STABILITY OF SPHERICALLY SYMMETRIC, CHARGED BLACK HOLES

and

MULTIPOLE MOMENTS FOR STATIONARY SYSTEMS

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ABSTRACT

This dissertation is written in two parts. Part I deals with the question of stability of a spherically symmetric, charged black hole against scalar, electromagnetic, and gravitational perturbations. It consists of two papers written in collaboration with Igor D. Novikov, Vernon D. Sandberg and A. A. Starobinsky. In these papers we describe the dynamical evolution of these perturbations on the interior of a Reissner-Nordstrom black hole. The instability of the hole's Cauchy horizon is discussed in detail in terms of the energy densities of the test fields as measured by a freely falling observer approaching the Cauchy horizon. We conclude that the Cauchy horizon of the analytically extended Reissner-Nordstrom solution is highly unstable and not a physical feature of a realistic gravitational collapse. Part II of this dissertation addresses two problems closely connected with multipole structure of stationary, asymptotically flat spacetimes. It consists of two papers written in collaboration with Kip S. Thorne despite the fact that his name does not appear on one of them. The first one (Paper III in this thesis) shows the equivalence of the moments defined by Kip S. Thorne and the moments defined by Robert Geroch and Richard Hansen. The second (Paper IV in this thesis) proves a conjecture by Kip S. Thorne: In the limit of "slow" motion, general relativistic gravity produces no changes whatsoever in the classical Euler equations of rigid body motion. We prove this conjecture by giving an algorithm for generating rigidly rotating solutions of Einstein's equations from nonrotating, static solutions.

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1. Introduction

A. Part I: Stability of the Interior of a Charged Black Hole

"Black hole" solutions of Einstein's equations are highly symmetric solutions. For example, the Schwarzschild solution is spherically symmetric, static, uncharged and asymptotically flat; the Reissner-Nordstrom solution is again spherically symmetric, static, charged and asymptotically flat; the Kerr solution is axisymmetric, stationary, uncharged and asymptotically flat; the Kerr-Newman solution is axisymmetric, stationary, charged and asymptotically flat. Each of these black holes has a "surface" called the event horizon outside of which is supposed to be the "real" world. What happens inside the event horizon depends on what kind of a black hole one is falling into. In the case of a Schwarzschild black hole, anybody who crosses the event horizon is never to be seen again by the people who live outside the hole. Depending on the size of the hole, he might have a reasonable amount of time to live before he eventually gets crushed by the ever-so-cruel singularity. The other black holes seem to offer better choices than simply being crushed. An observer (Physicist? Maybe. In any case, a rather adventurous fellow...) who takes a dive into a charged black hole in his rocket ship can actually maneuver his ship carefully to avoid hitting the singularity. What happens to him later is largely a matter of interpretation: He either comes back out to the universe he started his journey in, but he has the option of coming out at a different point than his starting point; or he ends up being in an entirely different universe (!). Wait, there is more! An examination of the "maximally extended" charged black hole solution reveals that there are actually an infinite number of such universes. Furthermore, these trips in and out of the hole take a very short time as measured by the clocks of the observer. Well, what more can one ask? This looks like the stuff dreams are made out of. This is THE "stargate" all science-fiction writers are after. You know what I am talking about: Instead of wasting time in interstellar travel by conventional means which might take tens of thousands of years to get anywhere, all you have to do is find one of these charged black holes and jump into it! Careful, you might hurt yourself...

By this time you probably think this graduate student is lost in darkness with no hope of recovery. Trust me, all I said is true as long as the solution above retains its perfect spherical symmetry. Not believing any of this far-out stuff, one might be inclined to think that keeping perfect spherical symmetry may not be the right thing to do. This charged black hole solution may be a "singularity" in the "solution space" which might turn into something else at the slightest provocation. It is rather unfortunate, but this actually is the case. The following lines are taken from the General Relativity Primer by Richard Price published in American Journal of Physics 50, 4 (1982): "If the more general black holes are primordial features of the universe, however, there are bridges across which timelike world lines can pass from one asymptotically flat region to another. It is perhaps a sign of the good taste built into the mathematical structure of General Relativity Theory that recent research indicates that such bridges are unstable and cannot exist even for a primordial black hole". The first part of this dissertation is a part of the "recent research" quoted above.

There are two ways to form black holes. One is through the collapse of a star which ran out of its fuel. The other kind of black hole forms shortly after the big bang due to inhomogeneities present in the spacetime, and not surprisingly, they are called primordial black holes. When a black hole is formed through the collapse of a star, not all of the "maximally extended" solution is present in the resultant black hole since the matter inside the star changes the regions of spacetime it occupies. Primordial black holes, on the other hand, are maximally extended, vacuum solutions of Einstein's equations. After the black hole is formed, these two solutions look identical to an observer located outside the hole. The difference between the two shows up when the observer decides to go down the hole and see what is going on.

In the case of Schwarzschild and Reissner-Nordstrom black holes, there are solutions of Einstein's equations with matter which will collapse to these black holes if they start out with the appropriate initial conditions. All of these solutions are spherically symmetric. A natural question to ask is whether the same spherically symmetric black hole will result if the collapsing star is not spherically symmetric initially. In the case of a primordial black hole, one can ask whether it is stable to small perturbations of the outside geometry. These perturbations can be induced, for example, by shining a small amount of light on the hole. In the case of Schwarzschild, Reissner-Nordstrom and Kerr black holes, these questions were answered during the 1970's. The answer is: The outside geometries of these black holes are stable to small perturbations. The answer is not known for the Kerr-Newman variety since nobody has managed to separate the perturbation equations yet.

In early 1978, Igor D. Novikov was visiting Caltech's relativity group. In a seminar he mentioned that he examined the interior geometry of a Schwarzschild black hole and found it stable against small perturbations. He suggested that we should do the same for the interior of a charged black hole since the geometry inside a charged black hole is far more interesting than the geometry inside an uncharged, spherically symmetric hole. We completed the work in early 1979 with the conclusion that the interior geometry of a spherically symmetric, charged black hole is unstable against small perturbations. In what follows, I will summarize the highlights of the first two papers without going into detailed mathematical descriptions. For precise details, the reader should consult the papers and references quoted there.

The Reissner-Nordstrom geometry is the unique, asymptotically flat, spherically symmetric solution to the Einstein-Maxwell equations that describes the spacetime outside of a spherical collapsing star with charge Q and mass M. This geometry may be analytically extended to an electro-vacuum solution representing a black hole for $0 < | Q | < M$. A part of the conformal Carter-Penrose diagram for this type of black hole is shown in Fig. 1. In this diagram, the coordinates are chosen so that the entire spacetime may be represented by a finite figure with rotational degrees of freedom suppressed. Since the spacetime is spherically symmetric, all the essential properties of the spacetime can be illustrated without loss of generality. The timelike coordinate runs along the long dimension of the paper, and the spacelike coordinate is in the plane of the paper and perpendicular to the timelike coordinate. In this diagram, the world lines of radial light rays are at an angle of 45° to the timelike direction (or to the spacelike direction). The level surfaces of the coordinates t (spatial coordinate inside the black hole) and r^* (temporal coordinate inside the black hole) shown by the dashed lines. The coordinates u and v are null coordinates related to t and r^* by $u = -r^* - t$ and $v = -r^* + t$. The Cauchy horizon is the null hypersurface $r = r_-$ with $u = \infty$ and $v = \infty$ on the left and right sides, respectively. The event horizon for the left exterior region is the null hypersurface $r = r_+$ with $u = -\infty$. Paths a and b represent timelike world lines beginning in the exterior, crossing the $u = -\infty$ event horizon, and crossing the $u = \infty$ and $v = \infty$ parts of the Cauchy horizon, respectively. While similar to the Schwarzschild black hole in the exterior region (i.e. outside the event horizon $r = r_+$), the charged black hole interior is dramatically different. The curvature singularity (the wavy line in Fig. 1) is timelike as opposed to the spacelike curvature singularity of a Schwarzschild black hole. It is this property of the charged black hole which enables an observer falling into the hole to avoid hitting the curvature singularity. Furthermore, the charged black hole has a Cauchy horizon inside of the event horizon at $r = r_-$. The Cauchy horizon or (as we will call it) r_- horizon has the peculiar property of being the boundary in spacetime where the Einstein-Maxwell equations lose their predictive power to describe the future evolution from prior data. However, the r_- horizon also has the property of being a null surface with infinite blue shift. An observer crossing it will see an arbitrarily large blue-shift of any incoming radiation and entire history of the exterior region in a finite lapse of his own proper time as he approaches the horizon. These properties suggest that r_- horizon will be unstable, small perturbations will develop into curvature singularities just before it. The future development then stops at the curvature singularity rather than the Cauchy horizon. The instability of the r_- horizon gives rise to the conjecture that in describing stellar collapse the development of a Cauchy horizon is a special feature arising from the assumption of perfect spherical symmetry. Previous studies suggest that nonsymmetric perturbations from symmetric stellar collapse develop into curvature singularities before the the formation of a Cauchy horizon.

In Part I of this dissertation we will consider the evolution of scalar, electromagnetic and gravitational test fields inside the black hole and the final state problem for the interior of the black hole left by a charged stellar collapse. Perturbation calculations can only show a solution to be *stable*. The unbounded growth of a perturbation *suggests* that (through the Einstein equations) a curvature singularity may develop, but higher-order nonlinear terms must be included to verify whether a singularity actually does form. For the electrovacuum solution we will show that in general massless scalar, electromagnetic and gravitational fields with arbitrary initial data on the past null infinity or on the event horizon $r = r_+$ will develop an unbounded energy density as measured by an observer freely falling from rest in the exterior in a neighborhood of the r_- horizon. The evolution of the fields proceed in two steps: *First*, the waves propagate in the exterior from the past null infinity to the event horizon at $r = r_+$. The detailed evolution of the fields outside the event horizon may be summarized for our purposes by a main wave traveling from the past null infinity to the horizon along a null ray and a sequence of waves scattered off of the background curvature from the main wave and subsequently from the scattered waves. The scattered fields superpose to fall off at late times as a power law. From earlier work, we know that this power-law falloff is a generic feature of wave propagation in the exterior. The field along the event horizon can be characterized by a main wave (which evolved directly from the initial data on the past null infinity) and a power-law tail. *Second*, the field on the event horizon evolves through the interior region $r_- < r < r_+$ to a neighborhood of the Cauchy horizon at $r = r_-$. We analyze in detail the evolution of the fields in the interior region and show how the data on the event horizon evolve to waves running along and across the r_- horizon. The waves running along the r_- horizon are then blue-shifted by an arbitrarily large amount at the r_- horizon.

The formation of power-law tails by waves propagating in the exterior and the blue-shift of the tails on the interior suggest that almost any perturbation in the exterior spacetime will grow into an instability on the r_- horizon. We also find that perturbations that develop inside the black hole (for example a momentary switching on and off of some scalar, electromagnetic or gravitational field on the surface of the star after it has crossed the event horizon) evolve subsequently to fields that are regular but with still divergent energy densities at the r_- horizon.

B. Part II: Multipole Moments for Stationary Systems

In this part we deal with two problems: *First*, we show that two families of multipole moments defined in rather different ways are actually equivalent to each other. *Second*, we prove that in the limit of slow motion, the rigid body motion is governed by the classical Euler equations of rigid body motion even in the case of bodies with arbitrarily strong internal gravity.

The basic idea behind the multipole moment formalisms in general relativity is the same as the one in classical physics: Define quantities which are called the multipole moments in such a way that given these objects one can construct the geometry of the spacetime in a well prescribed manner. This is completely analogous to multipole expansions for mass distributions, charge distributions, electromagnetic field, etc... It is much harder to define these quantities in a general spacetime, because the very idea of multipole moment expansions requires the existence of cartesian coordinate systems. Of course, in a general spacetime such coordinates do not exist. A certain class of spacetimes, known as asymptotically flat spacetimes, allow such coordinates to exist at "infinity". As one pulls away from the sources the effect of the matter on the spacetime diminishes leaving the spacetime flat at infinity. The spacetime metric is never exactly flat, but it gets closer and closer to the flat-space metric as the distance between the sources and the observer increases.

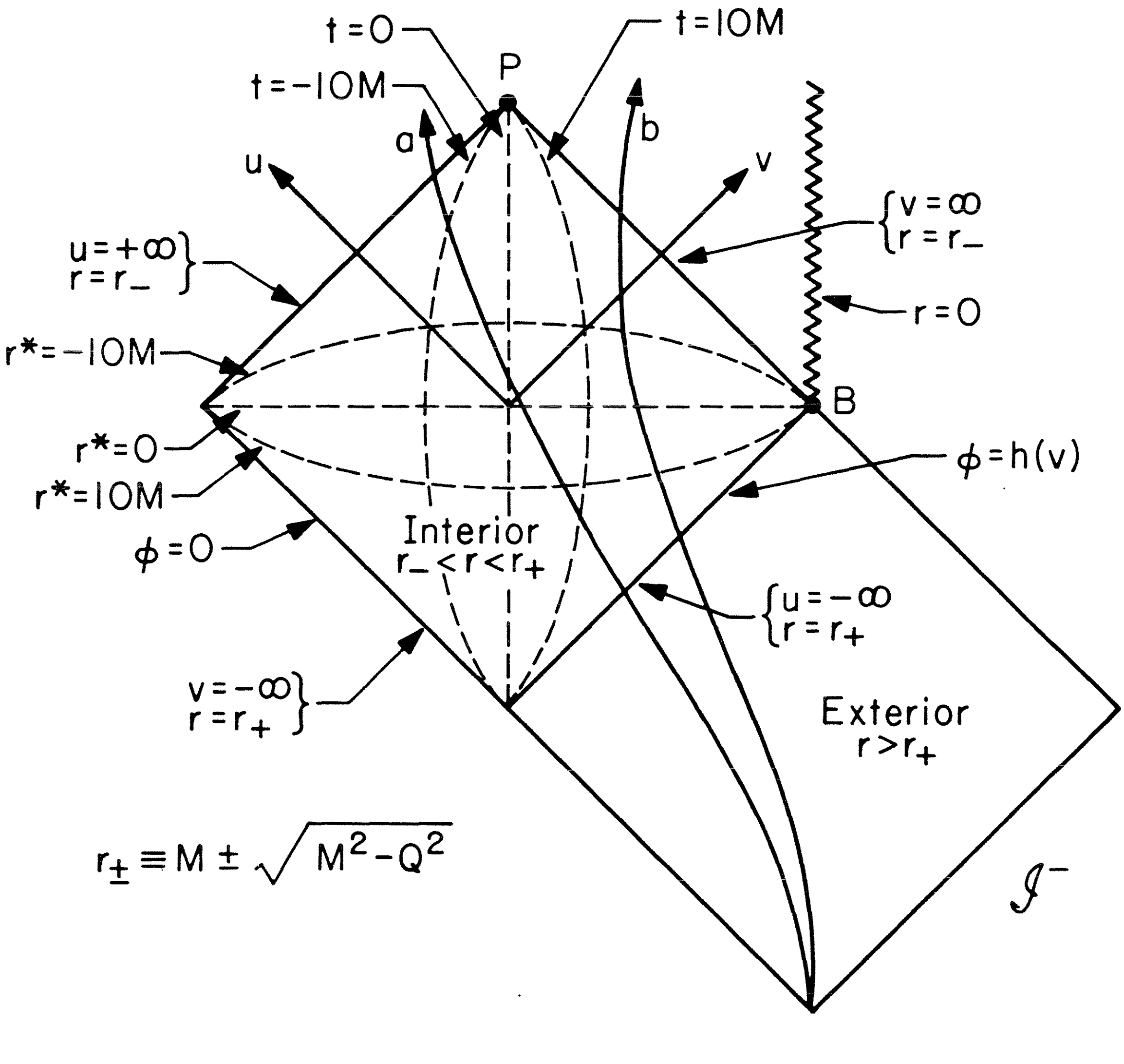
Two very different multipole moment formalisms are widely used in current research on asymptotically flat, stationary or slowly changing systems in general relativity. The first, which is rather elegant, was developed for precisely static systems by Geroch and was extended to precisely stationary systems by Hansen. The second, which is considered rather ugly by Kip S. Thorne but nevertheless has much computational power, was developed for slowly changing systems as well as precisely stationary ones by Thorne himself.

The Geroch-Hansen way of defining multipole moments, like any sophisticated and elegant mathematical theory, looks like magic. A stationary spacetime is invariant under time translations. Given the physical 4-dimensional stationary spacetime, Geroch and Hansen construct a 3-dimensional manifold which has its own metric which is induced by the physical metric of the 4-dimensional spacetime using the time invariance property of the stationary spacetime. Loosely speaking, they "factor out" the "time" degree of freedom which is not doing anything anyway. They then make a "conformal" transformation which brings the point at infinity to a "finite" location in this 3-dimensional manifold where it becomes a single point. They demand that the conformal factor and the metric of the 3-manifold be smooth at this point in the sense that there exist coordinates around the point at infinity in which both the conformal factor and the 3-metric are infinitely differentiable. Beig and Simon proved that there do exist a conformal factor and a coordinate system with this property for any stationary spacetime. Geroch and Hansen then define two "magic" scalar fields called the "mass" potential and the "current" potential on the 3-manifold. The "mass" and "current" multipole moments of the original physical spacetime are then obtained by repeatedly "differentiating" these potentials and evaluating their "derivatives" at the point at infinity.

Thorne's way of defining multipole moments is more on the intuitive side. His analysis is formulated entirely in the physical spacetime. It is a "straightforward" extension of the standard procedure of "reading out" the mass M of the source from the time-time part of the metric, and "reading out" the angular momentum $\vec{J}$ of the source from the time-space part of the metric. Naturally, any such read out requires the introduction of a coordinate system which becomes Minkowskii sufficiently rapidly at large radii. This is where the ugliness in Thorne's formalism lies: it requires the use of coordinates chosen from a special class called "*ACMC*" which stands for Asymptotically Cartesian and Mass Centered. In an *ACMC* coordinate system for any precisely stationary, vacuum, asymptotically flat spacetime the metric coefficients at order $1/r^{l+1}$, where r is the familiar radial coordinate, contain only multipoles of order $\leq l$, and the mass dipole vanishes entirely. One simply reads off the "mass" and the "current" multipoles of the spacetime from the leading terms at each order in $1/r$.

Paper III in this dissertation shows that these two seemingly different sets of multipole moments are in fact the same. The proof involves calculating "magic" Geroch-Hansen potentials in terms of Thorne's moments by choosing the "right" coordinate system and the "right" conformal factor. Precise details of the proof are given in the main body of paper III.

The last paper in this thesis tackles the problem of rigid-body motion in general relativity. We show that in the limit of slow rotation general relativistic gravity produces no changes whatsoever in the classical Euler equations of motion. Since this paper is written in a language accessible to people who know only a little relativity, the reader is urged to read the main body of the paper itself. An Appendix at the end of the paper justifies some of the assertions in the proof by giving an algorithm for generating rigidly rotating solutions of Einstein's equations from nonrotating solutions.



t=0
t=10M
t=-10M
P
a
b
u
v
{v=∞
r=r_
u=+∞
r=r_
r=0
r*=-10M
B
r*=0
φ=h(v)
r*=10M
φ=0
Interior
r_<r<r+
{u=-∞
r=r+
v=-∞
r=r+
Exterior
r>r+
r± ≡ M ± √(M²-Q²)
ℐ⁻

Fig. 1

PART I

A. PAPER I

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Evolution of scalar perturbations near the Cauchy horizon of a charged black hole

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We describe the evolution of a scalar test field on the interior of a Reissner-Nordström black hole. For a wide variety of initial field configurations the energy density in the scalar field is shown to develop singularities in a neighborhood of the geometry's Cauchy horizon, suggesting that for a stellar collapse curvature singularities will develop prior to encountering the Cauchy horizon. The extension to the interior of stationary perturbations due to exterior sources is shown not to disrupt the Cauchy horizon.

I. INTRODUCTION

The Reissner-Nordström geometry[1] is the unique,[2] asymptotically flat, spherically symmetric solution to the Einstein-Maxwell equations that describes the spacetime outside of a spherical star with charge Q and mass M. This geometry may be analytically extended to an electrovacuum solution representing a black hole for $0<|Q|<M$ (Ref. 3). While similar to the Schwarzschild black hole in the exterior region (i.e., outside of the event horizon $r=r_+$), the charged-black-hole interior (i.e., inside of the event horizon) is dramatically different. The Carter-Penrose diagram in Fig. 1 illustrates two distinguishing features: the timelike character of the curvature singularity (cf. the spacelike curvature singularity of a Schwarzschild black hole) and the Cauchy horizon inside of the event horizon at $r=r_-$. (See Hawking and Ellis[3] for the definitions of the global properties and Graves and Brill[4] for details on the analytic structure, coordinate systems, etc., for Reissner-Nordström black holes.)

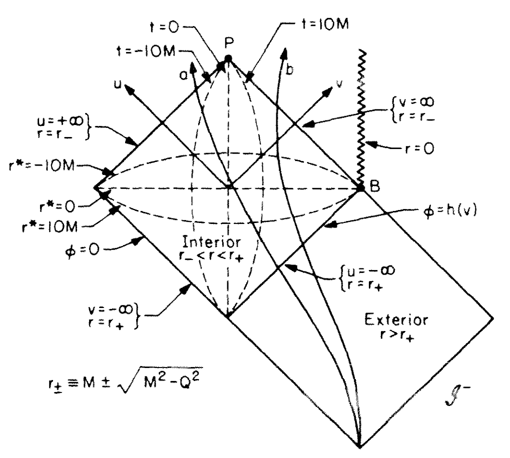


FIG. 1. Part of the conformal Carter-Penrose diagram for a Reissner-Nordström solution with $|Q|<M$. The level surfaces of the coordinates t and r^* are shown by the dashed lines. The coordinates u and v are null coordinates related to t and r^* by $u=-r^*-t$ and $v=-r^*+t$. The Cauchy horizon is the null hypersurface $r=r_-$ with $u=\infty$ and $v=\infty$ on the left and right sides, respectively. The event horizon for the left exterior region is the null hypersurface $r=r_+$ with $u=-\infty$. Paths a and b represent timelike world lines beginning in the exterior, crossing the $u=-\infty$ event horizon, and crossing the $u=\infty$ and $v=\infty$ parts of the Cauchy horizon, respectively.

The Cauchy horizon or (as we will call it) the r_- horizon has the peculiar global property of being the boundary in spacetime where the Einstein-Maxwell equations (or any other physical theory based on partial differential equations) lose their predictive power to describe the future evolution from prior data. However, the r_- horizon also has the property of being a null surface of infinite blue-shift. An observer crossing it will see an arbitrarily large blue-shift of any incoming radiation and the entire history of the exterior region in a finite lapse of her own proper time as she approaches the horizon. These properties suggested to Penrose[5] that the r_- horizon will be unstable, small perturbations will develop into curvature singularities just before it. The future development then stops at a curvature singularity rather than the Cauchy horizon.

The instability of the r_- horizon gives rise to the conjecture that in describing stellar collapse the development of a Cauchy horizon[6-8] is a special feature arising from the assumption of spherical symmetry. Previous studies[9,10] suggest that nonsymmetric perturbations from symmetric stellar collapse develop into curvature singularities before the formation of a Cauchy horizon.

Penrose and Simpson[9] have investigated numerically the evolution of a test massless vector field on a charged-black-hole background for a variety of initial field configurations. They found a general divergence of the field energy density as the evolution approached $r=r_-$ and they concluded that this divergence was a generic feature for this background geometry. McNamara[10] has demonstrated the existence of initial data for perturbations by a test scalar field which are bounded by power laws on $\mathscr{I}^-$ and evolve to have unbounded energy densities on $r=r_-$. In this paper we also consider the evolution of test scalar fields, discuss their detailed evolution inside of the black hole, and consider the implications for charged stellar collapse. In a sequel[11] we will discuss the evolution of the coupled electromagnetic and gravitational perturbations on and the final-state problem for the interior black hole left by a charged stellar collapse.

