t) \xi(s) \right. \\ &\quad \left. - \xi_0 \frac{\partial L}{\partial \dot{X}} (X_0, \dot{X}_0) + O(\xi^3) \right\} \chi(x'_0; x_0). \end{aligned} \quad (2.13)$$

Note that the real part of the phase includes the Euler-Lagrange equation of motion for the system, with the addition of a retarded force due to the interaction with the reservoir. The imaginary part is strictly non-negative, with a minimum at $\xi(t) \equiv 0$, and hence tends to suppress $D[x'(t), x(t)]$ for large ξ . This makes our expansion in terms of $\xi(t)$ seem reasonable, and also causes the system to decohere, at least approximately, since $\xi(t) \neq 0$ corresponds to off-diagonal terms. The ξ_0 term occurs because of an integration by parts.

B. The classical equivalent

We can now look at the classical system equivalent to the above quantum system, i.e., with the same action functional and distribution of oscillators. A coherent state is often characterized as a more “classical” state of an oscillator than the usual Fock states; it can be thought of as the state of an oscillator begun at a given initial position and momentum, within the limits imposed by the uncertainty principle.

We will begin by assuming knowledge of the trajectory of the system variable $x(t)$, and ask what the behavior of the reservoir of harmonic oscillators will be [6]. Assume that we start the oscillators in a definite state $Q^k(t = t_0) = q^k, \dot{Q}^k(t = t_0) = v^k$. The interaction potential is linear, so we can treat the trajectory of the system variable $x(t)$ as a simple driving force, giving us an equation of motion for the k th oscillator

$$\frac{d^2 Q^k}{dt^2} = -\omega_k^2 Q^k + (\gamma_k/m)x(t). \quad (2.14)$$

The solution to this equation is simply

$$Q^k(t) = q^k \cos(\omega_k(t - t_0)) + (v^k/\omega_k) \sin(\omega_k(t - t_0)) + \frac{\gamma_k}{m\omega_k} \int_{t_0}^t \sin(\omega_k(t - s))x(s)ds. \quad (2.15)$$

If the system is described by a Lagrangian $L(x, \dot{x})$, then we can write down the Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{X}}(x(t), \dot{x}(t)) - \frac{\partial L}{\partial X}(x(t), \dot{x}(t)) + \sum_k \gamma_k Q^k(t) = 0. \quad (2.16)$$

We can clearly substitute the above expression (2.15) for $Q^k(t)$ in the Euler-Lagrange equation to get

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial \dot{X}}(x(t), \dot{x}(t)) - \frac{\partial L}{\partial X}(x(t), \dot{x}(t)) + \sum_k \gamma_k \left(q^k \cos(\omega_k(t - t_0)) + (v^k/\omega_k) \sin(\omega_k(t - t_0)) \right. \\ \left. + \frac{\gamma_k}{m\omega_k} \int_{t_0}^t \sin(\omega_k(t - s)) x(s) ds \right) = 0. \end{aligned} \quad (2.17)$$

This expression is clearly closely related to the real part of the phase in the decoherence functional, if we make the identity

$$\sqrt{\frac{2\hbar}{m\omega_k}} \text{Re} \nu_k = q^k, \quad (2.18a)$$

$$\sqrt{\frac{2\hbar\omega_k}{m}} \text{Im} \nu_k = v^k. \quad (2.18b)$$

If we write the above classical equation as $e(t) = 0$, then the real part of the phase is just

$$\int_{t_0}^{t_f} \xi(t) e(t) dt.$$

How do we interpret the imaginary part of the phase, however? In treating reservoirs in an initial thermal state, we identified this term as the effect of a stochastic force $F(t)$ arising due to thermal noise. However, in this system, there *is* no noise classically. The persistence of this term indicates a fundamental difference between the quantum and classical systems. As Gell-Mann and Hartle point out [3], in the quantum system there is always noise from zero-point oscillations, unlike classical oscillators. So the actual equation of motion derived from the quantum theory is

$$\begin{aligned} 0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{X}}(x(t), \dot{x}(t)) - \frac{\partial L}{\partial X}(x(t), \dot{x}(t)) - F(t) \\ + \sum_k \gamma_k \left(q^k \cos(\omega_k(t - t_0)) + (v^k/\omega_k) \sin(\omega_k(t - t_0)) - \frac{\gamma_k}{m\omega_k} \int_{t_0}^t \sin(\omega_k(t - s)) x(s) ds \right). \end{aligned} \quad (2.19)$$

where $F(t)$ is a stochastic force with $\langle F(t) \rangle = 0$, and a two-time correlation function

$$\langle F(t)F(s) \rangle = \sum_k \frac{\gamma_k^2 \hbar}{4m\omega_k} \cos(\omega_k(t-s)). \quad (2.20)$$

C. The continuum limit

In order to consider the sorts of classical systems we are concerned with, we must go to the limit of a continuum of oscillator frequencies, both classically and quantum mechanically. In doing this, we replace our sums over oscillators with integrals over a distribution function $g(\omega)$. The usual choice for such a $g(\omega)$ is the Debye distribution [6,7]:

$$g(\omega) = \eta\omega^2 \exp(-\omega/\Omega), \quad (2.21)$$

where Ω is a cutoff frequency. The reservoir degrees of freedom will become a continuum, $Q^k(t) \rightarrow Q(\omega, t)$, and the eigenvalues ν_k will become a continuous complex function $\nu(\omega)$. In general, Ω must be taken to be fairly large. More precisely, we want $\Omega \gg 1/(t_f - t_0)$, so that the relaxation time of the reservoir is much less than the time-scale of the problem.

Let's consider now the various components of $W[X(t), \xi(t)]$ one at a time. In the continuum limit, we have

$$\sum_k \frac{\gamma_k^2}{m\omega_k} \int_{t_0}^t ds \sin(\omega_k(t-s))X(s) \rightarrow \frac{1}{m} \int_0^\Omega d\omega \int_{t_0}^t ds \frac{g(\omega)}{\omega} \sin(\omega(t-s))X(s). \quad (2.22)$$

We can invert the order of integration and do the ω integral, substituting (2.21) for $g(\omega)$:

$$\begin{aligned} \frac{1}{m} \int_{t_0}^t ds \int_0^\Omega d\omega \frac{g(\omega)}{\omega} \sin(\omega(t-s))X(s) &= \frac{\eta}{m} \int_{t_0}^t ds \int_0^\infty d\omega \omega \sin(\omega(t-s)) \exp(-\omega/\Omega) X(s) \\ &= \frac{\eta}{m} \int_{t_0}^t ds X(s) \frac{d}{ds} \int_0^\infty d\omega \cos(\omega(t-s)) \exp(-\omega/\Omega) \\ &= \frac{\eta}{m} \int_{t_0}^t ds X(s) \frac{d}{ds} \left(\frac{\Omega}{1 + \Omega^2(t-s)^2} \right) \\ &= \frac{\eta\Omega}{m} X(t) - \frac{\eta}{m} \int_{t_0}^t ds \dot{X}(s) \left(\frac{\Omega}{1 + \Omega^2(t-s)^2} \right) + O(1/\Omega). \end{aligned} \quad (2.23)$$

Now we use the fact that Ω is large. The second term becomes highly peaked about $s = t$, and the expression goes to

$$\frac{1}{m} \int_{t_0}^t ds \int_0^\infty d\omega \frac{g(\omega)}{\omega} \sin(\omega(t-s))X(s) \approx \frac{\eta\Omega}{m}X(t) - \frac{\pi\eta}{2m}\dot{X}(t) + O(\Omega^{-1}). \quad (2.24)$$

The second term has the form of a dissipation with constant $2\Gamma = \pi\eta/2mM$. The first term is a linear force. With Ω taken to be large, one might expect this term to diverge. This issue is discussed in detail by Caldeira and Leggett [7]. In essence, it is possible to take this term as an (admittedly large) modification to the system potential without actually being infinite. If the system were a harmonic oscillator, this would cause a shift in the oscillator frequency. More generally, we can absorb this term into the system action as an additional potential:

$$S_{\text{sys}}[x(t)] \rightarrow S'_{\text{sys}}[x(t)] = S_{\text{sys}}[x(t)] + \int_{t_0}^{t_f} \frac{\eta\Omega}{2m}x^2(t). \quad (2.25)$$

If our Lagrangian is the usual $L(x, \dot{x}) = \frac{1}{2}M\dot{x}^2 - U(x)$, then we effectively have a new Lagrangian

$$L(x, \dot{x}) \rightarrow L'(x, \dot{x}) = \frac{1}{2}M\dot{x}^2 - U'(x), \quad (2.26)$$

where

$$U'(x) = U(x) - \frac{\eta\Omega}{2m}x^2. \quad (2.27)$$

In subsequent analysis, it will be $U'(x)$ that we are interested in, as the effective potential.

The imaginary part of $W[X, \xi]$ is also of interest. Here we have

$$\begin{aligned} \sum_k \frac{\gamma_k^2}{4m\omega_k} \int_{t_0}^{t_f} ds \cos(\omega_k(t-s))\xi(t)\xi(s) &\rightarrow \frac{1}{4m} \int_0^\infty d\omega \int_{t_0}^{t_f} ds \frac{g(\omega)}{\omega} \cos(\omega(t-s))\xi(t)\xi(s), \\ &= \frac{\eta}{4m} \int_{t_0}^{t_f} ds \int_0^\infty d\omega \omega \exp(-\omega/\Omega) \cos(\omega(t-s))\xi(t)\xi(s), \\ &= -\frac{\eta}{4m} \int_{t_0}^{t_f} ds \xi(t)\xi(s) \frac{d}{ds} \int_0^\infty d\omega \exp(-\omega/\Omega) \sin(\omega(t-s)), \end{aligned}$$

$$= -\frac{\eta}{4m} \int_{t_0}^{t_f} ds \frac{d}{ds} \left(\frac{\Omega^2(t-s)}{1 + \Omega^2(t-s)^2} \right) \xi(t)\xi(s),$$

which in the limit as Ω becomes large is

$$= \frac{\eta}{4m} \int_{t_0}^{t_f} ds \frac{\xi(t)\xi(s)}{(t-s)^2}. \quad (2.28)$$

This, as Gell-Mann and Hartle argue [3], should be identified as the kernel of a two-time correlation function of the noise arising from the reservoir. Note that it is highly non-local in time. This is because the reservoir is essentially at absolute zero. At this low temperature, extremely long time correlations can be maintained, a quantum mechanical effect with no real classical analog. Classically, all noise would vanish.

Unfortunately, from a computational point of view, this is not very convenient. The non-local time correlation precludes any simple kind of differential formulation of this theory, i.e., no master equations. While some type of retarded equation might be introduced to model this system, to my knowledge this has not been done [8], and in any case such equations would almost certainly be unsolvable.

A slightly different reservoir initial condition can be introduced which is computationally somewhat more tractable, though perhaps not quite so interesting. This is the *displaced thermal state*,

$$\rho_{DT} = \hat{D}(\nu)\rho_T\hat{D}(\nu)^\dagger, \quad (2.29)$$

where $\hat{D}(\nu)$ is the coherent state displacement function defined before in (2.7) and T is the temperature of the thermal state ρ_T . The real part of $W[x'(t), x(t)]$ is the same with this initial condition as in the coherent state, but the imaginary part becomes very simple in the high temperature limit:

$$\text{Im}W[x'(t), x(t)] = iK \int_{t_0}^{t_f} dt \xi(t)^2, \quad (2.30)$$

where $K = 4M\Gamma k_B T/\hbar$. The derivation of this is carried out by Caldeira and Leggett, among others [7]. For convenience, I will use this initial condition for the rest of the paper.

Finally, we have the terms arising from the initial condition of the reservoir variables. When we go to the continuum limit here, the discrete sum in (2.10) becomes an integral. If we choose $\nu(\omega) \sim \delta(\omega - \omega_0)$, then we get

$$\sum_k \gamma_k \sqrt{\frac{2\hbar}{m\omega_k}} (\text{Re}\nu_k \cos(\omega_k t) + \text{Im}\nu_k \sin(\omega_k t)) \xi(t) \rightarrow Mq \cos(\omega_0 t + \phi_0) \xi(t), \quad (2.31)$$

where q can be set arbitrarily by adjusting the amplitude of $\text{Re}\nu(\omega)$. This term has the form of a periodic driving force. We can set the phase ϕ_0 to zero by making $\text{Im}\nu(\omega) = 0$. Physically, the presence of this term is equivalent to the system being driven by a plane wave at the frequency ω_0 .

Thus, our decoherence functional becomes

$$D[X(t), \xi(t)] = \exp \frac{i}{\hbar} \left\{ \int_{t_0}^{t_f} dt \left(-M\ddot{X}(t) - \frac{dU'}{dX}(X(t)) - 2M\Gamma\dot{X}(t) + Mq \cos(\omega_0 t) \right) \xi(t) - M\dot{X}_0 \xi_0 + iK \int_{t_0}^{t_f} dt \xi^2(t) + O(\xi^3) \right\} \chi(x'_0; x_0). \quad (2.32)$$

which gives us a quasiclassical equation of motion

$$\ddot{x} + \frac{1}{M} \frac{dU'}{dx}(x) + 2\Gamma\dot{x} = q \cos(\omega_0 t) + F(t)/M, \quad (2.33)$$

where $F(t)$ is a stochastic force with $\langle F(t) \rangle = 0$ and $\langle F(t)F(s) \rangle = \hbar K \delta(t - s)$. In a completely classical derivation, of course, this stochastic force would be absent. Thus, we have found a quantum system equivalent (in the appropriate limits) to a classical nonlinear oscillator with a periodic driving force and dissipation. All that remains now is to specialize to a chaotic example.

III. THE DAMPED, DRIVEN DUFFING OSCILLATOR

The quasiclassical equation of motion (2.33) is a fairly general expression for a one-dimensional damped, driven system. Many such systems exist which exhibit chaotic behavior for some values of the constants Γ and q . The ordinary pendulum is an example of such a system, where $U'(x) = -\cos(x)$. We will be examining

another system: the damped, driven Duffing oscillator. This nonlinear oscillator has a polynomial potential

$$\frac{1}{M}U'(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2. \quad (3.1)$$

We choose units to set $M = 1$. This system has the advantage of having been thoroughly studied and examined in the past and, also, since the potential is a polynomial, of not having an infinite number of nonzero derivatives. We will later see that this is convenient, though not vital.

The equation of motion is now

$$\ddot{x} + 2\Gamma\dot{x} + (x^3 - x) = q \cos(\omega_0 t) + F(t). \quad (3.2)$$

The potential is double-welled (see fig. 1). For some values of the constants, the oscillator undergoes periodic motion. By adjusting the frequency, one causes the system to undergo a series of bifurcations until eventually it enters into a region of chaotic behavior, typified by the presence of a strange attractor (see fig. 2). If one adjusts the driving force further and further, the chaotic region is eventually left, and periodic motion returns [9].

It is convenient to look at the long-term chaotic behavior in terms of a *constant phase map* or *surface of section*. That is, we consider the position x and momentum p at the discrete times $t_i = 2\pi i/\omega_0$. By then plotting the values x_i and p_i , we make the fractal structure of the strange attractor very clear (see fig. 3). We are also able to bring the mathematical toolbox of discrete dynamical system theory to bear on the problem. Formally, we define the constant phase map $x_i \rightarrow x_{i+1} = f_x(x_i, p_i)$, $p_i \rightarrow p_{i+1} = f_p(x_i, p_i)$, where \mathbf{f} is an operator which evolves the point (x_i, p_i) in phase space forward in time by $2\pi/\omega_0$.

We can now define a probability measure $P(x, p)$ on our phase space. In this discrete dynamics, it evolves according to the equation

$$P_{i+1}(x, p) = \int dx' \int dp' \delta(x - f_x(x', p')) \delta(p - f_p(x', p')) P_i(x', p'), \quad (3.3)$$

It is very useful then to consider an *invariant measure*, which gives a probability distribution on the strange attractor. This is defined by the equation

$$P_{i+1}(x, p) = P_i(x, p) = P_{\text{inv}}(x, p). \quad (3.4)$$

For a chaotic system such as the Duffing oscillator, P_{inv} will not be an analytic function; rather, it will be a generalized function. Also, it will not in general be unique; there are many invariant measures, most corresponding to unstable solutions, fixed points or periodic points. It has been shown that the inclusion of a small amount of noise removes both of these objections, eliminating the unstable solutions and making the function analytic [10]. The inclusion of noise effectively broadens the δ -functions in (3.3), making it impossible for probability measure to remain poised on an unstable fixed-point solution or periodic orbit, leaving only the strange attractor as a stable set. One can therefore define the classical P_{inv} as the limit of this unique P_{inv} as the noise goes to zero; it will still be a generalized function with a fractal structure, but now unique. Note that in the quantum system, the noise is *always* non-zero; we will see how this modifies our definitions in sections IV and V.

The structure of the strange attractor arises as a limit of repeated stretching and folding of phase space. As we look closer and closer at the component points of the attractor, we see repeated layers of substructure at every scale (see fig. 4). Such infinite substructure is commonly characterized by its *fractal dimension*. There are a number of ways of defining dimension, each of which has slightly different properties.

One common definition is that of the *capacity* or *Kolmogorov dimension*, D_C . This is calculated by means of a box-counting algorithm. Phase space is divided into small cells of linear size ϵ , and one counts the number of cells $N(\epsilon)$ which contain points of the attractor. The dimension is then

$$D_C = \lim_{\epsilon \rightarrow 0} -\frac{\ln N(\epsilon)}{\ln \epsilon}. \quad (3.5)$$

Though this definition is fairly easy to calculate numerically, it does not reflect the fact that an orbit may visit regions of the attractor with varying frequency. To

take this into account, one may instead use the *information dimension*, D_I . Again, phase space is divided into cells of linear dimension ϵ . The probability that a given point will fall in the i th cell is p_i . D_I is then

$$D_I = \lim_{\epsilon \rightarrow 0} \frac{\sum_i p_i \ln p_i}{\ln \epsilon}. \quad (3.6)$$

If p_i is equal for all cells that are visited, then $D_I = D_C$; otherwise $D_I < D_C$.

