# A KINETIC THEORY DESCRIPTION FOR EXTERNAL SPHERICAL FLOWS WITH ARBITRARY KNUDSEN NUMBER BY A MOMENT METHOD

Thesis by

Gary Duane Brinker

In Partial Fulfillment of the Requirements
For the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California
1969
(Submitted March 1969)

## ACKNOWLEDGMENTS

I would like to express my sincerest appreciation to Professor Lester Lees for his development of many of the basic concepts contained in this thesis and, more importantly, for his continued guidance, encouragement and patience throughout its course.

I would also like to thank Dr. Robert Seliger for many valuable discussions of the relevant mathematical aspects and Dr. P. N. Shankar for many mutual confrontations with the more reluctant formulations of the moment method. The excellent typing of the manuscript by Mrs. Virginia Conner is also greatly appreciated.

I wish to acknowledge the financial support of the U. S. Army Research Office and the Advanced Research Projects Agency, Contract No. DA-31-124-ARO(D)-33, and the Sloan Foundation. Personal support was also provided by the National Aeronautics and Space Administration.

A significant expression of gratitude is also due to my parents for understanding, support and encouragement shown throughout my education.

# ABSTRACT

The Maxwell integral equations of transfer are applied to a series of problems involving flows of arbitrary density gases about spheres. As suggested by Lees a two sided Maxwellian-like weighting function containing a number of free parameters is utilized and a sufficient number of partial differential moment equations is used to determine these parameters. Maxwell's inverse fifth-power force law is used to simplify the evaluation of the collision integrals appearing in the moment equations. All flow quantities are then determined by integration of the weighting function which results from the solution of the differential moment system. Three problems are treated: the heat-flux from a slightly heated sphere at rest in an infinite gas; the velocity field and drag of a slowly moving sphere in an unbounded space; the velocity field and drag torque on a slowly rotating sphere. Solutions to the third problem are found to both first and second-order in surface Mach number with the secondary centrifugal fan motion being of particular interest. Singular aspects of the moment method are encountered in the last two problems and an asymptotic study of these difficulties leads to a formal criterion for a "well posed" moment system. The previously unanswered question of just how many moments must be used in a specific problem is now clarified to a great extent.

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# LIST OF SYMBOLS

| | |
|---|---|
| A, B | constants in general Stokes solution |
| $\underline{\underline{A}}$ | transformation from parametric to canonical variables |
| $\underline{c}$ | relative particle velocity, $\underline{\xi} - \underline{u}$ |
| $\overline{c}$ | mean molecular speed, $(8kT/\pi m)^{\frac{1}{2}}$ |
| $C_D$ | drag coefficient |
| $C_n$ | integration constants |
| D | drag |
| f | velocity distribution function; moment weighting function |
| $f_1, f_2$ | components of two-stream Maxwellian or weighting function |
| $f_\infty$ | ambient Maxwellian |
| k | Boltzmann constant |
| $k_c$ | heat conductivity |
| Kn | Knudsen number, $Kn = \lambda/2R_0$ |
| m | particle mass |
| M | Mach number |
| n | particle number density |
| $n_1, n_2$ | number density functions in weighting function |
| $N_1, N_2$ | linearized number density parameters, $n_i = n_\infty(1+N_i)$ |
| p | pressure, $p = nkT$ |
| $p_{ii}$ | defined by $P_{ii} = -p+p_{ii}$ |
| $p_{ij} \equiv P_{ij}$ | shear stress, $p_{ij} = -m\int f c_i c_j \, d\underline{\xi}$, $i \neq j$ |
| $P_{ii}$ | normal stress, $P_{ii} = -\int f c_i^2 \, d\underline{\xi}$ |
| $P_{ij}^*$ | defined by $P_{ij}^* = -m\int f \xi_i^2 \, d\underline{\xi}$ |
| $P_{ijk}^*$ | defined by $P_{ijk}^* = m\int f \xi_i \xi_j \xi_k \, d\underline{\xi}$ |

## LIST OF SYMBOLS (Cont'd)

| | |
|---|---|
| $P^*_{ijk\ell}$ | defined by $P_{ijk\ell} = \frac{1}{2} m \int f \xi_i \xi_j \xi_k \xi_\ell d\underline{\xi}$ |
| $\dot{q}_i$ | heat flux, $\dot{q}_i = m \int f c_i (c^2/2) d\underline{\xi}$ |
| $\dot{q}_i^*$ | defined by $\dot{q}_i^* = m \int f \xi_i (\xi^2/2) d\underline{\xi}$ |
| Q | arbitrary function of velocity; total heat-flux |
| $\Delta Q$ | changes in Q produced by collisions |
| $\underline{R}$ | radius vector |
| R | physical coordinate |
| Re | Reynolds number |
| $R_0$ | sphere radius |
| s | free-molecule integration variable, $s = \sin(\nu + \sigma)$ |
| S | speed ratio, $S = U\beta_0$ |
| T | absolute temperature, $\frac{3}{2}nkT = m \int f (c^2/2) d\underline{\xi}$ |
| $T_1, T_2$ | temperature parameters in weighting function |
| $t_1, t_2$ | linearized parametric functions, $T_i = T_\infty(1+t_i)$ |
| $\underline{u}$ | mean velocity vector, $\rho\underline{u} = m \int f \underline{\xi}\, d\underline{\xi}$ |
| u, v, w | spherical components of $\underline{u}$ |
| $u_1, u_2$ | radial velocity parametric functions in weighting function |
| $\underline{U}$ | free stream velocity vector |
| $v_1, v_2$ | $\theta$ velocity parametric functions |
| $w_1, w_2$ | $\varphi$ velocity parametric functions |
| $\tilde{w}$ | parametric variable defined by $\tilde{w} = \tilde{u}_+ + \tilde{v}_+$ |
| x | defined by $x = \sqrt{1 - R_0^2/R^2}$ |
| y | defined by $y = R_0/R$ |
| $\underline{y}, \underline{Y}$ | vector of parametric functions |

LIST OF SYMBOLS (Cont'd)

| | |
|---|---|
| $Z$ | numerical integration variable, $Z = 1 - \sqrt{1-x}$ |
| $\alpha$ | defined by $\alpha = \cos^{-1}(Ro/R)$ |
| $\beta$ | defined by $\beta = (m/2kT)^{\frac{1}{2}}$; rarefaction parameter |
| $\beta_0$ | defined by $\beta_0 = (m/2kT_{\infty})^{\frac{1}{2}}$ |
| $\gamma$ | free-molecule integration angle, $\gamma = \nu + \sigma$; rarefaction parameter |
| $\Gamma_1, \Gamma_2$ | defined in Eq. (5.28) |
| $\delta_{ij}$ | Kronecker delta |
| $\delta$ | rarefaction parameter |
| $\epsilon$ | perturbation parameter |
| $\underline{\underline{\eta}}$ | general solution matrix |
| $\underline{\eta}$ | general solution vector (column of $\underline{\underline{\eta}}$) |
| $\theta$ | physical polar coordinate |
| $\lambda$ | Maxwell mean free path |
| $\lambda_{\infty}$ | ambient mean free path |
| $\mu$ | viscosity |
| $\nu$ | free-molecule integration angle (see Fig. 16) |
| $\underline{\xi}$ | vector particle velocity |
| $\xi_i$ | spherical component of $\underline{\xi}$ |
| $\rho$ | mass density, $\rho = m \int f \, d\underline{\xi}$ |
| $\sigma$ | moment integration angle, $\sigma = \cos^{-1}(\xi_R/\xi)$, see Fig. 2 |
| $\tau$ | moment integration angle, $\tau = \tan^{-1}(\xi_{\varphi}/\xi_{\theta})$, see Fig. 2 |
| $\varphi$ | physical azimuthal coordinate |
| $\underline{\omega}$ | angular velocity |

LIST OF SYMBOLS (Cont'd)

Superscripts

| | |
|---|---|
| — | nondimensionalized quantities, Eq. (3. 3) |
| ~ | indicates specific θ-dependence |
| ⌒ | indicates specific θ-dependence |
| ^ | indicates specific θ-dependence |
| (1) | first-order perturbation quantities |
| (2) | second-order perturbation quantities |

Subscripts

| | |
|---|---|
| $(\_)$ | indicates vector quantity |
| $( _{=} )$ | indicates tensor quantity |
| $(\ )_1$ | quantities in outgoing distribution |
| $(\ )_2$ | quantities in incoming distribution |
| $(\ )_+$ | equals $(\ )_1 + (\ )_2$ |
| $(\ )_-$ | equals $(\ )_1 - (\ )_2$ |
| c | continuum conditions |
| f. m. | free-molecule conditions |
| s | surface conditions |
| ∞ | ambient conditions |

# 1. INTRODUCTION

Although there is general understanding of the transition between highly rarefied gas flows and gas dynamics as described by the Navier-Stokes equations, a detailed theory is undeveloped because it involves the difficult problem of solving the Maxwell-Boltzmann transport equation subject to initial values and boundary conditions. Some insight into the nature of this problem is provided by the work of Willis, (1), (2), (3) who uses the Krook (4) model for the collision integral in the integro-differential equation for the single-particle velocity distribution function. Since the Krook model implies isotropic scattering, it is suspect when there are large mean velocity and mean temperature differences or surface curvature, especially in the low density high Knudsen number regime.

One would like to preserve the main features of the collision process, while retaining the ability to deal with nonlinear problems. With this goal in mind Lees (5) developed a moment method which is formally applicable for all gas densities. This moment method has previously been applied only to one dimensional nonsteady flows (6) and to internal flow problems such as steady plane compressible Couette flow (7) and conductive heat transfer between concentric cylinders. (8) In the present investigation we want to study the effects of surface curvature and other properties of external flows using this moment method. A series of problems in spherical geometry will be used to illustrate this approach.

The total external flow about a closed body cannot be simply characterized as continuum, transition, or free-molecule as is often done for gaseous flows contained within a finite region. Instead, these descriptions must be applied to specific regions in the flow. Within a distance much less than a mean free path from the body the velocity distribution function always exhibits the discontinuous character associated with collisionless flows. On the other hand, no matter how large the mean free path, the distribution function becomes continuous and approaches a local Maxwellian at large distances from the body. Thus, for large R the flow field always has a solution of the Navier-Stokes equations as a limiting form. At intermediate distances where the Knudsen number (defined as $\lambda/d$ where $\lambda$ is the mean free path in the gas and d is the sphere diameter) is of order one the distribution function naturally deviates from both of the above limiting solutions. The present attempt is to provide a description of this intermediate region which is not merely a partial extension of an asymptotic solution into the transition regime.

Lees suggested that the distribution function in the Maxwell integral equation of transfer be represented by a weighting function expressed in terms of a number of parametric functions of space and time, selected in such a way that essential physical features of the problem are introduced. The proper number of moments must be taken to insure that a complete set of first order partial differential equations is obtained for these undetermined functions. This weighting function is not to be considered as a solution of the governing Maxwell-Boltzmann equation.

A particular example of the type of weighting function proposed by Lees is the "two-sided Maxwellian," which is a natural generalization of the situation for free-molecule flow. In body coordinates, all outwardly directed particle velocity vectors lying within the "cone of body influence" (region 1 in Fig. 1) are described by the function $f = f_1$, where

$$f_1 = \frac{n_1(\underline{R}, t)}{[2\pi k T_1(\underline{R}, t)/m]^{3/2}} \exp\left\{-\frac{m[\underline{\xi} - \underline{u}_1(\underline{R}, t)]^2}{2kT_1(\underline{R}, t)}\right\} . \qquad (1.1a)$$

In region 2 (all other $\underline{\xi}$),

$$f = f_2 = \frac{n_2(\underline{R}, t)}{[2\pi k T_2(\underline{R}, t)/m]^{3/2}} \exp\left\{-\frac{m[\underline{\xi} - \underline{u}_2(\underline{R}, t)]^2}{2kT_2(\underline{R}, t)}\right\} , \qquad (1.1b)$$

where $n_1, \ldots, u_2$ are ten initially undetermined functions of $\underline{R}$ and t. One important advantage of the "two-sided Maxwellian" is that the surface boundary conditions are easily incorporated into the analysis. For example, in the case of completely diffuse reflection the re-emitted particles have a Maxwellian velocity distribution corresponding to the surface temperature $T_s$, and the mean velocity of the re-emitted particles is identical to the local surface velocity. Thus $\underline{u}_1(\underline{R}, t) = \underline{u}_s$, and $T_1(\underline{R}, t) = T_s$ when $R = R_0$. When there is no net mass transfer at the surface an additional boundary condition must be satisfied which is similar (except that $u_2 \neq u_\infty$) to the usual free-molecule flow condition. (5)

Once the form of the weighting function is selected, the collision integral appearing in the Maxwell moment equations can be evaluated for an arbitrary force law between the particles. However, Maxwell's well-known inverse fifth-power law provides an important simplification of the collision integrals because they can be evaluated once and for all in terms of simple combinations of lower order stresses and heat fluxes. (5) When the solutions for the parametric functions are inserted into Eqs. (1a) and (1b), the fluid field variables may be found by integration, thus completing the solution.

In particular, three external spherical flow problems will be considered. The first is a sphere at rest in an infinite gas of arbitrary density, but with a temperature different from that of the surrounding gas. The heat flux to the sphere is calculated as a function of this temperature difference and the density of the gas (Knudsen number). This example is used primarily as an introduction to the more difficult problems which follow.

The second situation is a sphere in uniform linear motion relative to a gas at rest at infinity. Since the weighting functions for this case are linearized for low Mach number, the sphere may be assumed adiabatic without any loss in generality. The nonadiabatic translating sphere solution is just a superposition of these first two cases. Whereas the heat conduction problem yields an analytic solution, the moving sphere necessitates a numerical integration of the resulting moment equations.

Finally, a stationary rotating sphere is investigated. As in the second case, the sphere is taken to be adiabatic and the weighting functions are expanded in powers of the equatorial surface Mach number. The first-order solution which determines the retarding torque on the sphere is not difficult to obtain analytically. More interesting, however, is the secondary motion which exhibits an influx at the two poles and an equatorial jet. The dependence of this "centrifugal fan" motion on the Knudsen number is determined by numerical integration of the second-order moment equations.

One of the primary results obtained in this investigation is a fuller realization of the importance of choosing the correct weighting function and moment equations. Specifically, the last two problems above illustrate the difficulties encountered when an unfortunate choice of systems is employed. A systematic approach is outlined for obtaining a reasonable weighting function and its associated moment equations for a more general class of problems. A criterion is also provided by which any moment model may be easily checked for the existence of various types of singular behavior.

# 2. MOMENT METHOD IN SPHERICAL GEOMETRY

## 2.1. General Remarks

A systematic formulation of the moment method in spherical geometry is the most efficient way of tying together the various problems discussed in the following chapters. Besides avoiding duplicated effort in describing these problems a general development offers a relatively easy extension of this method to other spherical problems.

Applying the moment method to a given physical situation involves, essentially, three distinct problems. First, a suitable weighting function must be chosen to exhibit the desired physical detail, remembering that the more parametric functions it contains, the greater is the difficulty likely encountered in solving the resulting moment equations. Some general suggestions for this choice will be discussed in Section 2.5.; however, a most useful rule is that gained from experience with various applications of the moment method.

Secondly, the many integral functions which appear in the system of moment equations must be determined in terms of these parametric functions. A table of integrals is provided in an Appendix which greatly reduces the effort required to obtain these functions for the special case of linearized weighting functions in spherical geometry.

The final and perhaps most difficult part is solving the system of partial differential equations subject to the two part boundary conditions which characterize the method. Under some circumstances nonphysical singular eigenfunctions appear which must be eliminated by finiteness arguments. This is a significant disadvantage when numerical integrations of the moment equations are sought.

## 2.2. Weighting Function and Mean Quantities

Consider a sphere of radius $R_0$ located at the center of a spherical coordinate system (Fig. 2a). The region about the sphere ($R > R_0$) is filled with monatomic gas at an arbitrary density level, which is characterized by the mean free path $\lambda_\infty$ evaluated at infinity. It is necessary to introduce a spherical velocity coordinate system at the field point (R, θ, φ) outside the sphere (Fig. 2b). As an intermediate step consider the rectangular components, $\xi_R$, $\xi_\theta$ and $\xi_\varphi$ of the velocity vector $\underline{\xi}$, which are directed along the three unit vectors of the original space $\underline{i}_R$, $\underline{i}_\theta$, and $\underline{i}_\varphi$. Then define the three spherical velocity components as

$$\xi = |\underline{\xi}|, \tag{2.1}$$

$$\sigma = \cos^{-1} (\xi_R/\xi), \tag{2.2}$$

and

$$\tau = \tan^{-1} (\xi_\varphi/\xi_\theta). \tag{2.3}$$

From these expressions it follows that

$$\xi_R = \xi \cos\sigma, \tag{2.4}$$

$$\xi_\theta = \xi \sin\sigma \cos\tau, \tag{2.5}$$

$$\xi_\varphi = \xi \sin\sigma \sin\tau. \tag{2.6}$$

As discussed in the Introduction, the simplest weighting function having a "two-sided" character and capable of giving a smooth transition between the highly rarefied and continuum regime consists of two Maxwellians, each containing several parametric functions. All outwardly directed molecules with velocity vector $\underline{\xi}$ lying inside the cone of influence (region 1 in Fig. 1) are characterized by one Maxwellian $f_1$, where

$$f = f_1 \text{ for } 0 < \sigma < \pi/2 - \alpha \quad , \tag{2.7}$$

in which

$$\alpha = \cos^{-1} (R_0/R) \ . \tag{2.8}$$

Then, all molecules with velocity $\underline{\xi}$ lying outside of region 1 are characterized by $f_2$, i.e.,

$$f = f_2 \text{ for } \pi/2 - \alpha < \sigma < \pi \ . \tag{2.9}$$

The requirement that f should be discontinuous on the edge of the "cone of body influence" is a basic feature of the present scheme.

The number of parametric functions which appear in the weighting function must be determined for each example to be studied. Although each parametric function has the dimensions of density, temperature, etc., it must be emphasized that no physical significance may be ascribed to it. For example, setting the temperature parameters equal to constants in the weighting function does not imply an isothermal flow. The choice is thus an arbitrary one (subject to conditions of Section 2.5.) reflecting the amount of detail desired and the insight into the physical problem.

The "two-sided Maxwellian" described in the introduction, Eqs. (1a) and (1b), is sufficiently general to include all of the weighting functions used for the examples in this paper.

Knowing the weighting function f, one can evaluate all mean quantities Q, which appear in the moment equations, by integrating over all velocity space,

$$\left\langle nQ \right\rangle = \int Qf d\underline{\xi} = \int_0^{\frac{\pi}{2}-\alpha} \int_0^{2\pi} \int_0^{\infty} Qf_1 \, \xi^2 \sin\sigma \, d\xi d\tau d\sigma + \int_{\frac{\pi}{2}-\alpha}^{\pi} \int_0^{2\pi} \int_0^{\infty} Qf_2 \xi^2 \sin\sigma d\xi d\tau d\sigma \ , \tag{2.10}$$

where Q represents various functions of the velocity $\underline{\xi}$ .

Since the above integrals are difficult to perform when mean motion exists in the gas, a perturbation expansion is utilized in the examples to be described. The perturbation parameter or parameters need not be determined at this time, and they, in fact, vary with the particular problem being considered. To effect this the exponential part of the weighting function is expanded in what is, essentially, a local Mach number expansion. That is, the parametric functions analagous to the mean velocity components are assumed small with respect to the local "sound speed" based on the parametric temperature.

Although the integrals can now be completed, before doing so the four density and temperature functions in Eqs. (1a) and (1b) are replaced by new parametric functions defined as follows:

$$
\begin{aligned}
N_1 &= (n_1 - n_\infty)/n_\infty, \\
N_2 &= (n_2 - n_\infty)/n_\infty, \\
t_1 &= (T_1 - T_\infty)/T_\infty, \\
t_2 &= (T_2 - T_\infty)/T_\infty .
\end{aligned}
\qquad (2.11)
$$

Of these, the temperature perturbations $t_1$ and $t_2$ are assumed to be much less than $T_\infty$. The weighting function may now be written as follows:

$$f_1 = f_\infty [1+N_1 - 3/2\, t_1+\beta_0^2 \xi^2 t_1+2\beta_0^2 \xi_R u_1+2\beta_0^2 \xi_\theta v_1+2\beta_0^2 \xi_\varphi w_1 ]$$

and (2.12)

$$f_2 = f_\infty [1+N_2-3/2\, t_2+\beta_0^2\xi^2 t_2+2\beta_0^2 \xi_R u_2+2\beta_0^2 \xi_\theta v_2+2\beta_0^2 \xi_\varphi w_2] ,$$

where

$$f_\infty = \frac{n_\infty \beta_0^3}{\pi^{\frac{3}{2}}} e^{-\beta_0^2 \xi^2} \qquad (2.13)$$

and

$$\beta_0^2 = m/2kT_\infty . \qquad (2.14)$$

To motivate this complete linearization of the weighting function notice that each moment integral Eq. (2.10) is now just a sum of a series of terms of the form,

$$\int f_\infty F(\underline{\xi}) d\underline{\xi}, \qquad (2.15)$$

since the parametric functions $N_1, \ldots, w_2$ are all functions of $\underline{R}$ only and independent of the velocity $\underline{\xi}$. An integral table containing a large selection of functions $F(\underline{\xi})$ is given in Appendix A, Table A. 4. Of course, this expansion can be extended to second order and higher order terms in a consistent manner.

The following moments and their defining functions Q will be encountered in the examples which follow:

$$
\begin{aligned}
&n && Q = 1, \\
&nu_i && Q = \xi_i \ , \\
&p && Q = \tfrac{1}{3}\, mc^2, \\
&P_{ij} && Q = -\, mc_i c_j, \\
&p_{ij} = P_{ij} + \delta_{ij} p, && \\
&P^*_{ij} && Q = -m\xi_i \xi_j \ , \\
&\dot{q}_i && Q = \tfrac{1}{2}\, mc_i c^2, \\
&\dot{q}^*_i && Q = \tfrac{1}{2}\, m\xi_i \xi^2, \\
&P^*_{ijk} && Q = m\xi_i \xi_j \xi_k, \\
&P^*_{ijkl} && Q = \tfrac{1}{2}\, m\xi_i \xi_j \xi_k \xi_l \ ,
\end{aligned}
\qquad (2.16)
$$

where $\underline{c}$ is the relative velocity $\underline{\xi} - \underline{u}$, $\delta_{ij}$ the Kronecker delta, and each subscript one of the spherical components R, $\theta$, $\varphi$. Notice that the starred quantities are each associated with a standard kinetic moment definition in which c is replaced by $\xi$.

## 2.3. Differential Equations

Maxwell's moment equation in orthogonal curvilinear coordinates including external forces is given by Lees (5), (6) in two reports, the second being more readily available. Neglecting external forces and making use of the axial symmetry, the steady state integral equation of transfer in spherical coordinates takes the following form:

$$
\begin{aligned}
&\frac{1}{R^2}\frac{\partial}{\partial R} R^2 \int f\xi_R Q d\underline{\xi} + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\sin\theta \int f\xi_\theta Q d\underline{\xi} - \int f\Big[\{\frac{1}{R}\xi_\theta^2 + \frac{1}{R}\xi_\varphi^2\}\frac{\partial Q}{\partial \xi_R} \\
&\quad + \{\frac{1}{R}\cot\theta\,\xi_\varphi^2 - \frac{1}{R}\xi_\theta\xi_R\}\frac{\partial Q}{\partial \xi_\theta} - \{\frac{1}{R}\xi_\varphi\xi_R + \frac{1}{R}\cot\theta\,\xi_\varphi\xi_\theta\}\frac{\partial Q}{\partial \xi_\varphi}\Big] d\underline{\xi} = \Delta Q,
\end{aligned}
\qquad (2.17)
$$

where $\Delta Q$ represents the rate of change of Q produced by binary collisions within the gas. Taking Q to be the collisional invariants m, $m\underline{\xi}$, $\frac{1}{2}m\xi^2$ successively, for which $\Delta Q = 0$, one obtains the continuity, momentum and energy equations of ordinary continuum theory. For the remaining Q's the collision integral $\Delta Q$ appearing in the integral transport equation can be evaluated using an arbitrary force law between the molecules. However, as in most all previous applications of the moment method, Maxwell's inverse fifth-power law of repulsion is used because of its simplicity, which results because the relative velocity is absent from the collision integral. The integrals for this model are given in the original report by Lees (5) for two of the most commonly used moments: $\Delta Q = (p/\mu)p_{ij}$ for $Q = m\xi_i\xi_j$ and $\Delta Q = (p/\mu)(-\frac{2}{3}\dot{q}_i + \sum_j p_{ij}\, u_j)$ for $Q = \frac{1}{2}\, m\xi_i\xi^2$. The viscosity in these may be replaced by the mean free path $\lambda$ through the classical relation for Maxwell molecules:

$$\lambda = \frac{\mu}{\rho\left(\frac{2kT}{\pi m}\right)^{\frac{1}{2}}} . \tag{2.18}$$

Therefore, for $Q = m\xi_i\xi_j$

$$\Delta Q = \sqrt{\frac{\pi kT}{2m}}\ \frac{p_{ij}}{\lambda} , \tag{2.19}$$

and for $Q = \frac{1}{2}\, m\xi_i\xi^2$

$$\Delta Q = \sqrt{\frac{\pi kT}{2m}}\ \frac{1}{\lambda}\left(-\frac{2}{3}\dot{q}_i + \sum_j p_{ij}\, u_j\right). \tag{2.20}$$

These results are independent of the form chosen for the weighting function and thus are evaluated once and for all without the need for further approximation.

The ten moment equations which will be used later are now written in terms of the moments defined in Section 2.2. and the collision integrals just given. Taken directly from Eq. (2.17), these equations are as follows:

Continuity ($Q = m$): (2.21)

$$\left(2 + R\frac{\partial}{\partial R}\right) nu_R + \left(\cot\theta + \frac{\partial}{\partial\theta}\right) nu_\theta = 0;$$

Radial momentum ($Q = m\xi_R$): (2.22)

$$\left(2 + R\frac{\partial}{\partial R}\right) P^*_{RR} + \left(\cot\theta + \frac{\partial}{\partial\theta}\right) P^*_{R\theta} - P^*_{\theta\theta} - P^*_{\varphi\varphi} = 0;$$

$\theta$ Tangential momentum ($Q = m\xi_\theta$): (2.23)

$$\left(3 + R\frac{\partial}{\partial R}\right) P^*_{R\theta} + \left(\cot\theta + \frac{\partial}{\partial\theta}\right) P^*_{\theta\theta} - \cot\theta\, P^*_{\varphi\varphi} = 0;$$

$\varphi$ Tangential momentum ($Q = m\xi_\varphi$): (2.24)

$$\left(3 + R\frac{\partial}{\partial R}\right) P^*_{R\varphi} + \left(\cot\theta + \frac{\partial}{\partial\theta}\right) P^*_{\theta\varphi} + \cot\theta\, P^*_{\theta\varphi} = 0;$$

Energy ($Q = \frac{1}{2} m\xi^2$): (2.25)

$$\left(2 + R\frac{\partial}{\partial R}\right) \dot{q}^*_R + \left(\cot\theta + \frac{\partial}{\partial\theta}\right) \dot{q}^*_\theta = 0;$$

Radial stress moment ($Q = m\xi_R^2$): (2.26)

$$\left(2 + R\frac{\partial}{\partial R}\right) P^*_{RRR} + \left(\cot\theta + \frac{\partial}{\partial\theta}\right) P^*_{RR\theta} - 2P^*_{R\theta\theta} - 2P^*_{R\varphi\varphi} = \sqrt{\frac{\pi k T}{2m}}\,\frac{R}{\lambda}\, p_{RR};$$

Tangential stress moment ($Q = m\xi_\theta^2$): (2.27)

$$\left(4 + R\frac{\partial}{\partial R}\right) P^*_{R\theta\theta} + \left(\cot\theta + \frac{\partial}{\partial\theta}\right) P^*_{\theta\theta\theta} - 2\cot\theta\, P^*_{\theta\varphi\varphi} = \sqrt{\frac{\pi k T}{2m}}\,\frac{R}{\lambda}\, p_{\theta\theta};$$

Shear stress moment ($Q = m\xi_R\xi_\theta$): (2.28)

$$\left(3 + R\frac{\partial}{\partial R}\right) P^*_{RR\theta} + \left(\cot\theta + \frac{\partial}{\partial\theta}\right) P^*_{R\theta\theta} - P^*_{\theta\theta\theta} - P^*_{\theta\varphi\varphi} - \cot\theta\, P^*_{R\varphi\varphi}$$

$$= \sqrt{\frac{\pi k T}{2m}}\,\frac{R}{\lambda}\, p_{R\theta};$$

Shear stress moment ($Q = m\xi_R \xi_\varphi$): (2.29)

$$(3+R\frac{\partial}{\partial R})P^*_{RR\varphi} + (\cot\theta + \frac{\partial}{\partial\theta})P^*_{R\theta\varphi} - P^*_{\varphi\varphi\varphi} + \cot\theta\, P^*_{R\theta\varphi} - P^*_{\theta\theta\varphi} = \sqrt{\frac{\pi kT}{2m}}\,\frac{R}{\lambda}\, p_{R\varphi}\,;$$

Radial heat-flux ($Q = \frac{1}{2} m\xi_R \xi^2$): (2.30)

$$(2+R\frac{\partial}{\partial R})P^*_{RRii} + (\cot\theta + \frac{\partial}{\partial\theta})P^*_{R\theta ii} - P^*_{\theta\theta ii} - P^*_{\varphi\varphi ii} = \sqrt{\frac{\pi kT}{2m}}\,\frac{R}{\lambda}\,(-\tfrac{2}{3}\dot{q}_R + p_{Ri}u_i).$$

In the last equation the repeated index i indicates a summation over the three component directions R, $\theta$, $\varphi$.

## 2.4. Boundary Conditions

The boundary conditions for the partial differential moment equations must be expressed as conditions on the parametric functions chosen for the weighting function. To do this, the analogy between the parametric function and the physical quantity in a Maxwellian distribution function must be stressed even more than in the derivation of the moment equations.