Perturbation calculations can only show a solution to be stable. The unbounded growth of a perturbation *suggests* that (through the Einstein equations) a curvature singularity may develop, but higher-order nonlinear terms must be included to provide sufficient conditions for instability. (The exterior of a Reissner-Nordström black hole has been shown to be stable to linear perturbations by Bičák,[12] Sibgatullin and Alekseev,[13] Moncrief,[14] and Zerilli.[15]) As McNamara[10] has done, we will consider the unbounded growth of linear perturbations to be indications of possible instabilities and will use the term "instability" in this sense.

For the electrovacuum solution we will show that in general a massless scalar test field ϕ with arbitrary initial data on $\mathscr{I}^-$ or on the event horizon $r=r_+$ will develop an unbounded energy density as measured by an observer freely falling from rest in the exterior (a "freely falling observer," FFO) in a neighborhood of the r_- horizon. The evolution of the field ϕ proceeds in two steps: First the wave propagates in the exterior from $\mathscr{I}^-$ to the event horizon $r=r_+$. The detailed evolution of ϕ outside the event horizon may be summarized for our purposes by a main wave traveling from $\mathscr{I}^-$ to the horizon along a null ray and a sequence of waves scattered off of the background curvature from the main wave and subsequently scattered waves. The scattered fields superpose to fall off at late times as a pwer law t^{-N}. From the work of Price[16] and Bičák[12] we know that this power-law falloff is a generic feature of wave propagation in the exterior. The field along the event horizon can be characterized by a main wave (which evolved directly along a null ray from data on $\mathscr{I}^-$) and a power-law tail. Second the field on the event horizon evolves through to the interior region $r_-<r<r_+$ to a neighborhood of the Cauchy horizon $r=r_-$. We analyze in detail the evolution of ϕ in the interior region and show how the data on the event horizon evolves to waves running along and across the r_- horizon. The waves running along the r_- horizon are then blue-shifted by an arbitrarily large amount at the r_- horizon.

The formation of power-law tails by waves propagating in the exterior and the blue-shift of the tails on the interior suggest that almost any perturbation in the exterior spacetime will grow into an instability on the r_- horizon. We also find that perturbations that develop inside the black hole (for example a momentary switching on and off of some scalar charge on the surface of a collapsing star after it has crossed the event horizon) evolve subsequently to fields that are regular but with still *divergent* energy densities at the r_- horizon.

In accord with Israel's theorem,[2] investigations of spherically symmetric collapsing shells of charge[6,8] and dust[7] have Reissner-Nordström geometries as their exterior solutions. Therefore, we may use the previous considerations supplemented with appropriate boundary conditions at the surface of the star to discuss the evolution of a charged collapse. Figure 1 shows two possible world lines for the surfaces of collapsing stars. These lines are to be interpreted as the boundaries of the stars, with the Reissner-Nordström solutions attached smoothly on the right and the stellar interior geometries (not shown in the figure) attached on the left. The development of curvature singularities along the r_- horizon then follows as in the previous case. (We note that McNamara's[10] analysis of instability depended upon the two-sphere P, the singularity in the Killing vector field. The collapse described by the world line to the right in Fig. 1 does not contain P in that spacetime and so his proof is not applicable to that case. Our discussion will show, however, that this section of the r_- horizon in that geometry is also unstable.)

In Sec. II we write the wave equation we will use for the propagation of the field ϕ on the Reissner-Nordström background, define the condition for stability at the r_- horizon, define the appropriate boundary conditions, and set up the characteristic initial-value problem to evolve data in the interior.

In Sec. III we solve for the evolved behavior of the field ϕ at the r_- horizon and present the result of a numerical integration. In Sec. IV we discuss the perturbations from exterior stationary sources. In Sec. V we summarize our results and briefly consider the effects of quantum-mechanical processes on our conclusions.

II. THE WAVE EQUATION

The Reissner-Nordström black-hole interior for $r_-<r<r_+$ is described by the metric

$$ds^2 = \frac{r^2}{(r_+-r)(r-r_-)}\,dr^2 - \frac{(r_+-r)(r-r_-)}{r^2}\,dt^2 - r^2(d\theta^2+\sin^2\theta d\varphi^2)\,, \quad (2.1)$$

where $r_\pm \equiv M \pm (M^2-Q^2)^{1/2}$, r is a temporal coordinate, and t is a spatial coordinate (see Fig. 1). It is convenient to define a "tortoise" coordinate r^* by the equation

$$r^* = -r - \frac{1}{\kappa_+}\ln(r_+-r) + \frac{1}{\kappa_-}\ln(r-r_-)\,, \quad (2.2)$$

where $\kappa_\pm \equiv (r_+-r_-)/r_\pm^2$ are the surface gravities at the two null surfaces. We will regard r as an implicit function of r^* with the asymptotic limits given by $r \sim r_\pm$ as $r^* \to \pm\infty$, and define the null coordinates u and v by the equations $u=-r^*-t$ and $v=-r^*+t$. These definitions for u and v are the natural extensions to the interior of Price's[16] exterior null coordinates. The event horizon is the null hypersurface $u=-\infty$ and the left and right r_- horizons are the null hypersurfaces $u=\infty$ and $v=\infty$, respectively.

The propagation of the scalar test field on this background is taken to be governed by the scalar wave equation

$$\phi_{;\alpha;\beta}g^{\alpha\beta}=0\,. \quad (2.3)$$

To exploit the symmetries of the background we expand ϕ in spherical harmonics and Fourier transform in t to obtain

$$\phi(r^*,t,\theta,\varphi) = \sum_{l,m}\int_{-\infty}^{\infty} dk\, e^{-ikt}Y_{lm}(\theta,\varphi)\frac{1}{r}\psi_{lmk}(r^*)\,. \quad (2.4)$$

Substituting this expression for ϕ, the wave equation is reduced to an ordinary differential equation in r^* for the modes ψ_{lmk} and given by

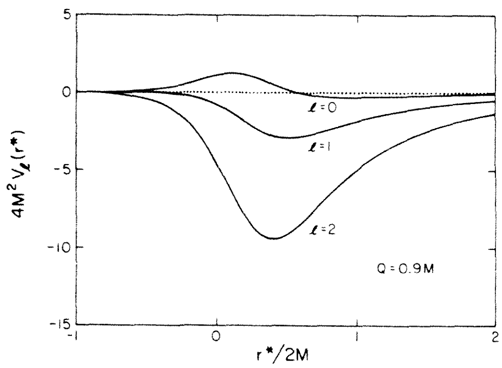


FIG. 2. The potential of the separated wave equation.

$$\frac{d^2\psi_{lmk}}{dr^{*2}} + [k^2 - V_l(r^*)]\psi_{lmk}(r^*) = 0\,, \quad (2.5)$$

where the scattering "potential" $V_l(r^*)$ is given by the equation

$$V_l(r^*) = \frac{-(r_+-r)(r-r_-)}{r^2} \times \left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3} - \frac{2Q^2}{r^4}\right]. \quad (2.6)$$

The potential $V_l(r^*)$ (shown in Fig. 2) is sharply localized in r^* and falls to zero exponentially with the asymptotic forms given by

$$V_l(r^*) \sim \exp(\mp\kappa_\pm r^*) \quad \text{as} \quad r^* \to \pm\infty\,. \quad (2.7)$$

The solutions to Eq. (2.5) as $r^* \to -\infty$ have the asymptotic forms given by

$$e^{-ikt}\psi_{lmk}(r^*) \sim \begin{bmatrix} e^{-ikv} \\ e^{iku} \end{bmatrix} [1 + O(e^{-\kappa_- r^*})]\,. \quad (2.8)$$

Near the r_- horizon these solutions may be described as left-going (e^{-ikv}) and right-going (e^{iku}) waves with exponentially vanishing corrections in r^*. Similar expressions hold for ψ near the r_+ horizon.

The energy density in the ϕ field (we are not considering the conformally invariant scalar field so the $\frac{1}{6}R$ term is absent) as measured by our FFO is a quadratic function of the quantity $\phi_{,\alpha}U^\alpha$ where U^α is the FFO's four-velocity. A FFO falling from rest in the exterior towards the left $u=\infty$ horizon has a four-velocity given by

$$\vec{U}_{(L)} = \left\{\frac{r^2}{-(r_+-r)(r-r_-)}\left[1-\left(1+\frac{(r_+-r)(r-r_-)}{r^2}\right)^{1/2}\right]\right\}\left(\frac{\partial}{\partial v}\right) - \left\{\frac{r^2}{-(r_+-r)(r-r_-)}\left[1+\left(1+\frac{(r_+-r)(r-r_-)}{r^2}\right)^{1/2}\right]\right\}\left(\frac{\partial}{\partial u}\right). \quad (2.9)$$

At $u=\infty$ the FFO will measure a $\phi_{,\alpha}U^\alpha$ given by

$$\phi_{,\alpha}U^\alpha_{(L)} \sim \frac{\partial\phi}{\partial u} e^{\kappa_-(v+u)/2} + (\text{const})\frac{\partial\phi}{\partial v} \quad \text{as} \quad u\to\infty. \tag{2.10}$$

Similarly a FFO falling to the right $v=\infty$ horizon will measure a $\phi_{,\alpha}U^\alpha$ given by

$$\phi_{,\alpha}U^\alpha_{(R)} \sim \frac{\partial\phi}{\partial v} e^{\kappa_-(v+u)/2} + (\text{const})\frac{\partial\phi}{\partial u} \quad \text{as} \quad v\to\infty. \tag{2.11}$$

Hence, for the FFO's to measure physically nonsingular fields near the r_- horizon, the appropriate derivatives of the field times the exponential blue-shift factor must be bound. [The condition Eq. (2.11) is identical with McNamara's[10] condition derived by an exponential boost to a nonsingular coordinate system at $v=\infty$.]

The higher-order terms in the solutions in Eq. (2.8) all fall off as fast or faster than $e^{\kappa_- r^*}$ as $r^*\to-\infty$ and the physical features described by these solutions are dominated by the leading terms $\psi\sim e^{-ikv}$ and $\psi\sim e^{iku}$. From Eqs. (2.11) and (2.10) we see that the e^{-ikv} waves are singular along the right $(v=\infty)$ r_- horizon and the e^{iku} waves are singular along the left $(u=\infty)$ r_- horizon.

The development of ϕ in the interior region from data on the r_+ horizon is most naturally stated as a characteristic initial-value problem. Since we are basically concerned with the history of a stellar collapse we take ϕ to be 0 on the left $(v=-\infty)$ r_+ horizon and take ϕ to be some initial-value function $h(v)$ on the right $(u=-\infty)$ r_+ horizon—the event horizon for the collapsing star. In the interior $(r_-<r<r_+)$ r^* is a timelike coordinate (as r^* goes from ∞ to $-\infty$ time increases in a positive sense) and Eq. (2.5) describes the temporal evolution between the horizons. In this respect the calculation is more like a cosmological problem than a scattering problem (cf. scattering in the exterior). The final evolution of ϕ is given by its values on the $u=\infty$ and $v=\infty$ r_- horizons.

III. EVOLUTION OF THE SCALAR FIELD

To impose the initial conditions on $r^*=+\infty$ it is convenient to write for a particular l,m-spherical harmonic mode the expression

$$\phi_{lm}(r^*,t)=\int_{-\infty}^{\infty} dk\, e^{-ikt}H_{lm}(k)\frac{1}{r}\,\psi^{(-)}_{lmk}(r^*), \tag{3.1}$$

where $\psi^{(-)}_{lmk}(r^*)$ is the solution to Eq. (2.5) with the asymptotic form at the r_+ horizons given by

$$e^{-ikt}\psi^{(-)}_{lmk}(r^*)\sim e^{-ikv} \quad \text{as} \quad r^*\to\infty. \tag{3.2}$$

[The absence of the conjugate function $\psi^{(+)}_{lmk}(r^*)$, which has the asymptotic behavior e^{iku} as $r^*\to\infty$, in Eq. (3.1) is due to the initial condition $\phi=0$ on the null surface $v=-\infty$.] At the r_- horizon $\psi^{(-)}_{lmk}(r^*)$ has the asymptotic form given by

$$e^{-ikt}\psi^{(-)}_{lmk}(r^*)\sim A_{lm}(k)e^{-ikv}+B_{lm}(k)e^{iku} \quad \text{as} \quad r^*\to-\infty, \tag{3.3}$$

where

$$|A_{lm}(k)|^2-|B_{lm}(k)|^2=1, \tag{3.4}$$

which follows from the Wronskian condition. In the sequel[11] we will show that $A_{lm}(k)$ and $B_{lm}(k)$ are analytic in k in a neighborhood of $k=0$, take the values at $k=0$ given by

$$A_{lm}(k=0)=\frac{(-1)^l}{2}\left(\frac{r_+}{r_-}+\frac{r_-}{r_+}\right),$$
$$B_{lm}(k=0)=-\frac{(-1)^l}{2}\left(\frac{r_+}{r_-}-\frac{r_-}{r_+}\right), \tag{3.5}$$

and as $k\to\infty$, $[A_{lm}(k)-1]$ and $B_{lm}(k)$ decay exponentially. [A strict proof of the regularity of $A_{lm}(k)$ and $B_{lm}(k)$ at all intermediate points for real k has been given by McNamara.[17]]

The function $H_{lm}(k)$ is determined from the initial data $h_{lm}(v)$ on the right $(u=-\infty)$ r_+ horizon [here $h_{lm}(v)$ are the multipole moments of the initial value data $h(v)$, i.e., $h(v)=\sum_{l,m}h_{lm}(v)Y_{lm}(\theta,\varphi)$]. Evaluating Eq. (3.1) as $u\to-\infty$, with the help of Eq. (3.2) and using the Fourier inversion theorem we may write

$$H_{lm}(k)=\frac{1}{2\pi}\int_{-\infty}^{\infty} h_{lm}(v)e^{ikv}\,dv. \tag{3.6}$$

The general structure of the solution in a neighborhood of the r_- horizon has the form

$$\phi_{lm}(r^*,t)=\frac{1}{r_-}\left[h_{lm}(v)+\phi_{1lm}(v)+\phi_{2lm}(u)+O(r-r_-)\right], \tag{3.7}$$

where

$$\phi_{1lm}(v)=\int_{-\infty}^{\infty} dk\, e^{-ikv}H_{lm}(k)[A_{lm}(k)-1], \tag{3.8}$$

$$\phi_{2lm}(u)=\int_{-\infty}^{\infty} dk\, e^{iku}H_{lm}(k)B_{lm}(k). \tag{3.9}$$

[Note that the integrals in Eqs. (3.8) and (3.9) are convergent for a wide class of data $H_{lm}(k)$ due to the exponential fall off of the $A_{lm}(k)-1$ and $B_{lm}(k)$ coefficients.] [A formal mathematical proof that the fourth term in Eq. (3.7) is $O(r-r_-)$ has also been given by McNamara.[17]]

If $H_{lm}(k)$ is regular on the real k axis, then since $A_{lm}(k)$ and $B_{lm}(k)$ have no irregular points in a neighborhood of the real k axis the contour of integration in Eq. (3.8) may be deformed into the lower half k plane in the case $v\to\infty$, and in Eq. (3.9) the contour may be deformed into the upper half plane in the case $u\to\infty$. Hence we obtain that $\phi_{1lm}(v)$ and $\phi_{2lm}(u)$ decay exponentially as v and $u\to\infty$, respectively.

We now specialize to an $h_{lm}(v)$ as a sum of a δ-function burst and a power-law tail after the burst, viz.,

$$h_{lm}(v)=\lambda\delta(v-v_0)+\mu\theta(v-v_0)v^{-\alpha},\quad \alpha>1 \qquad (3.10)$$

where λ, μ, α, and v_0 are all constants characterizing the initial data. Substituting into Eq. (3.6) we find $H_{lm}(k)$ to be given by

$$H_{lm}(k)=\frac{1}{2\pi}e^{ikv_0}\left[\lambda+\mu\int_0^\infty \frac{e^{ikz}}{(z+v_0)^\alpha}dz\right]. \qquad (3.11)$$

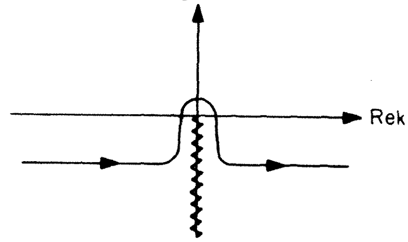


FIG. 3. Complex k-plane contour for integrating Eq. (3.8) for the contribution from the power-law tail of the initial-value data. There is a branch cut from $k=0$ to $k=-i\infty$ in the lower half plane along the imaginary k axis.

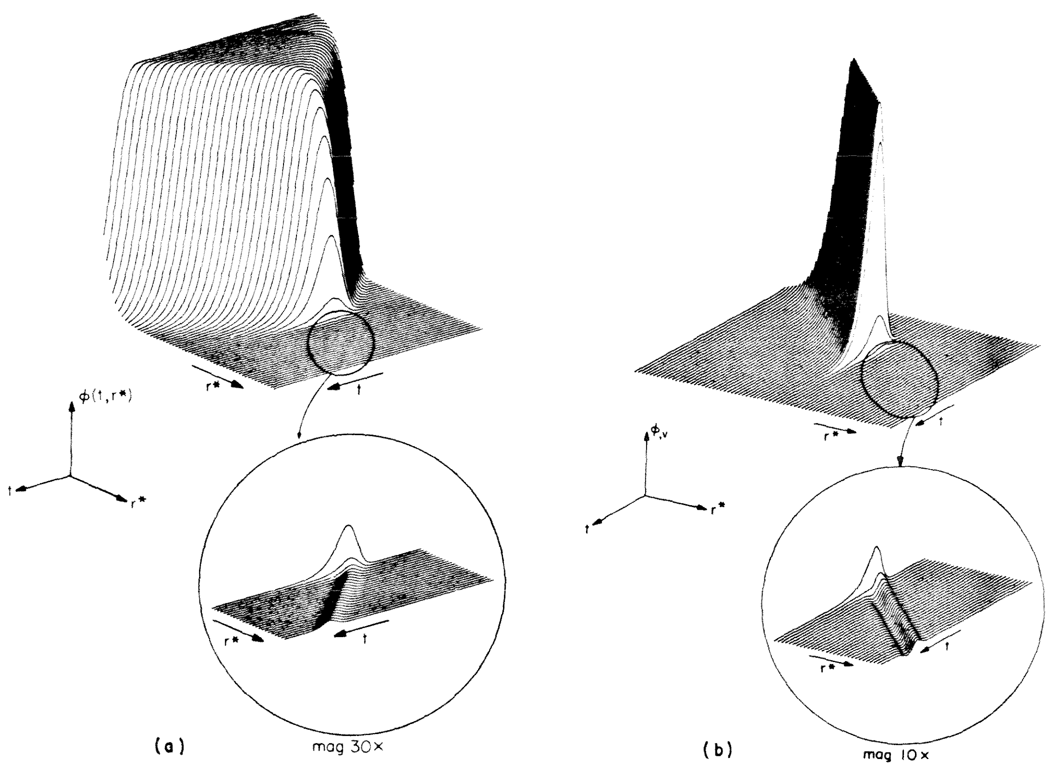


FIG. 4. These figures summarize the results of a numerical integration of the equation $\phi_{l,r^*r^*}-\phi_{l,tt}=V_l(r^*)\phi_l$. The results presented are for the particular case $Q=0.9M$ and $l=2$; however, additional integrations with alternative parameters support the qualitative features illustrated here. Figures 4(a) and 4(b) show ϕ_l and $\phi_{l,v}$ in relief as functions of r^* and t. The r^* coordinate ranges from $50M$ (near the event horizon) to $-60M$ (near the Cauchy horizon) and t ranges from $-50M$ to $50M$. The separation between lines of constant r^* is $2M$ for Fig. 4(a) and $1.6M$ for Fig. 4(b). The initial-wave form is taken to be a Gaussian of unit amplitude. These are shown in the inserts magnified $30\times$ for ϕ and $10\times$ for $\phi_{,v}$. The asymptotic exponential falloff of the derivatives of ϕ near the Cauchy horizon are shown in Figs. 4(c) and 4(d). Figure 4(c) shows the exponential behavior of $\phi_{,u}$ at constant v in the neighborhood of the right-hand side of the r_- horizon. Figure 4(d) shows the analogous behavior of $\phi_{,v}$ at constant u near the left-hand side of the r_- horizon.

In this case $H_{lm}(k)$ is not analytic at the point $k=0$. The nonanalytic part of $H_{lm}(k)$ is proportional to $k^{\alpha-1}$ if α is nonintegral and $k^{\alpha-1}\ln k$ if α is an integer. In either case $H_{lm}(k)$ has a cut in the complex k plane and we shall place this cut along the imaginary k axis in the lower half plane in the case of Eq. (3.6) and in the upper half plane in the case of Eq. (3.9).

We now consider the contribution to $\phi_{1lm}(v)$ and $\phi_{2lm}(u)$ from the power-law tail [*second term in* Eq. (3.11)]. Due to the cut, the main contribution to $\phi_{1lm}(v)$ and $\phi_{2lm}(u)$ for large values of their arguments comes from the integration in the vicinity of the origin. The general form of the contour for Eq. (3.8) is shown in Fig. 3. Owing to the analyticity of $A_{lm}(k)$ and $B_{lm}(k)$ at $k=0$ we can substitute for them their values at $k=0$ [Eqs. (3.5)] in Eqs. (3.8) and (3.9) and we obtain

$$\tilde{\phi}_{1lm}(v) = \mu v^{-\alpha}[A_{lm}(0) - 1] \text{ as } v \to \infty , \tag{3.12}$$

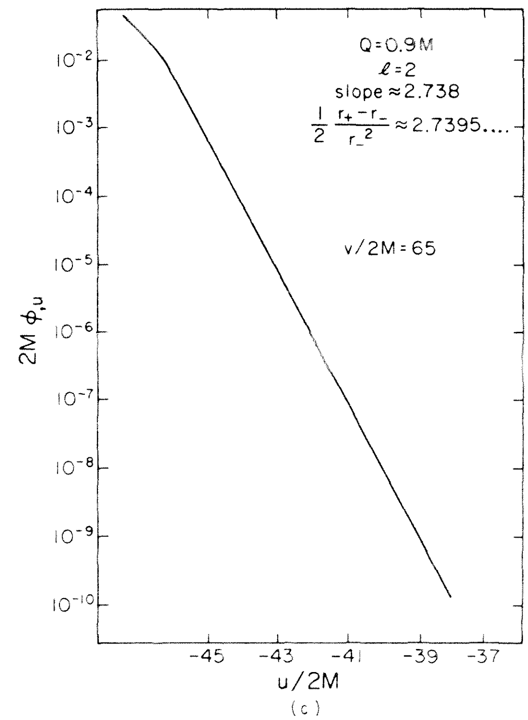
$$\tilde{\phi}_{2lm}(u) = \mu u^{-\alpha} B_{lm}(0) \text{ as } u \to \infty , \tag{3.13}$$

where the tilde means these are the contributions from the second term in Eq. (3.11). Therefore, in the case of a power-law tail ϕ is bounded at the r_- horizon and vanishes at the point P $(u\to\infty, v\to\infty)$, but the invariants in Eqs. (2.10) and (2.11) diverge according to the equation

$$\begin{bmatrix} v^{-(\alpha+1)} \\ u^{-(\alpha+1)} \end{bmatrix} \exp\left\{\frac{\kappa_-}{2}\begin{bmatrix} v \\ u \end{bmatrix}\right\} \tag{3.14}$$

on the respective horizons. Hence, the r_- horizon is unstable against scalar perturbations with power-law initial data at all points along the horizon except the point B $(u=-\infty, v=\infty)$. (The properties and behavior of perturbations at the point B will be discussed in the subsequent paper.[11])

It is interesting to note that the δ-function part of $h_{lm}(v)$ [i.e., the first term in Eq. (3.11)] also gives rise to singularities along the r_- horizon. In this case $H_{lm}(v) = (1/2\pi)e^{ikv_0}$ and hence it is analytic in the whole complex plane. Therefore the contour in Eqs. (3.8) and (3.9) can be deformed into the lower and upper half planes, respectively, until the contour intersects the nearest nonanalytic point of $A_{lm}(k)$ or $B_{lm}(k)$. Then we obtain the result that $\bar{\phi}_{1lm}(v)$ and $\bar{\phi}_{2lm}(u)$ [the corresponding contribution to $\phi_{1lm}(v)$ and $\phi_{2lm}(u)$ from the first term in Eq. (3.11)] decay exponentially for large values of their arguments. An estimate of the index γ for the exponential decay depends upon the



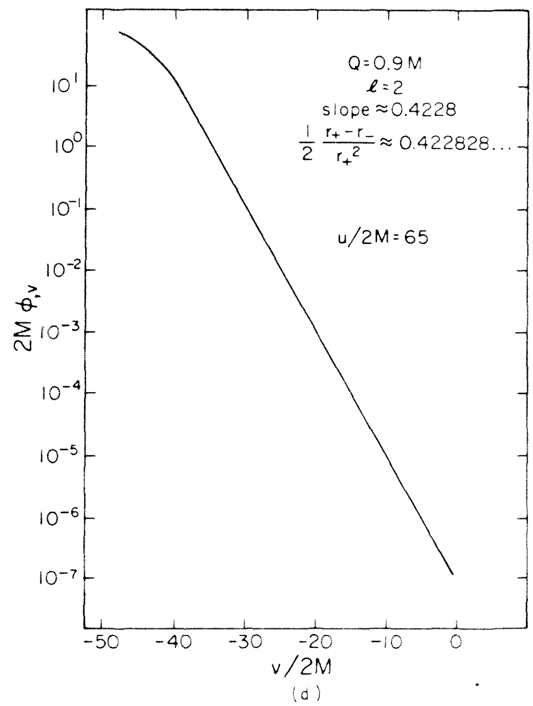


FIG. 4. (*Continued*)

imaginary value of the point where $A_{lm}(k)$ or $B_{lm}(k)$ becomes nonanalytic. Numerical computations of the evolution of a Gaussian wave packet are shown and described in Fig. 4. These computations indicate numerically that

$$\bar{\bar{\phi}}_{1lm}(v) \sim e^{-\kappa_+ v/2} \text{ as } v \to \infty ,$$
$$\bar{\bar{\phi}}_{2lm}(u) \sim e^{-\kappa_- u/2} \text{ as } u \to \infty . \tag{3.15}$$

We have not yet obtained an analytical proof of this result. Since $\kappa_+ < \kappa_-$ the invariants in Eqs. (2.11) and (2.10) diverge at the $v = \infty$ horizon.