If one examines a small cell of phase space evolve according to our equation, it will tend to be stretched along one dimension. The overall phase space volume, however, will contract, due to the effects of dissipation. This stretching is what provides the well known signature of chaotic systems, sensitivity to initial conditions. The contraction, together with the more global process of folding, is what leads to the fractal structure of the attractor. One can average these two effects over the entire attractor to calculate the Lyapunov exponents. In our two-dimensional phase space, this will be a pair of numbers λ_1 and λ_2 , with $\lambda_1 > 0$ characterizing the stretching and $\lambda_2 < 0$ characterizing the contraction. Since overall phase space volume is shrunk by this system, clearly $\lambda_1 + \lambda_2 < 0$. If our phase space were n dimensional, there would clearly be n characteristic exponents.

Lyapunov exponents are calculated by considering the time evolution of an infinitesimal frame of basis vectors in phase space. One can perform a Gram-Schmidt orthogonalization, separating out the most rapidly increasing direction from less rapidly increasing directions repeatedly until one has n orthogonal vectors. One then takes the logarithm of the rate of change in each of those directions. Allowing the frame to evolve for many driving cycles lets one follow a phase space cell as it samples all parts of an attractor. In this way one calculates the average values of the exponents. The values of the λ_i are global quantities, characterizing the attractor as a whole, or equivalently, the long-term behavior of orbits throughout the attractor.

Calculating the highest exponent, λ_1 , is not very difficult. Finding values for a full spectrum of exponents, however, is rather tricky, and requires a subtle touch. I

refer those interested to the papers of Wolf et al. and Brown et al. for details [10,11]. These definitions for fractal dimension and Lyapunov exponents run into trouble in the quantum case, where taking limits as $\epsilon \rightarrow 0$ is not very well-defined. We will see in section V how one might adjust these definitions appropriately for a quantum system.

IV. QUANTUM MAPS AND STRANGE ATTRACTORS

A. Quantum maps

In the classical case we went from continuous to discrete dynamics by going to the constant phase map of the Duffing oscillator. In considering a quantum equivalent, it is convenient to coarse-grain our selected trajectories $x(t)$ and $x'(t)$ by considering only their values at the times $t_i = 2\pi i/\omega_0$ of constant phase. The decoherence functional then becomes

$$D[\{x_i\}, \{x'_i\}] = \int_{\{x_i\}} \delta x \int_{\{x'_i\}} \delta x' D[x(t), x'(t)], \quad (4.1)$$

where the decoherence functional on continuous trajectories $D[x(t), x'(t)]$ is given by (2.32). The path integrals are over all paths of $x(t)$ and $x'(t)$ which pass through the points x_i and x'_i respectively at times t_i .

Such a coarse-graining is discussed by Gell-Mann and Hartle [3]. In general, in order for such a system to be sufficiently decoherent, we must also coarse-grain on the positions $\{x_i\}$ and $\{x'_i\}$. Instead of specifying the positions exactly, we instead require just that the positions fall in one of a group of short intervals $\Delta_{\alpha_i}^i$ at the times t_i . A history is then given by specifying the sequence of α_i 's. We'll use the shorthand notation α for this sequence.

We can estimate the minimum length d of such intervals by requiring that the off-diagonal terms of the decoherence functional be strongly suppressed for $|x_i - x'_i| =$

$|\xi_i| > d$. As Gell-Mann and Hartle show, this depends on the separation between times t_i , the strength of the coupling, and so forth. A rough estimate gives

$$d^2 \sim \frac{1}{\Gamma} \left[\frac{\hbar^2}{2MkT\Delta t} \right]. \quad (4.2)$$

When we are considering gaps of size $\Delta t = 2\pi/\omega_0$, it is clear that the intervals can be taken to be extremely tiny. For a pollen grain of mass 10^{-6} this would give an interval of size $\sim 10^{-17}$ cm, much smaller than the actual length scale characterizing the noise.

Since equation (2.32) is expressed in terms of the variables $X(t)$ and $\xi(t)$, it might be useful to change variables in our coarse-grained systems. As the decoherence functional is suppressed for large ξ , we can treat our integrand as being quadratic in $\xi(t)$ and carry out the ξ integration. This gives us

$$p(\alpha) = \sqrt{\frac{2\pi}{K}} \int_{\{\alpha_i\}} \delta X \exp \left\{ -\frac{1}{K\hbar} \int_{t_0}^{t_f} e^2(t) dt \right\} w(X_0, p_0), \quad (4.3)$$

where $e(t) = 0$ is the classical equation of motion as given above in (2.33) and $w(X_0, p_0)$ is the initial Wigner distribution, obtained by the integral over ξ_0 .

We see that if the $\{X_i\}$ do not lie along a classical trajectory $e(t) = 0$, then the functional will be suppressed. So the most probable histories are those which lie along the classical trajectory. The X path integral cannot be done exactly in most cases, but one can see that in general the $\{X_i\}$ must lie near the $\{x_i\}$ for some classical problem for the probability to be of reasonable magnitude.

The Wigner distribution $w(X, p)$ (not to be confused with the influence phase $W[X(t), \xi(t)]$, which is a functional!) is given by

$$w(X, p) = \frac{1}{\pi} \int_{-\infty}^{+\infty} e^{i\xi p/\hbar} \tilde{\rho}(X + \xi/2, X - \xi/2) d\xi, \quad (4.4)$$

where we see that X and ξ are our usual variables, and p has units of momentum. $w(X, p)$ is somewhat analogous to a probability distribution on phase space. It is real and integrates to a total of 1; its primary difference from a classical phase space distribution is that it is not strictly non-negative.

If we want to advance $w(X, p)$ in time, we can define a transition matrix \mathbf{T} such that $w(t_f) = \mathbf{T}w(t_0)$. More explicitly this is

$$w(X_f, p_f) = \int dX_0 \int dp_0 T(X_f, p_f; X_0, p_0) w(X_0, p_0) \quad (4.5)$$

where $T(X_f, p_f; X_0, p_0)$ is defined

$$T(X_f, p_f; X_0, p_0) = \frac{1}{\pi} \int d\xi_0 d\xi_f e^{i(\xi_f p_f - \xi_0 p_0)/\hbar} \tilde{T}(X_f + \xi_f/2, X_f - \xi_f/2; X_0 + \xi_0/2, X_0 - \xi_0/2), \quad (4.6a)$$

$$\tilde{T}(x_f, x'_f; x_0, x'_0) = \int \delta x \delta x' \exp \frac{i}{\hbar} \left\{ S_{\text{sys}}[x(t)] - S_{\text{sys}}[x'(t)] + W[x(t), x'(t)] \right\}. \quad (4.6b)$$

If we let $t_f - t_0 = 2\pi/\omega_0$ then time advancement can be performed by repeated applications of \mathbf{T} . This is a sort of *quantum map*,

$$w_i \rightarrow w_{i+1} = \mathbf{T}w_i. \quad (4.7)$$

We can ask if repeated applications of \mathbf{T} will tend to converge to some invariant Wigner distribution $w_{\text{inv}} = \mathbf{T}w_{\text{inv}}$, analogous to the invariant measure P_{inv} of section III. Preliminary numerical calculations seem to show that this is the case [12]. This w_{inv} appears unique, and should be analytic, thanks to the “blurring” effect of quantum noise.

To make closer contact with the classical system, we might wish instead to consider histories in which a trajectory passes through small cells in phase space, rather than just intervals in X . We can write such a history by considering projections onto intervals in X followed very briefly by projections onto intervals in p . Histories of this type have been considered by Gell-Mann and Hartle, and by Halliwell, who wrote down an explicit equation for such a history [3,13].

One cannot in general specify both the momentum and position of a particle at the same instant. One can, however, consider a measurement of position followed by a measurement of momentum, and let the time between them go to zero. Halliwell calculated the probability of such a history using approximate projections

$$P_{\bar{x}} = \frac{1}{\pi^{1/2}\sigma_{\bar{x}}} \int_{-\infty}^{\infty} dx \exp\left[-\frac{(x - \bar{x})^2}{\sigma_{\bar{x}}^2}\right] |x\rangle\langle x|, \quad (4.8a)$$

$$P_{\bar{p}} = \frac{1}{\pi^{1/2}\sigma_{\bar{p}}} \int_{-\infty}^{\infty} dp \exp\left[-\frac{(p - \bar{p})^2}{\sigma_{\bar{p}}^2}\right] |p\rangle\langle p|. \quad (4.8b)$$

These are approximate projections into intervals $\Delta_{\bar{x}}$ of width $\sigma_{\bar{x}}$ and $\tilde{\Delta}_{\bar{p}}$ of width $\sigma_{\bar{p}}$, centered on \bar{x} and \bar{p} , respectively. Halliwell shows [13] that for an initial Wigner distribution $w(X, p)$, the probability of finding a particle in the phase-space cell delimited by the two above projections is

$$p(\Delta_{\bar{x}}, \tilde{\Delta}_{\bar{p}}) = \int dX dp w(X, p) \exp[-a(p - \bar{p})^2 - b(X - \bar{x})^2], \quad (4.9)$$

where

$$a = \frac{\sigma_{\bar{x}}^2}{2[\hbar^2 + (1/4)\sigma_{\bar{x}}^2\sigma_{\bar{p}}^2]}, \quad b = \frac{2}{\sigma_{\bar{x}}^2} \quad (4.10)$$

when the X projection precedes the p projection, and

$$a = \frac{2}{\sigma_{\bar{p}}^2}, \quad b = \frac{\sigma_{\bar{p}}^2}{2[\hbar^2 + (1/4)\sigma_{\bar{x}}^2\sigma_{\bar{p}}^2]} \quad (4.11)$$

when the p projection precedes the X projection. There is a restriction on these projections that

$$0 < ab \leq \frac{1}{\hbar} \implies \sigma_{\bar{x}}^2\sigma_{\bar{p}}^2 > \hbar^2. \quad (4.12)$$

To calculate the probabilities of an orbit passing through a series of such cells $(\Delta_i, \tilde{\Delta}_i)$ at times t_i , we make use of the transition matrix \mathbf{T} . Let us assume that the X projection comes first. Then it turns out that

$$\begin{aligned} p(\{\Delta_i, \tilde{\Delta}_i\}) &= \int d\{X\}d\{p\}w(X_0, p_0) \\ &\times \exp\left[-\frac{2}{\sigma_{\bar{p}}^2}(p_1 - \bar{p}_0)^2 - \frac{2}{\sigma_{\bar{x}}^2}(X_0 - \bar{x}_0)^2 - \frac{\sigma_{\bar{x}}^2}{2\hbar^2}(p_0 - p_1)^2 - \frac{\sigma_{\bar{p}}^2}{2\hbar^2}(X_0 - X_1)^2\right] \\ &\times T(X_2, p_2; X_1, p_1) \\ &\times \exp\left[-\frac{2}{\sigma_{\bar{p}}^2}(p_3 - \bar{p}_1)^2 - \frac{2}{\sigma_{\bar{x}}^2}(X_2 - \bar{x}_1)^2 - \frac{\sigma_{\bar{x}}^2}{2\hbar^2}(p_2 - p_3)^2 - \frac{\sigma_{\bar{p}}^2}{2\hbar^2}(X_2 - X_3)^2\right] \\ &\times T(X_4, p_4; X_3, p_3) \\ &\times \dots \end{aligned} \quad (4.13)$$

where the interval Δ_i is centered on \bar{x}_i and $\tilde{\Delta}_i$ on \bar{p}_i . The final projection at time t_n will be of the form (4.9). The case where the p projection precedes the x is very similar to the above.

There are other ways of considering phase space projections, using coherent states or Gaussian combinations of coherent states. Phase space histories and their decoherence deserve a fuller discussion elsewhere [14].

B. The master equation

Another common method of studying systems such as this is by means of a master equation formalism. Caldeira and Leggett derive such an equation in the case of a harmonic oscillator interacting with a thermal bath at relatively high temperature [7]. Their result is readily adapted to the present case, yielding the equation

$$\begin{aligned} \frac{\partial \tilde{\rho}}{\partial t}(x, x') = & -\frac{K}{\hbar}(x - x')^2 \tilde{\rho} + \frac{iq}{\hbar}(x - x') \cos(\omega_0 t) \tilde{\rho} - \frac{i}{\hbar}(V(x) - V(x')) \tilde{\rho} \\ & + 2\Gamma(x - x') \left(\frac{\partial \tilde{\rho}}{\partial x'} - \frac{\partial \tilde{\rho}}{\partial x} \right) + \frac{i\hbar}{2m} \left(\frac{\partial^2 \tilde{\rho}}{\partial x^2} - \frac{\partial^2 \tilde{\rho}}{\partial x'^2} \right) \end{aligned} \quad (4.14)$$

In examining this equation, the meanings of the different terms are highly intuitive. The K/\hbar term is a diffusive effect resulting from the quantum noise; the Γ term includes the effects of dissipation; the $q \cos(\omega_0 t)$ is the driving force.

Changing to the variables X and ξ , the master equation becomes

$$\begin{aligned} \frac{\partial \tilde{\rho}}{\partial t}(X, \xi) = & -(4K/\hbar)\xi^2 \tilde{\rho} + (2iq/\hbar)\xi \cos(\omega_0 t) \tilde{\rho} - \frac{i}{\hbar}(V(X + \xi/2) - V(X - \xi/2)) \tilde{\rho} \\ & - 4\Gamma\xi \frac{\partial \tilde{\rho}}{\partial \xi} + \frac{i\hbar}{2m} \frac{\partial^2 \tilde{\rho}}{\partial \xi \partial X}. \end{aligned} \quad (4.15)$$

If ξ is small, then we can expand the potential term to give us

$$V(X + \xi/2) - V(X - \xi/2) \approx \xi \frac{\partial V}{\partial X}(X) + \frac{\xi^3}{24} \frac{\partial^3 V}{\partial X^3} + \dots \quad (4.16)$$

For the Duffing potential, of course, the higher-order terms vanish. This is a convenient benefit of dealing with a polynomial potential.

Transforming this equation by (4.4) gives us a new equation for the evolution of the Wigner distribution itself:

$$\begin{aligned} \frac{\partial w}{\partial t}(X, p) = & \hbar K \frac{\partial^2 w}{\partial p^2} - q \cos(\omega_0 t) \frac{\partial w}{\partial p} + \frac{\partial V}{\partial X}(X) \frac{\partial w}{\partial p} \\ & - \frac{\hbar^2}{24} \frac{\partial^3 V}{\partial X^3}(X) \frac{\partial^3 w}{\partial p^3} + 4\Gamma \frac{\partial}{\partial p} p w - \frac{p}{m} \frac{\partial w}{\partial X}. \end{aligned} \quad (4.17)$$

This is almost exactly the form of the Fokker-Planck equation for the classical equation of motion (2.33), with the diffusive $\hbar K$ term representing the effects of the random fluctuations on the “probability” distribution and the third-derivative term being a purely quantum-mechanical addition, enabling $w(X, p)$ to become negative in limited regions of phase space. To interpret this distribution as a probability, we must coarse-grain by averaging it over small volumes of phase space, producing the sort of “smeared” Wigner distribution discussed by Halliwell [13].

This is not, of course, the full story. In order to correctly describe this system, we need not only the time evolution of the Wigner distribution, but also to specify a set of decoherent histories, as discussed in the previous section. Without those histories, it is impossible to assign classical probabilities in a consistent manner. These two approaches can be made to complement each other, however, as the master equation can be solved to yield the transfer matrix \mathbf{T} , defined in the previous section as a path integral. In the case of chaos, one can in general only solve these equations numerically, and the master equation formalism then has a computational advantage over the path integral form.

V. INTERPRETATION OF CLASSICAL QUANTITIES

From the previous section, we see that the behavior of a system such as we are examining can be evaluated on many levels:

1. The Classical level. In the previous section we saw that all histories which deviate too far from the classical solution have their probabilities highly suppressed.

If a system is large enough in scale, with enough inertia that the quantum effects are lost in other sources of uncertainty, we can treat it as approximately classical. Clearly, in a chaotic system this quantum noise does cause large alterations in the overall behavior of the system over time, but it is often impossible to separate this from other sources of error, such as imprecision in measuring the initial conditions.

2. The Quasiclassical level. Here we again treat the system as essentially classical, but now explicitly include the noise arising from quantum effects, which is large enough to be noticed on the scale of the system; this is the system as described by equation (2.33). From a practical point of view, this is the level at which quantum effects are most easily calculated. This also overlaps the considerable work that has been done on dynamical systems with noise [10,15–17].

3. The Quantum level. Fundamentally, we can consider the system in terms of coarse-grainings and decoherent histories. Instead of treating a system as basically classical with added noise, we consider all possible histories and compute expectation values for classical quantities from the probabilities of those histories.

We've already discussed the classical (level 1) definitions of the Lyapunov exponents and fractal dimensions used to characterize chaotic systems and strange attractors in section III. As pointed out, these quantities are usually defined at least formally by calculating a quantity for the system at different levels of coarse-graining (i.e., different box sizes ϵ), and taking the limit as we go to finer and finer scales. While this has great mathematical power and consistency, in actual physical systems it inevitably breaks down. As Benoit Mandelbrot wrote on the problem of measuring coastlines with seemingly infinite levels of detail [18],

“To obtain a [fractal] Koch curve, the cascade of smaller and smaller new promontories is pushed to infinity, but in Nature every cascade must stop or change character. While endless promontories may exist, the notion that they are self-similar can apply only between certain limits. Below the lower limit, the concept of coastline ceases to belong to geography.

“It is therefore reasonable to view the real coastline as involving two *cutoff scales*. Its *outer cutoff* Ω might be the diameter of the smallest circle encompassing an island, or perhaps a continent, and the *inner cutoff* ϵ might be the twenty meters mentioned. . . Actual numerical values are hard to pinpoint, but the need for cutoffs is unquestionable.”

As we shall see, in the case of chaotic strange attractors, the underlying quantum physics effectively provides that lower cutoff.