At great distances from the sphere the weighting function becomes just $f_2$ because of the diminishing effect of the body through its "cone of influence." Therefore, the boundary condition at infinity is simply that $f_2$ approach a Maxwellian distribution function with temperature, density and mean velocity given by the physical boundary conditions at infinity. Clearly, this may be accomplished by having the parametric functions approach these physical values.

The situation near the sphere is much more complicated because it involves the physical details of the sphere-molecule interaction. Since a collision free (Knudsen layer) region exists adjacent to the body for any Knudsen number, the specification of the surface boundary conditions is very similar to that for the free-molecule problem. In general, the outgoing distribution function is dependent on the local incoming distribution function and the surface physics, characterized by some number of parameters such as energy accommodation coefficient, tangential momentum accomodation coefficient, etc. For simplicity in the present analyses it is assumed that the molecules suffer diffuse re-emission and complete energy accomodation. In terms of the moment weighting function this imposes conditions on the outgoing temperature parameter, $T_1 = T_s$ for $R = R_0$, and the outgoing velocity parameters, $u_1 = v_1 = w_1 = 0$ for $R = R_0$. One additional condition on the mass flux or heat flux must be given at the sphere to specify the density parameter in the outgoing part of the weighting function. In most cases a straightforward application of the above boundary conditions will yield a unique solution to the moment equations.

## 2.5. Criterion for a Well Posed Moment System

Although the moment method has been successfully applied to a variety of transition flows in the past, there have been occasional unexplained failures of seemingly reasonable moment formulations. Such difficulties were encountered in two of the problems in the present study, and their detailed investigation has led to an easily applied criterion for a well posed moment system. A number of considerations affecting the final choice of a weighting function and differential system are discussed in this Section, and a general procedure for obtaining a well posed moment formulation is presented.

Of major importance in prescribing a moment formulation is the specification of its order, the number of parametric variables in the weighting function. One accepted guideline originally proposed by Lees (5) is that this order must exceed the number of collisional invariants associated with the collision dynamics. This is clearly equivalent to requiring the collision integral to appear in the differential moment system. A further basic consideration is the relation between the number of moments and the number of independent physical quantities which exist naturally for each problem. In general, when these numbers are equal no difficulty is encountered in applying the moment method. Such is the case for the heat-flux problem and most previous moment studies.

In more complex problems it is often a matter of expediency to seek solutions using fewer moments than natural independent variables. Obviously, this procedure imposes a number of implicit relations among the physical flow variables. Depending upon the particular form of these relationships, various modes of singular behavior may appear. Both the translating and rotating sphere problems are examples of this type. A detailed description of the origin of these singular situations will now be described.

A linearized system of axisymmetric partial differential moment equations can always be reduced to one of ordinary differential equations by an expansion of the $\theta$-dependence in an orthogonal basis such as Legendre polynomials. For each term in the $\theta$-expansion the resulting system may be characterized by the nth-order system

$$\frac{d}{dR}\left[\underline{\underline{A}}(R)\cdot\underline{y}(R)\right] = \underline{\underline{B}}(R)\cdot\underline{y}(R) + \underline{c}(R) , \qquad (2.31)$$

where $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are known, $\underline{y}$ is the vector of parametric functions to be determined, and $\underline{c}$ may include cross coupling of terms with other θ-dependence. A necessary condition for a differential equation of this form to have a general nxn solution matrix is that $\underline{\underline{A}}$ be a reversible (nonsingular) transformation matrix. If this condition is not satisfied the solution is not sufficiently general to fit n arbitrary boundary conditions on the parametric functions.

Before proceeding it is useful to specify more precisely the solution to equation (2.31) when $\underline{\underline{A}}$ is singular and of rank (n - 1). No generality is lost if the elements of the nth row of $\underline{\underline{A}}$ are assumed to be zero since this can always be accomplished by appropriate linear operations on the vector equation (2.31) without altering its form. A very basic consequence of the singular nature of $\underline{\underline{A}}$ is now apparent, that is the reduction of the nth component differential equation (the nth row of Eq. (2.31) ) to an algebraic relationship among the parametric functions provided the elements of the nth row of $\underline{\underline{B}}$ are not all zero. In the rather special situation where all elements of the nth row of $\underline{\underline{B}}$ are also identically zero the system is either indeterminate (does not contain n independent equations) or inconsistent depending on whether $c_n$ is or is not zero. However, in most cases the algebraic equation is present and may be used to eliminate one of the parametric variables ($y_m$ where $B_{nm} \neq 0$) from the first n - 1 rows of the system, thus yielding a reduced (n - 1)-order differential system with a new transformation matrix $\underline{\underline{A}}'$. If this matrix is nonsingular the new system has an (n-1)-dimension solution space or, equivalently, a general solution containing n - 1 integration constants. The solution to the reduced system together with the algebraic expression for the mth parametric variable thus constitute the complete solution to the original moment system. It is also possible that the reduced transformation matrix may again be singular and of rank n - 2 in which case the above reduction is repeated as necessary. A similar discussion with obvious modification applies to a system whose original transformation matrix $\underline{\underline{A}}$ is of lower rank than n - 1. The important consideration here is that the general solution contains fewer than n integration constants and cannot satisfy n arbitrary boundary conditions on the parametric functions.

If a canonical variable associated with each moment equation is defined as the object of the radial differential operator in that equation, e.g. $nu_R$ in the continuity equation, then $\underline{\underline{A}}(R)$ represents the transformation from parametric to canonical variables. In this context the above nonsingular condition on $\underline{\underline{A}}$ is equivalent to the requirement that parametric variables be uniquely determined by the canonical variables associated with the selected moment equations. The converse is obviously true because of the integral definitions of the canonical variables.

A similar specification is applicable to the general nonlinear moment problem, where the condition that the determinant of $\underline{\underline{A}}$ be nonzero is replaced by a similar condition on the Jacobian of the vector transformation from parametric to canonical variables. The situation is complicated by the fact that the Jacobian is a function of the parametric variables $\underline{y}$, and "surfaces" may exist within the phase space of parametric variables on which the Jacobian vanishes. In this event the moment solution must be determined independently for different regions within this phase space. Because of the relatively large dimensions of this space the characterization of the singular points and the determination of their locci prove to be difficult problems.

Again considering the linearized moment system, it can be shown that the linear transformation matrix $\underline{\underline{A}}$ described above is the same for each perturbation order and is equal to the nonlinear Jacobian matrix evaluated at $\underline{y}_0$, the vector about which the parametric variables are perturbed. Therefore a singular transformation matrix $\underline{\underline{A}}$ indicates that the zeroth-order solution, usually a pure Maxwellian, represents a singular point in the parametric phase space for the particular moment model being investigated.

The significance of the transformation matrix from canonical to parametric variables has been emphasized in the preceding discussion of linear moment formulations. The fact that the determinant of this transformation is a function of the spatial variable R provides a final important statement about the behavior of a proposed moment system. Three significant situations may be encountered with respect to such singular transformations. First, when the matrix is nonsingular throughout space the moment formulation is well posed and conducive to relatively easy integration. Second, when the transformation is singular everywhere a general solution capable of satisfying n arbitrary boundary conditions is impossible. Such poorly posed formulations are briefly discussed in connection with both the translating and rotating sphere problems. Finally, the determinant may vanish at isolated points in R (usually the endpoints representing the sphere surface and infinity). In this case the moment system (2.31) may be transformed into a standard differential form for the parametric variables,

$$\frac{d}{dR}\, \underline{y}(R) = \underline{\underline{C}}(R) \cdot \underline{y}(R) + \underline{d}(R) , \tag{2.32}$$

where $\underline{\underline{C}}(R)$ contains at most isolated singularities. Singular solutions may arise from this formulation, and asymptotic means may be required for an analytic understanding of the resulting solution matrix. A detailed example of this third case is presented in Chapter 4 where the six-moment system for the translating sphere is solved.

It is now possible to describe a reasonably simple method for writing and testing a proposed moment formulation. Step number one is the determination of the independent natural fluid dynamic quantities for the flow in question. A Newtonian fluid is not assumed for this purpose, and consequently the stress tensor and the heat-flux vector are included in this count. However, many of these elements will be zero because of flow symmetries. The number obtained above provides a natural order for a trial moment system. A weighting function is then chosen containing the above number of parametric functions; the exact parameters are not necessarily determined. Also chosen at this time is a set of moment equations and its associated set of canonical variables. These canonical variables must be specified as functions of the parametric variables through integrations of the trial weighting function, and the determinant of this transformation may then be evaluated. If the resulting determinant has at most isolated zeros the moment system is capable of solution. On the other hand, if the transformation is singular throughout space, alterations must be made on either the weighting function or the set of moment equations until a well posed formulation is obtained. Details of this procedure are clarified in the specific problems of the following chapters.

The principal idea of the above is the avoidance of unrealistic coupling among the integrated physical flow quantities. A physical understanding of the problem will, in most instances, provide a proper system from the beginning, but occasional unexpected relationships are revealed by the above formal testing method. Although the above discussion is directed to flows in axisymmetrical spherical geometry the ideas may be carried over to a more general class of problems.

# 3. HEAT TRANSFER FROM A SPHERE

## 3.1. Description of the Problem

The problem of the heat transfer from a sphere to a monatomic gas at rest provides a simple and illustrative introduction to the moment method in spherical geometry. The related problem of heat transfer from a wire (or between two concentric cylinders) was investigated by Liu and Lees (8) using a four moment method with reasonable results, thus suggesting a similar four moment approach to the sphere problem. The solution for the heat transfer from a sphere, along with the cylindrical case, has been reported by Lees (6) in a discussion of the transition properties of kinetic flows. Here, this solution will be developed within the general framework of Chapter 2, which is slightly different from the formulation previously reported.

At normal gas density, heat transfer from the sphere is independent of the pressure in the gas, while at very low densities the heat flux becomes proportional to the pressure. At intermediate densities the heat conduction versus pressure relationship is more complex as is illustrated in Fig. 3. This intermediate region was studied both experimentally and analytically by Takao, (9) who utilized a more complex form containing polynomials for the weighting function. He found good agreement with experimentally measured heat transfer in air between concentric spheres. His values provide a good experimental check for the moment method discussed in this Chapter. There are no known experimental checks for the temperature and density fields about the sphere.

The sphere is assumed to have a high thermal conductivity and a temperature which is very near that of the surrounding gas. This temperature difference between the sphere and the gas at infinity (or the outer sphere) is small compared to $T_\infty$ and the ratio $\Delta T/T_\infty$ is the relevant perturbation parameter for this problem.

## 3.2. Weighting Function and Mean Quantities

In order to satisfy at least the three conservation equations and the heat-flux equation, one finds that the weighting function must contain a minimum of four parametric functions. Also, since the temperature and pressure gradients in this problem are quite small, it is reasonable to expect little contribution from the velocity parameters in Eq. (2.12). Consequently, a four moment system is chosen with the following weighting function:

$$f_1 = f_\infty\left[1 + N_1 - \tfrac{3}{2} t_1 + \beta_0^2 \xi^2 t_1\right]$$

and (3.1)

$$f_2 = f_\infty\left[1 + N_2 - \tfrac{3}{2} t_2 + \beta_0^2 \xi^2 t_2\right] .$$

This system incorporates sufficient freedom for the appropriate physical quantities to remain everywhere independent. The zeroth order part ($f_\infty$) of the weighting function may be ignored since it will identically satisfy all moment equations.

Now the mean quantities which appear in the moment equations must be integrated. For example, in the continuity Eq. (2.21) we need $nu_R = \int f\xi_R d\underline{\xi} = \int_{\text{Region 1}} f_1 \xi_R d\underline{\xi} + \int_{\text{Region 2}} f_2 \xi_R d\underline{\xi}$. Using Eq. (3.1) and Table A.4. of Appendix A, this results in

$$nu_R = \frac{n_\infty}{2\sqrt{\pi}\,\beta_0} (N_1 + \tfrac{1}{2}t_1 - N_2 - \tfrac{1}{2}t_2)(1-x^2),$$

where a new independent variable x is introduced for later convenience. It is defined as follows:

$$x = \sqrt{1 - R_0^2/R^2} \quad . \tag{3.2}$$

The other mean quantities appearing in the continuity, momentum, energy and heat-flux equations may be found similarly.

In order to bring out all pertinent parameters governing the problem we choose $n_\infty$, $T_\infty$, and $R_0$ as the characteristic number density, temperature, and length, respectively. The flow variables which occur in the moment equations are then non-dimensionalized as follows:

$$\begin{array}{ll}
n & \text{by } n_\infty ; \\
u, v, w & \text{by } \dfrac{1}{2\sqrt{\pi}\beta_0} = \sqrt{kT_\infty/2\pi m} \quad ; \\
P_{ij} & \text{by } n_\infty kT_\infty \quad ; \\
P_{ijk}, \dot{q}_i & \text{by } \dfrac{n_\infty kT_\infty}{2\sqrt{\pi}\,\beta_0} = n_\infty kT_\infty \sqrt{kT_\infty/2\pi m} \quad ; \\
P_{ijkl} & \text{by } \dfrac{n_\infty kT_\infty}{4\pi\beta_0^2} = mn_\infty/8\pi\beta_0^4 \quad .
\end{array} \tag{3.3}$$

Then, denoting all non-dimensionalized quantities by a bar superscript, the relevant moments are

$$\overline{nu_R} = (N_1 - N_2 + \tfrac{1}{2}t_1 - \tfrac{1}{2}t_2)(1 - x^2) , \tag{3.4}$$

$$\overline{P^*_{RR}} = -\tfrac{1}{2}(N_1 + t_1)(1 - x^3) - \tfrac{1}{2}(N_2 + t_2)(1 + x^3) , \tag{3.5}$$

$$\overline{P^*_{\theta\theta}} = -\tfrac{1}{2}(N_1+t_1)(1-3x/2+x^3/2)-\tfrac{1}{2}(N_2+t_2)(1+3x/2-x^3/2), \tag{3.6}$$

$$\overline{\dot{q}^*_R} = 2(N_1 - N_2 + 3t_1/2 - 3t_2/2)(1-x^2) , \tag{3.7}$$

$$\overline{P^*_{RRii}} = \frac{5\pi}{2}(N_1 + 2t_1)(1 - x^3) + \frac{5\pi}{2}(N_2 + 2t_2)(1+x^3), \tag{3.8}$$

$$\overline{P^*_{\theta\theta ii}} = \overline{P^*_{\varphi\varphi ii}} = \frac{5\pi}{2}(N_1 + 2t_1)(1 - 3/2\,x + 1/2\,x^3) + \frac{5\pi}{2}(N_2 + 2t_2)(1 + 3/2\,x - 1/2\,x^3), \tag{3.9}$$

$$\overline{\dot{q}_R} = \overline{\dot{q}^*_R} - 5/2\,\overline{nu_R} = (-\tfrac{1}{2}N_1 + \tfrac{1}{2}N_2 + 7/4\,t_1 - 7/4\,t_2)(1 - x^2). \tag{3.10}$$

## 3.3. Differential Equations and Boundary Conditions

In this problem the moment equations of Section 2.3. become ordinary differential equations because of the complete spherical symmetry. Thus, in non-dimensional form the system of equations for the first order solution is

Continuity:

$$(2 + R\frac{d}{dR})\,\overline{nu_R} = 0\,; \tag{3.11}$$

R-momentum:

$$(2 + R\frac{d}{dR})\,\overline{P^*_{RR}} - 2\,\overline{P^*_{\theta\theta}} = 0; \tag{3.12}$$

Energy:

$$(2 + R\frac{d}{dR})\,\overline{\dot{q}^*_R} = 0\,; \tag{3.13}$$

Heat-flux:

$$(2 + R\frac{d}{dR})\,\overline{P^*_{RRii}} - \overline{P^*_{\theta\theta ii}} - \overline{P^*_{\varphi\varphi ii}} = -\frac{2\pi R}{3\lambda_\infty}\,\overline{\dot{q}_R}\,. \tag{3.14}$$

Note that the zeroth order mean free path is used in the collision integral in Eq. (3.14).

The continuity and energy equations can be integrated directly giving

$$R^2\,\overline{nu_R} = N_1 - N_2 + 1/2\,t_1 - 1/2\,t_2 = \text{constant} = C_1\,, \tag{3.15}$$

and

$$R^2\,\overline{\dot{q}_R^*} = 2N_1 - 2N_2 + 3t_1 - 3t_2 = \text{constant} = C_2\,. \tag{3.16}$$

Using Eqs. (3.15) and (3.16) the momentum equation can also be integrated to give

$$N_1 + t_1 + N_2 + t_2 = \text{constant} = C_3\,. \tag{3.17}$$

Finally, using the above expressions, the heat-flux equation reduces to the simple form:

$$\frac{dt_1}{d\overline{R}} = -\frac{2}{15\mathrm{Kn}\overline{R}^2}\left(1/2\,C_2 - 5/4\,C_1\right), \tag{3.18}$$

where the Knudsen number is defined as $\mathrm{Kn} = \frac{\lambda_\infty}{2R_0}$.

This system is regular at both boundaries indicating the use of the simple boundary conditions described in Section 2.4. For this problem these conditions are

$$\begin{aligned} t_1 &= \epsilon = \frac{\Delta T}{T_\infty} && \text{at } \overline{R} = 1, \\ \overline{nu_R} &= 0 && \text{at } \overline{R} = 1, \\ N_2 &= 0 && \text{at } \overline{R} = \infty, \\ t_2 &= 0 && \text{at } \overline{R} = \infty. \end{aligned} \tag{3.19}$$

For the similar problem of heat-flux between two spheres it is only necessary to apply the last two conditions at a point $\overline{R}$ equal to the ratio of sphere diameters.

### 3.4. Moment Solutions

Integrating Eq. (3.18) with integration constant $C_4$ and applying these boundary conditions one can determine the four integration constants. These constants, which do depend on Knudsen number, are found to be

$$\begin{aligned} C_1 &= 0\,, \\ \tfrac{1}{2} C_2 = 2\,C_3 = C_4 &= \varepsilon / (1 + \frac{2}{15\mathrm{Kn}})\;. \end{aligned} \tag{3.20}$$

Thus, one obtains the following solutions for the four parametric functions:

$$\begin{aligned} N_1 &= -\,(1/2 + \frac{2}{15\mathrm{Kn}\bar{R}})\,\beta\;, \\ N_2 &= -\,2\beta/(15\mathrm{Kn}\bar{R})\,, \\ t_1 &= (1 + \frac{2}{15\mathrm{Kn}\bar{R}})\,\beta\,, \\ t_2 &= 2\beta/(15\mathrm{Kn}\bar{R})\;, \end{aligned} \tag{3.21}$$

where

$$\beta = \varepsilon/(1 + \frac{2}{15\mathrm{Kn}})\;.$$

All of the mean quantities may now be found by substituting Eqs. (3.21) into the expressions of Section 3.2. The most interesting of these quantities are the radial heat-flux and the temperature field. Using Eq. (3.10) the heat-flux is

$$\dot{q}_R = n_\infty k T_\infty \left(\frac{2kT_\infty}{\pi m}\right)^{\frac{1}{2}} \frac{\varepsilon}{(1 + \frac{2}{15\mathrm{Kn}})} \frac{R_0^2}{R^2}\;. \tag{3.22}$$

Using subscript c to denote quantities evaluated in the continuum limit, it is possible to represent the total heat-transfer ratio as

$$\frac{Q}{Q_c} = \frac{1}{1 + \frac{15\mathrm{Kn}}{2}}\;. \tag{3.23}$$

The temperature field is defined through the relation $p = nkT$, in which p and n can be found by integration of the now known weighting function. The resulting expression for the temperature is

$$\frac{T_s - T}{\Delta T} = \frac{\delta}{2}\left(1 + \sqrt{1 - R_0^2/R^2}\right) + (1-\delta)(1-R_0/R) , \tag{3.24}$$

where

$$\delta = \frac{1}{1 + \frac{2}{15Kn}} . \tag{3.25}$$

The first term in Eq. (3.24) represents a collisionless distribution weighted by the rarefaction parameter $\delta$, while the second term gives the slowly-varying, collision-dominated temperature field far from the sphere. In the next Section this dual character of the temperature field is pursued to give an alternative approach to the solution just found.

## 3.5. Solution by Matching

Kubota (10) has shown that the matching of a free-molecule "inner" and a Fourier "outer" temperature distribution using a method similar to that of Kaplun and Lagerstrom (11) can reproduce the entire four-moment solution for heat-transfer between concentric cylinders. The linear superposition of a free-molecule like and Fourier like temperature field in the sphere solution, Eq. (3.24), suggests a similar matching solution for the sphere problem. This procedure is not a true mathematical asymptotic matching, but a physically motivated analogy.

Here we take as an inner solution the Knudsen free-molecule expression (12) for the heat-flux from a sphere whose temperature differs slightly from that of the surrounding gas:

$$Q = \frac{16\pi k_c R_0^2}{15\lambda_\infty} \Delta T , \tag{3.26}$$

where $k_c$ is the heat conductivity of a monatomic gas defined by the kinetic relation:

$$k_c = \frac{15k}{4m}\mu = 15/4\ nk\lambda(\pi m/2kT)^{\frac{1}{2}} . \qquad (3.27)$$

The corresponding temperature distribution is given by

$$T_s - T = \tfrac{1}{2}(1 + \sqrt{1 - R_0^2/R^2})\,\Delta T \quad . \qquad (3.28)$$

Eliminating $\Delta T$ between Eqs. (3.26) and (3.28) gives the inner temperature field in terms of the total heat-flux:

$$T_s - T = \frac{15\lambda_\infty Q}{32\pi k_c R_0^2}(1 + \sqrt{1 - R_0^2/R^2}) \ . \qquad (3.29)$$

Far from the sphere, where collisions must eventually dominate the flow, the solution of the Fourier heat conduction equation is applicable:

$$T - T_\infty = Q/4\pi k_c R \ . \qquad (3.30)$$

The Q here must be the same as for the inner solution if these two temperature fields are to be matched.

The two solutions are each valid in different regions and the value of the heat-flux cannot be determined from either because of the lack of applicable boundary conditions. The condition which allows a solution is supplied by matching the inner and outer solutions as follows:

$$\lim_{R\to\infty} T_{inner} = \lim_{R\to R_0} T_{outer} \ .$$

Applying this condition to Eqs. (3.29) and (3.30) gives

$$T_s - \frac{15\lambda_\infty Q}{16\pi k_c R_0^2} = T_\infty + Q/4\pi k_c R_0. \qquad (3.31)$$

This may be solved for Q to give

$$\frac{Q}{Q_c} = \frac{1}{1 + \frac{15\lambda_\infty}{4R_0}} , \qquad (3.32)$$

where

$$Q_c = 4\pi k_c R_0 \Delta T . \qquad (3.33)$$

The uniformly valid temperature field is found by subtracting the common part, $T_\infty + Q/4\pi k_c R_0$, from the outer solution and adding the remainder to the inner solution, thereby giving

$$\frac{T_s - T}{\Delta T} = \tfrac{1}{2}\,\delta(1 + \sqrt{1 - R_0^2/R^2}\,) + (1-\delta)(1-R_0/R) , \qquad (3.34)$$

with

$$\delta = \frac{1}{1 + \frac{4R_0}{15\lambda_\infty}} . \qquad (3.35)$$

Therefore, matching in this manner produces both the temperature distribution and heat-flux exactly as in the four-moment method of the last Section.

## 3.6. Heat Transfer and Comparison with Experiment

The limiting moment solutions for the total heat-transfer from a sphere are easily found by integrating Eq. (3.22) over the sphere surface. In the limit of large Knudsen number this integration yields

$$Q = 4\pi R_0^2 n_\infty k \Delta T (2kT_\infty/\pi m)^{\frac{1}{2}} ,$$

which is just the Knudsen formula, Eq. (3.26). With the addition of a thermal accomodation coefficient to account for the imperfect accommodation, this formula is well established experimentally. In the limit of small Knudsen number the moment method readily reproduces the Fourier result,

$$Q = 4\pi k_c R_0 \Delta T ,$$

provided the definition (3. 27) is used for the heat conductivity of a monatomic gas.

Experimental values for the total heat conduction from a sphere at intermediate values of the Knudsen number are provided by Takao (9) who performed these measurements in air. In a monatomic gas the translational energy transferred to the gas by the sphere is proportional to $2k\Delta T$. For a diatomic gas at room temperature the additional rotational energy makes this energy transferral proportional to $3k\Delta T$. Therefore the heat-transfer ratio given by formula (3. 23) must be increased by a factor of 3/2 in the free-molecule limit to become

$$\frac{Q}{Q_c} = \frac{1}{1 + 5Kn} \tag{3. 36}$$

This expression is compared with Takao's experimental results in Fig. 4 in which excellent agreement is found to exist.

# 4. SLOW FLOW PAST A SPHERE

## 4.1. Description of the Problem

The problem is to determine the flow field about a sphere moving with uniform velocity $\underline{U}$ through an infinite homogeneous gas of arbitrary density. We restrict our attention to the low speed case where the speed ratio S, defined as $U(m/2kT_\infty)^{\frac{1}{2}}$, is very small (Mach number $M \ll 1$). The sphere is assumed to have a high thermal conductivity and a temperature which is very near that of the surrounding gas. This allows the flow dynamics to be decoupled from the thermodynamic temperature field, thus permitting each to be solved independently (neglecting coupling of order $S^2$). The companion problem of determining the temperature field about a stationary sphere with heat-flux was discussed in the last Chapter and may be added linearly to the dynamic solution if so desired.

It is a simple matter to show that flows about geometrically similar bodies are dynamically similar, provided only that their speed ratios and Knudsen numbers are equal. Thus the desired flow field solutions for linearized flow about a sphere will, when normalized by U, form one parameter families of curves depending only on Knudsen number. The nondimensionalized speed at infinity $\overline{U}$ is therefore the only perturbation quantity for this problem.

An alternative method for exhibiting the parameters involved in this problem is illustrated in Fig. 5 where the Reynolds number, $Re \sim M/Kn$, is introduced. This description more clearly gives the specific range of parameters for which the present solution is intended and, in particular, indicates the limiting solutions with which the moment values may be compared. These two limiting solutions for large and small Knudsen number are well known and provide excellent checks for any theory, although care must be exercised when assigning physical significance to them.

Consider the Mach number at infinity to be fixed at a value $M \ll 1$ and restrict the Reynolds number to the values, $0 < Re < 1$. The Knudsen number is then confined to the range, $M < Kn < \infty$, instead of the full range, $0 < Kn < \infty$, which one might expect for flow of arbitrary density. In this way the high Reynolds number flows, which are in the realm of continuum theory, are avoided, and the limiting solution for low Knudsen number is thus the Stokes solution (13) of the Navier-Stokes equations. Of course, the convective corrections due to Ossen will be needed for sufficiently large distances from the sphere as in the continuum theory.

Although $Kn = \infty$ represents the unrealistic situation of a totally collisionless flow, it closely resembles the solution near the sphere for $Kn \gg 1$. This free-molecule solution is described by two entirely independent Maxwellian distributions: one at sphere temperature and zero velocity for all molecules leaving the sphere, and a second at free stream conditions for all other particles.

There exists virtually no experimental data describing the velocity fields or stress patterns about a slowly moving sphere in any density regime. However, some excellent measurements by Millikan (14) in 1923 provide good experimental values for low speed sphere drag. Since the sphere drag is an integral of certain of the above stresses, Millikan's results may be used as a check on the present theory. Millikan's data cover a useful range of Knudsen numbers ($0.01 < Kn < 10$) at very low speed ratios ($S < 10^{-5}$).

Most previous theoretical investigations of this problem are, like the experiments, concerned only with integrated effects such as the sphere drag dependence on Knudsen number. They neglect the velocity and stress fields about the sphere except in the limiting solutions discussed above. Also, these theories are restricted to a partial range of Knudsen number.

Most of the theoretical work for low speed sphere drag has been done for the near-free-molecule regime ($Kn > 1$). In this regime the method of Knudsen iteration has been applied with various collision models to give the first order ($Kn^{-1}$) iteration of the Maxwell-Boltzmann equation. Szymanski (15) and Liu, et. al. (16) use this method with simplifying analytical approximations and Maxwellian molecules to determine the ratio, $C_D/C_{Dfm}$, of the drag coefficient to the free-molecule value. These two approaches are very similar with the exception that Liu, et. al. carry their analysis to higher order in the speed ratio S. Questioning the validity of the approximations used by Liu, et. al. on both theoretical and practical grounds, Willis (2) proposed that a simple statistical collision model be used to avoid the analytical difficulties. Hence, using the Krook model, (4) he performs a Knudsen iteration and finds results which agree well with Millikan's data.

One sphere theory in the near-continuum regime is that of Goldberg, (17) in which the Grad 13-moment equations are solved for linearized flow about a slowly moving sphere. Its validity is restricted to the near-continuum regime because this model fails to reproduce the discontinuous nature of the distribution function found in low density regimes.

Although the last two theories provide accurate drag values in the limiting regimes for which they are proposed, they both break down in the transition regime where the Knudsen number is of order one. The moment method is employed here in an attempt to bridge this gap and to provide drag, velocity and stress field values for flows at all Knudsen numbers.

## 4.2. Weighting Function and Mean Quantities

The number of parametric functions which appear in the weighting function for this problem must now be determined. Following the general procedure outlined in Section 2.5, the first step is the determination of the naturally independent physical quantities associated with the problem. In this example there are eight such quantities, $U_R$, $U_\theta$, $T$, $\rho$, $P_{RR}$, $P_{R\theta}$, $\dot{q}_R$ and $\dot{q}_\theta$, with only one independent normal stress rather than the usual two because of the axial symmetry. An eight-moment system would therefore be expected to exhibit minimum difficulty in solution.

The general weighting function (2.12) becomes a natural eight-parameter weighting function for this problem when the w's are eliminated by axial symmetry. Unfortunately, all combinations of eight-moment equations through third-order yield singular transformations from parametric to canonical variables throughout space and do not provide well posed moment systems (see Section 2.5.). The only hope for an eight-moment system is a different weighting function or the use of higher order moment equations with the attendant more complicated collision terms.