IV. STATIONARY EXTERNAL SOURCES

In this section we consider the nonradiative fields ϕ which are connected with external sources outside of the r_+ horizon. We assume that these sources are at rest in the exterior with respect to the charged black hole. This means that in the exterior part of the black-hole geometry $(r > r_+)$ the field ϕ is independent of the exterior time and is well behaved at the r_+ horizon (its behavior at $r = \infty$ is unimportant for this discussion). To extend this field inside $(r \leq r_+)$ we must solve Eq. (2.3) for the case that ϕ is independent of t on $r = r_+$. This means finding the solutions to Eq. (2.5) for $k = 0$.

The $k = 0$ solutions to Eq. (2.5) can be written in closed form as

$$\phi_{1l}(r) = P_l\left(\frac{2r - r_+ - r_-}{r_+ - r_-}\right) , \tag{4.1}$$

$$\phi_{2l}(r) = Q_l\left(\frac{2r - r_+ - r_-}{r_+ - r_-}\right) , \tag{4.2}$$

where P_l is the Legendre polynomial and Q_l is the Legendre function of the second kind.[18] Of these two independent solutions only P_l is regular at r_+ and hence describes the extension inside the hole of fields due to external sources. With this solution we see that near the r_- horizon the ϕ field behaves as $P_l(-1)$ and the energy density as measured by one of our FFO is proportional to the expression

$$\phi_{,\alpha} U^\alpha \sim \frac{\partial P_l}{\partial r^*} e^{\kappa_- r^*} + (\text{const}) \sim \frac{\partial P_l}{\partial r} + (\text{const}) ,$$

which is *finite*.

V. CONCLUSION

This work has described the dynamical development of a test scalar field with the aim of demonstrating that for a wide class of physically reasonable initial conditions (really, for all the conditions we considered) the energy density in the field grows singular along the Reissner-Nordström geometry's Cauchy horizon. Such behavior suggests that for the real collapse of a charged star curvature singularities will develop in the interior of the forming black hole before the Cauchy horizon and timelike singularity of the Reissner-Nordström black hole are encountered by the developing spacetime geometry of the collapse. These conclusions for the development of perturbations that began in the exterior are in agreement with the results of Penrose and Simpson[9] and McNamara.[10] An interesting feature of the present calculation is that even a δ-function initial distribution on the outer horizon leads to the unbounded growth in the energy density of the scalar field at the inner horizon.

In this paper we have restricted our attention to classical fields. To examine the physical evolution of a more realistic stellar collapse we must take into consideration quantum mechanical effects: viz., pair creation by the gravitational field (all particles are created, including massless particles) and the creation of charged particle pairs by the electromagnetic field. The former process takes place for all values of M and Q, the latter process is possible if $|eQ| > 4GmM$. If $|eQ| \gg G^2 mM/\hbar$ then the creation of charged pairs by the electromagnetic field is much more rapid than the creation of pairs by the gravitational field. We expect these and associated processes to drive perturbations that also disrupt the Cauchy horizon, but their contribution to the total energy-momentum tensor will be proportional to Planck's constant and in general small as compared to the classical perturbations. Work on these problems is currently underway.

ACKNOWLEDGMENTS

When completing the work described herein, the authors received a report of a similar calculation by McNamara.[17] We have included his paper as a reference and tried to note where his work has overlapped with ours in this paper.

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B. PAPER II

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Final state of the evolution of the interior of a charged black hole

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We describe the dynamical evolution of scalar, electromagnetic, and gravitational test fields on the interior of a Reissner-Nordström (spherically symmetric and electrically charged) black hole. The instability of the hole's Cauchy horizon is discussed in detail in terms of the divergences of the energy densities of the test fields as measured by a freely falling observer approaching the Cauchy horizon. The late-time development of the fields is discussed and a picture of the final state for the interior (in terms of classical fields) is developed. We conclude that the Cauchy horizon of the analytically extended Reissner-Nordström solution is highly unstable and not a physical feature of a realistic gravitational collapse.

I. INTRODUCTION

The Reissner-Nordström black hole represents the unique static exterior of a collapsed spherically symmetric distribution of charge and mass. However, the interior of the analytically extended solution possesses a Cauchy horizon prohibiting any deterministic future based on the hyperbolic Einstein field equations. In a previous paper[1] (hereafter referred to as I) we discussed the behavior of a test scalar field near the hole's Cauchy horizon where the field's energy density was shown to develop singularities and suggested a disruption of the horizon (e.g., through the back reaction of the singular energy density on the curvature). In this paper we continue the scalar field analysis, extend the development to include electromagnetic and gravitational perturbations, and investigate the problem of the final state of the evolution of the interior of a charged black hole for late times.

We assume that a nearly spherical star with a net electric charge has undergone a gravitational collapse with small deviations from spherical symmetry in the matter density and charge density at the moment when the surface of the star crossed the event horizon. All the perturbations from spherical symmetry are assumed to be weak enough so that we can neglect their back reaction on the spacetime metric at the moment of crossing. Mashhoon[2] has recently investigated the spherical charged collapse of a perfect fluid and found that while the exterior geometry was necessarily Reissner-Nordström type, the interior geometry collapsed behind an apparent horizon to a spacelike curvature singularity. Doroshkevich and Novikov[3] have investigated the final-state problem for perturbations inside a Schwarzschild black hole, where the evolution in time of a perturbation ceases at the spacelike curvature singularity. The geometry at the Cauchy horizon inside the Reissner-Nordström black hole is smooth and regular and gives rise to dramatically different features not found in the Schwarzschild interior (cf. paper I).

In Sec. II we solve the scalar wave equation for small wave number and use this solution to discuss the behavior of the scalar field in a "neighborhood" of the intersection of future timelike infinity, the event horizon, the Cauchy horizon, and the curvature singularity on the Carter-Penrose diagram of an analytically extended Reissner-Nordström black hole (point B in Fig. 1 of paper I). As will be shown approaching B along a spacelike hypersurface $r=\text{const}$ (for $r_-<r<r_+$), i.e., as $t\to\infty$ for r held fixed, a perturbation of multiple index l will decay as t^{-2l-2} and the field between the r_+ and r_- horizons (i.e., the interior as we have called it) does not develop any pathologies. However, if we approach B or the r_- horizon along the world line of a freely falling observer then the energy density as measured by the observer is blue-shifted and diverges exponentially in r^* [or as $(r-r_-)^{-1}$] near the r_- horizon. This suggests that a curvature singularity develops, topologically similar to that of the Schwarzschild black-hole interior.

In Sec. III we formulate and discuss the electromagnetic and gravitational perturbation problems. Using a Regge-Wheeler-type formalism[4-7] combined with the techniques developed previously to analyze the scalar case, we extend the above conclusions on instability.

In Sec. IV we consider the features of perturbations arising from stationary sources in the exterior region. As an example, we present the astrophysically interesting case of a hole in a uniform magnetic field and demonstrate that the "stationary" field which threads through the interior does not disrupt the r_- horizon (as also was the case with the stationary scalar field discussed in paper I).

In Sec. V we discuss our conclusions reached by treating the perturbations as classical fields. By a classical field we mean that the smallest values of our field amplitudes are still much larger than the corresponding quantum amplitudes. We do not consider quantum-mechanical processes here. While these processes undoubtedly influence the structure of the evolving singularity (see, e.g., Ref. 8), they do not alter our main conclusions.

II. SCALAR FIELD

In this section we conclude the scalar field analysis begun in paper I and compute the asymptotic behavior of the scalar test field in a neighborhood of the point B (cf. Fig. 1 of paper I) for the final-state analysis. We match the field to a power law on the r_+ horizon which had developed from the late-time field in the exterior due to the backscatter of radiation. Using a small-wave-number approximate solution, we evolve this field through the interior up to the r_- horizon. With this solution we investigate the asymptotic form of the field as point B is approached in a spacelike direction ($r=$const, $t\to\infty$) and in a null direction along the r_- horizon ($u=\infty$, $v\to\infty$).

Using the notation of paper I, we write the scalar field solution as

$$\phi(t,r,\theta,\varphi)=\sum_{l,m} Y_{lm}(\theta,\varphi)\int_{-\infty}^{\infty} dk\, e^{-ikt}\frac{1}{r}\psi_{lmk}(r)\,, \tag{1}$$

where $\psi_{lmk}(r)$ satisfies the evolution equation

$$\frac{d^2\psi_{lmk}}{dr^{*2}}+\left\{k^2+\frac{(r_+-r)(r-r_-)}{r^2}\left[\frac{l(l+1)}{r^2}+\frac{2M}{r^3}-\frac{2Q^2}{r^4}\right]\right\}\psi_{lmk}=0\,, \tag{2}$$

and r^* and r are related (on the interior) by the equation

$$r^*=-r-\frac{1}{\kappa_+}\ln\left(1-\frac{r}{r_+}\right)+\frac{1}{\kappa_-}\ln\left(\frac{r}{r_-}-1\right), \tag{3}$$

with $\kappa_\pm=(r_+-r_-)/r_\pm^2$. We will also use the null coordinates $u=-r^*-t$ and $v=-r^*+t$.

Let $\psi_k(r)$ be the left-going solution to Eq. (2) defined by

$$\psi_k(r)\sim e^{ikr^*} \text{ as } r^*\to\infty \text{ (i.e., as } r\to r_+)\,.$$

Then $\psi_k^*(r)$ is a second linearly independent solution. A general solution (for each set of multipole indices l, m which we suppress for brevity) may be written in the form

$$\psi(r,t)=\int_{-\infty}^{\infty} dk\, e^{-ikt}[a(k)\psi_k(r)+b(k)\psi_k^*(r)]\,. \tag{4}$$

The coefficients $a(k)$ and $b(k)$ are to be determined from the initial conditions on the r_+ horizon and the conditions on the collapsing star.

A comparison of the asymptotic form of Eq. (4) near the r_+ horizon with the late-time field in the exterior suggests (cf. Ref. 3) that we choose

$$a(k)=b(k)[-1+B(k)]\,, \tag{5}$$

where $[B(k)-1]$ is the reflection coefficient of a wave scattering off the static potential in the exterior of the black hole.[9] [The Fourier coefficients based on the (t,r) coordinates in the interior and exterior regions can be equated across the r_+ horizon for infalling waves only. This may be seen by comparing the (t,r)-based transforms with a transform based on, e.g., Eddington-Finkelstein coordinates (v,r) which are nonsingular at the r_+ horizon.] The form of $B(k)$ is such as to represent a power-law-type trail on the r_+ horizon for large values of t. The function $b(k)$ is determined by the details of the collapse as the star crosses the r_+ horizon (which we take to be at $v=0$). In what follows, all we will need are a few details on the analytic properties of $B(k)$ and $b(k)$. (For an analysis in the exterior, see Sibgatullin and Alekseev.[9])

The existence of power-law tails in the exterior region implies that $B(k)$ has, at least, a branch point at $k=0$. (This follows from a Weiner-Hopf-type analysis of the asymptotic behavior of Fourier transforms. Sibgatullin and Alekseev[9] have further demonstrated that the branch point has the character of a logarithmic branch point.) Using time dilation arguments, Price has shown that in the exterior the field on the surface of the star as the star crosses the r_+ horizon has the asymptotic form

$$\phi \sim \text{const} \times \theta(r_{\text{ext}}) + \text{const} \times e^{-u_{\text{ext}}/2r_+},$$

which when Fourier transformed implies that $b(k)$ has a simple pole at $k=0$.

As $t \to \infty$ (i.e., a neighborhood of point B), the dominant contribution to $\psi(r,t)$ in Eq. (4) comes from the modes in a neighborhood of $k=0$ and we may expect the solution for $|kM| \ll 1$ to Eq. (2) to apply. We use the now common technique[10–15] of matching the leading terms in k of the wavelike solutions at $r=r_+$ with the low-k solutions on the interior. This technique matches only the dominant terms and ignores, e.g., the logarithmic singularities which are of higher order in k. The $k=0$ solutions can be written in closed form and are given by

$$\psi_{k=0}(r) = \alpha r P_l\left(\frac{2r-r_+-r_-}{r_+-r_-}\right) + \beta r Q_l\left(\frac{2r-r_+-r_-}{r_+-r_-}\right), \tag{6}$$

where P_l is the Legendre polynomial, Q_l is the Legendre function of the second kind, and α and β are constants.

Near the r_+ horizon, $\psi_k(r)$ has the form of an ingoing wave of unit amplitude, viz., $\psi_k(r)=e^{ikr*}$. For $|krr^*| \ll 1$ we may neglect the k^2 term in Eq. (2) and match the leading terms in $\psi_k(r)$ in the region $-1/k \ll r^* \ll -M$ with the $\psi_{k=0}$ solution and we find to $O(k^2)$

$$\psi_k(r) = \frac{r}{r_+}\left[P_l\left(\frac{2r-r_+-r_-}{r_+-r_-}\right) + 2ik\frac{1}{\kappa_+}Q_l\left(\frac{2r-r_+-r_-}{r_+-r_-}\right)\right] \approx \left[1 - ik\frac{1}{\kappa_+}\ln\left(\frac{r-r_-}{r_+-r_-}\right)\right] \text{ as } r \to r_+. \tag{7}$$

Near the r_- horizon, $\psi_k(r)$ evolves to a solution of Eq. (2) with the asymptotic form given by

$$\psi_k(r) \approx \tilde{A}(k)\, e^{ikr^*} + \tilde{B}(k)\, e^{-ikr^*} \text{ as } r \to r_-. \tag{8}$$

The Wronskian for two solutions of Eq. (2) is independent of r and for $\psi_k(r)$ and $\psi_k^*(r)$ it has the value given by

$$W_{r^*}[\psi_k, \psi_k^*] = -2ik. \tag{9}$$

This in turn implies that $|\tilde{A}(k)|^2 - |\tilde{B}(k)|^2 = 1$ for all k. Comparing Eqs. (7) and (8) in the region $M \ll r^* \ll 1/k$, we find for $|kM| \ll 1$ the scattering amplitudes A and B are given by

$$\tilde{A}(k) = \frac{(-1)^l}{2}\left(\frac{r_+}{r_-} + \frac{r_-}{r_+}\right) + O(k), \tag{10}$$

$$\tilde{B}(k) = -\frac{(-1)^l}{2}\left(\frac{r_+}{r_-} - \frac{r_-}{r_+}\right) + O(k). \tag{11}$$

[Owing to the exponential decay of the potential in Eq. (2) as $|r^*| \to \infty$ (see paper I), the coefficients $\tilde{A}$ and $\tilde{B}$ are, in fact, analytic at $k=0$.[15]]

We now consider Eq. (4) with the Fourier coefficients related by Eq. (5), viz.,

$$\psi(r,t) = \int_{-\infty}^{\infty} e^{-ikt} b(k)[(B(k)-1)\psi_k(r) + \psi_k^*(r)]dk. \tag{12}$$

We take the branch cut [to define $B(k)$] to run from $k=0$ to $k=-i\infty$ along the negative imaginary axis and deform the contour of integration into the lower half of the complex k plane. The pole of $b(k)$ at $k=0$ produces a cancellation between the second and third terms $(-\psi_k + \psi_k^*)$ [cf. Eqs. (10) and (11)]. Any additional poles of $b(k)$ below the real axis will give terms that exponentially damp as $t \to \infty$. There remains the integral along the cut from $k=0$ to $k=-i\infty$. As previously discussed, the reflection coefficient $B(k)$ is chosen to reproduce the power-law tail on the r_+ horizon (expected from the late-time development of a collapse) with the asymptotic form given by

$$\psi(r,t) \sim \frac{\text{const}}{v^{2l+2}} = \frac{\text{const}}{(t-r^*)^{2l+2}} \text{ for } u \to -\infty,\ r \gg M. \tag{13}$$

At intermediate times (r=constant) using the $|kM| \ll 1$ solutions given in Eq. (7), we find

$$\psi(r=\text{const},t) = \text{const} \times \int_0^{\infty} dk\, e^{-kt}(k^{2l+1})\psi_k(r) = \frac{\text{const}}{t^{2l+2}} \times rP_l\left(\frac{2r-r_+-r_-}{r_+-r_-}\right) \times \left[1 + O\left(\left|\frac{r^*}{t}\right|\right)\right] \text{ as } t \to \infty. \tag{14}$$

Near the r_- horizon approaching the point B (i.e., for $v \to \infty$ with $u<0$) using Eq. (8), the Fourier synthesis gives the result

$$\psi(r,t) \sim \int_0^{\infty} dk\, k^{2l+1}[A(0)\, e^{-kv} + B(0)\, e^{ku}] = \text{const} \times \left[\frac{A(0)}{v^{2l+2}} + \frac{B(0)}{u^{2l+2}}\right]. \tag{15}$$

Recalling that near the r_+ horizon

$$\psi = \text{const} \times v^{-(2l+2)}, \tag{16}$$

we see that Eqs. (14)–(16) present the complete structure of the scalar field near the point B. (N.B. The regions in which each of these expressions are applicable overlap.)

In paper I we considered what a freely falling observer (with a four-velocity U^α) crossing the r_- horizon would observe for the energy density in the scalar field and demonstrated that the energy density was a function of $\phi_{,\alpha}U^\alpha$. For an observer crossing $v=\infty$ this is given by

$$\phi_{,\alpha}U^\alpha \sim \frac{\partial\phi}{\partial v}e^{(\kappa_-/2)(u+v)} + \text{const}\times\frac{\partial\phi}{\partial u} \text{ as } v\to\infty\,. \quad (17)$$

If the point B is approached along a surface of constant r, then since $u+v=-2r^*$ we find using Eqs. (14) and (17) that

$$\phi_{,\alpha}U^\alpha \sim \text{const}/t^{2l+3} \text{ as } t\to\infty\,.$$

Hence, there is no pathological behavior of ϕ approaching the point B in this direction. However, if the point B is approached along the r_- horizon, then using Eqs. (15) and (17) we find that

$$\phi_{,\alpha}u^\alpha \sim (\text{const}/v^{2l+3})\,e^{(\kappa_-/2)v} \text{ as } v\to\infty\,.$$

Hence, approaching the point B by running along the r_- horizon (i.e., by taking the order of the limits as $v\to\infty$, then $u\to-\infty$ in the previous expression) indicates that a singularity may develop near the Cauchy horizon in an arbitrarily small neighborhood of the point B. These conclusions when combined with the analysis in paper I indicate that the entire Cauchy horizon is unstable to perturbation produced by a scalar test field.

III. ELECTROMAGNETIC AND GRAVITATIONAL PERTURBATIONS

In this section we augment the previous analysis and extend our considerations to include electromagnetic and gravitational perturbations of the interior. Owing to the presence of the background electric field, the situation is mathematically more difficult. Electromagnetic and gravitational perturbations are described by coupled sets of wave equations which correspond physically to the conversion of electromagnetic perturbations into gravitational perturbations and vice versa by the catalytic action of the background fields. However, owing to the efforts of Zerilli,[5] Moncrief,[6,7] Sibgatullin and Alekseev,[9] and Chandrasekhar,[16,17] it is possible to obtain decoupled equations which upon separation satisfy potential-like equations similar to Eq. (2) (see Fig. 1).

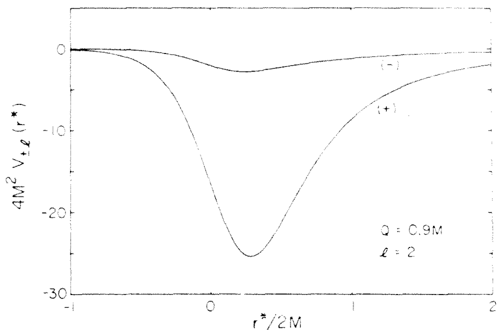

FIG. 1. The potential of the separated wave equation for electromagnetic and gravitational perturbations. The particular case $Q=0.9M$ and $l=2$ is shown. The potentials have the same overall features for other values of the parameters Q and l, as in the scalar case.