A. Lyapunov exponents

Classically, the Lyapunov exponents characterize the rate at which nearby trajectories diverge as they evolve according to the equations of motion. In a chaotic system, one expects any two trajectories, no matter how close they start, to eventually move on the strange attractor completely independently of each other. This is measured in the classical case by taking the limit as points start arbitrarily near each other and evolve for arbitrarily long lengths of time.

When we allow for the presence of quantum effects, however, this definition is no longer meaningful. As points begin closer and closer to each other, the effects of noise become larger and larger; one would expect the largest exponent to diverge in the limit as $\epsilon \rightarrow 0$. As we saw in section IV, the phase space cells in a decoherent history cannot be smaller than a certain size. This limit provides the lower cutoff mentioned above.

For most systems it is impossible to calculate the values of Lyapunov exponents exactly. Instead, one performs a numerical calculation. It is easiest to calculate the highest exponent; lower exponents are more difficult, as their effects tend to be swamped by λ_1 . In a numerical calculation small errors are unavoidable; each such error will add a small admixture of the most rapidly growing component, which will quickly drown out other effects.

Because of this, we'll first consider only the value of λ_1 . A simple way of estimating λ_1 is to numerically integrate equation (2.33) for a longer period of time, to generate a large number of points $\{x_i, p_i\}$ in phase space. One can then locate nearby points, closer than a certain cutoff ϵ , and trace their trajectories until they diverge further than an upper cutoff Δ . One then calculates the logarithm of the average divergence rate and averages it over many such pairs of points.

I have calculated this quantity in the quasiclassical case (see fig. 4). It turns out that the result one calculates is not very sensitive to the upper cutoff Δ , but is highly sensitive to the lower cutoff ϵ . In figure 4 we see the measured value of λ_1 as a function of ϵ for several different relative strengths of the quantum noise $\hbar K$. Notice how, for $\hbar K > 0$, λ_1 diverges as $\epsilon \rightarrow 0$.

Because of dissipation, the overall phase-space volume of an initial distribution tends to decrease with time. This indicates that, classically, $\lambda_1 + \lambda_2 < 0$. At very small length scales, however, the effects of noise counteract the effects of dissipation, causing phase-space volume to grow rather than shrink. Thus, at small length scales we expect to see the sum $\lambda_1 + \lambda_2$ become positive, and eventually approach $\lambda_1/\lambda_2 \approx 1$; the dimension at that length scales should also approach an integer (2 in this case).

We can try to define a quantum-mechanical analog of λ_1 . While I am not sure exactly what form such a definition should take, I can make a conjecture. Suppose that we start from the invariant Wigner distribution w_{inv} at $t = t_0$. We divide phase space into small cells $\{c_i\}$, centered on average positions $\{\vec{v}_i\}$ in phase space. These cells have a characteristic size ϵ (or area ϵ^2). Let d_{ij} be the distance between the centers of the i th and j th cells. We can define p_i to be the probability that the system is in cell c_i at time t_1 , using equation (4.9), and p_{ij} to be the probability that the system is in c_i at time t_1 and c_j at time t_2 , as shown in (4.13). Clearly

$$\sum_i p_i = 1, \quad \sum_j p_{ij} = p_i.$$

The probability of the system being in c_j at time t_2 *given* that it was in c_i at t_1 is

$$p(j|i) = \frac{p_{ij}}{p_i}. \quad (5.1)$$

A rough estimate of the rate of spreading is then given by

$$\lambda(\epsilon)_{\text{qm}} = \frac{1}{2} \sum_i p_i \log \left[\frac{\frac{1}{2} \sum_{j,k} p_{ij} p_{ik} d_{jk}^2}{p_i^2 \epsilon^2} \right]. \quad (5.2)$$

It isn't clear whether this will agree with the usual definition of λ_1 in the limit. This is much more a rate of expansion averaged over the attractor, whereas λ_1 is usually defined as the rate at which nearby solutions diverge when followed for a long period of time. This latter definition has serious problems in the quantum case, where it is impossible to start solutions arbitrarily close together, and hence equally impossible to follow them for arbitrarily long periods of time without global processes (such as folding) becoming important. When quantum effects are very small, one can approach this long-orbit definition, but in that case one is really doing a quasiclassical calculation (like the one above in figure 4), where the system can be treated as a classical stochastic equation.

Numerical experiments might serve to explore the connections, if any, between these classical and quantum ideas of Lyapunov exponents. I hope to do more such exploration soon. Also, it is not clear to me exactly what form quantum equivalents to lower Lyapunov exponents might take, nor even if such a concept is useful. These questions will soon, I hope, have at least tentative answers.

B. Information dimension

The information dimension is, as we saw in section IV, another number used to characterize strange attractors. It is usually defined by a box-counting algorithm of the type given in (3.6). As mentioned before, when one takes quantum mechanics into account, allowing the size of a box to go to zero no longer makes much sense.

Instead, let us consider the information dimension D_I as a function of box-size:

$$D_I(\epsilon) = \frac{\sum_i p_i \ln p_i}{\ln \epsilon}. \quad (5.3)$$

The probability p_i is defined as before. For non-zero ϵ , the exact value of $D_I(\epsilon)$ will vary slightly depending on how the boxes are chosen. This ambiguity can be eliminated by taking $D_I(\epsilon)$ to be the minimum possible value over all possible arrangements of boxes. In practice, this makes little difference. The usual classical limit is then just the limit of $D_I(\epsilon)$ as $\epsilon \rightarrow 0$.

Figure 5 shows the calculated values of $D_I(\epsilon)$ for different values of $\hbar K$. As we see, at large length scales the fractal nature of the attractor is not readily apparent; as we shrink our scale, the dimension decreases, until when dropping below the lower cutoff given by the quantum effects it abruptly turns upward again. This was calculated quasiclassically, using a long orbit with associated noise.

One can also do an analogous calculation using the complete quantum theory. Consider the invariant Wigner distribution $w_{\text{inv}}(X, p)$, defined in section IV. We can define the information dimension $D_I(\epsilon)_{\text{qm}}$ using the same definition (5.3). We divide phase space into evenly-sized cells $\{c_i\}$ of size ϵ , just as in the discussion of $\lambda(\epsilon)_{\text{qm}}$ above, and use the expression (4.9) for the probability p_i of being in the cell c_i . Again, we can eliminate ambiguity by minimizing $D_I(\epsilon)_{\text{qm}}$ over all possible divisions into cells. Clearly such a dimension will not even be well defined for cells of volume less than \hbar , and will in general depend on the scale of the coarse-graining, just as in the quasiclassical treatment.

Needless to say, it is much easier to extend the information dimension D_I to a probabilistic theory than it is to find an analogy for the capacity dimension D_C . The presence of noise will give a small but non-zero probability of finding a point in any cell, even if it is far from the classical strange attractor. So, for quantum dissipative chaos at any rate, D_I seems to be the more useful quantity.

VI. QUANTUM CHAOS

Since the discovery of chaos in the 1970's, there have been numerous attempts to look for the existence of chaos in quantum mechanical systems. Almost all of these have concentrated on quantized versions of non-integrable Hamiltonian systems. A seeming paradox was at the heart of the debate: chaos, as a classical phenomenon, depended entirely on the existence of nonlinear terms in the equations of motion; yet quantum systems are by their nature completely linear.

In fact, this argument is clearly invalid. While it is true that the linearity of the Schrödinger equation and its relativistic generalizations implies that one would not expect chaotic behavior in the wave function itself, this has little bearing on what one would actually see if one observed such a system. One does not measure wave functions; one measures particles.

An analogous classical treatment would be to consider probability distributions in phase space rather than values of position and momentum. One can then go from a set of nonlinear ordinary differential equations to a Fokker-Planck partial differential equation. The P.D.E. is completely linear. Does this then imply that chaos cannot exist in classical mechanics? Such a conclusion would be absurd.

Of course, wave functions are not probability distributions, so the comparison is a bit misleading. A sufficiently coarse-grained Wigner distribution, however, can be made to look very much like a probability distribution, and its master equation closely resembles the Fokker-Planck equation, as we have seen, so comparing the two is not completely inappropriate.

In fact, certain quantum models can exhibit chaos [19], but they are exceptional. Most systems which occupy a bounded volume in phase space do eventually exhibit *quantum recurrence* in which the expectation values of quantities such as energy are almost periodic [20–23]. A simple information-theory argument can be made for why this should be: a bounded volume in phase space V represents a finite number of

possible states $N \sim V/\hbar$, so one would expect the long-term behavior to be periodic or quasiperiodic. According to Ehrenfest's theorem, a narrow wave packet will tend to follow the classical chaotic trajectory for a time t_E until it has spread out to a size comparable to the phase space volume. Thus, it is argued, one should observe a long chaotic "transient," ending ultimately in periodic or almost periodic motion. As the system becomes more "macroscopic," the phase space volume becomes larger with respect to \hbar and t_E becomes longer.

Chirikov, Izrailev and Shepelyansky describe the usual approach to quantum chaos [19]. They separate the problem into two parts, the dynamics of the undisturbed wave function and the effects of measurement and wave function collapse. Obviously, this periodic behavior of expectation values says nothing about what an experimenter would actually observe upon measuring the system. They compare the former with deterministic behavior and the latter with randomness and "noise." Clearly, an actual series of measurements would not be periodic at all, but would instead resemble a random chaotic trajectory. They further dismiss the study of dissipative chaos as a mere phenomenological approximation to an underlying Hamiltonian system, e.g., in our case including both the system and reservoir degrees of freedom.

If the work of Gell-Mann, Hartle, and others is to be believed, we should consider only decoherent histories. Since Hamiltonian systems almost by definition do not interact with outside degrees of freedom, one cannot really talk of "measuring" them. If measurements are taken, the system is disturbed; if measurements are not taken, the system evolves undisturbed, but detailed histories of the motion will not decohere. In dissipative systems, by contrast, the chaotic system is interacting continually with the neglected degrees of freedom of the reservoir. These serve to provide a continual "measurement" of the system, in addition to causing dissipation and noise, so that histories of the system variables do decohere, as we have seen. Thus, considering these dissipative systems from a decoherence functional point of view is entirely appropriate. Also, any description of real macroscopic systems must allow for coarse-graining

over the many degrees of freedom of which we are ignorant. As one goes from the classical to the quantum realms, deterministic uncertainty or randomness is replaced by probabilistic uncertainty.

In the Duffing oscillator model studied here, an arbitrary initial distribution w_0 will tend to converge onto the invariant distribution, $\lim_{n \rightarrow \infty} \mathbf{T}^n w_0 \rightarrow w_{\text{inv}}$ [12]. This invariant distribution is periodic, with period $2\pi/\omega_0$, so that $\mathbf{T}w_{\text{inv}} = w_{\text{inv}}$. This resembles the idea of a long chaotic “transient” leading to a non-chaotic, periodic behavior.

This is not, however, very different from the behavior of classical systems evolving according to the Fokker-Planck equation. Most initial distributions P_0 of non-zero width evolve into the invariant probability measure P_{inv} in much the same way that w evolves into w_{inv} in the quantum case (see fig. 6). The classical case is complicated by the non-analyticity and non-uniqueness of P_{inv} , so that there can be a finite probability of sitting on top of some unstable equilibrium or periodic point. An initial distribution with non-zero width will never actually become the invariant measure, of course, always itself remaining an analytic function if it begins as one, but the difference rapidly becomes too small to measure. These difficulties do not exist in the quantum case, due to the presence of noise. This difference, plus the necessity for coarse-graining (and the requirement that an initial distribution have non-zero width, thanks to the uncertainty principle), is what distinguishes the classical and quantum cases, not the periodicity of the solution per se. The important idea is not the behavior of w_{inv} , but rather the probabilities of different possible histories of the system as described in (4.13).

Other authors have concentrated on the rate of expansion of uncertainty in quantum chaotic systems, and on how chaos leads to a form of “dissipation” in the quantum wave function itself [24–26].

A very interesting suggestion arose in work by Weinberg on possible nonlinear generalizations of quantum mechanics [27,28]. With true nonlinearity, chaos on the level

of the wave function could exist. Weinberg points out, however, that any nonlinearity in the theory must be very small to be consistent with experiment, and chaos is rare in such nearly-integrable systems. While such a generalization of quantum theory is certainly not ruled out by experiment, it poses a number of troubling problems. As pointed out by Polchinski [29], either faster-than-light transmission of information via an EPR-type experiment becomes possible, or different branches of histories can continue to interact with each other, effectively making decoherence impossible. Both of these, while certainly not impossible, have absolutely no basis in experimental evidence; indeed, the latter would imply that even classical probabilities could undergo a sort of “interference” with each other.

A few papers have been published on quantum dissipative chaos [30–34]. Davidson and Santhanam treat the problem by including a phenomenological term in the Schrödinger equation to model dissipation. Nörenberg and Milek treat a particular model in nuclear physics, while Savage considers a quantum optical system by numerically solving a master equation. The papers by Graham and Dittrich and Graham are particularly interesting; many ideas similar to those developed here are used (e.g., the study of systems as Wigner distributions, the importance of quantum noise). None, however, has the generality of the decoherence approach, in my opinion. While there is a fair bit of other work on the phenomenon of quantum dissipation itself [7,35–37], little of it has been applied to chaotic systems, as far as I am aware, and very little has been from the decoherence point of view.

I cannot do a broad survey of an extremely active field here. A number of good books now exist on quantum Hamiltonian chaos [38,39].

VII. CONCLUSIONS

In this paper, I have only treated a single model in depth: the forced, damped Duffing oscillator. It is clear, though, that the techniques used are easily applied to

any nonlinear oscillator problem. I believe that, in general, the formalism of Gell-Mann and Hartle provides a rigorous method for treating any classical chaotic system.

For some systems, of course, such a treatment may be inappropriate. Dissipative chaos was first discovered in attempts to model fluid dynamics; while in principle such problems could be treated in this way, any quantum effects are likely to be so small as to be irrelevant. For nonlinear oscillators, though, the application is quite reasonable, and it is possible that experiments could be done in quantum optical or electronic systems which would correspond to classically chaotic systems of this sort.

What is more, using this formalism provides a link between a quantum system and its “classical limit” which lets one define the idea of quantum chaos in a rigorous way. And such systems can be treated not only semi-classically, but with the full quantum laws as well.

Equally important, in this quasiclassical treatment we can see how chaotic systems can serve to amplify the effects of small quantum fluctuations. Sensitive dependence on initial conditions – the hallmark of chaos – is an important idea in understanding *measurement situations*, in which minor quantum effects can become correlated with a change in macroscopic variables. In complex systems with many degrees of freedom, sensitive dependence on initial conditions is probably the rule rather than the exception.

A great deal remains to be done. This formalism can readily be applied to a number of other systems besides the Duffing oscillator. Also, it would be valuable to develop numerical programs for solving the master equation, and calculating the quantities characterizing the attractor in the full quantum system as well as the quasiclassical limit. Some work on this has already been done [12], but it is still in a rather crude state. But the basic outlines of the theory are clear. Using the theory of Gell-Mann and Hartle, a rigorous treatment of quantum dissipative chaos is finally possible.

ACKNOWLEDGMENTS

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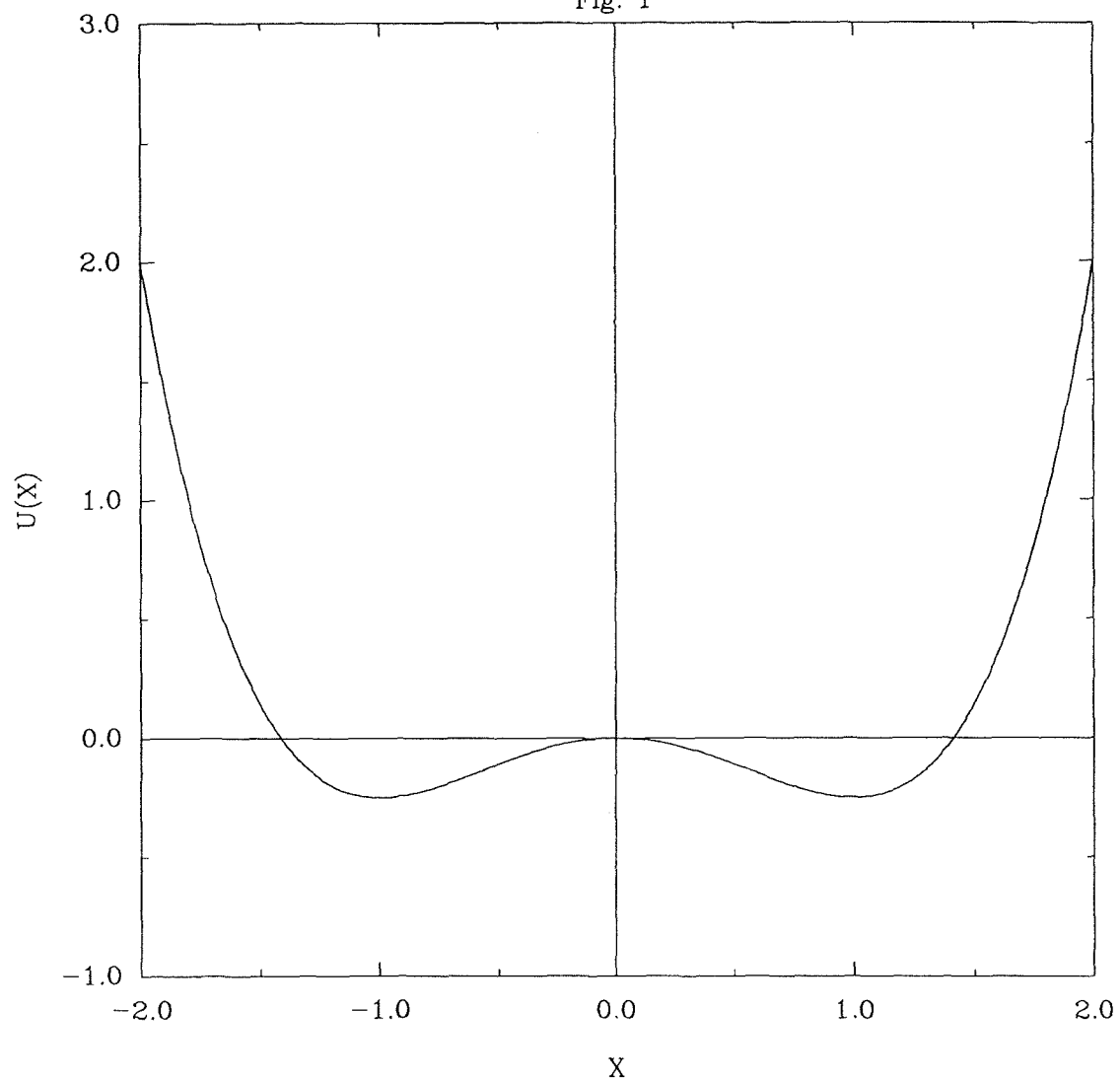
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Duffing Oscillator Potential