An alternative approach is the reduction of the number of moments, realizing that certain relationships among the eight physical moments are thereby introduced. In this way it is possible to obtain a system in which the transformation from parametric to canonical variables is singular only at isolated points. A six-parameter weighting function is formed by considering the temperatures in the Maxwellian weighting function to be constant and equal to $T_\infty$. The general rule proposed by Lees (5) that the number of parametric functions must exceed the number of collision invariants remains satisfied.

It was not realized at the time of solution that the obvious eight-moment system would not work, and the reduction to six moments was initially justified by the belief that the temperature variations in the weighting functions were second order in Mach number for an adiabatic sphere. This is not the case, however, and the justification for making the temperature parameters constant instead of the number densities, $N_1(\underline{R})$ and $N_2(\underline{R})$, is comparison with the known limiting solutions: the temperature and density are both constant in the continuum solution; the Maxwellian temperatures are constant in the free-molecule solution while the number densities are not.

The additional simplicity of the six-moment formulation further justifies it, and we proceed with the hope that the numerical differences are small. Thus the final forms for the weighting functions are chosen to be

$$f_1 = f_\infty\left[N_1 + 2\beta_0^2\xi_R u_1 + 2\beta_0^2\xi_\theta v_1\right]$$

and (4. 1)

$$f_2 = f_\infty\left[N_2 + 2\beta_0^2\xi_R u_2 + 2\beta_0^2\xi_\theta v_2\right] ,$$

where the zeroth-order part of the weighting function $f_\infty$ has been dropped as in the heat-flux problem.

Again, the mean quantities which appear in the moment equations must be found by integration of this weighting function. Using Table 4 of Appendix A and non-dimensionalizing all quantities by the expressions (3. 3), the required first order mean flow quantities are now collected with the notation $N_+ = N_1 + N_2$, $N_- = N_1 - N_2$, $\bar{u}_+ = \bar{u}_1 + \bar{u}_2$, etc., being used for convenience. They are

$$\overline{nu_R} = (1 - x^2)N_- + \tfrac{1}{2}\bar{u}_+ - \tfrac{1}{2}x^3\bar{u}_- , \tag{4. 2}$$

$$\overline{nu_\theta} = \tfrac{1}{2}\bar{v}_+ - (3/4\, x - 1/4\, x^3)\bar{v}_- , \tag{4. 3}$$

$$\overline{P^*_{RR}} = -\tfrac{1}{2}N_+ + \tfrac{1}{2}x^3 N_- - \frac{1}{\pi}(1 - x^4)\bar{u}_- , \tag{4. 4}$$

$$\overline{P^*_{\theta\theta}} = \overline{P^*_{\varphi\varphi}} = -\tfrac{1}{2}N_+ + (3/4\, x - 1/4\, x^3)N_- - \frac{1}{2\pi}(1 - x^2)^2\bar{u}_- , \tag{4. 5}$$

$$\overline{P^*_{R\theta}} = -\frac{1}{2\pi}(1 - x^2)^2\bar{v}_- , \tag{4. 6}$$

$$\overline{\dot{q}^*_R} = 2(1 - x^2)N_- + 5/4\, \bar{u}_+ - 5/4\, x^3\bar{u}_- , \tag{4. 7}$$

$$\overline{\dot{q}^*_\theta} = 5/4\, \bar{v}_+ - (15/8\, x - 5/8\, x^3)\bar{v}_- , \tag{4. 8}$$

$$\overline{P^*_{RRR}} = 2(1 - x^4)N_- + 3/2\, \bar{u}_+ - 3/2\, x^5\bar{u}_- , \tag{4. 9}$$

$$\overline{P^*_{RR\theta}} = 1/2\, \bar{v}_+ - (5/4\, x^3 - 3/4\, x^5)\bar{v}_- , \tag{4. 10}$$

$$\overline{P^*_{R\theta\theta}} = (1 - x^2)^2 N_- + 1/2\,\bar{u}_+ - (5/4\, x^3 - 3/4\, x^5)\bar{u}_- , \tag{4. 11}$$

$$\overline{P^*_{R\varphi\varphi}} = \overline{P^*_{R\theta\theta}} , \tag{4. 12}$$

$$\overline{P^*_{\theta\theta\theta}} = 3/2\ \bar{v}_+ - (45/16\ x - 15/8\ x^3 + 9/16\ x^5)\bar{v}_- , \tag{4.13}$$

$$\overline{P^*_{\theta\varphi\varphi}} = 1/3\ \overline{P^*_{\theta\theta\theta}} , \tag{4.14}$$

$$\overline{p_{R\theta}} = \overline{P^*_{R\theta}} , \tag{4.15}$$

$$\overline{p_{RR}} = -1/2\ (x - x^3)N_- - \frac{1}{3\pi}\ (1 + 2x^2 - 3x^4)\bar{u}_- . \tag{4.16}$$

4.3. Differential Equations

Unlike the heat conduction problem with its spherical symmetry, the system of moment equations governing this problem remains a partial differential system. The six lowest order independent equations of Section 2.3. which are not trivially satisfied are chosen for this system. In non-dimensional form these equations are

Continuity (2.21):

$$(2 + R\frac{\partial}{\partial R})\overline{nu_R} + (\cot\theta + \frac{\partial}{\partial\theta})\overline{nu_\theta} = 0 ; \tag{4.17}$$

Radial momentum (2.22):

$$(2 + R\frac{\partial}{\partial R})\overline{P^*_{RR}} + (\cot\theta + \frac{\partial}{\partial\theta})\overline{P^*_{R\theta}} - \overline{P^*_{\theta\theta}} - \overline{P^*_{\varphi\varphi}} = 0 ; \tag{4.18}$$

$\theta$ momentum (2.23):

$$(3 + R\frac{\partial}{\partial R})\overline{P^*_{R\theta}} + (\cot\theta + \frac{\partial}{\partial\theta})\overline{P^*_{\theta\theta}} - \cot\theta\,\overline{P^*_{\varphi\varphi}} = 0; \tag{4.19}$$

Energy (2.25):

$$(2 + R\frac{\partial}{\partial R})\overline{\dot{q}^*_R} + (\cot\theta + \frac{\partial}{\partial\theta})\overline{\dot{q}^*_\theta} = 0; \tag{4.20}$$

Radial stress moment (2.26):

$$(2+R\frac{\partial}{\partial R})\overline{P^*_{RRR}} + (\cot\theta + \frac{\partial}{\partial\theta})\overline{P^*_{RR\theta}} - 2\,\overline{P^*_{R\theta\theta}} - 2\,\overline{P^*_{R\varphi\varphi}} = \frac{\pi R}{\lambda_\infty}\overline{p_{RR}} ; \tag{4.21}$$

Shear stress moment (2.28)·

$$(3 + R\frac{\partial}{\partial R})\overline{P^*_{RR\theta}} + (\cot\ \theta + \frac{\partial}{\partial\theta})\overline{P^*_{R\theta\theta}} - \overline{P^*_{\theta\theta\theta}} - \overline{P^*_{\theta\varphi\varphi}} - \cot\theta\,\overline{P^*_{R\varphi\varphi}}$$

$$= \frac{\pi R}{\lambda_\infty}\ \overline{p_{R\theta}} \ . \qquad (4.22)$$

With the integrated expressions (4.2) through (4.16) inserted, these equations form a complete system governing the six parametric functions, $\overline{u}_1$, $\overline{u}_2$, $\overline{v}_1$, $\overline{v}_2$, $N_1$, and $N_2$.

One might expect that eliminating the two temperature parameters from the eight-moment formulation would indicate dropping the "thermodynamic" equations, i. e. the energy and heat-flux equations. This in fact is the case, however equations (2.25), (2.26) and (2.27) are linearly dependent under the θ-dependence assumed below and the simpler energy equation may be used instead of the equivalent tangential stress moment.

The flow symmetry now suggests a very important simplification, the reduction of the partial differential system to one of ordinary differential equations. This separation is accomplished by assuming that each parametric function is a product of either sin θ or cos θ and a function of R only. The following substitutions are thus made:

$$\overline{u}_1(\underline{R}) = \widetilde{u}_1(R)\cos\theta\ ;$$

$$\overline{v}_1(\underline{R}) = \widetilde{v}_1(R)\sin\theta\ ; \qquad (4.23)$$

$$N_1(\underline{R}) = \widetilde{N}_1(R)\cos\theta\ ;$$

$$\bar{u}_2(\underline{R}) = \tilde{u}_2(R)\cos\theta\ ;$$

$$\bar{v}_2(\underline{R}) = \tilde{v}_2(R)\sin\theta\ ;$$

$$N_2(\underline{R}) = \tilde{N}_2(R)\cos\theta\ .$$

The integrated flow variables which appear in the moment equations also separate in a similar manner with the tilde superscript having an analagous interpretation. Also, a simple change of variable to $x = \sin\alpha = \sqrt{1 - R_0^2/R^2}$ makes the domain of the independent variable, $0 \le x \le 1$, more convenient for subsequent numerical analysis. From this

$$R\frac{d}{dR} \equiv \frac{1 - x^2}{x}\frac{d}{dx}\ . \tag{4.24}$$

Making the above substitutions in the system of partial differential equations yields the following system of six ordinary differential equations:

Continuity:

$$\frac{1 - x^2}{x}\frac{d}{dx}\,\widetilde{nu}_R + 2\,\widetilde{nu}_R + 2\,\widetilde{nu}_\theta = 0\ ; \tag{4.25}$$

Radial momentum:

$$\frac{1 - x^2}{x}\frac{d}{dx}\,\tilde{P}^*_{RR} + 2\,\tilde{P}^*_{RR} + 2\,\tilde{P}^*_{R\theta} - 2\,\tilde{P}^*_{\theta\theta} = 0\ ; \tag{4.26}$$

$\theta$ Momentum:

$$\frac{1 - x^2}{x}\frac{d}{dx}\,\tilde{P}^*_{R\theta} + 3\,\tilde{P}^*_{R\theta} - \tilde{P}^*_{\theta\theta} = 0\ ; \tag{4.27}$$

Energy:

$$\frac{1-x^2}{x}\frac{d}{dx}\tilde{\dot{q}}^*_R + 2\,\tilde{\dot{q}}^*_R + 2\,\tilde{\dot{q}}^*_\theta = 0\;; \qquad (4.28)$$

Radial stress moment:

$$\frac{1-x^2}{x}\frac{d}{dx}\tilde{P}^*_{RRR} + 2\,\tilde{P}^*_{RRR} + 2\,\tilde{P}^*_{RR\theta} - 4\,\tilde{P}^*_{R\theta\theta} = \frac{\pi R}{\lambda_\infty}\tilde{p}_{RR}; \qquad (4.29)$$

Shear stress moment:

$$\frac{1-x^2}{x}\frac{d}{dx}\tilde{P}^*_{RR\theta} + 3\,\tilde{P}^*_{RR\theta} - \tilde{P}^*_{R\theta\theta} - 4\,\tilde{P}^*_{\theta\varphi\varphi} = \frac{\pi R}{\lambda_\infty}\,\tilde{p}_{R\theta}\,. \qquad (4.30)$$

## 4.4. Boundary Conditions

The determination of the proper boundary conditions for a numerical solution of the sixth-order system of linear ordinary differential equations is greatly complicated by the presence of singular points at both ends of the integration domain. These singular points correspond to the sphere surface and infinity, both points at which the boundary conditions are to be applied. A set of necessary conditions can be given at this time, but it may not be sufficient to determine a unique solution.

The physical properties of the surface-molecule interaction must first be determined and for simplicity in the present analysis it is assumed that the molecules suffer diffuse re-emission and complete energy accomodation. This stipulation imposes two conditions on the outgoing part of the weighting function at the surface of the sphere:

$$\tilde{u}_1(x) = 0 \qquad \text{at} \qquad x = \mathbf{0}\;; \qquad (4.31)$$

$$\tilde{v}_1(x) = 0 \quad \text{at} \quad x = 0. \tag{4.32}$$

A third condition at the surface of the sphere is that of no net mass-flux,

$$n\tilde{u}_R(x) = 0 \quad \text{at} \quad x = 0 , \tag{4.33}$$

which may be expressed in terms of the parametric functions, using Eq. (4.2), as

$$\tilde{N}_1(x) - \tilde{N}_2(x) + 1/2\, \tilde{u}_2(x) = 0 \quad \text{at} \quad x = 0 . \tag{4.34}$$

The remaining three conditions are obtained by forcing the weighting function to approach a Maxwellian distribution with uniform flow at infinity. Thus,

$$\tilde{u}_2(x) = -\overline{U} \quad \text{at} \quad x = 1 , \tag{4.35}$$

$$\tilde{v}_2(x) = \overline{U} \quad \text{at} \quad x = 1 , \tag{4.36}$$

and

$$\tilde{N}_2(x) = 0 \quad \text{at} \quad x = 1 . \tag{4.37}$$

For numerical purposes it is desirable to normalize the parametric functions by the perturbation parameter, $\epsilon = \overline{U}$, so that all boundary conditions are pure numbers. The conditions (4.35) and (4.36) then become

$$\tilde{u}_2(x) = -1 \quad \text{at} \quad x = 1 , \tag{4.38}$$

and

$$\tilde{v}_2(x) = 1 \qquad \text{at} \qquad x = 1 , \tag{4.39}$$

while the rest remain unchanged. This normalization is assumed in the following discussion of the numerical solution.

4.5. Moment Solutions

A number of attempts were made to integrate the sixth-order system subject to the boundary conditions of the last Section. All of these naive attempts failed with the exception of the special case where $Kn = \infty$ (no collision terms in the equations). This particular solution yielded velocity and stress fields which agreed with the exact free-molecule solution (See Appendix E) to a remarkable degree and provided much hope for the collisional moment solutions if they could be determined.

The general numerical approach which proved most promising involved writing the six differential equations as central finite difference equations for each of fifty one grid points in the interval $0 \le x \le 1$. The resulting three hundred algebraic equations plus the six boundary conditions then formed a $306 \times 306$ coefficient matrix whose inversion provided the values of the parametric functions at each of the grid points quite accurately. When collision terms were allowed in this scheme the resulting solution exhibited an instability near the sphere whose strength increased in proportion to the coefficient of the collision term, i.e. like $Kn^{-1}$.

Clearly, something more complex is involved than was found in the heat-transfer solution and a detailed analytic investigation of the differential system is called for. This involves the rather formidable task of classifying all singular points of the system and developing asymptotic representations for the solution matrix at these points. Fortunately, this is possible and many interesting aspects of this problem are clarified by this effort.

A number of simplifications may be made to the sixth-order system to facilitate the study of its singularities. Specifically, one quadrature and two integral relations can be found which reduce the required analysis to that of a third-order system. The first integral relation is suggested by the fact that the drag integral is constant for any spherical surface within the gas. With respect to the moment equations this result is found by subtracting two times the tangential momentum Eq. (4.27) from the radial momentum Eq. (4.26) to give

$$\frac{1-x^2}{x}\frac{d}{dx}(\tilde{P}^*_{RR} - 2\tilde{P}^*_{R\theta}) + 2\tilde{P}^*_{RR} - 4\tilde{P}^*_{R\theta} = 0 . \tag{4.40}$$

Direct integration of this equation gives

$$(\tilde{P}^*_{RR} - 2\tilde{P}^*_{R\theta}) = C_1(1 - x^2) , \tag{4.41}$$

which, expressed in terms of the parametric functions through Eqs. (4.4) and (4.6), is

$$-1/2\, \tilde{N}_+ + 1/2\, x^3 \tilde{N}_- - \frac{1}{\pi}(1-x^4)\tilde{u}_- + \frac{1}{\pi}(1-x^2)^2 \tilde{v}_- = C_1(1-x^2). \tag{4.42}$$

A second integral is found by subtracting five times the continuity equation (4.25) from the energy equation (4.28). Using expressions (4.2), (4.3), (4.7) and (4.8) the resulting differential equation is

$$\frac{1-x^2}{x}\frac{d}{dx}(1-x^2)\tilde{N}_- + 2(1-x^2)\tilde{N}_- = 0\,, \tag{4.43}$$

which may be integrated to show that

$$\tilde{N}_- = \text{constant} = C_2. \tag{4.44}$$

Finally, formal integration of the continuity equation yields the quadrature relation

$$\tilde{u}_+ = x^3\tilde{u}_- - 2\int^x \frac{x'}{1-x'^2}\left[\tilde{w} - x'^3\tilde{u}_- - (3/2\,x' - 1/2\,x'^3)\tilde{v}_-\right]dx' + C_3 \tag{4.45}$$

where a new variable is defined as

$$\tilde{w} = \tilde{u}_+ + \tilde{v}_+\,, \tag{4.46}$$

and $x'$ is a dummy integration variable.

The three relations, Eqs. (4.42), (4.44) and (4.45), may be used to replace the continuity, radial momentum and energy equations in the original sixth-order system. Using these relations to eliminate the parametric variables $\tilde{N}_+$, $\tilde{N}_-$ and $\tilde{u}_+$, the remaining three equations become:

$\theta$-Momentum:

$$\frac{d}{dx}(1-x^2)^2\tilde{v}_- + x(1+3x^2)\tilde{u}_- + x(1-x^2)\tilde{v}_- + 2\pi C_1 x + \frac{3\pi}{2}C_2 x^2 = 0\,; \tag{4.47}$$

Radial stress moment:

$$\frac{1-x^2}{x}\frac{d}{dx}\left[\frac{3}{2}x^3(1-x^2)\tilde{u}_-\right] + (8x^3-6x^5)\tilde{u}_- + \left(\frac{9}{2}x - 4x^3 + \frac{3}{2}x^5\right)\tilde{v}_-$$
$$- 2\tilde{w} + \frac{1}{6Kn}(1-x^2)^{\frac{1}{2}}(1+3x^2)\tilde{u}_- + \frac{\pi}{4Kn}x(1-x^2)^{\frac{1}{2}}C_2 = 0\,; \tag{4.48}$$

Shear stress moment:

$$\frac{1-x^2}{x}\frac{d}{dx}\left[\tilde{w} - x^3\tilde{u}_- - (\frac{5}{2}x^3 - \frac{3}{2}x^5)\tilde{v}_-\right] + \tilde{w} + (\frac{1}{2}x^3 - \frac{3}{2}x^5)\tilde{u}_-$$
$$+ (\frac{9}{2}x - \frac{23}{2}x^3 + 6x^5)\tilde{v}_- + \frac{1}{2Kn}(1-x^2)^{3/2}\tilde{v}_- - 2(1-x^2)^2 C_2 = 0. \tag{4.49}$$

These three equations exhibit the entire singular behavior of the original system.

A more convenient notation for the analysis of this system is afforded by the vector equation,

$$\frac{d}{dx}\underline{Y} = \underline{\underline{A}}\cdot\underline{Y} + \underline{a}, \tag{4.50}$$

where the vector $\underline{Y}$ is defined by

$$\underline{Y} \equiv \begin{Bmatrix} \tilde{w} \\ \tilde{u}_- \\ \tilde{v}_- \end{Bmatrix} .$$

It is relatively easy to show from equations (4.47), (4.48) and (4.49) that the coefficient matrix is

$$\underline{\underline{A}} = \frac{1}{(1-x^2)^2}\begin{bmatrix} \frac{x}{3}(1+3x^2) & -\frac{19}{3}x^4 - 2x^6 + 3x^8 & \frac{11}{3}x^4 - 8x^6 + 3x^8 \\ & -\frac{1}{9Kn}(1-x^2)^{\frac{1}{2}}(1+3x^2)x & \frac{-x}{2Kn}(1-x^2)^{5/2} \\ \frac{4}{3x^2} & \frac{-3}{x} + \frac{8}{3}x - x^3 & \frac{-3}{x} + \frac{8}{3}x - x^3 \\ & -\frac{1}{9Knx^2}(1-x^2)^{\frac{1}{2}}(1+3x^2) & \\ 0 & -x(1+3x^2) & 3x(1-x^2) \end{bmatrix}, \tag{4.51}$$

and that the inhomogeneous vector is

$$\underline{a} = \left\{ \begin{array}{l} \frac{\pi}{(1-x^2)^2}(3x^5-5x^3)(C_1 x+\frac{3}{4}x^2 C_2)+2x(1-x^2)C_2-\frac{\pi x^2}{6Kn(1-x^2)^{3/2}}C_2 \\ -\ \frac{\pi C_2}{6Knx(1-x^2)^{3/2}} \\ -\ \frac{2\pi}{(1-x^2)^2}(C_1 x+\frac{3}{4}x^2 C_2) \end{array} \right\} . \quad (4.52)$$

General methods for finding the asymptotic behavior of solutions to linear systems of ordinary differential equations are extensively described in a volume by Wasow. (18) The position and character of all singular points are determined by the matrix $\underline{\underline{A}}$. The singular points of the system occur at the singularities of the matrix elements which in this example are found at $x = 0$ and $x = 1$, the sphere surface and infinity. These singularities are related directly to the singular points of the transformation from parametric to canonical variables, although not realized at the time of solution. Three asymptotic expansions of this system, which explain the numerical difficulties encountered earlier, are now discussed with the details being given in the Appendices.

First, the behavior of the system is investigated at the sphere surface assuming that the Knudsen number is a bounded parameter. At $x = 0$ the matrix $\underline{\underline{A}}$ has a second order pole in two of its elements which makes this point an irregular singular point with rank one, according to the classification given by Wasow. The determination of the complete asymptotic expansion about an irregular singular point is quite involved and a detailed analysis is carried out in Appendix B. For the present purpose only the general solution matrix to the homogeneous form of equation (4.50) is needed. From the first column of this asymptotic solution matrix (B.24), one of the three general vector solutions to equation (4.50) is

$$\underline{n}_1 \sim \left\{ \begin{array}{l} \frac{1}{x^3} e^{1/9Knx}(0(x^3)) \\ \frac{1}{x^3} e^{1/9Knx}(1 - \frac{1}{2Kn}x - x^2 + \frac{1}{4Kn^2}x^2 + 0(x^3)) \\ \frac{1}{x^3} e^{1/9Knx}(0(x^3)) \end{array} \right\}, \qquad (4.53)$$

where $0(f(x))$ is used in the standard manner to represent any function of x such that the $\lim_{x \to 0} \frac{0(f(x))}{f(x)}$ is bounded. Clearly, this solution is very singular near $x = 0$ and contributes to the numerical instability whenever collisions are permitted ($Kn \neq \infty$) unless it is completely eliminated by the boundary conditions. The other two vector solutions, columns two and three of Eq. (B.24), are regular in x at the sphere surface and represent the physically real solution.

A curious behavior is found in the coefficients of the asymptotic expansions for the above solution matrix. The parameter Kn occurs in both the numerator and denominator of these coefficients, thus invalidating this representation for both the limiting cases of free-molecule and continuum flow. This behavior occurs because some of the similarity transformation matrices used in obtaining this solution become singular in these limits (see Appendix B). The implication is that the region of validity of the asymptotic representation depends on the parameter Kn in such a way as to shrink in about x=0 as the Knudsen number becomes either large or small. This dependence on a "stretched coordinate" is characteristic of a class of singular perturbation problems having "boundary layers" near the singular points of the independent variable. Physically, this behavior is expected in the continuum limit since a thin collisionless "Knudsen layer" always exists near the sphere for arbitrarily small values of Kn. On the other hand, no such singular behavior is expected in the physical flow for the large Knudsen number limit. Since the asymptotic representation exhibits this characteristic in both limits, one explanation is that the exact solution matrix consists of two solution vectors regular in (1/Kn) and a singular vector, which appears as a solution to a singular perturbation problem. Somehow the singular solution vector then affects the others in the asymptotic solution matrix when obtained by the methods of Appendix B. This explanation is borne out by the eventual numerical results. The consequences of the above are that the sphere surface is a special type of singularity called a "turning point" or "transition point" and that an extremely complex analysis of a third order turning point is required to fully support the above explanation. Fortunately, enough knowledge to numerically integrate the differential system can be determined without this effort.

In an attempt to further understand the behavior of this third order system a second asymptotic representation is found near the sphere, but for totally collisionless flow. Details of this calculation are given in Appendix C. Again, the matrix $\underline{\underline{A}}$ has a second order pole in one of its elements, suggesting an irregular singular point at $x = 0$. However, this point is what is often referred to as a "pseudo essential singular point" since a transformation (shearing) exists which takes $\underline{\underline{A}}$ into a matrix with a first order pole as its strongest singularity. Thus, for Kn = ∞, the system is really one about a regular singular point and has a solution matrix with, at most, poles at x = 0. This lack of an exponentially singular solution explains the early success in numerically integrating the free-molecule problem.

As a final step in the analytic investigation of this system an asymptotic representation is found about the point infinity. The vector differential equation (4.50) is recast with $\overline{R}$ as the independent variable and the matrix $\underline{\underline{A}}$ expanded in negative powers of $\overline{R}$. As in the first of the previous two expansions the point infinity is an essential singular point, but this time with rank two. The details of the expansion at this point are given in Appendix D and the resulting homogeneous asymptotic solution matrix (D.25) is

$$\underline{\underline{\eta}} \sim \begin{Bmatrix} 2 - \frac{9Kn}{2\overline{R}} + \ldots & \overline{R}^5 - \frac{27Kn}{2}\overline{R}^4 + \ldots & e^{\frac{-4\overline{R}}{9Kn}}(1/\overline{R} + \ldots) \\ \frac{3}{2\overline{R}^2} - \frac{9Kn}{2\overline{R}^3} + \ldots & -\frac{1}{2}\overline{R}^3 + \frac{27Kn}{8}\overline{R}^2 + \ldots & e^{\frac{-4\overline{R}}{9Kn}}(\frac{1}{9Kn\overline{R}^2} + \ldots) \\ 2 - \frac{9Kn}{2\overline{R}} + \ldots & \overline{R}^5 - \frac{27Kn}{2}\overline{R}^4 + \ldots & e^{\frac{-4\overline{R}}{9Kn}}(1/\overline{R} + \ldots) \end{Bmatrix}. \quad (4.54)$$

Again the exponential solution vector, characteristic of essential singular points, forms one of the three columns of the homogeneous matrix, but in this case it is a dying exponential for large $\overline{R}$ and causes no numerical difficulty. Also occurring is an algebraically singular vector, the second column of (4.54), which may be identified with a similar singular solution to the continuum Stokes equation. (13) The remaining solution corresponds to the regular incompressible Stokes solution for slow flow past a sphere. Although well behaved for small values of Kn, this asymptotic representation obviously fails in the free-molecule limit in a manner much like the collisional asymptotic solution does at the sphere surface. This behavior is an example of the well-known singular nature of unbounded collisionless flows as R approaches infinity.

A nearly complete understanding of the nature of all solutions to the original sixth-order differential system has been developed, and it is now possible to re-examine and reformulate the boundary conditions necessary to numerically integrate the collisional system. In view of the physical analogy involved it is reasonable to expect that all parametric functions remain finite throughout the flow field. However, as has been shown by the above asymptotic representations, one singular solution vector is present at each singular point, the sphere surface and infinity. It will now be shown that a finiteness condition is necessary to complete the numerical formulation of this problem.

Consider first the algebraic singular solution at infinity, column two of solution matrix (4.54). Substitution of this vector into the drag integral (4.42) yields a solution for $\widetilde{N}_2(R)$ which grows like R near infinity. Therefore the condition (4.37) that $\widetilde{N}_2(x=1) = 0$ is equivalent to a finiteness condition at infinity, and the original six boundary conditions remain intact. An inadequacy of this set of boundary conditions is, however, revealed through inspection of the quadrature relation (4.45). Specifically, since $\widetilde{u}_2(x=1) = -1$, the integral must be bounded and the part of the integrand in square brackets must consequently vanish for large R. Thus,

$$\widetilde{w} - \widetilde{u}_- - \widetilde{v}_- = 2\,\widetilde{u}_2 + 2\,\widetilde{v}_2 \sim 0(1/R^2) \tag{4.55}$$

for each of the six general solution vectors. This coupling between $\tilde{u}_2$ and $\tilde{v}_2$ at infinity makes the two boundary conditions on the velocity parameters, (4.38) and (4.39), equivalent, and only five conditions remain for the sixth-order system.

A new condition must now be provided to uniquely determine the complete six-moment solution. The obvious choice is to require all parametric functions to be bounded at the sphere surface. It cannot be proven that this finiteness condition is sufficient since the solution vector which is singular at the sphere may be eliminated by one of the boundary conditions at infinity. Analytically connecting the asymptotic representations about separated singular points is a very difficult problem. However, it can be shown numerically that the singular solution is not eliminated by the five remaining original boundary conditions. The finiteness condition therefore completes the formulation of this problem.

The finiteness condition must now be stated in a form convenient for numerical integration. The three general solution vectors to the collisional third-order system at the sphere are the columns of the matrix (B.24). One of these vectors is the exponentially singular solution $\underline{\eta}_1$ previously discussed, and the other two are

$$\underline{\eta}_2 \sim \begin{Bmatrix} \frac{1}{12Kn} \\ 1 \\ 0 \end{Bmatrix} + O(x) , \tag{4.56}$$

and

$$\underline{\eta}_3 \sim \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} + O(x) . \tag{4.57}$$

Also found in Appendix B is a particular integral for the third-order inhomogeneous system. Since this particular integral vanishes at the sphere the complete solution for the vector $\underline{Y}$ is

$$\underline{Y} = C_4 \underline{\eta}_1 + C_5 \underline{\eta}_2 + C_6 \underline{\eta}_3 + 0(x) . \tag{4.58}$$

In this context the finiteness condition is simply $C_4 = 0$. Therefore, from (4.56) and (4.57)

$$\tilde{w} = Y_1 = \frac{1}{12Kn} C_5 = \frac{1}{12Kn} \tilde{u}_- \text{ at } x = 0 \tag{4.59}$$

and, using the other boundary conditions,

$$(1 + \frac{1}{12Kn})\tilde{u}_2 + \tilde{v}_2 = 0 \text{ at } x = 0 \tag{4.60}$$

is an equivalent form for the finiteness condition at the sphere surface.