Following Moncrief and Zerilli, we expand the electromagnetic vector potential, the electromagnetic field tensor, and the metric perturbations of the Reissner-Nordström background in terms of the Regge-Wheeler spherical harmonics.[4,18] The Einstein-Maxwell equations for the perturbed quantities have been decoupled by Moncrief[6,7] into two second-order wave equations for the even-parity and odd-parity cases. Chandrasekhar,[16] using an approach based on the Newman-Penrose formalism,[12,19] has shown how to derive solutions of the even-parity Moncrief equations from the solutions of the much simpler odd-parity Moncrief equations. Hence it is sufficient to consider only the odd-parity Moncrief equations given by Ref. 20:

$$\left\{\frac{d^2}{dr^{*2}}+k^2-\frac{\Delta}{r^2}\left[l(l+1)-\frac{3M}{r}+\frac{4Q^2}{r^2}\right]\right\}\begin{pmatrix}\hat\pi_g\\ \hat\pi_f\end{pmatrix}=\frac{\Delta}{r^3}\begin{pmatrix}3M & 2Q[(l-1)(l+2)]^{1/2}\\ 2Q[(l-1)(l+2)]^{1/2} & -3M\end{pmatrix}\begin{pmatrix}\hat\pi_g\\ \hat\pi_f\end{pmatrix}, \quad (18)$$

where $\Delta=(r-r_+)(r-r_-)$ and where Moncrief's variables π_g and π_f are related to Zerilli's electromagnetic field f_{23} and metric perturbation h_1 (in the Regge-Wheeler odd-parity gauge although the variables π_g and π_f have gauge-invariant significance) by

$$\hat\pi_f=-\frac{2[(l-1)(l+2)]^{1/2}}{l(l+1)}f_{23}=[(l-1)(l+2)]^{1/2}\pi_f\,,$$

$$\hat\pi_g=(l-1)(l+2)\frac{\Delta}{ikr}h_1=\frac{\pi_g}{l(l+1)}\,.$$

This system is decoupled by the linear transformation that diagonalizes the right-hand side of Eq. (18) (see Matzner[20] for details) and the resulting combinations here called $R_\pm$ satisfy the equations

$$\frac{d^2R_\pm}{dr^{*2}}+\left\{k^2-\frac{(r-r_+)(r-r_-)}{r^2}\left[\frac{l(l+1)}{r^2}-\frac{3M}{r^3}\pm\frac{C}{r^3}+\frac{4Q^2}{r^2}\right]\right\}R_\pm=0\,, \tag{19}$$

where

$$C=[9M^2+4Q^2(l-1)(l+2)]^{1/2}\,. \tag{20}$$

The electromagnetic and gravitational perturbations are then extracted from the $R_\pm$ solutions by the relations

$$\hat{\pi}_f=\cos\psi R_+-\sin\psi R_-\,, \tag{21}$$

$$\hat{\pi}_g=\sin\psi R_++\cos\psi R_-\,, \tag{22}$$

where

$$\sin(2\psi)=+2\frac{Q[(l-1)(l+2)]^{1/2}}{C}\,. \tag{23}$$

The decoupled perturbation equations for $R_\pm$ [Eq. (19)] are very similar to Eq. (2) for the scalar field and the general qualitative features of the evolution of the scalar field also hold for $R_\pm$. For brevity we outline only the essential details since the scalar case was carried out in such detail. Near either horizon Eq. (19) has solutions of the form $R\sim e^{\pm ikr*}$. For the case $|kM|\ll 1$ we can match the leading terms of these wave solutions from the r_+ horizon to the r_- horizon through the use of the $k=0$ solution as was done with the scalar case, and obtain analytical expressions for the low-frequency fields between the horizon. The $k=0$ solutions are discussed in detail in the Appendix. Here we note that the solutions are generalizations of Eq. (6), one being a polynomial in r and one being a polynomial times a logarithm that diverges on either horizon [cf. Eq. (A7) (in the Appendix)].

We start with initially infalling waves ($R_\pm\sim e^{+ikr*}$) on the r_+ horizon. According to Sibgatullin and Alekseev's[19] analysis in the exterior region ($r>r_+$), the power-law tails, reflection coefficients, and general analytical features for the Fourier coefficients defined (as in the scalar field case) by expansions of the $R_\pm$ fields on the r_+ horizon are qualitatively similar to the scalar field case. Following a similar analysis that leads from Eq. (12) to Eqs. (14) and (15), we obtain the following picture for the development of gravitational and electromagnetic perturbations inside the black hole: On the spacelike surface $r=$const for $t\gg M$ we find

$$R_\pm(r=\text{const},t\gg M)\sim\frac{D_\pm}{t^{2l+2}}\left[1+O\left(\frac{|r^*|}{t}\right)\right], \tag{24}$$

and near the r_- horizon approaching the point B we find

$$R_\pm(v\to\infty,\ u<0)\sim D_\pm\left[\frac{A_\pm(0)}{v^{2l+2}}+\frac{B_\pm(0)}{u^{2l+2}}\right], \tag{25}$$

where the $D_\pm$ are constants. From Eqs. (21) and (22) it follows that the electromagnetic and metric perturbations have similar power-law developments near the r_- horizon. [The complete perturbed Maxwell field and the perturbed metric follow from a knowledge of π_f and π_g (see Moncrief and Zerilli for details).] Consequently, the energy density in the electromagnetic field and the energy density in the Landau-Lifshitz pseudotensor describing the energy density in the perturbed gravitational field will have similar power-law falloff relative to the frame stationary in (r,t) coordinates. Therefore, exactly as in the scalar field case, when the fields and the energy tensors are referred to a frame carried by a freely falling observer (cf. Gürsel *et al.*, paper I) the power-law falloff of the fields and energy densities are overcome by the exponential blue-shift factor [cf. Eq. (17)] of the observer's frame as she approaches the horizon.

IV. STATIONARY EXTERNAL SOURCES

For the special case of perturbations that are independent of t (i.e., stationary in the exterior and homogeneous in the interior), the Moncrief variable π_g is not well defined and the derivation that leads to Eq. (18) breaks down. However, it is possible to still obtain an appropriate set of decoupled equations. Using Zerilli's[5] notation in the odd-parity Regge-Wheeler gauge (i.e., $h_2=0$) Maxwell's equations for the t-independent case are given by

$$l(l+1)f_{12}=\frac{d}{dr}f_{23}\,, \tag{26}$$

$$\frac{d}{dr}\left(\frac{\Delta}{r^2}f_{12}\right)-\frac{1}{r^2}f_{23}=\frac{d}{dr}\left(\frac{Q}{r^2}h_0\right)\,, \tag{27}$$

$$f_{02}=0\,, \tag{28}$$

and the relevant Einstein equations are given by

$$\frac{d^2h_0}{dr^2}-\left[\frac{2}{r^2}+\frac{l(l+1)-2}{r^2}\left(\frac{r^2}{\Delta}\right)\right]h_0=\frac{4Q}{r^2}f_{12}=\frac{4Q}{r^2l(l+1)}\frac{d}{dr}f_{23}\,, \tag{29}$$

$$h_1=0\,, \tag{30}$$

where

$$\frac{\Delta}{r^2} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}.$$

Solving Eqs. (4.1) and (4.2) for f_{23}, we find

$$\frac{d}{dr}\left(\frac{\Delta}{r^2}\frac{df_{23}}{dr}\right) - \frac{l(l+1)}{r^2}f_{23} = l(l+1)\frac{d}{dr}\left(\frac{Q}{r^2}h_0\right) . \quad (31)$$

The problem is how to deal with h_0 and f_{23} in Eqs. (29) and (31). As Moncrief has pointed out,[6] Zerilli's odd-parity (what he calls "magnetic") equations for the functions

$$R_{LM}^{(m)} = \frac{\Delta}{r^3}h_1 \quad (32)$$

and

$$f_{LM}^{(m)} = \frac{1}{l(l+1)}f_{23} \quad (33)$$

may be decoupled by a linear transformation; however, for the t-independent case (i.e., $k \equiv \omega = 0$) the variable h_1 vanishes. This difficulty may be avoided and the decoupling of Eqs. (29) and (31) accomplished by considering Zerilli's Eqs. (14) and (16):

$$l(l+1)f_{02} = -i\omega f_{23} , \quad (34)$$

$$\left(i\omega - \frac{2i\lambda}{r^2}\frac{\Delta}{r^2}\frac{1}{\omega}\right)h_1 = \frac{4Q}{r^2}\frac{f_{23}}{l(l+1)} - r^2\frac{d}{dr}\left(\frac{h_0}{r^2}\right) . \quad (35)$$

Taking the $\omega \to 0$ limit, we find

$$\lim_{\omega\to 0}\left[2i\lambda\left(\frac{\Delta}{r^3}\frac{h_1}{\omega}\right)\right] = \lim_{\omega\to 0} 2i\lambda\left(\frac{R_{lM}^{(m)}}{\omega}\right) = r^3\frac{d}{dr}\left(\frac{h_0}{r^2}\right) - \frac{4Q}{r}f_{lM}^{(m)} . \quad (36)$$

Upon substitution of the new variable π_0 defined by

$$\pi_0 = r^3\frac{d}{dr}\left(\frac{h_0}{r^2}\right) - \frac{4Q}{r}f_{lM}^{(m)} \quad (37)$$

in Eqs. (29) and (31), we find a pair of suitable equations to describe the t-independent case, viz.,

$$\left\{r^3\frac{d}{dr}\left(\frac{\Delta}{r^2}\frac{d}{dr}\right) - \left[l(l+1)r + \frac{4Q^2}{r}\right]\right\}\begin{pmatrix}\pi_0\\ f_{lM}^{(m)}\end{pmatrix} = \begin{pmatrix}-6M & 8\lambda Q\\ Q & 0\end{pmatrix}\begin{pmatrix}\pi_0\\ f_{lM}^{(m)}\end{pmatrix} . \quad (38)$$

It is interesting to note that when Eq. (38) is decoupled, the resulting equations are identical to Eq. (19) with $k=0$.

We shall illustrate the case of an electromagnetic perturbation arising from a source current in the exterior that is at rest with respect to the black hole by the example of a charged black hole in a uniform static magnetic field. This situation is mathematically similar to the scalar case treated in paper I. The problem is to extend the field to the interior and solve for the behavior of the field near the r_- horizon.

To match onto a uniform field at $r=\infty$ we keep only the $l=1$ dipole term in the multipole expansion and align the external magnetic field along the z axis. For this case the Maxwell equations decouple from the gravitational perturbation and we find the solution that matches onto a uniform field for large r is given by

$$f_{23} = A\left(1 - \frac{r^2}{Q^2}\right), \quad (39)$$

where A is a constant determined by the magnitude of the external field. From Eq. (26) we deduce for f_{12} the value

$$f_{12} = -Ar/Q^2 , \quad (40)$$

and with these values the field two-form is given by

$$F = -B\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{1/2}\sin\theta\,\omega^{\hat r}\wedge\omega^{\hat\varphi} + B\frac{Q^2}{r^2}\left(-3 + \frac{r^2}{Q^2}\right)\cos\theta\,\omega^{\hat\theta}\wedge\omega^{\hat\varphi} , \quad (41)$$

where $B = (3/4\pi)^{1/2}A/Q^2$ and $[\omega^{\hat i}]$ is the orthonormal frame given by

$$\omega^{\hat t} = \frac{(|\Delta|)^{1/2}}{r}dt , \quad \omega^{\hat\theta} = r\,d\theta ,$$

$$\omega^{\hat r} = \frac{r}{(|\Delta|)^{1/2}}dr, \quad \omega^{\hat\varphi} = r\sin\theta\,d\varphi .$$

Notice that the field between r_- and r_+ contains an electric part ($F_{\hat r\hat\varphi}$) due to the fact that the orthonormal frame we have chosen is not stationary in this region.

This solution is finite at either horizon and like the scalar case (cf. paper I) does not disrupt the r_- horizon. The field does not depend on the time of an external observer, hence any observer falling into the hole from the outside at any time will always see the same electromagnetic field.

V. CONCLUSION

We have calculated the evolution of scalar, electromagnetic, and gravitational test fields in the interior of the Reissner-Nordström geometry near the "intersection" of the event horizon

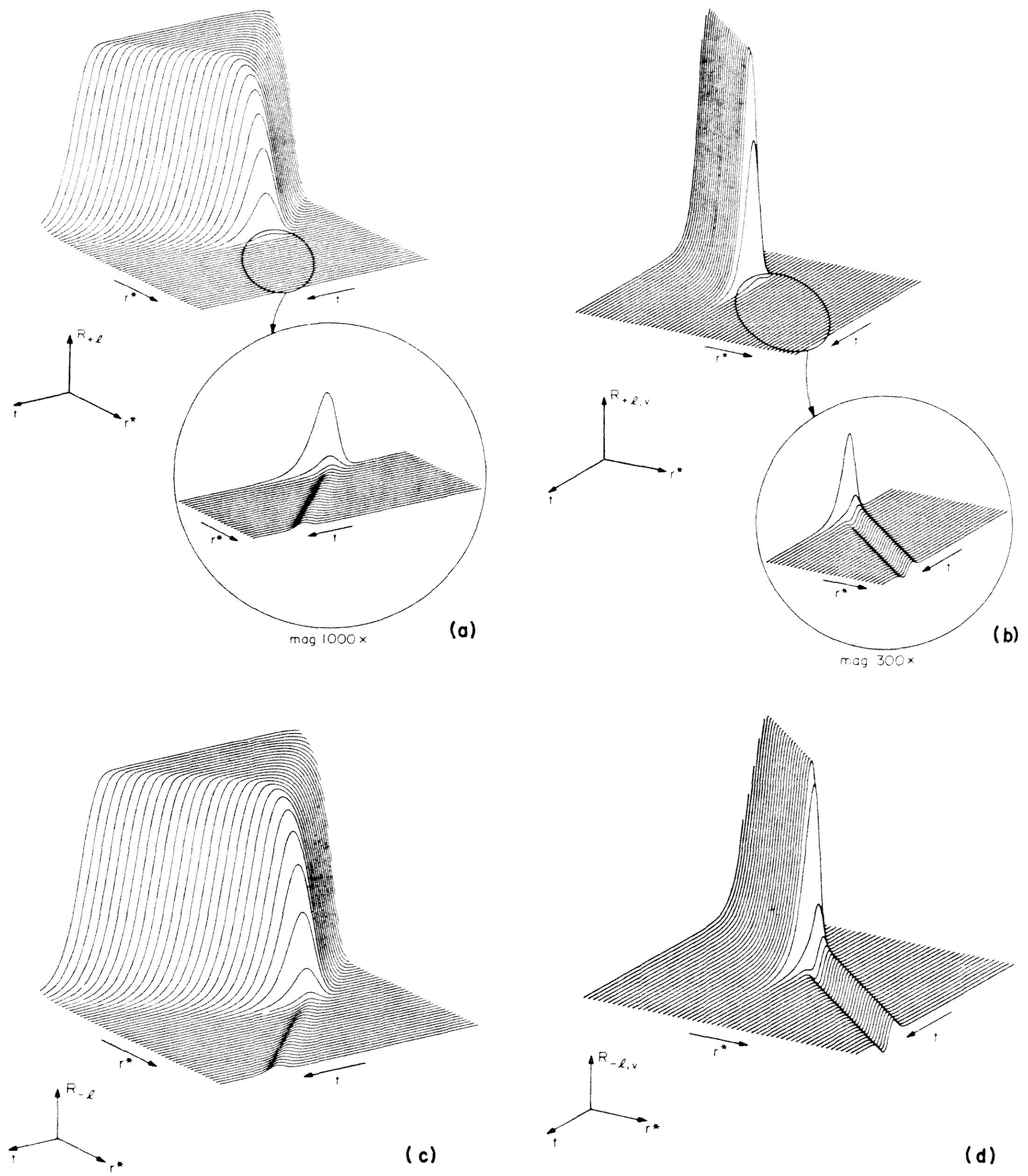

FIG. 2. These figures summarize the results of a numerical integration of the equation $R_{\pm l,r^*r^*} - R_{\pm l,tt} = V_{\pm l}(r^*)R_{\pm l}$. The results presented here are for the particular case $Q = 0.9M$ and $l = 2$; however, additional integrations with alternative parameters support the qualitative features illustrated here. (a)–(d) show R_{+l}, $R_{+l,v}$ and R_{-l}, $R_{-l,v}$, respectively, in relief as functions of r^* and t. The r^* coordinate ranges from 50M (near event horizon) to-48M (near Cauchy horizon) and t ranges from-50M to 50M. The separation between lines of constant r^* is 2M for (a) and (c) and 1.6 M for (b) and (d). The initial wave form is taken to be a Gaussian of unit amplitude. These are shown on or below the figures. The asymptotic exponential falloff of the derivatives of R_+ and R_- near the Cauchy horizon is shown in (e)–(h). (e) and (f) show the exponential behavior of $R_{+,u}$ and $R_{-,u}$, respectively, at constant v in the neighborhood of the right-hand side of the r_- horizon. (g) and (h) show the analogous behavior of $R_{+,v}$ and $R_{-,v}$, respectively, at constant u near the left-hand side of the r_- horizon.

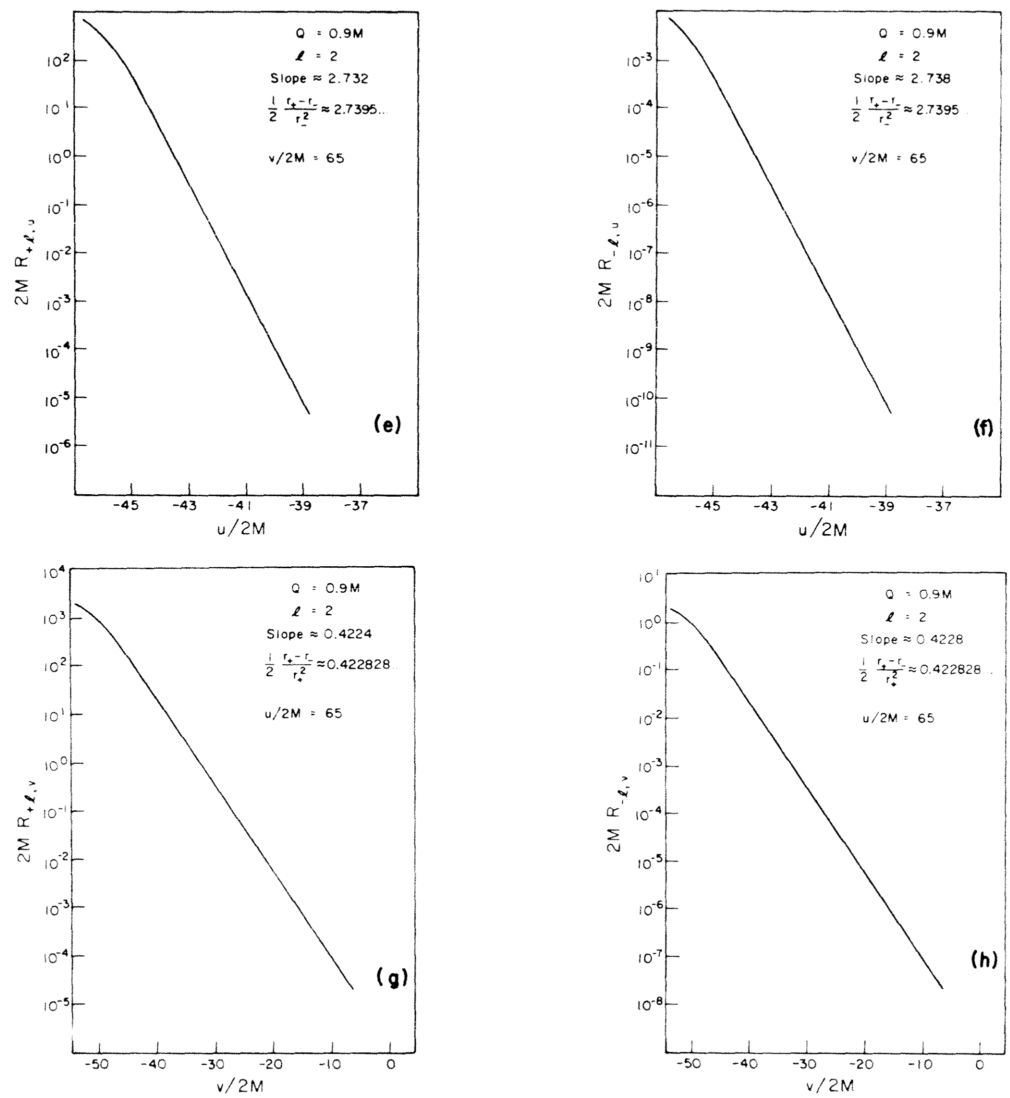

FIG. 2. *(Continued)*

($r = r_+$) and the Cauchy horizon ($r = r_-$) from initial data appropriate for long times after the formation of the charged black hole. The behavior of the fields in this region is qualitatively independent of details of the collapse process. In particular, the results are independent of whether the surface of the collapsing body approaches the left or right part of the r_- horizon (Fig. 2).

In this sense these results are universal and may be extended to the final stage of the evolution of a charged black hole, not just a collapsing body. The general picture is that at fixed times (r = constant) from the event horizon all perturbations damp according to a power-law-type behavior as we move away from the surface of a collapsing body (i.e., as $t \to \infty$). The metric becomes increasingly spherically symmetric, similar to the behavior of perturbations inside a Schwarzschild black hole (Ref. 3). However, freely falling observers will see an infinitely blue-shifted energy density or an infinite tidal shear as they approach the r_- horizon. This in turn suggests an instability in the geometry developing within the r_- horizon and its possible transformation into a spacelike curvature singularity.

We have not taken into account the quantum-mechanical process of pair creation by the classical electromagnetic and gravitational fields. In the exterior region ($r > r_+$) the Hawking process takes place on a characteristic time scale $G^2M^3/\hbar c^4$. The background electric charge Q may discharge itself by means of $e^+ - e^-$ pair creation in the outer region via electromagnetic interactions in a characteristic time $GM/c^3(e^2/\hbar c)^{-5/2}$ if $eQ \gg G^2 m_e^2 M^2/\hbar c$, but for smaller values of Q the process proceeds much more slowly (and even stops completely if $eQ < 4Gm_eM$). The Hawking effect will then dominate only long after the formation of the tails. Preliminary considerations[8] indicate that these effects also lead to a disruption of the r_- horizon. We hope to return to this question elsewhere.

ACKNOWLEDGMENTS

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APPENDIX: ZERO-FREQUENCY, ODD-PARITY SOLUTIONS OF THE ELECTROMAGNETIC AND GRAVITATIONAL PERTURBATION EQUATIONS

The electromagnetic-gravitational perturbations in the small-wave-number limit satisfy Eq. (19) with $k=0$. [Also, see Sec. IV Eq. (38).] Setting $k=0$ and changing the dependent variable by

$$G_\pm = rR_\pm$$

and using a dimensionless independent variable $x = r/2M$, Eq. (19) becomes

$$\frac{d^2G_\pm}{dx^2} + \left(\frac{-4}{x} + \frac{1}{x-r'_+} + \frac{1}{x-r'_-}\right)\frac{dG_\pm}{dx} + \left[\frac{-(l-1)(l+2)x + C_{\pm l} - 3}{x(x-r'_+)(x-r'_-)}\right]G_\pm = 0\,, \tag{A1}$$

where $r'_+ = r_+/2M$, $r'_- = r_-/2M$, and

$$C_{\pm l} = \frac{1}{2M}\left\{3M - (\pm)[9M^2 + 4Q^2(l-1)(l+2)]^{1/2}\right\}.$$

Notice that

$$C_{\pm l}(C_{\pm l} - 3) = -4\alpha^2[2 - l(l+1)]\,, \tag{A2}$$

where $\alpha = Q/2M$.

This equation is of the form

$$\frac{d^2y}{dq^2} + \left(\frac{\gamma}{q} + \frac{\delta}{q-1} + \frac{\epsilon}{q-a}\right)\frac{dy}{dq} + \left[\frac{\eta\beta q - p}{q(q-1)(q-a)}\right]y = 0\,, \tag{A3}$$

if we set $q = x/r'_-$, $a = r'_+/r'_-$, $p = -(C_{\pm l} - 3)/r'_-$, $\gamma = -4$, $\delta = 1$, $\epsilon = 1$, $\eta = -(l+2)$, and $\beta = (l-1)$. η, β, γ, δ, and ϵ satisfy the relation

$$\eta + \beta - \gamma - \delta - \epsilon + 1 = 0\,.$$

Such an equation is called Heun's equation[21] and represents a Fuchsian equation[22] of the second order with four singularities at $x = 0$, 1, a, and ∞. The constant p is called the accessory parameter whose presence is due to the fact that the solution to a Fuchsian equation of second order with four or more singularities is not completely determined by the positions of the singularities and the exponents at those singularities.[23]

Equation (A3) [or its equivalent Eq. (A1)] is known to admit polynomial solutions if both of the following conditions hold:[24]

(i) η or β should be a negative integer.

(ii) The accessory parameter should have one of its special values called the eigenvalues.

