λ-DESIGNS AND RELATED COMBINATORIAL CONFIGURATIONS

Thesis by

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In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1969

(Submitted April 17,1969)

ACKNOWLEDGMENTS

I wish to express my gratitude to Professor H. J. Ryser for his help in the preparation of this thesis.

ABSTRACT

This thesis deals with two problems. The first is the determination of λ-designs, combinatorial configurations which are essentially symmetric block designs with the condition that each subset be of the same cardinality negated. We construct an infinite family of such designs from symmetric block designs and obtain some basic results about their structure. These results enable us to solve the problem for $\lambda = 3$ and $\lambda = 4$. The second problem deals with configurations related to both λ-designs and (v, k, λ)-configurations. We have $(n-1)$ k-subsets of $\{1, 2, \ldots, n\}$, $S_1, \ldots, S_{n-1}$, such that $S_i \cap S_j$ is a λ-set for $i \neq j$. We obtain specifically the replication numbers of such a design in terms of n, k, and λ with one exceptional class which we determine explicitly. In certain special cases we settle the problem entirely.

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I. INTRODUCTION

For the purposes of this thesis, a λ-design is a (0, 1) square matrix A of order n such that

$$A^tA = \lambda J + \text{diag}[k_1-\lambda, \ldots, k_n-\lambda] , \qquad (1.1)$$

where A^t denotes the transpose of A, J is the $n \times n$ matrix of ones, $k_j > \lambda > 0$, and not all the k_j's are equal.

First definitively studied by de Bruijn and Erdös with $\lambda = 1$ [1], they have received new interest with the following theorem of H. J. Ryser [7]:

> A (0, 1) square matrix A satisfying (1.1) with $k_j > \lambda > 0$ either has all its row and column sums equal (and hence is a balanced incomplete block design) or has precisely two row sums r_1 and r_2 with $r_1 + r_2 = n + 1$.

Along with this result, Ryser also established that apart from row and column permutations there is precisely one 2-design. This design is of order 7 and is of a class of λ-designs, called H-designs, constructed from the symmetric block design with parameters $(4\lambda - 1, 2\lambda, \lambda)$.

The combinatorial interest in matrices of this type satisfying (1.1) is clear. They represent (i.e., are incidence matrices for) the following configuration: we have n subsets $S_1, S_2, \ldots, S_n$ of $\{1, 2, \ldots, n\}$ with the feature that $S_i \cap S_j$ is a λ-set for $i \neq j$ and the S_j's do not all have the same cardinality.

In Chapter II of the present work we will generalize Ryser's H-design construction to an arbitrary (v, k, λ)-configuration. In

Chapter III we will establish some properties of λ-designs which will enable us, in Chapters IV and V, to determine all 3-designs and all 4-designs.

Chapter VI then varies the problem slightly to consider the following combinatorial situation. We have n-1 subsets $S_1, S_2, \ldots, S_{n-1}$ of $\{1, 2, \ldots, n\}$ with the feature that $S_i \cap S_j$ is a λ-set for $i \neq j$ and each S_i is a k-set. We show here the representing matrices with one exceptional class have two row sums, determined explicitly in terms of n, k, λ. We can say much then about the structure of such configurations and in special cases ($k = 2\lambda$, $\lambda = 1$, $n = 2k$) determine all such designs modulo the determination of related (v, k, λ)-configurations. The λ-designs of Chapter I play a role here.

The exceptional class is determined explicitly modulo the determination of Hadamard matrices.

II. TYPE I λ-DESIGNS

Theorem 2.1

If there exists a (ν, k, λ')-configuration, not of the form $(4\lambda-1, 2\lambda-1, \lambda-1)$, then there exists a λ-design with $\lambda = k-\lambda'$ and row sums $\nu-k$ and $k+1$.

Proof: Let B be the incidence matrix of the (ν, k, λ')-configuration, written so that column one has its k ones in rows one through k, i.e.,

$$B = \left[\begin{array}{c|c} 1 & \\ \vdots & A_1 \\ 1 & \\ \hline 0 & \\ \vdots & A_2 \\ 0 & \end{array}\right]$$

where A_1 is of size $k \times \nu-1$ and A_2 is of size $\nu-k$ by $\nu-1$. Now form the matrix A:

$$A = \left[\begin{array}{c|c} 0 & \\ \vdots & A_1' \\ 0 & \\ \hline 1 & \\ \vdots & A_2 \\ 1 & \end{array}\right] .$$

Then A is a λ-design with $\lambda = k-\lambda'$ as follows. A_2 evidently has column sums $k-\lambda'$ so that column one of A has inner product $k-\lambda'$ with each of 2 through ν. Consider then columns i and j of A with $i \geq j \geq 2$. Suppose the corresponding columns in A_1 have inner product t, then these columns in A_2 have inner product $\lambda'-t$ and in A_1' $k-2\lambda'+t$ so that columns i and j of A have inner product $k-2\lambda'+t+\lambda'-t = k-\lambda'$. A has two column sums $\nu-k$ and $2(k-\lambda')$. These are distinct precisely if we have avoided the design with parameters $(4\lambda-1, 2\lambda-1, \lambda-1)$. The row sum claim in the theorem is obvious.

Definition 2. 1

A λ-design obtainable via the construction given in the proof of theorem 2. 1 will be called a type-I λ-design.

Remarks 2. 1

a) The type-I λ-design obtained from the $(4\lambda - 1, 2\lambda, \lambda)$-configuration is indeed one of Ryser's H-designs.

b) The above construction when applied to the (excluded) $(4\lambda - 1, 2\lambda - 1, \lambda - 1)$-configuration will simply produce the symmetric block design with parameters $(4\lambda - 1, 2\lambda, \lambda)$.

c) For a given λ there are at most a finite number of parameter sets for a type-I λ-design. This is because there are at most a finite number of (v, k, λ') triples with $k - \lambda' = \lambda$.

d) Let A be a type-I λ-design derived from the (v, k, λ') matrix B. A has $(n-1)$ columns with sum 2λ and one exceptional column. To obtain the λ-design corresponding to the complementary block design to B one need only replace this exceptional column by its complement.

III. PROPERTIES OF λ-DESIGNS

Throughout the discussion A will denote a λ-design of order n with entries a_{ij}. Its row sums will be denoted r_1 and r_2 with $r_1 > \frac{n+1}{2}$, $r_2 < \frac{n+1}{2}$. e_1 will denote the number of rows of A with row sum r_1 and $e_2 \equiv n-e_1$. k_j' will denote the number of ones in column j of A which occur in rows with sum r_1, and $k_j^* \equiv k_j - k_j'$ where k_j is the j^{th} column sum of A. Following Ryser we set

$$\rho = \frac{r_1-1}{r_2-1}, \quad x_i = \frac{r_i-1}{n-1}, \quad u = -\lambda + \sum_{i=1}^{n} x_i^2 . \qquad (3.1)$$

The proof of Ryser's theorem is essentially the establishment of the relation:

$$x_i^2 - x_i - u = 0 , \qquad (3.2)$$

from which we have

$$x_1 + x_2 = 1 \quad \text{and} \quad x_1 x_2 = -u . \qquad (3.3)$$

From (3.2) $u = x_i(x_i-1)$ or

$$u = \frac{(r_1-1)(r_1-n)}{(n-1)^2} = -\frac{(r_1-1)(r_2-1)}{(n-1)^2}$$

so that $\rho u = -x_1^2$ and then from (3.3) $\rho x_2^2 = -u$. Also note

$$\rho+1 = \frac{r_1-1+r_2-1}{r_2-1} = \frac{n-1}{r_2-1} = \frac{1}{x_2} ,$$

so that $x_1 = \frac{\rho}{\rho+1}$ and $u = -x_1 x_2 = -\frac{\rho}{(\rho+1)^2}$. We list these relations as

$$-\frac{x_1^2}{u} = \rho , \quad -\frac{x_2^2}{u} = \frac{1}{\rho} , \quad x_1 = \frac{\rho}{\rho+1} , \quad x_2 = \frac{1}{1+\rho} , \quad u = \frac{-\rho}{(\rho+1)^2} . \qquad (3.4)$$

Now

$$\frac{\rho n+1}{\rho+1} = \frac{\rho}{\rho+1} n + \frac{1}{\rho+1} = x_1 n + x_2 = x_1(n-1) + x_1 + x_2 = r_1 - 1 + 1 = r_1$$

using (3.3) and (3.4). From this we obtain the following relations which we list for future reference:

$$r_1 = \frac{\rho n+1}{\rho+1}, \qquad r_1 - 1 = \frac{\rho(n-1)}{\rho+1},$$
$$r_2 = \frac{\rho+n}{\rho+1}, \qquad r_2 - 1 = \frac{n-1}{\rho+1}. \tag{3.5}$$

We note in addition the following relations established in Ryser's paper:

$$k_j^* = \lambda - \rho(k_j' - \lambda) \tag{3.6}$$

$$\sum_{j=1}^{n} \frac{1}{k_j - \lambda} = -\frac{1}{\lambda} - \frac{1}{u} = \frac{\lambda(1+\rho)^2 - \rho}{\lambda\rho} \tag{3.7}$$

using (3.4).

$$\sum_{j=1}^{n} \frac{a_{ij}a_{ej}}{k_j - \lambda} = \delta_{ie} - \frac{x_i x_e}{u} \tag{3.8}$$

where δ_{ie} is Kronecker's delta.

$$\sum_{j=1}^{n} \frac{a_{ij}}{k_j - \lambda} = 1 - \frac{x_i^2}{u} = \frac{-x_i}{u}. \tag{3.9}$$

From the relation

$$e_1 r_1(r_1 - 1) + e_2 r_2(r_2 - 1) = \lambda n(n-1)$$

we obtain, using (3.5) and $e_1 + e_2 = n$:

$$e_1 \frac{(\rho n+1)(n-1)\rho}{(1+\rho)^2} + (n - e_1)\frac{(\rho+n)(n-1)}{(1+\rho)^2} = \lambda n(n-1).$$

Hence,

$$e_1\{(\rho n+1)\rho - (n+\rho)\} = \lambda n(1+\rho)^2 - n(\rho+n)$$

and

$$e_1 n(\rho^2 - 1) = \lambda n(1+\rho)^2 - n(\rho+n),$$

so that

$$e_1 = \frac{\lambda(1+\rho)^2 - (\rho+n)}{\rho^2 - 1} \quad . \tag{3.10}$$

Finally, if $\Delta = \det A$, Δ is integral and

$$\Delta^2 = \left[\prod_{j=1}^{n} (k_j - \lambda)\right] \left[1 + \lambda \sum_{j=1}^{n} \frac{1}{k_j - \lambda}\right] \quad . \tag{3.11}$$

Type-I λ-designs with $\lambda > 1$ all have $e_1 \geq 3$. The next two theorems show this to be true of λ-designs in general.

Theorem 3.1

A λ-design with $e_1 = 1$ has $\lambda = 1$.

Proof: With $e_1 = 1$ the matrix A has two column types:

$$\begin{aligned} k_1' &= 1 \ , & k_1^* &= \lambda\rho - \rho + \lambda \ , \\ k_2' &= 0 \ , & k_2^* &= \lambda(1+\rho) \ , \end{aligned} \tag{3.12}$$

as seen from (3.6). Now (3.10) yields

$$n - 1 = (\rho+1)(\lambda\rho - \rho + \lambda) \ , \tag{3.13}$$

and we compute from (3.5) and (3.13)

$$r_2 = \lambda(1+\rho) - \rho + 1 \ . \tag{3.14}$$

From (3.12) we note that $\rho = k_2^* - k_1^*$ is an integer, while (3.12) and (3.14) indicate that $r_2 = k_1$.

We now normalize the matrix A to the form

$$A = \begin{array}{c|c} 1 \dots 1 & 0 \dots 0 \\ \hline B & C \end{array}$$

and use (3.8) with $i = 1$, $\ell > 1$ to deduce

$$\sum_{j=1}^{r_1} \frac{a_{\ell j}}{k_1 - \lambda} = - \frac{x_1 x_2}{u} = 1$$

or

$$\sum_{j=1}^{r_1} a_{\ell j} = k_1 - \lambda ,$$

i.e., B has constant row sums $k_1 - \lambda$. Since $r_2 = k_1$, C has row sums λ.