The system of finite difference equations discussed at the beginning of this Section can now be solved. With the finiteness condition (4.60) replacing the redundant boundary condition (4.39) the band matrix inversion proceeds quickly and gives well behaved solutions for the six parametric functions. It is easier to invert the original sixth-order system than to work with the inhomogeneous third-order system and its attendant quadrature and integral relations. An additional benefit results since these integral relations then provide excellent checks on the accuracy of the numerical procedure, which was found to be quite good. The numerical values for the six parametric functions are tabulated in Appendix G for a representative collection of Knudsen numbers. The flow fields and integral properties of these moment solutions will be discussed in succeeding Sections.

## 4.6. Solution by Matching

The successful attempt in Section 3.5 to obtain the heat-transfer solution through a matching procedure suggests a similar approach for the more complex flow problem. For this flow problem the drag integral is constant throughout both regions instead of the heat-flux, and the velocity fields are matched instead of the temperatures.

The free-molecule solution for slow flow past a sphere is chosen as the inner solution for this matching procedure. This linearized collisionless solution is developed in Appendix E where the drag is found to be

$$D = \pi R_0^2 n_\infty U_0 \left( \frac{8mkT_\infty}{\pi} \right)^{\frac{1}{2}} \left( \frac{4}{3} + \frac{\pi}{6} \right). \tag{4.61}$$

In this expression $U_0$ represents a uniform flow at some point away from the sphere (not necessarily at $\infty$). The corresponding velocity fields are

$$u_R = -U_0 \cos\theta \, f_1(R) \tag{4.62}$$

and

$$u_\theta = U_0 \sin\theta \, f_2(R), \tag{4.63}$$

where

$$f_1(R) = \frac{3}{8} - \frac{R_0^2}{8R^2} - \frac{R_0^3}{4R^3} + \frac{1}{2}\left(1 - \frac{R_0^2}{R^2}\right)^{3/2} + \frac{R}{16R_0}\left(1 - \frac{R_0^2}{R^2}\right)^2 \log\left(\frac{R+R_0}{R-R_0}\right), \tag{4.64}$$

and

$$f_2(R) = \frac{5}{16} + \frac{R_0^2}{16R^2} + \frac{R_0^3}{8R^3} + \left(\frac{1}{2} + \frac{R_0^2}{4R^2}\right)\left(1 - \frac{R_0^2}{R^2}\right)^{\frac{1}{2}} + \frac{R}{32R_0}\left(3 - 2\frac{R_0^2}{R^2} - \frac{R_0^4}{R^4}\right)\log\left(\frac{R+R_0}{R-R_0}\right). \quad (4.65)$$

These velocity field functions are found by integration in Appendix E. Using Eq. (4.61) to eliminate $U_0$, a complete inner velocity field is determined as a function of the drag integral only.

Utilized in the region far from the sphere is a solution of Stokes equation which satisfies only the boundary condition at infinity:

$$u_R = - U_\infty \cos\theta(1 - A/R + B/R^3) ; \quad (4.66)$$

$$u_\theta = U_\infty \sin\theta (1 - \tfrac{1}{2}A/R - \tfrac{1}{2}B/R^3) . \quad (4.67)$$

The no slip conditions at the sphere which give the usual Stokes solution are not applied, thus leaving two free constants to be determined by the matching, A and B in the above expressions. Integration of the forces due to this flow pattern gives a relation between the drag and one of the constants,

$$D = 4\pi\mu U_\infty A. \quad (4.68)$$

These inner and outer solutions are each valid in different regions and the drag integral cannot be determined from either except by matching in an appropriate manner. The matching condition discussed earlier takes the vector form

$$\lim_{R\to\infty} \underline{u}_{inner} = \lim_{R\to R_0} \underline{u}_{outer} , \quad (4.69)$$

and provides values for both of the undetermined constants. Applying this condition to each component of the velocity field gives

$$U_\infty (1 - A/R_0 + B/R_0^3) = U_0 \tag{4.70}$$

$$U_\infty (1 - \tfrac{1}{2} A/R_0 - \tfrac{1}{2} B/R_0^3) = U_0 . \tag{4.71}$$

Subtraction of Eq. (4.71) from (4.70) then provides the relationship

$$B = \frac{1}{3} R_0^2 A . \tag{4.72}$$

Finally, substitution of Eqs. (4.61), (4.68) and (4.72) into either (4.70) or (4.71) yields an expression for the drag which is analagous to the heat-flux relation (3.32),

$$\frac{D}{D_c} = \frac{1}{1 + \dfrac{6\mu}{R_0 n_\infty \left(\dfrac{8mkT}{\pi}\right)^{\frac{1}{2}} \left(\dfrac{4}{3} + \dfrac{\pi}{6}\right)}} , \tag{4.73}$$

where

$$D_c = 6\pi R_0 \mu U_\infty . \tag{4.74}$$

Of course, $\mu$ may be replaced by $\lambda$ through the relation

$$\lambda = \frac{\mu}{\rho \left(\dfrac{2kT}{\pi m}\right)^{\frac{1}{2}}} \tag{4.75}$$

so the drag may be represented as a function of the Knudsen number,

$$\frac{D}{D_c} = \frac{1}{1 + \dfrac{Kn}{\left(\dfrac{2}{9} + \dfrac{\pi}{36}\right)}} . \tag{4.76}$$

The uniformly valid radial velocity field is found by subtracting the common part, $-U_\infty \cos\theta\,(1 - A/R_0 + B/R_0^3)$, from the outer solution and adding the remainder to the inner solution, giving

$$u_R = -U_\infty \cos\theta \left(\frac{2}{3R_0} - \frac{1}{R} + \frac{R_0^2}{3R^3}\right) A - U_0 \cos\theta f_1(R) , \qquad (4.77)$$

where account is taken of Eq. (4.72). Then, eliminating A and $U_0$ with the aid of equations (4.61), (4.68), (4.74) and (4.76), this expression becomes

$$u_R = -U_\infty \cos\theta \left[(1 - \delta)(1 - \frac{3R_0}{2R} + \frac{R_0^3}{2R^3}) + \delta f_1(R)\right] , \qquad (4.78)$$

where

$$\delta = \frac{1}{1 + \frac{\left(\frac{2}{9} + \frac{\pi}{36}\right)}{Kn}} . \qquad (4.79)$$

Similarly, the tangential velocity is found to be

$$u_\theta = U_\infty \sin\theta \left[(1 - \delta)(1 - \frac{3R_0}{4R} - \frac{R_0^3}{4R^3}) + \delta f_2(R)\right] . \qquad (4.80)$$

Inspection of these velocity functions reveals that the flow determined by this matching procedure is composed of two parts: one, identical to the free-molecule solution weighted by a function of Knudsen number; a second which is exactly the continuum Stokes solution also weighted by a function of Kn. This dual character is qualitatively very similar to the behavior found for the heat-transfer solution, although the rarefaction parameter $\delta$ is numerically different in the two problems. The velocity fields found here by matching methods will now be illustrated and compared with the numerical moment solutions obtained in the previous Section.

## 4.7. Velocity Field

Once the values of the six parametric functions are determined by the moment method it is possible to calculate all physical properties of the flow about the sphere. Of particular interest is the velocity distribution which is given explicitly by Eqs. (4.2) and (4.3) and the known parametric functions. The radial velocity is shown in Fig. 6 and the related tangential velocity in Fig. 7, both plotted as functions of $x^2 = 1 - R_0^2/R^2$. Both of these curves illustrate the dual character of the flow field for transitional values of the Knudsen number. Specifically, the inner portions of the transition velocities have the general shape of the upper free-molecule solution while the velocities far from the sphere have the character of the lower Stokes solution. The shear-stress is also shown in Fig. 8 as an example of the higher moments which can easily be found.

In the velocity plots the exact free-molecule solution given in Appendix E is shown as a dotted line, and the collisionless moment solution is not shown since it lies within one percent of the exact free-molecule solution at all points in the flow field. The other limiting solution could not be calculated exactly by the moment method as formulated, but the solution for Kn = .001 is virtually identical to the Stokes solution which is also shown as a dotted line.

Three representative tangential velocity fields which result from the matching procedure of the last Section are plotted in Fig. 9. These curves, taken directly from equation (4.80), are each compared with a moment solution for the same value of the Knudsen number. The plots resulting from these two methods appear to have nearly identical shapes, although the matched solutions are shifted slightly as if representing flows at somewhat different Knudsen number. However, the numerical differences are small and the near congruence of the two solutions is encouraging. The favorable comparison of the moment solutions with curves composed of linearly superimposed Stokes and free-molecule patterns is even stronger evidence of the dual character of the physical flow field as modeled by the moment equations. Plots of the radial velocity field show a very similar behavior.

No experimental values for any of the flow field variables are known for comparison with the distributions predicted. Also, no theoretical results providing transitional velocity details are known. However, a velocity field for the slip regime (low Knudsen number) was obtained originally by Basset (19) for low Reynolds number flows. Basset's solution may be reproduced in the present context very simply by applying appropriate boundary conditions to the general Stokes solution discussed in the last Section. The necessary conditions are

$$u_R = 0 \qquad \text{at} \qquad R = R_0 \tag{4.81}$$

and the Maxwell slip condition for diffuse re-emission

$$u_{gas} - u_{wall} = \left(\frac{2\mu}{\rho\bar{c}}\frac{du}{dy}\right)_{wall} . \tag{4.82}$$

Using the relation (4.75) and the expression for the mean velocity,

$$\bar{c} = (8kT/\pi m)^{\frac{1}{2}} , \tag{4.83}$$

the second boundary condition above becomes

$$u_\theta = \lambda\frac{du_\theta}{dR} \qquad \text{at} \qquad R = R_0 . \tag{4.84}$$

Applying these conditions to the velocity expressions (4.66) and (4.67), the velocity fields found by Basset become:

$$u_R = -U_\infty \cos\theta \left[ 1 - \frac{3}{2}\left(\frac{1 + 2Kn}{1 + 4Kn}\right)\frac{R_0}{R} + \frac{1}{2}\left(\frac{1-2Kn}{1+4Kn}\right)\frac{R_0^3}{R^3}\right]; \tag{4.85}$$

$$u_\theta = U_\infty \sin\theta \left[ 1 - \frac{3}{4}\left(\frac{1 + 2Kn}{1 + 4Kn}\right)\frac{R_0}{R} - \frac{1}{4}\left(\frac{1-2Kn}{1+4Kn}\right)\frac{R_0^3}{R^3}\right]. \tag{4.86}$$

The expression (4.68) for the drag of the general Stokes solution then gives

$$D = 6\pi\mu U_\infty R_0 \left(\frac{1 + 2Kn}{1 + 4Kn}\right) . \tag{4.87}$$

as the drag for the Basset slip solution.

Obviously, this slip solution is a reasonable approximation for low Knudsen numbers only. A plot of the tangential velocity (4.86) is presented in Fig. 9 for Kn = 0.05 along with the velocities found by matching and the moment method. The slip solution is qualitatively in error near the sphere where a Knudsen layer is evident in the other two solutions.

## 4.8. Sphere Drag and Comparison with Experiment

Although very few investigations of the velocity and pressure distributions exist, a number of results are available which give the drag integral for a slowly moving sphere in various flow regimes. A comparison of these results with the drag values found in this Chapter will now be presented.

The drag predicted by the moment method is easily found by integrating the normal stress and shear stress over the sphere surface. A convenient representation for this drag is the ratio of the drag coefficient to the coefficient for collisionless flow $C_D/C_{Dfm}$, which in terms of the normalized moments is

$$\frac{C_D}{C_{Dfm}} = \frac{-\tilde{P}^*_{RR}(Kn) + 2\,\tilde{P}^*_{R\theta}(Kn)}{-\tilde{P}^*_{RR}(\infty) + 2\,\tilde{P}^*_{R\theta}(\infty)} \quad . \tag{4.88}$$

A plot of this moment drag as a function of Knudsen number is presented in Fig. 10 along with the experimental values by Millikan. (14) Also shown is the drag ratio, expression (4.76), which results from the matched solution. The values from both the moment method and the matching method are seen to pass smoothly from the continuum to the free-molecule limit, but both fall somewhat below the experimental points. The quite good agreement between these two curves further suggests a relationship between the two methods, originally found in the heat-transfer problem where they gave identical results.

Some other theoretical results are also presented in Fig. 10, although all are valid only in limited Knudsen number regimes. The agreement with Goldberg's thirteen-moment solution (17) is excellent in the near continuum regime, but the thirteen-moment solution approaches the wrong limit for free-molecule flow. In the low density regime the Knudsen iteration curves of Willis (2) and Liu, et. al. (16) are seen to give better agreement with Millikan's values than the moment method.

# 5. SLOWLY ROTATING SPHERE

## 5.1. Description of the Problem

In this Chapter the sphere is again at rest in an infinite homogeneous gas of arbitrary density, but it is now spinning with an angular velocity $\underline{\omega}$ about an axis fixed in this space. The rotational speed is restricted to low values of the surface equatorial speed ratio $S_{eq}$, defined as $\omega R_0(m/2kT_\infty)^{\frac{1}{2}}$. Of particular interest in this problem is the flow which is of second-order in the surface Mach number. Consequently, the weighting function is expanded in this parameter and both first and second-order moment solutions are determined.

As before, the sphere is assumed to have a high thermal conductivity and the sphere temperature is taken equal to the gas temperature at infinity to eliminate all heat-flux other than that produced by the spinning sphere. Without this last assumption the presence of two perturbation parameters, $\omega R_0$ and $\Delta t$, would unnecessarily complicate the problem, especially the portion of the solution which is second-order in $\omega R_0$.

In determining the limiting solutions for the rotating sphere, the discussion of Section 4.1. remains relevant. In accord with this the moment solution should approach in the continuum limit the solution of Stokes equation about a rotating sphere, and the more complex high Reynolds number solutions should not be reproduced. The relatively simple first-order Stokes solution was obtained by Lamb (20) who also knew of the existence of the second-order "centrifugal fan" motion as did Stokes. However, the formulation and solution of this second-order problem remained for Bickley (21) and still later for

Collins, (22) both of whom used an expansion of the Navier-Stokes equation in powers of Reynolds number. This motion is characterized by a radial influx at both poles of the sphere which is balanced by an equal outflux at the equator.

As in the case of the translating sphere the collisionless limit is described by one Maxwellian distribution representing the stationary ambient conditions and another reflecting the effect of the sphere. Note that the velocity in the sphere influenced part of the distribution function depends on the point of the sphere from which the particle in question originates. The collisionless velocity fields are then determined by integration after expanding the sphere Maxwellian in powers of surface Mach number. The first-order velocity field is readily integrated and is described, for example, by Willis. (3) The second-order integration is more difficult because of the varying mean velocity within the sphere's cone of influence described above, and no known reference to this motion is available. This integration is presented in Appendix F for the velocity fields and some higher moments such as the stress tensor. The "centrifugal fan" motion is also found in the totally collisionless flow pattern which results from this integration. Further, the velocity fields are very nearly the same shape as in the continuum low Reynolds number solution, but of different magnitude.

Some excellent measurements by Lord, Bowden and Harbour (23), (24), (25), (26) of the drag torque on a revolving sphere in various Knudsen number regimes constitute nearly the entire experimental effort devoted to this problem. No measurements of the velocity field for either the first or second-order motion are known. This is not surprising considering the very small induced secondary flow, particularly in the low density regime.

Existing theoretical studies of the rotating sphere are also quite limited. Willis (3) has investigated the first-order motion by means of a Knudsen iteration technique through which he determined the drag torque in the near free-molecule regime. In the same report Willis compares the first-order torque found by the moment method with proposed interpolation formulas and the above data by Lord and Harbour. No investigations of the second-order velocity patterns are available other than the continuum solutions by Bickley and Collins previously mentioned. Consequently, the second-order moment solution obtained in the following Sections cannot be compared with anything but the known limiting solutions for large and small Knudsen number.

## 5.2. First-Order Moment Solution

The first-order moment solution was originally presented by Willis (3) for the case of a slowly rotating sphere within a fixed concentric sphere of larger diameter. This solution is quite simple and since it provides the basis for the second-order development to follow, it will be repeated here.

In order to provide sufficient freedom in the formulation of the second-order moment equations a general weighting function must be chosen to represent the true distribution function. The "two-sided Maxwellian" described in the introduction, Eqs. (1a) and (1b), consists of ten parametric functions and is appropriate for this purpose.

The linearized form of this weighting function is given by Eq. (2.12)

Although it will not be shown in detail, the substitution of the weighting function (2.12) into the moment equations results in the total separation of the two tangential velocity parameters, $w_1$ and $w_2$, from other parametric functions. Specifically, the tangential momentum equation (2.24) and the shear stress moment equation (2.29) contain only the w's while the remaining equations involve only the other eight parameters. A great simplification now follows from the homogeneous boundary conditions appropriate to the eight functions, $N_1$, $N_2$, $t_1$, $t_2$, $u_1$, $u_2$, $v_1$, $v_2$. The only solution satisfying these conditions is the trivial solution with all eight parameters identically zero. The problem is in reality a two moment system for the parameters, $w_1$ and $w_2$, which will now be integrated to complete the first-order solution.

The reduced form of the first-order weighting function is

$$f_1 = f_\infty \, [1+2\beta_0^2 \xi_\varphi w_1]$$

and (5.1)

$$f_2 = f_\infty \, [1 + 2\beta_0^2 \xi_\varphi w_2] \; .$$

The mean quantities which appear in the $\varphi$ momentum equation (2.24) and the shear stress equation (2.29) can now be integrated using Table 4 of Appendix A. Nondimensionalizing all quantities by the expressions (3.3), the relevant moments are

$$\overline{P^*_{R\varphi}} = - \frac{1}{2\pi} (1 - x^2)^2 \; \overline{w}_- , \tag{5.2}$$

$$\overline{P^*_{\theta\varphi}} = 0 , \tag{5.3}$$

$$\overline{P^*_{RR\varphi}} = 1/2\,\overline{w}_+ - (5/4\,x^3 - 3/4\,x^5)\overline{w}_-, \tag{5.4}$$

$$\overline{P^*_{R\theta\varphi}} = 0, \tag{5.5}$$

$$\overline{P^*_{\theta\theta\varphi}} = 1/2\,\overline{w}_+ - (15/16\,x - 5/8\,x^3 + 3/16\,x^5)\overline{w}_-, \tag{5.6}$$

$$\overline{P^*_{\varphi\varphi\varphi}} = 3\,\overline{P^*_{\theta\theta\varphi}}\;. \tag{5.7}$$

As a consequence of expressions (5.3) and (5.5) above the partial differential moment equations become ordinary differential equations and the velocity parameters may be written as

$$\overline{w}_+ = \tilde{w}_+(R)g(\theta),$$

and (5.8)

$$\overline{w}_- = \tilde{w}_-(R)g(\theta),$$

where $g(\theta)$ is an arbitrary function of $\theta$ which will be determined later through application of the appropriate boundary conditions.

In a nondimensional form the system of equations for the first order solution is

$$(3 + R\frac{d}{dR})\tilde{P}^*_{R\varphi} = 0, \tag{5.9}$$

and

$$(3 + R\frac{d}{dR})\tilde{P}^*_{RR\varphi} - \tilde{P}^*_{\theta\theta\varphi} - \tilde{P}^*_{\varphi\varphi\varphi} = \frac{\pi R}{\lambda_\infty}\tilde{p}_{R\varphi} = \frac{\pi R}{\lambda_\infty}\tilde{P}^*_{R\varphi}\,. \tag{5.10}$$

The first of these equations may be immediately integrated to give

$$\tilde{P}^*_{R\varphi} = C_1/\overline{R}^3, \tag{5.11}$$

from which

$$\tilde{w}_- = -2\pi\overline{R}\,C_1. \tag{5.12}$$

Finally, using the moment integrals and the above solution for $\tilde{w}_-$, the shear stress equation (5.10) reduces to the simple form:

$$\frac{d}{d\bar{R}}(\tilde{w}_+/\bar{R}) = \frac{\pi C_1}{Kn\bar{R}^4} , \tag{5.13}$$

which upon integration yields

$$\tilde{w}_+ = C_2\bar{R} - \frac{\pi C_1}{3Kn\bar{R}^2} . \tag{5.14}$$

This completes the first order solution except for the application of boundary conditions, which for this problem are

$$w_1 = \tfrac{1}{2}(w_+ + w_-) = \omega R_0 \sin\theta \quad \text{at} \quad \bar{R} = 1 , \tag{5.15}$$

and

$$w_2 = \tfrac{1}{2}(w_+ - w_-) = 0 \quad \text{at} \quad \bar{R} = \infty . \tag{5.16}$$

The integration constants can now be determined with the aid of these conditions to be

$$C_1 = -\frac{\overline{\omega R_0}}{2\pi}\left(\frac{1}{1+\frac{1}{12Kn}}\right) \tag{5.17}$$

and

$$C_2 = \overline{\omega R_0}\left(\frac{1}{1+\frac{1}{12Kn}}\right) . \tag{5.18}$$

The $\theta$ dependence is now also specified to be

$$g(\theta) = \sin\theta . \tag{5.19}$$

The final solution to the first order moment equations is thus

$$w_1 = \left(\bar{R} + \frac{1}{12Kn\bar{R}^2}\right)\left(\frac{\omega R_0 \sin\theta}{1+\frac{1}{12Kn}}\right) , \tag{5.20}$$

and

$$w_2 = \frac{1}{12Kn\overline{R}^2}\left(\frac{\omega R_0 \sin\theta}{1+\frac{1}{12Kn}}\right) .$$

The mean quantities of interest are the tangential velocity and the shear stress which determines the drag torque on the sphere. The mean velocity, as specified by a particular integral of the weighting function (5.1), is

$$u_\varphi = 1/2\ w_+ - (3/4\ x - 1/4\ x^3)\,w_- , \tag{5.21}$$

which becomes

$$u_\varphi = \left[\frac{R_0^3}{12KnR^3} + 1/2 - (1/2 + \frac{R_0^2}{4R^2})(1 - R_0^2/R^2)^{\frac{1}{2}}\right] \left(\frac{\omega R \sin\theta}{1+\frac{1}{12Kn}}\right) . \tag{5.22}$$

The shear stress moment, Eq. (5.2), becomes

$$P_{R\varphi} = -\frac{n_\infty kT_\infty}{2\pi R^3}\left(\frac{\omega R_0^4 \sin\theta}{1+\frac{1}{12Kn}}\right) . \tag{5.23}$$

Both the tangential velocity and the shear stress approach the proper values in the limits of large and small Knudsen number.

The drag torque on the sphere is simply an integral of expression (5.23) over the surface of the sphere. A convenient representation of the drag dependence on density is the ratio

$$D/D_c = \frac{1}{1 + 12Kn} , \tag{5.24}$$

where the subscript c indicates the continuum limit value. This simple Knudsen number dependence is identical to a general interpolation formula suggested by Sherman. (27)

One comment is in order concerning the above moment solution in which $w_1$ (5.20) is found to grow like R at large distances from the sphere. Initially, this behavior appears to contradict the earlier low Mach number assumption which permitted the expansion of the exponential in the weighting function. However, this growth is due to a geometrical effect reflecting the diminishing range of the $\xi_\varphi$ integration far from the sphere. The consequence of this is that the mean $\varphi$ velocity of the weighting function is not represented directly by $w_1$. For example, consider the expansion of the exact collisionless distribution function performed in Appendix F. There it is shown that to first order in surface Mach number

$$f_{FM} = f_\infty \left[ 1 + 2\beta_0^2 \underline{\xi} \cdot (\underline{\omega} \times \underline{R_0}) \right] = f_\infty \left[ 1 + 2\beta_0^2 \xi_\varphi \omega R \sin\theta \right] . \qquad (5.25)$$

Note that in the last expression $\omega R \sin\theta$, which corresponds to $w_1$ in the moment formulation, is unbounded even though the magnitude of the mean surface velocity $(\underline{\omega} \times \underline{R_0})$ is always much less than the mean molecular speed.

## 5.3. Second-Order Weighting Function and Mean Quantities

A ten-parameter second-order weighting function follows naturally from an expansion of the "two-sided Maxwellian" described in the Introduction. To effect this expansion, the parametric functions are assumed to have the following form:

$$n_i = n_\infty (1 + N_i^{(2)} + \ldots ) ;$$

$$T_i = T_\infty (1 + t_i^{(2)} + \ldots ) ;$$

$$u_i = u_i^{(2)} + \ldots ;$$
$$v_i = v_i^{(2)} + \ldots ; \qquad (5.26)$$
$$w_i = w_i^{(1)} + w_i^{(2)} + \ldots ,$$

where account is taken of the first-order solution found in the previous Section. All quantities with bracketed superscripts are of order $(\varepsilon^n)$ where $\varepsilon$ is the perturbation parameter, the equatorial surface Mach number in this case. With these parameters inserted into Eq. (1.1a) and (1.1b) the order $(\varepsilon^2)$ part of the weighting function becomes

$$f_i^{(2)} = f_\infty \Big[ N_i^{(2)} - 3/2\, t_i^{(2)} - \beta_0^2 w_i^{(1)2} + 2\beta_0^2 u_i^{(2)} \xi_R + 2\beta_0^2 v_i^{(2)} \xi_\theta + 2\beta_0^2 w_i^{(2)} \xi_\varphi + 2\beta_0^4 w_i^{(1)2} \xi_\varphi^2 + \beta_0^2 t_i^{(2)} \xi^2 \Big] \qquad (5.27)$$

where i = 1, 2.

The moment integrals appearing in the moment equations must now be determined using the above second-order weighting function. To simplify notation in the ensuing development of the second-order solution the new quantities,

$$\Gamma_+ = \bar{w}_1^{(1)2} + \bar{w}_2^{(1)2}$$

and (5.28)

$$\Gamma_- = \bar{w}_1^{(1)2} - \bar{w}_2^{(1)2} ,$$

are introduced. With the only first-order parameters thus eliminated, the bracketed superscripts may be dropped and the remaining parameters will henceforth be understood to be of second-order. Again using Table 4 of Appendix A and non-dimensionalizing all quantities by the expressions (3. 3), the necessary second-order moments are as follows:

$$\overline{nu_R}^{(2)} = (1-x^2)(N_- + 1/2\, t_-) + 1/2\, \overline{u}_+ - 1/2\, x^3\, \overline{u}_- - \frac{x^2}{4\pi}(1 - x^2)\Gamma_- \; ; \tag{5.29}$$

$$\overline{nu_\theta}^{(2)} = 1/2\, \overline{v}_+ - (3/4\, x - 1/4\, x^3)\overline{v}_- \; ; \tag{5.30}$$

$$\overline{nu_\varphi}^{(2)} = 1/2\, \overline{w}_+ - (3/4\, x - 1/4\, x^3)\overline{w}_- \; ; \tag{5.31}$$

$$\overline{P^*_{RR}}^{(2)} = -1/2\, N_+ + 1/2\, x^3 N_- - 1/2\, t_+ + 1/2\, x^3 t_- - \frac{1}{\pi}(1 - x^4)\overline{u}_- + \frac{3x^3}{16\pi}(1 - x^2)\Gamma_- \; ; \tag{5.32}$$

$$\overline{P^*_{\theta\theta}}^{(2)} = -1/2\,(N_+ + t_+) + (3/4\, x - 1/4\, x^3)(N_- + t_-) - \frac{1}{2\pi}(1 - x^2)^2\overline{u}_- + \frac{3x}{64\pi}(1 - x^2)^2\,\Gamma_- \; ; \tag{5.33}$$

$$\overline{P^*_{\varphi\varphi}}^{(2)} = -1/2\,(N_+ + t_+) + (3/4\, x - 1/4\, x^3)(N_- + t_-) - \frac{1}{2\pi}(1 - x^2)^2\,\overline{u}_- - \frac{1}{4\pi}\Gamma_+ + \frac{1}{64\pi}(33x - 26x^3 + 9x^5)\Gamma_- ; \tag{5.34}$$

$$\overline{P^*_{R\theta}}^{(2)} = -\frac{1}{2\pi}(1 - x^2)^2\overline{v}_- \; ; \tag{5.35}$$

$$\overline{P^*_{R\varphi}}^{(2)} = -\frac{1}{2\pi}(1 - x^2)^2\overline{w}_- \; ; \tag{5.36}$$

$$\overline{\dot{q}^*_R}^{(2)} = (1 - x^2)(2N_- + 3t_-) + 5/4\, \overline{u}_+ - 5/4\, x^3\overline{u}_- + \frac{1}{4\pi}(1 - 4x^2 + 3x^4)\Gamma_- \; ; \tag{5.37}$$

$$\overline{\dot{q}^*_\theta}^{(2)} = 5/4\, \overline{v}_+ - (15/8\, x - 5/8\, x^3)\overline{v}_- \; ; \tag{5.38}$$

$$\overline{\dot{q}_R}^{(2)} = (1-x^2)(7/4\ t_- - 1/2\ N_-) + \frac{1}{4\pi}(1-3/2\ x^2 + 1/2\ x^4)\Gamma_- ; \qquad (5.39)$$

$$\overline{P^*_{RRR}}^{(2)} = (1-x^4)(2N_- + 3t_-) + 3/2\,\overline{u}_+ - 3/2\,x^5\,\overline{u}_- - \frac{1}{\pi}(1-x^2)x^4\Gamma_- ; \qquad (5.40)$$

$$\overline{P^*_{RR\theta}}^{(2)} = 1/2\ \overline{v}_+ - (5/4\ x^3 - 3/4\ x^5)\overline{v}_- ; \qquad (5.41)$$

$$\overline{P^*_{RR\varphi}}^{(2)} = 1/2\ \overline{w}_+ - (5/4\ x^3 - 3/4\ x^5)\overline{w}_- ; \qquad (5.42)$$

$$\overline{P^*_{R\theta\theta}}^{(2)} = (1 - x^2)^2 (N_- + 3/2\ t_-) + 1/2\ \overline{u}_+$$

$$- (5/4\ x^3 - 3/4\ x^5)\overline{u}_- - \frac{x^2}{4\pi}(1 - x^2)^2\ \Gamma_- ; \qquad (5.43)$$

$$\overline{P^*_{R\varphi\varphi}}^{(2)} = (1 - x^2)^2 (N_- + 3/2\ t_-) + 1/2\ \overline{u}_+$$

$$- (5/4\ x^3 - 3/4\ x^5)\overline{u}_- + \frac{1}{4\pi}(2 - 3x^2)(1 - x^2)^2\ \Gamma_- ; \qquad (5.44)$$

$$\overline{P^*_{\theta\varphi\varphi}}^{(2)} = 1/2\ \overline{v}_+ - (15/16\ x - 5/8\ x^3 + 3/16\ x^5)\overline{v}_- ; \qquad (5.45)$$

$$\overline{P^*_{\theta\theta\theta}}^{(2)} = 3\ \overline{P^*_{\theta\varphi\varphi}} ; \qquad (5.46)$$

$$\overline{P^*_{\theta\theta\varphi}}^{(2)} = 1/2\ \overline{w}_+ - (15/16\ x - 5/8\ x^3 + 3/16\ x^5)\overline{w}_- ; \qquad (5.47)$$

$$\overline{P^*_{\varphi\varphi\varphi}}^{(2)} = 3\ \overline{P^*_{\theta\theta\varphi}} ; \qquad (5.48)$$

$$\overline{P^*_{RRii}}^{(2)} = 5\pi(N_+ + 2t_+) - 5\pi x^3(N_- + 2t_-) + 12(1 - x^4)\overline{u}_-$$

$$+ 1/2\ \Gamma_+ - (25/8\ x^3 - 21/8\ x^5)\Gamma_- ; \qquad (5.49)$$

$$\overline{P^*_{\theta\theta ii}}^{(2)} = 5\pi(N_+ + 2t_+) - \frac{5\pi}{2}(3x - x^3)(N_- + 2t_-) + 6(1 - x^2)^2\,\overline{u}_-$$

$$+ 1/2\ \Gamma_+ - (45/32\ x - 25/16\ x^3 + 21/32\ x^5)\Gamma_- ; \qquad (5.50)$$

$$\overline{P^{*}_{\varphi\varphi ii}}^{(2)} = 5\pi(N_{+} + 2t_{+}) - \frac{5\pi}{2}(3x - x^{3})(N_{-} + 2t_{-}) + 6(1 - x^{2})^{2}\,\overline{u}_{-}$$

$$+ 4\Gamma_{+} - (255/32\, x - 95/16\, x^{3} + 63/32\, x^{5})\Gamma_{-}; \qquad (5.51)$$

$$\overline{P^{*}_{R\theta ii}}^{(2)} = 6(1 - x^{2})^{2}\,\overline{v}_{-} \quad . \qquad (5.52)$$

## 5.4. Differential Equations

Clearly, full utilization of the ten-parameter weighting function just described requires a system of ten partial differential moment equations. The ten moment equations which were collected in Section 2.3. represent a complete sequence of moments through second order in $\xi$, plus one third order moment, $Q = \frac{1}{2} m\, \xi_R\, \xi^2$, and provide a natural initial system with which to attempt a solution of this problem. The equations missing in this sequence are linear combinations of the ten given equations, a consequence of the physical symmetries present in the stress tensor, and thus provide no useful information.