Condition (i) is clearly satisfied, but at this point one is not sure about (ii). In what follows, we will examine the solutions and the nature of the singularities of Eq. (A1). We will drop the $\pm$ subscript on $G_\pm$ and $C_{\pm l}$ with the understanding that one takes the appropriate root of (A2) for the case one has. Set

$$G = \sum_n a_n x^{n+s}\,. \tag{A4}$$

One then obtains the indicial equation

$$s(s-5) = 0 \text{ or } s_1 = 5 \text{ and } s_2 = 0\,.$$

We will try to obtain the solutions of (A1) by the method of Frobenius.[25]

In this method, one chooses the smaller root of the indicial equation, namely $s_2 = 0$. Taking a_0 a constant different from zero, we find $a_1 = ((C_l - 3)/4\alpha^2)\, a_0$ and the remaining coefficients are given by the recursion relation

$$\alpha^2(n+2)(n-3)a_{n+2} + [n(2-n) + C_l]a_{n+1} + (n-l-2)(n+l-1)a_n = 0 \,. \quad \text{(A5)}$$

When $n=3$, Eq. (A5) implies a consistency relation between a_4 and a_3 which were calculated before. This is a fifth-degree equation in C_l and is of the form

$$\{C_l(C_l - 3) + 4\alpha^2[2 - l(l+1)]\}\{\text{a cubic in } C_l\} = 0 \,. \quad \text{(A6)}$$

Equation (A2) implies that this is always satisfied. [Incidentally, the cubic shares no common roots with Eq. (A2).] Since Eq. (A6) is always satisfied, the coefficient a_5 is left totally arbitrary. Two linearly independent solutions of Eq. (A1) can be obtained by setting $a_0 = 1$, $a_5 = 0$ and $a_0 = 0$, $a_5 = 1$. Call these solutions G_1 and G_2. From Eq. (A4) it follows that both G_1 and G_2 are regular at $x=0$, i.e., the point $x=0$ is an "apparent" singular point of (A1). [An apparent singular point of the differential equation $y'' + P(x)y' + Q(x)y = 0$ is the one at which the coefficients $P(x)$ and $Q(x)$ blow up but the solutions do not.]

The following tests[25] can be made to show that $x=0$ is an apparent singularity:

(i) Write the equation as $y'' + P(x)y' + Q(x)y = 0$.

(ii) Set $P(x) = p(x)/x$; $p(0)$ should be a negative integer (−4 in our case).

(iii) Calculate $s_1 - s_2$. It should be a nonzero integer (5 in our case).

(iv) Finally, the case with two arbitrary constants (as in the case of G_1 and G_2 above) should result.

If any of these fail, the singularity is real.

We will use these tests to show that the singularities at $x = r'_+$ and $x = r'_-$ are real [i.e., one of the solutions at $x = r'_+$ and $x = r'_-$ contains $\ln(r'_+ - x)$ and $\ln(x - r'_-)$, respectively]. Consider now the recursion relation (A5). Notice that when $n = l+2$, the coefficient multiplying a_n vanishes. This fact can be used to generate a polynomial solution to Eq. (A1). The prescription is the following:

(i) Set $a_0 = 1$, leave a_5 arbitrary.

(ii) Calculate a_{l+3}. Choose a_5 so that $a_{l+3} = 0$. (This is always possible since $a_{l+3} = 0$ is a linear equation in a_5.)

Then a_{l+4} will be zero because of Eq. (A5) and so will $a_{l+5}, \ldots$. Hence the series will terminate and one will obtain a polynomial of order $l+2$ as a solution. This procedure corresponds to taking the right linear combination of G_1 and G_2 above to get a polynomial.

The first few of the solutions are given below:

$l=1$: (special case)

$$G = 1 - \frac{3}{4\alpha^2}x + \frac{6}{24\alpha^4}x^3 \,,$$

$l=2$:

$$G = 1 + \frac{C_l - 3}{4\alpha^2}x - \frac{C_l - 3}{4\alpha^4}x^3 - \frac{1}{\alpha^4}x^4 \,,$$

$l=3$:

$$G = 1 + \frac{C_l - 3}{4\alpha^2}x - \frac{C_l - 3}{2\alpha 4}x^3 - \frac{5}{\alpha 4}x^4 - \frac{30}{(C_l - 8)\alpha 4}x^5 \,,$$

$l=4$:

$$G = 1 + \frac{C_l - 3}{4\alpha^2}x - \frac{5(C_l - 3)}{6\alpha 4}x^3 - \frac{15}{\alpha 4}x^4 + \frac{21(C_l - 15)}{2\alpha^4(C_l - 6 - 6\alpha^2)}x^5 + \frac{84}{\alpha 4}\frac{1}{(C_l - 6 - 6\alpha^2)}x^6 \,.$$

The region of interest in our case is $r'_- \leq x \leq r'_+$. But since the solutions above are polynomials, they will serve as well as any other regular solution of Eq. (A1) between r'_- and r'_+. (Indeed, these are the solutions that match to the waves outside properly.) We should now determine the solution of Eq. (A1) which is not regular at r'_+ or r'_-. The singularities at these points are real. This can be seen as follows. (We will take r'_- as an example; the results are identical for r'_+.)

Try an expansion of the form

$$G = \sum_n b_n x'^{n+s} \,,$$

where $x' = x - r'_-$. One obtains the indicial equation $s^2 = 0$, i.e., $s_1 = s_2 = 0$. Write the equation as $\ddot{y} + P'(x')\dot{y} + Q'(x')y = 0$, where the single overdot denotes differentiation with respect to x'.

Now set $P'(x') = p'(x')/x'$, then $p'(0) = 1$. Using the method of Frobenius[25] does not lead to the case with two arbitrary constants. It is clear that all of the tests mentioned before give negative results. Hence the solution at r'_- will contain $\ln(x') = \ln(x - r'_-)$. If G' is the polynomial found above, then the irregular solution at $x = r'_-$ is of the form

$$G'' = AG'\ln(x - r'_-) + \sum_{n=0}^{\infty} b_n x'^n \,, \quad \text{(A7)}$$

where A is a constant and the b_n's can be determined by the method of undetermined coefficients. The precise form of the b_n's is not important in the calculations we do in the main portion of the paper.

From the expressions written above, it follows that for the regular solution

$$R'_\pm = \frac{G'_\pm}{r} \,, \quad \frac{R'_\pm(r_+)}{R'_\pm(r_-)} = (-1)^{l+1}\frac{r_+}{r_-} \,.$$

Let us normalize this solution so that $R'_{\pm}(r_+) = 1$, then

$$R'_{\pm}(r_-) = (-1)^{l+1}\frac{r_-}{r_+}.$$

If $R''_{\pm}$ denotes the irregular solution normalized in such a manner that

$$R''_{\pm}(\text{as } r \to r_+) = \ln\left(\frac{r_+ - r}{r_+}\right) + \text{const}$$

then

$$R''_{\pm}(\text{as } r \to r_-) = A\ln\left(\frac{r - r_-}{r_-}\right) + \text{const}$$

The constant A can be determined by the Wronskian of Eq. (A1),

$$W = \frac{dR'_{\pm}}{dr}R''_{\pm} - \frac{dR''_{\pm}}{dr}R'_{\pm} = \text{const},$$

near the points $r^* \to -\infty$ and $r^* \to +\infty$. We obtain

$$A = -R'_{\pm}(r_-) = (-1)^l\frac{r_-}{r_+}.$$

From this one can determine the zero-k behavior of the coefficients $\tilde{A}(k)$ and $\tilde{B}(k)$ defined in the main portion of the paper. The answer has the same form as in the scalar case.

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PART II

A. PAPER III

Multipole Moments for Stationary Systems : The Equivalence of the Geroch-Hansen Formulation and the Thorne Formulation*

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ABSTRACT

It is proved that the multipole moments of a stationary, asymptotically flat system in general relativity theory as defined by Thorne are identical, aside from normalization, to those defined by Geroch and Hansen:

$$\mathcal{I}_{a_1 \ldots\ldots a_l} = \frac{1}{(2l-1)!!}\,\mathcal{M}_{a_1 \ldots\ldots a_l} \;; \qquad \mathcal{S}_{a_1 \ldots\ldots a_l} = \frac{(l+1)}{2l\,(2l-1)!!}\,\mathcal{J}_{a_1 \ldots\ldots a_l}$$

Here $\mathcal{I}_{a_1 \ldots\ldots a_l}$ is Thorne's mass moment of order l; $\mathcal{M}_{a_1 \ldots\ldots a_l}$ is the Geroch-Hansen mass moment; $\mathcal{S}_{a_1 \ldots\ldots a_l}$ is Thorne's current moment of order l, and $\mathcal{J}_{a_1 \ldots\ldots a_l}$ is Hansen's current moment.

The mathematical techniques of Thorne are combined with those of Geroch and Hansen to prove several new theorems about multipole moments, and to give new proofs to some of the old theorems.

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§(1). Introduction and Summary

Two very different multipole moment formalisms are widely used in current research on asymptotically flat, stationary or slowly changing systems in general relativity. The first, which is beautifully elegant, was developed for precisely static systems by Geroch [1] and was extended to precisely stationary systems by Hansen [2]. The second, which is rather ugly but nevertheless has much computational power, was developed for slowly changing systems as well as precisely stationary ones by Thorne [3].

A. The Geroch-Hansen Formalism

The Geroch-Hansen analysis [1,2] is formulated in terms of the 3-dimensional manifold $\mathfrak{M}$ of time-translation Killing vector trajectories. A 3-metric h_{ab} is induced on $\mathfrak{M}$ by the spacetime metric $g_{\alpha\beta}$

$$h_{\alpha\beta} = -\vec{\xi}^2 g_{\alpha\beta} + \xi_\alpha \xi_\beta \quad , \quad \vec{\xi} = \text{Killing vector;} \tag{1a}$$

$$h_{ab} = -g_{00}\, g_{ab} + g_{0a}\, g_{0b} \quad \text{in coordinates } (t, x^a) \text{ where } \vec{\xi} = \frac{\partial}{\partial t}. \tag{1b}$$

(Here and below the notational conventions of MTW [4] and of Thorne [3] are used, including "geometrized units" with $c = G = 1$). Geroch and Hansen conformally transform $\mathfrak{M}$ to obtain a new 3-manifold $\widetilde{\mathfrak{M}}$ with metric

$$\tilde{h}_{ab} = \Omega^2 \; h_{ab} \;, \tag{2a}$$

$$\Omega = (\text{conformal factor}) \approx \frac{1}{r^2} \quad \text{at large } r, \tag{2b}$$

where r is the distance from the gravitating source. This conformal transformation brings the spatial infinity of physical spacetime and of $\mathfrak{M}$ into a "finite" location in $\widetilde{\mathfrak{M}}$, where it becomes a single point Λ. Geroch and Hansen demand that the conformal factor Ω and the metric $\tilde{h}_{ab}$ be smooth at Λ in the sense that there exist coordinates $x^{\tilde{a}}$ there, in which Ω and $\tilde{h}_{\tilde{a}\tilde{b}}$ are infinitely differentiable at Λ. Beig and Simon [5] and independently Kundu [6] have proved that for any stationary spacetime, there do exist $(\Omega, x^{\tilde{a}})$ with this smoothness property, and that $\tilde{h}_{\tilde{a}\tilde{b}}$ and Ω can even be made analytic at Λ. Geroch and Hansen introduce into $\tilde{\mathfrak{M}}$ scalar fields

$$\tilde{\Phi}_M = \Omega^{-\frac{1}{2}} \Phi_M \quad , \quad \Phi_M = \left(\frac{1}{4\lambda}\right) (\lambda^2 + \omega^2 - 1) \quad , \tag{3a}$$

$$\tilde{\Phi}_J = \Omega^{-\frac{1}{2}} \Phi_J \quad , \quad \Phi_J = \left(\frac{1}{2\lambda}\right) \omega \ , \tag{3b}$$

where λ and ω are scalar fields given by

$$\lambda = \vec{\xi}^2 \ , \tag{3c}$$

$$\vec{\nabla}\omega = *(\vec{\xi} \wedge d\vec{\xi}) \ . \tag{3d}$$

Here $*(\vec{\xi} \wedge d\vec{\xi})$ is the *twist* of $\vec{\xi}$ written in the notation of exterior calculus. They then define symmetric, trace-free (" *STF* ") tensor fields $P^M{}_{a_1 \ldots\ldots a_l}$ and $P^J{}_{a_1 \ldots\ldots a_l}$ on $\tilde{\mathfrak{M}}$ by

$$P^A = \tilde{\Phi}^A \quad , \tag{4a}$$

$$P^A{}_{a_1 \ldots\ldots a_{l+1}} = \left[\tilde{D}_{a_{l+1}} P^A{}_{a_1 \ldots\ldots a_l} - \left(\frac{1}{2}\right) l \, (2l-1) \, \tilde{R}_{a_l a_{l+1}} \, P^A{}_{a_1 \ldots\ldots a_{l-1}} \right]^{STF} \quad , \tag{4b}$$

where $\tilde{D}_a$ is the covariant derivative on $\tilde{\mathfrak{M}}$, $\tilde{R}_{ab}$ is the Ricci tensor of $\tilde{\mathfrak{M}}$ and the superscript " *STF* " means "take the symmetric, trace-free part" (cf. equation (2.2) of Thorne [3]). Geroch and Hansen then define the mass l-pole moment $\mathscr{M}_{a_1 \ldots\ldots a_l}$ and the current l-pole moment $\mathscr{J}_{a_1 \ldots\ldots a_l}$ by

$$\mathcal{M}_{a_1 \ldots\ldots a_l} = P^M{}_{a_1 \ldots\ldots a_l}(\Lambda) \quad , \tag{5a}$$

$$\mathcal{J}_{a_1 \ldots\ldots a_l} = P^J{}_{a_1 \ldots\ldots a_l}(\Lambda) \quad . \tag{5b}$$

The general idea of this formalism was invented by Geroch [1], and he gave a precise formulation for static spacetimes, which differs in some details from the above, but which yields precisely the same mass moments as the above. The above version of the formalism is due to Hansen [2].

The beauty of this formalism lies in its "general covariance", i.e. in the fact that it does not rely on any special choice of coordinate system. This beauty carries with itself considerable power for proving general theorems: Xanthopoulos [7] has proved that a stationary, vacuum, asymptotically flat spacetime is static if and only if all its current moments vanish, and also that an asymptotically flat, vacuum, static spacetime is flat if and only if all of its mass moments vanish. Beig and Simon [5] have proved that the multipole moments given by Geroch and Hansen are unique up to isometries, and that any stationary, axially symmetric, asymptotically flat, vacuum solution of Einstein's equations approaches the Kerr solution at infinity [8].

B. Thorne's Formalism

Thorne's analysis [3] is formulated entirely in the physical spacetime. It is a straightforward extension of the standard procedure of "reading out" the mass M of a source from the time-time part of the metric, and "reading out" the angular momentum $\vec{J}$ from the time-space part of the metric.

$$g_{00} = -\left[1 - \frac{2M}{r} + O\left(\frac{1}{r^2}\right)\right] , \tag{6a}$$

$$g_{0j} = -2\, \varepsilon_{jkl} J_k \frac{x^l}{r^3} + O\left(\frac{1}{r^3}\right) , \tag{6b}$$

$$g_{jk} = \delta_{jk} + O\left(\frac{1}{r}\right) , \qquad (6c)$$

where $r = \left[(x^1)^2 + (x^2)^2 + (x^3)^2 \right]^{\frac{1}{2}}$ (cf. equation (19.13) of MTW [4]). Of course, any such read out requires the introduction of a coordinate system which becomes Minkowskii sufficiently rapidly at large radii. Herein lies the ugliness of the Thorne's formalism: it requires the use of coordinates chosen from a special class, called "*ACMC*" (Asymptotically Cartesian and Mass Centered). De Donder coordinates are in this class.

In an *ACMC* coordinate system for any precisely stationary, vacuum, asymptotically flat spacetime the metric coefficients at $O(1/r^{l+1})$ contain only multipoles of order $\leq l$, and the mass dipole vanishes entirely:

$$g_{00} = -1 + \frac{2M}{r} + \sum_{l=1}^{\infty} \frac{1}{r^{l+1}} \left[\frac{2(2l-1)!!}{l!} \mathscr{I}_{A_l} N_{A_l} + S_{l-1} \right] , \qquad (7a)$$

$$\mathscr{I}_a = 0 , \qquad (7b)$$

$$g_{0j} = \sum_{l=1}^{\infty} \frac{1}{r^{l+1}} \left[-\frac{4l(2l-1)!!}{(l+1)!} \varepsilon_{jka_l} \mathscr{S}_{kA_{l-1}} N_{A_l} + S_{l-1} \right] , \qquad (7c)$$

$$g_{jk} = \delta_{jk} + \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} S_{l-1} . \qquad (7d)$$

Here S_l is a symbol denoting a quantity which is independent of r and has angular dependence with spherical harmonics of order $l, l-1, \ldots, 0$; also the subscript A_l is shorthand for $a_1 \ldots a_l$; N_{A_l} is shorthand for $n_{a_1} \ldots n_{a_l}$ where $n_a = x^a / r$; ε_{ijk} is the flat space Levi-Civita tensor, and there is here and below an implied summation over repeated Latin indices. The quantities $\mathscr{I}_{A_l} = \mathscr{I}_{a_1 \ldots a_l}$ and $\mathscr{S}_{A_l} = \mathscr{S}_{a_1 \ldots a_l}$ are the source's mass moment and the source's current moment of order l.

Thorne's formalism has a number of properties which make it powerful for astrophysical calculations: *First*, it is a specialization to the stationary case of Thorne's multipole moment formalism for asymptotically flat systems which change dynamically on time scales $\tau \gg L =$ (size of the source), where

$$L \equiv Max \left\{ M \, , \left| \frac{\mathcal{I}_{a_1 a_2}}{M} \right|^{\frac{1}{2}} , \ldots\ldots , \left| \frac{\mathcal{I}_{A_l}}{M} \right|^{\frac{1}{l}} , \left| \frac{\mathcal{S}_{A_l}}{M} \right|^{\frac{1}{l}} \right\} . \tag{8}$$

(See § T,IX; i.e. § IX of Thorne's paper [3].) Such "slow motion" sources radiate gravitational waves, and the multipole moments which one reads off the near zone form (7) of the metric are identical to those which characterize the gravitational radiation field in the "local wave zone" (§ T,IV):

$$h^{TT}{}_{jk} = \left[\sum_{l=2}^{\infty} \frac{1}{r} \frac{4}{(l+1)!} \, {}^{(l)}\mathcal{I}_{jkA_{l-2}}(t-r) \;\; N_{A_{l-2}} \right. \tag{9}$$

$$\left. + \sum_{l=2}^{\infty} \frac{1}{r} \frac{8l}{(l+1)!} \varepsilon_{pq(j} \;\; {}^{(l)}\mathcal{S}_{k)A_{l-2}}(t-r) \;\; n_q \;\; N_{A_{l-2}} \right]^{TT} .$$

(Here " TT " means "take the transverse, traceless part", " $(t-r)$ " means "evaluated at the retarded time $t-r$ ", and a prefix superscript (l) means "take l time derivatives".) *Second*, there exists an explicit prescription for computing from $\mathcal{I}_{A_l}$ and $\mathcal{S}_{A_l}$ the metric coefficients $g_{\alpha\beta}$ in de Donder gauge to any desired order in $1/r$ (§ § T,IX and T,X; also see Appendix). *Third*, for any source whose non-vacuum interior can be covered by de Donder coordinates, the moments $\mathcal{I}_{A_l}$ and $\mathcal{S}_{A_l}$ can be computed from integrals over the source (§ T,V). In the precisely stationary case, the source integrals in the de Donder coordinates are:

$$_{A_l} = \left[\int \tau^{00} \, X_{A_l} \, d^3x \, \right]^{STF} , \tag{10a}$$

$$\mathcal{S}_{A_l} = \left[\int \varepsilon_{a_l pq}\, x^p\, \tau^{0q}\, X_{A_{l-1}}\, d^3x \right]^{STF} . \tag{10b}$$

where τ^{00} and τ^{0j} are the "effective" energy density and momentum density of the de Donder formalism (equations T,5.3), and $X_{A_l} = x^{a_1} \ldots\ldots x^{a_l}$. For weakly gravitating sources τ^{00} becomes the mass density and $\mathcal{I}_{A_l}$ is its l'th moment as usually defined in flat space, while $\varepsilon_{a_l pq}\, x^p\, \tau^{0q}$ becomes " $\rho\, \vec{r} \times \vec{v}$ " = (angular momentum density) and $\mathcal{S}_{A_l}$ is its $(l-1)$'th moment.

Recent astrophysical applications of the slow motion version of Thorne's formalism include analyses of g-mode pulsations of neutron stars [9], torsional oscillations of neutron star crusts [10], the precession of a slowly, rigidly rotating system with arbitrary shape and arbitrarily strong internal gravity [11], and the gravitationally forced precession of a rapidly rotating black hole [12].

C. The New Results in this Paper

The main objective of this paper is to show that for precisely stationary systems, the moments of Thorne are equivalent to those of Geroch and Hansen; more precisely, that

$$\mathcal{M}_{A_l} = (2l-1)!! \quad \mathcal{I}_{A_l} , \tag{11a}$$

$$\mathcal{J}_{A_l} = \left[\frac{2l}{(l+1)} \right] (2l-1)!! \quad \mathcal{S}_{A_l} . \tag{11b}$$

The proof is given in §(2). This equivalence is then used in §(3) to prove new theorems about multipole moments. In this section we also review and sketch new proofs to several old theorems and pose an important conjecture about multipole moments in general relativity.

§(2): Equivalence of Geroch-Hansen and Thorne Moments

In this section we prove the equivalence of the Geroch-Hansen moments and the Thorne moments; i.e. we derive equations (11). As underpinnings of our proof we first present, in the next two paragraphs, two mathematical lemmas:

A. Mathematical Lemmas

Lemma 1: Let $\left\{ h_{ab} \, , x^{a} \right\}$ be a metric and a coordinate system which are harmonic on a 3-dimensional manifold, i.e. $\left[(h)^{\frac{1}{2}} h^{ab} \right]_{,b} = 0$ where $h \equiv \det \| h_{jk} \|$. Let ω be a conformal factor and $x^{\tilde{a}}$ be a new coordinate system so that

$$\tilde{h}_{\tilde{a}\tilde{b}} = \omega^2 \, h_{cd} \, \frac{\partial x^{\tilde{a}}}{\partial x^{c}} \, \frac{\partial x^{\tilde{b}}}{\partial x^{d}} \quad , \tag{12}$$

where $\tilde{h}_{\tilde{a}\tilde{b}}$ is the conformally transformed metric in the tilded coordinates. Then $\left\{ \tilde{h}_{\tilde{a}\tilde{b}} \, , x^{\tilde{a}} \right\}$ are harmonic, that is to say $\left[(\tilde{h})^{\frac{1}{2}} \tilde{h}^{\tilde{a}\tilde{b}} \right]_{,\tilde{b}} = 0$, if and only if $x^{\tilde{a}}(x^{b})$ are solutions of the differential equation

$$D_b \left[\omega \, D^{b} \, x^{\tilde{a}} \right] = 0 \quad , \tag{13}$$

where D_b is the covariant derivative with respect to h_{ab}, and $x^{\tilde{a}}$ is regarded as a scalar function of x^{b}. This equation can be written as

$$h^{ab} \, \frac{\partial}{\partial x^{a}} \left[\omega \, \frac{\partial x^{\tilde{c}}}{\partial x^{b}} \right] = 0 \tag{14}$$

since $\left\{ h_{ab} , x^{a} \right\}$ are harmonic. Proof: Denote $\partial x^{\tilde{a}} / \partial x^{b} = X^{\tilde{a}}{}_{b}$. Then,

$$\tilde{h} = \det \| \tilde{h}_{\tilde{a}\tilde{b}} \| = \det \| \omega^2 h_{cd} X^c{}_{\tilde{a}} X^d{}_{\tilde{b}} \| = \omega^6 h \left[\det \| X^{\tilde{a}}{}_b \| \right]^{-2} , \tag{15}$$

where $h = \det \| h_{ab} \|$. Then

$$\left[(\tilde{h})^{\frac{1}{2}} \tilde{h}^{\tilde{a}\tilde{b}} \right]_{,\tilde{b}} = (h)^{\frac{1}{2}} h^{cd} \left[\omega X^{\tilde{a}}{}_c X^{\tilde{b}}{}_d \left[\det \| X^{\tilde{e}}{}_f \| \right]^{-1} \right]_{,\tilde{b}} . \tag{16}$$

One can show that

$$X^{\tilde{a}}{}_b = \frac{1}{2} \left[\det \| X^{\tilde{e}}{}_f \| \right] \delta^{\tilde{a}\tilde{c}\tilde{d}}{}_{bgh} X^g{}_{\tilde{c}} X^h{}_{\tilde{d}} . \tag{17}$$

This leads to

$$\left[(\tilde{h})^{\frac{1}{2}} \tilde{h}^{\tilde{a}\tilde{b}} \right]_{,\tilde{b}} = \frac{(h)^{\frac{1}{2}}}{\det \| X^{\tilde{e}}{}_f \|} \frac{1}{(h)^{\frac{1}{2}}} \left[\omega (h)^{\frac{1}{2}} h^{cd} \frac{\partial x^{\tilde{a}}}{\partial x^c} \right]_{,d} , \tag{18}$$

which gives

$$\left[(\tilde{h})^{\frac{1}{2}} \tilde{h}^{\tilde{a}\tilde{b}} \right]_{,\tilde{b}} = \frac{(h)^{\frac{1}{2}}}{\det \| X^{\tilde{e}}{}_f \|} D_b \left[\omega D^b x^{\tilde{a}} \right] . \tag{19}$$

So, in general, $\left[(\tilde{h})^{\frac{1}{2}} \tilde{h}^{\tilde{a}\tilde{b}} \right]_{,\tilde{b}} = 0$ if and only if $D_b \left[\omega D^b x^{\tilde{a}} \right] = 0$. QED.

