\(\lambda\)-DESIGNS AND RELATED

COMBINATORIAL CONFIGURATIONS

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ABSTRACT

This thesis deals with two problems. The first is the determination of \( \lambda \)-designs, combinatorial configurations which are essentially symmetric block designs with the condition that each subset be of the same cardinality negated. We construct an infinite family of such designs from symmetric block designs and obtain some basic results about their structure. These results enable us to solve the problem for \( \lambda = 3 \) and \( \lambda = 4 \). The second problem deals with configurations related to both \( \lambda \)-designs and \((v, k, \lambda)\)-configurations. We have \((n-1)k\) subsets of \(\{1, 2, \ldots, n\}\), \(S_1, \ldots, S_{n-1}\), such that \(S_i \cap S_j\) is a \(\lambda\)-set for \(i \neq j\). We obtain specifically the replication numbers of such a design in terms of \(n, k, \) and \(\lambda\) with one exceptional class which we determine explicitly. In certain special cases we settle the problem entirely.
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I. INTRODUCTION

For the purposes of this thesis, a $\lambda$-design is a $(0, 1)$ square matrix $A$ of order $n$ such that

$$A^t A = \lambda J + \text{diag}[k_1 - \lambda, \ldots, k_n - \lambda],$$

(1.1)

where $A^t$ denotes the transpose of $A$, $J$ is the $n \times n$ matrix of ones, $k_j > \lambda > 0$, and not all the $k_j$'s are equal.

First definitively studied by de Bruijn and Erdős with $\lambda = 1$ [1], they have received new interest with the following theorem of H. J. Ryser [7]:

A $(0, 1)$ square matrix $A$ satisfying (1.1) with $k_j > \lambda > 0$ either has all its row and column sums equal (and hence is a balanced incomplete block design) or has precisely two row sums $r_1$ and $r_2$ with $r_1 + r_2 = n + 1$.

Along with this result, Ryser also established that apart from row and column permutations there is precisely one 2-design. This design is of order 7 and is of a class of $\lambda$-designs, called H-designs, constructed from the symmetric block design with parameters $(4\lambda - 1, 2\lambda, \lambda)$.

The combinatorial interest in matrices of this type satisfying (1.1) is clear. They represent (i.e., are incidence matrices for) the following configuration: we have $n$ subsets $S_1, S_2, \ldots, S_n$ of $\{1, 2, \ldots, n\}$ with the feature that $S_i \cap S_j$ is a $\lambda$-set for $i \neq j$ and the $S_j$'s do not all have the same cardinality.

In Chapter II of the present work we will generalize Ryser's H-design construction to an arbitrary $(v, k, \lambda)$-configuration. In
Chapter III we will establish some properties of \( \lambda \)-designs which will enable us, in Chapters IV and V, to determine all 3-designs and all 4-designs.

Chapter VI then varies the problem slightly to consider the following combinatorial situation. We have \( n-1 \) subsets \( S_1, S_2, \ldots, S_{n-1} \) of \( \{1, 2, \ldots, n\} \) with the feature that \( S_i \cap S_j \) is a \( \lambda \)-set for \( i \neq j \) and each \( S_i \) is a \( k \)-set. We show here the representing matrices with one exceptional class have two row sums, determined explicitly in terms of \( n, k, \lambda \). We can say much then about the structure of such configurations and in special cases (\( k = 2\lambda, \lambda = 1, n = 2k \)) determine all such designs modulo the determination of related \( (v, k, \lambda) \) - configurations. The \( \lambda \)-designs of Chapter I play a role here.

The exceptional class is determined explicitly modulo the determination of Hadamard matrices.
II. TYPE I $\lambda$-DESIGNS

Theorem 2.1

If there exists a $(v, k, \lambda')$-configuration, not of the form $(4\lambda - 1, 2\lambda - 1, \lambda - 1)$, then there exists a $\lambda$-design with $\lambda = k - \lambda'$ and row sums $v - k$ and $k + 1$.

Proof: Let $B$ be the incidence matrix of the $(v, k, \lambda')$-configuration, written so that column one has its $k$ ones in rows one through $k$, i.e.,

$$B = \begin{bmatrix}
1 & \cdots & A_1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & A_2 \\
0 & \cdots & 0
\end{bmatrix}$$

where $A_1$ is of size $k \times v - 1$ and $A_2$ is of size $v - k$ by $v - 1$. Now form the matrix $A$:

$$A = \begin{bmatrix}
0 & \cdots & A'_1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & A_2 \\
1
\end{bmatrix}$$

Then $A$ is a $\lambda$-design with $\lambda = k - \lambda'$ as follows. $A_2$ evidently has column sums $k - \lambda'$ so that column one of $A$ has inner product $k - \lambda'$ with each of $2$ through $v$. Consider then columns $i$ and $j$ of $A$ with $i \geq j \geq 2$. Suppose the corresponding columns in $A_1$ have inner product $t$, then these columns in $A_2$ have inner product $\lambda' - t$ and in $A'_1$ $k - 2\lambda' + t$ so that columns $i$ and $j$ of $A$ have inner product $k - 2\lambda' + t + \lambda' - t = k - \lambda'$. $A$ has two column sums $v - k$ and $2(k - \lambda')$. These are distinct precisely if we have avoided the design with parameters $(4\lambda - 1, 2\lambda - 1, \lambda - 1)$. The row sum claim in the theorem is obvious.
Definition 2.1

A λ-design obtainable via the construction given in the proof of theorem 2.1 will be called a type-I λ-design.

Remarks 2.1

a) The type-I λ-design obtained from the (4λ - 1, 2λ, λ)-configuration is indeed one of Ryser's H-designs.

b) The above construction when applied to the (excluded) (4λ - 1, 2λ - 1, λ - 1)-configuration will simply produce the symmetric block design with parameters (4λ - 1, 2λ, λ).

c) For a given λ there are at most a finite number of parameter sets for a type-I λ-design. This is because there are at most a finite number of (ν, k, λ') triples with k - λ' = λ.

d) Let A be a type-I λ-design derived from the (ν, k, λ') matrix B. A has (n-1) columns with sum 2λ and one exceptional column. To obtain the λ-design corresponding to the complementary block design to B one need only replace this exceptional column by its complement.
III. PROPERTIES OF $\lambda$-DESIGNS

Throughout the discussion $\Lambda$ will denote a $\lambda$-design of order $n$ with entries $a_{ij}$. Its row sums will be denoted $r_1$ and $r_2$ with $r_1 > \frac{n+1}{2}$, $r_2 < \frac{n+1}{2}$. $e_1$ will denote the number of rows of $\Lambda$ with row sum $r_1$ and $e_2 = n - e_1$. $k_j^*$ will denote the number of ones in column $j$ of $\Lambda$ which occur in rows with sum $r_1$, and $k_j = k_j - k_j^*$ where $k_j$ is the $j$th column sum of $\Lambda$. Following Ryser we set

$$\rho = \frac{r_1 - 1}{r_2 - 1}, \quad x_i = \frac{r_i - 1}{n - 1}, \quad u = -\lambda + \sum_{i=1}^{n} x_i^2. \quad (3.1)$$

The proof of Ryser's theorem is essentially the establishment of the relation:

$$x_i^2 - x_i - u = 0, \quad (3.2)$$

from which we have

$$x_1 + x_2 = 1 \quad \text{and} \quad x_1 x_2 = -u. \quad (3.3)$$

From (3.2) $u = x_1(x_1 - 1)$ or

$$u = \frac{(r_1 - 1)(r_1 - n)}{(n - 1)^2} = -\frac{(r_1 - 1)(r_2 - 1)}{(n - 1)^2}$$

so that $\rho u = -x_1^2$ and then from (3.3) $\rho x_2^2 = -u$. Also note

$$\rho + 1 = \frac{r_1 - 1 + r_2 - 1}{r_2 - 1} = \frac{n - 1}{r_2 - 1} = \frac{1}{x_2^2},$$

so that $x_1 = \frac{\rho}{\rho + 1}$ and $u = -x_1 x_2 = -\frac{\rho}{(\rho + 1)^2}$. We list these relations as

$$-\frac{x_1^2}{u} = \rho, \quad -\frac{x_2^2}{u} = \frac{1}{\rho}, \quad x_1 = \frac{\rho}{\rho + 1}, \quad x_2 = \frac{1}{1 + \rho}, \quad u = \frac{-\rho}{(\rho + 1)^2}. \quad (3.4)$$

Now

$$\frac{\rho n + 1}{\rho + 1} = \frac{\rho}{\rho + 1} n + \frac{1}{\rho + 1} = x_1 n + x_2 = x_1(n - 1) + x_1 + x_2 = r_1 - 1 + 1 = r_1$$
using (3.3) and (3.4). From this we obtain the following relations which we list for future reference:

\[ r_1 = \frac{\rho n+1}{\rho+1}, \quad r_1-1 = \frac{\rho(n-1)}{\rho+1}, \]
\[ r_2 = \frac{\rho n}{\rho+1}, \quad r_2-1 = \frac{n-1}{\rho+1}. \]  \hspace{1cm} (3.5)

We note in addition the following relations established in Ryser's paper:

\[ k_j^* = \lambda - \rho(k_j! - \lambda) \]  \hspace{1cm} (3.6)
\[ \sum_{j=1}^{n} \frac{1}{k_j^{-\lambda}} = -\frac{1}{\lambda} - \frac{1}{u} = \frac{\lambda(1+\rho)^2 - \rho}{\lambda \rho} \]  \hspace{1cm} (3.7)

using (3.4).

\[ \sum_{j=1}^{n} \frac{a_{ij}a_{ej}}{k_j^{-\lambda}} = \delta_{ie} - \frac{x_i x_e}{u} \]  \hspace{1cm} (3.8)

where \( \delta_{ie} \) is Kronecker's delta.

\[ \sum_{j=1}^{n} \frac{a_{ij}}{k_j^{-\lambda}} = 1 - \frac{x_i}{u} = \frac{-x_i}{u} \]  \hspace{1cm} (3.9)

From the relation

\[ e_1 r_1(r_1-1) + e_2 r_2(r_2-1) = \lambda n(n-1) \]

we obtain, using (3.5) and \( e_1 + e_2 = n \):

\[ e_1 \frac{(\rho n+1)(n-1)\rho}{(1+\rho)^2} + (n-e_1) \frac{(\rho n)(n-1)}{(1+\rho)^2} = \lambda n(n-1). \]

Hence,

\[ e_1(\rho n+1 - (n+\rho)) = \lambda n(1+\rho)^2 - n(\rho+n) \]

and

\[ e_1 n(\rho^2-1) = \lambda n(1+\rho)^2 - n(\rho+n), \]

so that
\[ e_1 = \frac{\lambda(1+\rho)^2-(\rho+n)}{\rho^2-1}. \]  

(3.10)

Finally, if \( \Delta = \det A \), \( \Delta \) is integral and

\[ \Delta^2 = \left[ \prod_{j=1}^{n} (k_j - \lambda) \right] \left[ 1 + \lambda \sum_{j=1}^{n} \frac{1}{k_j - \lambda} \right]. \]  

(3.11)

Type-I \( \lambda \)-designs with \( \lambda > 1 \) all have \( e_1 \geq 3 \). The next two theorems show this to be true of \( \lambda \)-designs in general.

**Theorem 3.1**

A \( \lambda \)-design with \( e_1 = 1 \) has \( \lambda = 1 \).

**Proof:** With \( e_1 = 1 \) the matrix \( A \) has two column types:

\[ k'_1 = 1, \quad k'_1^* = \lambda \rho - \rho + \lambda, \]

\[ k'_2 = 0, \quad k'_2^* = \lambda (1+\rho), \]  

as seen from (3.6). Now (3.10) yields

\[ n-1 = (\rho+1)(\lambda \rho - \rho + \lambda), \]  

(3.13)

and we compute from (3.5) and (3.13)

\[ r_2 = \lambda (1+\rho) - \rho + 1. \]  

(3.14)

From (3.12) we note that \( \rho = k'_2^* - k'_1^* \) is an integer, while (3.12) and (3.14) indicate that \( r_2 = k'_1 \).