Unfortunately, this ten-moment formulation is improperly posed in the sense described in Section 2.5. and cannot be solved to satisfy ten arbitrary boundary conditions. To demonstrate that the linear transformation from parametric to canonical variables is singular it is necessary to consider only the three canonical variables $\overline{\dot{q}^{*}_{R}}^{(2)}$, $\overline{P^{*}_{RRR}}^{(2)}$ and $\overline{P^{*}_{R\theta\theta}}^{(2)}$ associated respectively with the moment equations (2.25), 2.26) and (2.27). Using the integral definitions of these variables provided by expressions (5.37), (5.40) and (5.43) it is easy to verify that

$$2\,\overline{q^{*}_{R}}^{(2)} - \overline{P^{*}_{RRR}}^{(2)} - 2\,\overline{P^{*}_{R\theta\theta}}^{(2)} = \frac{\Gamma_{-}}{2\pi R^{6}} \qquad (5.53)$$

A relationship thus exists between these three canonical variables, and appropriate manipulations of the associated moment equations will produce a row of zeros in the transformation matrix $\underline{\underline{A}}$. Fundamentally, this reflects the coupling between $P^*_{R\theta\theta}$ and $P^*_{R\varphi\varphi}$ through symmetries in the chosen weighting function.

A mathematically interesting possibility is that the ten natural boundary conditions might be satisfied by a solution containing fewer integration constants. This conjecture is not pursued now since the complete general solution, which is difficult to obtain, would be required for verification and other avenues of approach are open.

An alternative formulation which provides for a nonsingular transformation to canonical variables everywhere except at $\infty$ will now be described. Unless the form of the weighting function is changed quite radically, perhaps by introducing an anisotropic temperature parameter, the coupling among the canonical variables of Eqs. (2. 25), (2. 26) and (2. 27) will remain and the transformation matrix will be singular throughout physical space. Another approach is the replacement of one of these three equations by a higher moment equation, which avoids the difficulty by the employment of a new canonical variable. One such possibility is the tangential heat-flux equation, but its canonical variable $P^*_{R\theta ii}$ is directly related to $P^*_{R\theta}$ of the tangential momentum equation indicating that another singular system would result. Although higher order moment equations undoubtedly exist which satisfy the nonsingular criterion of Section 2. 5., all involve more complicated expressions for the collision terms than those previously given. For this reason a ten moment formulation offering reasonable hope of solution appears unlikely and efforts are redirected to a system containing fewer parametric functions and moment equations. This simplification is very analagous to the elimination of the temperature parameters which was carried out in Chapter 4 for the first-order solution of flow past a sphere.

It has been found generally desirable to maintain the symmetry of the weighting function by incorporating even numbers of parametric functions. Therefore, in the present example eight parameters and eight moment equations are used, which fluid dynamically corresponds to assigning $u_R$, $u_\theta$, $u_\varphi$, $P_{R\theta}$ $P_{R\varphi}$ $P_{RR}$, $\rho$, and T as independent variables. The eight moment equations are Continuity, Radial Momentum, $\theta$ Momentum, $\varphi$ Momentum, Energy, $R\theta$ Shear Stress, $R\varphi$ Shear Stress and Radial Heat-flux. Eliminated are the two normal stress equations, (2.26) and (2.27).

## 5.5. Boundary Conditions

Once again a singular point of the moment system at infinity presents possible complications in the specification of appropriate boundary conditions. However, a set of necessary conditions is readily available from Section 2.4. where diffuse re-emission and complete energy accomodation were assumed at the sphere surface. With these assumptions all temperature perturbations must vanish and, since the sphere motion is entirely accomodated by the first-order velocity solutions, all higher order velocities must satisfy homogeneous boundary conditions at the sphere. The surface conditions on the second-order parameters are therefore

$$t_1(x) = 0 \quad \text{at} \quad x = 0 \quad , \tag{5.54}$$

$$\bar{u}_1(x) = 0 \quad \text{at} \quad x = 0 \quad , \tag{5.55}$$

$$\bar{v}_1(x) = 0 \quad \text{at} \quad x = 0 \quad , \tag{5.56}$$

and

$$\bar{w}_1(x) = 0 \quad \text{at} \quad x = 0 \quad . \tag{5.57}$$

Completing the conditions at the sphere is the specification of no net mass-flux,

$$\overline{nu_R}^{(2)}(x) = 0 \quad \text{at} \quad x = 0 , \tag{5.58}$$

which with Eq. (5.29) becomes

$$N_-(x) + 1/2\; t_-(x) + 1/2\; \bar{u}_+(x) = 0 \quad \text{at} \quad x = 0. \tag{5.59}$$

The remaining conditions result from forcing the weighting function to approach a Maxwellian distribution at rest at infinity. These conditions on the second-order parametric functions are

$$\bar{u}_2(x) = 0 \quad \text{at} \quad x = 1 , \tag{5.60}$$

$$\bar{v}_2(x) = 0 \quad \text{at} \quad x = 1 , \tag{5.61}$$

$$\bar{w}_2(x) = 0 \quad \text{at} \quad x = 1 , \tag{5.62}$$

$$N_2(x) = 0 \quad \text{at} \quad x = 1 , \tag{5.63}$$

and

$$t_2(x) = 0 \quad \text{at} \quad x = 1 . \tag{5.64}$$

Note that ten conditions are given since the specific form of the eight-parameter weighting function is still undetermined.

5.6. Moment Solution

The eight-moment formulation of the second-order flow about a rotating sphere does not involve a singular transformation to canonical variables provided that $v_+$, $v_-$, $w_+$, $w_-$ and any four of the remaining six parametric functions are retained. Since the choice of which four additional parameters to incorporate is arbitrary at this point, all will be carried until the detailed specification is required. This choice is dictated more by ease of solution than by any fundamental property of the resulting system. In fact, solutions having small numerical variations may be obtained using different forms of the weighting function, a result which is typical of moment methods. The solutions now to be described are found in part by analytical methods but must be completed by a numerical integration.

An immediate simplification results from substituting the integral definitions, (5.29) to (5.52), into the $\varphi$ momentum and shear moment equations, which are expressed in nondimensional form by Eqs. (5.9) and (5.10). Since the relevant moment integrals are identical to the first-order expressions the resulting solutions for the second-order parameters $w_+^{(2)}$ and $w_-^{(2)}$ are also the same as the first-order solutions given by Eqs. (5.12) and (5.14). However, only the trivial solution,

$$w_1^{(2)} = 0 \qquad \text{and} \qquad w_2^{(2)} = 0 \quad , \tag{5.65}$$

satisfies the homogeneous second-order boundary conditions, which indicates that no secondary azimuthal motion exists. Also, the second-order drag torque is zero for all Knudsen number regimes, a result which agrees with both the continuum and free-molecule limiting solutions. It is therefore necessary to solve the third-order moment system to obtain corrections to the simple drag torque relation (5.24) of Section 5.2.

In the other examples considered the partial differential moment equations have been reduced to ordinary differential equations by assuming appropriate $\theta$-dependence for the parametric variables. The separation has always been reasonably obvious from flow symmetries or boundary conditions, but in this case the situation is not nearly so clear. Of course, a general expansion in some orthogonal basis of the $\theta$-dependence of all parameters must work, but this involves more effort than is necessary if the proper separation can be guessed. Fortunately, a relatively simple $\theta$-dependence is appropriate and its development will now be described.

The initial indications of the $\theta$-dependence were obtained from the velocity fields of the known limiting solutions. The radial velocity has a $(3 \cos^2\theta - 1)$ dependence and the tangential velocity behaves as $\sin\theta \cos\theta$ in both the continuum and free-molecule solutions. It was also believed reasonable that the thermodynamic part of the flow, temperature and density, would be driven entirely by the first-order velocity field which has $\sin\theta$ behavior, and the N's and t's were assumed to have $\sin^2\theta$ dependence. However, although they permitted separation of the equations, these assumptions were overly restrictive and the resulting solutions could not satisfy the required boundary conditions.

The clue for correcting the assumed $\theta$-dependence is provided by the second-order free-molecule distribution function of Appendix F, which may be represented by the general form,

$$f_{FM}^{(2)} = G_1(\underline{\xi})\sin^2\theta + G_2(\underline{\xi})(\sin^2\theta\cos^2\tau - \cos^2\theta) + G_3(\underline{\xi})\cos\theta\sin\theta\cos\tau, \tag{5.66}$$

where the G's are not functions of $\tau$ (see Fig. 2). Thus all moment integrals involving odd powers of $\xi_\theta = \xi\cos\sigma\cos\tau$ have only $\sin\theta\cos\theta$ dependence as was assumed ($\tau$ is integrated from 0 to $2\pi$). However, all other moments have in general two components which behave as $\sin^2\theta$ and $\cos^2\theta$, not just $(3\cos^2\theta - 1)$ as does the radial velocity. Consequently, one appropriate though not unique separation is as follows:

$$N_+(\underline{R}) = \tilde{N}_+(R)\sin^2\theta + \hat{N}_+(R)(3\cos^2\theta - 1) \;; \tag{5.67}$$

$$N_-(\underline{R}) = \tilde{N}_-(R)\sin^2\theta + \hat{N}_-(R)(3\cos^2\theta - 1) \;; \tag{5.68}$$

$$t_+(\underline{R}) = \tilde{t}_+(R)\sin^2\theta + \hat{t}_+(R)(3\cos^2\theta - 1) \;; \tag{5.69}$$

$$t_-(\underline{R}) = \tilde{t}_-(R)\sin^2\theta + \hat{t}_-(R)(3\cos^2\theta - 1) \;; \tag{5.70}$$

$$\overline{u}_+(\underline{R}) = \tilde{u}_+(R)\sin^2\theta + \hat{u}_+(R)(3\cos^2\theta - 1) \;; \tag{5.71}$$

$$\overline{u}_-(\underline{R}) = \tilde{u}_-(R)\sin^2\theta + \hat{u}_-(R)(3\cos^2\theta - 1) \;; \tag{5.72}$$

$$\overline{v}_+(\underline{R}) = \hat{v}_+(R)\sin\theta\cos\theta \;; \tag{5.73}$$

$$\overline{v}_-(\underline{R}) = \hat{v}_-(R)\sin\theta\cos\theta \,, \tag{5.74}$$

with similar behavior for the moment integrals. This separation is also appropriate for the two equations dropped from the original system of ten moment equations.

The remaining six partial differential moment equations are now collected using the above superscript notation to indicate the appropriate $\theta$-dependence of the moment integrals. The resulting ten nondimensional ordinary differential equations obtained through separation of these moment equations are as follows:

Continuity, $\sin^2\theta$:

$$(2 + R\frac{d}{dR})\, \widetilde{nu}_R = 0 \quad ; \tag{5.75}$$

Continuity, $3\cos^2\theta - 1$:

$$(2 + R\frac{d}{dR})\, \hat{nu}_R + \widehat{nu}_\theta = 0 \quad ; \tag{5.76}$$

R momentum, $\sin^2\theta$:

$$(2 + R\frac{d}{dR})\, \widetilde{P}^*_{RR} - \widetilde{P}^*_{\theta\theta} - \widetilde{P}^*_{\varphi\varphi} = 0 \quad ; \tag{5.77}$$

R momentum, $3\cos^2\theta - 1$:

$$(2 + R\frac{d}{dR})\, \hat{P}^*_{RR} + \widehat{P}^*_{R\theta} - \hat{P}^*_{\theta\theta} - \hat{P}^*_{\varphi\varphi} = 0 \quad ; \tag{5.78}$$

$\theta$ momentum, $\sin\theta\cos\theta$ :

$$(3 + R\frac{d}{dR})\, \widehat{P}^*_{R\theta} + 3\,\widetilde{P}^*_{\theta\theta} - \widetilde{P}^*_{\varphi\varphi} - 6\,\hat{P}^*_{\theta\theta} = 0 \quad ; \tag{5.79}$$

Energy, $\sin^2\theta$ :

$$(2 + R\frac{d}{dR})\, \widetilde{\dot{q}}^*_R = 0 \quad ; \tag{5.80}$$

Energy, $3\cos^2\theta - 1$ :

$$(2 + R\frac{d}{dR})\,\hat{\dot{q}}^*_R + \hat{\dot{q}}^*_\theta = 0 \quad ; \tag{5.81}$$

Shear stress, $\sin\theta\cos\theta$ :

$$(3 + R\frac{d}{dR})\,\widehat{P}^*_{RR\theta} - \widehat{P}^*_{\theta\theta\theta} - \widehat{P}^*_{\theta\varphi\varphi} + 3\,\widetilde{P}^*_{R\theta\theta} - \widetilde{P}^*_{R\varphi\varphi} - 6\,\hat{P}^*_{R\theta\theta} = \frac{\pi R}{\lambda_\infty}\,\widehat{p}_{R\theta}; \tag{5.82}$$

R heat-flux, $\sin^2\theta$ :

$$(2 + R\frac{d}{dR})\,\widetilde{P}^*_{RRii} - \widetilde{P}^*_{\theta\theta ii} - \widetilde{P}^*_{\varphi\varphi ii} = \frac{\pi R}{\lambda_\infty}(p^{(1)}_{R\varphi}\,u^{(1)}_\varphi - 2/3\,\widetilde{\dot{q}}_R); \tag{5.83}$$

R heat-flux, $3\cos^2\theta - 1$ :

$$(2 + R\frac{d}{dR})\hat{P}^*_{RRii} - \hat{P}^*_{\theta\theta ii} - \hat{P}^*_{\varphi\varphi ii} + \widehat{P}^*_{R\theta ii} = -\frac{2\pi R}{3\lambda_\infty}\,\hat{\dot{q}}_R\,. \tag{5.84}$$

The exact form of the reduced weighting function must now be specified to provide a closed system of ordinary differential equations. One possibility is obtained by eliminating the two temperature parameters from expression (5.27) as was done for flow past a sphere. This function then provides a system of ten ordinary differential equations governing the ten remaining parameters on the right hand side of expressions (5.67) to (5.74). However, because of a singular point at infinity, the solution to this system is not uniquely determined by application of the boundary conditions of Section 5.5. One integration constant remains which must be determined through some specification of the asymptotic nature of $\widetilde{u}_2(R)$ at infinity. Investigation of this condition is not pursued since an appropriate value for the free constant makes this solution nearly identical to one provided by a slightly different formulation now to be described.

Consider now a weighting function with the parameters $\hat{t}_+(R)$, $\hat{t}_-(R)$, $\tilde{u}_+(R)$ and $\tilde{u}_-(R)$ of Eqs. (5.69) to (5.72) set equal to zero. Eliminating $\tilde{u}_2(R)$ removes the above singular behavior and allows a unique solution. The resulting $\theta$-dependence of the parametric functions is very much as originally thought with the exception of the two density parameters which now have an additional $(3\cos^2\theta - 1)$ dependence. This addition is necessary to satisfy the mass-flux boundary condition (5.59) which is actually two conditions, one on $\widetilde{nu}_R$ and one on $\widehat{nu}_R$.

The final formulation is now complete and involves a system of ten ordinary differential equations, (5.75) to (5.84), governing the ten parametric functions, $\tilde{N}_+$, $\tilde{N}_-$, $\hat{N}_+$, $\hat{N}_-$, $\tilde{t}_+$, $\tilde{t}_-$, $\hat{u}_+$, $\hat{u}_-$, $\hat{v}_+$ and $\hat{v}_-$. It may appear that the above ordinary differential system derives from a six moment partial differential system containing eight parametric functions. This apparent contradiction arises from the desire to label the weighting function parameters with a physically meaningful notation. In other words, there exist two implied relations among the eight parameters, which result from the assumed $\theta$-dependence, and the weighting function actually contains only six free parametric functions at this point. As is typical of moment methods, the final weighting function is itself the ultimate justification of the entire procedure. A weighting function will now be developed which satisfies the originally prescribed eight moment equations and the associated boundary conditions.

The ten ordinary differential equations must now be integrated to complete the eight moment solution for the rotating sphere. A great simplification is available in that these ten equations are separable and may be integrated as a succession of lower order systems. Specifically, the four equations exhibiting $\sin^2\theta$ behavior contain only four parametric functions and can be integrated independently of the other equations. The $\sin^2\theta$ part of the continuity equation (5.75) is directly integrable to yield

$$\widetilde{nu}_R = C_1/\bar{R}^2 , \tag{5.85}$$

which with Eq. (5.29) becomes

$$\widetilde{N}_- + 1/2\,\widetilde{t}_- - \frac{x^2}{4\pi}\,\Gamma_- = C_1 . \tag{5.86}$$

Similarly, from the energy equation (5.80) and the heat-flux definition (5.37) it follows that

$$2\,\widetilde{N}_- + 3\,\widetilde{t}_- + \frac{1}{4\pi}(1 - 3x^2)\Gamma_- = C_2 . \tag{5.87}$$

The intermixing of independent variables R and $x^2 = 1 - R_0^2/R^2$ is somewhat unfortunate, but this seems preferable to the proliferation of radical expressions in the following discussion. Since

$$\Gamma_- = \bar{R}^{-2}\left(1 + \frac{1}{6Kn\bar{R}^3}\right)\gamma , \tag{5.88}$$

with

$$\gamma = \left(1 + \frac{1}{12Kn}\right)^{-2} , \tag{5.89}$$

is known from the first-order solutions (5.20) and (5.21) these last two expressions provide immediate solutions for two of the ten parametric functions,

$$\tilde{t}_- = -C_1 + 1/2\, C_2 - \frac{1}{8\pi}\left(1 + \frac{1}{6\mathrm{Kn}\bar{R}^3}\right)\gamma , \tag{5.90}$$

and

$$\tilde{N}_- = 3/2\, C_1 - 1/4\, C_2 + \frac{1}{16\pi}\left(4 - \frac{3}{\bar{R}^2}\right)\left(\bar{R}^2 + \frac{1}{6\mathrm{Kn}\bar{R}}\right)\gamma . \tag{5.91}$$

The remaining $\sin^2\theta$ parameters are only slightly more difficult to obtain from the radial momentum equation (5.77) and heat-flux equation (5.83). When the moment definitions of Section 5.3. and the above solutions for $\tilde{N}_-$ and $\tilde{t}_-$ are inserted, both of these equations simply become quadratures for linear combinations of $\tilde{N}_+$ and $\tilde{t}_+$. The necessary integrations are straightforward and the parametric solutions which result are

$$\tilde{N}_+ = C_3 - \frac{C_2}{15\mathrm{Kn}\bar{R}} + \frac{\gamma\bar{R}^2}{4\pi} - \frac{\gamma}{80\pi\mathrm{Kn}}\left[\frac{14}{3\bar{R}} + \frac{1}{18\mathrm{Kn}\bar{R}^4} + \frac{111}{32}\cos^{-1}\left(\frac{1}{\bar{R}}\right) - \frac{(1 - 1/\bar{R}^2)^{\frac{1}{2}}}{32\,\bar{R}}\left(145 - 26/\bar{R}^2 - 8/\bar{R}^4\right)\right], \tag{5.92}$$

and

$$\tilde{t}_+ = C_4 + \frac{C_2}{15\mathrm{Kn}\bar{R}} - \frac{\gamma}{80\pi\mathrm{Kn}}\left[\frac{2}{\bar{R}} + \frac{1}{12\mathrm{Kn}\bar{R}^4} + \frac{7}{16}\cos^{-1}\left(\frac{1}{\bar{R}}\right) - \frac{(1 - 1/\bar{R}^2)^{\frac{1}{2}}}{48\,\bar{R}}\left(75 - 46/\bar{R}^2 - 8/\bar{R}^4\right)\right]. \tag{5.93}$$

The boundary conditions (5.54), (5.58), (5.63) and (5.64) may now be applied to these solutions to determine the four integration constants and complete this portion of the system. The resulting constants are

$$C_1 = 0 , \tag{5.94}$$

$$C_2 = \left(\frac{1}{4\pi} - \frac{7}{2560\mathrm{Kn}} + \frac{11}{240\pi\mathrm{Kn}} + \frac{1}{960\pi\mathrm{Kn}^2}\right)\frac{\gamma}{\left(1 + \frac{1}{15\mathrm{Kn}}\right)} , \tag{5.95}$$

$$C_3 = -1/4\, C_2 - \frac{3\gamma}{16\pi} + \frac{111\gamma}{5120 Kn} , \qquad (5.96)$$

$$C_4 = 1/2\, C_2 + \frac{7\gamma}{2560\ Kn} - \frac{\gamma}{8\pi} . \qquad (5.97)$$

Two of the remaining six parameters can now be determined from the equations having $(3 \cos^2\theta - 1)$ angular behavior. In particular, five-halves of the continuity equation (5.76) minus the energy equation (5.81) is

$$(2 + R\frac{d}{dR})(5/2\ \hat{nu}_R - \hat{\dot{q}}^*_R) + 5/2\ \widehat{nu}_\theta - \widehat{\dot{q}}^*_\theta = 0 , \qquad (5.98)$$

which with the integral definitions of Section 5.3. becomes

$$(2 + R\frac{d}{dR})\left[(1 - x^2)\hat{N}_-\right] = 0 . \qquad (5.99)$$

This equation readily integrates to give

$$\hat{N}_- = \text{constant} = C_5 . \qquad (5.100)$$

Similarly, consider $12\pi$ times the R-momentum equation (5.78) plus the R-heat-flux equation (5.84). Again using the integral definitions of Section 5.3. the relevant combinations are as follows:

$$12\pi\ \hat{P}^*_{RR} + \hat{P}^*_{RRii} = -\pi\hat{N}_+ + \pi x^3\hat{N}_- ; \qquad (5.101)$$

$$12\pi\ \widehat{P}^*_{R\theta} + \widehat{P}^*_{R\theta ii} = 0 ; \qquad (5.102)$$

$$12\pi\ \hat{P}^*_{\theta\theta} + \hat{P}^*_{\theta\theta ii} = -\pi\hat{N}_+ + \pi(3/2\ x - 1/2\ x^3)\hat{N}_- ; \qquad (5.103)$$

$$12\pi\ \hat{P}^*_{\varphi\varphi} + \hat{P}^*_{\varphi\varphi ii} = -\pi\hat{N}_+ + \pi(3/2\ x - 1/2\ x^3)\hat{N}_- ; \qquad (5.104)$$

and

$$\hat{\dot{q}}_R = -1/2\ (1 - x^2)\hat{N}_- . \tag{5.105}$$

After inserting these expressions and the above solution for $\hat{N}_-$ the combined equation reduces to the form,

$$\frac{d}{d\overline{R}} \hat{N}_+ = - \frac{C_5}{6Kn\overline{R}^2} , \tag{5.106}$$

which is easily integrated to give

$$\hat{N}_+ = C_6 + C_5/6Kn\overline{R} . \tag{5.107}$$

The boundary condition (5.63) may now be applied to show that

$$C_5 = C_6 , \tag{5.108}$$

and consequently

$$\hat{N}_+ = C_5(1 + 1/6\,Kn\overline{R}) . \tag{5.109}$$

The integration constant $C_5$ cannot be determined until the remaining parameters, $\hat{u}_2$ in particular, are found. The solution for the six parameters which can be determined analytically is now complete.

The four remaining parameters are found by simultaneous numerical integration of a system of four ordinary differential equations even though this system may be further separated into two second order differential equations. This reduction is helpful, however, in understanding the analytic behavior of the solutions at the singular points of the system and will therefore now be described. Of the undetermined parameters only $\hat{u}_-$ and $\hat{\nabla}_-$ appear in the R-momentum equation (5.78) and $\theta$-momentum equation (5.79). After a solution for $\hat{u}_-$ and $\hat{v}_-$ is determined the continuity equation (5.76) and the shear stress equation (5.82) form a system governing the parametric functions $\hat{u}_+$ and $\hat{v}_+$ and allow the full completion of the eight moment formulation.

The radial momentum equation with the substitution of the integral moments (5.32) through (5.35) becomes

$$(2\overline{R}^3 - \overline{R}^2)\frac{d}{d\overline{R}}\hat{u}_- + \hat{u}_- + 1/2\,\hat{v}_- = F_1(\overline{R}) \,, \tag{5.110}$$

where $F_1(\overline{R})$ is known from the first-order solution and the previously completed portion of the second-order solution. Similarly, the $\theta$-momentum equation may be written as

$$\overline{R}\frac{d}{d\overline{R}}\hat{v}_- - \hat{v}_- - 6\,\hat{u}_- = F_2(\overline{R}) \,. \tag{5.111}$$

These two equations readily combine to yield

$$(2\overline{R}^4 - \overline{R}^2)\frac{d^2}{d\overline{R}^2}\hat{v}_- + \overline{R}\frac{d}{d\overline{R}}\hat{v}_- + 2\hat{v}_- = 6F_1 + F_2 + (2\overline{R}^3 - \overline{R})\frac{d}{d\overline{R}}F_2 \,, \tag{5.112}$$

from which $\hat{u}_-$ is determined by the expression,

$$\hat{u}_- = (\overline{R}\frac{d}{d\overline{R}}\hat{v}_- - \hat{v}_- - F_2)/6 \,. \tag{5.113}$$

The equation (5.112) has a regular singular point at infinity but no finite singular points within the range of integration ($\overline{R} \geq 1$). Although the indicial roots at this singular point differ by an integer ($\alpha$ = 0, -1) no logarithmic solutions exist and the general homogeneous solution may be written

$$\hat{v}_- = a_0(\bar{R} - 3/4\,\bar{R}^{-1} - 1/32\,\bar{R}^{-3} + \dots) + \Sigma a_n \bar{R}^{1-n}$$

$$+ b_0(1 - 1/6\,\bar{R}^{-2} - 1/40\,\bar{R}^{-4} + \dots) + \Sigma b_n \bar{R}^{-n}, \tag{5.114}$$

where

$$a_{n+2} = \frac{(n^2 - 3)a_n}{2(1+n)(2+n)} \tag{5.115}$$

and

$$b_{n+2} = \frac{(n^2 + 2n - 2)b_n}{2(2+n)(3+n)}. \tag{5.116}$$

A series expression could also be given for a particular integral of this equation but the algebraic complexity of the F's makes this very tedious and the result is unnecessary for the following numerical integration.

Now consider the other second-order differential equation which results from combining the continuity equation with the shear stress equation. The continuity equation (5.76) may be written as

$$\bar{R}\frac{d}{d\bar{R}}\hat{u}_+ + 2\,\hat{u}_+ + \hat{v}_+ = F_3(\bar{R}) \tag{5.117}$$

and the shear stress equation (5.82) as

$$\bar{R}\frac{d}{d\bar{R}}\hat{v}_+ - \hat{v}_+ - 6\,\hat{u}_+ = F_4(\bar{R}), \tag{5.118}$$

where $F_3$ and $F_4$ now involve the above solutions for $\hat{u}_-$ and $\hat{v}_-$ as well as the parametric functions found earlier. From these it is not difficult to obtain the equation,

$$\bar{R}^2\frac{d^2}{d\bar{R}^2}\hat{v}_+ + 2\bar{R}\frac{d}{d\bar{R}}\hat{v}_+ + 4\,\hat{v}_+ = 6\,F_3 + 2\,F_4 + \bar{R}\frac{d}{d\bar{R}}F_4, \tag{5.119}$$

for $\hat{v}_+$ and the auxiliary expression for $\hat{u}_+$ ,

$$\hat{u}_+ = (\overline{R} \frac{d}{d\overline{R}} \hat{v}_+ - \hat{v}_+ - F_4)/6. \qquad (5.120)$$

Equation (5.119) is of a type known as an Euler-Cauchy differential equation and has the analytic homogeneous solutions,

$$\hat{v}_+ = \overline{R}^{(\pm i\sqrt{15/4} - \frac{1}{2})} \ . \qquad (5.121)$$

The difficulty which prevents a complete analytic solution for the parameters $\hat{u}_+$ and $\hat{v}_+$ is the presence of nonanalytic expressions in the inhomogeneous terms $F_3$ and $F_4$. Because of these quantities which include $\hat{u}_-$ and $\hat{v}_-$ a particular integral cannot be determined by variation of parameters except as an infinite series.