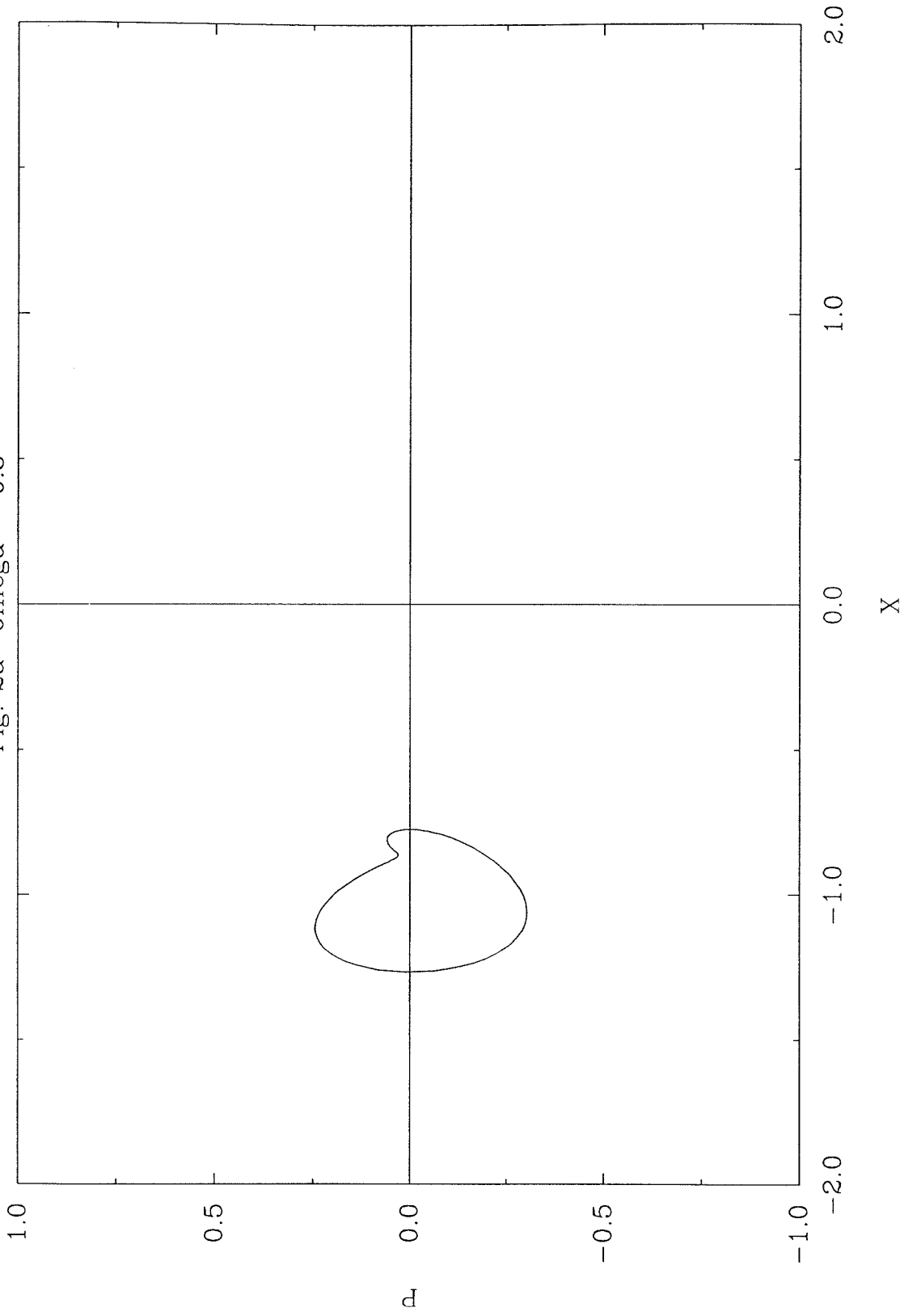
Fig. 1



Damped, Driven Duffing Oscillator

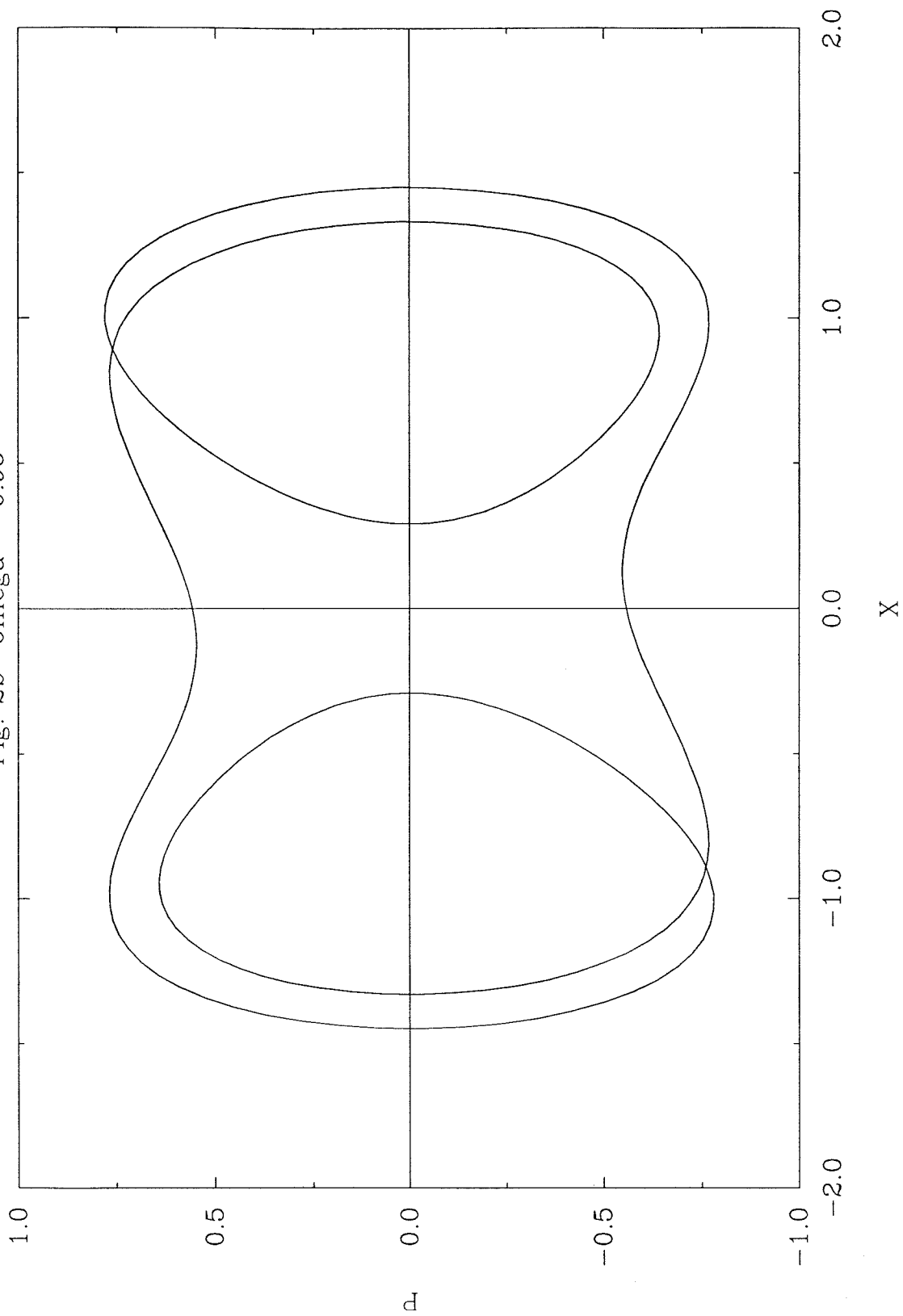
II-37

Fig. 2a $\omega = 0.8$



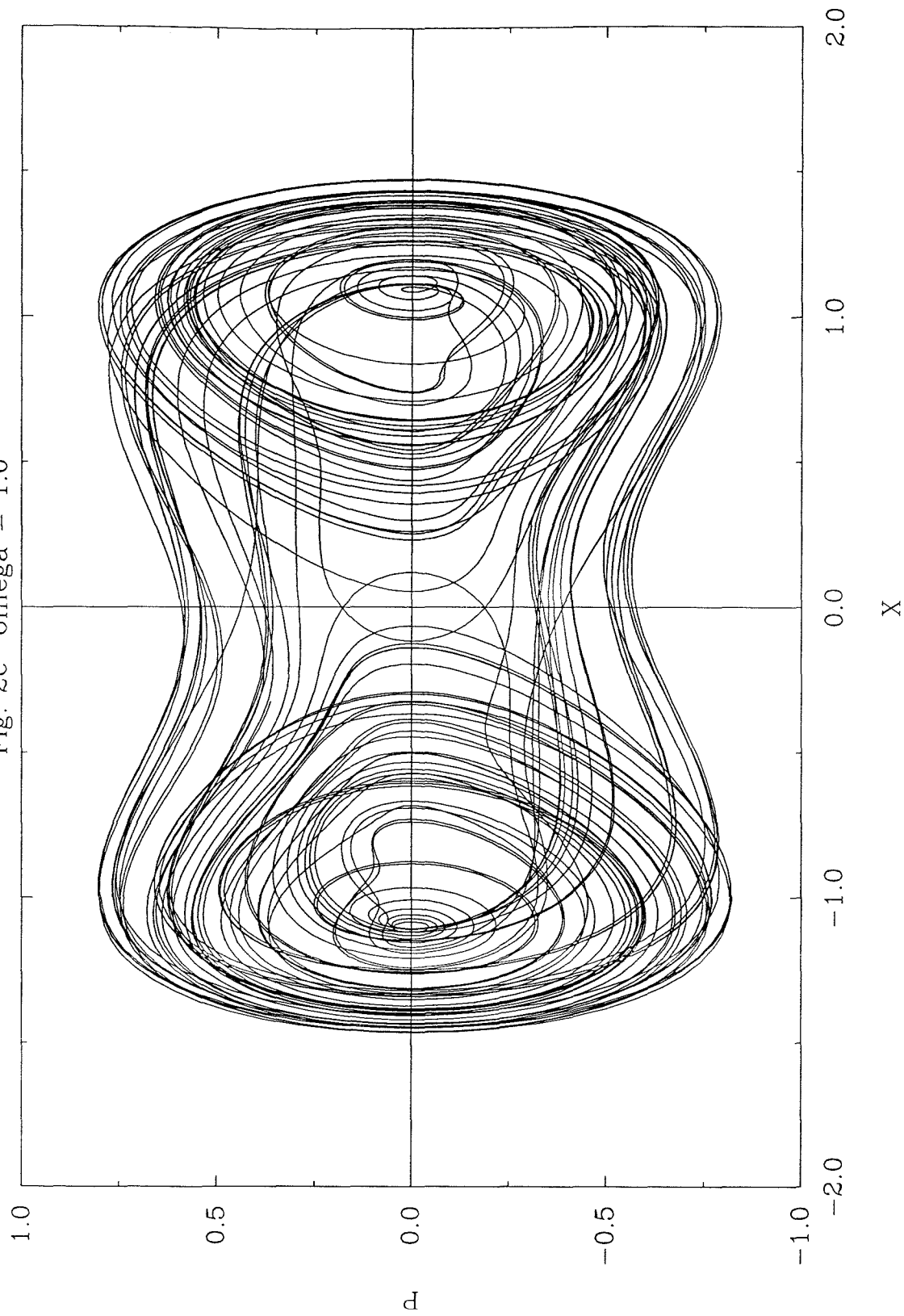
Damped, Driven Duffing Oscillator

Fig. 2b $\omega = 0.95$



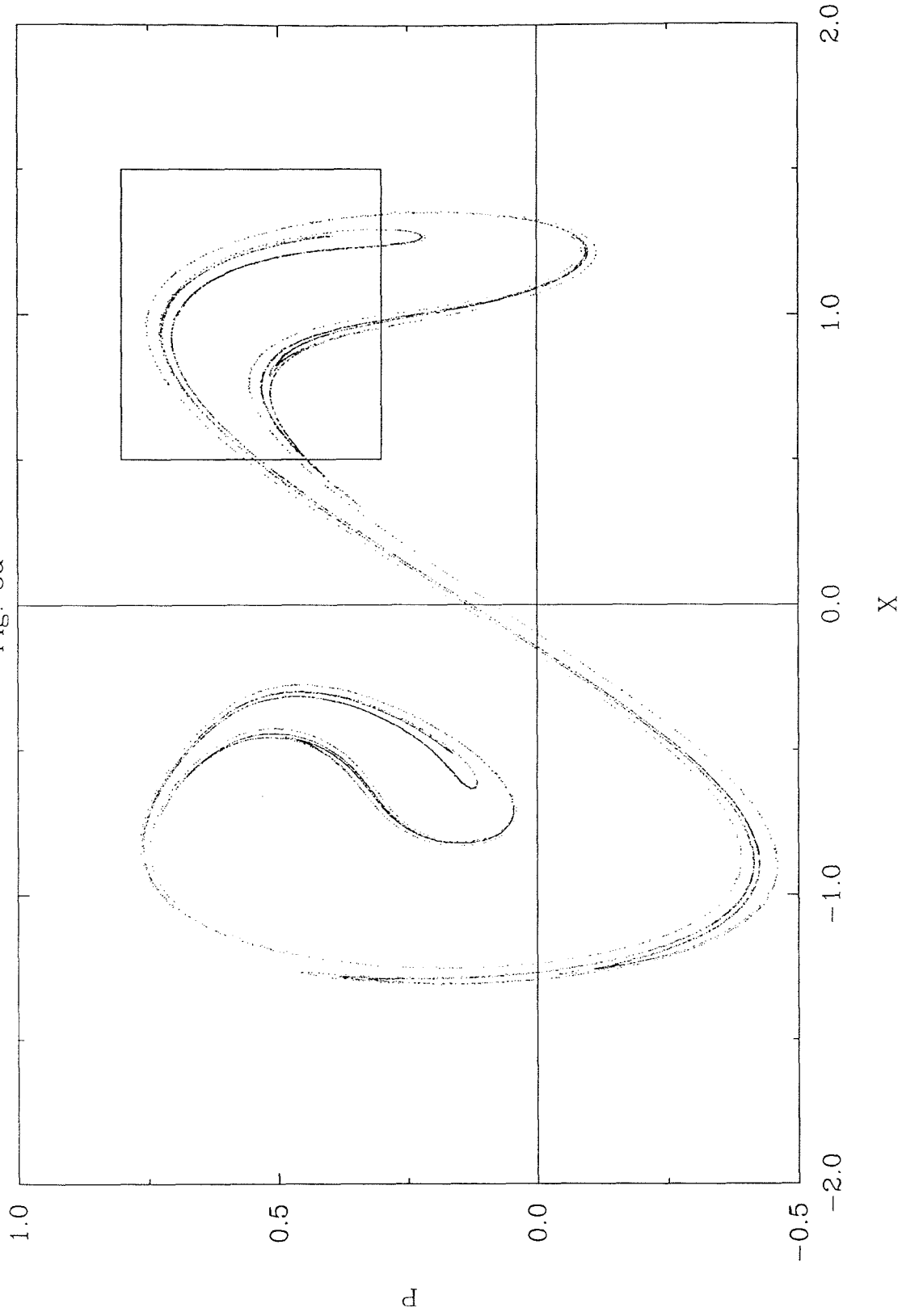
Damped, Driven Duffing Oscillator

Fig. 2c $\omega = 1.0$



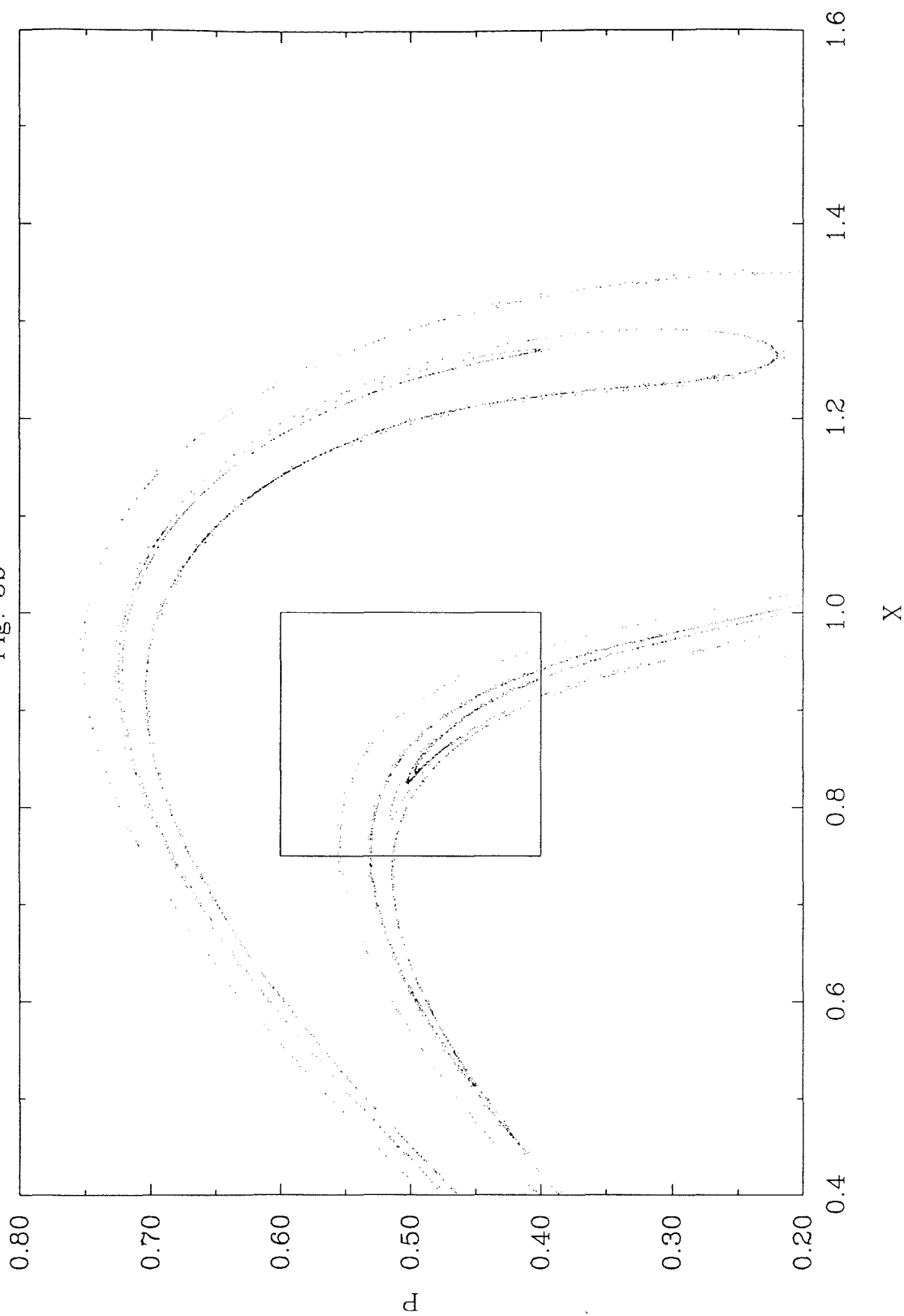
Poincare Section of Duffing Attractor

Fig. 3a



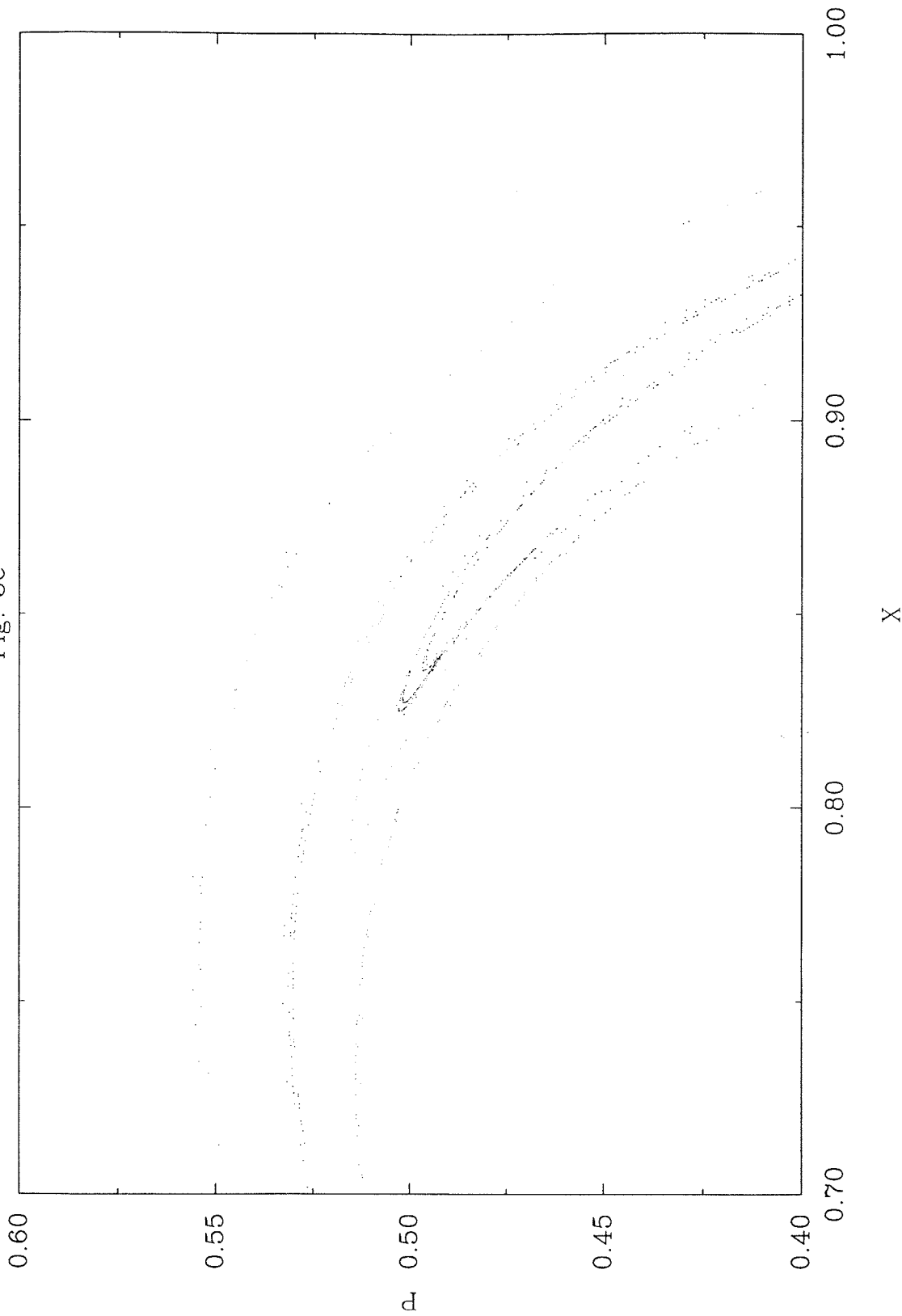
Poincare Section of Duffing Attractor

Fig. 3b



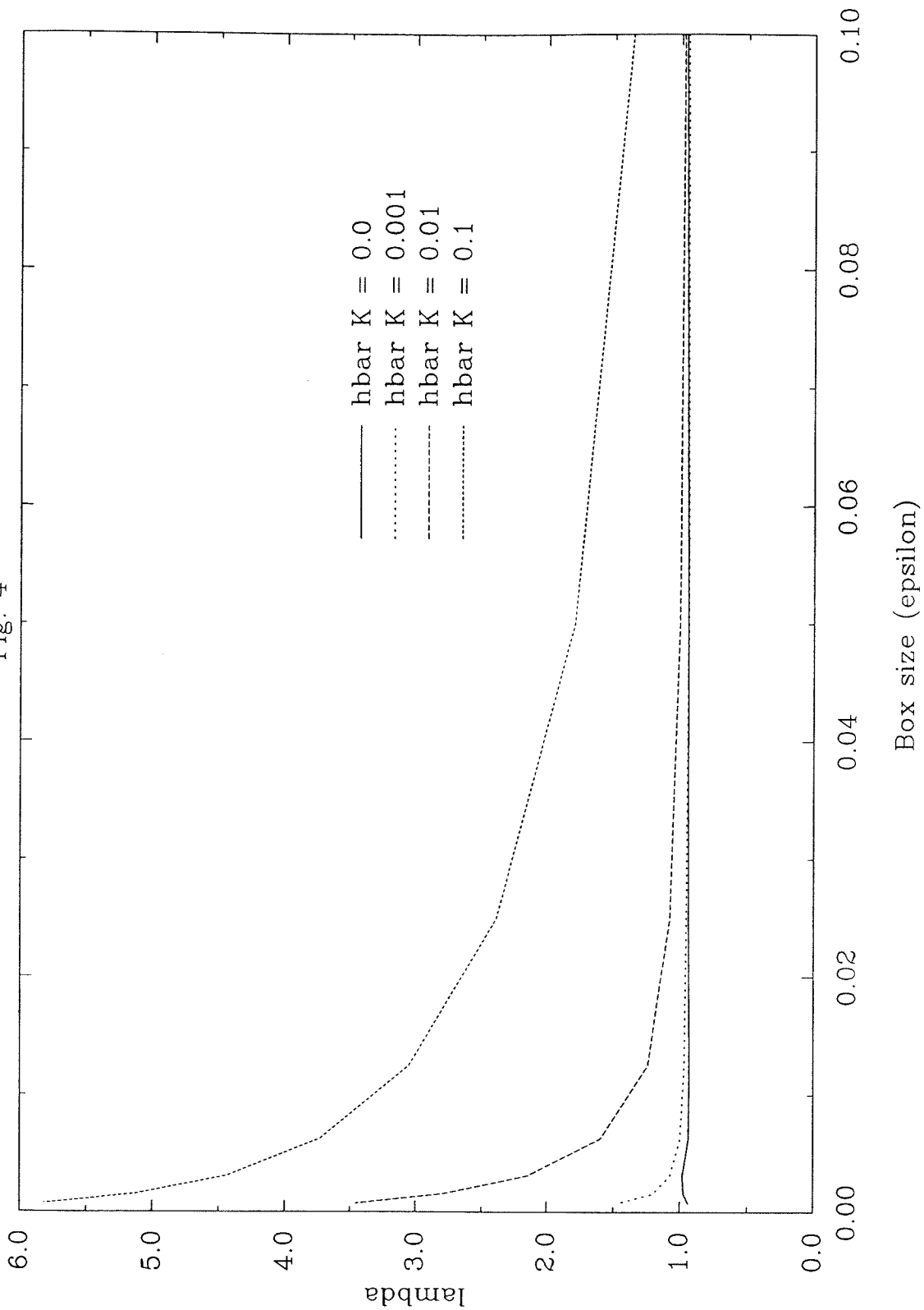
Poincare Section of Duffing Attractor

Fig. 3c



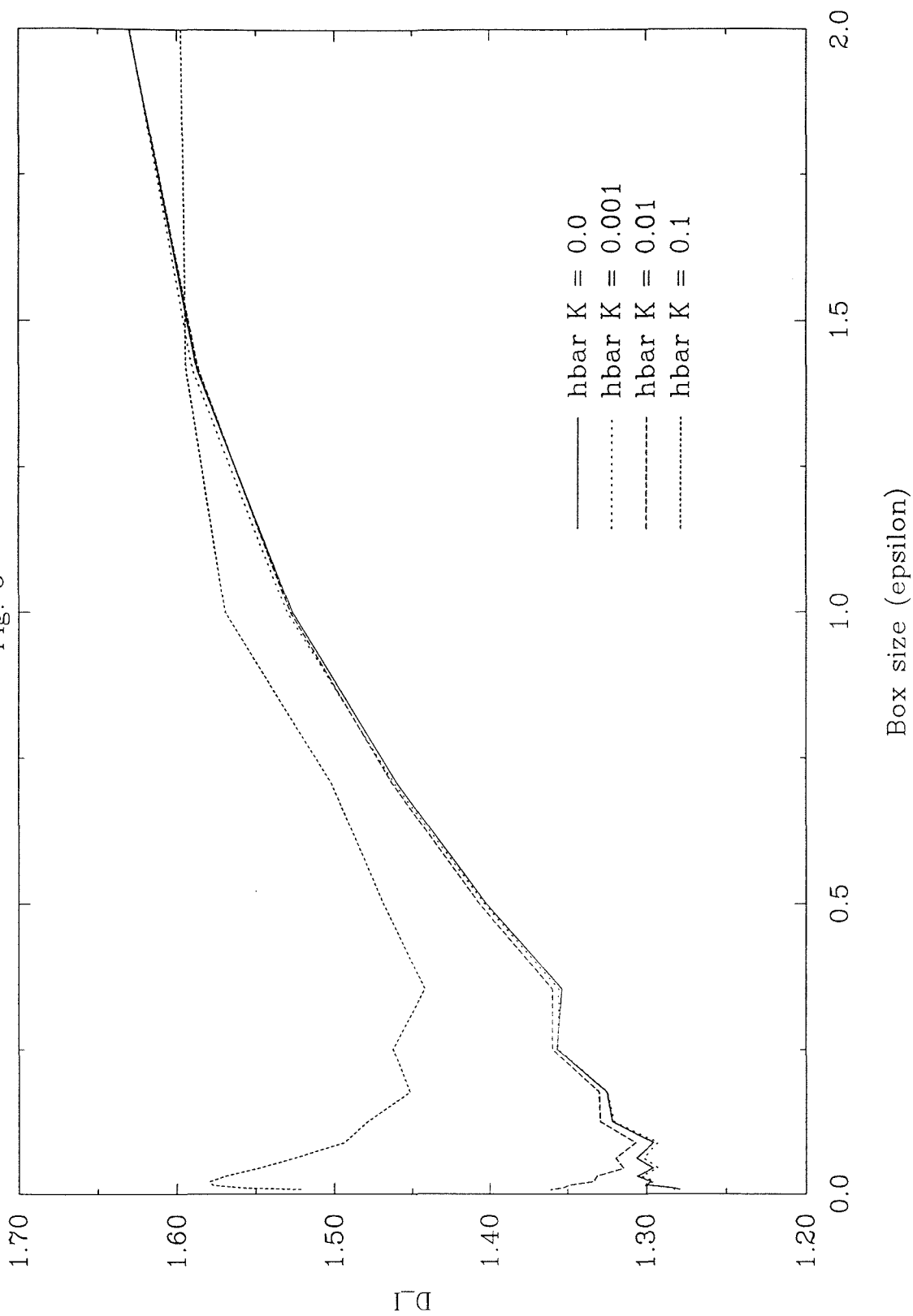
Lyapunov Exponent

Fig. 4

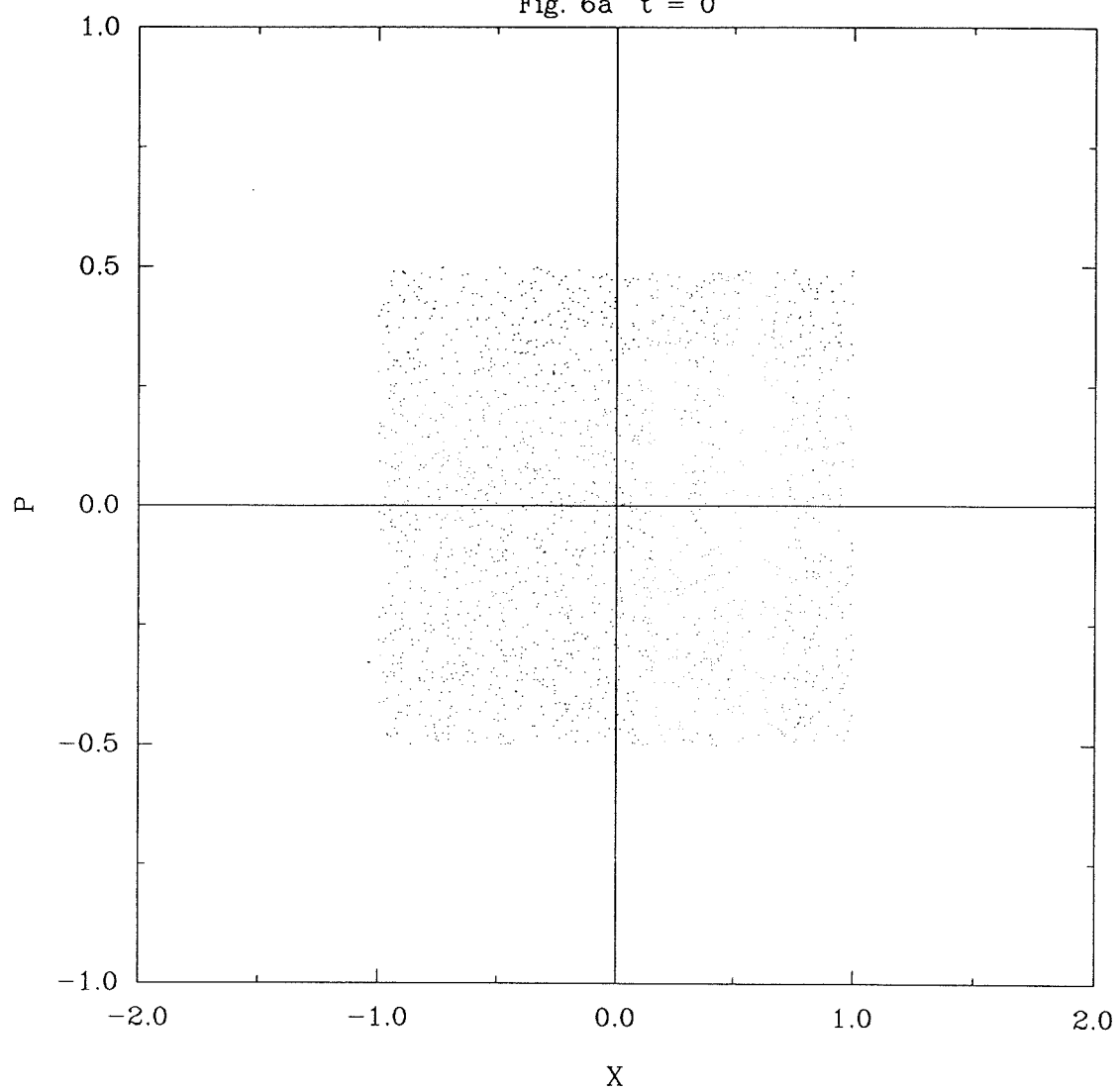


Information Dimension of attractor

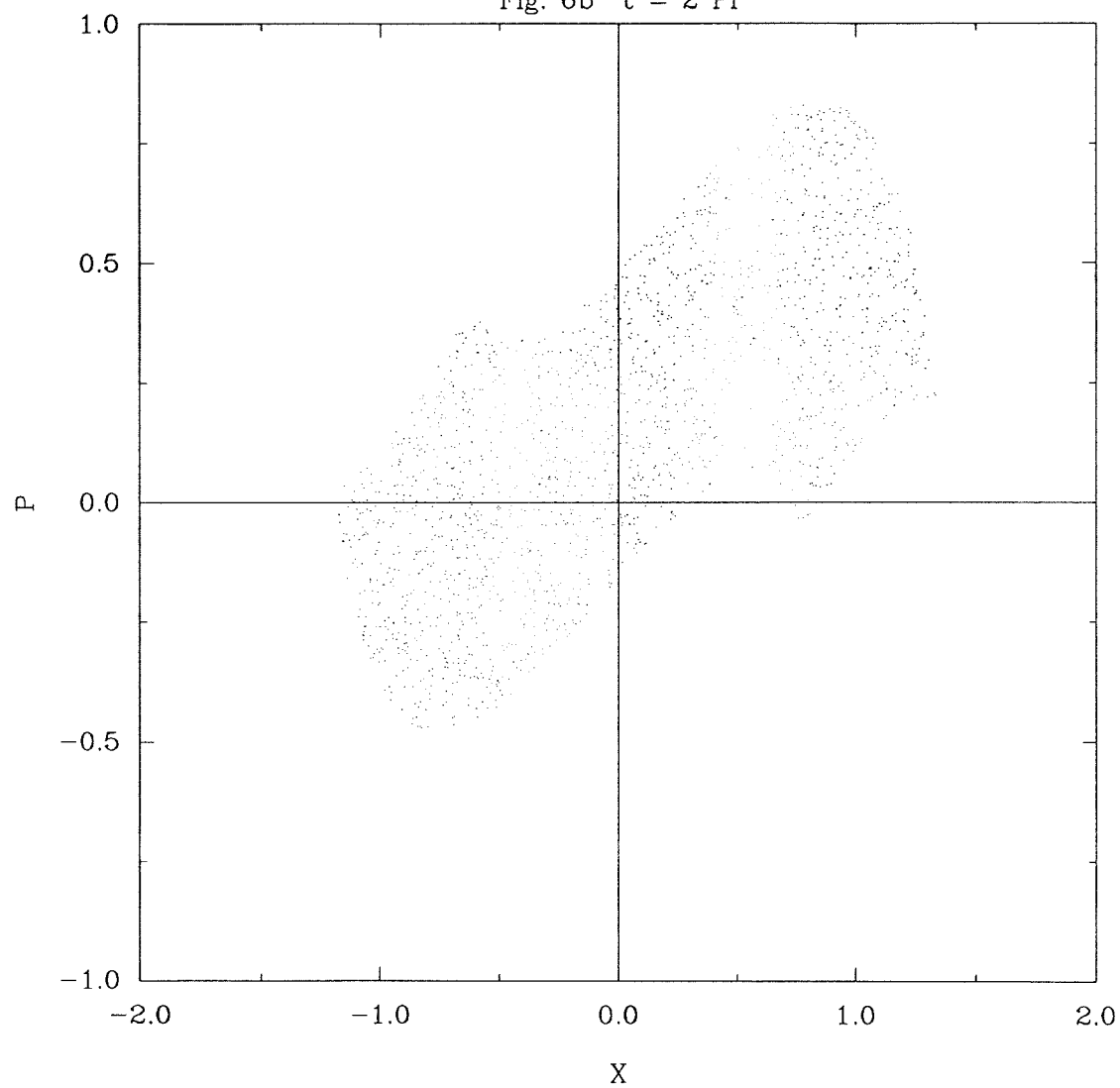
Fig. 5



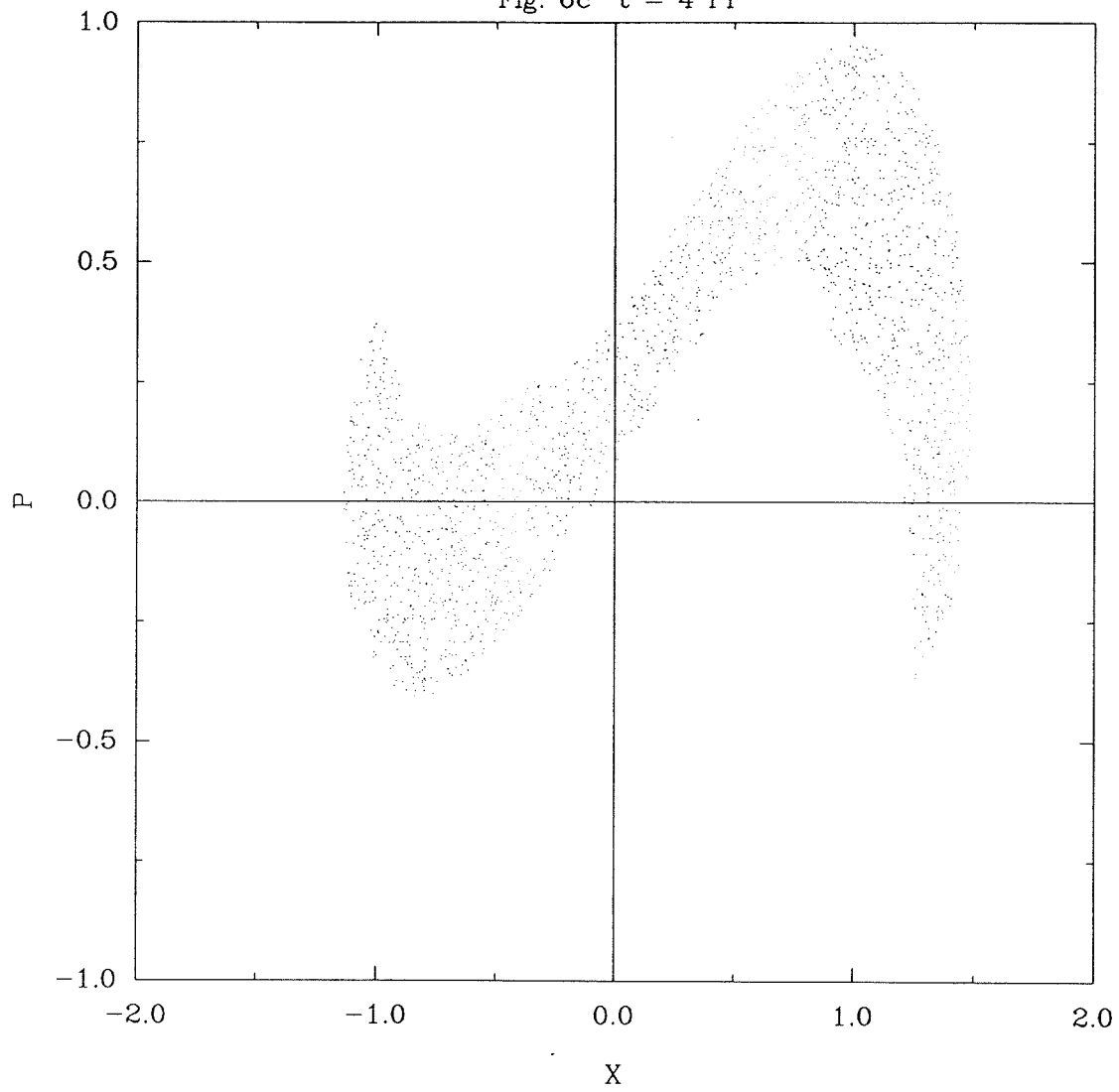
Evolution of Damped, Driven Duffing Oscillator

Fig. 6a $t = 0$ 

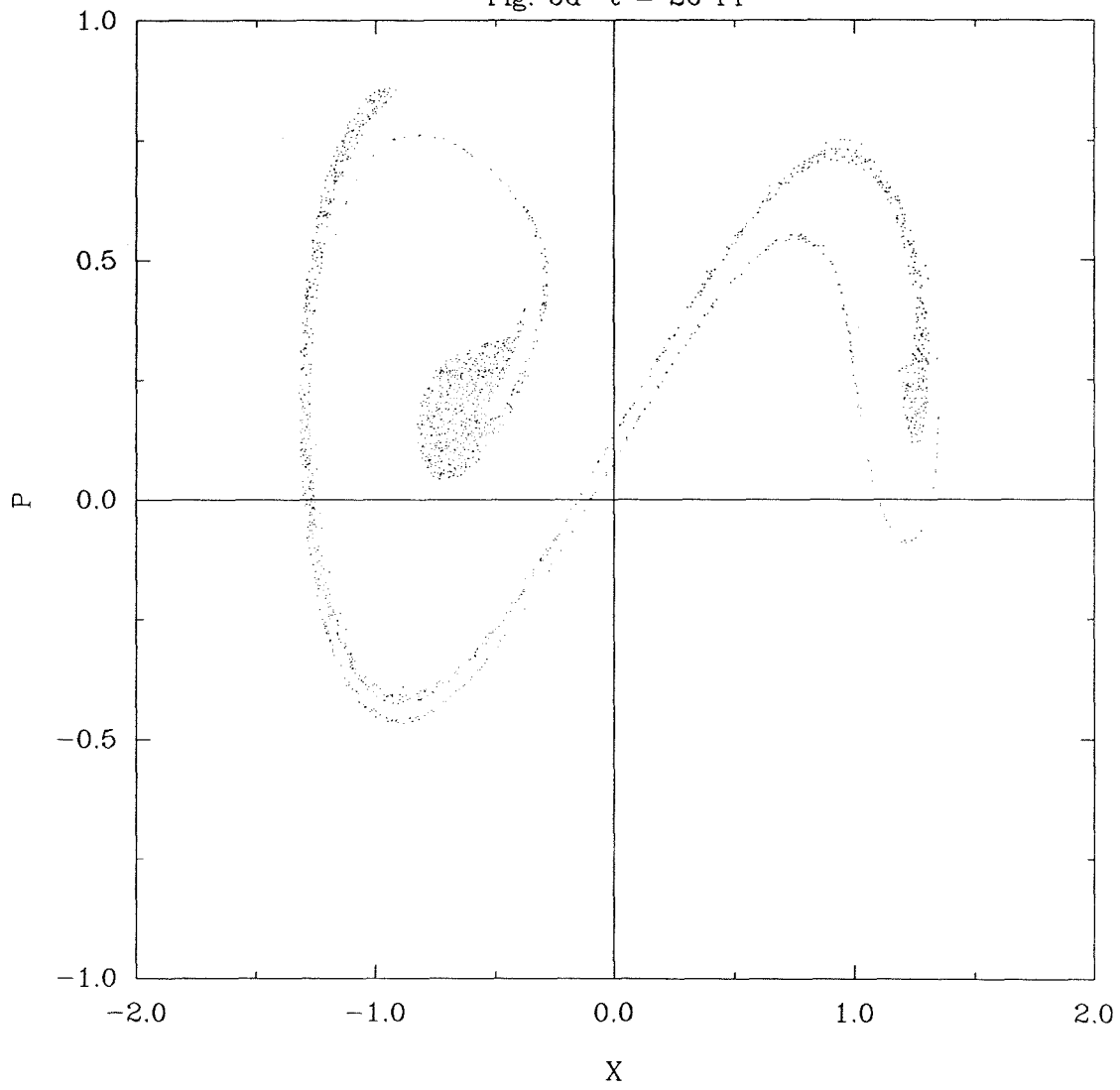
Evolution of Damped, Driven Duffing Oscillator

Fig. 6b $t = 2\pi$ 

Evolution of Damped, Driven Duffing Oscillator

Fig. 6c $t = 4 \text{ Pi}$ 

Evolution of Damped, Driven Duffing Oscillator

Fig. 6d $t = 20 \text{ Pi}$ 

Chapter III

The Decoherence of Phase Space Histories

The Decoherence of Phase Space Histories

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(April 27, 1994)

Abstract

In choosing a family of histories for a system, it is often convenient to choose a succession of locations in phase space, rather than configuration space, for comparison to classical histories. Although there are no good projections onto phase space, several approximate projections have been used in the past; three of these are examined in this paper. Expressions are derived for the probabilities of histories containing arbitrary numbers of projections into phase space, and the conditions for the decoherence of these histories are studied.

I. INTRODUCTION

A great deal of work has been done recently on the use of the decoherence formalism to describe quantum mechanical systems [1–8]. These systems can be described in terms of *decoherent histories*, which can be assigned probabilities obeying classical probability sum rules. While, in principle, a history could be described in terms of any set of variables, most of the work has focussed on histories of particles in configuration space. The simplest and most fine-grained such history is just the classical trajectory of a particle, specifying its exact position at every moment in time. Such histories do not decohere, however. Instead, one must consider considerably coarse-grained histories, in which a position is given only at certain discrete times, and only within certain finite intervals. A history can then be specified by a string of indices α_i , stating which interval the particle is in at time t_i .