We now normalize the matrix \( A \) to the form

\[ A = \begin{pmatrix} 1 & \ldots & 1 & 0 & \ldots & 0 \\ & \vdots & & \ddots & & \vdots \\ & & & & B & \\ & & & & & C \end{pmatrix} \]

and use (3.8) with \( i = 1 \), \( t > 1 \) to deduce

\[ \sum_{j=1}^{r_1} \frac{a_{t,j}}{k'_1 - \lambda} = -\frac{x_1 x_2}{u} = 1. \]
or

\[ \sum_{j=1}^{r_1} a_{tj} = k_1 - \lambda , \]

i.e., B has constant row sums \( k_1 - \lambda \). Since \( r_2 = k_1 \), C has row sums \( \lambda \).

We now further normalize within the matrices B and C to bring A to the form

\[
\begin{array}{cccc|ccc}
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[ k_2 = \lambda (1 + \rho) \]

(3.15)

where \( C_1 \) has an initial zero column. We suppose \( C_1 \) is not vacuous. Let \( \sigma \) denote the sum of row 1 of \( B_1 \), \( \tau \) the sum of row 1 of \( C_1 \), and note from (3.8) with \( i = 2, \mu = k_2 + 2 \)

\[
\frac{\sigma}{k_1 - \lambda} + \frac{\tau}{k_2 - \lambda} = \frac{x_2}{u} = \frac{1}{\rho}
\]

in view of (3.4). We write this more conveniently as

\[
\frac{\sigma}{\lambda \rho - \rho + 1} + \frac{\tau}{\lambda \rho} = \frac{1}{\rho} .
\]

(3.16)

Thus, we have

\[
\lambda \rho \sigma + \tau (\lambda \rho - \rho + 1) = \lambda (\lambda \rho - \rho + 1)
\]

or

\[
\lambda \rho (\sigma + \tau) = \lambda^2 \rho + (\rho - 1)(\tau - \lambda) .
\]

But \( \rho > 1 \) and \( \tau < \lambda \) so that

\[ (\sigma + \tau) < \lambda . \]

(3.17)
We now write (3.16) as
\[ \rho \left( \lambda^2 - \lambda (\sigma + \tau + 1) + \tau \right) = \tau - \lambda < 0, \]
so that
\[ \lambda^2 - \lambda (\sigma + \tau + 1) + \tau < 0. \quad (3.18) \]
But then \( \lambda^2 + \tau < \lambda (\sigma + \tau + 1) \leq \lambda (\lambda) \) because of (3.17). This means that \( \tau = 0 \), but then (3.16) gives \( \sigma = \lambda - 1 + \frac{1}{\rho} \). Hence, we are forced to conclude that \( C_1 \) is vacuous, and thus from (3.15) we see that \( k_2 = (n-1) \), or from (3.13)
\[ \lambda (1 + \rho) = (1 + \rho)(\lambda \rho - \rho + \lambda), \]
whence \( \lambda = 1 \) as asserted.

**Theorem 3.2**

A \( \lambda \)-design has \( e_1 \neq 2 \).

**Proof:** From (3.5) and (3.10) with \( e_1 = 2 \) we have
\[ n = (\lambda - 2)\rho^2 + (2\lambda - 1)\rho + \lambda + 2 \]
\[ r_1 = (\lambda - 2)\rho + (\lambda + 2) \]
\[ r_2 = (\lambda - 2)\rho^2 + (\lambda + 1)\rho + 1. \quad (3.19) \]
The possibilities for \( k^j \) are 0, 1, 2 and the corresponding column types are displayed in the following table:

<table>
<thead>
<tr>
<th>( k^j )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
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<td>( k^* )</td>
<td>( \lambda + \lambda \rho )</td>
<td>( \lambda + \lambda \rho - \rho )</td>
<td>( \lambda + \lambda \rho - 2\rho )</td>
</tr>
<tr>
<td>( k_j )</td>
<td>( \lambda + \lambda \rho )</td>
<td>( \lambda + \lambda \rho - \rho + 1 )</td>
<td>( \lambda + \lambda \rho - 2\rho + 2 )</td>
</tr>
<tr>
<td>no. of columns</td>
<td>( w )</td>
<td>( x )</td>
<td>( y )</td>
</tr>
</tbody>
</table>

(3.20)

We thus have the relations
\[ w + x + y = (\lambda - 2)\rho^2 + (2\lambda - 1)\rho + \lambda + 2 \]  
\[ x + 2y = 2(\lambda - 2)\rho^2 + 2(\lambda + 1)\rho + 2 \]  
(3.21)

from \( w + x + y = n, \sum_{j=1}^{n} k_j^1 = e_1 r_1 \) and (3.19). We explicitly determine \( w, y, \) and \( y \) as follows. Normalize the first two rows of \( A: \)

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
y & x/2 & x/2 & w \\
\end{array}
\]

and use (3.8) on these rows obtaining

\[
\frac{y}{\lambda\rho - 2\rho + 2} = -\frac{x_1^2}{u} = \rho
\]

or

\[
y = (\lambda - 2)\rho^2 + 2\rho.
\]  
(3.22)

Now from (3.21) we may compute

\[
x = 2(\lambda - 1)\rho + 2
\]

\[
w = \lambda - \rho.
\]  
(3.23)

Thus, \( \rho \) is integral and \( \rho \leq \lambda \). Further, normalize \( A \) so that its first three rows appear so:

\[
\begin{array}{cccccccc}
y & x & w \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
\sigma & \tau & \tau' & \alpha \\
\end{array}
\]

With \( \sigma, \tau, \tau', \) and \( \alpha \) defined by this diagram, use (3.8) on rows one and three and also on two and three. This will provide the information
\[
\frac{\sigma}{\lambda \rho - 2 \rho + 2} + \frac{\tau}{\lambda \rho - \rho + 1} = 1 \quad (3.24)
\]

so that necessarily \( \tau = \tau' \). Using (3.8) on row three with itself:

\[
\frac{\sigma}{\lambda \rho - 2 \rho + 2} + \frac{2\tau}{\lambda \rho - \rho + 1} + \frac{\alpha}{\lambda \rho} = 1 + \frac{1}{\rho} \quad (3.25)
\]

Then (3.24) and (3.25) imply

\[
\frac{\tau}{\lambda \rho - \rho + 1} + \frac{\alpha}{\lambda \rho} = \frac{1}{\rho} \quad (3.26)
\]

which when solved for \( \tau \) becomes

\[
\tau = \frac{(\lambda - \alpha)(\lambda \rho - \rho + 1)}{\lambda \rho} \quad (3.27)
\]

Solving (3.24) for \( \sigma \) gives

\[
\sigma = (\lambda \rho - 2 \rho + 2)\left(1 - \frac{\tau}{\lambda \rho - \rho + 1}\right)
\]

\[
= (\lambda \rho - 2 \rho + 2)\left(1 - \left\{\frac{1}{\rho} - \frac{\alpha}{\lambda \rho}\right\}\right) ;
\]

\[
\sigma = \frac{(\lambda \rho - 2 \rho + 2)(\lambda \rho - \lambda + \alpha)}{\lambda \rho} \quad (3.28)
\]

Now (3.27) and (3.28) mean

\[
\sigma + \tau = \lambda \rho - 2 \rho + 3 - \left(\frac{\lambda + \alpha \rho - \alpha}{\lambda \rho}\right) ,
\]

so that evidently \( m = [\lambda + \alpha(\rho - 1)]/\lambda \rho \) is a positive integer. But

\[
\alpha \leq \lambda - \rho < \lambda \quad \text{and} \quad \rho \geq 2 ,
\]

so

\[
(\rho - 1)\alpha < (\rho - 1)\lambda .
\]

Hence,

\[
\lambda + (\rho - 1)\alpha < \lambda \rho \quad \text{and} \quad 0 < m < 1 .
\]

This contradiction denies the existence of a \( \lambda \)-design with \( e_1 = 2 \).
We remark that the corresponding statements to theorems 3.1 and 3.2 for the parameter $e_2$ are almost immediate. For $e_2 \leq 2$ we have $k_j^* = 0, 1, 2$, correspondingly $k_j^! = \lambda / \rho$, $\lambda - 1 / \rho$, $\lambda - 2 / \rho$. Since $1 / \rho$ is not integral, the only compatible pair of these is $(\lambda / \rho$, $\lambda - 2 / \rho)$, whence $e_2 = 2$, $\rho = 2$. But (3.8) then used on the last two rows of $A$ would say that $r_2 = (\lambda + 2) / 4$. But (3.10) becomes $n - 2 = \frac{9 \lambda - n - 2}{3}$ or $n = \frac{9 \lambda}{4} + 1$. But then 4 divides $\lambda$ and $r_2$ is not integral.

Since 1-designs have $e_1 = 1$, Theorem 3.1 offers a characterization of these configurations. The next theorem characterizes 1-designs in a different way.

Theorem 3.3

A $\lambda$-design may be permuted to a normal matrix if and only if $\lambda = 1$.

Proof: Since a 1-design may be permuted to a symmetric matrix, this part of the implication is clear.

Conversely, suppose $A$ is normal. Permute the rows and columns of $A$ so that the first $e_1$ rows (columns) have row (column) sum $r_1$. We work with this permuted matrix viewed as

$$
\begin{bmatrix}
A_1 & A_2 \\
A_3 & A_4
\end{bmatrix}
$$

where $A_1$ is $e_1 \times e_1$. From (3.6) we have

$k_j^! = \frac{[\lambda(n-1) - k_j(r_2-1)](r_1-r_2)}{r_1-r_2}$.

Since $A$ is a $\lambda$-design with respect to rows and columns, $A_1$ has constant row and column sums, $x$, given by:
\[ x = \frac{\lambda(n-1) - r_1}{(r_1 - r_2)} \quad (3.30) \]

\[ A_2 \text{ has row sums } r_1 - x \text{ and column sums } c \text{ where} \]
\[ c = \frac{\lambda(n-1) - r_2}{(r_1 - r_2)} \quad (3.31) \]

From (3.30) and (3.31) we have \( c - x = r_2 - 1 \) and hence
\[ r_2 - c = 1 - x \geq 0. \quad (3.32) \]

Thus, \( x = 0 \) or \( x = 1 \). If \( x = 1 \), then from (3.32) \( r_2 = c \), so \( A_4 \) is a zero block and columns one and \( e_1 + 1 \) meet in \( \lambda \leq 1 \) positions. If \( x = 0 \), \( A_1 \) is a zero block and (3.32) shows \( A_4 \) to have column sums one. Here again columns \( e_1 + 1 \) and one meet in less than two positions. Hence in either event, \( \lambda = 1 \) as asserted.

The last three results contain the following theorem due to Majumdar [5]:

**Corollary 3.4**

Let \( A \) be a \((0, 1)\) matrix of size \( v \times v \). Suppose both
\[ AA^t = \lambda'J + \text{diag}[r_1 - \lambda', \ldots, r_v - \lambda'] \]
and
\[ A^tA = \lambda J + \text{diag}[k_1 - \lambda, \ldots, k_v - \lambda], \]
\( 0 < \lambda' < r_1 \), \( 0 < \lambda < k_j \). Then \( A \) is either a \((v, k, \lambda)\)-configuration or a \(1\)-design.