Lemma 2: Let $\left\{ h^{ab} , x^a \right\}$ be a positive definite metric and a coordinate system on a 3-dimensional manifold, with h^{ab} harmonic and analytic functions of x^a in some neighborhood of a point Λ, and with $h_{ab}(\Lambda) = \delta_{ab}$. Let ω be a conformal factor which is an analytic function of the x^a with $\omega(\Lambda) = 1$, and introduce new coordinates $x^{\tilde{a}}$ in which the conformally transformed metric

$$\tilde{h}^{\tilde{a}\tilde{b}} = \omega^2 h^{cd} \frac{\partial x^{\tilde{a}}}{\partial x^c} \frac{\partial x^{\tilde{b}}}{\partial x^d} \tag{20}$$

is harmonic, and which have the form

$$x^{\tilde{a}} = x^a + O[\,(x^b)^2\,] \quad \text{near } \Lambda . \tag{21}$$

Then the new coordinates $x^{\tilde{a}}$ are analytic functions of the old ones x^a; and $\tilde{h}^{\tilde{a}\tilde{b}}$ and ω are analytic functions of $x^{\tilde{a}}$ in some neighborhood of Λ. Proof: By Lemma 1 $x^{\tilde{a}}(x^b)$ must satisfy the differential equation (15). This differential equation is elliptic in some neighborhood of Λ since h^{ab} and ω are nonzero at Λ. The analyticity of h^{ab} and of ω together with Morrey's theorem [13] then implies that $x^{\tilde{a}}$ are analytic functions of x^b; and this, with definition (20) of $\tilde{h}^{\tilde{a}\tilde{b}}$, implies that ω and $\tilde{h}^{\tilde{a}\tilde{b}}$ are analytic functions of $x^{\tilde{a}}$. QED.

B. Proof of Equivalence

We now turn from mathematical lemmas to the proof of the equivalence of the Geroch-Hansen moments and the Thorne moments. Our proof begins by introducing into the physical spacetime manifold $\mathfrak{X}$ the stationary, de Donder coordinate system (x^0 , x^a)of Thorne's formalism. Because $\partial/\partial x^0$ is the Killing vector, the de Donder spatial coordinates x^a of $\mathfrak{X}$ are constant on Killing trajectories and thus can be used as coordinates for the manifold $\mathfrak{M}$ of Killing trajectories. Theorem: When this is done, the resulting $\left\{ h_{ab} \, , x^a \right\}$ are harmonic. Proof: $\vec{\xi} = \partial/\partial x^0$ is the Killing vector. So,

$$\xi^\alpha = \delta^\alpha{}_0 \quad , \quad \delta^\alpha{}_\beta \equiv \text{Kronecker } \delta \ , \tag{22}$$

and

$$\xi_\alpha = g_{0\alpha} \ , \tag{23}$$

$$\lambda = \vec{\xi}^2 = \xi^\alpha \xi_\alpha = g_{00} \ . \tag{24}$$

Hence,

$$h_{ab} = -g_{00}\, g_{ab} + g_{0a}\, g_{0b} \quad , \tag{25}$$

$$h^{ab} = \frac{g^{ab}}{-g_{00}} \quad , \tag{26}$$

$$h = \det \| h_{ab} \| = (-g_{00})^2 \left[-\det \| g_{\alpha\beta} \| \right] \; . \tag{27}$$

The de Donder gauge condition is:

$$\left[(-g)^{\frac{1}{2}} g^{\alpha\beta} \right]_{,\beta} = 0 \quad , \tag{28}$$

where $g = det \, \| g_{\alpha\beta} \|$. Equations (26), (27), (28) imply

$$\left[(h)^{\frac{1}{2}} h^{ab} \right]_{,b} = 0 \tag{29}$$

QED.

The metric h_{ab} of $\mathfrak{M}$ and Hansen's potentials Φ_M and Φ_J are easily computed from Thorne's spacetime metric (7). By combining (7) with equations (25), (26) and (3) we obtain

$$h_{ab} = \delta_{ab} + \sum_{l=2}^{\infty} \frac{1}{r^{l+1}} S_{l-1} \quad , \tag{30}$$

$$h^{ab} = \delta^{ab} + \sum_{l=2}^{\infty} \frac{1}{r^{l+1}} S_{l-1} \quad , \tag{31}$$

$$\lambda = g_{00} = -1 + \frac{2M}{r} + \Psi + \sum_{l=2}^{\infty} \frac{1}{r^{l+1}} S_{l-1} \quad , \tag{32a}$$

where

$$\Psi = \sum_{l=2}^{\infty} \frac{1}{r^{l+1}} \frac{2(2l-1)!!}{l!} \quad \mathscr{I}_{A_l} N_{A_l} \quad . \tag{32b}$$

Equations (3c), $\xi^\alpha = \delta^\alpha{}_0$, and the spacetime metric (7) imply that

$$\omega_0 = 0 \ , \tag{33}$$

$$\omega_i = \varepsilon_{0ijk}\, g^{jl}\, g^{km} \left[g_{0l,m} - \frac{g_{00,m}}{g_{00}} g_{0l} \right] \tag{34}$$

$$= \sum_{l=1}^{\infty} -\frac{4l(2l-1)!!}{(l+1)!} \varepsilon_{ijk}\, \varepsilon_{jma_l}\ \mathcal{S}_{mA_{l-1}} \left\{ \frac{1}{r^{l+1}} N_{A_l} \right\}_{,k} + \frac{1}{r^{l+2}} S_{l-1} \ ,$$

where ε_{0ijk} is the curved-spacetime Levi-Civita tensor, while ε_{ijk} and ε_{jma_l} are flat space antisymmetric symbols. From this expression, $\omega_\alpha = \omega_{,\alpha}$, and the fact that all the tensors $\mathcal{S}_{A_l}$ are symmetric and trace free, it follows that

$$\omega = \sum_{l=1}^{\infty} \frac{1}{r^{l+1}} \left[-\frac{4l(2l-1)!!}{(l+1)!}\ \mathcal{S}_{A_l} N_{A_l} + S_{l-1} \right] \ . \tag{35}$$

Equations (32) and (35) give

$$\Phi_M = \frac{M}{r} + \sum_{l=2}^{\infty} \frac{1}{r^{l+1}} \left[\frac{(2l-1)!!}{l!}\ \mathcal{I}_{A_l} N_{A_l} + S_{l-1} \right] \ , \tag{36}$$

$$\Phi_J = \sum_{l=1}^{\infty} \frac{1}{r^{l+1}} \left[\frac{2l(2l-1)!!}{(l+1)!}\ \mathcal{S}_{A_l} N_{A_l} + S_{l-1} \right] \ . \tag{37}$$

We must now choose a conformal factor Ω to take us from $\mathfrak{M}$ to $\tilde{\mathfrak{M}}$ (eqs. 2 above). One attractive choice might be that of Beig and Simon [5]:

$$\Omega_{BS} = (1/2B^2) \left[(\ 1 + 4\Phi_M{}^2 + 4\Phi_J{}^2\ )^{\frac{1}{2}} - 1 \right] \ , \tag{38}$$

$$B \equiv \text{constant chosen to make } \tilde{D}_a\, \tilde{D}_b\, \Omega = 2\tilde{h}_{ab} \ \text{ at } \Lambda \ ,$$

which has the beautiful property that, if $x^{\tilde{a}}{}_{BS}$ are coordinates in which $\tilde{h}_{\tilde{a}\tilde{b}}{}^{BS} = \Omega^2\, h_{\tilde{a}\tilde{b}}$ is harmonic, then Ω_{BS}, $\tilde{h}_{\tilde{a}\tilde{b}}{}^{BS}$, $\tilde{\Phi}_M{}^{BS}$, and $\tilde{\Phi}_J{}^{BS}$ are all analytic functions of $x^{\tilde{\alpha}}{}_{BS}$; see Appendix. Lemma 2 enables us to generate other attractive choices: We just multiply Ω_{BS} by any analytic function ω of $x^{\tilde{\alpha}}{}_{BS}$ with $\omega(\Lambda) = 1$ to obtain our new $\Omega = \omega\,\Omega_{BS}$, and then change to coordinates $x^{\tilde{\alpha}}$ in which the new $\tilde{h}_{\tilde{a}\tilde{b}} = \Omega^2 h_{\tilde{a}\tilde{b}} = \omega^2 \tilde{h}_{\tilde{a}\tilde{b}}{}^{BS}$ is harmonic; Lemma 2 guarantees that the resulting $\tilde{h}_{\tilde{a}\tilde{b}}$, Ω, $\tilde{\Phi}_M$, and $\tilde{\Phi}_J$ will all be analytic functions of the new coordinates $x^{\tilde{\alpha}}$. One such choice is that of Kundu [6]:

$$\Omega_K = \frac{{\Phi_M}^2 + {\Phi_J}^2}{M^2 + J^2} \,, \quad J \text{ is the current monopole} \,, \tag{39}$$

which is related to Ω_{BS} by $\Omega_K = \omega\,\Omega_{BS}$ with

$$\omega = \frac{2B^2}{M^2 + J^2} \frac{{\Phi_M}^2 + \quad {\Phi_J}^2}{(1 + 4{\Phi_M}^2 + 4{\Phi_J}^2)^{\frac{1}{2}} - 1} = \frac{{\tilde{\Phi}_M}{}^{BS2} + {\tilde{\Phi}_J}{}^{BS2}}{M^2 + J^2} \,, \tag{40}$$

which is an analytic function of $x^{\tilde{\alpha}}{}_{BS}$. Another such choice is that of Geroch [1]:

$$\Omega_G = {\Phi_M}^2 / \; M^2 \,, \tag{41}$$

which is related to Ω_{BS} by $\Omega_G = \omega\,\Omega_{BS}$ with

$$\omega = \frac{2B^2}{M^2} \frac{{\Phi_M}^2}{(1 + 4{\Phi_M}^2 + 4{\Phi_J}^2)^{\frac{1}{2}} - 1} = \frac{{\tilde{\Phi}_M}{}^{BS2}}{M^2} \,, \tag{42}$$

which again is an analytic function of $x^{\tilde{\alpha}}{}_{BS}$. Because Ω_G is the simplest of the three, we tentatively select it. Equation (36) for Φ_M implies that

$$\Omega_G = \frac{1}{r^2}\left\{1 + \sum_{l=2}^{\infty} \frac{1}{r^l}\left[\quad \mathcal{Y}_{A_l} N_{A_l} + S_{l-1} \right]\right\} \,, \tag{43a}$$

where

$$\mathcal{Y}_{A_l} = \frac{4(2l-1)!!}{l!}\left(\frac{\mathcal{I}_{A_l}}{2M} \right) + \tag{43b}$$

$$\sum_{m=2}^{l-2} \frac{2(2m-1)!!}{m!} \frac{2(2m'-1)!!}{m'!} \frac{\mathcal{I}_{A_m} \mathcal{I}_{a_{m+1}\cdots\cdots a_l}}{4M^2}$$

and $m + m' = l$. The coordinates $x^{\tilde{a}}$ in which $\tilde{h}_{ab} = \Omega_G{}^2 h_{ab}$ is harmonic and analytic are related to the harmonic coordinates x^a of $\mathfrak{M}$ by the differential equation (14) of Lemma 1

$$h^{ab} \frac{\partial}{\partial x^a} \left[\Omega_G \frac{\partial x^{\tilde{c}}}{\partial x^b} \right] = 0 \ , \tag{44}$$

which can be solved analytically using the ansatz

$$x^{\tilde{a}} = \frac{x^a}{r^2} \left[1 + \sum_{l=2}^{\infty} \frac{1}{r^l} \mathcal{A}_{A_l} N_{A_l} \right] + \sum_{l=2}^{\infty} \frac{1}{r^{l+1}} \left[\mathcal{B}_{aA_{l-1}} N_{A_{l-1}} + S_{l-1} \right] \ . \tag{45}$$

By combining equations (30), (43), (44), and (45) one can obtain explicit expressions for the STF constants $\mathcal{A}_{A_l}$ and $\mathcal{B}_{A_l}$ of this ansatz in terms of Thorne's moments $\mathcal{I}_{A_l}$ and $\mathcal{S}_{A_l}$. Fortunately, we shall not need these complicated expressions.

To simplify subsequent steps of the calculation we shall now modify our conformal factor slightly. In place of Ω_G we use

$$\Omega' = \omega \, \Omega_G \ , \ \omega \equiv 1 + \sum_{l=2}^{\infty} \mathcal{C}_{A_l} X^{\tilde{A}_l} \ , \tag{46}$$

where $\mathcal{C}_{A_l}$ are STF constants to be fixed below and ω is obviously an analytic function of $x^{\tilde{a}}$. We also modify slightly our coordinates by means of an analytic transformation:

$$x^{a'} = x^{\tilde{a}} \left[1 + \sum_{l=2}^{\infty} \mathcal{D}_{A_l} X^{\tilde{A}_l} \right] + \sum_{l=2}^{\infty} \mathcal{E}_{aA_{l-1}} X^{\tilde{A}_{l-1}} \tilde{r}^2 \ . \tag{47}$$

We choose the three families of STF constants $\mathscr{C}_{A_l}$, $\mathscr{D}_{A_l}$, and $\mathscr{E}_{A_l}$ in such a way that (43), (45), and (47) combine to give

$$x^{a'} = \frac{x^a}{r^2} + \frac{1}{r}\sum_{l=2}^{\infty}\frac{S_{l-1}}{r^l} \quad . \tag{48}$$

$$\Omega' = \omega\,\Omega_G = \frac{1}{r}\left[1 + \sum_{l=2}^{\infty}\frac{S_{l-1}}{r^l}\right] \quad . \tag{49}$$

Then the conformally transformed metric (eqs. 30, 48, 49) is

$$\widetilde{h}'_{a'b'} = \Omega^2\, h_{cd}\,(\,\partial x^c / \partial x^{a'}\,)\,(\,\partial x^d / \partial x^{b'}\,) \tag{50a}$$

$$= \omega^2\, \widetilde{h}_{\widetilde{c}\widetilde{d}}\,(\,\partial x^{\widetilde{c}} / \partial x^{a'}\,)\,(\,\partial x^{\widetilde{d}} / \partial x^{b'}\,)$$

$$= \delta_{a'b'} + \sum_{l=2}^{\infty} r'^{l}\, S_{l-1}$$

and the transformed fields Φ_M and Φ_J (eqs. 36, 37, and 49) are

$$\widetilde{\Phi}'_M = \Omega^{-\frac{1}{2}}\,\Phi_M = \omega^{-\frac{1}{2}}\,\widetilde{\Phi}_M \tag{50b}$$

$$= M + \sum_{l=2}^{\infty}\left[\frac{(2l-1)!!}{l!}\,\mathscr{I}_{A'_l}\, X^{A'_l} \qquad + r'^{l}\, S_{l-1}\right] \quad .$$

$$\widetilde{\Phi}'_J = \Omega^{-\frac{1}{2}}\,\Phi_J = \omega^{-\frac{1}{2}}\,\widetilde{\Phi}_J \tag{50c}$$

$$= \sum_{l=1}^{\infty}\left[\frac{2l\,(2l-1)!!}{(l+1)!}\,\mathscr{S}_{A'_l}\, X^{A'_l} \quad + r'^{l}\, S_{l-1}\right] \quad .$$

Moreover, because ω, $x^{a'}$, $\widetilde{h}_{\widetilde{a}\widetilde{b}}$, $\widetilde{\Phi}_M$, $\widetilde{\Phi}_J$ are all analytic functions of $x^{\widetilde{a}}$ near Λ, equations (50)-(52) guarantee that $\widetilde{h}'_{a'b'}$, $\widetilde{\Phi}'_M$, and $\widetilde{\Phi}'_J$ are all analytic functions of $x^{a'}$.

The computation of the Geroch-Hansen moments from equations (4), (5), and (50) is then straightforward. Because the r'^{l} part of $\tilde{h}'_{a'b'}$ involves spherical harmonics only of order $l-1$, $l-2$,......., 0 (eq. 50a), the connection coefficients of $\tilde{D}'_{a'}$ and the Ricci tensor terms contribute nothing to the moments. And because of the analyticity the unknown terms in (50) denoted S_{l-1} never give any problems in the computation. The result of the computation is

$$\mathcal{M}_{A_l} = l\text{'th mass multipole of Hansen} = (2l-1)!! \quad \mathcal{I}_{A_l} \tag{51}$$

$$\mathcal{J}_{A_l} = l\text{'th current multipole of Hansen} = \frac{2l}{(l+1)}(2l-1)!! \quad \mathcal{S}_{A_l} \; . \tag{52}$$

§(3). Theorems and Conjectures About Multipole Moments of Stationary, Asymptotically Flat Systems

In their pioneering work Geroch [1] and Hansen [2] posed several conjectures about multipole moments of stationary, asymptotically flat systems. Some of those conjectures have been proved by Xanthopoulos [7], Kundu [6], and Beig and Simon [5] using the Geroch-Hansen formalism. Others of those conjectures have evaded proof by Geroch-Hansen methods but are trivial and obvious in Thorne's formalism. Now that the equivalence of the two formalisms is known, we can combine their associated theorems thereby fleshing out our understanding of multipole moments in general relativity. Below we list the various theorems and conjectures and briefly describe the status of their proofs. Note that in Theorems (i)-(v) we tacitly restrict attention to spacetimes for which the multipole expansion of the metric converges; while in conjectures (vi) and (vii) we address the issue of convergence.

(i) Theorem: A stationary spacetime is static if and only if all of its current moments vanish. Proofs: Xanthopoulos [7] has given an elegant proof using Geroch-Hansen methods that if the current moments vanish, than the spacetime is static; and the proof of this would be difficult in Thorne's formalism. That a static spacetime has vanishing current moments is trivial to prove in both formalisms.

(ii) Theorem: A static spacetime is flat if and only if all of its mass moments vanish. Proofs: Xanthopoulos [7] has given an elegant proof by Geroch-Hansen methods. The proof in Thorne's formalism follows trivially from equations (7) for the moments in terms of the metric and from Thorne's [3] unique and precise algorithm (see Appendix) for generating the metric in de Donder coordinates from its multipole moments.

(iii) Theorem: A stationary spacetime is axisymmetric if and only if all of its moments are axisymmetric. Proof: Again the proof in Thorne's formalism follows trivially from equation (7) for the moments in terms of the metric and from Thorne's algorithm for the metric in terms of the moments.

(iv) Theorem: Two spacetimes with the same multipole moments have the same spacetime geometry at large radii, where the multipole expansion of the metric converges. Proofs: Beig-Simon [5] and Kundu [6] proved this theorem using Geroch-Hansen methods. Again the proof in Thorne's formalism follows easily from equation (7) for the moments in terms of the metric and from Thorne's algorithm for the metric in terms of the moments.

(v) Theorem: Any stationary, axially symmetric, asymptotically flat solution of Einstein's equations approaches the Kerr solution near infinity. Proofs: This was proved by Beig and Simon using Geroch-Hansen methods and expanding the gravitational field to quadratic order in $1/r$. This follows trivially from the explicit form (7) of the metric in Thorne's formalism; cf. §XI.D of Thorne

[3].

(vi) Conjecture: Given any two sets $\left\{ \mathcal{J}_{A_l}, \mathcal{S}_{A_l} \right\}$ of symmetric, trace-free tensors satisfying $\left| \mathcal{J}_{A_l} N_{A_l} \right| < \frac{ML^l}{(2l-1)!!}$ and $\left| \mathcal{S}_{A_l} N_{A_l} \right| < \frac{(l+1)}{2l} \frac{ML^l}{(2l-1)!!}$ for all N_{A_l} and for some constants M and L, the metric generated as an infinite series by Thorne's algorithm will converge; and consequently, there exists a unique spacetime geometry with these moments. Status of Proof: It is not difficult to verify that allthe pieces $\gamma_{\mu\nu}{}^{p}$ of Thorne's metric (as defined in his §X of [3]) individually converge; a proof that the series $\mathfrak{g}_{\alpha\beta} = \sum_{p} \gamma_{\alpha\beta}{}^{p}$ converges has not yet been found.

(vii) Conjecture: Given any stationary, asymptotically flat, vacuum solution of Einstein's equations, the moments computed by Thorne's method will satisfy the conditions in (vi) for some M (the mass of the "source") and L (its characteristic "size"). The moments calculated by the Geroch-Hansen method will satisfy a similar relationship as determined by eqs. 51-52. Status of Proof: A proof has not yet been attempted.

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Appendix

Analyticity of the Beig-Simon Metric $\tilde{h}_{\tilde{a}\tilde{b}}$

and

Explicit Form of Thorne's Metric $g_{\alpha\beta}$

In their proof for the analyticity of the metric $\tilde{h}_{\tilde{a}\tilde{b}}$, Beig and Simon [5] require that $\tilde{h}_{\tilde{a}\tilde{b}}$ be a $C^{4,\alpha}$ metric on $\tilde{\mathfrak{M}}$. (A function is said to be of class $C^{n,\alpha}$ if and only if its n-th derivatives exist and are Hölder-continuous with exponent $0 < \alpha < 1$) In this Appendix we will show that this condition is satisfied by Thorne's metric by explicitly calculating the first few terms (up to order $(1/r^5)$) of Thorne's metric.