We now further normalize within the matrices B and C to bring A to the form

$$\begin{array}{cc|cc}
\multicolumn{2}{c|}{1 \text{———} 1} & \multicolumn{2}{c}{0 \text{———} 0} \\ \hline
\multicolumn{1}{c|}{1 \text{—} 1} & 0 \text{—} 0 & \multicolumn{1}{c|}{1 \text{—} 1} & 0 \text{—} 0 \\
\multicolumn{1}{c|}{} & & \multicolumn{1}{c|}{|} & \\
\multicolumn{1}{c|}{} & & \multicolumn{1}{c|}{1} & \\ \hline
\multicolumn{1}{c|}{B_1} & & \multicolumn{1}{c|}{C_1} &
\end{array}
\left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} k_2 = \lambda(1+\rho) \qquad (3.15)$$

where C_1 has an initial zero column. <u>We suppose C_1 is not vacuous.</u> Let σ denote the sum of row 1 of B_1, τ the sum of row 1 of C_1, and note from (3.8) with $i = 2$, $\ell = k_2+2$

$$\frac{\sigma}{k_1-\lambda} + \frac{\tau}{k_2-\lambda} = -\frac{x_2^2}{u} = \frac{1}{\rho}$$

in view of (3.4). We write this more conveniently as

$$\frac{\sigma}{\lambda\rho-\rho+1} + \frac{\tau}{\lambda\rho} = \frac{1}{\rho} . \qquad (3.16)$$

Thus, we have

$$\lambda\rho\sigma + \tau(\lambda\rho-\rho+1) = \lambda(\lambda\rho-\rho+1)$$

or

$$\lambda\rho(\sigma+\tau) = \lambda^2\rho + (\rho-1)(\tau-\lambda) .$$

But $\rho > 1$ and $\tau < \lambda$ so that

$$(\sigma+\tau) < \lambda . \qquad (3.17)$$

We now write (3.16) as

$$\rho\{\lambda^2 - \lambda(\sigma+\tau+1) + \tau\} = \tau - \lambda < 0 ,$$

so that

$$\lambda^2 - \lambda(\sigma+\tau+1) + \tau < 0 . \qquad (3.18)$$

But then $\lambda^2+\tau < \lambda(\sigma+\tau+1) \le \lambda(\lambda)$ because of (3.17). This means that $\tau = 0$, but then (3.16) gives $\sigma = \lambda - 1 + \frac{1}{\rho}$. Hence, we are forced to conclude that C_1 is vacuous, and thus from (3.15) we see that $k_2 = (n-1)$, or from (3.13)

$$\lambda(1+\rho) = (1+\rho)(\lambda\rho-\rho+\lambda) ,$$

whence $\lambda = 1$ as asserted.

Theorem 3.2

A λ-design has $e_1 \neq 2$.

Proof: From (3.5) and (3.10) with $e_1 = 2$ we have

$$\begin{aligned} n &= (\lambda-2)\rho^2+(2\lambda-1)\rho+\lambda+2 \\ r_1 &= (\lambda-2)\rho+(\lambda+2) \\ r_2 &= (\lambda-2)\rho^2+(\lambda+1)\rho+1 . \end{aligned} \qquad (3.19)$$

The possibilities for k_j' are 0, 1, 2 and the corresponding column types are displayed in the following table:

k_j'	0	1	2
k_j^*	$\lambda+\lambda\rho$	$\lambda+\lambda\rho-\rho$	$\lambda+\lambda\rho-2\rho$
k_j	$\lambda+\lambda\rho$	$\lambda+\lambda\rho-\rho+1$	$\lambda+\lambda\rho-2\rho+2$
no. of columns	w	x	y

(3.20)

We thus have the relations

$$
\begin{aligned}
w+x+y &= (\lambda-2)\rho^2+(2\lambda-1)\rho+\lambda+2 \\
x+2y &= 2(\lambda-2)\rho^2+2(\lambda+1)\rho+2
\end{aligned}
\qquad (3.21)
$$

from $w+x+y = n$, $\sum_{j=1}^{n} k_j^! = e_1 r_1$ and (3.19). We explicitly determine w, y, and y as follows. Normalize the first two rows of A:

$$
\begin{array}{cccc}
1 \text{---} 1 & 1 \text{---} 1 & 0 \text{---} 0 & 0 \text{---} 0 \\
\underbrace{1 \text{---} 1}_{y} & \underbrace{0 \text{---} 0}_{x/2} & \underbrace{1 \text{---} 1}_{x/2} & \underbrace{0 \text{---} 0}_{w}
\end{array}
$$

and use (3.8) on these rows obtaining

$$\frac{y}{\lambda\rho-2\rho+2} = -\frac{x_1^2}{u} = \rho$$

or

$$y = (\lambda-2)\rho^2 + 2\rho \ . \qquad (3.22)$$

Now from (3.21) we may compute

$$
\begin{aligned}
x &= 2(\lambda-1)\rho+2 \\
w &= \lambda-\rho \ .
\end{aligned}
\qquad (3.23)
$$

Thus, ρ is integral and $\rho \le \lambda$. Further, normalize A so that its first three rows appear so:

$$
\begin{array}{cccccccccc}
\overbrace{\begin{matrix}1 \text{---} 1 & 1 \text{---} 1\end{matrix}}^{y} & \overbrace{\begin{matrix}1 \text{---} 1 & 1 \text{---} 1 & 0 \text{---} 0 & 0 \text{---} 0\end{matrix}}^{x} & \overbrace{\begin{matrix}0 \text{---} 0 & 0 \text{---} 0\end{matrix}}^{w} \\
\begin{matrix}1 \text{---} 1 & 1 \text{---} 1\end{matrix} & \begin{matrix}0 \text{---} 0 & 0 \text{---} 0 & 1 \text{---} 1 & 1 \text{---} 1\end{matrix} & \begin{matrix}0 \text{---} 0 & 0 \text{---} 0\end{matrix} \\
\begin{matrix}\underbrace{1 \text{---} 1}_{\sigma} & 0 \text{---} 0\end{matrix} & \begin{matrix}\underbrace{1 \text{---} 1}_{\tau} & 0 \text{---} 0 & \underbrace{1 \text{---} 1}_{\tau'} & 0 \text{---} 0\end{matrix} & \begin{matrix}\underbrace{1 \text{---} 1}_{\alpha} & 0 \text{---} 0\end{matrix}
\end{array}
$$

With σ, τ, τ', and α defined by this diagram, use (3.8) on rows one and three and also on two and three. This will provide the information

$$\frac{\sigma}{\lambda\rho-2\rho+2}+\frac{\tau}{\lambda\rho-\rho+1} = 1$$
$$\frac{\sigma}{\lambda\rho-2\rho+2}+\frac{\tau'}{\lambda\rho-\rho+1} = 1 \qquad (3.24)$$

so that necessarily $\tau = \tau'$. Using (3.8) on row three with itself:

$$\frac{\sigma}{\lambda\rho-2\rho+2}+\frac{2\tau}{\lambda\rho-\rho+1}+\frac{\alpha}{\lambda\rho} = 1+\frac{1}{\rho} \quad . \qquad (3.25)$$

Then (3.24) and (3.25) imply

$$\frac{\tau}{\lambda\rho-\rho+1}+\frac{\alpha}{\lambda\rho} = \frac{1}{\rho} \, , \qquad (3.26)$$

which when solved for τ becomes

$$\tau = \frac{(\lambda-\alpha)(\lambda\rho-\rho+1)}{\lambda\rho} \quad . \qquad (3.27)$$

Solving (3.24) for σ gives

$$\sigma = (\lambda\rho-2\rho+2)\left(1-\frac{\tau}{\lambda\rho-\rho+1}\right)$$
$$= (\lambda\rho-2\rho+2)\left(1-\left\{\frac{1}{\rho}-\frac{\alpha}{\lambda\rho}\right\}\right) \quad ;$$

$$\sigma = \frac{(\lambda\rho-2\rho+2)(\lambda\rho-\lambda+\alpha)}{\lambda\rho} \quad . \qquad (3.28)$$

Now (3.27) and (3.28) mean

$$\sigma+\tau = \lambda\rho-2\rho+3-\left(\frac{\lambda+\alpha\rho-\alpha}{\lambda\rho}\right) \, , \qquad (3.29)$$

so that evidently $m = [\lambda+\alpha(\rho-1)]/\lambda\rho$ is a positive integer. But

$$\alpha \le \lambda-\rho < \lambda \quad \text{and} \quad \rho \ge 2 \, ,$$

so

$$(\rho-1)\alpha < (\rho-1)\lambda \quad .$$

Hence,

$$\lambda+(\rho-1)\alpha < \lambda\rho \qquad \text{and} \qquad 0 < m < 1 \quad .$$

This contradiction denies the existence of a λ-design with $e_1 = 2$.

We remark that the corresponding statements to theorems 3.1 and 3.2 for the parameter e_2 are almost immediate. For $e_2 \le 2$ we have $k_j^* = 0, 1, 2$, correspondingly $k_j' - \lambda = \lambda/\rho$, $\lambda - 1/\rho$, $\lambda - 2/\rho$. Since $1/\rho$ is not integral, the only compatible pair of these is $(\lambda/\rho$, $\lambda - 2/\rho)$, whence $e_2 = 2$, $\rho = 2$. But (3.8) then used on the last two rows of A would say that $r_2 = (\lambda+2)/4$. But (3.10) becomes $n-2 = \frac{9\lambda - n - 2}{3}$ or $n = \frac{9\lambda}{4} + 1$. But then 4 divides λ and r_2 is not integral.

Since 1-designs have $e_1 = 1$, Theorem 3.1 offers a characterization of these configurations. The next theorem characterizes 1-designs in a different way.

<u>Theorem 3.3</u>

A λ-design may be permuted to a normal matrix if and only if $\lambda = 1$.

<u>Proof</u>: Since a 1-design may be permuted to a symmetric matrix, this part of the implication is clear.

Conversely, suppose A is normal. Permute the rows and columns of A so that the first e_1 rows (columns) have row (column) sum r_1. We work with this permuted matrix viewed as

$$\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}$$

where A_1 is $e_1 \times e_1$. From (3.6) we have

$$k_j' = [\lambda(n-1) - k_j(r_2 - 1)]/(r_1 - r_2) .$$

Since A is a λ-design with respect to rows <u>and</u> columns, A_1 has constant row and column sums, x, given by:

$$x = [\lambda(n-1)-r_1(r_2-1)]/(r_1-r_2) \ . \qquad (3.30)$$

A_2 has row sums r_1-x and column sums c where

$$c = [\lambda(n-1)-r_2(r_2-1)]/(r_1-r_2) \ . \qquad (3.31)$$

From (3.30) and (3.31) we have $c-x = r_2-1$ and hence

$$r_2 - c = 1 - x \geq 0 \ . \qquad (3.32)$$

Thus, $x = 0$ or $x = 1$. If $x = 1$, then from (3.32) $r_2 = c$, so A_4 is a zero block and columns one and e_1+1 meet in $\lambda \leq 1$ positions. If $x = 0$, A_1 is a zero block and (3.32) shows A_4 to have column sums one. Here again columns e_1+1 and one meet in less than two positions. Hence in either event, $\lambda = 1$ as asserted.

The last three results contain the following theorem due to Majumdar [5]:

Corollary 3.4

Let A be a (0, 1) matrix of size $\nu \times \nu$. Suppose both

$$AA^t = \lambda' J + \text{diag}[r_1-\lambda', \ldots, r_\nu-\lambda']$$

and

$$A^tA = \lambda J + \text{diag}[k_1-\lambda, \ldots, k_\nu-\lambda] \ ,$$

$0 < \lambda' < r_i$, $0 < \lambda < k_j$. Then A is either a (ν, k, λ)-configuration or a 1-design.