**Proof:** We suppose \( A \) is not a \((v, k, \lambda)\)-configuration so that \( A \) is a \(\lambda\)-design and \( A^t \) is a \(\lambda'\)-design. We specialize (3.11) to the case where \( A \) has two column sums \( k_1 \) and \( k_2 \) occurring respectively \( f_1 \) and \( f_2 \) times:
\[ \Delta^2 = \begin{cases} f_1 - 1 & f_2 - 1 \\ (k_1 - \lambda) & (k_2 - \lambda) \\ (k_1 - \lambda)(k_2 - \lambda) + \lambda e_1(k_2 - \lambda) + \lambda e_2(k_1 - \lambda) \end{cases} \]
\[ (3.33) \]
From (3.33) we see the characteristic polynomial of $A^tA$ to be
\[
P_1(x) = (k_1 - \lambda - x)^{f_1-1} (k_2 - \lambda - x)^{f_2-1} g_1(x),
\]
where
\[
g_1(x) = (k_1 - \lambda - x)(k_2 - \lambda - x) + \lambda f_1(k_2 - \lambda - x) + \lambda f_2(k_1 - \lambda - x).
\]
"Similarly," the characteristic polynomial of $AA^t$:
\[
P_2(x) = (r_1 - \lambda' - x)^{e_1-1} (r_2 - \lambda' - x)^{e_2-1} g_2(x)
\]
where
\[
g_2(x) = (r_1 - \lambda' - x)(r_2 - \lambda' - x) + \lambda 'e_1(r_2 - \lambda' - x) + \lambda 'e_2(r_1 - \lambda' - x).
\]
If $e_1$ or $f_1$ is 1, we have $\lambda = \lambda' = 1$. Hence we may take $e_i \geq 3$, $f_i \geq 3$ by Theorems 3.1, 3.2 and the following remarks. Since, e.g., $g_1(k_i - \lambda) \neq 0$, $f_i - 1$ is the precise multiplicity of $k_i - \lambda$ with similar remarks for $P_2(x)$. Further, $g_1(x)$ has distinct roots -- this may be seen directly or by noting that $P_i(x)$ must have a root of multiplicity one by the classical theorem of Perron-Frobenius [2]. Now since $AA^t$ and $A^tA$ are similar, $P_1(x) \equiv P_2(x)$. But the above remarks show that $g_1(x) = g_2(x)$ and hence $\lambda = \lambda'$, $k_i = r_i$, and $e_i = f_i$ whence $A$ may be permuted to a normal matrix and by Theorem 3.3
$\lambda = \lambda' = 1$.

Our next few results, though of a general nature, are developed explicitly for considering the nature of $\lambda$-designs for specified $\lambda$ (particularly here $\lambda = 3$ and $\lambda = 4$).

Lemma 3.5

(1) A $\lambda$-design with a column with $k_j' = 2\lambda - 1$ has $\rho = \frac{\lambda}{\lambda - 1}$. 
(2) A $\lambda$-design with $\rho = \frac{\lambda}{\lambda - 1}$ is an H-design. †

Proof: (1) The corresponding $k^*_j$ is $\lambda - \rho(\lambda - 1)$; hence, $\lambda - \rho(\lambda - 1) \geq 0$ or $\rho \leq \frac{\lambda}{\lambda - 1}$. Further, $\rho(\lambda - 1)$, and since $\rho > 1$, $\rho(\lambda - 1) \geq \lambda$ or $\rho \geq \frac{\lambda}{\lambda - 1}$. Hence, $\rho = \frac{\lambda}{\lambda - 1}$ as asserted.

(2) From (3.5) we have

$$r_{1} - 1 = \frac{\lambda(n-1)}{2\lambda - 1}, \quad r_{2} - 1 = \frac{(n-1)(\lambda - 1)}{2\lambda - 1}, \quad (3.38)$$

and since $\lambda$ and $2\lambda - 1$ are relatively prime, we have for a positive integer $t$

$$n - 1 = t(2\lambda - 1) \quad (3.39)$$

and we may rewrite (3.38) as

$$r_{1} - 1 = \lambda t, \quad r_{2} - 1 = t(\lambda - 1). \quad (3.40)$$

Now from (3.10)

$$e_1 = \frac{\lambda \left( \frac{2\lambda - 1}{\lambda - 1} \right)^2 - \left( \frac{\lambda}{\lambda - 1} \right) - n}{\left( \frac{\lambda}{\lambda - 1} \right)^2 - 1},$$

i.e.,

$$e_1 = -t(\lambda - 1)^2 - (\lambda - 1) + \lambda(2\lambda - 1). \quad (3.41)$$

Now theorems 3.1 and 3.2 imply $e_1 \geq 3$, so that

$$t(\lambda - 1)^2 = \lambda(2\lambda - 1) - (\lambda - 1) - 3,$$

$$t \leq \frac{2\lambda(\lambda - 1)^2(\lambda - 1)^2}{(\lambda^2 - 1)^2} \leq 2 + \frac{2}{\lambda - 1} - \frac{2}{(\lambda - 1)^2}.$$

Hence, $t = 1$ or $t = 2$. If $t = 1$ from (3.41), $e_1 = \lambda^2$ and from (3.39) $n = 2\lambda$. Hence, $\lambda^2 < 2\lambda$ or $\lambda < 2$. Thus, $t = 2$ and $r_1 = 2\lambda + 1$, $r_2 = 2\lambda - 1$ from (3.40), and Ryser has shown that a $\lambda$-design with

† See Remark (2.1.a).
these parameters is necessarily an H-design.

Lemma 3.6

Let \( A \) be a \( \lambda \)-design with two column sums \( k_1 \) and \( k_2 \). Suppose further that there is precisely one column with sum \( k_1 \). Then \( A \) is a type-I \( \lambda \)-design.

Proof: Supposing \( A \) has two column sums, write \( A \) in the form

\[
\begin{bmatrix}
A_1 & A_2 \\
A_3 & A_4
\end{bmatrix}
\]

where \( [A_1 \ A_2] \) has row sums \( r_1 \) and \( [A_3 \ A_4] \) has row sums \( r_2 \), and \( [A_1] \) has column sums \( k_1 \) while \( [A_2] \) has column sums \( k_2 \). Let \( \sigma_i \) be an arbitrary row sum of \( A_1 \). Then from (3.9) we have

\[
\frac{\sigma_i}{k_1-\lambda} + \frac{r_1-\sigma_i}{k_2-\lambda} = 1 - \frac{x_1^2}{u} = 1 + \rho,
\]

whence, since \( k_1 \neq k_2 \), \( \sigma_i \) does not depend on \( i \), i.e., \( A_1 \) has constant row sums \( \sigma \). Similarly, \( A_3 \) has constant row sums \( \tau \). In the present case, \( A_1 \) and \( A_3 \) are column vectors, and since surely \( \sigma \neq \tau \), we have \( \sigma = 0 \), \( \tau = 1 \) or \( \sigma = 1 \), \( \tau = 0 \). In either case, all remaining columns are of the \( k^!_j = k^*_j = \lambda \) type. We suppose \( \sigma = 0 \), \( \tau = 1 \) and form

\[
B = \begin{bmatrix}
1 \\
\vdots \\
1 \\
0 \\
\vdots \\
0
\end{bmatrix}
\begin{bmatrix}
A_1^2 \\
A_2 \\
A_3 \\
0
\end{bmatrix}
\]

where \( A_2^! \) denotes the complement of \( A_2 \). Column one of \( B \) has sum \( e_1 \). Column \( j \) of \( A_2 \) has sum \( \lambda \), column \( j \) of \( A_2^! \) has sum \( e_1-\lambda \),
and column \( j \) of \( A_4 \) has sum \( \lambda \); thus, \( B \) has constant column sums \( e_1 \). If two columns of \( A_4 \) meet in \( t \) positions the corresponding columns of \( A_2 \) meet in \( \lambda - 1 \) locations, so those columns in \( A_2^1 \) meet in \( e_1 - \lambda - t \) for a total column inner product in \( B \) of \( e_1 - \lambda \). Thus, \( B \) is an \((n, e_1, e_1 - \lambda)\)-balanced, incomplete symmetric block design yielding our matrix \( A \) as a type-I \( \lambda \)-design.

We remark that it is easy to show that a \( \lambda \)-design cannot have two columns with one sum and the remaining \( n - 2 \) with another sum.

We will need one further lemma for our discussion of 3- and 4-designs.

Lemma 3.7

A \( \lambda \)-design with \( e_1 = \lambda \) has \( \rho \leq \lambda \) with \((2\lambda - 1)\rho \) integral.

Proof: Let \( x \) denote the number of columns with \( k^*_j = k_j = \lambda \). Then

\[ \lambda x \leq e_2 = n - \lambda \quad \text{or} \quad x \leq \frac{n - \lambda}{\lambda} \quad \text{(3.42)} \]

Now (3.10) becomes

\[ n - 1 = (2\lambda - 1)((1 + \rho) \), \quad \text{(3.43)} \]

so that from (3.5) \( r_2 = 2\lambda \) and thus \( r_1 = n + 1 - 2\lambda \). Thus, the first \( \lambda \) rows of \( A \) contain \( \lambda(2\lambda - 1) \) zeros, and if \( n \geq \lambda(2\lambda - 1) \), surely

\[ x \geq n - \lambda(2\lambda - 1) \], which, together with (3.43), forces

\[ n - \lambda(2\lambda - 1) \leq \frac{n - \lambda}{\lambda} \]

\[ n(\lambda - 1) \leq \lambda^2(2\lambda - 1) - \lambda = \lambda(2\lambda + 1)(\lambda - 1) \]

Hence,

\[ n \leq \lambda(2\lambda + 1) \quad \text{(3.44)} \]

Thus, in any event \( n \leq \lambda(2\lambda + 1) \), and from (3.43) we have
\[ \rho = \frac{n - 2\lambda}{2\lambda - 1} \leq \lambda, \quad (3.45) \]

noting (3.44).

From (3.45), of course, \( \rho(2\lambda - 1) \) is integral.
IV. 3-DESIGNS

Type-I 3-Designs

$(\nu, k, \lambda)$-configurations with $k-\lambda = 3$ have one of the following parameter sets: $(13, 4, 1); (13, 9, 6); (11, 5, 2); (11, 6, 3)$. Each of these excepting the $(11, 5, 2)$ will produce via Theorem 2.1 a 3-design. We illustrate each type:

\[
\begin{align*}
0 & 1 1 1 1 1 1 1 1 1 0 0 0 \quad 1 1 1 1 1 1 1 1 1 1 0 0 0 \\
0 & 1 1 1 1 1 1 0 0 0 1 1 1 \quad 1 1 1 1 1 1 1 0 0 0 1 1 1 \\
0 & 1 1 1 0 0 0 1 1 1 1 1 1 \quad 1 1 1 1 0 0 0 1 1 1 1 1 1 \\
0 & 0 0 0 1 1 1 1 1 1 1 1 1 \quad 1 0 0 0 1 1 1 1 1 1 1 1 1 \\
1 & 1 0 0 1 0 0 1 0 0 1 0 0 \quad 0 1 0 0 1 0 0 1 0 0 1 0 0 \\
1 & 1 0 0 0 1 0 0 1 0 0 1 0 \quad 0 1 0 0 0 1 0 0 1 0 0 1 0 \\
1 & 1 0 0 0 0 1 0 0 1 0 0 1 \quad 0 1 0 0 0 0 1 0 0 1 0 0 1 \\
1 & 0 1 0 1 0 0 0 1 0 1 0 1 0 \quad 0 0 1 0 1 0 0 0 1 0 1 0 1 0 \\
1 & 0 1 0 0 1 0 1 0 0 0 1 \quad 0 0 1 0 0 1 0 1 0 0 0 1 \\
1 & 0 1 0 0 0 1 0 1 0 1 0 0 \quad 0 0 1 0 0 0 1 0 1 0 1 0 0 \\
1 & 0 0 1 1 0 0 0 1 0 0 0 1 \quad 0 0 0 1 1 0 0 0 1 0 0 0 1 \\
1 & 0 0 1 0 1 0 0 0 1 1 0 0 \quad 0 0 0 1 0 1 0 0 0 1 1 0 0 \\
1 & 0 0 1 0 0 1 1 0 0 0 1 0 \quad 0 0 0 1 0 0 1 1 0 0 0 1 0 \\
\end{align*}
\]

3-design from $(13, 4, 1)$; $e_1 = 4, \ r_1 = 9, \ r_2 = 5, \ p = 2$.

From $(13, 9, 6)$; $e_1 = 4$, $r_1 = 10, \ r_2 = 4, \ p = 3$.

**Theorem 4.1**

All 3-designs are type-I designs.

**Proof:** In view of Lemma 3.5, we may take $k_j' \leq 4$. If some $k_j' = 4$, then $p = 2$ or $p = 3$. From (3.10) $e_1 = \frac{25-n}{3}$ so that, since $e_1 \leq n-3$ and $4 \leq e_1 \leq 11$ ($e_1 < 4\lambda$), we have $n = 10, 13$. If $n = 10$, $e_1 = 5$, and $r_1 = 7$. The remaining columns have $k_j' \geq 2$ since $k_j' = 4$ and $p = 2$ implies $k_j^* = 1$. Let $f_i$ denote the number of columns with $k_j' = i$. 