Another important class of histories, though, would be descriptions of a system as being in cells of phase space at successive points in time. A small amount of work has been done on this subject [9–13], but they have not been tackled in full generality.

A. The decoherence functional

The decoherence functional is a functional on *pairs of histories* of a quantum mechanical system. One simple description of the functional (though not the most general) has the form:

$$D[\alpha, \alpha'] = \text{Tr} \left\{ P_{\alpha_n}^n(t_n) \cdots P_{\alpha_1}^1(t_1) \rho P_{\alpha'_1}^1(t_1) \cdots P_{\alpha'_n}^n(t_n) \right\}. \quad (1.1)$$

In this expression, ρ is the initial density matrix of the system. The $P_{\alpha_i}^i(t_i)$ are Heisenberg *projection operators* onto Hilbert space. At each time t_i these projection operators represent different alternatives for the system. In terms of Schrödinger projections P , these time dependent projections can be written $P(t) = e^{iHt/\hbar} P e^{-iHt/\hbar}$. An exhaustive set of histories has an exhaustive set of alternatives at each time,

$$\sum_{\alpha_i} P_{\alpha_i}^i(t_i) = 1. \quad (1.2)$$

A particular choice of the $\{\alpha_i\}$ represents one particular history, which we will denote α for brevity. Thus, saying that a given history α occurs implies that alternative α_1 occurs at time t_1 , α_2 at time t_2 , and so forth.

The probability of a given history occurring is just given by the diagonal elements of $D[\alpha, \alpha']$:

$$p(\alpha) = D[\alpha, \alpha]. \quad (1.3)$$

In order for these histories to obey the classical probability sum rules, we must require that the set of histories *decoheres*. The usual requirement for this is that the off-diagonal terms of the decoherence function vanish,

$$D[\alpha, \alpha'] = 0, \quad \alpha \neq \alpha'. \quad (1.4)$$

This is actually a sufficient, but not a necessary condition for decoherence. All that is truly required is that the *real parts* of the off-diagonal terms vanish. Most physically decoherent systems, however, display this stronger form of decoherence; we will see this in the cases that we consider.

B. The Transition Matrix

The most common type of problem treated at present is that in which the variables are divided into a *system* and a *reservoir*, or *environment*. In this case one traces over the reservoir variables and is left with a reduced density matrix on only the system variables. Instead of the simple time evolution operator $e^{-iHt/\hbar}$, the system evolves according to a somewhat more complicated *transition matrix* or *propagator* \mathbf{T} . In terms of path integrals, this is

$$T(x_f, x'_f, t_f; x_i, x'_i, t_i) = \int \delta x \delta x' \exp \frac{i}{\hbar} \left\{ S[x(t)] - S[x'(t)] + W[x(t), x'(t)] \right\}, \quad (1.5)$$