With this general understanding of the behavior of these four parametric functions in hand a numerical procedure can now be prescribed. The integration is performed over the interval 0 to 1 of the independent variable x with the only difficulty occurring at the singular point x = 1 (R = infinity) where two of the boundary conditions are to be applied. From the homogeneous solution (5.114) it can be seen that a forward integration in the direction of increasing x will in general diverge making a simple application of these conditions impossible. A solution to this system could be obtained by matching a numerical integration over most of the range to an asymptotic, convergent in this case, expansion in a small region near x = 1. The determination of this expansion is quite difficult however, and a much simpler method was employed to satisfy the boundary conditions at infinity. Initializing with the three boundary conditions at the sphere

(5.55), (5.56) and (5.58) (remember that $C_5$ is undetermined from the previous solutions) a Runge-Kutta-Gill integration scheme was forward integrated to the point $x^2 = .95$ where the slopes of the calculable quantities $\hat{u}_R$ and $\hat{u}_\theta$ were then matched to continuum like solutions of the same magnitude. Justification of this procedure is two-fold: at large distances from the sphere all transition flow fields become continuum in character and secondly the free-molecule velocity fields agree with the continuum to dominant order in expansions about infinity. Thus a reasonably quick and efficient integration scheme provides the final four parametric functions, $\hat{u}_+$, $\hat{u}_-$, $\hat{v}_+$ and $\hat{v}_-$.

In summary, a second-order weighting function of the general form (5.27) has been developed which satisfies a reasonable choice of eight partial differential moment equations. This weighting function reduces to the appropriate Maxwellian forms at the sphere surface and at infinity thus satisfying the most basic specification of the boundary conditions. The parameters appearing in this weighting function have the following $\theta$ separation:

$$N_\pm(\underline{R}) = \tilde{N}_\pm(R)\sin^2\theta + \hat{N}_\pm(R)(3\cos^2\theta - 1) \quad ; \tag{5.122}$$

$$t_\pm(\underline{R}) = \tilde{t}_\pm(R)\sin^2\theta \quad ; \tag{5.123}$$

$$\bar{u}_\pm(\underline{R}) = \hat{u}_\pm(R)\,(3\cos^2\theta - 1) \quad ; \tag{5.124}$$

$$\bar{v}_\pm(\underline{R}) = \hat{v}_\pm(R)\sin\theta\cos\theta \quad ; \tag{5.125}$$

$$\bar{w}_\pm(\underline{R}) = 0 \quad . \tag{5.126}$$

A mixed analytic and numerical solution is provided for the ten nonzero quantities above and tabulations of these results are presented in Appendix H for a representative collection of Knudsen numbers.

Once the values of the parametric functions are determined by the moment method it is possible to calculate all physical properties of the flow about the sphere using Equations (5.29) to (5.52). Of particular interest are the velocity fields and the total drag torque on the sphere, which may be found by an integration over the sphere surface. Notice that because of the perturbation procedure, all second-order physical quantities must also be multiplied by the factor, $4\pi\beta_0^2\omega^2R_0^2$, to obtain the actual numerical values.

5.7. Velocity Field

The description of the velocity field is greatly simplified because of its alternating character with succeeding terms in the surface Mach number expansion. All velocity fields of odd power in Mach number have only $\varphi$ components and those with even powers have only R and $\theta$ components. The first-order $\varphi$ velocity field is given analytically by equation (5.22) and is shown for a series of representative Knudsen numbers in Fig. 11, plotted versus $x^2 = 1 - R_0^2/R^2$. The slip nature of the free-molecule flow at the sphere surface is clearly evident and the dual character of the transition solutions is again illustrated as in the earlier problems.

The second-order velocity field, which involves both radial and tangential components in planes through the sphere axis, is more difficult to illustrate because it is normalized by a function of Knudsen number and not just by the sphere surface speed as in the first-order solution. Further understanding of this behavior may arise from an examination of the limiting solutions of continuum and free-molecule flow. The continuum solution given by Bickley (21) has the form,

$$u_R = \frac{-1}{8\overline{R}^2}(1 - 1/\overline{R})^2(3\cos^2\theta - 1)\omega R_0 Re , \tag{5.127}$$

$$u_\theta = \frac{1}{4\overline{R}^3}(1 - 1/\overline{R}) \sin\theta \cos\theta\, \omega R_0 Re , \tag{5.128}$$

where the Reynolds number is given by

$$Re = (\pi m/8kT)^{\frac{1}{2}} \omega R_0/Kn . \tag{5.129}$$

Thus, when normalized by the sphere surface speed, the secondary flow field may be written as

$$\underline{u}_c / \omega R_0 = \underline{f}(\underline{R})\ M/Kn . \tag{5.130}$$

In contrast the free-molecule secondary flow obtained in Appendix F is of the form,

$$\underline{u}_{fm}/\omega R_0 = \underline{g}(\underline{R})\ M . \tag{5.131}$$

The induced secondary flow is consequently of much greater magnitude in the high density regime.

Because of the difference in magnitude between free-molecule and continuum secondary flow, the solutions are illustrated in two parts. In Fig. 12 the magnitude of the maximum radial velocity is plotted as a function of Knudsen number and is seen to approach the limiting solutions in a reasonable manner. It is somewhat surprising to find that this curve has a minimum for a transition value of Kn and lies below the continuum line for most of the low Kn solutions. This result cannot be substantiated, however, since no other solutions exist for this problem. A curve of the maximum tangential velocity exhibits essentially the same behavior.

The second characteristic of the secondary flow which must be described is the velocity pattern as a function of Knudsen number. For this, all velocity moment solutions are normalized by their maximum value and shown along with normalized distributions of the free-molecule and continuum velocities. Fig. 13 gives the R-dependence of the radial velocity field for a sequence of Kn values with the most noticeable feature being the similarity of the patterns over the extreme range of densities. The moment solution for infinite Kn provides a better approximation of the limiting solution than do the low Kn evaluations whose maximum values appear to be shifted slightly away from the sphere.

The R-dependence of the normalized tangential velocity fields is shown in Fig. 14, again with the two limiting patterns as dashed lines. As in the radial velocity fields the low Kn solutions appear shifted away from the sphere surface, but the general shape is similar throughout the Knudsen number range. Of particular interest is the existence of velocity slip at the sphere surface in all of the moment solutions, while the limiting solutions show no evidence of slip for reasons which are easily understood. For example, the integrated free-molecule solution of Appendix F exhibits no second-order slip because the distribution function of the ambient particles has only a zeroth-order component and the second-order distribution function of sphere molecules must approach zero at the surface. However, there is no reason to expect no slip for transition values of Kn. The slip found for the infinite Kn moment solution must be due to the averaging properties of the moment method coupled with the relatively high radial slopes near the surface.

5.8. Sphere Drag Torque and Comparison with Experiment

Although only qualitative flow visualization investigations of the velocity fields exist, a series of experiments are available which give the drag torque for a slowly revolving sphere in a variety of density regimes. The drag torque found by the moment method is now compared with these experiments and with other theoretical results.

In the moment solution it has been found that no second-order contribution to the sphere drag is present and the only comparison possible is with the first-order expression (5. 24). A plot of this moment drag as a function of Knudsen number is presented in Fig. 15 along with the above mentioned experimental values by Lord and Harbour. (24) As for the translating sphere drag the moment torque values are seen to pass smoothly from the continuum to the free-molecule limit, but fall somewhat below the experimental points.

Two other theoretical results are also presented in Fig. 15, although each is valid only in a limited Knudsen number regime. A slip calculation by Lord and Harbour fits the experimental data somewhat better than the moment solution for small values of Kn. Also shown is a Knudsen iteration calculation by Willis (3) which is restricted to large Knudsen number flows. A number of other results are also available, but being essentially experimental curve fits these are of little interest in this discussion.

Additional experiments by Bowden and Harbour (25) provide excellent values for the drag torque on a sphere revolving at surface Mach numbers of up to five. A noticeable drop in measured transition drag was found for the higher values of M with no satisfactory explanation being given. The present moment method presents an interesting possibility for further study of this Mach number dependence. A third-order solution about the rotating sphere would reduce to a system of only two partial differential moment equations in the same way as the first-order solution. Again the existence of an immediate integral of the $\varphi$ momentum equation simplifies the problem to one requiring only a quadrature for completion. Unfortunately, a very large number of inhomogeneous terms from the lower order solutions makes this numerical integration quite difficult. Such a solution might, however, be valid for surprisingly large values of the surface Mach number because of the very small numerical values found for the second-order parametric functions, which dominate the inhomogeneous driving terms for the third-order solution.

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# APPENDIX A

## INTEGRAL TABLES FOR MOMENTS OF THE WEIGHTING FUNCTION AND EXACT FREE-MOLECULE SOLUTIONS

In Section 2. 2 it was shown that for a completely linearized weighting function each moment integral consists of a sum of terms of the form,

$$\int f_{\infty}\, F(\underline{\xi})\, d\underline{\xi} \tag{A. 1}$$

A collection of integrals of this type is provided in Table A. 4 for all quantities $F(\underline{\xi})$ required in the problems discussed in the text. In component form these integrals are

$$\int_{\text{Region 1}} f_{\infty}\, F(\underline{\xi}) d\underline{\xi} = \frac{n_{\infty}\beta_0^3}{\pi^{3/2}} \int_0^{\frac{\pi}{2}-\alpha} \int_0^{2\pi} \int_0^{\infty} F(\underline{\xi})\, e^{-\beta_0^2 \xi^2}\, \xi^2 \sin\sigma\, d\xi\, d\tau\, d\sigma, \tag{A. 2}$$

$$\int_{\text{Region 2}} f_{\infty}\, F(\underline{\xi}) d\underline{\xi} = \frac{n_{\infty}\beta_0^3}{\pi^{3/2}} \int_{\frac{\pi}{2}-\alpha}^{\pi} \int_0^{2\pi} \int_0^{\infty} F(\underline{\xi})\, e^{-\beta_0^2 \xi^2}\, \xi^2 \sin\sigma\, d\xi\, d\tau\, d\sigma, \tag{A. 3}$$

where $\sigma$, $\tau$ and $\xi$ are defined by expressions (2. 1) through (2. 3) and illustrated in Fig. 2. Of course, $F(\underline{\xi})$ must also be expressed in component form as

$$F(\underline{\xi}) = F(\xi_R, \xi_\theta, \xi_\varphi) = F(\xi, \sigma, \tau) \quad , \tag{A. 4}$$

with

$$\xi_R = \xi \cos\sigma \quad , \tag{A. 5}$$

$$\xi_\theta = \xi \sin\sigma \cos\tau \,, \tag{A. 6}$$

$$\xi_\varphi = \xi \sin\sigma \sin\tau \quad . \tag{A. 7}$$

Tables A. 1, A. 2 and A. 3 provide the definite integrals required to evaluate (A. 2) and (A. 3), and in combination they provide the very useful results of Table A. 4.

Also collected in Table A. 5 are the definite integrals needed to obtain the exact free molecule results described in Appendices E and F.

TABLE A. 1

| Number | $F(\xi)$ | $\int_0^\infty F(\xi) e^{-\beta_0^2 \xi^2} d\xi$ |
|---|---|---|
| A. 1. 1 | 1 | $\frac{1}{2\beta_0}\sqrt{\pi}$ |
| A. 1. 2 | $\xi$ | $\frac{1}{2\beta_0^2}$ |
| A. 1. 3 | $\xi^2$ | $\frac{1}{4\beta_0^3}\sqrt{\pi}$ |
| A. 1. 4 | $\xi^3$ | $\frac{1}{2\beta_0^4}$ |
| A. 1. 5 | $\xi^4$ | $\frac{3}{8\beta_0^5}\sqrt{\pi}$ |
| A. 1. 6 | $\xi^5$ | $\frac{1}{\beta_0^6}$ |
| A. 1. 7 | $\xi^6$ | $\frac{15}{16\beta_0^7}\sqrt{\pi}$ |
| A. 1. 8 | $\xi^7$ | $\frac{3}{\beta_0^8}$ |
| A. 1. 9 | $\xi^8$ | $\frac{105}{32\beta_0^9}\sqrt{\pi}$ |

TABLE A. 2

| Number | $F(\tau)$ | $\int_0^{2\pi} F(\tau) d\tau$ |
|---|---|---|
| A. 2. 1 | 1 | $2\pi$ |
| A. 2. 2 | $\sin^2 \tau$ | $\pi$ |
| A. 2. 3 | $\sin^4 \tau$ | $3\frac{\pi}{4}$ |
| A. 2. 4 | $\sin^2\tau \cos^2\tau$ | $\frac{\pi}{4}$ |

TABLE A. 3
($x = \sin \alpha$)

| Number | $F(\sigma)$ | $\int_0^{\pi/2-\alpha} F(\sigma)d\sigma$ | $\int_{\pi/2-\alpha}^{\pi} F(\sigma)\,d\sigma$ |
|---|---|---|---|
| A. 3. 1 | $\sin \sigma$ | $1 - x$ | $1 + x$ |
| A. 3. 2 | $\sin^3\sigma$ | $\frac{2}{3} - x + \frac{1}{3} x^3$ | $\frac{2}{3} + x - \frac{1}{3} x^3$ |
| A. 3. 3 | $\sin^5\sigma$ | $\frac{8}{15} - x + \frac{2}{3}x^3 - \frac{1}{5}x^5$ | $\frac{8}{15} + x - \frac{2}{3} x^3 + \frac{1}{5} x^5$ |
| A. 3. 4 | $\sin\sigma \cos\sigma$ | $\frac{1}{2} (1 - x^2)$ | $- \frac{1}{2} (1 - x^2)$ |
| A. 3. 5 | $\sin\sigma \cos^2\sigma$ | $\frac{1}{3} (1 - x^3)$ | $\frac{1}{3} (1 + x^3)$ |
| A. 3. 6 | $\sin\sigma \cos^3\sigma$ | $\frac{1}{4} (1 - x^4)$ | $- \frac{1}{4} (1 - x^4)$ |
| A. 3. 7 | $\sin\sigma \cos^4\sigma$ | $\frac{1}{5} (1 - x^5)$ | $\frac{1}{5} (1 + x^5)$ |
| A. 3. 8 | $\sin\sigma \cos^5\sigma$ | $\frac{1}{6} (1 - x^6)$ | $- \frac{1}{6} (1 - x^6)$ |
| A. 3. 9 | $\sin^3\sigma \cos\sigma$ | $\frac{1}{4} (1 - x^2)^2$ | $- \frac{1}{4} (1 - x^2)^2$ |
| A. 3. 10 | $\sin^3\sigma \cos^2\sigma$ | $\frac{2}{15} - \frac{1}{3} x^3 + \frac{1}{5} x^5$ | $\frac{2}{15} + \frac{1}{3} x^3 - \frac{1}{5} x^5$ |
| A. 3. 11 | $\sin^3\sigma \cos^3\sigma$ | $\frac{1}{12} - \frac{1}{4} x^4 + \frac{1}{6} x^6$ | $- \frac{1}{12} + \frac{1}{4} x^4 - \frac{1}{6} x^6$ |
| A. 3. 12 | $\sin^5\sigma \cos\sigma$ | $\frac{1}{6} (1 - x^2)^3$ | $- \frac{1}{6} (1 - x^2)^3$ |

## TABLE A. 4

| Number | $F(\underline{\xi})$ | $\int F(\underline{\xi})\, f_{\infty} d\underline{\xi}$ Region 1 | $\int F(\underline{\xi})\, f_{\infty}\, d\underline{\xi}$ Region 2 |
|---|---|---|---|
| A. 4. 1 | 1 | $\frac{n_{\infty}}{2}(1-x)$ | $\frac{n_{\infty}}{2}(1+x)$ |
| A. 4. 2 | $\xi_R$ | $\frac{n_{\infty}}{2\sqrt{\pi}\beta_0}(1-x^2)$ | $-\frac{n_{\infty}}{2\sqrt{\pi}\beta_0}(1-x^2)$ |
| A. 4. 3 | $\xi_R^{\ 2}$ | $\frac{n_{\infty}}{4\beta_0^2}(1-x^3)$ | $\frac{n_{\infty}}{4\beta_0^2}(1+x^3)$ |
| A. 4. 4 | $\xi_{\theta}^{\ 2}, \xi_{\varphi}^{\ 2}$ | $\frac{n_{\infty}}{8\beta_0^2}(2-3x+x^3)$ | $\frac{n_{\infty}}{8\beta_0^2}(2+3x-x^3)$ |
| A. 4. 5 | $\xi^2$ | $\frac{3n_{\infty}}{4\beta_0^2}(1-x)$ | $\frac{3n_{\infty}}{4\beta_0^2}(1+x)$ |
| A. 4. 6 | $\xi_R^{\ 3}$ | $\frac{n_{\infty}}{2\sqrt{\pi}\beta_0^3}(1-x^4)$ | $-\frac{n_{\infty}}{2\sqrt{\pi}\beta_0^3}(1-x^4)$ |
| A. 4. 7 | $\xi_R\xi_{\theta}^{\ 2}, \xi_R\xi_{\varphi}^{\ 2}$ | $\frac{n_{\infty}}{4\sqrt{\pi}\beta_0^3}(1-x^2)^2$ | $-\frac{n_{\infty}}{4\sqrt{\pi}\beta_0^3}(1-x^2)^2$ |
| A. 4. 8 | $\xi_R\xi^2$ | $\frac{n_{\infty}}{\sqrt{\pi}\beta_0^3}(1-x^2)$ | $-\frac{n_{\infty}}{\sqrt{\pi}\beta_0^3}(1-x^2)$ |
| A. 4. 9 | $\xi_R^{\ 4}$ | $\frac{3n_{\infty}}{8\beta_0^4}(1-x^5)$ | $\frac{3n_{\infty}}{8B_0^4}(1+x^5)$ |
| A. 4. 10 | $\xi_{\theta}^{\ 4}, \xi_{\varphi}^{\ 4}$ | $\frac{3n_{\infty}}{8\beta_0^4}(1-\frac{15}{8}x+\frac{5}{4}x^3-\frac{3}{8}x^5)$ | $\frac{3n_{\infty}}{8\beta_0^4}(1+\frac{15}{8}x-\frac{5}{4}x^3+\frac{3}{8}x^5)$ |
| A. 4. 11 | $\xi_R^{\ 2}\xi_{\theta}^{\ 2}, \xi_R^{\ 2}\xi_{\varphi}^{\ 2}$ | $\frac{n_{\infty}}{16\beta_0^4}(2-5x^3+3x^5)$ | $\frac{n_{\infty}}{16\beta_0^4}(2+5x^3-3x^5)$ |

TABLE A. 4 (Cont.)

| Number | $F(\underline{\xi})$ | $\int_{\text{Region 1}} F(\underline{\xi}) f_\infty d\underline{\xi}$ | $\int_{\text{Region 2}} F(\underline{\xi}) f_\infty d\underline{\xi}$ |
|---|---|---|---|
| A. 4. 12 | $\xi_\theta^2 \xi_\varphi^2$ | $\frac{n_\infty}{64\beta_0^4}(8-15x+10x^3-3x^5)$ | $\frac{n_\infty}{64\beta_0^4}(8+15x-10x^3+3x^5)$ |
| A. 4. 13 | $\xi_R^2 \xi^2$ | $\frac{5n_\infty}{8\beta_0^4}(1-x^3)$ | $\frac{5n_\infty}{8\beta_0^4}(1+x^3)$ |
| A. 4. 14 | $\xi_\theta^2 \xi^2, \xi_\varphi^2 \xi^2$ | $\frac{5n_\infty}{16\beta_0^4}(2-3x+x^3)$ | $\frac{5n_\infty}{16\beta_0^4}(2+3x-x^3)$ |
| A. 4. 15 | $\xi^4$ | $\frac{15n_\infty}{8\beta_0^4}(1-x)$ | $\frac{15n_\infty}{8\beta_0^4}(1+x)$ |
| A. 4. 16 | $\xi_R^5$ | $\frac{n_\infty}{\sqrt{\pi}\beta_0^5}(1-x^6)$ | $-\frac{n_\infty}{\sqrt{\pi}\beta_0^5}(1-x^6)$ |
| A. 4. 17 | $\xi_R^3 \xi_\theta^2, \xi_R^3 \xi_\varphi^2$ | $\frac{n_\infty}{4\sqrt{\pi}\beta_0^5}(1-3x^4+2x^6)$ | $-\frac{n_\infty}{4\sqrt{\pi}\beta_0^5}(1-3x^4+2x^6)$ |
| A. 4. 18 | $\xi_R^3 \xi^2$ | $\frac{3n_\infty}{2\sqrt{\pi}\beta_0^5}(1-x^4)$ | $-\frac{3n_\infty}{2\sqrt{\pi}\beta_0^5}(1-x^4)$ |
| A. 4. 19 | $\xi_R \xi_\theta^4, \xi_R \xi_\varphi^4$ | $\frac{3n_\infty}{8\sqrt{\pi}\beta_0^5}(1-x^2)^3$ | $-\frac{3n_\infty}{8\sqrt{\pi}\beta_0^5}(1-x^2)^3$ |
| A. 4. 20 | $\xi_R \xi_\theta^2 \xi_\varphi^2$ | $\frac{n_\infty}{8\sqrt{\pi}\beta_0^5}(1-x^2)^3$ | $-\frac{n_\infty}{8\sqrt{\pi}\beta_0^5}(1-x^2)^3$ |
| A. 4. 21 | $\xi_R \xi_\theta^2 \xi^2, \xi_R \xi_\varphi^2 \xi^2$ | $\frac{3n_\infty}{4\sqrt{\pi}\beta_0^5}(1-x^2)^2$ | $-\frac{3n_\infty}{4\sqrt{\pi}\beta_0^5}(1-x^2)^2$ |
| A. 4. 22 | $\xi_R \xi^4$ | $\frac{3n_\infty}{\sqrt{\pi}\beta_0^5}(1-x^2)$ | $-\frac{3n_\infty}{\sqrt{\pi}\beta_0^5}(1-x^2)$ |
| A. 4. 23 | $\xi_R^2 \xi_\varphi^2 \xi^2$ | $\frac{7n_\infty}{32\beta_0^6}(2-5x^3+3x^5)$ | $\frac{7n_\infty}{32\beta_0^6}(2+5x^3-3x^5)$ |

TABLE A. 4 (Cont. )

| Number | $F(\underline{\xi})$ | $\int F(\underline{\xi}) f_\infty d\underline{\xi}$ Region 1 | $\int F(\underline{\xi}) f_\infty d\underline{\xi}$ Region 2 |
|---|---|---|---|
| A. 4. 24 | $\xi_R^2 \xi^4$ | $\frac{35 n_\infty}{16 \beta_0^6}(2-5x^3+3x^5)$ | $\frac{35 n_\infty}{16 \beta_0^6}(2+5x^3-3x^5)$ |
| A. 4. 25 | $\xi_\theta^4 \xi^2, \xi_\varphi^4 \xi^2$ | $\frac{21 n_\infty}{128 \beta_0^6}(8-15x+10x^3-3x^5)$ | $\frac{21 n_\infty}{128 \beta_0^6}(8+15x-10x^3+3x^5)$ |
| A. 4. 26 | $\xi_\theta^2 \xi^4, \xi_\varphi^2 \xi^4$ | $\frac{35 n_\infty}{32 \beta_0^6}(2-3x+x^3)$ | $\frac{35 n_\infty}{32 \beta_0^6}(2+3x-x^3)$ |
| A. 4. 27 | $\xi_\theta^2 \xi_\varphi^2 \xi^2$ | $\frac{7 n_\infty}{128 \beta_0^6}(8-15x+10x^3-3x^5)$ | $\frac{7 n_\infty}{128 \beta_0^6}(8+15x-10x^3+3x^5)$ |
| A. 4. 28 | $\xi^6$ | $\frac{105 n_\infty}{16 \beta_0^6}(1-x)$ | $\frac{105 n_\infty}{16 \beta_0^6}(1+x)$ |

All integrals involving odd powers of either $\xi_\theta$ or $\xi_\varphi$ are equal to zero.

TABLE A. 5

($x = \sin\alpha$, $y = \cos\alpha = \sqrt{1-x^2}$)

| Number | F(x, s) | $\int_0^1 (F(x, s)ds$ |
|---|---|---|
| A. 5. 1 | $s(1-s^2)^{\frac{1}{2}}$ | $\frac{1}{3}$ |
| A. 5. 2 | $s^3(1-s^2)^{\frac{1}{2}}$ | $\frac{2}{15}$ |
| A. 5. 3 | $s^5(1-s^2)^{\frac{1}{2}}$ | $\frac{8}{105}$ |
| A. 5. 4 | $s(1-s^2)^{-\frac{1}{2}}$ | $1$ |
| A. 5. 5 | $s^3(1-s^2)^{-\frac{1}{2}}$ | $\frac{2}{3}$ |
| A. 5. 6 | $s^5(1-s^2)^{-\frac{1}{2}}$ | $\frac{8}{15}$ |
| A. 5. 7 | $s^7(1-s^2)^{-\frac{1}{2}}$ | $\frac{16}{35}$ |
| A. 5. 8 | $s(1-y^2s^2)^{\frac{1}{2}}$ | $\frac{1}{3}(1-x^3)/y^2$ |
| A. 5. 9 | $s^3(1-y^2s^2)^{\frac{1}{2}}$ | $\frac{1}{15}(2-5x^3+3x^5)/y^4$ |
| A. 5. 10 | $s^5(1-y^2s^2)^{\frac{1}{2}}$ | $\frac{1}{105}(8-35x^3+42x^5-15x^7)/y^6$ |
| A. 5. 11 | $s(1-y^2s^2)^{-\frac{1}{2}}$ | $1/(1+x)$ |
| A. 5. 12 | $s^3(1-y^2s^2)^{-\frac{1}{2}}$ | $\frac{1}{3}(2-3x+x^3)/y^4$ |

TABLE A. 5 (Cont. )

| Number | $F(x, s)$ | $\int_0^1 F(x, s)ds$ |
|---|---|---|
| A. 5. 13 | $s^5(1-y^2s^2)^{-\frac{1}{2}}$ | $\frac{1}{15}(8-15x+10x^3-3x^5)/y^6$ |
| A. 5. 14 | $s^7(1-y^2s^2)^{-\frac{1}{2}}$ | $\frac{1}{35}(16-35x+35x^3-21x^5+5x^7)/y^8$ |
| A. 5. 15 | $s(1-s^2)^{\frac{1}{2}}(1-y^2s^2)^{\frac{1}{2}}$ | $\frac{2-x^2}{8y^2} - \frac{x^4}{16} \log\left(\frac{1+y}{1-y}\right)$ |
| A. 5. 16 | $s^3(1-s^2)^{\frac{1}{2}}(1-y^2s^2)^{\frac{1}{2}}$ | $\frac{(2-x^2)^2}{16y^4} - \frac{1}{6y^2} - \frac{x^4(2-x^2)}{32y^5} \log\left(\frac{1+y}{1-y}\right)$ |
| A. 5. 17 | $s^5(1-s^2)^{\frac{1}{2}}(1-y^2s^2)^{\frac{1}{2}}$ | $-\frac{5(2-x^2)}{48y^4} + \frac{(5+6y^2+5y^4)}{16y^4}\left[\frac{2-x^2}{8y^2} - \frac{x^4}{16y^3}\log\left(\frac{1+y}{1-y}\right)\right]$ |
| A. 5. 18 | $s(1-s^2)^{-\frac{1}{2}}(1-y^2s^2)^{-\frac{1}{2}}$ | $\frac{1}{2y}\log\left(\frac{1+y}{1-y}\right)$ |
| A. 5. 19 | $s(1-s^2)^{-\frac{1}{2}}(1-y^2s^2)^{\frac{1}{2}}$ | $\frac{1}{2} + \frac{x^2}{4y}\log\left(\frac{1+y}{1-y}\right)$ |
| A. 5. 20 | $s^3(1-s^2)^{-\frac{1}{2}}(1-y^2s^2)^{\frac{1}{2}}$ | $\frac{2-3x^2}{8y^2} + \frac{x^2(4-3x^2)}{16y^3}\log\left(\frac{1+y}{1-y}\right)$ |
| A. 5. 21 | $s(1-s^2)^{\frac{1}{2}}(1-y^2s^2)^{-\frac{1}{2}}$ | $\frac{1}{2y^2} - \frac{x^2}{4y^3}\log\left(\frac{1+y}{1-y}\right)$ |
| A. 5. 22 | $s^3(1-s^2)^{\frac{1}{2}}(1-y^2s^2)^{-\frac{1}{2}}$ | $\frac{2+x^2}{8y^4} - \frac{x^2(4-x^2)}{16y^5}\log\left(\frac{1+y}{1-y}\right)$ |
| A. 5. 23 | $s^5(1-s^2)^{\frac{1}{2}}(1-y^2s^2)^{-\frac{1}{2}}$ | $\frac{8+10x^2-3x^4}{48y^6} - \frac{x^2}{32y^7}(8-4x^2+x^4)\log\left(\frac{1+y}{1-y}\right)$ |

# APPENDIX B

## ASYMPTOTIC EXPANSION OF COLLISIONAL SOLUTION NEAR SPHERE SURFACE

A brief description of the asymptotic expansions about the singular points of the third-order vector equation governing the flow past a sphere is given in this and the next two Appendices. Considered here is the singular point at the sphere surface (x=0) for the case of a bounded but otherwise arbitrary Knudsen number flow. The solution for the totally collisionless system is fundamentally different and will be described in a separate Appendix. Since the analysis follows rather closely the development provided by Wasow, (18) any relevant proofs or detail procedural motivations may be found in that text. The principal ideas involved in each of the expansions and the appropriate results will be presented in the following pages.