...
From (11, 6, 3) -- "H-design with $\lambda = 3"

$e_1 = 5, r_1 = 7, r_2 = 5, \rho = 3/2.$

Then since $k_j' = 2$ implies $k_j^* = 5, f_2 = 0, 1$ with $f_2 + f_3 + f_4 = 10$ and $2f_2 + 3f_3 + 4f_4 = 35$. Clearly then $f_2 = 0, f_3 = f_4 = 5$. Now (3.9) cannot hold with $i = 6$. If $n = 13, e_1 = 4, r_1 = 9$. Then $f_4 = 1,$ which forces $f_2 = 4, f_3 = 8$. But then from (3.11) we have the contradiction $\Delta^2 = 2^8 \cdot 3^{11}$. Hence, we must have $\rho = 3$ and $k_j' = 4$ means $k_j^* = 0$, so all remaining columns have $k_j' = k_j^* = 3$. Again, (3.10) and $e_1 \geq 4$ force $n \leq 13$ and we must have $n = 13$ for $e_1$ to be integral, but then $e_1 = 4$ so that $f_4 = 1$ and we have a type-I design by Lemma 3.6.

For all the remaining 3-designs we have then $k_j' \leq 3$, and Table 4.1 displays the column possibilities.

<table>
<thead>
<tr>
<th>$k_j'$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_j^*$</td>
<td>$3+3\rho$</td>
<td>$3+2\rho$</td>
<td>$3+\rho$</td>
<td>3</td>
</tr>
<tr>
<td>$k_j$</td>
<td>$3+3\rho$</td>
<td>$4+2\rho$</td>
<td>$5+\rho$</td>
<td>6</td>
</tr>
</tbody>
</table>

Table 4.1
Suppose we have a 3-design with $e_1 \geq 6$. Then from (3.10) we deduce $-n \geq 3\rho^2 - 5\rho - 9$. But Table 4.1 makes it clear that $n \geq 10$ whence $1 < \rho < 3/2$. But then $2\rho$ is not integral and $3\rho$ must be, which means $\rho = 4/3$. This forces $n = 10$ and $e_1$ would not be integral.

With $e_1 = 5$, (3.10) becomes $n = -2\rho^2 + 5\rho + 8$ so that $2\rho$ is integral and $\rho \leq 2$. Hence, $\rho = 2, 3/2$; $\rho = 3/2$ can only yield the $H$-design by Lemma 3.5; and $\rho = 2$ means $n = 10$, $r_p = 7$. But $e_2 = 5$ forces one column with $k_j^* = 5$ and the remaining with $k_j^* = 3$; hence, Lemma 3.6 applies.

If $e_1 = 4$, we have $n = -\rho^2 + 5\rho + 7$, so that $\rho = 2, 3, 4$.

1. $\rho = 2$, $n = 13$, $r_1 = 9$, $e_2 = 9$; now $f_0 = 0, 1$. If $f_0 = 0$ we would have $f_1 + f_2 + f_3 = 13$, $f_1 + 2f_2 + 3f_3 = 36$, and $\frac{1}{5}f_1 + \frac{1}{4}f_2 + \frac{1}{3}f_3 = \frac{25}{6}$, which has no integral solution. Hence, $f_0 = 1$, $f_3 = 12$, and we have a type-I design. (2) $\rho = 3$, $n = 13$, $r_1 = 10$; $e_1 r_1 = 40$, but $k_j \leq 3$ denies this. (3) $\rho = 4$, $n = 11$, $r_1 = 9$, $e_2 = 7$; $e_2 = 7$ means $k_j^*$ must be 3 or 7; hence, $f_2 = 1$, $f_3 = 12$, and Lemma 3.6 applies.

The final case is $e_1 = 3$. Here, Lemma 3.7 gives $\rho \leq 3$, (3.10) becomes $n = 5\rho + 6$ so that $\rho = 2, 3$. (1) $\rho = 2$, $n = 16$, $r_1 = 11$; the column structure is uniquely determined and we obtain $f_1 = 0$, $f_2 = 12$, $f_3 = 3$, from which $\Delta^2 = 3^7 \cdot 2^{24}$. (2) $\rho = 3$, $n = 21$, $r_1 = 16$; here, the proof of Lemma 3.7 shows that $f_3 = 6$ so that surely $f_0 = 0$ and from $f_0 + f_1 + f_2 = 15$ and $f_1 + 2f_2 = 30$ we have $f_1 = 0$, $f_2 = 15$. But then $\Delta^2 = 2^4 \cdot 3^6 \cdot 5^{15}$.

Thus, all 3-designs are type-I designs.
V. 4-DESIGNS

All $(v, k, \lambda)$-triples with $k-\lambda = 4$ are listed below [excluding the $(15, 7, 3)$], together with the parameters of the derived 4-designs:

1. $(21, 5, 1)$: $n = 21$, $r_1 = 16$, $e_1 = 5$, $\rho = 3$
2. $(21, 16, 12)$: $n = 21$, $r_1 = 17$, $e_1 = 5$, $\rho = 4$
3. $(16, 6, 2)$: $n = 16$, $r_1 = 10$, $e_1 = 6$, $\rho = 3/2$
4. $(16, 10, 6)$: $n = 16$, $r_1 = 11$, $e_1 = 6$, $\rho = 2$
5. $(15, 8, 4)$: $n = 15$, $r_1 = 9$, $e_1 = 7$, $\rho = 4/3$

Table 5.1

Theorem 5.1

All 4-designs are type-I.

Proof: We proceed as in the case of 3-designs to note we may take $k_j^! \leq 6$, eliminating H-designs from consideration. The column possibilities are then displayed:

<table>
<thead>
<tr>
<th>$k_j^!$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_j^*$</td>
<td>4+4$\rho$</td>
<td>4+3$\rho$</td>
<td>4+2$\rho$</td>
<td>4+$\rho$</td>
<td>4</td>
<td>4-$\rho$</td>
<td>4-2$\rho$</td>
</tr>
</tbody>
</table>

Table 5.2

Suppose a 4-design has a column with $k_j^! = 6$. Then $2\rho$ is integral and in fact $\rho = 3/2$ or $\rho = 2$. From (3.10) and $6 \leq e_1 \leq n-3$ we have

$$7 + \frac{7}{\rho} + \frac{1}{\rho^2} \leq n \leq -2\rho^2 + 7\rho + 10;$$

further, $(\rho n + 1)/(\rho + 1) = r_1$ must be integral. So for $\rho = 3/2$ we have $n = 16$, $e_1 = 6$, $r_1 = 10$. Since $k_j^! = 6$ now means $k_j^* = 1$, all remaining $k_j$'s are 3 or 4. But 3 is not possible since $\rho$ is not integral. Thus, Lemma 3.6 applies, and the design does not exist.
With \( p = 2 \) from (5.1) we have \( 11 \leq n \leq 16 \) with \( \frac{2n+1}{3} \) integral, i.e., \( n = 13, 16 \). If \( n = 13 \), \( r_1 = 9 \), \( e_1 = 7 \), we can have only one \( k_j^! = 6 \) with the remaining columns of the form \( k_j^! = 4 \) or \( k_j^! = 5 \). With \( f_i \) the number of columns with \( k_j^! = i \), we have \( f_4 + f_5 = 12 \) and \( 4f_4 + 5f_5 = 57 \). This means \( f_4 = 3 \) and \( f_5 = 9 \), but then from (3.11) we obtain the contradiction \( \Delta^2 = 2^8 \cdot 3^{11} \). With \( n = 16 \), \( r_1 = 11 \), and \( e_1 = 6 \). Here, \( k_j^! = 6 \) having \( k_j^* = 0 \) means that all remaining columns have \( k_j^! = k_j^* = 4 \) and we have the type-I design from line 4 of Table 5.1.

Next, suppose we have a 4-design with \( k_j^! = 5 \) occurring. Then \( p = 2, 3, \) or \( 4 \). We have here \( 5 \leq e_1 \leq n-3 \) or

\[
7 + 7/\rho + 1/\rho^2 \leq n \leq -\rho^2 + 7\rho + 9 .
\]

(1) \( \rho = 2 \). \( 11 \leq n \leq 19 \), \( n \equiv 1 \mod 3 \); hence, \( n = 13, 16, 19 \). Note that \( k_j^! = 5 \) means \( k_j^* = 2 \); hence, all remaining columns have \( k_j^! \geq 2 \), so we are working with the following column table:

<table>
<thead>
<tr>
<th>( k_j^! )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_j^* )</td>
<td>8</td>
<td>6</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>( k_j )</td>
<td>10</td>
<td>9</td>
<td>8</td>
<td>7</td>
</tr>
</tbody>
</table>

Table 5.3

\( n = 13 \), \( r_1 = 9 \), \( e_1 = 7 \): \( e_2 = 6 \) forces \( f_2 = 0 \), \( f_3 \leq 1 \). If \( f_3 = 1 \) then \( f_4 + f_5 = 12 \), \( 4f_4 + 5f_5 = 60 \). Hence, \( f_5 = 12 \), \( f_4 = 0 \), and Lemma 3.6 shows this design does not exist.

\( n = 16 \), \( r_1 = 11 \), \( e_1 = 6 \): \( e_2 = 10 \) forces \( f_2 \leq 1 \) and

\[
f_2 + f_3 + f_4 + f_5 = 16 \, ,
\]
\[
2f_2 + 3f_3 + 4f_4 + 5f_5 = 66 \, .
\]
Thus, \( f_2 = 1 \) gives \( t_3 = 0, \ f_5 = 4, \ f_4 = 11 \) and we obtain the contradiction \( \Delta^2 = 3^5 \cdot 2^{24} \), while \( f_2 = 0 \) forces the absurdity \( f_3 = 5/2, \ f_4 = 9, \ f_5 = 9/2 \).

\[ n = 19, \ r_1 = 13, \ e_1 = 5; \ e_1 = 5 \text{ gives } f_5 = 1 \text{ and } f_2 + f_3 + f_4 = 18,
\]
\[ 2f_2 + 3f_3 + 4f_4 = 60, \]
\[ \frac{1}{3} + \frac{1}{6} f_2 + \frac{1}{5} f_3 + \frac{1}{4} f_4 = \frac{17}{4}, \]

which has the unique solution \( f_2 = 1, \ f_3 = 10, \ f_4 = 7 \). Here, \( \Delta^2 = 3^4 \cdot 2^{16} \cdot 5^{10} \) does not exclude this possibility. Consider a row of \( A \) with sum \( r_1 = 13 \) and a zero in the column with \( k_j = 2 \). Let \( \tau \) be the number of ones in this row in columns with \( k_j = 3 \) and use (3.8) with \( i = \tau \) obtaining:

\[ \frac{1}{3} + \frac{\tau}{5} + \frac{12-\tau}{4} = 3 \]

or \( \tau = 20/3 \).

(2) \( p = 3 \). We have from (5.2) \( 10 \leq n \leq 21 \) and from (3.10) \( e_1 = (61-n)/8 \) so that \( n \equiv 5 \) (mod 8). Thus, \( n = 13 \) or \( n = 21 \). Since \( k_j = 5 \) and \( p = 3 \) gives \( k_j^* = 1 \), all remaining \( k_j \) values are either 3 or 4. In case \( n = 13, \ e_1 = 6, \ r_1 = 10 \) and the column structure of \( A \) is determined by the system:

\[ f_3 + f_4 + f_5 = 13, \]
\[ 3f_3 + 4f_4 + 5f_5 = 60, \]
\[ \frac{1}{6} f_3 + \frac{1}{4} f_4 + \frac{1}{2} f_5 = \frac{61}{12}, \]

which has the unique (and unacceptable) solution \( f_3 = -1, \ f_4 = f_5 = 7 \).
In the case of \( n = 21 \), \( e_1 = 5 \), \( r_1 = 16 \). Surely then \( f_5 = 1 \) and \( f_3 + f_4 = 20 \), while \( 3f_3 + 4f_4 = 75 \), i.e., \( f_3 = 5 \), \( f_4 = 15 \). We write rows one and two of \( A \):

\[
\begin{array}{cccc}
1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0
\end{array}
\]

\[
\begin{array}{cccc}
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0
\end{array}
\]

We use (3.8) with \( i = \ell = 1 \) to obtain

\[
\frac{1}{2} + \frac{\sigma}{6} + \frac{15-\sigma}{4} = 4
\]

where \( \sigma \) is the partial sum of row one occurring in columns with \( k_j^1 = 3 \). We see then that \( \sigma = 3 \); this means if \( \beta \) denotes the intersection of rows one and two in the last 15 columns that \( \beta \geq 9 \). With \( \alpha \) denoting the intersection in columns with \( k_j^1 = 3 \) we have \( \alpha \geq 1 \), and from (3.8) with \( i = 1 \), \( \ell = 2 \),

\[
\frac{1}{2} + \frac{\alpha}{6} + \frac{\beta}{4} = 3
\]

or

\[
\alpha = \frac{3}{2} (10-\beta)
\]

Since \( \alpha \geq 1 \) we have \( \beta = 9 \), but then \( \alpha = 3/2 \).