where the integral is over all paths $x(t)$ and $x'(t)$ from t_i to t_f which begin at x_i and x'_i and end at x_f and x'_f , respectively. $S[x(t)]$ is the action of the system variables independent of the reservoir, and $W[x(t), x'(t)]$ is the Feynman-Vernon *influence phase* arising due to the interactions with the reservoir [14]. The model most commonly considered is one that was developed in the study of Brownian motion [15], in which a one-dimensional particle described by a single variable x interacts with an infinite bath of harmonic oscillators via a linear or weakly nonlinear potential, but this formalism is quite general.

The reduced density matrix $\rho(x; x')$ evolves straightforwardly:

$$\rho(x; x')|_{t_f} = \int dx_i dx'_i T(x_f, x'_f, t_f; x_i, x'_i, t_i) \rho(x_i; x'_i)|_{t_i}. \quad (1.6)$$

Thus, the decoherence functional can now be written

$$D[\alpha, \alpha'] = \text{Tr}_x \left\{ P_{\alpha_n}^n \mathbf{T}(\cdots \mathbf{T}(P_{\alpha_1}^1 \rho P_{\alpha'_1}^1) \cdots) P_{\alpha'_n}^n \right\}. \quad (1.7)$$

With the projections P being onto intervals of coordinate space, it is very easy to write the decoherence functional as a constrained path integral over x and x' . For phase-space projections, the form of the decoherence function is more complicated, as we shall see. While there are no true projectors onto cells of phase space (as there are for intervals of coordinate space), there are a number of approximate projectors, and we shall consider these one at a time.

In dealing with phase space, it is natural to consider other representations of the density matrix, most obviously the Wigner distribution:

$$w(X, p) = \frac{1}{\pi} \int d\xi e^{i\xi p/\hbar} \rho(X + \xi/2; X - \xi/2). \quad (1.8)$$

This distribution acts in many ways like a probability distribution in phase space, with the major exception that it can be negative in localized regions. The time evolution of $w(X, p)$ is also described by a transition matrix:

$$w(X_f, p_f)|_{t_f} = \int \int dX_i dp_i T_w(X_f, p_f, t_f; X_i, p_i, t_i) w(X_i, p_i)|_{t_i}. \quad (1.9)$$

$$T_w(X_f, p_f, t_f; X_i, p_i, t_i) = \int \int d\xi_f d\xi_i e^{i(p_f \xi_f - p_i \xi_i)/\hbar} \\ \times T(X_f + \frac{\xi_f}{2}, X_f - \frac{\xi_f}{2}, t_f; X_i + \frac{\xi_i}{2}, X_i - \frac{\xi_i}{2}, t_i), \quad (1.10)$$

where this T is the same transition matrix defined above (1.5). We shall see that the expressions for the probabilities of phase space histories are described very naturally in terms of Wigner distributions.

II. PROBABILITIES OF PHASE SPACE HISTORIES

As mentioned above, there are no true projections onto cells of phase space [10]. This is essentially a side-effect of the uncertainty principle, which prevents both x and p from being localized simultaneously. However, for cells larger in area than \hbar , we can approximate projections reasonably well.