Proof: We suppose A is not a (ν, k, λ)-configuration so that A is a λ-design and A^t is a λ'-design. We specialize (3.11) to the case where A has two column sums k_1 and k_2 occurring respectively f_1 and f_2 times:

$$\Delta^2 = (k_1-\lambda)^{f_1-1}(k_2-\lambda)^{f_2-1}\{(k_1-\lambda)(k_2-\lambda)+\lambda e_1(k_2-\lambda)+\lambda e_2(k_1-\lambda)\} \ . \qquad (3.33)$$

From (3.33) we see the characteristic polynomial of A^tA to be

$$P_1(x) = (k_1-\lambda-x)^{f_1-1}(k_2-\lambda-x)^{f_2-1}g_1(x) , \tag{3.34}$$

where

$$g_1(x) = (k_1-\lambda-x)(k_2-\lambda-x)+\lambda f_1(k_2-\lambda-x)+\lambda f_2(k_1-\lambda-x) . \tag{3.35}$$

"Similarly," the characteristic polynomial of AA^t :

$$P_2(x) = (r_1-\lambda'-x)^{e_1-1}(r_2-\lambda'-x)^{e_2-1}g_2(x) \tag{3.36}$$

where

$$g_2(x) = (r_1-\lambda'-x)(r_2-\lambda'-x)+\lambda' e_1(r_2-\lambda'-x)+\lambda' e_2(r_1-\lambda'-x) . \tag{3.37}$$

If e_1 or f_1 is 1 , we have $\lambda = \lambda' = 1$. Hence we may take $e_i \geq 3$, $f_i \geq 3$ by Theorems 3.1, 3.2 and the following remarks. Since, e.g. $g_1(k_i-\lambda) \neq 0$, f_i-1 is the precise multiplicity of $k_i-\lambda$ with similar remarks for $P_2(x)$. Further, $g_i(x)$ has distinct roots -- this may be seen directly or by noting that $P_i(x)$ must have a root of multiplicity one by the classical theorem of Perron-Frobenius [2]. Now since AA^t and A^tA are similar, $P_1(x) \equiv P_2(x)$. But the above remarks show that $g_1(x) = g_2(x)$ and hence $\lambda = \lambda'$, $k_i = r_i$, and $e_i = f_i$ whence A may be permuted to a normal matrix and by Theorem 3.3 $\lambda = \lambda' = 1$.

Our next few results, though of a general nature, are developed explicitly for considering the nature of λ-designs for specified λ (particularly here $\lambda = 3$ and $\lambda = 4$).

<u>Lemma 3.5</u>

(1) A λ-design with a column with $k_j' = 2\lambda-1$ has $\rho = \frac{\lambda}{\lambda-1}$.

(2) A λ-design with $\rho = \frac{\lambda}{\lambda-1}$ is an H-design.[†]

Proof: (1) The corresponding k_j^* is $\lambda-\rho(\lambda-1)$; hence, $\lambda-\rho(\lambda-1) \geq 0$ or $\rho \leq \frac{\lambda}{\lambda-1}$. Further, $\rho(\lambda-1)$, and since $\rho > 1$, $\rho(\lambda-1) \geq \lambda$ or $\rho \geq \frac{\lambda}{\lambda-1}$. Hence, $\rho = \frac{\lambda}{\lambda-1}$ as asserted.

(2) From (3.5) we have

$$r_1 - 1 = \frac{\lambda(n-1)}{2\lambda-1}, \qquad r_2 - 1 = \frac{(n-1)(\lambda-1)}{2\lambda-1}, \tag{3.38}$$

and since λ and $2\lambda-1$ are relatively prime, we have for a positive integer t

$$n-1 = t(2\lambda-1) \tag{3.39}$$

and we may rewrite (3.38) as

$$r_1 - 1 = \lambda t, \qquad r_2 - 1 = t(\lambda-1). \tag{3.40}$$

Now from (3.10)

$$e_1 = \frac{\lambda\left(\frac{2\lambda-1}{\lambda-1}\right)^2 - \left(\frac{\lambda}{\lambda-1}\right) - n}{\left(\frac{\lambda}{\lambda-1}\right)^2 - 1},$$

i.e.,

$$e_1 = -t(\lambda-1)^2 - (\lambda-1) + \lambda(2\lambda-1). \tag{3.41}$$

Now theorems 3.1 and 3.2 imply $e_1 \geq 3$, so that

$$t(\lambda-1)^2 = \lambda(2\lambda-1) - (\lambda-1) - 3,$$

$$t \leq \frac{2\lambda(\lambda-1)-2}{(\lambda-1)^2} \leq 2 + \frac{2}{\lambda-1} - \frac{2}{(\lambda-1)^2}.$$

Hence, $t = 1$ or $t = 2$. If $t = 1$ from (3.41), $e_1 = \lambda^2$ and from (3.39) $n = 2\lambda$. Hence, $\lambda^2 < 2\lambda$ or $\lambda < 2$. Thus, $t = 2$ and $r_1 = 2\lambda+1$, $r_2 = 2\lambda-1$ from (3.40), and Ryser has shown that a λ-design with

[†] See Remark (2.1.a).

these parameters is necessarily an H-design.

Lemma 3.6

Let A be a λ-design with two column sums k_1 and k_2. Suppose further that there is precisely one column with sum k_1. Then A is a type-I λ-design.

Proof: Supposing A has two column sums, write A in the form

$$\begin{matrix} A_1 & A_2 \\ A_3 & A_4 \end{matrix}$$

where $[A_1A_2]$ has row sums r_1 and $[A_3A_4]$ has row sums r_2, and $\begin{bmatrix} A_1 \\ A_3 \end{bmatrix}$ has column sums k_1 while $\begin{bmatrix} A_2 \\ A_4 \end{bmatrix}$ has column sums k_2. Let σ_i be an arbitrary row sum of A_1. Then from (3.9) we have

$$\frac{\sigma_i}{k_1-\lambda} + \frac{r_1-\sigma_i}{k_2-\lambda} = 1 - \frac{x_1^2}{u} = 1+\rho ,$$

whence, since $k_1 \neq k_2$, σ_i does not depend on i, i.e., A_1 has constant row sums σ. Similarly, A_3 has constant row sums τ. In the present case, A_1 and A_3 are column vectors, and since surely $\sigma \neq \tau$, we have $\sigma = 0$, $\tau = 1$ or $\sigma = 1$, $\tau = 0$. In either case, all remaining columns are of the $k_j' = k_j^* = \lambda$ type. We suppose $\sigma = 0$, $\tau = 1$ and form

$$B = \left[\begin{array}{c|c} 1 & \\ \vdots & A_2' \\ 1 & \\ \hline 0 & \\ \vdots & A_4 \\ 0 & \end{array}\right]$$

where A_2' denotes the complement of A_2. Column one of B has sum e_1. Column j of A_2 has sum λ, column j of A_2' has sum $e_1-\lambda$, and column j of A_4 has sum λ ; thus, B has constant column sums e_1. If two columns of A_4 meet in t positions the corresponding columns of A_2 meet in $\lambda-1$ locations, so those columns in A_2' meet in $e_1-\lambda-t$ for a total column inner product in B of $e_1-\lambda$. Thus, B is an $(n, e_1, e_1-\lambda)$-balanced, incomplete symmetric block design yielding our matrix A as a type-I λ-design.

We remark that it is easy to show that a λ-design cannot have two columns with one sum and the remaining n-2 with another sum.

We will need one further lemma for our discussion of 3- and 4-designs.

Lemma 3.7

A λ-design with $e_1 = \lambda$ has $\rho \le \lambda$ with $(2\lambda-1)\rho$ integral.

Proof: Let x denote the number of columns with $k_j' = k_j^* = \lambda$. Then $\lambda x \le e_2 = n-\lambda$ or

$$x \le \frac{n-\lambda}{\lambda} \quad . \tag{3.42}$$

Now (3.10) becomes

$$n-1 = (2\lambda-1((1+\rho) \quad , \tag{3.43}$$

so that from (3.5) $r_2 = 2\lambda$ and thus $r_1 = n+1-2\lambda$. Thus, the first λ rows of A contain $\lambda(2\lambda-1)$ zeros, and if $\underline{n \ge \lambda(2\lambda-1)}$, surely $x \ge n-\lambda(2\lambda-1)$, which, together with (3.43), forces

$$n - \lambda(2\lambda-1) \le \frac{n-\lambda}{\lambda} \quad ,$$

$$n(\lambda-1) \le \lambda^2(2\lambda-1)-\lambda = \lambda(2\lambda+1)(\lambda-1) \ .$$

Hence,

$$\underline{n \le \lambda(2\lambda+1)} \ . \tag{3.44}$$

Thus, in any event $n \le \lambda(2\lambda+1)$, and from (3.43) we have

$$\rho = \frac{n-2\lambda}{2\lambda-1} \leq \lambda \quad , \tag{3.45}$$

noting (3.44).

From (3.45), of course, $\rho(2\lambda-1)$ is integral.

IV. 3-DESIGNS

Type-I 3-Designs

(ν, k, λ)-configurations with $k-\lambda = 3$ have one of the following parameter sets: (13, 4, 1); (13, 9, 6); (11, 5, 2); (11, 6, 3). Each of these excepting the (11, 5, 2) will produce via Theorem 2.1 a 3-design. We illustrate each type:

0	1	1	1	1	1	1	1	1	1	0	0	0
0	1	1	1	1	1	1	0	0	0	1	1	1
0	1	1	1	0	0	0	1	1	1	1	1	1
0	0	0	0	1	1	1	1	1	1	1	1	1
1	1	0	0	1	0	0	1	0	0	1	0	0
1	1	0	0	0	1	0	0	1	0	0	1	0
1	1	0	0	0	0	1	0	0	1	0	0	1
1	0	1	0	1	0	0	0	0	1	0	1	0
1	0	1	0	0	1	0	1	0	0	0	0	1
1	0	1	0	0	0	1	0	1	0	1	0	0
1	0	0	1	1	0	0	0	1	0	0	0	1
1	0	0	1	0	1	0	0	0	1	1	0	0
1	0	0	1	0	0	1	1	0	0	0	1	0

3-design from (13, 4, 1); $e_1 = 4$, $r_1 = 9$, $r_2 = 5$, $\rho = 2$.

1	1	1	1	1	1	1	1	1	1	0	0	0
1	1	1	1	1	1	1	0	0	0	1	1	1
1	1	1	1	0	0	0	1	1	1	1	1	1
1	0	0	0	1	1	1	1	1	1	1	1	1
0	1	0	0	1	0	0	1	0	0	1	0	0
0	1	0	0	0	1	0	0	1	0	0	1	0
0	1	0	0	0	0	1	0	0	1	0	0	1
0	0	1	0	1	0	0	0	0	1	0	1	0
0	0	1	0	0	1	0	1	0	0	0	0	1
0	0	1	0	0	0	1	0	1	0	1	0	0
0	0	0	1	1	0	0	0	1	0	0	0	1
0	0	0	1	0	1	0	0	0	1	1	0	0
0	0	0	1	0	0	1	1	0	0	0	1	0

From (13, 9, 6); $e_1 = 4$, $r_1 = 10$, $r_2 = 4$, $\rho = 3$.

Theorem 4.1

All 3-designs are type-I designs.

Proof: In view of Lemma 3.5, we may take $k_j' \le 4$. If some $k_j' = 4$, then $\rho = 2$ or $\rho = 3$. From (3.10) $e_1 = \frac{25-n}{3}$ so that, since $e_1 \le n-3$ and $4 \le e_1 \le 11$ $(e_1 < 4\lambda)$, we have $n = 10, 13$. If $n = 10$, $e_1 = 5$, and $r_1 = 7$. The remaining columns have $k_j' \ge 2$ since $k_j' = 4$ and $\rho = 2$ implies $k_j^* = 1$. Let f_i denote the number of columns with $k_j' = i$.