(3) \( \rho = 4 \). Here, \( e_1 = (96-n)/15 \) and \( 9 \leq n \leq 21 \) with \( n \equiv 6 \text{ mod } 15 \). This means \( n = 21 \), \( e_1 = 5 \). Surely \( f_5 = 1 \), \( f_4 = 20 \), and this is the type-I design from line 2 of Table 5.1.

We have thus shown that all 4-designs with some \( k_j^1 > 4 \) are type-I. For the remaining designs we have then the abbreviated column table:
Note that this table makes it clear that $n \geq 12$.

We now suppose we have a 4-design with $e_1 \geq 7$. Since

$$e_1 = \frac{4p^2 + 7p + 4 - n}{p^2 - 1},$$

we have $4p^2 + 7p + 4 - n \geq 7p^2 - 7$, or $3p^2 - 7p - 11 \leq -n \leq -12$; hence,

$$3p^2 - 7p + 1 \leq 0$$

or

$$p \leq \frac{7 + \sqrt{49 - 12}}{6} < 2 \frac{13}{72}.$$

Thus, if $p$ is integral, $p = 2$, while one of $2p$, $3p$, or $4p$ must be integral so that $p$ must take one of the values $2$, $3/2$, $4/3$, $5/3$, $5/4$, $7/4$. Since we have $n \leq -3p^2 + 7p + 11$ and $e_1$ must be integral, this leaves only three candidates: (1) $n = 13$, $p = 2$, $e_1 = 7$; (2) $n = 15$, $p = 4/3$, $e_1 = 7$; (3) $n = 12$, $p = 7/4$, $e_1 = 8$.

(1) Since $p = 2$, Table 5.4 makes the column structure clear: precisely one column with $k^j = 3$, $k^*_j = 6$, and 12 with $k^j = k^*_j = 4$.

Hence, Lemma 3.6 applies.

(2) Here again, the column structure is forced. One column has $k^j = 1$ and the remaining have $k^j = k^*_j = 4$. Hence, Lemma 3.6 applies.

(3) There is only one admissible column here.
We now consider 4-designs with \( e_1 = 6 \). We obtain as usual

\[
\begin{align*}
n &= -2p^2 + 7p + 10 \geq 12, \\
\end{align*}
\]

(5.3)

from which we deduce

\[
\rho < 3 \frac{1}{5}.
\]

Further, (5.3) shows that \( 2\rho \) is integral, so that we obtain the following candidates for a 4-design with \( e_1 = 6 \).

\[
\begin{array}{cccccc}
\text{Case} & \rho & n & r_1 & r_2 & e_2 \\
1 & 2 & 16 & 11 & 6 & 10 \\
2 & 3 & 13 & 10 & 4 & 7 \\
3 & 3/2 & 16 & 10 & 7 & 10 \\
4 & 5/2 & 15 & 11 & 5 & 9 \\
\end{array}
\]

Case 1. We are supposing \( k_j^* \leq 4 \); then surely \( e_1 r_1 \leq 64 \), but evidently \( e_1 r_1 = 66 \).

Case 2. From Table 5.4 we see only 2 column types are admissable: \( k_j^* = 3 \), \( k_j^* = 7 \) and \( k_j^* = k_j^* = 4 \). We must have one of the former and 12 of the latter so that Lemma 3.6 excludes this design.

Case 3. The column possibilities here are

\[
\begin{array}{ccc}
k_j' & 0 & 2 & 4 \\
k_j^* & 10 & 7 & 4 \\
k_j & 10 & 9 & 8 \\
\end{array}
\]

With \( f_i \) the number of columns with \( k_j^* = i \), we have

\[
\begin{align*}
f_0 + f_2 + f_4 &= 16 , \\
2f_2 + 4f_4 &= 60 , \\
\frac{f_0}{6} + \frac{f_2}{5} + \frac{f_4}{4} &= \frac{47}{12} ,
\end{align*}
\]
yielding the unique solution \( f_0 = 1, f_2 = 0, f_4 = 15 \). This is the type-I design from the \((16, 6, 2)\) configuration.

**Case 4.** Here again, with \( k_j^! = 4 \) we cannot have \( e_1 r_1 = 66 \).

This brings us to 4-designs with \( e_1 = 5 \). We have

\[
    n = -p^2 + 7p + 9
\]

so that \( p \) is an integer and \( p \leq 6 \). We systematically exclude the five possibilities.

**Case 1.** \( p = 2, n = 19, r_1 = 13, r_2 = 7, e_2 = 14 \). Here, \( f_1, f_0 \leq 1 \).

If \( f_0 = 1 \), the remaining \( k_j^* \)'s satisfy \( k_j^* \leq 6 \) so that \( f_1 = f_2 = 0 \), \( f_4 = 11, f_3 = 7 \). These values violate (3.7). Hence, \( f_0 = 0 \). If \( f_1 = 1 \) we have

\[
    f_2 + f_3 + f_4 = 18,
    2f_2 + 3f_3 + 4f_4 = 64,
    \frac{1}{6}f_2 + \frac{1}{5}f_3 + \frac{1}{4}f_4 = \frac{115}{28},
\]

which has no integral solution. Thus, \( f_1 = 0 \) and the conditions are

\[
    f_2 + f_3 + f_4 = 18,
    2f_2 + 3f_3 + 4f_4 = 65,
    \frac{1}{6}f_2 + \frac{1}{5}f_3 + \frac{1}{4}f_4 = \frac{17}{4},
\]

yielding the inadmissible values \( f_2 = 6, f_4 = 17, f_3 = -5 \).

**Case 2.** \( p = 3, n = 21, r_1 = 16, r_2 = 6, e_2 = 16 \). Here, note if \( f_0 = 1 \) we have the type-I design from the \((21, 5, 1)\)-configuration.

Since \( f_0 \leq 1 \) we suppose \( f_0 = 0 \). Surely also \( f_1 \leq 1 \) and if \( f_1 = 1 \) necessarily \( f_2 = 0 \) with \( f_3 = 0, 1 \) and respectively \( f_4 = 20, 19 \). The former alternative is excluded by Lemma 3.6 and the latter by (3.7)

\[
    \frac{1}{10} + \frac{1}{6} + \frac{19}{4} = \frac{301}{60} \neq \frac{61}{12}.
\]
Thus, we have $f_1 = 0$ and
\[ f_2 + f_3 + f_4 = 21, \]
\[ 2f_2 + 3f_3 + 4f_4 = 80, \]
\[ \frac{1}{8} f_2 + \frac{1}{6} f_3 + \frac{1}{4} f_4 = \frac{61}{12}, \]
but this forces $f_4 = 21$, $f_3 = -4$, $f_2 = 4$, so that only the type-I design occurs.

**Case 3.** $\rho = 4$, $n = 21$, $r_1 = 13$, $r_2 = 9$, $e_2 = 16$. Table 5.4 shows here $f_0 = 0$, $f_1 \leq 1$, $f_2 \leq 1$. If $f_1 = 1$, $f_2 = f_3 = 0$ and $f_4 = 20$, and Lemma 3.6 excludes this possibility. If $f_1 = 0$ we have
\[ f_2 + f_3 + f_4 = 21, \]
\[ 2f_2 + 3f_3 + 4f_4 = 65, \]
\[ \frac{1}{10} f_2 + \frac{1}{7} f_3 + \frac{1}{4} f_4 = 6, \]
yielding the absurdity $f_2 = 130/3$, $f_3 = -203/3$, $f_4 = 136/3$.

**Case 4.** $\rho = 5$, $n = 19$, $e_2 = 14$, $r_1 = 16$, $r_2 = 4$. Here, $e_1 r_1 = 80$ denies $k_j! \leq 4$.

**Case 5.** $\rho = 6$, $n = 15$, $r_1 = 13$, $r_2 = 3$. Again, $e_1 r_1 = 65$ forbids $k_j! \leq 4$.

We now take the case $e_1 = 4$. Lemma 3.7 and Table 5.4 make it clear that $\rho$ is integral and $\rho \leq 4$. Indeed
\[ n = 7\rho + 8, \quad r_2 = 8, \quad r_1 = 7\rho + 1, \]
so that there are three possible 4-designs with $e_1 = 4$.

(1) $\rho = 4$, $n = 36$, $r_1 = 29$. For reference we note the column possibilities are given in the following table:
Now since \( e_2 = 32 \), \( f_4 \leq 8 \), but if \( f_4 \leq 7 \) we have \( e_1 r_1 = 116 = \sum k_j = 28 + 3 \cdot 29 = 115 \). Thus, \( f_4 = 8 \) and

\[
\sum_{k_j \leq 3} k_j = 84 \leq 3 \cdot 28
\]

forces \( f_3 = 28 \) with the remaining \( f_i \)'s zero. In a row with sum \( r_2 = 8 \), let \( \sigma \) be the sum of the entries in columns with \( k_j = k_j^* = 4 \).

From (3.9) we must have

\[
\frac{\sigma}{4} + \frac{8-\sigma}{7} = 1 + \frac{1}{4}
\]

or \( \sigma = 1 \). Now if \( \alpha \) is the inner product of this row with, say, row one, we have from (3.8)

\[
\frac{1}{4} + \frac{\alpha - 1}{7} = 1
\]

But this says \( \alpha - 1 = 21/4 \), and thus the design cannot exist.

(2) \( p = 3 \), \( n = 29 \), \( r_1 = 22 \). Here, the column table is

\[
\begin{array}{cccccc}
k_j^! & 0 & 1 & 2 & 3 & 4 \\
k_j^* & 20 & 16 & 12 & 8 & 4 \\
k_j & 20 & 17 & 14 & 11 & 8
\end{array}
\]

Note that \( f_0 \leq 1 \). If \( f_0 = 1 \) then \( f_1 \leq 1 \). We have in general the constraints:

\[
\begin{align*}
f_0 + f_1 + f_2 + f_3 + f_4 &= 29 , \\
f_1 + 2f_2 + 3f_3 + 4f_4 &= 88 , \\
\frac{1}{6}f_0 + \frac{1}{5}f_1 + \frac{1}{4}f_2 + \frac{1}{3}f_3 + \frac{1}{2}f_4 &= \frac{61}{6} .
\end{align*}
\]
With \( f_0 = 1, f_1 = 1 \) this system has a unique solution with \( f_2 = -12/5 \).

If \( f_0 = 1, f_1 = 0 \), we obtain \( f_2 = 0, f_3 = 24, f_4 = 4 \). We can write this design so that rows 4 and 5 appear as:

\[
\begin{array}{cccccc}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 \\
\hline
18 & \sigma & 6
\end{array}
\]

where column one has \( k_j^* = 0 \), columns 2 through 5 have \( k_j^* = 4 \).

Now using (3.8) with these two rows, we must have

\[
\frac{1}{4} + \frac{\sigma}{6} = 1 ,
\]

which is not possible with \( \sigma \) integral. Thus, \( f_0 = 0 \) and the system (5.4) becomes a rank 3 system with the one parameter solution:

\[
\begin{align*}
f_2 &= 8f_4 - 26 , \\
f_1 &= -7/2 f_4 + 25/2 , \\
f_3 &= -11/2 f_4 + 85/2 .
\end{align*}
\]

Since \( e_2 = 25, f_4 \leq 6 \), but the above equations show a contradiction for \( f_2 \geq 0 \) means \( f_4 \geq 4 \), while \( f_1 \geq 0 \) forces \( f_4 \leq 3 \).