In making these calculations, we will find that it is useful to work in terms of $w(X, P)$ and \mathbf{T}_w . For this we use the inverses of (1.8) and (1.10), namely

$$\rho(x; x') = \int dp e^{-ip(x-x')/\hbar} w((x+x')/2, p) \quad (2.1)$$

and

$$T(x_f, x'_f, t_f; x_i, x'_i, t_i) = \frac{1}{\pi} \int \int dp_i dp_f e^{ip_i(x_i - x'_i) - ip_f(x_f - x'_f)} \\ \times T_w((x_f + x'_f)/2, p_f, t_f; (x_i + x'_i)/2, p_i, t_i). \quad (2.2)$$

A. Consecutive X and P projections

While there are no good projections onto phase space cells, projections onto intervals in coordinate or momentum space are perfectly well-defined and straightforward. They are just

$$P_{x_i} = \int_{x_i - \Delta x/2}^{x_i + \Delta x/2} dx |x\rangle\langle x|, \quad (2.3a)$$

$$P_{p_i} = \int_{p_i - \Delta p/2}^{p_i + \Delta p/2} dp |p\rangle\langle p|. \quad (2.3b)$$

We can imagine using a projection to determine which interval of x the system is in, followed a short time later by a projection onto an interval of p . If we let the time between these two “measurements” go to zero, we can make use of the relation

$$\langle x|p\rangle = e^{ipx/\hbar}. \quad (2.4)$$

This is unsatisfactory in a number of ways. The most obvious is that x and p are non-commuting variables, so that P_x and P_p are also non-commuting. The order in which one makes these measurements matters, particularly if the intervals are fairly small (compared to \hbar). If we are interested in fairly large cells in phase space, this is of less importance; for classical and quasiclassical systems, this is often the case.

Measurements of this type were treated by Halliwell [10]. He also considered another type of two-projection measurement: a pair of successive position measurements, separated by a small time interval Δt , with the momentum determined by the *time of flight* between x_1 and x_2 . I have not considered this type of measurement, as it is ill-defined as $\Delta t \rightarrow 0$, and therefore requires non-trivial consideration of the system’s time evolution between the two position projections. For a system with complex dynamics, this is difficult.

The exact projections (2.3) used above are less convenient for the purposes of calculation, though they are certainly more correct. For ease of computation, therefore, it is customary to use approximate Gaussian projections,

$$P_{\bar{x}_i} = \frac{1}{\sqrt{\pi\Delta x}} \int_{-\infty}^{\infty} dx e^{-(x-\bar{x}_i)^2/\Delta x^2} |x\rangle\langle x|, \quad (2.5a)$$

$$P_{\bar{p}_i} = \frac{1}{\sqrt{\pi\Delta p}} \int_{-\infty}^{\infty} dp e^{-(p-\bar{p}_i)^2/\Delta p^2} |p\rangle\langle p|. \quad (2.5b)$$

Using these projections, the probabilities just reduce to a product of Gaussian integrals, which can, with a little algebra, be easily solved. Using the expressions (2.1) and (2.2) for a history with N measurements of cells of phase space centered on $(\bar{x}_1, \bar{p}_1), (\bar{x}_2, \bar{p}_2), \dots, (\bar{x}_N, \bar{p}_N)$, we get

$$\begin{aligned}
P_\alpha &= \frac{1}{(\pi\Delta x\Delta p)^{2N}} \int d\{X_i\}d\{p_i\}d\{z_i\}d\{k_i\}d\{z'_i\}d\{k'_i\}d\{\xi_i\} \\
&\times W(X_0, p_0)\delta(z_1 - (X_0 + \xi_0/2))\delta(z'_1 - (X_0 - \xi_0/2)) \\
&\times \exp\left[-\frac{(z_1 - \bar{x}_1)^2 + (z'_1 - \bar{x}_1)^2}{\Delta x^2} - \frac{(k_1 - \bar{p}_1)^2 + (k'_1 - \bar{p}_1)^2}{\Delta p^2}\right. \\
&\quad \left. + \frac{i}{\hbar}\left((k'_1 z'_1 - k_1 z_1) + X_1(k_1 - k'_1) + \xi_1(p_1 - k_1/2 - k'_1/2) - \xi_0 p_0\right)\right] \\
&\times T(X_2, p_2, t_2; X_1, p_1, t_1) \\
&\times \dots
\end{aligned} \tag{2.6}$$

The integrals over z , z' , k , k' , and ξ are all simple, and yield

$$\begin{aligned}
P_\alpha &= \frac{1}{(2\pi)^{N-1}} \int d\{X_i\}d\{p_i\}w(X_0, p_0) \\
&\times \exp\left[-\frac{2}{\Delta p^2}(p_1 - \bar{p}_1)^2 - \frac{2}{\Delta x^2}(X_0 - \bar{x}_1)^2 - \frac{\Delta x^2}{2\hbar^2}(p_0 - p_1)^2 - \frac{\Delta p^2}{2\hbar^2}(X_0 - X_1)^2\right] \\
&\times T(X_2, p_2, t_2; X_1, p_1, t_1) \\
&\times \exp\left[-\frac{2}{\Delta p^2}(p_3 - \bar{p}_2)^2 - \frac{2}{\Delta x^2}(X_2 - \bar{x}_2)^2 - \frac{\Delta x^2}{2\hbar^2}(p_2 - p_3)^2 - \frac{\Delta p^2}{2\hbar^2}(X_2 - X_3)^2\right] \\
&\times T(X_4, p_4, t_3; X_3, p_3, t_2) \\
&\times \dots \\
&\times \exp\left[-\frac{2}{\Delta x^2}(X_{2N-2} - \bar{x}_N)^2 - \frac{\Delta x^2(p_{2N-2} - \bar{p}_N)^2}{2[\hbar^2 + \Delta x^2\Delta p^2/4]}\right].
\end{aligned} \tag{2.7}$$

Note that the expression for the probability behaves very reasonably, i.e., the evolution after a “measurement” continues to be centered about the measured values of X and p , with a spread determined by the size of the phase space cell.

A measurement of p followed by a measurement of X produces an expression very similar to the above, and is readily evaluated by the same methods. The differences are chiefly notable when the cell size is small compared to \hbar .

B. Coherent state projections

The closest thing to a true projection onto a cell in phase space is probably the *coherent state* projection $|\bar{p}, \bar{x}\rangle\langle\bar{p}, \bar{x}|$ centered on (\bar{p}, \bar{x}) . While these are true

projections, they are not orthogonal:

$$|\langle \bar{p}, \bar{x} | \bar{p}', \bar{x}' \rangle|^2 = \exp[-(\bar{x} - \bar{x}')^2/2\sigma - \sigma(\bar{p} - \bar{p}')^2/2\hbar^2]. \quad (2.8)$$

Also, these states are overcomplete. Thus, phase space histories built from coherent states cannot be truly decoherent, and can only be even approximately decoherent if a discrete sample of them (e.g., the states corresponding to a lattice of points in phase space) is taken. In a coordinate basis we can represent a coherent state as

$$\langle x | \bar{p}, \bar{x} \rangle = \frac{1}{(\pi\sigma)^{1/4}} \exp\left[-\frac{(x - \bar{x})^2}{2\sigma} + \frac{i\bar{p}x}{\hbar}\right]. \quad (2.9)$$

This expression is useful in evaluating the probability of a coherent state history.

When we consider a history of N “measurements” in phase space using coherent state projections, we get an expression analogous to (2.7), which can (again) be solved for the probability:

$$\begin{aligned} P_\alpha &= \frac{4^N}{2\pi\sigma} \int d\{X_i\} d\{p_i\} w(X_0, p_0) \\ &\times \exp\left[-\frac{(X_0 - \bar{x}_1)^2}{\sigma} - \frac{(X_1 - \bar{x}_1)^2}{\sigma} - \frac{(p_0 - \bar{p}_1)^2\sigma}{\hbar^2} - \frac{(p_1 - \bar{p}_1)^2\sigma}{\hbar^2}\right] \\ &\times T(X_2, p_2, t_2; X_1, p_1, t_1) \\ &\times \dots \\ &\times T(X_{2N-2}, p_{2N-2}, t_N; X_{2N-3}, p_{2N-3}, t_{N-1}) \\ &\times \exp\left[-\frac{(X_{2N-2} - \bar{x}_N)^2}{\sigma} - \frac{(p_{2N-2} - \bar{p}_N)^2\sigma}{\hbar^2}\right]. \end{aligned} \quad (2.10)$$

The general behavior of the probabilities is very similar to that in the first case we considered, but even cleaner and easier to see. Coherent states are an excellent way of representing phase space histories.

There is one other kind of approximate projection that we could consider. It is not, in my opinion, a very attractive one, but it has been used in the literature, and so might as well be treated here. Consider approximate “projections” of the form

$$P_{(\bar{p}, \bar{x})} = \frac{1}{\pi\Delta p\Delta x} \int dp dx |p, x\rangle e^{-(p-\bar{p})^2/\Delta p^2 - (x-\bar{x})^2/\Delta x^2} \langle p, x|. \quad (2.11)$$

If we consider N measurements of this form, the probability becomes

$$\begin{aligned}
P_\alpha = & \frac{4\pi\sigma\Delta p\Delta x}{2\sigma + \sigma^2\Delta p^2 + \Delta x^2 + \sigma\Delta p^2\Delta x^2} \int d\{X_i\}d\{p_i\} w(X_0, p_0) \\
& \times \exp\left[-\frac{(X_0 - \bar{x}_1)^2 + (X_1 - \bar{x}_1)^2}{\sigma + \Delta x^2} - \sigma\frac{(p_0 - \bar{p}_1)^2 + (p_1 - \bar{p}_1)^2}{\hbar^2 + \sigma\Delta p^2}\right. \\
& \left. - (\sigma^2\Delta p^2/\hbar^2 + \Delta x^2 + \sigma\Delta p^2\Delta x^2/\hbar^2)\left(\frac{(X_0 - X_1)^2}{2\sigma(\sigma + \Delta x^2)} + \frac{(p_0 - p_1)^2}{2(\hbar^2 + \sigma\Delta p^2)}\right)\right] \\
& \times T(X_2, p_2, t_2; X_1, p_1, t_1) \\
& \times \dots \\
& \times T(X_{2N-2}, p_{2N-2}, t_N; X_{2N-3}, p_{2N-3}, t_{N-1}) \\
& \times \exp\left[-2\frac{\sigma(\sigma + \Delta x^2)(p_{2N-2} - \bar{p}_N)^2 + (\hbar^2 + \sigma\Delta p^2)(X_{2N-2} - \bar{x}_N)^2}{2\sigma\hbar^2 + \sigma^2\Delta p^2 + \Delta x^2\hbar^2 + \sigma\Delta p^2\Delta x^2}\right]. \quad (2.12)
\end{aligned}$$

Again, the same sort of qualitative behavior, but a much uglier expression.

III. DECOHERENCE OF PHASE SPACE HISTORIES

While the above expressions are highly intuitive in their qualitative behavior, we have (in a sense) been putting the cart before the horse. It is meaningless to assign a probability to a history without first being assured that the set of histories described is decoherent. There is nothing in the expressions above to prevent one from choosing extremely tiny cells in phase space, with areas small compared to \hbar ; yet such histories are certainly not decoherent, as they flagrantly violate the uncertainty principle.

Unfortunately, while we can write *expressions* for the probabilities without having to know much about the physics of the system (i.e., the actual behavior of the transition matrix \mathbf{T}), in order to actually calculate them, or to say much about decoherence, we need to know something about the path integrals.

Except in the case of quadratic systems, these integrals are not exactly solvable. Limited treatments of this case have been considered elsewhere [13,16]. Most interesting systems, however, include nonlinearities. This can be handled in one of three ways: numerically; in perturbation theory; or in the semiclassical limit, where

solutions are peaked about the “classical trajectory.” The first approach is robust, but does not lend itself to general arguments. While almost any approach will eventually have to be treated numerically to calculate actual values for probabilities or decoherence functional elements, one would hope to get a rough idea as to a systems behavior before invoking that numerical machinery. The second approach, perturbation theory, is the most commonly adopted. When the nonlinearities are weak, the path integrals can be approximated with considerable precision. Unfortunately, many interesting cases (e.g., chaotic systems) cannot be treated in this fashion; for them, their nonlinearities are intrinsically important. The last approach is limited to systems with sufficient mass and inertia to resist quantum fluctuations [3]. This is useful in considering either the classical limit of quantum systems, or in estimating quantum effects in otherwise classical systems, and is the approach we will adopt here.

Earlier work has concentrated on distinguished systems interacting with a large reservoir or environment whose degrees of freedom can be neglected. As has been shown, such systems give rise to decoherence functionals with probabilities peaked about classical trajectories. The transition matrix for such a system has the form (in the limit of a large thermal reservoir)

$$T(X_1, \xi_1, t_1; X_0, \xi_0, t_0) = \int \delta X \delta \xi \exp \frac{i}{\hbar} \left\{ - \int_{t_0}^{t_1} \left(M \ddot{X}(t) + dV/dX(X(t)) + 2M\gamma \dot{X}(t) - g(t) \right) \xi(t) dt + \frac{iM\gamma kT}{\hbar} \int_{t_0}^{t_1} \xi^2(t) dt + M\xi_1 \dot{X}_1 - M\xi_0 \dot{X}_0 + O(\xi^3) \right\}, \quad (3.1)$$

where X and ξ are variables defined by

$$X = \frac{1}{2}(x + x'), \quad (3.2a)$$

$$\xi = x - x', \quad (3.2b)$$

and the reservoir temperature is T . This is basically a toy system, consisting of a single one-dimensional particle of mass M moving in an arbitrary potential $V(X)$.

The interaction with the reservoir provides the dissipative term and a thermal noise; it is this noise which causes the system to decohere. $|\xi|$ is a measure of how far “off-diagonal” the decoherence functional is; for large $|\xi|$ it will clearly be strongly suppressed.

Since large ξ is suppressed, we can neglect the higher order terms in ξ with good accuracy. This makes the ξ path integral purely quadratic, and therefore solvable. Doing this integral yields

$$T(X_1, \xi_1, t_1; X_0, \xi_0, t_0) = \sqrt{\frac{\pi \hbar^2}{M\gamma kT}} \int \delta X \exp \left\{ -\frac{1}{M\gamma kT} \int_{t_0}^{t_1} \left(M\ddot{X}(t) + dV/dX(X(t)) + 2M\gamma\dot{X}(t) - g(t) \right)^2 dt + (i/\hbar)M(\xi_1\dot{X}_1 - \xi_0\dot{X}_0) \right\}, \quad (3.3)$$

which is clearly peaked about the solution to the classical equation of motion

$$M\ddot{X}(t) + dV/dX(X(t)) + 2M\gamma\dot{X}(t) = g(t), \quad (3.4)$$

more and more strongly in the limit of large M .

Let $X_{cl}(t)$ be the solution to the above classical equation with the boundary conditions $X_{cl}(t_0) = X_0$ and $X_{cl}(t_1) = X_1$. We can then define a new variable $\eta(t)$

$$\eta(t) = X(t) - X_{cl}(t). \quad (3.5)$$

Clearly $\eta(t)$ has boundary conditions $\eta(t_1) = \eta(t_0) = 0$. As M becomes large, we can treat $\eta(t)$ as a small deviation, and approximate the path integral as

$$T(X_1, \xi_1, t_1; X_0, \xi_0, t_0) = \sqrt{\frac{\pi \hbar^2}{M\gamma kT}} \int \delta \eta \exp \left\{ -\frac{1}{M\gamma kT} \int_{t_0}^{t_1} \left(M\ddot{\eta}(t) + d^2V/dX^2(X_{cl}(t))\eta(t) + 2M\gamma\dot{\eta}(t) \right)^2 dt + (i/\hbar)M(\xi_1\dot{\eta}_1 - \xi_0\dot{\eta}_0) + (i/\hbar)M(\xi_1\dot{X}_{cl}(t_1) - \xi_0\dot{X}_{cl}(t_0)) \right\}. \quad (3.6)$$

This path integral is quadratic in η and therefore solvable, at least in principle. This principle runs into a few problems in practice. It assumes that you know $X_{cl}(t)$

as a function of the boundary conditions. This is true only in very simple cases. In chaotic cases, it may be difficult to determine this function even numerically. Also, this integral contains (in essence) 4th derivatives of η , which complicate the calculation in some ways. Still, by making a few assumptions about the behavior of $X_{\text{cl}}(t)$, we can still extract some useful information from this expression.

Since for the purposes of determining decoherence we are really only interested in the ξ dependence of \mathbf{T} , it is straightforward, albeit tedious, to show that

$$T(X_1, \xi_1, t_1; X_0, \xi_0, t_0) = K \exp \left\{ -\frac{M\gamma kT}{\hbar^2} (\lambda_1 \xi_0^2 + \lambda_2 \xi_0 \xi_1 + \lambda_3 \xi_1^2) + (i/\hbar) M (\xi_1 \dot{X}_{\text{cl}}(t_1) - \xi_0 \dot{X}_{\text{cl}}(t_0)) \right\}. \quad (3.7)$$

Note that $\lambda_i = \lambda_i(X_1, X_0, t_1, t_0)$ and $K = K(X_1, X_0, t_1, t_0)$. These functions are not especially easy to calculate, but can be computed numerically if necessary. Simple calculations along those lines seem to show that $\lambda_i/(t_1 - t_0)$ is relatively constant for $(t_1 - t_0)$ short compared to the dynamical time of the system and long compared to the decoherence time, at least for high-probability paths. For longer times, comparable to the dynamic timescale of the system in question, the λ_i vary enormously in magnitude; numerical results showed a variability of more than four orders of magnitude, though most results for $\lambda_1/(t_1 - t_0)$ and $\lambda_3/(t_1 - t_0)$ clustered around certain values, and never became negligibly small. $\lambda_1 \xi_0^2 + \lambda_2 \xi_0 \xi_1 + \lambda_3 \xi_1^2$ is, in any case, always a strictly non-negative quantity. For details of these calculations, see the Appendix.