The starting point for each of the asymptotic developments is the third-order vector differential equation (4.50) along with the defining expressions (4.51) and (4.52). The singular nature of this system is determined entirely by the homogeneous solution matrix and consequently by the character of the matrix $\underline{\underline{A}}$ given by (4.51). When expanded about $x = 0$, the highest order element in this matrix is of order $(1/x^2)$ and the system thus has an irregular singular point at $x = 0$, according to the classification employed by Wasow. Since most equations with irregular singularities have them occurring at infinity the present problem is made to parallel the general development by the transformation $z = 1/x$, which gives the following equation,

$$\frac{d}{dz}\,\underline{Y}(z) = \underline{\underline{B}}(z)\cdot\underline{Y}(z). \tag{B.1}$$

The matrix $\underline{\underline{B}}$ is represented by a series,

$$\underline{\underline{B}}(z) = \Sigma B_n z^{-n}, \tag{B.2}$$

in which the $B_n$'s are determined directly from expression (4.51) to be

$$B_0 = \begin{bmatrix} 0 & 0 & 0 \\ -4/3 & 1/9\,Kn & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{B.3}$$

$$B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \tag{B.4}$$

etc. The actual analysis was carried out to much higher order than shown and occasionally results will appear which do not follow from the expressions provided in the previous step.

As a preliminary operation the equation is put into standard form through the application of a similarity transformation,

$$\underline{\underline{T}} = \begin{bmatrix} 0 & 1/12Kn & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{B.5}$$

which yields the system,

$$\frac{d}{dz}\,\underline{Y}_1(z) = \underline{\underline{C}}(z)\cdot\underline{Y}_1(z). \tag{B.6}$$

The new dependent variable $\underline{Y}_1$ is defined by the relation,

$$\underline{Y}(z) = \underline{\underline{T}}\cdot\underline{Y}_1(z), \tag{B.7}$$

and the matrix $\underline{\underline{C}}(z) = \Sigma C_n z^{-n}$ by

$$\underline{\underline{C}}(z) = \underline{\underline{T}}^{-1} \cdot \underline{\underline{B}}(z) \cdot \underline{\underline{T}} , \tag{B. 8}$$

from which

$$C_0 = \begin{bmatrix} 1/9Kn & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{B. 9}$$

$$C_1 = \begin{bmatrix} 3 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{B. 10}$$

etc. This similarity transformation is applied so that $C_0$ is a block diagonal matrix, which is assumed in the general development of the theory of irregular singular points.

The fundamental idea is now to find a matrix function $\underline{\underline{P}}(z)$, holomorphic and with nonvanishing determinant at z = infinity, which transforms the system (B. 6) into one more amenable to solution. Formally making the transformation

$$\underline{Y}_1(z) = \underline{\underline{P}}(z) \cdot \underline{Y}_2(z) , \tag{B. 11}$$

the differential equation (B. 6) becomes an equation for the new dependent variable $\underline{Y}_2(z)$,

$$\frac{d}{dz} \underline{Y}_2(z) = \underline{\underline{D}}(z) \cdot \underline{Y}_2(z) , \tag{B. 12}$$

in which

$$\underline{\underline{D}}(z) = \underline{\underline{P}}^{-1}(z) \cdot \underline{\underline{C}}(z) \cdot \underline{\underline{P}}(z) - \underline{\underline{P}}^{-1}(z) \cdot \frac{d}{dz} \underline{\underline{P}}(z). \tag{B. 13}$$

Tentatively assuming that

$$\underline{\underline{P}}(z) = \Sigma P_n z^{-n} , \tag{B. 14}$$

$$\underline{\underline{D}}(z) = \Sigma D_n z^{-n} , \tag{B. 15}$$

equation (B. 13) provides the following relations:

$$C_0 \cdot P_0 - P_0 \cdot D_0 = 0 , \tag{B. 16}$$

$$C_0 \cdot P_r - P_r \cdot D_0 = \sum_{s=0}^{r-1} (P_s \cdot D_{r-s} - C_{r-s} \cdot P_s) - (r-1) P_{r-1}, \; r>0. \tag{B. 17}$$

If these two equations can be satisfied, then the series (B. 14) and (B. 15) formally satisfy equation (B. 13) and a new equation is obtained which may be easier to solve. Flexibility is gained in that $\underline{\underline{P}}(z)$ and $\underline{\underline{D}}(z)$ are both partially free to be chosen in a most advantageous manner, however $\underline{\underline{P}}(z)$ will in general not be convergent, thereby providing the restriction to asymptotic solutions.

An obvious solution to equation (B. 16) is

$$P_0 = \underline{\underline{I}} , \tag{B. 18}$$

and

$$D_0 = C_0 . \tag{B. 19}$$

With these two expressions equation (B. 17) may be written in the form,

$$C_0 \cdot P_r - P_r \cdot C_0 = D_r + H_r, \tag{B. 20}$$

where $H_r$ depends only on $P_j$ and $D_j$ with $j < r$. Fortunately a matrix equation of this form has unique solutions for specific simple forms of the matrices $P_r$ and $D_r$. In particular, $D_r$ is assumed to be of the same block diagonal form as the matrix $C_0$, and all elements of $P_r$ $(r > 0)$ which occur in this block diagonal pattern are assumed to be zero. Substitution of matrices of this form into equation (B. 20) provides an algorithm by which successive terms in the expansion of $\underline{\underline{P}}(z)$ and $\underline{\underline{D}}(z)$ may be determined. This procedure gives the following expressions:

$$\underline{\underline{P}} \sim \begin{bmatrix} 1 & -27Kn/z + (972Kn^2 - 5/2)/z^2 & -27Kn/z+972Kn^2/z^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (B.21)$$

$$\underline{\underline{D}} \sim \begin{bmatrix} 1/9Kn + 3/z + 1/2Knz^2 + 2/z^3 & 0 & 0 \\ 0 & 1/z^3 & 6/z^3 \\ 0 & 1/z^3 & -3/z^3 \end{bmatrix} . \quad (B.22)$$

The equation (B. 12) has now been reduced to two simple differential systems: one a scalar equation for the first component of $Y_2(z)$, and a second-order system about an ordinary point which may be solved by a straight-forward series approach. Consequently, one solution matrix for equation (B. 12) is

$$\underline{\underline{Y}}_2 \sim \begin{bmatrix} e^{z/9Kn}(z^3 - z^2/Kn - z + \ldots) & 0 & 0 \\ 0 & 1 - 1/2z^2 \ldots & -3/z^2 + \ldots \\ 0 & -1/2z^2 + \ldots & 1+3/2z^2+\ldots \end{bmatrix}. \quad (B.23)$$

Obtaining the asymptotic representation for the original third-order differential system (4. 50) is merely an algebraic exercise in reversing the various transformations described above. The final homogeneous solution matrix is

$$\underline{\eta} \sim \begin{bmatrix} e^{1/9Knx}(0(1)) & 1/12Kn - x^2/12Kn & -x^2/4Kn \\ \dfrac{e^{1/9Knx}}{x^3}(1 - x/2Kn - x^2 + \ldots) & 1 - 27Knx & -27Knx \\ e^{1/9Knx}(0(1)) & -x^2/2 & 1 + 3x^2/2 \end{bmatrix}. \quad (B.24)$$

The solution is completed by finding a particular integral of the inhomogeneous equation (4.50). This may be accomplished by a matrix form of the variation of parameters method or more simply by a trial series solution. Expanding the inhomogeneous vector (4.52) for small x and using either method, a particular integral is

$$\underline{Y}_0 \sim \begin{Bmatrix} C_2 x^2 + \ldots \\ \frac{-3\pi}{2} C_2 x + (54\pi Kn + 12Kn) C_2 x^2 + \ldots \\ -\pi C_1 x^2 + \ldots \end{Bmatrix} . \quad (B.25)$$

This particular integral plus the solution matrix (B.24) times an arbitrary constant vector completes the general asymptotic representation for flow near the sphere.

# APPENDIX C

## ASYMPTOTIC EXPANSION OF COLLISIONLESS SOLUTION NEAR SPHERE SURFACE

In Appendix B a general asymptotic representation about the sphere surface was developed for the third-order differential system (4. 50) of Chapter Four. Unfortunately this expansion does not fully reveal the singular nature of the solution near the sphere. Investigation of the asymptotic homogeneous solution matrix (B. 24) shows an unexpected characteristic of the second two column vectors which represent the physically meaningful solutions. The Knudsen number parameter appears in both the numerator and denominator of the coefficients in these asymptotic expressions, thus rendering them useless as representations for the collisionless flow.

In an effort to understand this behavior the previous study is repeated with the Knudsen number set equal to infinity in the original differential system. As before the singular character is determined by the elements of the matrix $\underline{\underline{A}}$ in (4. 51), and the transformation $z = 1/x$ is again made to place the system in standard form. This transformation gives equation (B. 1) with $\underline{\underline{B}}(z)$ now represented by a series expansion with the elements,

$$B_0 = \begin{bmatrix} 0 & 0 & 0 \\ -4/3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{C. 1}$$

$$B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \tag{C. 2}$$

etc. This system appears to have an irregular singular point at $z = \infty$ as was found in Appendix B for the full system, but it is actually what is often termed a "pseudo essential" singular point. This means that transformations exist which take the system into one with a regular singular point, and a solution matrix with at most poles at that point results. Because of the complexity of the transformations involved, the development of the collisionless asymptotic representation is merely sketched without full motivation for many of the steps.

The present development parallels that of Appendix B for awhile with one of the preliminary operations being the application of the similarity transformation,

$$\underline{\underline{T}} = \begin{bmatrix} 0 & 1 & 0 \\ -4/3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \tag{C. 3}$$

to the matrix $\underline{\underline{B}}(z)$. This step is utilized to obtain a Jordan canonical form for the first term in the series, $B_0$, and leads to a new differential equation,

$$\frac{d}{dz} \underline{Y}_1(z) = \underline{\underline{C}}(z) \cdot \underline{Y}_1(z) , \tag{C. 4}$$

with

$$\underline{Y}(z) = \underline{\underline{T}} \cdot \underline{Y}_1(z) , \tag{C. 5}$$

and

$$C_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \tag{C. 6}$$

$$C_1 = \begin{bmatrix} 3 & 0 & -9/4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{C.7}$$

etc. Notice that $C_0$ is now in block diagonal form and consider a transformation to a new variable defined by

$$\underline{Y}_1(z) = \underline{\underline{P}}(z) \cdot \underline{Y}_2(z) , \tag{C.8}$$

which must satisfy the equation,

$$\frac{d}{dz} \underline{Y}_2(z) = \underline{\underline{D}}(z) \cdot \underline{Y}_2(z) . \tag{C.9}$$

$\underline{\underline{P}}(z)$ and $\underline{\underline{D}}(z)$ must satisfy the same conditions, (B.16) and (B.17), as in Appendix B, and again an obvious solution for the first of these is

$$P_0 = \underline{\underline{I}} , \tag{C.10}$$

and

$$D_0 = C_0 . \tag{C.11}$$

The process now diverges from that used in Appendix B because all the eigenvalues of $C_0$ are zero and $D_r$ $(r > 0)$ can no longer be chosen to be block diagonal. A slightly more complex algorithm (See Wasow (18) for details) can, however, be found which yields

$$\underline{\underline{P}}(z) \sim \begin{bmatrix} 1 & 0 & 0 \\ -3/z + 8/3z^3 & 1 - 2/z^2 & 9/4z - 2/z^3 \\ 0 & 0 & 1 \end{bmatrix} + 0(z^{-4}), \tag{C.12}$$

and

$$\underline{\underline{D}}(z) \sim \begin{bmatrix} 0 & 1 & 0 \\ -3/z^2 & 3/z - 1/z^3 & 9/4z^2 \\ -4/3z^3 & 0 & -3/z^3 \end{bmatrix} + 0(z^{-4}) . \tag{C.13}$$

A shearing transformation, defined as

$$\underline{\underline{S}}(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/z & 0 \\ 0 & 0 & 1/z^2 \end{bmatrix} , \qquad (C.14)$$

now gives a vector differential equation,

$$\frac{d}{dz} \underline{Y}_3(z) = \underline{\underline{E}}(z) \cdot \underline{Y}_3(z) , \qquad (C.15)$$

in which

$$\underline{Y}_2(z) = \underline{\underline{S}}(z) \cdot \underline{Y}_3(z) , \qquad (C.16)$$

Performing the necessary algebra then shows that

$$\underline{\underline{E}}(z) \sim \begin{bmatrix} 0 & 1/z & 0 \\ -3/z + 6/z^3 & 4/z - 1/z^3 & 9/4z^3 \\ -4/3z - 20/3z^3 & 0 & 2/z - 3/z^3 \end{bmatrix} + 0(z^{-4}), \qquad (C.18)$$

and the system has been transformed to one with $E_0 = 0$. The problem has consequently been reduced to one with a regular singular point for which it is relatively easy to obtain asymptotic representations of the solution matrix.

Consider now the differential system recast in the usual form associated with regular singular points,

$$x \frac{d}{dx} \underline{Y}_3(x) = \underline{\underline{F}}(x) \cdot \underline{Y}_3(x) , \qquad (C.19)$$

where

$$\underline{\underline{F}}(x) = - \frac{1}{x} \underline{\underline{E}}(1/x) . \qquad (C.20)$$

The eigenvalues of $F_0$ are -1, -2 and -3 and it is convenient to diagonalize $F_0$ with the similarity transformation,

$$\underline{\underline{T}}_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 3 \\ 4/3 & 1 & -4/3 \end{bmatrix} , \tag{C.21}$$

which yields the new differential equation,

$$x \frac{d}{dx} \underline{Y}_4(x) = \underline{\underline{G}}(x) \cdot \underline{Y}_4(x) \quad . \tag{C.22}$$

The new variables are defined by the expressions ,

$$\underline{Y}_3(x) = \underline{\underline{T}}_1 \cdot \underline{Y}_4(x) \quad , \tag{C.23}$$

and

$$\underline{\underline{G}}(x) \sim \begin{bmatrix} -1+4x^2 & 9x^2/8 & 0 \\ 0 & -2 & 8x^2/3 \\ -4x^2 & -9x^2/8 & -3 \end{bmatrix} + 0(x^4) \, . \tag{C.24}$$

The two shearing transformations,

$$\underline{\underline{S}}_1(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/x \end{bmatrix} , \tag{C.25}$$

and

$$\underline{\underline{S}}_2(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/x & 0 \\ 0 & 0 & 1/x \end{bmatrix} , \tag{C.26}$$

are now applied in succession to give $\underline{Y}_5$ defined by the relation,

$$\underline{Y}_4(x) = \underline{\underline{S}}_1(x) \cdot \underline{\underline{S}}_2(x) \cdot \underline{Y}_5(x) \quad . \tag{C.27}$$

The differential equation,

$$x \frac{d}{dx} \underline{Y}_5(x) = \underline{\underline{H}}(x) \cdot \underline{Y}_5(x) \quad , \tag{C.28}$$

results in which

$$\underline{\underline{H}}(x) \sim \begin{bmatrix} -1 + 4x^2 & 9x/8 & 9x^2/2 \\ 0 & -1 & 8x/3 \\ 0 & 0 & -1 \end{bmatrix} + 0(x^3). \qquad \text{(C. 29)}$$

Only one more transformation is necessary to complete this analysis and it is formally very similar to the transformation (C. 8) used earlier. A new variable $\underline{Y}_6(x)$ is defined by the expression,

$$\underline{Y}_5(x) = \underline{\underline{Q}}(x) \cdot \underline{Y}_6(x), \qquad \text{(C. 30)}$$

and the resulting differential equation is

$$x \frac{d}{dx} \underline{Y}_6(x) = \underline{\underline{J}}(x) \cdot \underline{Y}_6(x) , \qquad \text{(C. 31)}$$

in which

$$\underline{\underline{J}}(x) = \underline{\underline{Q}}^{-1}(x) \cdot \underline{\underline{H}}(x) \cdot \underline{\underline{Q}}(x) - x\underline{\underline{Q}}^{-1}(x) \cdot \frac{d}{dx} \underline{\underline{Q}}(x) . \qquad \text{(C. 32)}$$

The aim is, of course, to determine $\underline{\underline{Q}}(x)$ in such a way that $\underline{\underline{J}}(x)$ in (C. 31) becomes as simple as possible, preferably so simple that an explicit solution is possible. Insertion of assumed series expansions into equation (C. 32) and performance of the necessary algebra provide the following equivalent formulation,

$$H_0 \cdot Q_0 - Q_0 \cdot J_0 = 0 , \qquad \text{(C. 33)}$$

$$(H_0 - r\underline{\underline{I}}) \cdot Q_r - Q_r \cdot J_0 = \sum_{s=0}^{r-1} (Q_s \cdot J_{r-s} - H_{r-s} \cdot Q_s), \quad r>0. \qquad \text{(C. 34)}$$

These expressions are analagous to equations (B. 16) and (B. 17) in the essential singular point analysis of Appendix B.

Again an obvious choice for satisfying Eq. (C. 33) is

$$Q_0 = \underline{\underline{I}} , \qquad \text{(C. 35)}$$

and

$$J_0 = H_0. \qquad \text{(C. 36)}$$

Finally, it may be shown that independently of the right hand side a matrix equation of the form (C. 34) has a unique solution for $Q_r$ provided that $H_0$ and $H_0 - r\underline{\underline{I}}$ have no eigenvalues in common. This is the motivation for applying the two immediately preceding shearing transformations which made the eigenvalues of $H_0$ identical. Since the right hand side of (C. 34) is free at this point, the simplest possible choice for $\underline{\underline{J}}(x)$ is made, namely $J_0 = H_0$ and $J_r = 0$ for $r > 0$. Equation (C. 34) then provides a convenient formula for calculating successive terms in the expansion of the transformation $\underline{\underline{Q}}(x)$ to give

$$\underline{\underline{Q}}(x) \sim \begin{bmatrix} 1 + 2x^2 & 9x/8 & 15x^2/4 \\ 0 & 1 & 8x/3 \\ 0 & 0 & 1 \end{bmatrix} + 0(x^3) . \qquad \text{(C. 37)}$$

The problem has now been reduced to solving equation (C. 31) with $\underline{\underline{J}}(x) = H_0 = -\underline{\underline{I}}$, which quite simply provides the solution matrix,

$$\underline{\underline{Y}}_6(x) = \frac{1}{x}\underline{\underline{I}} . \qquad \text{(C. 38)}$$

Finding the asymptotic representation for the original third-order differential system (4. 50) is merely an algebraic exercise in reversing the above transformations. The resulting homogeneous solution matrix for the collisionless system is

$$\underline{\underline{\eta}} \sim \begin{bmatrix} -2 - x^2/3 & 0 & -19/3+3x^2/2 \\ -4/3x - 4x/3 & -1-6x^2/5 & -4/3x^3-5x/2 \\ 4x/3 & 1+2x^2 & -4/3x+8x/3 \end{bmatrix} +0(x^3). \qquad \text{(C. 39)}$$

The solution is completed by the addition of a particular integral of the inhomogeneous system (4. 50). One such integral as determined by series substitution is

$$\underline{Y}_0 \sim \left\{ \begin{array}{l} C_2x^2 + \ldots \\ C_2x/3 + 3\pi C_1x^2/5 + (2/9 + \pi/4)C_2x^3 + \ldots \\ -\pi C_1x^2 - (1/9 + \pi/2)C_2x^3 + \ldots \end{array} \right\} . \qquad (C.40)$$

# APPENDIX D

# ASYMPTOTIC EXPANSION OF COLLISIONAL SOLUTION AT INFINITY

A general understanding of the singular nature of the sixth-order moment system near the sphere surface has been obtained in the previous two Appendices, but for a global description of the six eigenfunctions it is necessary to look also at the other singular point of this system. In this Appendix the reduced third-order system (4.50) is written with R as the independent variable and the analysis of Wasow is applied to the study of the point $R = \infty$.

The starting point for this development is again the homogeneous formulation of the differential equation (4.50) and specifically the expression (4.51) for the matrix $\underline{\underline{A}}$. Recasting with R as the independent variable, equation (4.50) becomes

$$\frac{d}{dR} \underline{Y}(R) = R\, \underline{\underline{B}}(R) \cdot \underline{Y}(R) , \qquad (D.1)$$

where R is assumed to be the nondimensional variable previously denoted by a bar superscript. The matrix $\underline{\underline{B}}(R)$ may be represented by the series,

$$\underline{\underline{B}}(R) = \Sigma B_n R^{-n} , \qquad (D.2)$$

in which the $B_n'$s are determined directly from expression (4.51) to be

$$B_0 = \begin{bmatrix} 4/3 & -16/3 & -4/3 \\ 4/3 & -4/3 & -4/3 \\ 0 & -4 & 0 \end{bmatrix} , \qquad (D.3)$$

$$B_1 = \begin{bmatrix} 0 & -4/9Kn & 0 \\ 0 & -4/9Kn & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (D.4)$$

etc. This system has an irregular singular point with rank two at infinity, according to the classification provided by Wasow.

The initial step is the transformation of this system into one with rank one which can then be solved in the same way as the system of Appendix B. The similarity transformation,

$$\underline{\underline{T}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1/2 & 0 \\ 1 & 1/2 & 0 \end{bmatrix}, \quad (D.5)$$

is first applied to yield the system,

$$\frac{d}{dR}\,\underline{Y}_1(R) = R\,\underline{\underline{C}}(R)\cdot\underline{Y}_1(R) \;, \quad (D.6)$$

containing the new variable,

$$\underline{Y}_1(R) = \underline{\underline{T}}^{-1}\cdot\underline{Y}(R). \quad (D.7)$$

The resulting matrix $\underline{\underline{C}}(R) = \Sigma\, C_n R^{-n}$ is given by

$$\underline{\underline{C}}(R) = \underline{\underline{T}}^{-1}\cdot\underline{\underline{B}}(R)\cdot\underline{\underline{T}} \;, \quad (D.8)$$

from which

$$C_0 = \begin{bmatrix} 0 & -2 & -4/3 \\ 0 & 0 & 8/3 \\ 0 & 0 & 0 \end{bmatrix}, \quad (D.9)$$

$$C_1 = \begin{bmatrix} 0 & 2/9Kn & 0 \\ 0 & -4/9Kn & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (D.10)$$

etc. The purpose of this transformation is to provide an upper diagonal form for the first term $C_0$ which then permits the shearing transformation,

$$\underline{\underline{S}}(R) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/R & 0 \\ 0 & 0 & 1/R^2 \end{bmatrix}, \tag{D. 11}$$

to be used to lower the rank of the system. The resulting rank one system is

$$\frac{d}{dR} \underline{Y}_2(R) = \underline{\underline{D}}(R) \cdot \underline{Y}_2(R), \tag{D. 12}$$

in which

$$\underline{Y}_2(R) = \underline{\underline{S}}^{-1}(R) \cdot \underline{Y}_1(R), \tag{D. 13}$$

and

$$\underline{\underline{D}}(R) = R\, \underline{\underline{S}}^{-1}(R) \cdot \underline{\underline{C}}(R) \cdot \underline{\underline{S}}(R) - \underline{\underline{S}}^{-1}(R) \cdot \frac{d}{dR} \underline{\underline{S}}(R) . \tag{D. 14}$$

Performing the necessary algebra reveals $\underline{\underline{D}}(R)$ to have the components,

$$D_0 = \begin{bmatrix} 0 & -2 & 0 \\ 0 & -4/9Kn & 8/3 \\ 0 & 0 & 0 \end{bmatrix}, \tag{D. 15}$$

$$D_1 = \begin{bmatrix} 3 & 2/9Kn & -4/3 \\ 0 & 1 & 0 \\ 0 & 1/3Kn & -1 \end{bmatrix}, \tag{D. 16}$$

etc.

The system (D. 12) is now in a form directly analagous to equation (B. 1), and the remaining analysis follows step by step that described in Appendix B. The matrix $D_0$ is first block diagonalized through application of the similarity transformation,

$$\underline{\underline{T}}_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2/9Kn \\ 0 & 1/6Kn & 0 \end{bmatrix} , \tag{D. 17}$$

which yields the differential equation,

$$\frac{d}{dR} \underline{Y}_3(R) = \underline{\underline{E}}(R) \cdot \underline{Y}_3(R) , \tag{D. 18}$$

for the variable,

$$\underline{Y}_3(R) = \underline{\underline{T}}_1^{-1} \cdot \underline{Y}_2(R) . \tag{D. 19}$$

As before, the next step is to transform $\underline{\underline{E}}(R)$ into a matrix $\underline{\underline{F}}(R)$ which is fully block diagonal in the pattern of $E_0$. The formal transformation $\underline{\underline{P}}(R)$ which accomplishes this may be found term by term from a series of matrix equations of the form (B. 16) and (B. 17). The resulting differential system is

$$\frac{d}{dR} \underline{Y}_4(R) = \underline{\underline{F}}(R) \cdot \underline{Y}_4(R) , \tag{D. 20}$$

in which

$$\underline{Y}_4(R) = \underline{\underline{P}}^{-1}(R) \cdot \underline{Y}_3(R) . \tag{D. 21}$$

The matrix $\underline{\underline{F}}(R)$, which is given by

$$\underline{\underline{F}}(R) = \underline{\underline{P}}^{-1}(R) \cdot \underline{\underline{E}}(R) \cdot \underline{\underline{P}}(R) - \underline{\underline{P}}^{-1}(R) \cdot \frac{d}{dR} \underline{\underline{P}}(R) , \tag{D. 22}$$

is easily shown to be

$$\underline{\underline{F}}(R) \sim \begin{bmatrix} \frac{3}{R} - \frac{9Kn}{R^2} & -2 + (\frac{3}{2} + \frac{405Kn^2}{4})\frac{1}{R^2} & 0 \\ \frac{-3}{R^2} & \frac{1}{R} + \frac{45Kn}{2R^2} & 0 \\ 0 & 0 & \frac{-4}{9Kn} - \frac{1}{R} - (\frac{1}{3Kn} + \frac{27Kn}{2})\frac{1}{R^2} \end{bmatrix} + 0(R^{-3}) \ . \tag{D. 23}$$

The differential system (D. 20) has now been reduced to two simpler differential systems: one a scalar equation for the third component of the vector $\underline{Y}_4$, and a second-order system which still contains a regular singular point at $R = \infty$. Although similar to the preceding discussion, the second-order analysis is relatively simple and the details will not be presented here. Combining the results of the reduced systems gives

$$\underline{Y}_4(R) \sim \begin{bmatrix} 2 - \frac{9Kn}{2R} + \ldots & R^5 - \frac{27Kn}{2} R^4 + \ldots & 0 \\ \frac{3}{R} - \frac{18Kn}{R^2} + \ldots & -R^4 + \frac{9Kn}{4} R^3 + \ldots & 0 \\ 0 & 0 & e^{-4R/9Kn}\left\{\frac{1}{R} + \ldots\right\} \end{bmatrix} \tag{D. 24}$$

as a solution matrix for the system (D. 20).

The asymptotic representation for the original third-order differential system (4. 50) is once again found by inverting the series of transformations described above. The resulting homogeneous solution matrix valid in the neighborhood of $R = \infty$ is

$$\underline{\underline{\eta}} \sim \begin{bmatrix} 2 - \frac{9Kn}{2R} + \ldots & R^5 - \frac{27Kn}{2} R^4 + \ldots & e^{-4R/9Kn}\{\frac{1}{R} + \ldots\} \\ \frac{3}{2R^2} - \frac{9Kn}{2R^3} + \ldots & -\frac{1}{2}R^3 + \frac{27Kn}{8} R^2 + \ldots & e^{-4R/9Kn}\left\{\frac{1}{9KnR^2} + \ldots\right\} \\ 2 - \frac{9Kn}{2R} + \ldots & R^5 - \frac{27Kn}{2} R^4 + \ldots & e^{-4R/9Kn}\left\{\frac{1}{R} + \ldots\right\} \end{bmatrix} . \tag{D.25}$$

Note that only one regular solution vector (column one) exists at infinity.

The solution is completed by the addition of a particular integral of the inhomogeneous equation (4.50). Expanding the inhomogeneous vector (4.52) in negative powers of R and substituting a trial series provides the following particular integral,

$$\underline{Y}_0 \sim \begin{Bmatrix} 0(R^{-3}) \\ -\frac{\pi}{2}(C_1 + \frac{3}{4}C_2) + \frac{3\pi C_2}{16R^2} + 0(R^{-3}) \\ \frac{\pi}{2}(C_1 + \frac{3}{4}C_2) + \frac{\pi C_1}{6KnR} - \frac{3\pi C_2}{16R^2} + 0(R^{-3}) \end{Bmatrix} . \qquad (D.26)$$

This vector plus the solution matrix (D.25) times an arbitrary constant vector completes the asymptotic representation of the third-order system at infinity.

# APPENDIX E

## FREE-MOLECULE SOLUTION ABOUT TRANSLATING SPHERE

An exact collisionless solution for slow flow past a sphere provides one limiting theoretical evaluation with which to compare the moment solution for Kn = $\infty$. All free-molecule integral moments are readily evaluated by velocity space integrations of a simple "two-sided Maxwellian" distribution function. Details of these integrations and a table of several resulting flow quantities are presented in this Appendix.