(3) \( p = 2, n = 22, r_1 = 15 \). With these parameters we have the following rank 3 system on the variables \( f_1 \):

\[
\begin{align*}
\sum_{i=0}^{4} f_i &= 22 , \\
\sum_{i=0}^{4} if_i &= 60 , \\
\frac{1}{8} f_0 + \frac{1}{7} f_1 + \frac{1}{6} f_2 + \frac{1}{5} f_3 + \frac{1}{4} f_4 &= \frac{17}{4} .
\end{align*}
\]

This system has the 2-parameter solution
\[ f_0 = 34 - \frac{8}{5} f_3 - 6 f_4, \]
\[ f_1 = -84 + \frac{21}{5} f_3 + 14 f_4, \]
\[ f_2 = 72 - \frac{18}{5} f_3 - 9 f_4. \]

We require integer values, non-negative with \( f_0 \leq 1 \) and \( f_4 \leq 4 \). This yields precisely one possibility:

\[ (f_0, f_1, f_2, f_3, f_4) = (0, 0, 9, 10, 3). \]

However, this is not an acceptable column structure, as it yields from (3.11) \( \Delta^2 = 2^{16} \cdot 3^{11} \cdot 5^{10} \). This completes consideration of the case \( e_1 = 4 \).

We take the final case \( e_1 = 3 \). We have from (3.10) and (3.5)

\[ n = \rho^2 + 7\rho + 7, \]
\[ r_1 = \rho^2 + 6\rho + 1, \]
\[ r_2 = \rho + 7. \]

From Table 5.4 we see our usual constraints are

\[ \frac{f_0}{4\rho} + \frac{f_1}{3\rho+1} + \frac{f_2}{2\rho+2} + \frac{f_3}{\rho+3} = \frac{4\rho^2+7\rho+4}{4\rho} \]

\[ f_0 + f_1 + f_2 + f_3 = \rho^2 + 7\rho + 7 \quad (5.5) \]

\[ f_1 + 2f_2 + 3f_3 = 3\rho^2 + 18\rho + 3, \]

noting that \( k^j_i \leq 3 \).

Now

\[
\det \left( \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 2 & 3 \\ \frac{1}{3\rho+1} & \frac{1}{2\rho+1} & \frac{1}{\rho+3} \end{array} \right) = \frac{2(\rho+1)}{(3\rho+1)} \left( \frac{1}{\rho+3} - \frac{1}{2\rho+2} \right) \neq 0.
\]
for $p \neq 1$. So the system (5.5) has the 1-parameter solution:

$$f_0 = \frac{4p^3 + 7p^2 - (11 + 4f_3)p + 12}{p+3}, \quad (5.6)$$

$$f_1 = \frac{-3(3p + 1)(p^2 + 2p - 3 - f_3)}{p+3}, \quad (5.7)$$

$$f_2 = \frac{6(p + 1)(p^2 + 3p - f_3)}{p+3}. \quad (5.8)$$

From (5.7), $(11+4f_3)p \geq 4p^3 + 8p^2 - p$. On the other hand, from (5.6),
$(11+4f_3)p \leq 4p^3 + 7p^2 + 12$. Hence, we have

$$4p^3 + 8p^2 - p \leq 4p^3 + 7p^2 + 12$$
or

$$p^2 - p - 12 \leq 0,$$

whence $p \leq 4$.

(1) $p = 4$. The relations (5.6), (5.7), and (5.8) become

$$f_2 = \frac{30}{7} (28-f_3), \quad f_1 = \frac{39}{7} (f_3-21), \quad f_0 = \frac{1}{7} (336-16f_3).$$

$f_1 \geq 0$ implies $f_3 \geq 21$, but $f_0 \geq 0$ forces $f_3 \leq 21$. Thus, $f_3 = 21$,
$f_2 = 30$, $f_0 = f_1 = 0$. But then (3.11) gives $\Delta^2 = 2^{30} \cdot 5^{32} \cdot 7^{21}$.

(2) $p = 2$. We have $f_2 = \frac{18}{5} (10-f_3)$, $f_1 = \frac{21}{5} (f_3-5)$, $f_0 = \frac{1}{5} (50-8f_3)$
so that $f_3 = 5$, $f_1 = 0$, $f_2 = 18$, $f_0 = 2$. Here again, (3.11) excludes the design for $\Delta^2 = 3^{20} \cdot 5^{5} \cdot 2^{25}$.