If we make the (admittedly highly questionable) assumption that the λ_i are roughly constant for constant $(t_1 - t_0)$, then we can estimate the level of decoherence achievable with phase-space projections. For simplicity, we will only look at the projections at a single time:

$$\text{Tr} \left\{ \cdots \mathbf{T}(P_{\alpha_i}^i(t_i) \mathbf{T}(\cdots) P_{\alpha_i'}^i(t_i)) \cdots \right\} \quad (3.8)$$

A. Consecutive X and P Projections

Using the approximate X and P projections described in (2.5b) above, we can examine decoherence by looking at the off-diagonal elements, where the projections are centered on (\bar{x}, \bar{p}) and (\bar{x}', \bar{p}') respectively. A single pair of projections at time t_i will multiply the decoherence functional by a factor

$$\begin{aligned} & \int dX_1 dX_2 d\xi_1 d\xi_2 dp dp' T(X_3, \xi_3, t_{i+1}; X_2, \xi_2, t_i) \\ & \times \exp \left\{ -\frac{(X_1 + \xi_1/2 - \bar{x})^2}{\Delta x^2} - \frac{(X_1 - \xi_1/2 - \bar{x}')^2}{\Delta x^2} - \frac{(p - \bar{p})^2}{\Delta p^2} - \frac{(p' - \bar{p}')^2}{\Delta p^2} \right. \\ & \left. + i(p - p')(X_2 - X_1) + \frac{i}{2}(p + p')(\xi_2 - \xi_1) \right\} \\ & \times T(X_1, \xi_1, t_i; X_0, \xi_0, t_{i-1}) \end{aligned} \quad (3.9)$$

$$\begin{aligned} & \sim \int dX_1 dX_2 K^2 \exp \left\{ -\left(\frac{(X_1 - \bar{x})^2 + (X_1 - \bar{x}')^2 + (\Delta p^2 \Delta x^2 / 2 \hbar^2)(X_2 - X_1)^2}{\Delta x^2} \right) \right. \\ & + \left(\frac{(8\hbar^2 M \gamma k T \lambda + \hbar^2 \Delta p^2)(\bar{x} - \bar{x}')^2}{2\Delta x^2 (8\hbar^2 M \gamma k T \lambda + \hbar^2 \Delta p^2 + 16(M \gamma k T \lambda)^2 \Delta x^2 + 4M \gamma k T \lambda \Delta p^2 \Delta x^2)} \right) \\ & - \left(\frac{(\hbar^2 + 2M \gamma k T \lambda \Delta x^2)[(p_2 - \bar{p})^2 + (p_2 - \bar{p}')^2]}{8\hbar^2 M \gamma k T \lambda + \hbar^2 \Delta p^2 + 16(M \gamma k T \lambda)^2 \Delta x^2 + 4M \gamma k T \lambda \Delta p^2 \Delta x^2} \right) \\ & - \left(\frac{(2M \gamma k T \lambda \Delta x^2)[(p_1 - \bar{p})^2 + (p_1 - \bar{p}')^2]}{8\hbar^2 M \gamma k T \lambda + \hbar^2 \Delta p^2 + 16(M \gamma k T \lambda)^2 \Delta x^2 + 4M \gamma k T \lambda \Delta p^2 \Delta x^2} \right) \\ & + \left(\frac{(\hbar^2 + 4M \gamma k T \lambda \Delta x^2)(\bar{p} - \bar{p}')^2 - \Delta p^2 (p_2 - p_1)^2}{2(8\hbar^2 M \gamma k T \lambda + \hbar^2 \Delta p^2 + 16(M \gamma k T \lambda)^2 \Delta x^2 + 4M \gamma k T \lambda \Delta p^2 \Delta x^2)} \right) \\ & - i \left(\frac{(4\hbar M \gamma k T \lambda)(\bar{x} - \bar{x}')(\bar{p} + \bar{p}' - 2p_1)}{8\hbar^2 M \gamma k T \lambda + \hbar^2 \Delta p^2 + 16(M \gamma k T \lambda)^2 \Delta x^2 + 4M \gamma k T \lambda \Delta p^2 \Delta x^2} \right) \\ & \left. + \frac{i(\bar{p} - \bar{p}')(X_2 - X_1)}{\hbar} \right\}, \end{aligned} \quad (3.10)$$

where p_1 and p_2 are $M\dot{X}_{cl}$ for boundary conditions $\{X(t_{i-1}) = X_0, X(t_i) = X_1\}$ and $\{X(t_i) = X_2, X(t_{i+1}) = X_3\}$ respectively.

A formidable expression indeed! One can, with difficulty, see that in general this factor will be suppressed for off-diagonal terms. If we simplify matters by taking the semiclassical, high-temperature limit, an examination of the real terms of the exponent show that for $|\bar{x} - \bar{x}'|^2 \sim \delta x^2$ this expression is suppressed by a minimum factor of

$$\exp\left\{-\frac{\delta x^2}{\Delta x^2}\left(\frac{1}{2}-\frac{\hbar^2}{4M\gamma kT\lambda\Delta x^2}\right)\right\}.$$

A similar examination of the real p terms gives no similar comfort, for we find that there the minimum level of suppression is none at all! This doesn't mean that histories with differing p 's do not decohere; the last imaginary term in the exponent oscillates extremely rapidly, and will tend to suppress all off-diagonal terms as X_1 and X_2 are integrated over. This will work, in general, if $|\bar{p}-\bar{p}'|^2 \sim \delta p^2$ is large compared to Δp^2 and $\hbar^2/\Delta x^2$. To suppress the off-diagonal terms of the decoherence functional by a factor ϵ , where $\epsilon \ll 1$, we must have

$$\begin{aligned}\delta x^2 &\geq -\Delta x^2 \ln \epsilon, \\ \delta p^2 &\geq -\Delta p^2 \ln \epsilon, \\ \Delta x^2 \Delta p^2 &\geq \hbar^2.\end{aligned}\tag{3.11}$$

B. Coherent state projections

The results from coherent state projections are similar, but somewhat cleaner and easier to see. In this case the factor from the projections at one time t_i goes as

$$\begin{aligned}\int dX_1 dX_2 K^2 \exp\left\{-\left(\frac{(X_1 - \bar{x})^2 + (X_1 - \bar{x}')^2 + (X_2 - \bar{x})^2 + (X_2 - \bar{x}')^2}{2\sigma}\right)\right. \\ \left. + \left(\frac{\hbar^2(\bar{x} - \bar{x}')^2}{2\hbar^2\sigma + 8\hbar M\gamma kT\lambda\sigma^2}\right)\right. \\ \left. - \left(\frac{(p_1 - \bar{p})^2 + (p_1 - \bar{p}')^2 + (p_2 - \bar{p})^2 + (p_2 - \bar{p}')^2 - (\bar{p} - \bar{p}')^2}{2\hbar^2/\sigma + 8\hbar M\gamma kT\lambda}\right)\right. \\ \left. - \frac{i\hbar(\bar{x} - \bar{x}')(p_2 - p_1)}{\hbar^2 + 4M\gamma kT\lambda\sigma} + \frac{i(\bar{p} - \bar{p}')(X_2 - X_1)}{\hbar}\right\}.\end{aligned}\tag{3.12}$$

Here again, we see that in the semiclassical limit this reduces to a minimal level of suppression

$$\exp\left\{-\frac{\delta x^2}{\sigma}\left(\frac{1}{2}-\frac{\hbar^2}{8M\gamma kT\lambda\sigma}\right)\right\}$$

for the x terms, and that δp^2 must be large compared to $\Delta p^2 \geq \hbar^2/\sigma$.

IV. CONCLUSIONS

While no set of approximate phase space projections treated in this paper is completely satisfactory, they do serve to illustrate certain traits that phase space histories should possess. Highly discontinuous trajectories are suppressed, and as one goes to the semiclassical limit, the probabilities of histories become peaked about the classical solutions. While precise statements about decoherence are hard to make, given the difficulty of solving the problem for highly general systems, rough arguments indicate that the size of phase space cells needed for decoherence is much larger than that naively indicated by the uncertainty principle ($\Delta x \Delta p \sim \hbar$).

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APPENDIX: PATH INTEGRAL FOR THE TRANSITION MATRIX

The path integral in (3.6) is somewhat unusual in that it has only two boundary conditions ($\eta(t_0) = \eta(t_1) = 0$) for an integrand with four derivatives! Thus, the usual prescription for solving quadratic path integrals is not immediately applicable.

This procedure can still be used, however, by the simple expedient of imposing two more boundary conditions, $\dot{\eta}(t_0) = v_0$ and $\dot{\eta}(t_1) = v_1$, and solving the path integral, then integrating the result over all values of v_0 and v_1 .

The path integral to be solved is then

$$\int \delta\eta \exp\left\{S[\eta(t)] + i(M/\hbar)(\xi_1 v_1 - \xi_0 v_0)\right\} = F(t_1, t_0) \exp\{S_{\text{cl}} + i(M/\hbar)(\xi_1 v_1 - \xi_0 v_0)\}, \quad (\text{A1})$$

where

$$S[\eta(t)] = -\frac{1}{M\gamma kT} \int_{t_0}^{t_1} \varepsilon^2(t) dt, \quad (\text{A2})$$

$$\varepsilon(t) = M(\ddot{\eta} + 2\gamma\dot{\eta} + f(t)\eta), \quad (\text{A3})$$

$$f(t) = \frac{1}{M} \frac{d^2 V}{dX^2}(X_{\text{cl}}(t)), \quad (\text{A4})$$

and $F(t_1, t_0)$ is an undetermined multiplier independent of the boundary conditions of η . (Of course, since $f(t)$ is defined in terms of $X_{\text{cl}}(t)$, this whole solution is dependent on X_0 and X_1 . This dependence is complicated, as we will see.)

The classical action S_{cl} is the action of the path $\eta(t)$ that obeys the classical equation of motion. For

$$S = \int_{t_0}^{t_1} L(\eta, \dot{\eta}, \ddot{\eta}) dt$$

the Euler-Lagrange equation is

$$\frac{d^2}{dt^2} \left(\frac{\partial L}{\partial \ddot{\eta}} \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\eta}} \right) + \left(\frac{\partial L}{\partial \eta} \right) = 0. \quad (\text{A5})$$

Plugging in our definitions for $S[\eta(t)]$ and $\varepsilon(t)$, we get the equations

$$\ddot{\eta} + 2\gamma\dot{\eta} + f(t)\eta = \varepsilon(t)/M, \quad (\text{A6a})$$

$$\ddot{\varepsilon} - 2\gamma\dot{\varepsilon} + f(t)\varepsilon = 0. \quad (\text{A6b})$$

The first of these follows from the definition (A3), the second from (A5).

The solution to the first equation is

$$\varepsilon(t) = M e^{\gamma t} (A a_1(t) + B a_2(t)), \quad (\text{A7})$$

where a_1 and a_2 are two independent solutions of the equation

$$\ddot{a} = (\gamma^2 - f(t))a. \quad (\text{A8})$$

This equation is not easily solved analytically in most cases. For slowly varying $f(t)$ one can approximate the solution

$$a(t) \approx \exp\left\{\pm \int_{t_0}^t \sqrt{\gamma^2 - f(s)} ds\right\}; \quad (\text{A9})$$

in any case, (A8) is readily solvable numerically. While any independent boundary conditions will work for a_1 and a_2 , a convenient choice is

$$\begin{aligned} a_1(0) &= 1, \dot{a}_1(0) = 0; \\ a_2(0) &= 0, \dot{a}_2(0) = 1. \end{aligned} \quad (\text{A10})$$

This is then plugged into the equation for η to give the solution

$$\begin{aligned} \eta(t) &= e^{-\gamma t}(C a_1(t) + D a_2(t)) \\ &+ (e^{-\gamma t}/M) \int_{t_0}^t \left(\frac{a_1(t)a_2(s) - a_2(t)a_1(s)}{\dot{a}_1(s)a_2(s) - \dot{a}_2(s)a_1(s)} \right) \varepsilon(s) e^{2\gamma s} ds. \end{aligned} \quad (\text{A11})$$

Imposing the boundary conditions on η then gives us a set of equations involving A , B , C , and D :

$$C = 0, \quad (\text{A12a})$$

$$D = v_0, \quad (\text{A12b})$$

$$v_0 a_2(t_1) + A(a_1(t_1)I_2 - a_2(t_1)I_1) + B(a_1(t_1)I_3 - a_2(t_1)I_2) = 0, \quad (\text{A12c})$$

$$v_0 \dot{a}_2(t_1) + A(\dot{a}_1(t_1)I_2 - \dot{a}_2(t_1)I_1) + B(\dot{a}_1(t_1)I_3 - \dot{a}_2(t_1)I_2) = v_1 e^{\gamma t_1}, \quad (\text{A12d})$$

where

$$I_1 = \int_{t_0}^{t_1} \frac{a_1(t)^2 e^{2\gamma t} dt}{\dot{a}_1(t)a_2(t) - \dot{a}_2(t)a_1(t)}, \quad (\text{A13})$$

$$I_2 = \int_{t_0}^{t_1} \frac{a_1(t)a_2(t) e^{2\gamma t} dt}{\dot{a}_1(t)a_2(t) - \dot{a}_2(t)a_1(t)}, \quad (\text{A14})$$

$$I_3 = \int_{t_0}^{t_1} \frac{a_2(t)^2 e^{2\gamma t} dt}{\dot{a}_1(t)a_2(t) - \dot{a}_2(t)a_1(t)}. \quad (\text{A15})$$

Again, these integrals are usually only solvable numerically.

Solving for A and B then yields

$$\begin{aligned} A &= \frac{(\dot{a}_1(t_1)a_2(t_1) - a_1(t_1)\dot{a}_2(t_1))I_3v_0 + e^{\gamma t_1}(a_1(t_1)I_3 - a_2(t_1)I_2)v_1}{(a_1(t_1)\dot{a}_2(t_1) - \dot{a}_1(t_1)a_2(t_1))(I_2^2 - I_1I_3)} \\ &= A_0v_0 + A_1v_1, \end{aligned} \quad (\text{A16})$$

$$\begin{aligned} B &= \frac{-(\dot{a}_1(t_1)a_2(t_1) - a_1(t_1)\dot{a}_2(t_1))I_2v_0 - e^{\gamma t_1}(a_1(t_1)I_2 - a_2(t_1)I_1)v_1}{(a_1(t_1)\dot{a}_2(t_1) - \dot{a}_1(t_1)a_2(t_1))(I_2^2 - I_1I_3)} \\ &= B_0v_0 + B_1v_1. \end{aligned} \quad (\text{A17})$$

Plugging these results back into the definitions of $\varepsilon(t)$ and $S[\eta(t)]$ gives a value for the path integral

$$\begin{aligned} \exp\left\{S_{cl} + (iM/\hbar)(\xi_1v_1 - \xi_0v_0)\right\} &= \\ \exp\left\{-\frac{M}{\gamma kT}(A^2I_4 + 2ABI_5 + B^2I_6) + i(M/\hbar)(\xi_1v_1 - \xi_0v_0)\right\}, \end{aligned} \quad (\text{A18})$$

where

$$I_4 = \int_{t_0}^{t_1} a_1(t)^2 e^{2\gamma t} dt, \quad (\text{A19})$$

$$I_5 = \int_{t_0}^{t_1} a_1(t)a_2(t)e^{2\gamma t} dt, \quad (\text{A20})$$

$$I_6 = \int_{t_0}^{t_1} a_2(t)^2 e^{2\gamma t} dt. \quad (\text{A21})$$

Clearly, the exponent is quadratic in v_0 and v_1 . Integrating over these two boundary conditions, we perform two gaussian integrals and arrive at our final result

$$\begin{aligned} \int dv_0 dv_1 \int \delta\eta \exp\left\{S[\eta(t)] + i(M/\hbar)(\xi_1v_1 - \xi_0v_0)\right\} &= \\ K \exp\left\{-\frac{M\gamma kT}{\hbar^2}(\lambda_1\xi_0^2 + \lambda_2\xi_0\xi_1 + \lambda_3\xi_1^2)\right\}. \end{aligned} \quad (\text{A22})$$

where

$$\lambda_1 = \frac{A_1^2I_4 + 2A_1B_1I_5 + B_1^2I_6}{4(A_0B_1 - A_1B_0)^2(I_5^2 - I_4I_6)}, \quad (\text{A23a})$$

$$\lambda_2 = \frac{A_0A_1I_4 + (A_1B_0 + A_0B_1)I_5 + B_0B_1I_6}{2(A_0B_1 - A_1B_0)^2(I_5^2 - I_4I_6)}, \quad (\text{A23b})$$

$$\lambda_3 = \frac{A_0^2 I_4 + 2A_0 B_0 I_5 + B_0^2 I_6}{4(A_0 B_1 - A_1 B_0)^2 (I_5^2 - I_4 I_6)}, \quad (\text{A23c})$$

$$K = \frac{\pi \gamma k T}{M \sqrt{(A_1 B_0 - A_0 B_1)^2 (I_4 I_6 - I_5^2)}} F(t_1, t_0). \quad (\text{A23d})$$

Clearly, these quantities are dependent on X_0 and X_1 : $\lambda_i = \lambda_i(X_0, X_1, t_0, t_1)$ and $K = K(X_0, X_1, t_0, t_1)$. In principle, K can be calculated, but in practice it is not necessary to do so in order to make arguments about decoherence. It would be necessary to do so in order to actually compute the probability of a history.

The derivations in this section, tedious as they are, can nevertheless be readily automated. Once one determines the classical trajectory X_{cl} which corresponds to the boundary conditions X_0 and X_1 , determining the λ_i numerically is straightforward. I have used this technique to examine the values of the λ_i for the forced, damped Duffing oscillator model. For long times ($t_1 - t_0$) the λ_i varied enormously in magnitude as a function of X_0 and X_1 . For times short compared to the dynamical time of the system, however, the λ_i were nearly linear: $\lambda/(t_1 - t_0) \approx 0.085$. For some boundary conditions they might become considerably larger, but in what was, admittedly, not an exhaustive sampling, none got much smaller.

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