1	1	0	0	0	1	1	1	0	1	1
1	1	1	0	1	0	0	0	1	1	1
1	1	1	1	0	1	1	0	1	0	0
1	0	1	1	1	0	1	1	0	1	0
1	0	0	1	1	1	0	1	1	0	1
0	0	0	1	0	0	1	0	1	1	1
0	0	1	0	1	1	1	0	0	0	1
0	0	1	0	0	1	0	1	1	1	0
0	1	0	1	1	1	0	0	0	1	0
0	1	0	0	1	0	1	1	1	0	0
0	1	1	1	0	0	0	1	0	0	1

From (11, 6, 3) -- "H-design with $\lambda = 3$"
$e_1 = 5$, $r_1 = 7$, $r_2 = 5$, $\rho = 3/2$.

Then since $k_j^! = 2$ implies $k_j^* = 5$, $f_2 = 0, 1$ with $f_2+f_3+f_4 = 10$ and $2f_2 + 3f_3 + 4f_4 = 35$. Clearly then $f_2 = 0$, $f_3 = f_4 = 5$. Now (3.9) cannot hold with $i = 6$. If $n = 13$, $e_1 = 4$, $r_1 = 9$. Then $f_4 = 1$, which forces $f_2 = 4$, $f_3 = 8$. But then from (3.11) we have the contradiction $\Delta^2 = 2^8 \cdot 3^{11}$. Hence, we must have $\rho = 3$ and $k_j^! = 4$ means $k_j^* = 0$, so all remaining columns have $k_j^! = k_j^* = 3$. Again, (3.10) and $e_1 \geq 4$ force $n \leq 13$ and we must have $n = 13$ for e_1 to be integral, but then $e_1 = 4$ so that $f_4 = 1$ and we have a type-I design by Lemma 3.6.

For all the remaining 3-designs we have then $k_j^! \leq 3$, and Table 4.1 displays the column possibilities.

$k_j^!$	0	1	2	3
k_j^*	$3+3\rho$	$3+2\rho$	$3+\rho$	3
k_j	$3+3\rho$	$4+2\rho$	$5+\rho$	6

Table 4.1

Suppose we have a 3-design with $\underline{e_1 \geq 6}$. Then from (3.10) we deduce $-n \geq 3\rho^2-5\rho-9$. But Table 4.1 makes it clear that $n \geq 10$ whence $1 < \rho < 3/2$. But then 2ρ is not integral and 3ρ must be, which means $\rho = 4/3$. This forces $n = 10$ and e_1 would not be integral.

With $\underline{e_1 = 5}$, (3.10) becomes $n = -2\rho^2+5\rho+8$ so that 2ρ is integral and $\rho \leq 2$. Hence, $\rho = 2, 3/2$; $\rho = 3/2$ can only yield the H-design by Lemma 3.5; and $\rho = 2$ means $n = 10$, $r_p = 7$. But $e_2 = 5$ forces one column with $k_j^* = 5$ and the remaining with $k_j^* = 3$; hence, Lemma 3.6 applies.

If $\underline{e_1 = 4}$, we have $n = -\rho^2+5\rho+7$, so that $\rho = 2, 3, 4$. (1) $\rho = 2$, $n = 13$, $r_1 = 9$, $e_2 = 9$; now $f_0 = 0, 1$. If $f_0 = 0$ we would have $f_1 + f_2 + f_3 = 13$, $f_1 + 2f_2 + 3f_3 = 36$, and $\frac{1}{5} f_1 + \frac{1}{4} f_2 + \frac{1}{3} f_3 = \frac{25}{6}$, which has no integral solution. Hence, $f_0 = 1$, $f_3 = 12$, and we have a type-I design. (2) $\rho = 3$, $n = 13$, $r_1 = 10$; $e_1 r_1 = 40$, but $k_j' \leq 3$ denies this. (3) $\rho = 4$, $n = 11$, $r_1 = 9$, $e_2 = 7$; $e_2 = 7$ means k_j^* must be 3 or 7; hence, $f_2 = 1$, $f_3 = 12$, and Lemma 3.6 applies.

The final case is $\underline{e_1 = 3}$. Here, Lemma 3.7 gives $\rho \leq 3$, (3.10) becomes $n = 5\rho+6$ so that $\rho = 2, 3$. (1) $\rho = 2$, $n = 16$, $r_1 = 11$; the column structure is uniquely determined and we obtain $f_1 = 0$, $f_2 = 12$, $f_3 = 3$, from which $\Delta^2 = 3^7 \cdot 2^{24}$. (2) $\rho = 3$, $n = 21$, $r_1 = 16$; here, the proof of Lemma 3.7 shows that $f_3 = 6$ so that surely $f_0 = 0$ and from $f_0+f_1+f_2 = 15$ and $f_1+2f_2 = 30$ we have $f_1 = 0$, $f_2 = 15$. But then $\Delta^2 = 2^4 3^6 5^{15}$.

Thus, all 3-designs are type-I designs.

V. 4-DESIGNS

All (v, k, λ)-triples with $k-\lambda = 4$ are listed below [excluding the (15, 7, 3)], together with the parameters of the derived 4-designs:

1.	(21, 5, 1) :	$n = 21$,	$r_1 = 16$,	$e_1 = 5$,	$\rho = 3$
2.	(21, 16, 12):	$n = 21$,	$r_1 = 17$,	$e_1 = 5$,	$\rho = 4$
3.	(16, 6, 2) :	$n = 16$,	$r_1 = 10$,	$e_1 = 6$,	$\rho = 3/2$
4.	(16, 10, 6) :	$n = 16$,	$r_1 = 11$,	$e_1 = 6$,	$\rho = 2$
5.	(15, 8, 4) :	$n = 15$,	$r_1 = 9$,	$e_1 = 7$,	$\rho = 4/3$.

Table 5. 1

Theorem 5. 1

All 4-designs are type-I.

Proof: We proceed as in the case of 3-designs to note we may take $k_j' \le 6$, eliminating H-designs from consideration. The column possibilities are then displayed:

k_j'	0	1	2	3	4	5	6
k_j^*	$4+4\rho$	$4+3\rho$	$4+2\rho$	$4+\rho$	4	$4-\rho$	$4-2\rho$

Table 5. 2

Suppose a 4-design has a column with $k_j' = 6$. Then 2ρ is integral and in fact $\rho = 3/2$ or $\rho = 2$. From (3. 10) and $6 \le e_1 \le n-3$ we have

$$7 + \frac{7}{\rho} + \frac{1}{\rho^2} \le n \le -2\rho^2+7\rho+10 \quad ; \qquad (5.1)$$

further, $(\rho n+1)/(\rho+1) = r_1$ must be integral. So for $\rho = 3/2$ we have $n = 16$, $e_1 = 6$, $r_1 = 10$. Since $k_j' = 6$ now means $k_j^* = 1$, all remaining k_j's are 3 or 4. But 3 is not possible since ρ is not integral. Thus, Lemma 3. 6 applies, and the design does not exist.

With $\rho = 2$ from (5.1) we have $11 \le n \le 16$ with $\frac{2n+1}{3}$ integral, i.e., $n = 13, 16$. If $n = 13$, $r_1 = 9$, $e_1 = 7$, we can have only one $k_j^! = 6$ with the remaining columns of the form $k_j^! = 4$ or $k_j^! = 5$. With f_i the number of columns with $k_j^! = i$, we have $f_4 + f_5 = 12$ and $4f_4 + 5f_5 = 57$. This means $f_4 = 3$ and $f_5 = 9$, but then from (3.11) we obtain the contradiction $\Delta^2 = 2^8 \cdot 3^{11}$. With $n = 16$, $r_1 = 11$, and $e_1 = 6$. Here, $k_j^! = 6$ having $k_j^* = 0$ means that all remaining columns have $k_j^! = k_j^* = 4$ and we have the type-I design from line 4 of of Table 5.1.

Next, suppose we have a **4**-design with $k_j^! = 5$ occurring. Then $\rho = 2, 3,$ or 4. We have here $5 \le e_1 \le n-3$ or

$$7 + 7/\rho + 1/\rho^2 \le n \le -\rho^2 + 7\rho + 9 \quad . \tag{5.2}$$

(1) $\rho = 2$. $11 \le n \le 19$, $n \equiv 1 \bmod 3$; hence, $n = 13, 16, 19$. Note that $k_j^! = 5$ means $k_j^* = 2$; hence, all remaining columns have $k_j^! \ge 2$, so we are working with the following column table:

$k_j^!$	2	3	4	5
k_j^*	8	6	4	2
k_j	10	9	8	7

Table 5.3

n = 13, $r_1 = 9$, $e_1 = 7$: $e_2 = 6$ forces $f_2 = 0$, $f_3 \le 1$. If $f_3 = 1$ then $f_4 + f_5 = 12$, $4f_4 + 5f_5 = 60$. Hence, $f_5 = 12$, $f_4 = 0$, and Lemma 3.6 shows this design does not exist.

n = 16, $r_1 = 11$, $e_1 = 6$: $e_2 = 10$ forces $f_2 \le 1$ and

$$f_2+f_3+f_4+f_5 = 16 \quad ,$$

$$2f_2+3f_3+4f_4+5f_5 = 66 \quad ,$$

$$\frac{1}{6} f_2 + \frac{1}{5} f_3 + \frac{1}{4} f_4 + \frac{1}{3} f_5 = \frac{17}{4} .$$

Thus, $f_2 = 1$ gives $f_3 = 0$, $f_5 = 4$, $f_4 = 11$ and we obtain the contradiction $\Delta^2 = 3^7 \cdot 2^{24}$, while $f_2 = 0$ forces the absurdity $f_3 = 5/2$, $f_4 = 9$, $f_5 = 9/2$.

<u>n = 19</u>, $r_1 = 13$, $e_1 = 5$: $e_1 = 5$ gives $f_5 = 1$ and

$$f_2 + f_3 + f_4 = 18 ,$$
$$2f_2 + 3f_3 + 4f_4 = 60 ,$$
$$\frac{1}{3} + \frac{1}{6} f_2 + \frac{1}{5} f_3 + \frac{1}{4} f_4 = \frac{17}{4} ,$$

which has the unique solution $f_2 = 1$, $f_3 = 10$, $f_4 = 7$. Here, $\Delta^2 = 3^4 \cdot 2^{16} \cdot 5^{10}$ does <u>not</u> exclude this possibility. Consider a row of A with sum $r_1 = 13$ and a zero in the column with $k_j^! = 2$. Let τ be the number of ones in this row in columns with $k_j^! = 3$ and use (3.8) with $i = \ell$ obtaining:

$$\frac{1}{3} + \frac{\tau}{5} + \frac{12-\tau}{4} = 3$$

or $\tau = 20/3$.

(2) $\rho = 3$. We have from (5.2) $10 \le n \le 21$ and from (3.10) $e_1 = (61-n)/8$ so that $n \equiv 5 \pmod 8$. Thus, $n = 13$ or $n = 21$. Since $k_j^! = 5$ and $\rho = 3$ gives $k_j^* = 1$, all remaining $k_j^!$ values are either 3 or 4. In case $n = 13$, $e_1 = 6$, $r_1 = 10$ and the column structure of A is determined by the system:

$$f_3 + f_4 + f_5 = 13 ,$$
$$3f_3 + 4f_4 + 5f_5 = 60 ,$$
$$\frac{1}{6} f_3 + \frac{1}{4} f_4 + \frac{1}{2} f_5 = \frac{61}{12} ,$$

which has the unique (and unacceptable) solution $f_3 = -1$, $f_4 = f_5 = 7$.

In the case of $n = 21$, $e_1 = 5$, $r_1 = 16$. Surely then $f_5 = 1$ and $f_3 + f_4 = 20$, while $3f_3 + 4f_4 = 75$, i.e., $f_3 = 5$, $f_4 = 15$. We write rows one and two of A :

$$\begin{array}{lll}
\overbrace{}^{k'_j = 5} & \overbrace{1\ .\ .\ .\ 1\quad 0\ .\ .\ .\ 0}^{k'_j = 3} & \overbrace{1\ .\ .\ .\ 1\quad 0\ .\ .\ .\ 0}^{k'_j = 4} \\
1 & \underbrace{1 \ldots 1}_{\alpha}\ 0 \ldots 0\ 1 \ldots 1\ 0 \ldots 0 & \underbrace{1 \ldots 1}_{\beta}\ 0 \ldots 0\ 1 \ldots 1\ 0 \ldots 0
\end{array}$$

We use (3.8) with $i = \ell = 1$ to obtain

$$\frac{1}{2} + \frac{\sigma}{6} + \frac{15-\sigma}{4} = 4$$

where σ is the partial sum of row one occurring in columns with $k'_j = 3$. We see then that $\sigma = 3$; this means if β denotes the intersection of rows one and two in the last 15 columns that $\beta \geq 9$. With α denoting the intersection in columns with $k'_j = 3$ we have $\alpha \geq 1$, and from (3.8) with $i = 1$, $\ell = 2$,

$$\frac{1}{2} + \frac{\alpha}{6} + \frac{\beta}{4} = 3 \qquad \text{or} \qquad \alpha = \frac{3}{2}(10-\beta) .$$

Since $\alpha \geq 1$ we have $\beta = 9$, but then $\alpha = 3/2$.