(3) $p = 3$. $f_2 = 4(18-f_3)$, $f_1 = 5(f_3-12)$, $f_0 = 25-2f_3$. Surely then
$f_3 = 12$, $f_2 = 24$, $f_1 = 0$, $f_0 = 1$. (3.11) will not exclude this possibility, so we write rows one and twenty as

```
0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0
0 1 1 0 0 1 1 0 0 1 1 0 0 1 0 0 1 0 0
```

```
Noting $n = 37$, $r_1 = 28$, $r_2 = 10$, and $k_j^! = 0$ gives $k_j^* = 16 < e_2$. In the diagram, columns 2 through 13 have $k_j^! = 3$. Now (3.9) gives

$$\frac{\sigma}{6} + \frac{10-\sigma}{8} = \frac{4}{3} \quad \text{or} \quad \sigma = 2.$$ 

Then (3.8) with these two rows gives

$$\frac{\sigma}{6} + \frac{\tau}{8} = 1 \quad \text{or} \quad \tau = \frac{16}{3},$$

which is not possible. This completes the discussion of $e_1 = 3$ and hence also the tabulation of 4-designs.
VI. $\lambda$-MATRICES

1. Introduction

In this section we shall be interested in the following combinatorial situation: $S_1, \ldots, S_{n-1}$ are to be subsets of $\{1, 2, \ldots, n\}$ with the feature that each $S_i$ is a $k$-set and for $i \neq j$, $S_i \cap S_j$ is a $\lambda$-set. Our results will show that with one exceptional class, there are precisely two replication numbers for such a configuration and we can find them explicitly in terms of $n$, $k$, and $\lambda$. We can describe quite completely the structure of such configurations, and in certain cases describe all such designs. Further, we can completely list the exceptional designs, modulo the problem of the determination of all Hadamard matrices. The device used here is similar to that used for the problem of $\lambda$-designs, and we deal exclusively with the incidence matrix of such configurations. This matrix, $A$, is $(0, 1)$, of size $n \times (n-1)$ with the feature

$$A^t A = (k-\lambda)I + \lambda J,$$

where $I$ and $J$ are the usual matrices of order $(n-1)$.

For ease in stating our results, we make the following formal definition.

**Definition.** Let $n, k, \lambda$ be integers with $n > k > \lambda > 0$. $\Lambda(n, k, \lambda)$ will denote the class of all $n \times (n-1)$ $(0, 1)$-matrices $A$ such that

$$A^t A = (k-\lambda)I + \lambda J$$

where $I$ is the identity matrix of order $n-1$ and $J$ is the matrix of ones of order $n-1$. We call the elements of $\Lambda(n, k, \lambda)$ $\lambda$-matrices.
2. The Structure of $\lambda$-Matrices

Examples:

1. Adjoining a row of zeros or a row of ones to a $(v, k, \lambda)$-configuration produces elements of $\Lambda(v+1, k, \lambda)$ or $\Lambda(v+1, k+1, \lambda+1)$, respectively.

2. Removing a column from a $(v, k, \lambda)$-configuration produces an element of $\Lambda(v, k, \lambda)$.

3. Removing the exceptional column $(k, j \neq 2\lambda)$ from a type-I $\lambda$-design of order $n$ gives an element of $\Lambda(n, 2\lambda, \lambda)$.

4. Let $B$ be the incidence matrix of a $(4\lambda-1, 2\lambda, \lambda)$-symmetric block design and $0 \leq e \leq 4\lambda-1$. Choose $e$ columns of $B$ and replace them by their complementary vectors. Now adjoin a row vector with ones in precisely those chosen columns and zeros elsewhere. The resultant matrix lies in $\Lambda(4\lambda, 2\lambda, \lambda)$. We prove this last assertion. The complemented columns had sum $4\lambda-1-2\lambda = 2\lambda-1$ and adjoining a one brings this sum to $2\lambda$. These complemented columns have inner product $\lambda-1$ among themselves and the additional row vector augments this to $\lambda$. The unaltered columns meet in $\lambda$ positions, and adjoining zeros has not changed this. Finally, a complemented column of a $(v, k, \lambda)$-configuration meets a non-complemented column in $k-\lambda$ positions; here, this is $\lambda$ and the bordering row vector does not affect this count. We will call matrices constructed in this way $\mathbb{H}_\lambda$-matrices.

Before proceeding to the discussion of the properties of $\lambda$-matrices, we list a few remarks:
(1) \( \lambda \)-matrices never have constant row sums, for if \( r \) were the row sum we would have \( rn = k(n-1) \), and hence \( n \) divides \( k \), denying \( k < n \).

(2) The construction of \( H_\lambda \)-matrices given in example 4 allows the choice of 0 columns or all the columns to be complemented. This amounts to adjoining a zero row to the \((4\lambda-1, 2\lambda, \lambda)\)-configuration and a row of ones to the complementary \((4\lambda-1, 2\lambda-1, \lambda-1)\)-design, respectively.

(3) \( H_\lambda \) matrices may have more than 2 row sums, as the following example constructed from the \((7, 4, 2)\)-design shows:

\[
\begin{array}{cccccc}
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0
\end{array}
\]

(4) \( H_\lambda \) matrices may be viewed as constructed directly from Hadamard matrices as follows: normalize the Hadamard matrix \( H \) of order \( 4\lambda \) so that its initial column contains positive ones. Delete this column, obtaining \( H_1 \) of order \( 4\lambda \times 4\lambda-1 \). Now let \( A = \frac{1}{2}(H_1+J) \) and note that \( A \) is \((0, 1)\), and since \( JH_1 = 0 \), \( A^tA = \lambda I + \lambda J \).

(5) The class \( \Lambda(4\lambda, 2\lambda, \lambda) \). We have shown that this class contains the so-called \( H_\lambda \) matrices. In fact, these are all its members. For take \( A \in \Lambda(4\lambda, 2\lambda, \lambda) \) and write the first row with ones initially placed, say, in columns \( 1, 2, \ldots, r \). Complement the first \( r \) columns
and remove row one. The resulting matrix $B$ is square, $(4\lambda - 1) \times (4\lambda - 1)$ has $j^{th}$ column sum $4\lambda - 2\lambda = 2\lambda$ for $j = 1, \ldots, r$, and of course $2\lambda$ for $j > r$. Viewing two columns of $A$ we observe that $11$, $10$, $01$, and $00$ each occur precisely $\lambda$ times, so that $B$ has column inner products $\lambda$ and indeed is the incidence matrix of a $(4\lambda - 1, 2\lambda, \lambda)$ block design evidently yielding $A$ as an $H_\lambda$ matrix.

(6) If $A \in \Lambda(n, k, \lambda)$, then the complement of $A$ lies in $\Lambda(n, n-k, n-2k+\lambda)$.

(7) The class $\Lambda(n, k, \lambda)$ with $k(k-1) = \lambda(n-1)$ consists precisely of the examples (2) above. This will come out in our discussion of $\lambda$-matrices, but of course follows e.g. from the rational completion theorem of Hall and Ryser [4].

**Theorem 6.2.1**

Let $A \in \Lambda(n, k, \lambda)$. Then either (1) $\lambda n = k^2$ and $A$ is an $H_\lambda$-matrix, or (2) $\lambda n \neq k^2$ and $A$ has two row sums given by the roots of the quadratic equation:

$$x^2 - \left[n - \frac{(n-k)(k-2\lambda)}{\lambda n-k^2}\right] x + (k-\lambda)(n-1)\left[1 + \frac{k-2\lambda}{\lambda n-k^2}\right] = 0.$$ 

**Proof:** Taking the case (2) first, we form the matrix

$$B = \begin{bmatrix}
\frac{\lambda}{k} & a_{11} & a_{12} & \cdots & a_{1,n-1} \\
\frac{\lambda}{k} & a_{21} & a_{22} & \cdots & a_{2,n-1} \\
\vdots \\
\frac{\lambda}{k} & a_{n,1} & a_{n,2} & \cdots & a_{n,n-1}
\end{bmatrix}$$

and argue that it is non-singular as follows: since $(k-\lambda)I + \lambda J$ with $k \neq \lambda$ is non-singular, $A$ has rank $n-1$, so if $B$ is singular, the
vector \((\lambda/k, \ldots, \lambda/k)^t\) lies in the column space of \(A\), i.e., there is some real vector \(X\) such that \(AX = (\lambda/k, \ldots, \lambda/k)^t\). Now this means \(A^tAX = A^t(\lambda/k, \ldots, \lambda/k)^t = (\lambda, \lambda, \ldots, \lambda)^t\). If \(X = (x_1, \ldots, x_{n-1})\), this gives \((k-\lambda)x_i + \sum_{j=1}^{n-1} x_j = \lambda\). Thus, since \(k \neq \lambda\), all the \(x_i\) are equal, which would imply that \(A\) has constant row sums, contrary to the remark (1) above. Now let \(Y = (y_1, y_2, \ldots, y_n)^t\) be the unique solution to \(B^tY = (\lambda, \ldots, \lambda)^t\) and set 
\[
u = \lambda(\lambda n-k^2)/k^2, \quad w = -\lambda + \sum_{i=1}^{n} y_i^2.\]

Now form the matrix of order \((n+1) \times (n+1)\):

\[
C = \begin{bmatrix}
y_1 & \lambda/k & a_1, 1 & \ldots & a_{1, n-1} 
\vdots & & \ddots & & \vdots 
y_n & \lambda/k & a_{n, 1} & \ldots & a_{n, n-1} 
\sqrt{-\lambda} & \sqrt{-\lambda} & \sqrt{-\lambda} & \ldots & \sqrt{-\lambda}
\end{bmatrix}
\]

Note that \(C^tC = \text{diag}[w, u, k-\lambda, \ldots, k-\lambda]\). Since \(\lambda n \neq k^2\), \(u \neq 0\) and \(C\) is singular if \(w = 0\). The last \(n\)-columns of \(C\) are independent since \(B\) was non-singular so that if \(C\) were singular, the vector \(\tilde{Y}_1 = (y_1, \ldots, y_n, \sqrt{-\lambda})^t\) would lie in the column space of \(B_1 = \begin{bmatrix} B \\
\sqrt{-\lambda} & \ldots & \sqrt{-\lambda}
\end{bmatrix}\) i.e., we would have a vector \(\tilde{\phi}\) such that \(\tilde{Y}_1 = B_1\tilde{\phi}\). Hence, 
\[
B_1^t\tilde{Y}_1 = B_1^tB_1\tilde{\phi} = \text{diag}[u, k-\lambda, \ldots, k-\lambda]\tilde{\phi}.
\]
But indeed, \(B_1^t\tilde{Y}_1 = \vec{0}\), so then \(\tilde{\phi} = \vec{0}\), \(\tilde{Y}_1 = \vec{0}\), forcing \(\lambda = 0\). Hence, \(C\) is non-singular and \(w \neq 0\), and we can form
\[ K = C \text{ diag } [1/\sqrt{w}, 1/\sqrt{u}, 1/\sqrt{k-\lambda}, \ldots, 1/\sqrt{k-\lambda}] \]

and note

\[ K^t K = \text{ diag } [1/\sqrt{w}, 1/\sqrt{u}, \ldots, 1/\sqrt{k-\lambda}]^2 \text{ diag } [w, u, k-\lambda, \ldots, k-\lambda] = I. \]

Hence, \( KK^t = I \) and we obtain the relations:

\[
\begin{align*}
\frac{1}{w} + \frac{1}{u} + \frac{n-1}{k-\lambda} &= -\frac{1}{\lambda} \quad (6.1) \\
y_i \left( \frac{\lambda}{ku} + \frac{r_i}{k-\lambda} \right) &= 0 \quad (6.2) \\
y_i^2 \left( \frac{\lambda}{k^2 u} + \frac{r_i}{k-\lambda} \right) &= 1 \quad (6.3) \\
\frac{y_i y_j}{w} \left( \frac{\lambda}{k^2 u} + \frac{\alpha_{ij}}{k-\lambda} \right) &= \delta_{ij} \quad (6.4)
\end{align*}
\]

where \( AA^t = (\alpha_{ij}) \) and \( \alpha_{ii} = r_i \). From (6.2) and (6.3) we obtain

\[
y_i^2 - y_i + \frac{w}{u} \left( \frac{\lambda}{k^2} - \frac{\lambda}{k-\lambda} \right) = 0. \quad (6.5)
\]

Since from (6.2) we see not all the \( y_i \) are equal, (6.5) shows there are precisely two values \( y_1 \) and \( y_2 \) with \( y_1 + y_2 = 1 \). (Note here if \( k(k-1) = \lambda(n-1) \), (6.5) forces \( y_i = 0, 1 \).) So from (6.2) there are two values for \( r_i \) with:

\[
\frac{1}{w} + \frac{2\lambda}{ku} + \frac{r_1 + r_2}{k-\lambda} = 0
\]

\[
r_1 + r_2 = (k-\lambda) \left( -\frac{1}{w} - \frac{2\lambda}{ku} \right) \quad (6.6)
\]

\[
= (k-\lambda) \left( \frac{1}{\lambda} + \frac{1}{u} + \frac{n-1}{k-\lambda} - \frac{2\lambda}{ku} \right)
\]

\[
= \frac{(k-\lambda)}{\lambda} \left( 1 + \frac{k^2}{\lambda(n-k^2)} - \frac{2\lambda k}{\lambda n-k^2} \right) + n - 1
\]

\[
= \frac{(k-\lambda)(n-2k)}{\lambda n-k^2} + n - 1
\]
\[ r_1 + r_2 = n - \frac{(n-k)(k-2\lambda)}{\lambda n-k^2} \quad (6.7) \]

We have thus only to check the value of \( r_1 r_2 \). The calculation is a bit messy, and we introduce

\[ \delta = \lambda n - k^2 = k^2 u / \lambda \quad , \quad N = k - \lambda \quad . \quad (6.8) \]

Now (6.2) then gives

\[ y_i^2 = w^2 \left( \frac{r_i^2}{N^2} + \frac{2r_1 \lambda}{k u N} + \frac{\lambda^2}{k^2 u^2} \right) \]

so that (6.3) becomes

\[ w \left( \frac{r_i^2}{N^2} + \frac{r_i}{\delta N} + \frac{\lambda}{\delta} \right) + \frac{r_i}{N} = 1 \quad . \quad (6.9) \]

Viewing (6.9) as a quadratic in \( r_i / N \) we see that

\[ \frac{r_1 r_2}{N^2} = \frac{k^2}{\delta^2} + \frac{1}{w} \left( \frac{\lambda}{\delta} - 1 \right) \quad . \quad (6.10) \]

From (6.1) and the definition of \( u \) we observe that \(-\frac{1}{w} = k^2 / \lambda \delta + \frac{n-1}{N} + \frac{1}{\lambda}\); hence, (6.10) becomes