A. Algorithm for calculating the metric to any desired order

The foundation for the computation is

$$\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} - \bar{h}^{\alpha\beta} = \eta^{\alpha\beta} - \sum_{p=1}^{\infty} \gamma_p{}^{\alpha\beta} \tag{A.1}$$

where $\mathfrak{g}^{\alpha\beta} \equiv (-g)\, g^{\alpha\beta}$, $(-g) = (-\det \| g^{\alpha\beta} \|)^{-1}$, $g^{\alpha\beta}$ is Thorne's metric and p is a "nonlinearity parameter". Define $\gamma^{\alpha\beta} = \sum_{p=1}^{\infty} \gamma_p{}^{\alpha\beta}$. One raises and lowers indices on $\gamma_p{}^{\alpha\beta}$ with $\eta_{\alpha\beta}$ which is the flat space metric. The linear terms $\gamma_1{}^{\alpha\beta}$ are (see §T,X)

$$\gamma^1{}_{00} = \gamma_1{}^{00} = \frac{4M}{r} + \sum_{l=2}^{\infty} \frac{4(2l-1)!!}{l\,!} \frac{\mathscr{I}_{A_l} N_{A_l}}{r^{l+1}} \tag{A.2}$$

$$\gamma^1{}_{0j} = -\gamma_1{}^{0j} = -\sum_{l=1}^{\infty} \frac{4l\,(2l-1)!!}{(l+1)!} \frac{\varepsilon_{jpq} \; \mathscr{S}_{pA_{l-1}}}{r^{l+1}} N_{A_{l-1}} n_q \tag{A.3}$$

$$\gamma^1{}_{jk} = \gamma_1{}^{jk} = 0 \tag{A.4}$$

One then calculates next order $\gamma_p{}^{\alpha\beta}$'s by integrating the differential equations

$$\gamma_p{}^{\alpha\beta}{}_{,jj} = W_p{}^{\alpha\beta} \tag{A.5}$$

where $W_p{}^{\alpha\beta}$ is made entirely out of lower p γ's than the one in question and is given by

$$W_p{}^{\alpha\beta} = \Big\{ -\frac{1}{2}\ \mathfrak{g}^{\alpha\beta}\ \mathfrak{g}_{\lambda\mu}\,\gamma^{\lambda\nu}{}_{,\rho}\,\gamma^{\rho\mu}{}_{,\nu} + \ \mathfrak{g}^{\alpha\lambda}\ \mathfrak{g}_{\mu\nu}\,\gamma^{\beta\nu}{}_{,\rho}\,\gamma^{\mu\rho}{}_{,\lambda} \tag{A.6}$$

$$+\ \mathfrak{g}^{\beta\lambda}\ \mathfrak{g}_{\mu\nu}\,\gamma^{\alpha\nu}{}_{,\rho}\,\gamma^{\mu\rho}{}_{,\lambda} - \ \mathfrak{g}_{\lambda\mu}\ \mathfrak{g}^{\nu\rho}\,\gamma^{\alpha\lambda}{}_{,\nu}\,\gamma^{\beta\mu}{}_{,\rho}$$

$$+\frac{1}{8}\Big[-2\ \mathfrak{g}^{\alpha\lambda}\ \mathfrak{g}^{\beta\mu} + \ \mathfrak{g}^{\alpha\beta}\ \mathfrak{g}^{\lambda\mu}\Big]\Big[2\ \mathfrak{g}_{\nu\rho}\ \mathfrak{g}_{\sigma\tau} - \ \mathfrak{g}_{\rho\sigma}\ \mathfrak{g}_{\nu\tau}\Big]\gamma^{\nu\tau}{}_{,\lambda}\,\gamma^{\rho\sigma}{}_{,\mu}$$

$$-\ \gamma^{\alpha\mu}{}_{,\nu}\,\gamma^{\beta\nu}{}_{,\mu} + \gamma^{\alpha\beta}{}_{,\mu\nu}\,\gamma^{\mu\nu}\Big\}^{p\text{-order part}}$$

Here $\mathfrak{g}_{\alpha\beta}$ is the matrix inverse of $\mathfrak{g}^{\alpha\beta}$,

$$\mathfrak{g}_{\alpha\beta} = \eta_{\alpha\beta} + \gamma_{\alpha\beta} + \gamma_\alpha{}^\mu\,\gamma_{\mu\beta} + \gamma_\alpha{}^\mu\,\gamma_\mu{}^\nu\,\gamma_{\nu\beta} + \ldots\ldots , \tag{A.7}$$

and all the differentiation indices in equation (A.6) must be spatial in order to give a non-zero result. Equations (A.1) through (A.7) are all that is needed to calculate $\mathfrak{g}^{\alpha\beta}$ to any desired order. One convenient way of keeping track of orders is to associate (or just multiply) $\gamma_1{}^{\alpha\beta}$ with a parameter λ which will be set to 1 when the calculation is finished. Then $\gamma_2{}^{\alpha\beta}$ will involve only the terms with λ^2, $\gamma_3{}^{\alpha\beta}$ will involve only the terms with λ^3, ... etc. One starts with $\gamma_1{}^{\alpha\beta}$ and goes up the chain one order at a time. This algorithm can easily be implemented on an algebraic manipulator. The hardest part to get through is the integration step, equation (A.5), but all the integrations are reasonably simple. The author was able, with little difficulty, to implement the algorithm described on the algebraic manipulator MACSYMA with its indicial tensor manipulation package.

B. Explicit form of Thorne's metric

Here we list most of the relevant functions up to order $(1/r^4)$. We will also give a complete list of $\gamma_p{}^{\alpha\beta}$ which will enable the reader to calculate all these functions up to order $(1/r^5)$.

$$\mathfrak{g}^{00} = -1 - \frac{4M}{r} - \frac{7M^2}{r^2} - \frac{6\ \mathcal{I}_{ab}\ n_a\ n_b}{r^3} - \frac{8M^3}{r^3} + O(\frac{1}{r^4})$$

$$\mathfrak{g}^{0j} = \frac{2\ \varepsilon_{jkl}\ \mathcal{S}_k\ n_l}{r^2}\left[1 + \frac{M}{r}\right] + \frac{4\ \varepsilon_{jkl}\ \mathcal{S}_{km}\ n_l\ n_m}{r^3} + O(\frac{1}{r^4})$$

$$\mathfrak{g}^{jk} = \delta^{jk} - \frac{M^2}{r^2} n_j\ n_k + O(\frac{1}{r^4})$$

$$\mathfrak{g}_{00} = -1 + \frac{4M}{r} - \frac{9M^2}{r^2} + \frac{16M^3}{r^3} + \frac{6\ \mathcal{I}_{ab}\ n_a\ n_b}{r^3} + O(\frac{1}{r^4})$$

$$\mathfrak{g}_{0j} = -\frac{2\ \varepsilon_{jkl}\ \mathcal{S}_k\ n_l}{r^2}\left[1 + \frac{3M}{r}\right] - \frac{4\ \varepsilon_{jkl}\ \mathcal{S}_{km}\ n_l\ n_m}{r^3} + O(\frac{1}{r^4})$$

$$\mathfrak{g}_{jk} = \delta_{jk} + \frac{M^2}{r^2} n_j\ n_k + O(\frac{1}{r^4})$$

$$g_{00} = -1 + \frac{2M}{r} - \frac{2M^2}{r^2} + \frac{2M^3}{r^3} + \frac{3\ \mathcal{I}_{ab}\ n_a\ n_b}{r^3} + O(\frac{1}{r^4})$$

$$g_{0j} = -\frac{2\ \varepsilon_{jkl}\ \mathcal{S}_k\ n_l}{r^2} + \frac{2\ \varepsilon_{jkl}\ M\ \mathcal{S}_k\ n_l}{r^3} - \frac{4\ \varepsilon_{jkl}\ \mathcal{S}_{km}\ n_l\ n_m}{r^3} + O(\frac{1}{r^4})$$

$$g_{jk} = \delta_{jk}\left[1 + \frac{2M}{r}\right] + \frac{M^2}{r^2}\left[\delta_{jk} + n_j\ n_k\right] + \frac{2M^3}{r^3} n_j\ n_k$$

$$+ \frac{3\ \mathcal{I}_{ab}\ n_a\ n_b}{r^3}\delta_{jk} + O(\frac{1}{r^4})$$

$$g^{00} = -1 - \frac{2M}{r} - \frac{2M^2}{r^2} - \frac{2M^3}{r^3} - \frac{3\ \mathcal{I}_{ab}\ n_a\ n_b}{r^3} + O(\frac{1}{r^4})$$

$$g^{0j} = \frac{2\ \varepsilon_{jkl}\ \mathcal{S}_k\ n_l}{r^2} - \frac{2\ \varepsilon_{jkl}\ M\ \mathcal{S}_k\ n_l}{r^3} + \frac{4\ \varepsilon_{jkl}\ \mathcal{S}_{km}\ n_l\ n_m}{r^3} + O(\frac{1}{r^4})$$

$$g^{jk} = \delta^{jk}\left[1 - \frac{2M}{r}\right] + \frac{M^2}{r^2}\left(\delta^{jk} - n_j\ n_k\right] - \frac{M^3}{r^3}\left(4\ \delta^{jk} - 2\ n_j\ n_k\right)$$

$$- \frac{3\ \mathcal{I}_{ab}\ n_a\ n_b}{r^3}\delta^{jk} + O(\frac{1}{r^4})$$

$$\lambda = g_{00}$$

$$\omega_i = -\frac{2\ \mathcal{S}_i}{r^3} + \frac{6\ \mathcal{S}_k\ n_k\ n_i}{r^3} + O(\frac{1}{r^4})$$

$$\omega = -\frac{2\ \mathcal{S}_k\ n_k}{r^2} + O(\frac{1}{r^3})$$

$$\Phi_M = \frac{M}{r} + \frac{M^3}{r^3} + \frac{3}{2}\ \frac{\mathcal{I}_{ab}\ n_a\ n_b}{r^3} + O(\frac{1}{r^4})$$

$$\Phi_J = \frac{\mathcal{S}_k\ n_k}{r^2} + O(\frac{1}{r^3})$$

$$\Omega_G = \frac{1}{r^2}\left[1 + \frac{2M^2}{r^2} + \frac{3\ \mathcal{I}_{ab}\ n_a\ n_b}{Mr^2} + O(\frac{1}{r^3})\right]$$

$$\Omega_{BS} = \frac{1}{r^2}\left[1 + \frac{2M^2}{r^2} + \frac{3\ \mathcal{I}_{ab}\ n_a\ n_b}{Mr^2} + \frac{\mathcal{S}_a\ \mathcal{S}_b\ n_a\ n_b}{M^2\ r^2} + O(\frac{1}{r^3})\right]$$

$$h_{ab} = \delta_{ab}\left[1 - \frac{M^2}{r^2}\right] + \frac{M^2}{r^2}n_a\ n_b + O(\frac{1}{r^4})$$

$$h^{ab} = \delta^{ab}\left[1 + \frac{M^2}{r^2}\right] - \frac{M^2}{r^2}n_a\ n_b + O(\frac{1}{r^4})$$

Here we list all the γ's up to $O(\frac{1}{r^5})$.

$$\gamma_1{}^{00} = \frac{4M}{r} + \frac{6\ \mathscr{I}_{ab}\ n_a\ n_b}{r^3} + \frac{10\ \mathscr{I}_{abc}\ n_a\ n_b\ n_c}{r^4} + O(\frac{1}{r^5})$$

$$\gamma_1{}^{0j} = \frac{2\ \varepsilon_{jkl}\ \mathscr{S}_k\ n_l}{r^2} + \frac{4\ \varepsilon_{jkl}\ \mathscr{S}_{km}\ n_k\ n_m}{r^3} + \frac{15}{2}\frac{\varepsilon_{jkl}\ \mathscr{S}_{kmn}\ n_k\ n_m\ n_n}{r^4} + O(\frac{1}{r^5})$$

$$\gamma_1{}^{jk} = 0$$

$$\gamma_2{}^{00} = \frac{7M^2}{r^2} + \frac{21\ M\ \mathscr{I}_{ab}\ n_a\ n_b}{r^4} + \frac{7\ \mathscr{S}_a\ \mathscr{S}_b\ n_a\ n_b}{r^4} - \frac{4\ \mathscr{S}_a\ \mathscr{S}_a}{r^4} + O(\frac{1}{r^5})$$

$$\gamma_2{}^{0j} = \frac{2\ \varepsilon_{jpq}\ M\ \mathscr{S}_p\ n_q}{r^3} + \frac{16}{3}\frac{M\ \varepsilon_{jpq}\ \mathscr{S}_{pa}\ n_a\ n_q}{r^4} + O(\frac{1}{r^5})$$

$$\gamma_2{}^{jk} = \frac{M^2}{r^2} n_j\ n_k + \frac{15}{2}\frac{M\ \mathscr{I}_{ab}\ n_a\ n_b\ n_j\ n_k}{r^4} - \frac{3M\ \mathscr{I}_{aj}\ n_a\ n_k}{r^4}$$

$$- \frac{3M\ \mathscr{I}_{ak}\ n_a\ n_j}{r^4} + \frac{1}{2}\frac{M\ \mathscr{I}_{ab}\ n_a\ n_b\ \delta_{jk}}{r^4} + \frac{M\ \mathscr{I}_{jk}}{r^4}$$

$$- \frac{7}{2}\frac{\mathscr{S}_a\ \mathscr{S}_a\ n_j\ n_k}{r^4} - \frac{2\ \mathscr{S}_k\ \mathscr{S}_j}{r^4} - \frac{5}{2}\frac{\delta_{jk}\ \mathscr{S}_p\ n_p\ \mathscr{S}_q\ n_q}{r^4}$$

$$+ \frac{\mathscr{S}_j\ \mathscr{S}_q\ n_q\ n_k}{r^4} + \frac{\mathscr{S}_k\ \mathscr{S}_q\ n_q\ n_j}{r^4} + \frac{5}{2}\frac{\delta_{jk}\ \mathscr{S}_a\ \mathscr{S}_a}{r^4}$$

$$+ \frac{9}{2}\frac{\mathscr{S}_p\ \mathscr{S}_q\ n_p\ n_q\ n_j\ n_k}{r^4} - \frac{2\ \varepsilon_{klm}\ \varepsilon_{jpb}\ \mathscr{S}_l\ \mathscr{S}_p\ n_m\ n_b}{r^4} + O(\frac{1}{r^5})$$

$$\gamma_3{}^{00} = \frac{8M^3}{r^3} + O(\frac{1}{r^5})$$

$$\gamma_3{}^{0j} = -\frac{2M^2\ \varepsilon_{jmn}\ \mathscr{S}_m\ n_n}{r^4} + O(\frac{1}{r^5})$$

$$\gamma_3{}^{jk} = 0 + O(\frac{1}{r^5})$$

$$\gamma_4{}^{00} = \frac{M^4}{r^4} + O(\frac{1}{r^5})$$

$$\gamma_4{}^{0j} = 0 + O(\frac{1}{r^5})$$

$$\gamma_4{}^{jk} = 0 + O(\frac{1}{r^5})$$

C. Proof of continuity

Given the explicit form of the metric, the proof that $\tilde{h}_{\tilde{a}\tilde{b}}$ is at least $C^{4,\alpha}$ is straightforward if not tedious. Rather than going into the details, we will outline the calculation below.

The first step is to solve

$$h_{ab} \frac{\partial}{\partial x^a} \left[\Omega_{BS} \frac{\partial x^{\tilde{c}}}{\partial x^b} \right] = 0$$

up to the desired order. We use the ansatz (45) terminated at the desired order to determine the form of $x^{\tilde{a}}$. Then the equation above will give the *STF* constants in $x^{\tilde{a}}$ since everything else in that equation is explicitly known. Once the new coordinate system is determined up to the desired order, we then use

$$\tilde{h}_{ab} = \Omega_{BS}{}^2 \, h_{ab}$$

to determine $\tilde{h}_{ab}$ from the known explicit forms of Ω_{BS} and h_{ab}. We use

$$\tilde{h}_{\tilde{a}\tilde{b}} = \tilde{h}_{cd} \frac{\partial x^c}{\partial x^{\tilde{a}}} \frac{\partial x^d}{\partial x^{\tilde{b}}}$$

to calculate the form of $\tilde{h}_{\tilde{a}\tilde{b}}$ in the Beig-Simon coordinates up to the desired order. The final step in the proof is simply looking at the result to see whether it is differentiable the desired number of times.

We have (believe it or not) calculated everything above up to order $(1/r^6)$ and determined that $\widetilde{h}_{\widetilde{a}\widetilde{b}}$ *is* at least C^5. With this result, all the requirements of the Beig-Simon analyticity theorem are satisfied. Their result is then applicable to our case. A final remark: The algorithm described in Thorne[3] to transfer metrics to *ACMC−N* coordinate systems can also be implemented on an algebraic manipulator. This coupled with the algorithm above makes Thorne's formalism a little easier to deal with and in the author's opinion, a little less ugly. This view is not shared by Thorne (private communication).

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B. PAPER IV

The Free Precession of Slowly Rotating Neutron Stars: Rigid-Body Motion in General Relativity*

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ABSTRACT

This paper discusses the free precession of slowly rotating neutron stars, idealized as rigidly rotating, fully general relativistic bodies. It is shown that, in the limit of slow rotation, general relativistic gravity produces no changes whatsoever in the classical Euler equations of rigid body motion. The proof is sketched in the body of the paper in language accessible to people who know only a little general relativity. An appendix justifies some assertions in the proof by giving an algorithm for generating slowly and rigidly rotating solutions of Einstein's equations from nonrotating, static solutions.

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The free precession of neutron stars has played a role of some importance in theories of temporal fluctuations of pulsar periods (see, e.g. Pandharipande, Pines, and Smith 1976), though as yet no free precessional motion has been identified definitely in any pulsar timing data. While radio astronomers continue their observational search for free precession, physicists are designing "third generation" gravitational wave detectors which have some hope of seeing gravity waves from pulsars (see, e.g. Drever et. al. 1982), gravity waves whose "line" spectrum would carry detailed information about the precession (Zimmermann and Szedenits 1979, Zimmermann 1980). In anticipation of the day when free precession is seen by one method or the other, we ask the question: What effects, if any, will general relativistic gravity have on the precession? We are not prepared to discuss this question in general; but for the idealized case of a slowly and rigidly rotating star this paper will give the answer.

Consider a rigidly rotating, general relativistic body. "Rigid rotation" means, physically and mathematically, that the distance between every pair of neighboring points in the body is forever fixed (in the jargon of relativists, the "rate of expansion" Θ and "rate of shear" σ_{jk} of the body's matter vanish). This definition is due to Max Born (1909, 1910), who (along with Ehrenfest 1909) pointed out that a rigidly rotating body so defined can never change its angular velocity - and thus can never precess - because such changes would alter the Lorentz contractions of the distances between neighboring points and thereby would deform the body. However, Born's Lorentz-contraction deformations have magnitudes $\delta l / l \sim (\Omega L / c)^2 \alpha$ where Ω is the body's angular velocity, L is its linear size, c is the speed of light, and α is the precession angle or the fractional change of Ω. For a slowly rotating body ($\Omega L / c \ll 1$) these deformations can be negligible, and (to first order in $\Omega L / c$) the precessing body can rotate rigidly. We shall confine attention to this case.

The space around our slowly and rigidly rotating body consists of three regions:

$$\text{the "wave zone": radii} \quad r \gtrsim c/\Omega$$

$$\text{the "weak-field near zone": radii} \quad c/\Omega \gg r \gg L \sim GM/c^2 \equiv M_* \,, \tag{1}$$

$$\text{the "strong-field region": radii} \quad r \lesssim L \sim M_* \,.$$

Here M is the body's mass; and because the body is presumed to be relativistic, its size L is not much bigger than (e.g. not 30 times bigger than) its "gravitational radius" $2GM/c^2 \equiv 2M_*$. In the strong-field region gravity is relativistically strong, i.e. spacetime is highly curved. In the weak-field near zone gravity is weak (spacetime is nearly flat), and the gravitational field is quasistationary (retardation is negligible). In the wave zone gravity is weak, but retardation is important and the quadrupole and higher multipole parts of the gravitational field are carried by gravitational waves. For further discussion see §III of Thorne (1980) [cited henceforth as "RMP"].

As discussed in §IX of RMP, the body's multipole moments are flat-space-type tensors which reside in the weak-field near zone, and which can be moved about with mathematical impurity there (no significant changes due to "parallel transport around closed curves," because spacetime is very nearly flat in the weak-field near zone). The body has two families of multipole moments (analogs of electric and magnetic moments of electromagnetism):

$$\left.\begin{array}{ll}\text{mass moments} & \mathcal{I}_{a_1 \ldots\ldots a_l}\,, \\ \text{current moments} & \mathcal{S}_{a_1 \ldots\ldots a_l}\,,\end{array}\right\} \; l = 0,1,2,\ldots., \tag{2}$$

which are symmetric and trace-free on all pairs of indices. Among these are the mass dipole $\mathcal{I}_j$ which vanishes identically, and the mass quadrupole $\mathcal{I}_{jk}$ and the current dipole $\mathcal{S}_j$ ($\equiv$ angular momentum) which for a nonrelativistic body would be expressible as volume integrals over the mass density ρ and velocity $\vec{v}$

$$\vec{\mathcal{S}} = \int (\vec{r} \times \rho \, \vec{v} \,) \, d^3x \, ,$$

$$\mathcal{I}_{jk} = \int \rho \, (\, x_j \, x_k - \frac{1}{3} \delta_{jk} \, r^2 \,) \, d^3x \, ,$$

but which for a relativistic body cannot be so expressed. The mass and current moments are fully determined by and fully determine the metric (i.e. spacetime curvature) outside the body. For example, in a wide class of coordinate systems (t , x_j) the metric in the weak-field near zone has the form

$$ds^2 = \left[-1 + \frac{2M_*}{r} + \frac{3 \, \mathcal{I}_{ab} \, x_a \, x_b}{r^5} + const \, \frac{M_*^2}{r^2} + const \, \frac{M_*^3}{r^3} + O(\frac{1}{r^4}) \right] c^2 \, dt^2$$
$$- \left[\frac{4 \varepsilon_{jkl} \, \mathcal{S}_k \, x_l}{r^3} + O(\frac{1}{r^3}) \right] dt \, dx_j \qquad (3)$$
$$+ \left[\delta_{jk} + const \, \frac{M_*}{r} + const \, \frac{M_*^2}{r^2} + O(\frac{1}{r^3}) \right] dx_j \, dx_k \, ,$$

where the constants "*const*" depend on the precise choice of coordinates, δ_{jk} is the Kronecker delta, ε_{ijk} is the completely antisymmetric symbol which produces a vector cross product, $r = \left[x_1^2 + x_2^2 + x_3^2 \right]^{\frac{1}{2}}$, and there is an implicit summation over repeated indices. As here, so in general, the mass moments $\mathcal{I}_{a_1 \ldots\ldots a_l}$ can be "read off of" the metric coefficients g_{00}, and the current moments $\mathcal{S}_{a_1 \ldots\ldots a_l}$ can be read off of g_{0j}.