(3) $\rho = 4$. Here, $e_1 = (96-n)/15$ and $9 \leq n \leq 21$ with $n \equiv 6 \bmod 15$. This means $n = 21$, $e_1 = 5$. Surely $f_5 = 1$, $f_4 = 20$, and this is the type-I design from line 2 of Table 5.1.

We have thus shown that all 4-designs with some $k'_j > 4$ are type-I. For the remaining designs we have then the abbreviated column table:

k'_j	0	0	0	3	4
k^*_j	$4+4\rho$	$4+3\rho$	$4+2\rho$	$4+\rho$	4
k_j	$4+4\rho$	$5+3\rho$	$6+2\rho$	$7+\rho$	8

Table 5.4

Note that this table makes it clear that $n \geq 12$.

We now suppose we have a 4-design with $e_1 \geq 7$. Since

$$e_1 = \frac{4\rho^2 + 7\rho + 4 - n}{\rho^2 - 1} ,$$

we have $4\rho^2 + 7\rho + 4 - n \geq 7\rho^2 - 7$, or $3\rho^2 - 7\rho - 11 \leq -n \leq -12$; hence,

$$3\rho^2 - 7\rho + 1 \leq 0$$

or

$$\rho \leq \frac{7 + \sqrt{49 - 12}}{6} < 2\frac{13}{72} .$$

Thus, if ρ is integral, $\rho = 2$, while one of 2ρ, 3ρ, or 4ρ must be integral so that ρ must take one of the values 2, 3/2, 4/3, 5/3, 5/4, 7/4. Since we have $n \leq -3\rho^2 + 7\rho + 11$ and e_1 must be integral, this leaves only three candidates: (1) $n = 13$, $\rho = 2$, $e_1 = 7$; (2) $n = 15$, $\rho = 4/3$, $e_1 = 7$; (3) $n = 12$, $\rho = 7/4$, $e_1 = 8$.

(1) Since $\rho = 2$, Table 5.4 makes the column structure clear: precisely one column with $k'_j = 3$, $k^*_j = 6$, and 12 with $k'_j = k^*_j = 4$. Hence, Lemma 3.6 applies.

(2) Here again, the column structure is forced. One column has $k'_j = 1$ and the remaining have $k'_j = k^*_j = 4$. Hence, Lemma 3.6 applies.

(3) There is only one admissable column here.

We now consider 4-designs with $\underline{e_1 = 6}$. We obtain as usual

$$n = -2\rho^2 + 7\rho + 10 \geq 12 , \tag{5.3}$$

from which we deduce

$$\rho < 3\frac{1}{5} .$$

Further, (5.3) shows that 2ρ is integral, so that we obtain the following candidates for a 4-design with $e_1 = 6$.

Case	ρ	n	r_1	r_2	e_2
1	2	16	11	6	10
2	3	13	10	4	7
3	3/2	16	10	7	10
4	5/2	15	11	5	9

Case 1. We are supposing $k_j^! \leq 4$; then surely $e_1 r_1 \leq 64$, but evidently $e_1 r_1 = 66$.

Case 2. From Table 5.4 we see only 2 column types are admissable: $k_j^! = 3$, $k_j^* = 7$ and $k_j^! = k_j^* = 4$. We must have one of the former and 12 of the latter so that Lemma 3.6 excludes this design.

Case 3. The column possibilities here are

$k_j^!$	0	2	4
k_j^*	10	7	4
k_j	10	9	8

With f_i the number of columns with $k_j^! = i$, we have

$$f_0 + f_2 + f_4 = 16 ,$$

$$2f_2 + 4f_4 = 60 ,$$

$$\frac{f_0}{6} + \frac{f_2}{5} + \frac{f_4}{4} = \frac{47}{12} ,$$

yielding the unique solution $f_0 = 1$, $f_2 = 0$, $f_4 = 15$. This is the type-I design from the (16, 6, 2) configuration.

Case 4. Here again, with $k_j' = 4$ we cannot have $e_1 r_1 = 66$.

This brings us to 4-designs with $e_1 = 5$. We have

$$n = -\rho^2 + 7\rho + 9$$

so that ρ is an integer and $\rho \leq 6$. We systematically exclude the five possibilities.

Case 1. $\rho = 2$, $n = 19$, $r_1 = 13$, $r_2 = 7$, $e_2 = 14$. Here, $f_1, f_0 \leq 1$. If $f_0 = 1$, the remaining k_j^*'s satisfy $k_j^* \leq 6$ so that $f_1 = f_2 = 0$, $f_4 = 11$, $f_3 = 7$. These values violate (3.7). Hence, $f_0 = 0$. If $f_1 = 1$ we have

$$f_2 + f_3 + f_4 = 18 ,$$
$$2f_2 + 3f_3 + 4f_4 = 64 ,$$
$$\frac{1}{6} f_2 + \frac{1}{5} f_3 + \frac{1}{4} f_4 = \frac{115}{28} ,$$

which has no integral solution. Thus, $f_1 = 0$ and the conditions are

$$f_2 + f_3 + f_4 = 18 ,$$
$$2f_2 + 3f_3 + 4f_4 = 65 ,$$
$$\frac{1}{6} f_2 + \frac{1}{5} f_3 + \frac{1}{4} f_4 = \frac{17}{4} ,$$

yielding the inadmissable values $f_2 = 6$, $f_4 = 17$, $f_3 = -5$.

Case 2. $\rho = 3$, $n = 21$, $r_1 = 16$, $r_2 = 6$, $e_2 = 16$. Here, note if $f_0 = 1$ we have the type-I design from the (21, 5, 1)-configuration. Since $f_0 \leq 1$ we suppose $f_0 = 0$. Surely also $f_1 \leq 1$ and if $f_1 = 1$ necessarily $f_2 = 0$ with $f_3 = 0, 1$ and respectively $f_4 = 20, 19$. The former alternative is excluded by Lemma 3.6 and the latter by (3.7)

$$\frac{1}{10} + \frac{1}{6} + \frac{19}{4} = \frac{301}{60} \neq \frac{61}{12} .$$

Thus, we have $f_1 = 0$ and

$$f_2 + f_3 + f_4 = 21 ,$$

$$2f_2 + 3f_3 + 4f_4 = 80 ,$$

$$\frac{1}{8} f_2 + \frac{1}{6} f_3 + \frac{1}{4} f_4 = \frac{61}{12} ,$$

but this forces $f_4 = 21$, $f_3 = -4$, $f_2 = 4$, so that only the type-I design occurs.

<u>Case 3</u>. $\rho = 4$, $n = 21$, $r_1 = 13$, $r_2 = 9$, $e_2 = 16$. Table 5.4 shows here $f_0 = 0$, $f_1 \leq 1$, $f_2 \leq 1$. If $f_1 = 1$, $f_2 = f_3 = 0$ and $f_4 = 20$, and Lemma 3.6 excludes this possibility. If $f_1 = 0$ we have

$$f_2 + f_3 + f_4 = 21 ,$$

$$2f_2 + 3f_3 + 4f_4 = 65 ,$$

$$\frac{1}{10} f_2 + \frac{1}{7} f_3 + \frac{1}{4} f_4 = 6 ,$$

yielding the absurdity $f_2 = 130/3$, $f_3 = -203/3$, $f_4 = 136/3$.

<u>Case 4</u>. $\rho = 5$, $n = 19$, $e_2 = 14$, $r_1 = 16$, $r_2 = 4$. Here, $e_1 r_1 = 80$ denies $k_j' \leq 4$.

<u>Case 5</u>. $\rho = 6$, $n = 15$, $r_1 = 13$, $r_2 = 3$. Again, $e_1 r_1 = 65$ forbids $k_j' \leq 4$.

We now take the case $e_1 = 4$. Lemma 3.7 and Table 5.4 make it clear that ρ is integral and $\rho \leq 4$. Indeed

$$n = 7\rho + 8 , \quad r_2 = 8 , \quad r_1 = 7\rho + 1 ,$$

so that there are three possible 4-designs with $e_1 = 4$.

(1) $\rho = 4$, $n = 36$, $r_1 = 29$. For reference we note the column possibilities are given in the following table:

k'_j	0	1	2	3	4
k^*_j	20	16	12	8	4
k_j	20	17	14	11	8

Now since $e_2 = 32$, $f_4 \le 8$, but if $f_4 \le 7$ we have $e_1 r_1 = 116 = \Sigma k'_j \le 28 + 3 \cdot 29 = 115$. Thus, $f_4 = 8$ and

$$\sum_{k'_j \le 3}' k'_j = 84 \le 3 \cdot 28$$

forces $f_3 = 28$ with the remaining f_i's zero. In a row with sum $r_2 = 8$, let σ be the sum of the entries in columns with $k'_j = k^*_j = 4$. From (3.9) we must have

$$\frac{\sigma}{4} + \frac{8-\sigma}{7} = 1 + \frac{1}{4} ,$$

or $\sigma = 1$. Now if α is the inner product of this row with, say, row one, we have from (3.8)

$$\frac{1}{4} + \frac{\alpha - 1}{7} = 1 .$$

But this says $\alpha - 1 = 21/4$, and thus the design cannot exist.

(2) $\rho = 3$, $n = 29$, $r_1 = 22$. Here, the column table is

k'_j	0	1	2	3	4
k^*_j	16	13	10	7	4
k_j	16	14	12	10	8

Note that $f_0 \le 1$. If $f_0 = 1$ then $f_1 \le 1$. We have in general the constraints:

$$\begin{aligned} f_o + f_1 + f_2 + f_3 + f_4 &= 29 , \\ f_1 + 2f_2 + 3f_3 + 4f_4 &= 88 , \qquad (5.4) \\ \frac{1}{6} f_0 + \frac{1}{5} f_1 + \frac{1}{4} f_2 + \frac{1}{3} f_3 + \frac{1}{2} f_4 &= \frac{61}{6} . \end{aligned}$$

With $f_0 = 1$, $f_1 = 1$ this system has a unique solution with $f_2 = -12/5$. If $f_0 = 1$, $f_1 = 0$, we obtain $f_2 = 0$, $f_3 = 24$, $f_4 = 4$. We can write this design so that rows 4 and 5 appear so:

$$\begin{array}{llllll} 0 & 1\,1\,1\,1 & \overbrace{1 \text{------------} 1}^{18} & & \overbrace{0 \text{------------} 0}^{6} & \\ 1 & 1\,0\,0\,0 & \underbrace{1 \text{---} 1}_{\sigma} & 0 \text{---} 0 & 1 \text{---} 1 & 0 \text{---} 0 \end{array}$$

where column one has $k_j^* = 0$, columns 2 through 5 have $k_j^* = 4$. Now using (3.8) with these two rows, we must have

$$\frac{1}{4} + \frac{\sigma}{6} = 1 ,$$

which is not possible with σ integral. Thus, $f_0 = 0$ and the system (5.4) becomes a rank 3 system with the one parameter solution:

$$f_2 = 8f_4 - 26 ,$$

$$f_1 = -7/2\, f_4 + 25/2 ,$$

$$f_3 = -11/2\, f_4 + 85/2 .$$

Since $e_2 = 25$, $f_4 \leq 6$, but the above equations show a contradiction for $f_2 \geq 0$ means $f_4 \geq 4$, while $f_1 \geq 0$ forces $f_4 \leq 3$.