\[ \frac{r_1 r_2}{N^2} = \frac{k^2}{\delta^2} + \frac{k^2}{\lambda \delta} + \frac{n-1}{N} + \frac{1}{\lambda} - \frac{k^2}{\delta^2} - \frac{\lambda(n-1)}{N \delta} - \frac{1}{\delta} \]

\[ = \frac{(n-1)}{N} \left( 1 - \frac{\lambda}{\delta} \right) + \frac{1}{\lambda \delta} \left( k^2 + \delta - \lambda \right) = \frac{(n-1)}{N} \left( 1 - \frac{\lambda}{\delta} \right) + \frac{(n-1)}{\delta} \quad . \]

Thus,

\[ r_1 r_2 = (n-1)N \left\{ 1 - \frac{\lambda}{\delta} + \frac{N}{\delta} \right\} = (n-1)(k-\lambda) \left\{ 1 + \frac{k-2\lambda}{\lambda n-k^2} \right\} \]

precisely as desired.

We now treat the case (1), i.e., we assume \( \lambda n = k^2 \) and show \( A \) is an \( H \) matrix. In view of remark (5) above, this means we must show \( n = 4\lambda \), \( k = 2\lambda \). From \( \sum_{i=1}^{n} a_{ij} a_{ik} = \lambda \) (\( j \neq k \)) we note that
where \( r_i \) is the \( i \)th row sum of \( A \). Hence if

\[
x_i = \frac{r_i - 1}{n - 2} \quad i = 1, 2, \ldots, n \tag{6.11}
\]

we have

\[
\sum_{i=1}^{n} a_{ij} x_i = \lambda \quad j = 1, 2, \ldots, n-1 . \tag{6.12}
\]

Now set

\[
u = -\lambda + \sum_{i=1}^{n} x_i^2 \tag{6.13}
\]

and suppose \( u \geq 0 \). Then (6.13) and (6.11) give

\[
\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} \left( \frac{r_i - 1}{n - 2} \right)^2 \geq \lambda
\]
or

\[
\sum_{i=1}^{n} r_i (r_i - 1) - \sum_{i=1}^{n} r_i + n \geq \lambda(n-2)^2 . \tag{6.14}
\]

But summing (6.12) over \( j \) we see that

\[
\sum_{i=1}^{n} r_i (r_i - 1) = \lambda(n-1)(n-2) .
\]

So (6.14) is \( \lambda(n-1)(n-2) - k(n-1) + n \geq \lambda(n-2)^2 \), which we may write

\[
(k-\lambda)(n-2) \leq (n-k) ,
\]

which, since \( k-\lambda \geq 1 \), forces \( k = 2, \lambda = 1 \). Now \( \lambda n = k^2 \) means \( n = 4 \), so we have the \( H_\lambda \) matrices of order 4. Thus, we take \( n > 4 \) and have \( u < 0 \). Now the matrix

\[
B = \begin{bmatrix} x_1 & a_1, 1 & \cdots & a_1, n-1 \\ \vdots \\ x_n & a_n, 1 & \cdots & a_n, n-1 \end{bmatrix}
\]
is non-singular as follows. As before, if $B$ is singular with 
$X_t = (x_1, \ldots, x_n)^t$, there is some $Y$ such that 
$X = AY$. Then 
$A^tX = A^tAY$ which if $Y = (y_1, \ldots, y_{n-1})^t$ says 
\[(k-\lambda)y_j + \lambda \sum_{i=1}^{n} y_i = \lambda \quad j = 1, \ldots, r-1.
\]
Since $k \neq \lambda$ this means $y_j = y_j \quad j = 1, \ldots, n-1$. But then 
\[\frac{(r-1)}{(n-2)} = \frac{r}{y_i} \quad i = 1, \ldots, n,\] 
which would imply $A$ has constant row sums. We may then choose 
$Z = (z_1, \ldots, z_n)^t$ such that 
$B^tZ = (\lambda, \lambda, \ldots, \lambda)^t$.

With $W = -\lambda + \sum_{i=1}^{n} Z_i^2$, we show $W \neq 0$. Suppose not. Then 
\[||Z||^2 = \lambda \quad \text{and} \quad ||X-Z||^2 = ||X||^2 + ||Z||^2 - 2X \cdot Z = u + \lambda + \lambda - 2\lambda = u.
\]
But $u < 0$.

We thus form, as in the previous theorem, the matrix $K$ of order $(n+1) \times (n+1)$:

\[
K = \begin{bmatrix}
\frac{y_1}{\sqrt{w}} & \frac{x_1}{\sqrt{u}} & \frac{a_1, 1}{\sqrt{k-\lambda}} & \ldots & \frac{a_1, n-1}{\sqrt{k-\lambda}} \\
\frac{y_n}{\sqrt{w}} & \frac{x_n}{\sqrt{u}} & \frac{a_n, 1}{\sqrt{k-\lambda}} & \ldots & \frac{a_n, n-1}{\sqrt{k-\lambda}} \\
\sqrt{-\frac{\lambda}{w}} & \sqrt{-\frac{\lambda}{u}} & \sqrt{-\frac{\lambda}{k-\lambda}} & \ldots & \sqrt{-\frac{\lambda}{k-\lambda}}
\end{bmatrix}
\]

As we have arranged $K^tK = I$ so we have as a bonus $KK^t = I$ or specifically:

\[
\frac{n-1}{k-\lambda} + \frac{1}{w} + \frac{1}{\lambda} = 0
\] 
(6.15)
Now from (6.17) we have
\[ y_i = -w \left( \frac{x_i}{u} + \frac{r_i}{k-\lambda} \right), \]  
(6.18)

and from (6.11)
\[ \frac{r_i}{k-\lambda} = \frac{n-2}{k-\lambda} x_i + \frac{1}{k-\lambda}, \]
so that (6.18) gives
\[ y_i = -w(\beta x_i + \frac{1}{k-\lambda}), \]  
(6.19)

where
\[ \beta = \frac{1}{u} + \frac{n-2}{k-\lambda}. \]

Now (6.16) and (6.17) give
\[ \frac{x_i^2}{u} - \frac{x_i}{u} + \frac{y_i^2}{w} - \frac{y_i}{w} = 1, \]
so with (6.19),
\[ \frac{x_i^2}{u} - \frac{x_i}{u} + \beta x_i + \frac{1}{k-\lambda} + w \left( \frac{x_i^2}{u} \beta^2 + \frac{2\beta x_i}{k-\lambda} + \frac{1}{(k-\lambda)^2} \right) = 1, \]  
(6.20)
\[ x_i \left( \frac{1}{u} + w\beta^2 \right) + x_i \left( \frac{2\beta w}{k-\lambda} \right) + \left( \frac{w}{(k-\lambda)^2} + \frac{1}{k-\lambda} - 1 \right) = 0. \]

Now observe that
\[ u(n-2)^2 = -\lambda(n-2)^2 + \sum_{i=1}^{n} (r_i-1)^2 = -\lambda(n-2)^2 + \sum_{i=1}^{n} r_i(r_i-1) - \sum_{i=1}^{n} r_i + n \]
\[ = \lambda(n-2) - k(n-1) + n, \]
and for ease of computation set \( \tau = u(n-2)^2. \) We first compute \( \beta \)
more explicitly:

\[
\beta = \frac{1}{\lambda} + \frac{n-2}{k-\lambda} = \frac{(n-2)^2}{\tau} + \frac{(n-2)}{k-\lambda}
\]

\[
= \frac{(n-2)}{(k-\lambda)\tau} \{k(n-2)-\lambda(n-2)+\lambda(n-2)-k(n-1)+n\}
\]

\[
\beta = \frac{(n-2)(n-k)}{(k-\lambda)\tau}
\]  

(6.21)

The observation here is that \(\beta \neq 0\) so that (6.19) shows that the number of distinct \(y_i\) is the same as the number of distinct \(x_i\) and hence \(r_i\) as well. We now compute the coefficient of \(x_i^2\) in the quadratic (6.20). This number is \(\beta(\frac{1}{\lambda \beta} + \omega \beta)\).

\[
u \beta = \frac{\tau}{(n-2)^2} \frac{(n-2)(n-k)}{(k-\lambda)\tau} = \frac{(n-k)}{(n-2)(k-\lambda)}
\]  

(6.22)

From (6.15)

\[
\frac{1}{w} = \frac{1}{\lambda} + \frac{1}{\mu} - \frac{n-1}{k-\lambda} = \frac{1}{\lambda} - \frac{(n-2)^2}{\tau} - \frac{(n-1)}{k-\lambda}
\]

\[
= \frac{\tau(k-\lambda) + (n-2)^2 \lambda(k-\lambda) + \lambda(n-1) \lambda \tau}{-\lambda \tau(k-\lambda)}
\]

Computing the numerator we obtain, since \(\lambda n = k^2\),

\[
\tau(k-\lambda) + (n-2)^2 \lambda(k-\lambda) + \lambda(n-1) \tau = k(n-k).
\]

Thus,

\[
w = \frac{-\lambda \tau(k-\lambda)}{k(n-k)}
\]  

(6.23)

Now

\[
\beta(\frac{1}{\lambda \beta} + \omega \beta) = \beta \left\{ \frac{(n-2)(k-\lambda)}{n-k} - \frac{\lambda \tau(k-\lambda)}{k(n-k)} \frac{(n-2)(n-k)}{(k-\lambda)\tau} \right\}
\]

\[
= \frac{\beta(n-2)}{k(n-k)} \left\{ k^2 - \lambda \cdot k - \lambda(n-k) \right\} = \frac{\beta(n-2)}{k(n-k)} \{ k^2 - \lambda n \} = 0.
\]

This means that (6.20) is not a quadratic, but then if the coefficient of \(x_i\) is not zero, we would arrive at the contradiction of constant
row sums. We therefore may assert that a $\lambda$-matrix with $\lambda n = k^2$
also has

$$-2\beta w = n-2.$$  

We then use (6.21) and (6.23), obtaining

$$+2\left(\frac{(n-2)(n-k)}{(k-\lambda)\tau}\right)\left(\frac{\lambda\tau(k-\lambda)}{k(n-k)}\right) = n-2,$$

$$\frac{2\lambda}{k} = 1,$$

$$k = \frac{2\lambda}{k}.$$  

Since $n = k^2/\lambda$ we have $n = 4\lambda$ and $A$ is an $H_\lambda$-matrix. This completes the proof of Theorem 6.2.1.

Having completely settled the classes $\Lambda(n, k, \lambda)$ with $\lambda n = k^2$ we discuss the case $\lambda n \neq k^2$ and note that our proof of Theorem 6.2.1 gives us some information on the structure of these $\lambda$-matrices.

Let us write the $\lambda$-matrix $A$ so that its first $e$ rows have sum $r_1$ and the remaining $(n-e)$ have sum $r_2$. Note that the $e \times (n-1)$ submatrix $A_1$ with row sums $r_1$ has constant column sums $k'$ where

$$k'(r_1-1) + (k-k')(r_2-1) = \lambda(n-2)$$

or

$$k' = \frac{\lambda(n-2) - k(r_2-1)}{r_1 - r_2}. \quad (6.24)$$

In view of (6.4) with this normalization

$$AA^t = \begin{bmatrix}
  r_1 & \cdots & \lambda_1 \\
  \lambda_1 & \cdots & r_1 \\
  \vdots & & \vdots \\
  \lambda' & & \lambda'
\end{bmatrix}$$

$$\lambda' = \begin{bmatrix}
  r_2 & \cdots & \lambda_2 \\
  \lambda_2 & \cdots & r_2
\end{bmatrix}. \quad (6.25)$$
where
\[
\frac{\lambda_1}{k-\lambda} = \frac{-y_1^2}{w} - \frac{\lambda}{\lambda n-k^2} = -1 + \frac{r_1}{k-\lambda} + \frac{\lambda}{\lambda n-k^2} = \frac{r_1-k+\lambda}{k-\lambda}.
\]
So
\[
\lambda_1 = r_1 - k + \lambda,
\]
and similarly,
\[
\lambda_2 = r_2 - k + \lambda.
\]
We may compute also \( \lambda' \) from (6.4) and (6.5):
\[
\frac{\lambda'}{k-\lambda} = -\frac{y_1 y_2}{w} - \frac{\lambda}{\lambda n-k^2} = -\frac{1}{uk^2} (\lambda^2 - \lambda k - uk^2) - \frac{\lambda}{\lambda n-k^2}
\]
\[
= \frac{-\lambda + k + \lambda n - k^2 - \lambda}{\lambda n-k^2}
\]
\[
= 1 + \frac{k-2\lambda}{\lambda n-k^2}
\]
\[
\lambda' = (k-\lambda) \left\{ 1 + \frac{k-2\lambda}{\lambda n-k^2} \right\} = \frac{r_1 r_2}{(n-1)}.
\]
We finally note that our choice of the \( y_i \) gives
\[
k' y_1 + (k-k') y_2 = \lambda.
\]
With these remarks on the general structure of \( \lambda \)-matrices, we discuss some special classes \( \Lambda(n,k,\lambda) \).

3. \( \Lambda(n,2\lambda,\lambda) \)

**Theorem 6.3.1**

\( A \in \Lambda(n,2\lambda,\lambda) \) if and only if

(a) \( n = 4\lambda \) and \( A \) is an \( H_\lambda \)-matrix, or

(b) \( n = 4\lambda-1 \) and \( A \) is a partial \((4\lambda-1,2\lambda,\lambda)\)-configuration, or

(c) \( A \) is completeable to a type-I \( \lambda \)-design.

**Proof:** If \( n = 4\lambda \), we have already remarked that \( A \) is an \( H_\lambda \)-
matrix. If \( n \neq 4 \lambda \), then \( \lambda n \neq k^2 \) and we may use (6.28):

\[
k'y_1 + (2\lambda - k')y_2 = \lambda ;
\]

since \( y_1 + y_2 = 1 \) this becomes

\[
2y_1(k' - \lambda) = k' - \lambda.
\]

Now if \( k' \neq \lambda \) we would have \( y_1 = y_2 = \frac{1}{2} \), which would force \( r_1 = r_2 \). Hence, \( k' = \lambda \), and adjoining a column with ones in positions one through \( e \) and zeros elsewhere completes \( A \) to a \((v, k, \lambda)\) with \( k = 2\lambda \), i.e., \((4\lambda - 1, 2\lambda, \lambda)\) or by Lemma 3.6 to a type-I \( \lambda \)-design.

4. \( \Lambda(n, k, 1) \)

Theorem 6.4.1

A \( \in \Lambda(n, k, 1) \) if and only if

(a) \( A \) is a \( 4 \times 4 \) \( H_1 \) matrix, or

(b) \( A \) is an \( (n-1) \times (n-1) \) permutation matrix bordered with a row of ones (a partial I-design), or

(c) \( A \) is a partial projective plane, or

(d) \( A \) is a projective plane with a row of zeros added.

Proof: Since \( \lambda = 1 \) forces row inner products to be zero or one, we have, from (6.25) and (6.26), either \( e = 1 \) or \( e > 1 \) and say \( \lambda_1 = 1 \), \( \lambda_2 = 0 \). If \( e = 1 \), then \( k' = 0, 1 \). If \( k' = 0 \), we clearly have the possibility (d) of the theorem. If \( k' = 1 \), we surely have the case (b).

With \( e > 1 \), note that if \( r_i = 0 \) we have case (d) and \( r_i \neq 0 \) with (6.27) and the above remarks \( \lambda' = 1 \). If \( \lambda_1 = 1 \), \( \lambda_2 = 0 \), the matrix obtained by adjoining a column with zeros in positions one through \( e \) yields a projective plane.
5. $\Lambda(2k, k, \lambda)$ and $\Lambda(4\lambda, k, \lambda)$

Theorem 6.5.1

A $\lambda$-matrix with $n = 2k$ or $n = 4\lambda$ is an $H_\lambda$-matrix.

Proof: If $n = 2k$ and $\lambda n \neq k^2$, we apply Theorem 6.2.1. In particular, we compute the product $r_1 r_2$:

$$r_1 r_2 = (k-\lambda)(2k-1)\left\{1 + \frac{k-2\lambda}{k(2\lambda-k)}\right\},$$

$$r_1 r_2 = \frac{(k-\lambda)(2k-1)(k-1)}{k},$$

but this would mean $k$ divides $\lambda$, denying $k > \lambda$. Thus, we conclude $\lambda n = k^2$ and $A$ is an $H_\lambda$-matrix.

If $n = 4\lambda$ and $\lambda n \neq k^2$, we compute

$$r_1 + r_2 = 4\lambda - \frac{(4\lambda-k)(k-2\lambda)}{4\lambda^2 - k^2},$$

$$r_1 + r_2 = 4\lambda + \frac{4\lambda - k}{2\lambda + k}.$$

Now since $k < n$ we must have

$$4\lambda - k \geq 2\lambda + k,$$

but this says $k \leq \lambda$. Hence, we conclude in this case also that $\lambda n = k^2$ and $A$ is an $H_\lambda$-matrix.

The preceding theorem then shows that the exceptional class of $H_\lambda$-matrices is characterized by any one of the conditions $\lambda n = k^2$, $n = 2k$, or $n = 4\lambda$. 

REFERENCES