The angular velocity vector Ω_j of a rigidly rotating body, like the multipole moments, is a flat-space-type object which resides in the weak-field near zone. The time units used in defining Ω_j are those of an observer at rest in the weak-field near zone (no "gravitational redshift" effects). It seems intuitively obvious, and we prove mathematically in the Appendix, that the body's mass moments all rotate, in the weak-field near zone, with angular velocity Ω_j:

$$\frac{d\,\mathcal{I}_{jk}}{dt} = \varepsilon_{jpq}\,\Omega_p\,\mathcal{I}_{qk} + \varepsilon_{kpq}\,\Omega_p\,\mathcal{I}_{jq}\,, \tag{4}$$

$$\frac{d\,\mathcal{I}_{a_1 \ldots\ldots a_l}}{dt} = \varepsilon_{a_1 pq}\,\Omega_p\,\mathcal{I}_{q a_2 \ldots\ldots a_l} + \ldots\ldots + \varepsilon_{a_l pq}\,\Omega_p\,\mathcal{I}_{a_1 \ldots\ldots a_{l-1} q}$$

(vector cross product of $\vec{\Omega}$ with each index of $\mathcal{I}$). By contrast, the angular momentum (current dipole moment) $\mathcal{S}_j$ is conserved, aside from gravitational-wave losses which are negligible on the timescales of rotation and precession

$$\frac{d\,\mathcal{S}_j}{dt} = 0. \tag{5a}$$

It also seems intuitively obvious, and we prove mathematically in the Appendix, that the body's angular momentum and other current moments are linear functions of its angular velocity

$$\mathcal{S}_j = I_{jk}\,\Omega_k\,. \tag{5b}$$

Since $\mathcal{S}_j$ and Ω_k are flat-space-type vectors residing in the weak-field near zone, I_{jk} is a flat-space-type tensor also residing there. Again, it seems intuitively obvious and we prove in the Appendix that the moment of inertia tensor rotates with angular velocity Ω_j, just as do the the mass multipole moments:

$$\frac{dI_{jk}}{dt} = \varepsilon_{jpq}\,\Omega_p\,I_{qk} + \varepsilon_{kpq}\,\Omega_p\,I_{jq}\,. \tag{5c}$$

For a classical, nonrotating body the moment of inertia tensor I_{jk} is symmetric in its two indices. The following thought experiment proves that general relativity does not break this symmetry:

$$I_{jk} = I_{kj} \ . \tag{5d}$$

[This thought experiment is a variant of an argument which was introduced into general relativity by Zel'dovich, via private discussions with Thorne in Moscow in 1969, and has subsequently been a powerful tool for research on relativistic stars and black holes; see, e.g. Carter (1973)]. Insert into the rotating body a long, rigid pole with your end in the weak-field near zone and the other end at the body's center. For a short time $\delta t \ll 1/\Omega$ apply a force $\vec{F}$ to your end of the pole and thereby, through the associated torque, change the body's angular momentum and total mass-energy by

$$\delta \vec{\mathcal{S}} = \vec{r} \times \vec{F}\, \delta t \ , \tag{6a}$$

$$\delta M\, c^2 = (\, \vec{\Omega} \times \vec{r}\,) \cdot \vec{F}\, \delta t = \vec{\Omega} \cdot (\, \vec{r} \times \vec{F}\, \delta t\,) = \vec{\Omega} \cdot \delta \vec{\mathcal{S}} . \tag{6b}$$

Here $\vec{r}$ is the location on the pole where you apply the force; $\vec{\Omega} \times \vec{r}$ is the velocity with which the pole is moving at that location; the mathematical expressions $\vec{r} \times \vec{F}\, \delta t$ and $(\, \vec{\Omega} \times \vec{r}\,) \cdot \vec{F}\, \delta t$ are the angular momentum and the energy you put into the pole; and the conservation laws of general relativity (e.g. chapters 19 and 20 of Misner, Thorne, and Wheeler (1973), cited below as MTW) guarantee that this energy and angular momentum, when transmitted down the pole and into the body, show up as changes of the gravitationally defined $\vec{\mathcal{S}}$ and M (equation 3). Now, it seems intuitively obvious and we shall prove in the Appendix that the body's mass-energy M differs from that of a nonrotating body with the same rigid form by an amount quadratic in the angular velocity

$$M\, c^2 = M_{nonrot}\, c^2 + \frac{1}{2} K_{jk}\, \Omega_j\, \Omega_k \ , \qquad K_{jk} = K_{kj} \ . \tag{7}$$

By combining this $M-\Omega$ relation with $\mathcal{S}_j = I_{jk}\,\Omega_k$ and $\delta M\,c^2 = \Omega_j\,\delta\,\mathcal{S}_j$ we see that

$$K_{jk}\,\Omega_j\,\delta\Omega_k = I_{jk}\,\Omega_j\,\delta\Omega_k$$

for all Ω_j, $\delta\Omega_k$ - which in turn means that

$$I_{jk} = K_{jk}\ ; \qquad (8)$$

and since K_{jk} is symmetric, I_{jk} must be symmetric. QED.

Equations (5a,b,c,d) are precisely equivalent to the Euler equations which govern the precession of a rigidly rotating, nongravitating body (e.g. Goldstein 1980). They guarantee that our relativistic body will undergo a free precession which is identical to that of a nongravitating body with the same moment of inertia I_{jk} and angular momentum $\mathcal{S}_j$. The only influence of general relativity will be through its effects on the values of the components of the moment of inertia tensor I_{jk}; cf. Hartle (1973).

As the body rotates and precesses, it will emit gravitational waves which are predominantly quadrupolar. §XII of RMP proves that the gravitational waves are governed by the body's time changing quadrupole moment through formulas which are independent of whether the body has strong internal gravity or weak. For example, the transverse-traceless gravitational wave field has the standard quadrupole-moment form

$$h^{TT}{}_{jk} = \frac{2}{r}\,\frac{d^2}{dt^2}\,\mathcal{I}^{TT}{}_{jk}(t-r)\ , \qquad (9)$$

where "TT" means "take the transverse-traceless part in the manner of Box 35.1 of MTW". Since the time evolution of the quadrupole moment is produced by rotation in the same manner as for a nonrelativistic body (equation 4), it might appear that no gravitational-wave observations of a rigidly rotating body can reveal whether the body has strong internal gravity or weak. However, for weakly gravitating bodies there is a key relation between the quadrupole moment and the moment of inertia:

$$\mathcal{J}_{jk} = I_{jk} - \frac{1}{3}\delta_{jk} I_{ii} \quad \text{for weakly gravitating bodies.} \tag{10}$$

We suspect, but have not proved, that this relation is violated by strongly gravitating bodies and that in principle its violation could be detected by gravitational wave observations and be used to determine whether the gravitational-wave source has strong internal gravity or weak.

Unfortunately, most observed neutron stars are likely to deviate significantly from rigid-body rotation as they precess, because of a noninfinite shear modulus and a superfluid mantle or core. Those deviations will modify substantially the conclusions of this paper. However, it may well turn out that, as for rigid rotation, so also in the nonrigid case general relativity introduces no qualitatively new effects except the breakdown in $\mathcal{J}_{jk} = I_{jk} - \frac{1}{3}\delta_{jk} I_{ii}$.

Acknowledgements: At one stage of this research we thought the moment of inertia tensor I_{jk} might have an antisymmetric part. When we mentioned this to Martin Rees he raised objections which triggered us to find the energy-angular-momentum argument by which we here prove symmetry. We thank him.

APPENDIX

ALGORITHM FOR GENERATING SLOWLY AND RIGIDLY ROTATING SOLUTIONS OF EINSTEIN'S EQUATIONS FROM NONROTATING, STATIC SOLUTIONS

In this Appendix we take the solution of Einstein's equations for a static (nonrotating) body, set it into slow and rigid rotation, and give an algorithm for computing all rotation-induced changes in the static solution at first order in the angular velocity Ω. From our algorithm we deduce the "intuitively obvious" claims made in the text.[Note that our analysis ignores deviations from the static solution not only of order $(\Omega L/c)^2$, but also of order $\Omega^2 (GM/L^3)^{-1} = (L/M_*)(\Omega/Lc)^2$; for example, the ratio of centrifugally induced stress to the stresses in the static body could be this large and are ignored.]

Our analysis is carried out in de Donder (harmonic) coordinates where the gauge conditions and Einstein's equations, expressed in terms of the metric density $\mathfrak{g}^{\alpha\beta} = (-g)^{\frac{1}{2}} g^{\alpha\beta}$ are

$$\mathfrak{g}^{\alpha\beta}{}_{,\beta} = 0 \tag{A.1a}$$

$$\left[-\left(\frac{\partial}{\partial t}\right)^2 + \nabla^2 \right] \mathfrak{g}^{\alpha\beta} = -16\pi (-g)(T^{\alpha\beta} + W^{\alpha\beta}), \tag{A.1b}$$

$$W^{\alpha\beta} = t^{\alpha\beta} + \left[16\pi(-g)\right]^{-1} \left[\mathfrak{g}^{\alpha\mu}{}_{,\nu}\, \mathfrak{g}^{\beta\nu}{}_{,\mu} + \mathfrak{g}^{\alpha\beta}{}_{,\mu\nu} (\mathfrak{g}^{\mu\nu} - \eta^{\mu\nu}) \right], \tag{A.1c}$$

$$\begin{aligned} 16\pi(-g)\, t^{\alpha\beta} = {} & \frac{1}{2} \mathfrak{g}^{\alpha\beta}\, \mathfrak{g}_{\lambda\mu}\, \mathfrak{g}^{\lambda\nu}{}_{,\rho}\, \mathfrak{g}^{\rho\mu}{}_{,\nu} + \mathfrak{g}_{\lambda\mu}\, \mathfrak{g}^{\nu\rho}\, \mathfrak{g}^{\alpha\lambda}{}_{,\nu}\, \mathfrak{g}^{\beta\mu}{}_{,\rho} \\ & - (\mathfrak{g}^{\alpha\lambda}\, \mathfrak{g}_{\mu\nu}\, \mathfrak{g}^{\beta\nu}{}_{,\rho}\, \mathfrak{g}^{\mu\rho}{}_{,\lambda} + \mathfrak{g}^{\beta\lambda}\, \mathfrak{g}_{\mu\nu}\, \mathfrak{g}^{\alpha\nu}{}_{,\rho}\, \mathfrak{g}^{\mu\rho}{}_{,\lambda}) \qquad \text{(A.1d)} \\ & + \frac{1}{8}(2\, \mathfrak{g}^{\alpha\lambda}\, \mathfrak{g}^{\beta\mu} - \mathfrak{g}^{\alpha\beta}\, \mathfrak{g}^{\lambda\mu})(2\, \mathfrak{g}_{\nu\rho}\, \mathfrak{g}_{\sigma\tau} - \mathfrak{g}_{\rho\sigma}\, \mathfrak{g}_{\nu\tau})\, \mathfrak{g}^{\nu\tau}{}_{,\lambda}\, \mathfrak{g}^{\rho\sigma}{}_{,\mu} ; \end{aligned}$$

see, e.g. §II of Kovacs and Thorne (1975) and §100 of Landau and Lifshitz (1962). Here $\mathfrak{g}_{\alpha\beta}$ is the matrix inverse of $\mathfrak{g}^{\alpha\beta}$, i.e. $\mathfrak{g}_{\alpha\beta} = (-g)^{-\frac{1}{2}} g_{\alpha\beta}$; $T^{\alpha\beta}$ is the material stress-energy tensor; $t^{\alpha\beta}$ is the "Landau-Lifshitz pseudotensor"; ∇^2 is the flat-space Laplacian; we set the speed of light to unity; and other notational conventions are those of MTW (or of Landau and Lifshitz 1962 with Greek and Latin indices interchanged).

We begin with an arbitrary, static (time independent with $\mathfrak{g}^{0j} = T^{0j} = 0$) solution of equations (A.1a) describing an isolated, nonrotating body. Our coordinates are "mass centered", so the mass dipole moment vanishes and the metric $g_{\alpha\beta}$ in the weak-field near zone has the form (3) with $\mathcal{S}_j = 0$ [or, in more full detail, equation (10.6) of RMP with $g_{0j} = 0$ and all $\mathcal{S}_{a_1......a_l} = 0$]. We denote this static solution by an index (0) and we denote the static de Donder coordinates by primes:

$$T_{(0)}{}^{\alpha'\beta'}(x^{j'}) , \quad \mathfrak{g}_{(0)}{}^{\alpha'\beta'}(x^{j'}) , \quad \mathfrak{g}^{(0)}{}_{\alpha'\beta'}(x^{j'}) . \tag{A.2}$$

We then, as a tool in our algorithm, rotate our spatial de Donder coordinates through an infinitesimal angle - Θ^k to obtain new static coordinates

$$t = t' , x^j = x^{j'} + \varepsilon_{jkl}\, \Theta^k\, x^{l'} . \tag{A.3}$$

Here ε_{jkl} is the flat-space Levi-Civita tensor; and although the space is highly curved, this rotation is performed mathematically as though space were flat and the de Donder coordinates were Cartesian. Because the de Donder gauge conditions (A.1a) involve a flat space divergence which is invariant under flat-space-type rotations, the new x^α coordinates, like the old $x^{\alpha'}$ ones, are de Donder. Our static solution in the rotated coordinates, and linearized in Θ^k , we shall denote by

$$T_{(0)}{}^{\alpha\beta}(\, x^j \,,\, \Theta^k \,)\,, \quad \mathfrak{g}_{(0)}{}^{\alpha\beta}(\, x^j \,,\, \Theta^k \,)\,, \text{ etc.} \tag{A.4a}$$

For example,

$$\mathfrak{g}_{(0)}{}^{00}(\, x^j \,,\, \Theta^k \,) = \mathfrak{g}_{(0)}{}^{0'0'}(\, x^{j'} = x^j - \varepsilon_{jkl}\, \Theta^k\, x^l \,) \tag{A.4b}$$

$$= \mathfrak{g}_{(0)}{}^{0'0'}(\, x^j \,) - \mathfrak{g}_{(0)}{}^{0'0'}{}_{,i'}(\, x^j \,)\, \varepsilon_{ikl}\, \Theta^k\, x^l \,.$$

Note the absence of any time dependence and also note that because the solution is static

$$T_{(0)}{}^{0j} = 0\,, \quad \mathfrak{g}_{(0)}{}^{0j} = 0\,, \quad \mathfrak{g}^{(0)}{}_{0j} = 0\,, \quad W_{(0)}{}^{0j} = 0\,, \quad t_{(0)}{}^{0j} = 0\,. \tag{A.4c}$$

Now make the rotation dynamical and rigid, let the primed coordinates rotate with the body, and keep the unprimed coordinates de Donder and at rest relative to distant inertial frames. At time $t = 0$ and for infinitesimal time t thereafter the rigid angular velocity is Ω^j, the rotation angle is $\Theta^j = \Omega^j\, t$, and a zero-order solution of the gauge conditions and field equations is

$$T_{(0)}{}^{\alpha\beta}(\, x^j \,,\, \Omega^k t \,)\,, \quad \mathfrak{g}_{(0)}{}^{\alpha\beta}(\, x^j \,,\, \Omega^k t \,)\,, \text{ etc.} \tag{A.5}$$

We seek first-order corrections [corrections linear in Ω^k, and denoted by subscripts (1)] to this zero-order solution. Our full solution

$$T^{\alpha\beta} = T_{(0)}{}^{\alpha\beta} + T_{(1)}{}^{\alpha\beta}\,, \quad \mathfrak{g}^{\alpha\beta} = \mathfrak{g}_{(0)}{}^{\alpha\beta} + \mathfrak{g}_{(1)}{}^{\alpha\beta} \tag{A.6}$$

must not only satisfy the gauge conditions and field equations (A.1) to first order in Ω; it must also have vanishing momentum density as measured by observers who rotate with the body ($dx^j / dt = \varepsilon_{jkl}\, \Omega^k\, x^l$) -- which means that the momentum density in the nonrotating, unprimed coordinate frame must be

$$T_{(1)}{}^{0j} = T_{(0)}{}^{00}\, \varepsilon_{jkl}\, \Omega^k\, x^l + T_{(0)}{}^{jk}\, \mathfrak{g}^{(0)}{}_{kl}\, (\, \mathfrak{g}_{(0)}{}^{l0} - \mathfrak{g}_{(0)}{}^{00}\, \varepsilon_{lmn}\, \Omega^m\, x^n \,) \tag{A.7}$$

Once the solution (A.6) for infinitesimal times t is known and from it we have learned the time derivative of Ω^k, the solution for all times can be obtained by "e-folding".

Lemma 1: There are no first-order corrections to quantities with time-time or space-space indices :

$$T_{(1)}{}^{00} = T_{(1)}{}^{jk} = t_{(1)}{}^{00} = t_{(1)}{}^{jk} = W_{(1)}{}^{00} = W_{(1)}{}^{jk} = \mathfrak{g}_{(1)}{}^{00} = \mathfrak{g}_{(1)}{}^{jk} = 0. \quad \text{(A.8)}$$

Proof: The gauge condition (A.1a) at first order will be satisfied as a result of the structure of $\mathfrak{g}_{(1)}{}^{0j}$ to be discussed in Lemma 2 below, so they do not constrain $\mathfrak{g}_{(1)}{}^{00}$ or $\mathfrak{g}_{(1)}{}^{jk}$. In the field equations (A.1b) the second time derivative is $O(\Omega^2)$ and can be ignored, so at first order the field equations say

$$\nabla^2 \mathfrak{g}_{(1)}{}^{\alpha\beta} = -16\pi\,(-g)\,(T_{(1)}{}^{\alpha\beta} + W_{(1)}{}^{\alpha\beta})\,. \quad \text{(A.9)}$$

The quantities $W_{(1)}{}^{00}$ and $W_{(1)}{}^{jk}$, as computed from (A.1c,d), must be linear in $\mathfrak{g}_{(1)}{}^{00}$, $\mathfrak{g}_{(1)}{}^{jk}$, $\mathfrak{g}_{(1)}{}^{0j}$, $\mathfrak{g}_{(0)}{}^{00}{}_{,0}$, $\mathfrak{g}_{(0)}{}^{jk}{}_{,0}$ and their spatial derivatives [the time derivatives in $\mathfrak{g}_{(0)}{}^{\alpha\beta}$ coming from the dependence on $\Omega^k t$ in (A.5)]. All other contributors to $W_{(1)}{}^{00}$ and $W_{(1)}{}^{jk}$ must be $\mathfrak{g}_{(0)}{}^{00}$, $\mathfrak{g}_{(0)}{}^{jk}$ and their spatial derivatives. But $W_{(1)}{}^{00}$ and $W_{(1)}{}^{jk}$ have an even number of temporal indices (2 and 0 respectively); and because of the way that indices must "line up" in equations (A.1c,d) this means that each of the terms that are added together to make up $W_{(1)}{}^{00}$ and $W_{(1)}{}^{jk}$ must have an even number of temporal indices; and this in turn means that $\mathfrak{g}_{(1)}{}^{0j}$, $\mathfrak{g}_{(0)}{}^{00}{}_{,0}$ and $\mathfrak{g}_{(0)}{}^{jk}{}_{,0}$ with their odd temporal indices must be absent from $W_{(1)}{}^{00}$ and $W_{(1)}{}^{jk}$; and this in turn means that every term on the right hand side of $\nabla^2 \mathfrak{g}_{(1)}{}^{00}$ and $\nabla^2 \mathfrak{g}_{(1)}{}^{jk}$ in (A.9) contains as a factor $T_{(1)}{}^{00}$, $T_{(1)}{}^{jk}$, $\mathfrak{g}_{(1)}{}^{00}$, or $\mathfrak{g}_{(1)}{}^{jk}$. Thus, the "00" and "jk" field equations are satisfied by $T_{(1)}{}^{00} = T_{(1)}{}^{jk} = \mathfrak{g}_{(1)}{}^{00} = \mathfrak{g}_{(1)}{}^{jk} = 0$ which in turn implies that $t_{(1)}{}^{00} = t_{(1)}{}^{jk} = W_{(1)}{}^{00} = W_{(1)}{}^{jk} = 0$. We must also verify that the resulting solution satisfies Born's criterion for rigid rotation to first order in Ω. Indeed, it does: infinitesimal distances between neighboring points in the body are preserved to first order in Ω by the rotational motion $dx^j / dt = \varepsilon_{jkl}\, \Omega^k x^l$ when the metric density is $\mathfrak{g}^{jk} = \mathfrak{g}_{(0)}{}^{jk} =$ (eq. A.5), $\mathfrak{g}^{00} = \mathfrak{g}_{(0)}{}^{00} =$ (eq. A.5), $\mathfrak{g}^{0j} = \mathfrak{g}_{(1)}{}^{0j} =$ (anything of order Ω). QED.

<u>Lemma</u> 2: The first-order correction to $\mathfrak{g}^{0j}$ is determined by the two equations

$$\nabla^2 \mathfrak{g}_{(1)}{}^{0j} = A^j{}_k\, \Omega^k + B^j{}_k\, \mathfrak{g}_{(1)}{}^{0k} + C^j{}_k{}^l\, \mathfrak{g}_{(1)}{}^{0k}{}_{,l} + D^{kl}\, \mathfrak{g}_{(1)}{}^{0j}{}_{,kl}\,, \qquad \text{(A.10a)}$$

$$\mathfrak{g}_{(1)}{}^{0j}{}_{,j} = \mathfrak{g}_{(0)}{}^{00}{}_{,j}\, \varepsilon_{jkl}\, \Omega^k x^l\,, \qquad \text{(A.10b)}$$

which are compatible because the right hand side of (A.10b) has vanishing Laplacian. Here $A^j{}_k$, $B^j{}_k$, $C^j{}_k{}^l$, and D^{kl} are functions of x^i determined by the nonrotating solution (i.e. by $T_{(0)}{}^{\alpha\beta}$, $\mathfrak{g}_{(0)}{}^{\alpha\beta}$, and $\mathfrak{g}^{(0)}{}_{\alpha\beta}$ with Ω^j set to zero).

<u>Proof:</u> Equation (A.10b) is the gauge condition $\mathfrak{g}_{(1)}{}^{0j}{}_{,j} = -\,\mathfrak{g}_{(0)}{}^{00}{}_{,0}$ where the time dependence in $\mathfrak{g}_{(0)}{}^{00}$ comes from equation (A.4b) with $\Theta^k = \Omega^k t$. The other gauge condition, $\mathfrak{g}_{(1)}{}^{0j}{}_{,0} = -\,\mathfrak{g}_{(0)}{}^{jk}{}_{,k} \equiv 0$ is automatically satisfied to first order in Ω by solutions of (A.10). Equation (A.10a) is the "0j" field equation (A.1b). The $T_{(1)}{}^{0j}$ term in (A.1b) contributes to $A^j{}_k\, \Omega^k$ and to $B^j{}_k\, \mathfrak{g}_{(1)}{}^{0k}$ (cf. equation A.7). Every term in W^{0j} of (A.1c,d) must include an odd number of temporal indices and must thus be proportional to $\mathfrak{g}_{(0)}{}^{00}{}_{,0}$ or $\mathfrak{g}_{(0)}{}^{jk}{}_{,0}$ [which give terms of the form $A^j{}_k\, \Omega^k$ in (A.10a)], or to $\mathfrak{g}_{(1)}{}^{0k}$ and its spatial derivatives, or to $\mathfrak{g}^{(1)}{}_{0k}$ which in turn is proportional to $\mathfrak{g}_{(1)}{}^{0i}$. This dictates the form of the right hand side of (A.10a). QED.

The solution of the equations (A.10) will be linear in Ω^k:

$$\mathfrak{g}^{0j} = E^j{}_k\, \Omega^k \qquad \text{(A.11)}$$

with x^i dependent coefficients $E^j{}_k$ which, like $\mathfrak{g}_{(0)}{}^{\alpha\beta}$ and $T_{(0)}{}^{\alpha\beta}$ from which they come, are time independent in the "body-centered" coordinates $x^{j'}$ - i.e. they rotate with the body. Comparison with $\mathfrak{g}^{0j} = -g_{0j} = 2\,\varepsilon_{jkl}\,\mathcal{S}^k\,x^l / r^3$ in the weak-field near zone (equation 3) reveals that $\mathcal{S}_i = I_{ij}\,\Omega_j$ where I_{ij} is time-independent in the body-centered coordinates and thus rotates with the body; and similarly for all the other current moments $\mathcal{S}_{a_1......a_l}$, which one also "reads off" of $\mathfrak{g}^{0j}$. This verifies two "intuitively obvious" assertions made in the text: equations (5b,c) and associated discussion. Because $\mathfrak{g}^{00} = \mathfrak{g}_{(0)}{}^{00}$ rotates with the body and the mass multipole moments $\mathcal{I}_{a_1......a_l}$ can be read off of $\mathfrak{g}^{00}$ in the weak-field near zone in a manner analogous to equation (3) (see RMP), those moments must also rotate with the body. This verifies another of the text's "intuitively obvious" assertions: equation (4). The total mass-energy of the rotating body can be computed as the volume integral

$$M\,c^2 = \int (-g)\,(T^{00} + t^{00})\,d^3x \tag{A.12}$$

(§100 of Landau and Lifshitz 1962). To first order in Ω and at time $t = 0$ the energy density $(-g)\,(T^{00} + t^{00}) = (-g)\,(T_{(0)}{}^{00} + t_{(0)}{}^{00})$ is equal to that in the nonrotating body , and thus $M\,c^2 = M_{nonrot}\,c^2$. Changes in the mass-energy $M\,c^2$ must thus be second order in Ω, which verifies the text's last "intuitively obvious" assertion: equation (7).

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