(3) $\rho = 2$, $n = 22$, $r_1 = 15$. With these parameters we have the following rank 3 system on the variables f_i:

$$\sum_{i=0}^{4} f_i = 22 ,$$

$$\sum_{i=0}^{4} i f_i = 60 ,$$

$$\frac{1}{8} f_0 + \frac{1}{7} f_1 + \frac{1}{6} f_2 + \frac{1}{5} f_3 + \frac{1}{4} f_4 = \frac{17}{4} .$$

This system has the 2-parameter solution

$$f_0 = 34 - \frac{8}{5} f_3 - 6f_4 \ ,$$

$$f_1 = -84 + \frac{21}{5} f_3 + 14f_4 \ ,$$

$$f_2 = 72 - \frac{18}{5} f_3 - 9f_4 \ .$$

We require integer values, non-negative with $f_0 \leq 1$ and $f_4 \leq 4$. This yields precisely one possibility:

$$(f_0, f_1, f_2, f_3, f_4) = (0, 0, 9, 10, 3) \ .$$

However, this is not an acceptable column structure, as it yields from (3.11) $\Delta^2 = 2^{16} \cdot 3^{11} \cdot 5^{10}$. This completes consideration of the case $e_1 = 4$.

We take the final case $\underline{e_1 = 3}$. We have from (3.10) and (3.5)

$$n = \rho^2 + 7\rho + 7 \ ,$$

$$r_1 = \rho^2 + 6\rho + 1 \ ,$$

$$r_2 = \rho + 7 \ .$$

From Table 5.4 we see our usual constraints are

$$\frac{f_0}{4\rho} + \frac{f_1}{3\rho+1} + \frac{f_2}{2\rho+2} + \frac{f_3}{\rho+3} = \frac{4\rho^2+7\rho+4}{4\rho}$$

$$f_0 + f_1 + f_2 + f_3 = \rho^2 + 7\rho + 7 \qquad (5.5)$$

$$f_1 + 2f_2 + 3f_3 = 3\rho^2 + 18\rho + 3 \ ,$$

noting that $k_j^! \leq 3$.

Now

$$\det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ \frac{1}{3\rho+1} & \frac{1}{2\rho+1} & \frac{1}{\rho+3} \end{pmatrix}$$

$$= \frac{2(\rho+1)}{(3\rho+1)} \left(\frac{1}{\rho+3} - \frac{1}{2\rho+2} \right) \neq 0$$

for $\rho \neq 1$. So the system (5.5) has the 1-parameter solution:

$$f_0 = \frac{4\rho^3 + 7\rho^2 - (11 + 4f_3)\rho + 12}{\rho+3}, \tag{5.6}$$

$$f_1 = \frac{-3(3\rho + 1)(\rho^2 + 2\rho - 3 - f_3)}{\rho+3}, \tag{5.7}$$

$$f_2 = \frac{6(\rho + 1)(\rho^2 + 3\rho - f_3)}{\rho+3}. \tag{5.8}$$

From (5.7), $(11+4f_3)\rho \geq 4\rho^3+8\rho^2-\rho$. On the other hand, from (5.6), $(11+4f_3)\rho \leq 4\rho^3+7\rho^2+12$. Hence, we have

$$4\rho^3 + 8\rho^2 - \rho \leq 4\rho^3 + 7\rho^2 + 12$$

or

$$\rho^2 - \rho - 12 \leq 0,$$

whence $\rho \leq 4$.

(1) $\rho = 4$. The relations (5.6), (5.7), and (5.8) become

$$f_2 = \frac{30}{7}(28-f_3), \quad f_1 = \frac{39}{7}(f_3-21), \quad f_0 = \frac{1}{7}(336-16f_3).$$

$f_1 \geq 0$ implies $f_3 \geq 21$, but $f_0 \geq 0$ forces $f_3 \leq 21$. Thus, $f_3 = 21$, $f_2 = 30$, $f_0 = f_1 = 0$. But then (3.11) gives $\Delta^2 = 2^{30} \cdot 5^{32} \cdot 7^{21}$.

(2) $\rho = 2$. We have $f_2 = \frac{18}{5}(10-f_3)$, $f_1 = \frac{21}{5}(f_3-5)$, $f_0 = \frac{1}{5}(50-8f_3)$ so that $f_3 = 5$, $f_1 = 0$, $f_2 = 18$, $f_0 = 2$. Here again, (3.11) excludes the design for $\Delta^2 = 3^{20} \cdot 5^5 \cdot 2^{25}$.

(3) $\rho = 3$. $f_2 = 4(18-f_3)$, $f_1 = 5(f_3-12)$, $f_0 = 25-2f_3$. Surely then $f_3 = 12$, $f_2 = 24$, $f_1 = 0$, $f_0 = 1$. (3.11) will not exclude this possibility, so we write rows one and twenty as

$$\begin{array}{ccccc}
0 & \overbrace{1 \text{———————} 1}^{12} & \overbrace{1 \text{———————} 1}^{16} & \overbrace{0 \text{———————} 0}^{8} \\
0 & \underbrace{1 \text{—} 1}_{\sigma}\ 0 \text{—} 0 & \underbrace{1 \text{—} 1}_{\tau}\ 0 \text{—} 0 & 1 \text{—} 1\ 0 \text{—} 0
\end{array}$$

Noting $n = 37$, $r_1 = 28$, $r_2 = 10$, and $k_j^! = 0$ gives $k_j^* = 16 < e_2$. In the diagram, columns 2 through 13 have $k_j^! = 3$. Now (3.9) gives

$$\frac{\sigma}{6} + \frac{10-\sigma}{8} = \frac{4}{3} \qquad \text{or} \qquad \sigma = 2 \ .$$

Then (3.8) with these two rows gives

$$\frac{\sigma}{6} + \frac{\tau}{8} = 1 \qquad \text{or} \qquad \tau = \frac{16}{3} \ ,$$

which is not possible. This completes the discussion of $e_1 = 3$ and hence also the tabulation of 4-designs.

VI. λ-MATRICES

1. Introduction

In this section we shall be interested in the following combinatorial situation: $S_1, \ldots, S_{n-1}$ are to be subsets of $\{1, 2, \ldots, n\}$ with the feature that each S_i is a k-set and for $i \neq j$, $S_i \cap S_j$ is a λ-set. Our results will show that with one exceptional class, there are precisely two replication numbers for such a configuration and we can find them explicitly in terms of n, k, and λ. We can describe quite completely the structure of such configurations, and in certain cases describe all such designs. Further, we can completely list the exceptional designs, modulo the problem of the determination of all Hadamard matrices. The device used here is similar to that used for the problem of λ-designs, and we deal exclusively with the incidence matrix of such configurations. This matrix, A, is (0, 1), of size $n \times (n-1)$ with the feature

$$A^tA = (k-\lambda)I + \lambda J ,$$

where I and J are the usual matrices of order $(n-1)$.

For ease in stating our results, we make the following formal definition.

<u>Definition.</u> Let n, k, λ be integers with $n > k > \lambda > 0$. $\Lambda(n, k, \lambda)$ will denote the class of all $n \times (n-1)$ (0, 1)-matrices A such that

$$A^tA = (k-\lambda)I + \lambda J$$

where I is the identity matrix of order n-1 and J is the matrix of ones of order n-1. We call the elements of $\Lambda(n, k, \lambda)$ <u>λ-matrices.</u>

2. The Structure of λ-Matrices

Examples:

1. Adjoining a row of zeros or a row of ones to a (ν, k, λ)-configuration produces elements of $\Lambda(\nu+1, k, \lambda)$ or $\Lambda(\nu+1, k+1, \lambda+1)$, respectively.

2. Removing a column from a (ν, k, λ)-configuration produces an element of $\Lambda(\nu, k, \lambda)$.

3. Removing the exceptional column $(k_j \neq 2\lambda)$ from a type-I λ-design of order n gives an element of $\Lambda(n, 2\lambda, \lambda)$.

4. Let B be the incidence matrix of a $(4\lambda-1, 2\lambda, \lambda)$-symmetric block design and $0 \le e \le 4\lambda-1$. Choose e columns of B and replace them by their complementary vectors. Now adjoin a row vector with ones in precisely those chosen columns and zeros elsewhere. The resultant matrix lies in $\Lambda(4\lambda, 2\lambda, \lambda)$. We prove this last assertion. The complemented columns had sum $4\lambda-1-2\lambda = 2\lambda-1$ and adjoining a one brings this sum to 2λ. These complemented columns have inner product $\lambda-1$ among themselves and the additional row vector augments this to λ. The unaltered columns meet in λ positions,and adjoining zeros has not changed this. Finally, a complimented column of a (ν, k, λ)-configuration meets a non-complemented column in $k-\lambda$ positions; here, this is λ and the bordering row vector does not affect this count. We will call matrices constructed in this way H_λ-matrices.

Before proceeding to the discussion of the properties of λ-matrices, we list a few remarks:

(1) λ-matrices never have constant row sums, for if r were the row sum we would have $rn = k(n-1)$, and hence n divides k, denying $k < n$.

(2) The construction of H_λ-matrices given in example 4 allows the choice of 0 columns or all the columns to be complemented. This amounts to adjoining a zero row to the $(4\lambda-1, 2\lambda, \lambda)$- configuration and a row of ones to the complementary $(4\lambda-1, 2\lambda-1, \lambda-1)$ design, respectively.

(3) H_λ matrices may have more than 2 row sums, as the following example constructed from the (7, 4, 2)-design shows:

$$\begin{matrix} 1&1&0&0&0&0&0\\ 1&1&0&1&1&1&1\\ 1&0&1&1&1&0&0\\ 1&0&1&0&0&1&1\\ 0&0&0&1&0&1&0\\ 0&0&0&0&1&0&1\\ 0&1&1&1&0&0&1\\ 0&1&1&0&1&1&0 \end{matrix}$$

(4) H_λ matrices may be viewed as constructed directly from Hadamard matrices as follows: normalize the Hadamard matrix H of order 4λ so that its initial column contains positive ones. Delete this column, obtaining H_1 of order $4\lambda \times 4\lambda-1$. Now let $A = \frac{1}{2}(H_1+J)$ and note that A is (0, 1), and since $JH_1 = 0$, $A^tA = \lambda I+\lambda J$.

(5) The class $\Lambda(4\lambda, 2\lambda, \lambda)$. We have shown that this class contains the so-called H_λ matrices. In fact, these are all its members. For take $A \in \Lambda(4\lambda, 2\lambda, \lambda)$ and write the first row with ones initially placed, say, in columns $1, 2, \ldots, r$. Complement the first r columns

and remove row one. The resulting matrix B is square, $(4\lambda-1) \times (4\lambda-1)$ has j^{th} column sum $4\lambda - 2\lambda = 2\lambda$ for $j = 1, \ldots, r$, and of course 2λ for $j > r$. Viewing two columns of A we observe that 1 1 , 1 0 , 0 1 , and 0 0 each occur precisely λ times, so that B has column inner products λ and indeed is the incidence matrix of a $(4\lambda-1, 2\lambda, \lambda)$ block design evidently yielding A as an H_λ matrix.

(6) If $A \in \Lambda(n, k, \lambda)$, then the complement of A lies in $\Lambda(n, n-k, n-2k+\lambda)$.

(7) The class $\Lambda(n, k, \lambda)$ with $k(k-1) = \lambda(n-1)$ consists precisely of the examples (2) above. This will come out in our discussion of λ-matrices, but of course follows e. g. from the rational completion theorem of Hall and Ryser [4].

Theorem 6. 2. 1

Let $A \in \Lambda(n, k, \lambda)$. Then either (1) $\lambda n = k^2$ and A is an H_λ-matrix, or (2) $\lambda n \neq k^2$ and A has two row sums given by the roots of the quadratic equation:

$$x^2 - \left[n - \frac{(n-k)(k-2\lambda)}{\lambda n - k^2}\right] x + (k-\lambda)(n-1)\left[1 + \frac{k-2\lambda}{\lambda n - k^2}\right] = 0 .$$

Proof: Taking the case (2) first, we form the matrix

$$B = \begin{bmatrix} \lambda/k & a_{11} & a_{12} & \cdots & a_{1,n-1} \\ \lambda/k & a_{21} & a_{22} & \cdots & a_{2,n-1} \\ \vdots & & & & \\ \lambda/k & a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} \end{bmatrix}$$

and argue that it is non-singular as follows: since $(k-\lambda)I + \lambda J$ with $k \neq \lambda$ is non-singular, A has rank $n-1$, so if B is singular, the vector $(\lambda/k, \ldots, \lambda/k)^t$ lies in the column space of A, i.e., there is some real vector $\vec{X}$ such that $A\vec{X} = (\lambda/k, \ldots, \lambda/k)^t$. Now this means $A^tA\vec{X} = A^t(\lambda/k, \ldots, \lambda/k)^t = (\lambda, \lambda, \ldots, \lambda)^t$. If $\vec{X} = (x_1, \ldots, x_{n-1})$, this gives $(k-\lambda)x_i + \lambda \sum_{j=1}^{n-1} x_j = \lambda$. Thus, since $k \neq \lambda$, all the x_i are equal, which would imply that A has constant row sums, contrary to the remark (1) above. Now let $\vec{Y} = (y_1, y_2, \ldots, y_n)^t$ be the unique solution to $B^t\vec{Y} = (\lambda, \ldots, \lambda)^t$ and set

$$u = \lambda(\lambda n - k^2)/k^2 ,$$

$$w = -\lambda + \sum_{i=1}^{n} y_i^2 .$$

Now form the matrix of order $(n+1)\times(n+1)$:

$$C = \begin{bmatrix} y_1 & \lambda/k & a_{1,1} & \cdots & a_{1,n-1} \\ y_2 & \lambda/k & a_{2,1} & \cdots & a_{2,n-1} \\ \vdots & & & & \\ y_n & \lambda/k & a_{n,1} & \cdots & a_{n,n-1} \\ \sqrt{-\lambda} & \sqrt{-\lambda} & \sqrt{-\lambda} & \cdots & \sqrt{-\lambda} \end{bmatrix}$$

Note that $C^tC = \text{diag}[w, u, k-\lambda, \ldots, k-\lambda]$. Since $\lambda n \neq k^2$, $u \neq 0$ and C is singular if $w = 0$. The last n-columns of C are independent since B was non-singular so that if C were singular, the vector $\vec{Y}_1 = (y_1, \ldots, y_n, \sqrt{-\lambda})^t$ would lie in the column space of

$$B_1 = \begin{bmatrix} & B & \\ \sqrt{-\lambda} & \cdots & \sqrt{-\lambda} \end{bmatrix}$$

i.e., we would have a vector $\vec{\phi}$ such that $\vec{Y}_1 = B_1\vec{\phi}$. Hence,

$$B_1^t\vec{Y}_1 = B_1^tB_1\vec{\phi} = \text{diag}[u, k-\lambda, \ldots, k-\lambda]\vec{\phi} .$$

But indeed, $B_1^t\vec{Y}_1 = \vec{0}$, so then $\vec{\phi} = \vec{0}$, $\vec{Y}_1 = \vec{0}$, forcing $\lambda = 0$. Hence, C is non-singular and $w \neq 0$, and we can form

$$K = C\,\mathrm{diag}[\,1/\sqrt{w}\,,\ 1/\sqrt{u}\,,\ 1/\sqrt{k-\lambda}\,, \ldots,\ 1/\sqrt{k-\lambda}\,]$$

and note

$$K^tK = \mathrm{diag}[\,1/\sqrt{w},\ 1/\sqrt{u}, \ldots, 1/\sqrt{k-\lambda}\,]^2\,\mathrm{diag}[\,w, u, k-\lambda, \ldots, k-\lambda\,] = I.$$

Hence, $KK^t = I$ and we obtain the relations:

$$\frac{1}{w} + \frac{1}{u} + \frac{n-1}{k-\lambda} = -\frac{1}{\lambda} \tag{6.1}$$

$$\frac{y_i}{w} + \frac{\lambda}{ku} + \frac{r_i}{k-\lambda} = 0 \tag{6.2}$$

$$\frac{y_i^2}{w} + \frac{\lambda^2}{k^2u} + \frac{r_i}{k-\lambda} = 1 \tag{6.3}$$

$$\frac{y_iy_j}{w} + \frac{\lambda^2}{k^2u} + \frac{\alpha_{ij}}{k-\lambda} = \delta_{ij} \tag{6.4}$$

where $AA^t = (\alpha_{ij})$ and $\alpha_{ii} = r_i$. From (6.2) and (6.3) we obtain

$$y_i^2 - y_i + \frac{w}{u}\left(\frac{\lambda^2}{k^2} - \frac{\lambda}{k} - u\right) = 0\ . \tag{6.5}$$

Since from (6.2) we see not all the y_i are equal, (6.5) shows there are precisely two values y_1 and y_2 with $y_1+y_2 = 1$. (Note here if $k(k-1) = \lambda(n-1)$, (6.5) forces $y_i = 0, 1$.) So from (6.2) there are two values for r_i with:

$$\frac{1}{w} + \frac{2\lambda}{ku} + \frac{r_1 + r_2}{k-\lambda} = 0$$

$$\begin{aligned} r_1 + r_2 &= (k-\lambda)\left(-\frac{1}{w} - \frac{2\lambda}{ku}\right) \qquad (6.6)\\ &= (k-\lambda)\left(\frac{1}{\lambda} + \frac{1}{u} + \frac{n-1}{k-\lambda} - \frac{2\lambda}{ku}\right)\\ &= \frac{(k-\lambda)}{\lambda}\left(1 + \frac{k^2}{\lambda n-k^2} - \frac{2\lambda k}{\lambda n-k^2}\right) + n - 1\\ &= \frac{(k-\lambda)(n-2k)}{\lambda n-k^2} + n - 1 \end{aligned}$$

$$r_1 + r_2 = n - \frac{(n-k)(k-2\lambda)}{\lambda n - k^2} . \qquad (6.7)$$

We have thus only to check the value of $r_1 r_2$. The calculation is a bit messy, and we introduce

$$\delta \equiv \lambda n - k^2 = k^2 u/\lambda \quad , \quad N \equiv k - \lambda . \qquad (6.8)$$

Now (6.2) then gives

$$y_i^2 = w^2 \left(\frac{r_i^2}{N^2} + \frac{2 r_i \lambda}{kuN} + \frac{\lambda^2}{k^2 u^2} \right)$$

so that (6.3) becomes

$$w \left(\frac{r_i^2}{N^2} + \frac{2 r_i k}{\delta N} + \frac{k^2}{\delta^2} \right) + \frac{\lambda}{\delta} + \frac{r_i}{N} = 1 . \qquad (6.9)$$

Viewing (6.9) as a quadratic in r_i/N we see that

$$\frac{r_1 r_2}{N^2} = \frac{k^2}{\delta^2} + \frac{1}{w} \left(\frac{\lambda}{\delta} - 1 \right) . \qquad (6.10)$$

From (6.1) and the definition of u we observe that $-\frac{1}{w} = \frac{k^2}{\lambda\delta} + \frac{n-1}{N} + \frac{1}{\lambda}$; hence, (6.10) becomes

$$\frac{r_1 r_2}{N^2} = \frac{k^2}{\delta^2} + \frac{k^2}{\lambda\delta} + \frac{n-1}{N} + \frac{1}{\lambda} - \frac{k^2}{\delta^2} - \frac{\lambda(n-1)}{N\delta} - \frac{1}{\delta}$$

$$= \frac{(n-1)}{N} \left(1 - \frac{\lambda}{\delta} \right) + \frac{1}{\lambda\delta} \left(k^2 + \delta - \lambda \right) = \frac{(n-1)}{N} \left(1 - \frac{\lambda}{\delta} \right) + \frac{(n-1)}{\delta} .$$

Thus,

$$r_1 r_2 = (n-1) N \left\{ 1 - \frac{\lambda}{\delta} + \frac{N}{\delta} \right\} = (n-1)(k-\lambda) \left\{ 1 + \frac{k-2\lambda}{\lambda n - k^2} \right\} ,$$

precisely as desired.

We now treat the case (1), i.e., we assume $\lambda n = k^2$ and show A is an H_λ matrix. In view of remark (5) above, this means we must show $n = 4\lambda$, $k = 2\lambda$. From $\sum_{i=1}^{n} a_{ij} a_{ik} = \lambda$ $(j \neq k)$ we note that

$$\sum_{i=1}^{n} a_{ij}(r_i - 1) = \lambda(n-2) \qquad j = 1, 2, \ldots, n-1$$

where r_i is the i^{th} row sum of A. Hence if

$$x_i = \frac{r_i - 1}{n - 2} \qquad i = 1, 2, \ldots, n \qquad (6.11)$$

we have

$$\sum_{i=1}^{n} a_{ij}\, x_i = \lambda \qquad j = 1, 2, \ldots, n-1 \ . \qquad (6.12)$$

Now set

$$u = -\lambda + \sum_{i=1}^{n} x_i^2 \qquad (6.13)$$

and suppose $u \geq 0$. Then (6.13) and (6.11) give

$$\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} \left(\frac{r_i - 1}{n-2}\right)^2 \geq \lambda$$

or

$$\sum_{i=1}^{n} r_i(r_i - 1) - \sum_{i=1}^{n} r_i + n \geq \lambda(n-2)^2 \ . \qquad (6.14)$$

But summing (6.12) over j we see that

$$\sum_{i=1}^{n} r_i(r_i - 1) = \lambda(n-1)(n-2) \ .$$

So (6.14) is $\lambda(n-1)(n-2) - k(n-1) + n \geq \lambda n-2)^2$, which we may write

$$(k-\lambda)(n-2) \leq (n-k) \ ,$$

which, since $k-\lambda \geq 1$, forces $k = 2$, $\lambda = 1$. Now $\lambda n = k^2$ means $n = 4$, so we have the H_λ matrices of order 4. Thus, we take $n > 4$ and have $u < 0$. Now the matrix

$$B = \begin{bmatrix} x_1 & a_{1,1} & \cdots & a_{1,n-1} \\ \vdots & & & \\ x_n & a_{n,1} & \cdots & a_{n,n-1} \end{bmatrix}$$

is non-singular as follows. As before, if B is singular with $\vec{X} = (x_1, \ldots, x_n)^t$, there is some $\vec{Y}$ such that $\vec{X} = A\vec{Y}$. Then $A^t\vec{X} = A^tA\vec{Y}$ which if $\vec{Y} = (y_1, \ldots, y_{n-1})^t$ says

$$(k-\lambda)y_j + \lambda \sum_{i=1}^{n} y_i = \lambda \qquad j = 1, \ldots, r-1 .$$

Since $k \neq \lambda$ this means $y_j = y \quad j = 1, \ldots, n-1$. But then $(r_i - 1)/(n-2) = r_i y \quad i = 1, \ldots, n$, which would imply A has constant row sums. We may then choose $\vec{Z} = (z_1, \ldots, z_n)^t$ such that

$$B^t\vec{Z} = (\lambda, \lambda, \ldots, \lambda)^t .$$

With $W = -\lambda + \sum_{i=1}^{n} Z_i^2$, we show $W \neq 0$. Suppose not. Then $||\vec{Z}||^2 = \lambda$ and

$$\|\vec{X}-\vec{Z}\|^2 = \|X\|^2 + \|Z\|^2 - 2\vec{X}\cdot\vec{Z} = u+\lambda+\lambda-2\lambda = u .$$

But $u < 0$.

We thus form, as in the previous theorem, the matrix K of order $(n+1)\times(n+1)$:

$$K = \begin{bmatrix} \frac{y_1}{\sqrt{w}} & \frac{x_1}{\sqrt{u}} & \frac{a_{1,1}}{\sqrt{k-\lambda}} & \cdots & \frac{a_{1,n-1}}{\sqrt{k-\lambda}} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{y_n}{\sqrt{w}} & \frac{x_n}{\sqrt{u}} & \frac{a_{n,1}}{\sqrt{k-\lambda}} & \cdots & \frac{a_{n,n-1}}{\sqrt{k-\lambda}} \\ \sqrt{\frac{-\lambda}{w}} & \sqrt{\frac{-\lambda}{u}} & \sqrt{\frac{-\lambda}{k-\lambda}} & \cdots & \sqrt{\frac{-\lambda}{k-\lambda}} \end{bmatrix}$$

As we have arranged $K^tK = I$ so we have as a bonus $KK^t = I$ or specifically:

$$\frac{n-1}{k-\lambda} + \frac{1}{u} + \frac{1}{w} + \frac{1}{\lambda} = 0 \qquad (6.15)$$

$$\frac{r_i}{k-\lambda} + \frac{x_i^2}{u} + \frac{y_i^2}{w} = 1 \qquad (6.16)$$

$$\frac{r_i}{k-\lambda} + \frac{x_i}{u} + \frac{y_i}{w} = 0 \ . \qquad (6.17)$$

Now from (6.17) we have

$$y_i = -w\left[\frac{x_i}{u} + \frac{r_i}{k-\lambda}\right], \qquad (6.18)$$

and from (6.11)

$$\frac{r_i}{k-\lambda} = \frac{n-2}{k-\lambda}x_i + \frac{1}{k-\lambda} ,$$

so that (6.18) gives

$$y_i = -w(\beta x_i + \frac{1}{k-\lambda}) \qquad (6.19)$$

where

$$\beta = \frac{1}{u} + \frac{n-2}{k-\lambda} \ .$$

Now (6.16) and (6.17) give

$$\frac{x_i^2}{u} - \frac{x_i}{u} + \frac{y_i^2}{w} - \frac{y_i}{w} = 1 ,$$

so with (6.19),

$$\frac{x_i^2}{u} - \frac{x_i}{u} + \beta x_i + \frac{1}{k-\lambda} + w\left(x_i^2\beta^2 + \frac{2\beta x_i}{k-\lambda} + \frac{1}{(k-\lambda)^2}\right) = 1 ,$$

$$x_i^2\left(\frac{1}{u} + w\beta^2\right) + x_i\left(\frac{2\beta w}{k-\lambda} + \frac{n-2}{k-\lambda}\right) + \left(\frac{w}{(k-\lambda)^2} + \frac{1}{k-\lambda} - 1\right) = 0. \qquad (6.20)$$

Now observe that

$$u(n-2)^2 = -\lambda(n-2)^2 + \sum_{i=1}^{n}(r_i-1)^2 = -\lambda(n-2)^2 + \sum_{i=1}^{n} r_i(r_i-1) - \sum_{i=1}^{n} r_i + n$$

$$= \lambda(n-2) - k(n-1) + n ,$$

and for ease of computation set $\tau = u(n-2)^2$. We first compute β

more explicitly:

$$\beta = \frac{1}{u} + \frac{n-2}{k-\lambda} = \frac{(n-2)^2}{\tau} + \frac{(n-2)}{k-\lambda}$$

$$= \frac{(n-2)}{(k-\lambda)\tau} \{k(n-2)-\lambda(n-2)+\lambda(n-2)-k(n-1)+n\} ,$$

$$\beta = \frac{(n-2)(n-k)}{(k-\lambda)\tau} . \qquad (6.21)$$

The observation here is that $\beta \neq 0$ so that (6.19) shows that the number of distinct y_i is the same as the number of distinct x_i and hence r_i as well. We now compute the coefficient of x_i^2 in the quadratic (6.20). This number is $\beta(\frac{1}{u\beta} + w\beta)$.

$$u\beta = \frac{\tau}{(n-2)^2} \frac{(n-2)(n-k)}{(k-\lambda)\tau} = \frac{(n-k)}{(n-2)(k-\lambda)} . \qquad (6.22)$$

From (6.15)

$$\frac{1}{w} = -\frac{1}{\lambda} - \frac{1}{u} - \frac{n-1}{k-\lambda} = -\frac{1}{\lambda} - \frac{(n-2)^2}{\tau} - \frac{(n-1)}{k-\lambda}$$

$$= \frac{\tau(k-\lambda) + (n-2)^2\lambda(k-\lambda) + (n-1)\lambda\tau}{-\lambda\tau(k-\lambda)} .$$

Computing the numerator we obtain, since $\lambda n = k^2$, $\tau(k-\lambda) + (n-2)^2\lambda(k-\lambda) + \lambda(n-1)\tau = k(n-k)$. Thus,

$$w = \frac{-\lambda\tau(k-\lambda)}{k(n-k)} . \qquad (6.23)$$

Now

$$\beta(\frac{1}{u\beta} + w\beta) = \beta\left\{ \frac{(n-2)(k-\lambda)}{n-k} - \frac{\lambda\tau(k-\lambda)}{k(n-k)} \frac{(n-2)(n-k)}{(k-\lambda)\tau} \right\}$$

$$= \frac{\beta(n-2)}{k(n-k)} \{k^2-\lambda k-\lambda(n-k)\} = \frac{\beta(n-2)}{k(n-k)} \{k^2-\lambda n\} = 0 .$$

This means that (6.20) is <u>not</u> a quadratic, but then if the coefficient of x_i is not zero, we would arrive at the contradiction of constant row sums. We therefore may assert that a λ-matrix with $\lambda n = k^2$ also has

$$-2\beta w = n-2 .$$

We then use (6.21) and (6.23), obtaining

$$+2\left(\frac{(n-2)(n-k)}{(k-\lambda)\tau}\right)\left(\frac{\lambda\tau(k-\lambda)}{k(n-k)}\right) = n-2 ,$$

$$\frac{2\lambda}{k} = 1 ,$$

$$\underline{k = 2\lambda} .$$

Since $n = k^2/\lambda$ we have $n = 4\lambda$ and A is an H_λ-matrix. This completes the proof of Theorem 6.2.1.

Having completely settled the classes $\Lambda(n,k,\lambda)$ with $\lambda n = k^2$ we discuss the case $\lambda n \neq k^2$ and note that our proof of Theorem 6.2.1 gives us some information on the structure of these λ-matrices.

Let us write the λ-matrix A so that its first e rows have sum r_1 and the remaining (n-e) have sum r_2. Note that the $e \times (n-1)$ submatrix A_1 with row sums r_1 has constant column sums k' where $k'(r_1-1) + (k-k')(r_2-1) = \lambda(n-2)$ or

$$k' = \frac{\lambda(n-2) - k(r_2-1)}{r_1 - r_2} . \tag{6.24}$$

In view of (6.4) with this normalization

$$AA^t = \left[\begin{array}{c|c} \overbrace{\begin{matrix} r_1 & & \lambda_1 \\ & \ddots & \\ \lambda_1 & & r_1 \end{matrix}}^{e} & \lambda' \\ \hline \lambda' & \begin{matrix} r_2 & & \lambda_2 \\ & \ddots & \\ \lambda_2 & & r_2 \end{matrix} \end{array}\right] \tag{6.25}$$

where

$$\frac{\lambda_1}{k-\lambda} = \frac{-y_1^2}{w} - \frac{\lambda}{\lambda n-k^2} = -1 + \frac{\lambda}{\lambda n-k^2} + \frac{r_1}{k-\lambda} - \frac{\lambda}{\lambda n-k^2} = \frac{r_1-k+\lambda}{k-\lambda} .$$

So

$$\lambda_1 = r_1 - k + \lambda ,$$

and similarly, (6.26)

$$\lambda_2 = r_2 - k + \lambda .$$

We may compute also λ' from (6.4) and (6.5):

$$\frac{\lambda'}{k-\lambda} = -\frac{y_1 y_2}{w} - \frac{\lambda}{\lambda n-k^2} = -\frac{1}{uk^2}(\lambda^2-\lambda k-uk^2) - \frac{\lambda}{\lambda n-k^2}$$

$$= \frac{-\lambda+k+\lambda n-k^2-\lambda}{\lambda n-k^2} = 1 + \frac{k-2\lambda}{\lambda n-k^2}$$

$$\lambda' = (k-\lambda)\left\{1 + \frac{k-2\lambda}{\lambda n-k^2}\right\} = \frac{r_1 r_2}{(n-1)} . \qquad (6.27)$$

We finally note that our choice of the y_i gives

$$k'y_1 + (k-k')y_2 = \lambda . \qquad (6.28)$$

With these remarks on the general structure of λ-matrices, we discuss some special classes $\Lambda(n, k, \lambda)$.

3. $\Lambda(n, 2\lambda, \lambda)$

Theorem 6.3.1

$A \in \Lambda(n, 2\lambda, \lambda)$ if and only if

(a) $n = 4\lambda$ and A is an H_λ-matrix, or

(b) $n = 4\lambda - 1$ and A is a partial $(4\lambda-1, 2\lambda, \lambda)$-configuration, or

(c) A is completeable to a type-I λ-design.

Proof: If $n = 4\lambda$, we have already remarked that A is an H_λ-

matrix. If $n \neq 4\lambda$, then $\lambda n \neq k^2$ and we may use (6.28):

$$k'y_1 + (2\lambda - k')y_2 = \lambda \;;$$

since $y_1 + y_2 = 1$ this becomes

$$2y_1(k'-\lambda) = k'-\lambda \;.$$

Now if $k' \neq \lambda$ we would have $y_1 = y_2 = \frac{1}{2}$, which would force $r_1 = r_2$. Hence, $k' = \lambda$, and adjoining a column with ones in positions one through e and zeros elsewhere completes A to a (ν, k, λ) with $k = 2\lambda$, i.e., $(4\lambda-1, 2\lambda, \lambda)$ or by Lemma 3.6 to a type-I λ-design.

4. $\Lambda(n, k, 1)$

Theorem 6.4.1

$A \in \Lambda(n, k, 1)$ if and only if

(a) A is a 4×4 H_1 matrix, or

(b) A is an $(n-1) \times (n-1)$ permutation matrix bordered with a row of ones (a partial I-design), or

(c) A is a partial projective plane, or

(d) A is a projective plane with a row of zeros added.

Proof: Since $\lambda = 1$ forces row inner products to be zero or one, we have, from (6.25) and (6.26), either $e = 1$ or $e > 1$ and say $\lambda_1 = 1$, $\lambda_2 = 0$. If $e = 1$, then $k' = 0, 1$. If $k' = 0$, we clearly have the possibility (d) of the theorem. If $k' = 1$, we surely have the case (b). With $e > 1$, note that if $r_i = 0$ we have case (d) and $r_i \neq 0$ with (6.27) and the above remarks $\lambda' = 1$. If $\lambda_1 = 1$, $\lambda_2 = 0$, the matrix obtained by adjoining a column with zeros in positions one through e yields a projective plane.

5. $\Lambda(2k, k, \lambda)$ and $\Lambda(4\lambda, k, \lambda)$

Theorem 6.5.1

A λ-matrix with $n = 2k$ or $n = 4\lambda$ is an H_λ-matrix.

Proof: If $n = 2k$ and $\lambda n \neq k^2$, we apply Theorem 6.2.1. In particular, we compute the product $r_1 r_2$:

$$r_1 r_2 = (k-\lambda)(2k-1)\left\{1 + \frac{k-2\lambda}{k(2\lambda-k)}\right\} ,$$

$$r_1 r_2 = \frac{(k-\lambda)(2k-1)(k-1)}{k} ,$$

but this would mean k divides λ, denying $k > \lambda$. Thus, we conclude $\lambda n = k^2$ and A is an H_λ-matrix.

If $n = 4\lambda$ and $\lambda n \neq k^2$, we compute

$$r_1 + r_2 = 4\lambda - \frac{(4\lambda-k)(k-2\lambda)}{4\lambda^2 - k^2} ,$$

$$r_1 + r_2 = 4\lambda + \frac{4\lambda - k}{2\lambda + k} .$$

Now since $k < n$ we must have

$$4\lambda - k \geq 2\lambda + k ,$$

but this says $k \leq \lambda$. Hence, we conclude in this case also that $\lambda n = k^2$ and A is an H_λ-matrix.

The preceding theorem then shows that the exceptional class of H_λ-matrices is characterized by any one of the conditions $\lambda n = k^2$, $n = 2k$, or $n = 4\lambda$.

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