The necessary velocity space integrations for the collisionless flow are similar to those required for the moment method, and the geometry and notation described in Section 2. 2. may again be used (see Fig. 2). The distribution function is discontinuous on the sphere grazing cone as before and is characterized by

$$f = f_1 \quad \text{for} \quad 0 < \sigma < \frac{\pi}{2} - \alpha$$
$$f = f_2 \quad \text{for} \quad \frac{\pi}{2} - \alpha < \sigma < \pi \,, \tag{E. 1}$$

where

$$\alpha = \cos^{-1} (R_0/R). \tag{E. 2}$$

The moment integrals are then given by the expression,

$$\langle nQ\rangle = \int_0^{\frac{\pi}{2}-\alpha} \int_0^{2\pi} \int_0^{\infty} Q f_1 \xi^2 \sin\sigma \, d\xi \, d\tau \, d\sigma + \int_{\frac{\pi}{2}-\alpha}^{\pi} \int_0^{2\pi} \int_0^{\infty} Q f_2 \xi^2 \sin\sigma \, d\xi \, d\tau \, d\sigma \,. \tag{E. 3}$$

As a prerequisite for the evaluation of the moment integrals it is necessary to express the distribution function f explicitly as a function of $\underline{R}$ and $\underline{\xi}$. This is quite easy in the case of $f_2$, the ambient part of the distribution, since it is a Maxwellian with mean velocity $\underline{U}$. Thus, the linearized form is

$$f_2 = f_\infty \left[ 1 - 2\beta_0^2 \xi_R U \cos\theta + 2\beta_0^2 \xi_\theta U \sin\theta \right] , \qquad \text{(E. 4)}$$

which may also be found from Eq. (2. 12) by setting

$$u_2 = -U \cos\theta , \qquad \text{(E. 5)}$$

$$v_2 = U \sin\theta \qquad \text{(E. 6)}$$

and the other parameters equal to zero.

The correct expression for the sphere influenced portion of the distribution function is more difficult to obtain because the effective number density is not constant over the sphere as was assumed in the moment method. With the assumptions of diffuse re-emission and total energy accomodation, the sphere influenced distribution may be written as

$$f_1 = f_\infty [1 + N_1(\underline{R}, \underline{\xi})] \ . \qquad \text{(E. 7)}$$

The outgoing number density $N_1(\underline{R}, \underline{\xi})$ is most easily evaluated through the application of a local continuity condition at the sphere surface where $N_1(\underline{R_0}, \underline{\xi}) = N_1(\underline{R_0})$ is independent of $\underline{\xi}$. Local surface continuity is assured by requiring the radial mass-flux to vanish at the sphere. Thus

$$nu_R = N_1(\underline{R_0}) - \tfrac{1}{2}\,\bar{U}\cos\theta = 0 \quad \text{at} \quad R = R_0 \,, \tag{E. 8}$$

where $\theta$ is the polar angle to the vector $\underline{R_0}$. The distribution on the sphere surface is therefore

$$f_1 = f_\infty[1 + \tfrac{1}{2}\,\bar{U}\cos\theta] \qquad \text{at} \qquad R = R_0 \,. \tag{E. 9}$$

One very important characteristic of free-molecule flows is the constant nature of the velocity distribution function along particle paths. This feature allows the distribution at any point in space to be related to that just found for the sphere surface. In fact, the distribution function at any point $\underline{R}$ is of the similar form,

$$f_1(\underline{R}, \underline{\xi}) = f_\infty[1 + \tfrac{1}{2}\,\bar{U}\cos\delta] \,, \tag{E. 10}$$

where $\delta$ is the polar angle of the point at which the velocity vector $\underline{\xi}$ intersects the sphere (see Fig. 16). It only remains to express $\delta$ as an explicit function of $\underline{R}$ and $\underline{\xi}$, which may be accomplished through the spherical trigonometric relation,

$$\cos\delta = \cos\nu\cos\theta + \sin\nu\sin\theta\cos\tau \,. \tag{E.11}$$

With the introduction of

$$\gamma = \nu + \sigma \tag{E. 12}$$

and

$$s = \sin\gamma = \sin(\nu + \sigma) \,, \tag{E. 13}$$

the angle $\nu$ may be eliminated from Eq. (E. 11) by the expressions,

$$\cos\nu = \cos(\gamma - \sigma) = \cos\gamma \, \cos\sigma + \sin\gamma \, \sin\sigma$$
$$\sin\nu = \sin(\gamma - \sigma) = \sin\gamma \, \cos\sigma - \cos\gamma \sin\sigma \; . \qquad \text{(E. 14)}$$

Finally, $\sigma$ is related to s through the law of sines,

$$y = \sqrt{1-x^2} = \cos\alpha = \frac{R_0}{R} = \frac{\sin\sigma}{\sin(\pi - \nu - \sigma)} = \frac{\sin\sigma}{s} \; , \qquad \text{(E. 15)}$$

and $\delta$ is given by the expression ,

$$\cos\delta = \left[(1-y^2s^2)^{\frac{1}{2}}(1-s^2)^{\frac{1}{2}} + ys^2\right]\cos\theta$$
$$+ \left[s(1-y^2s^2)^{\frac{1}{2}} - ys(1-s^2)^{\frac{1}{2}}\right]\sin\theta\,\cos\tau \; . \qquad \text{(E. 16)}$$

Before evaluating the integral moments it is convenient to replace $\sigma$ by s in the first integral of Eq. (E. 3). The range of integration, $0 < \sigma < \frac{\pi}{2} - \alpha$ , becomes simply $0 < s < 1$, and from Eq. (E.15)

$$d\sigma = \frac{\cos\alpha}{\cos\sigma}\, ds = (1-y^2s^2)^{-\frac{1}{2}}\, y ds \; . \qquad \text{(E. 17)}$$

Thus, with $f_1$ and $f_2$ given by Eqs. (E. 10) and (E. 4) the required result is

$$\langle nQ\rangle = \int_0^1\int_0^{2\pi}\int_0^{\infty} Q\, f_1\, \xi^2\, sy^2(1-y^2s^2)^{-\frac{1}{2}}\, d\xi d\tau ds$$
$$+ \int_{\frac{\pi}{2}-\alpha}^{\pi}\int_0^{2\pi}\int_0^{\infty} Q\, f_2\, \xi^2 \sin\sigma\, d\xi\, d\tau\, d\sigma \; . \qquad \text{(E. 18)}$$

Table A. 5 contains a number of definite integrals in s which simplify the evaluation of the first part of Eq. (E. 18). The second part is very similar to the integrations needed in the moment method and may consequently be determined with the aid of Table A. 4. A sequence of collisionless integral moments is collected in the following Table.

TABLE E. 1

$(x = \sqrt{1 - R_0^2/R^2}, \; y = R_0/R)$

| Number | Integral Moment |
|---|---|
| E. 1. 1 | $n = n_\infty + n_\infty \bar{U}\left[\left(\frac{1}{2\pi} + \frac{1}{12}\right)y^2 - \frac{xy}{4} + \frac{1 - x^3}{6y}\right]\cos\theta$ |
| E. 1.2 | $nu_R = n_\infty U\left[-\frac{3}{8} + \frac{y^2}{8} + \frac{y^3}{4} - \frac{x^3}{2} - \frac{x^4}{16y}\log\left(\frac{1+y}{1-y}\right)\right]\cos\theta$ |
| E. 1. 3 | $nu_\theta = n_\infty U\left[\frac{3}{8} + \frac{3x}{4} - \frac{x^2}{16} - \frac{x^3}{4} + \frac{y^3}{8} + \frac{(4-x^2)x^2}{32y}\log\left(\frac{1+y}{1-y}\right)\right]\sin\theta$ |
| E. 1. 4 | $P_{RR} = -n_\infty kT\left[1 + \frac{\bar{U}}{4}\left\{y^2 - \frac{2y^4}{5} + \frac{(2-5x^3+3x^5)}{5y} + \frac{4}{\pi}(1-x^4)\right\}\cos\theta\right]$ |
| E. 1. 5 | $P_{\theta\theta} = P_{\varphi\varphi} = -n_\infty kT\left[1 + \frac{\bar{U}}{8}\left\{\frac{2y^4}{5} + \frac{(8-15x+10x^3-3x^5)}{5y} + \frac{4y^4}{\pi}\right\}\cos\theta\right]$ |
| E. 1. 6 | $P_{R\theta} = p_{R\theta} = n_\infty kT\,\frac{\bar{U}}{8}\left[\frac{2y^4}{5} - \frac{(2-5x^3+3x^5)}{5y} + \frac{4y^4}{\pi}\right]\sin\theta$ |

# APPENDIX F

## FREE-MOLECULE SOLUTION ABOUT ROTATING SPHERE

The collisionless flow induced by a sphere rotating in an unbounded gas is represented exactly by a "two stream Maxwellian" as was that about the translating sphere. The evaluation of the velocity field and other integral moments for this flow parallels the development of Appendix E, and the geometry and trigonometric relationships discussed therein are assumed (see Figs. 2 and 16). The distribution function is expanded in powers of the surface equatorial Mach number, and both first and second-order integral moments are evaluated. The secondary velocity field is of particular interest as a limiting solution with which to compare the second-order moment method solution.

Although the basic specification of the distribution function is not conceptually difficult, the explicit form needed for the moment evaluation is rather long and the integrations are consequently somewhat tedious. Since the gas is assumed stagnant at infinity the ambient part of the distribution is simply

$$f_2 = f_\infty \quad , \qquad \text{(F. 1)}$$

a steady Maxwellian at ambient conditions.

Because of the steady ambient distribution, the incoming mass flux is constant over the sphere surface, and the outstreaming number density is therefore constant everywhere and equal to $n_\infty$. The temperature in $f_1$ is also constant from the assumption of complete energy accomodation. Again using the constant character of collisionless distribution functions along particle paths, the sphere influenced portion of the velocity distribution is

$$f_1 = \frac{n_\infty \beta_0^3}{\pi^{\frac{3}{2}}} \exp\left[-\beta_0^2 (\underline{\xi} - \underline{\omega} \times \underline{R_0})^2\right] , \qquad \text{(F. 2)}$$

where $\underline{R_0}$ is the point at which the velocity vector $\underline{\xi}$ intersects the sphere. With $\beta_0 |\underline{\omega} \times \underline{R_0}| << 1$ the exponential expression in Eq. (F. 2) may be expanded through second order to give

$$f_1 = f_\infty \left[1 + 2\beta_0^2 \underline{\xi} \cdot (\underline{\omega} \times \underline{R_0}) - \beta_0^2 (\underline{\omega} \times \underline{R_0})^2 + 2\beta_0^4 \{\underline{\xi} \cdot (\underline{\omega} \times \underline{R_0})\}^2\right] . \qquad \text{(F. 3)}$$

This distribution may now be simplified through the colinearity of $\underline{R} - \underline{R_0}$ and $\underline{\xi}$ to give

$$f_1 = f_\infty \left[1 + 2\beta_0^2 \underline{\xi} \cdot (\underline{\omega} \times \underline{R}) - \beta_0^2 (\underline{\omega} \times \underline{R_0})^2 + 2\beta_0^4 \{\underline{\xi} \cdot (\underline{\omega} \times \underline{R})\}^2\right] , \qquad \text{(F. 4)}$$

which may be written as

$$f_1 = f_\infty \left[1 + 2\beta_0^2 \xi_\varphi \omega R \sin\theta - \beta_0^2 \omega^2 R_0^2 \sin^2\delta + 2\beta_0^4 \xi_\varphi^2 \omega^2 R^2 \sin^2\theta\right] . \qquad \text{(F.5)}$$

The zeroth-order velocity distribution is thus $f_\infty$ throughout velocity space, and all higher orders exist only within the "cone of body influence."

The first-order integral moments are very easily integrated with the help of Table A. 4. For example,

$$nu_\varphi = \int_{\text{Region 1}} f_1^{(1)} \xi_\varphi \, d\underline{\xi} = 2\beta_0^2 \omega R \sin\theta \int_{\text{Region 1}} f_\infty \xi_\varphi^2 \, d\underline{\xi}$$

$$= n_\infty \omega R \sin\theta \left(\frac{1}{2} - \frac{3}{4}x + \frac{1}{4}x^3\right) , \qquad \text{(F. 6)}$$

where integral A. 4. 4 is utilized.

The first-order integral moments are collected in Table F. 1.

From Eq. (F. 5) the second-order velocity distribution is

$$f_1^{(2)} = f_\infty \left[2\beta_0^4 \xi_\varphi^2 \omega^2 R^2 \sin^2\theta - \beta_0^2 \omega^2 R_0^2 \sin^2\delta\right] \qquad \text{(F. 7)}$$

The first term in this expression is also integrable directly from Table A. 4, but the second term requires a bit more effort. The expression (E. 16) for cos δ may be used to give

$$\begin{aligned} \sin^2\delta = 1 - \cos^2\delta &= \sin^2\theta \\ &+ [2y^2s^4 - s^2 - y^2s^2 + 2ys^2(1-s^2)^{\frac{1}{2}}(1-y^2s^2)^{\frac{1}{2}}](\sin^2\theta\cos^2\tau - \cos^2\theta) \\ &+ [(2s-4s^3)y(1-y^2s^2)^{\frac{1}{2}} - (2s-4y^2s^3)(1-s^2)^{\frac{1}{2}}]\cos\theta\sin\theta\cos\tau\,. \end{aligned}$$

(F. 8)

The second part of any second-order integral moment is thus given by

$$-\beta_0^2\,\omega^2 R_0^2 \int_0^1 \int_0^{2\pi} \int_0^\infty Q f_\infty \sin^2\delta (1-s^2y^2)^{-\frac{1}{2}} sy^2 d\xi\, d\tau\, ds\,, \qquad \text{(F. 9)}$$

where $\sin^2\delta$ is substituted from above. Although tedious, these second-order integral moments can now be evaluated directly with the assistance of Table A. 5 for the s integrals. A listing of the simpler secondary moments is provided in Table F. 2.

TABLE F. 1

$(x = \sqrt{1 - R_0^2/R^2})$

| Number | Integral Moment |
|---|---|
| F. 1. 1 | $nu_\varphi^{(1)} = n_\infty \omega R \sin\theta \left(\frac{1}{2} - \frac{3}{4}x + \frac{1}{4}x^3\right)$ |
| F. 1. 2 | $P_{R\varphi}^{(1)} = -\frac{n_\infty kT_\infty}{2\pi}\omega \frac{R_0^4}{R^3} \sin\theta$ |

TABLE F.2

$(x = \sqrt{1 - R_0^2/R^2},\ y = R_0/R)$

| Number | Integral Moment |
|---|---|
| F.2.1 | $\overline{nu}_R^{(2)} = -\frac{1}{4\pi}\left[\frac{y^2}{4} - \frac{y^4}{12} + \frac{y}{3} - \frac{(1+y^2)^2}{8y} + \frac{x^4(1+y^2)}{16y^2}\log\left(\frac{1+y}{1-y}\right)\right] \cdot (3\cos^2\theta - 1)$ |
| F.2.2 | $\overline{nu}_\theta^{(2)} = \frac{1}{4\pi}\left[\frac{y^3}{4} + \frac{y}{12} - \frac{1}{2y} + \frac{y^4}{6} + \frac{x^2}{8y^2}(2+y^2+y^4)\log\left(\frac{1+y}{1-y}\right)\right] \cdot \cos\theta\sin\theta$ |
| F.2.3 | $\overline{\dot{q}_R^*}^{(2)} = \frac{R_0^2}{4\pi R^2}\sin^2\theta + 2\,\overline{nu}_R^{(2)}$ |
| F.2.4 | $\overline{\dot{q}_\theta^*}^{(2)} = 2\,\overline{nu}_\theta^{(2)}$ |
| F.2.5 | $\overline{P_{RR}^*}^{(2)} = -\frac{x^2}{8\pi y^2}\left(1 - \frac{3}{2}x + \frac{1}{2}x^3\right)\sin^2\theta + \frac{1}{8\pi}\left[\frac{1}{35y^2} - \frac{1}{5} + \frac{2y^3}{5} - \frac{8y^5}{35} + \frac{x^5}{10y^2} - \frac{9x^7}{70y^2}\right](1-3\cos^2\theta)$ |
| F.2.6 | $\overline{P_{\theta\theta}^*}^{(2)} = \frac{1}{8\pi y^2}\left(\frac{3}{8}x - x^2 + \frac{3}{4}x^3 - \frac{1}{8}x^5\right)\sin^2\theta + \frac{1}{8\pi}\left[\frac{8}{7y^2} - \frac{8}{5} + \frac{16y^5}{35} - \frac{x^3}{y^2} - \frac{2x^5}{5y^2} + \frac{9x^7}{35y^2}\right]\left(\frac{3}{8}\sin^2\theta - \frac{1}{2}\cos^2\theta\right)$ |
| F.2.7 | $\overline{P_{\varphi\varphi}^*}^{(2)} = \frac{-1}{8\pi y^2}\left(2 - \frac{33}{8}x + x^2 + \frac{7}{4}x^3 - \frac{5}{8}x^5\right)\sin^2\theta + \frac{1}{8\pi}\left[\frac{8}{7y^2} - \frac{8}{5} + \frac{16y^5}{35} - \frac{x^3}{y^2} - \frac{2x^5}{5y^2} + \frac{9x^7}{35y^2}\right]\left(\frac{1}{8}\sin^2\theta - \frac{1}{2}\cos^2\theta\right)$ |
| F.2.8 | $\overline{P_{R\theta}^*}^{(2)} = \frac{1}{8\pi}\left[\frac{2}{5} - \frac{2}{5}y^3 + \frac{16}{35}y^5 - \frac{16}{35y^2} + \frac{x^3}{y^2} - \frac{4x^5}{5y^2} + \frac{9x^7}{35y^2}\right] \cdot \cos\theta\sin\theta$ |

# APPENDIX G

## NUMERICAL VALUES OF MOMENT PARAMETERS FOR TRANSLATING SPHERE

An analytic solution of the sixth-order moment system consisting of Eqs. (4.17) to (4.22) was impossible, and a numerical evaluation of the six parametric quantities was necessary. The above six linear differential equations were written as central finite difference equations for each of fifty cells in the internal $0 < Z < 1$, where the new independent variable,

$$Z = 1 - \sqrt{1 - x} = 1 - \sqrt{1 - \sqrt{1 - R_0^2/R^2}} \ , \qquad (G.1)$$

was used for numerical convenience. This transformation eliminated large values for the slopes of some of the parameters near $x = Z = 1$. The resulting system of algebraic equations and the six boundary conditions formed a $306 \times 19$ band matrix whose inversion provided the parametric functions at each of the fifty one grid points. The integral relations described in Section 4.5 provided useful checks on the entire procedure. The numerical solutions are tabulated on the following pages for representative values of the Knudsen number.

KNUDSEN NUMBER = INFINITY

TABLE G.1

| Z | $\tilde{N}_1$ | $\tilde{U}_1$ | $\tilde{V}_1$ | $\tilde{N}_2$ | $\tilde{U}_2$ | $\tilde{V}_2$ |
|---|---|---|---|---|---|---|
| 0. | 0.5036 | 0. | 0. | -0.0027 | -1.0126 | 1.0126 |
| 0.02 | 0.4997 | 0.0062 | 0.0010 | -0.0066 | -1.0188 | 1.0124 |
| 0.04 | 0.4998 | 0.0060 | 0.0038 | -0.0065 | -1.0186 | 1.0120 |
| 0.06 | 0.4988 | 0.0074 | 0.0082 | -0.0075 | -1.0201 | 1.0113 |
| 0.08 | 0.4981 | 0.0085 | 0.0141 | -0.0082 | -1.0212 | 1.0105 |
| 0.10 | 0.4976 | 0.0091 | 0.0213 | -0.0087 | -1.0219 | 1.0097 |
| 0.12 | 0.4973 | 0.0094 | 0.0296 | -0.0090 | -1.0222 | 1.0088 |
| 0.14 | 0.4972 | 0.0093 | 0.0391 | -0.0092 | -1.0223 | 1.0080 |
| 0.16 | 0.4971 | 0.0089 | 0.0497 | -0.0092 | -1.0222 | 1.0072 |
| 0.18 | 0.4973 | 0.0082 | 0.0612 | -0.0091 | -1.0218 | 1.0065 |
| 0.20 | 0.4975 | 0.0072 | 0.0738 | -0.0089 | -1.0212 | 1.0058 |
| 0.22 | 0.4977 | 0.0060 | 0.0873 | -0.0086 | -1.0206 | 1.0051 |
| 0.24 | 0.4981 | 0.0045 | 0.1018 | -0.0083 | -1.0197 | 1.0045 |
| 0.26 | 0.4984 | 0.0027 | 0.1173 | -0.0079 | -1.0189 | 1.0039 |
| 0.28 | 0.4988 | 0.0007 | 0.1338 | -0.0075 | -1.0179 | 1.0034 |
| 0.30 | 0.4993 | -0.0015 | 0.1513 | -0.0071 | -1.0169 | 1.0029 |
| 0.32 | 0.4997 | -0.0039 | 0.1700 | -0.0066 | -1.0158 | 1.0025 |
| 0.34 | 0.5002 | -0.0065 | 0.1897 | -0.0061 | -1.0148 | 1.0022 |
| 0.36 | 0.5006 | -0.0094 | 0.2106 | -0.0057 | -1.0137 | 1.0018 |
| 0.38 | 0.5011 | -0.0124 | 0.2329 | -0.0052 | -1.0126 | 1.0015 |
| 0.40 | 0.5015 | -0.0157 | 0.2565 | -0.0048 | -1.0116 | 1.0013 |
| 0.42 | 0.5020 | -0.0191 | 0.2816 | -0.0044 | -1.0105 | 1.0010 |
| 0.44 | 0.5024 | -0.0228 | 0.3084 | -0.0039 | -1.0095 | 1.0008 |
| 0.46 | 0.5028 | -0.0267 | 0.3369 | -0.0035 | -1.0086 | 1.0007 |
| 0.48 | 0.5032 | -0.0307 | 0.3675 | -0.0032 | -1.0077 | 1.0005 |
| 0.50 | 0.5035 | -0.0349 | 0.4003 | -0.0028 | -1.0068 | 1.0004 |
| 0.52 | 0.5039 | -0.0394 | 0.4355 | -0.0025 | -1.0060 | 1.0003 |
| 0.54 | 0.5042 | -0.0440 | 0.4736 | -0.0021 | -1.0052 | 1.0002 |
| 0.56 | 0.5045 | -0.0488 | 0.5148 | -0.0019 | -1.0045 | 1.0002 |
| 0.58 | 0.5047 | -0.0538 | 0.5596 | -0.0016 | -1.0039 | 1.0001 |
| 0.60 | 0.5050 | -0.0589 | 0.6085 | -0.0014 | -1.0033 | 1.0001 |
| 0.62 | 0.5052 | -0.0642 | 0.6622 | -0.0011 | -1.0028 | 1.0000 |
| 0.64 | 0.5054 | -0.0697 | 0.7214 | -0.0009 | -1.0023 | 1.0000 |
| 0.66 | 0.5056 | -0.0753 | 0.7872 | -0.0008 | -1.0019 | 1.0000 |
| 0.68 | 0.5057 | -0.0811 | 0.8607 | -0.0006 | -1.0015 | 1.0000 |
| 0.70 | 0.5058 | -0.0871 | 0.9435 | -0.0005 | -1.0012 | 1.0000 |
| 0.72 | 0.5059 | -0.0932 | 1.0376 | -0.0004 | -1.0009 | 1.0000 |
| 0.74 | 0.5060 | -0.0994 | 1.1456 | -0.0003 | -1.0007 | 1.0000 |
| 0.76 | 0.5061 | -0.1058 | 1.2710 | -0.0002 | -1.0005 | 1.0000 |
| 0.78 | 0.5062 | -0.1123 | 1.4185 | -0.0002 | -1.0004 | 1.0000 |
| 0.80 | 0.5062 | -0.1189 | 1.5948 | -0.0001 | -1.0003 | 1.0000 |
| 0.82 | 0.5062 | -0.1257 | 1.8095 | -0.0001 | -1.0002 | 1.0000 |
| 0.84 | 0.5063 | -0.1325 | 2.0770 | -0.0000 | -1.0001 | 1.0000 |
| 0.86 | 0.5063 | -0.1395 | 2.4201 | -0.0000 | -1.0001 | 1.0000 |
| 0.88 | 0.5063 | -0.1466 | 2.8766 | -0.0000 | -1.0000 | 1.0000 |
| 0.90 | 0.5063 | -0.1538 | 3.5146 | -0.0000 | -1.0000 | 1.0000 |
| 0.92 | 0.5063 | -0.1611 | 4.4706 | -0.0000 | -1.0000 | 1.0000 |
| 0.94 | 0.5063 | -0.1685 | 6.0628 | -0.0000 | -1.0000 | 1.0000 |
| 0.96 | 0.5063 | -0.1762 | 9.2431 | -0.0000 | -1.0000 | 1.0000 |
| 0.98 | 0.5062 | -0.1842 | 18.5052 | -0.0000 | -1.0000 | 1.0000 |
| 1.00 | 0.4888 | -0.1791 | 117.5373 | 0. | -1.0000 | 1.0000 |

KNUDSEN NUMBER = 3.000 TABLE G. 2

| Z | $\tilde{N}_1$ | $\tilde{U}_1$ | $\tilde{V}_1$ | $\tilde{N}_2$ | $\tilde{U}_2$ | $\tilde{V}_2$ |
|---|---|---|---|---|---|---|
| 0. | 0.4451 | 0. | 0. | -0.0066 | -0.9034 | 0.9285 |
| 0.02 | 0.4429 | 0.0034 | 0.0009 | -0.0088 | -0.9068 | 0.9284 |
| 0.04 | 0.4425 | 0.0040 | 0.0036 | -0.0092 | -0.9075 | 0.9282 |
| 0.06 | 0.4416 | 0.0053 | 0.0078 | -0.0101 | -0.9090 | 0.9280 |
| 0.08 | 0.4410 | 0.0061 | 0.0133 | -0.0107 | -0.9102 | 0.9279 |
| 0.10 | 0.4406 | 0.0065 | 0.0201 | -0.0111 | -0.9111 | 0.9279 |
| 0.12 | 0.4403 | 0.0065 | 0.0280 | -0.0114 | -0.9118 | 0.9280 |
| 0.14 | 0.4402 | 0.0061 | 0.0370 | -0.0114 | -0.9122 | 0.9282 |
| 0.16 | 0.4403 | 0.0054 | 0.0470 | -0.0114 | -0.9125 | 0.9286 |
| 0.18 | 0.4405 | 0.0043 | 0.0578 | -0.0112 | -0.9127 | 0.9291 |
| 0.20 | 0.4407 | 0.0028 | 0.0696 | -0.0110 | -0.9128 | 0.9298 |
| 0.22 | 0.4410 | 0.0011 | 0.0823 | -0.0107 | -0.9129 | 0.9305 |
| 0.24 | 0.4414 | -0.0010 | 0.0958 | -0.0103 | -0.9130 | 0.9314 |
| 0.26 | 0.4418 | -0.0033 | 0.1102 | -0.0099 | -0.9131 | 0.9325 |
| 0.28 | 0.4423 | -0.0060 | 0.1254 | -0.0094 | -0.9132 | 0.9336 |
| 0.30 | 0.4428 | -0.0090 | 0.1415 | -0.0089 | -0.9133 | 0.9349 |
| 0.32 | 0.4433 | -0.0123 | 0.1585 | -0.0084 | -0.9136 | 0.9362 |
| 0.34 | 0.4438 | -0.0159 | 0.1764 | -0.0079 | -0.9139 | 0.9376 |
| 0.36 | 0.4443 | -0.0198 | 0.1952 | -0.0074 | -0.9144 | 0.9391 |
| 0.38 | 0.4448 | -0.0240 | 0.2151 | -0.0068 | -0.9149 | 0.9406 |
| 0.40 | 0.4454 | -0.0285 | 0.2360 | -0.0063 | -0.9156 | 0.9422 |
| 0.42 | 0.4459 | -0.0333 | 0.2581 | -0.0058 | -0.9164 | 0.9439 |
| 0.44 | 0.4463 | -0.0383 | 0.2815 | -0.0053 | -0.9174 | 0.9457 |
| 0.46 | 0.4468 | -0.0437 | 0.3062 | -0.0049 | -0.9185 | 0.9474 |
| 0.48 | 0.4473 | -0.0494 | 0.3323 | -0.0044 | -0.9198 | 0.9492 |
| 0.50 | 0.4477 | -0.0554 | 0.3601 | -0.0040 | -0.9213 | 0.9511 |
| 0.52 | 0.4481 | -0.0616 | 0.3896 | -0.0036 | -0.9229 | 0.9530 |
| 0.54 | 0.4485 | -0.0681 | 0.4212 | -0.0032 | -0.9247 | 0.9549 |
| 0.56 | 0.4489 | -0.0749 | 0.4550 | -0.0028 | -0.9266 | 0.9568 |
| 0.58 | 0.4492 | -0.0820 | 0.4912 | -0.0025 | -0.9287 | 0.9587 |
| 0.60 | 0.4495 | -0.0894 | 0.5304 | -0.0022 | -0.9310 | 0.9607 |
| 0.62 | 0.4498 | -0.0970 | 0.5727 | -0.0019 | -0.9334 | 0.9626 |
| 0.64 | 0.4501 | -0.1048 | 0.6188 | -0.0016 | -0.9360 | 0.9646 |
| 0.66 | 0.4503 | -0.1129 | 0.6692 | -0.0014 | -0.9387 | 0.9666 |
| 0.68 | 0.4505 | -0.1212 | 0.7246 | -0.0012 | -0.9416 | 0.9685 |
| 0.70 | 0.4507 | -0.1298 | 0.7859 | -0.0010 | -0.9446 | 0.9705 |
| 0.72 | 0.4509 | -0.1385 | 0.8543 | -0.0008 | -0.9477 | 0.9725 |
| 0.74 | 0.4510 | -0.1475 | 0.9311 | -0.0007 | -0.9510 | 0.9744 |
| 0.76 | 0.4512 | -0.1567 | 1.0181 | -0.0005 | -0.9544 | 0.9764 |
| 0.78 | 0.4513 | -0.1660 | 1.1178 | -0.0004 | -0.9578 | 0.9784 |
| 0.80 | 0.4514 | -0.1755 | 1.2333 | -0.0003 | -0.9614 | 0.9803 |
| 0.82 | 0.4514 | -0.1852 | 1.3688 | -0.0002 | -0.9650 | 0.9823 |
| 0.84 | 0.4515 | -0.1950 | 1.5305 | -0.0002 | -0.9688 | 0.9843 |
| 0.86 | 0.4516 | -0.2050 | 1.7267 | -0.0001 | -0.9725 | 0.9862 |
| 0.88 | 0.4516 | -0.2150 | 1.9704 | -0.0001 | -0.9764 | 0.9882 |
| 0.90 | 0.4516 | -0.2252 | 2.2807 | -0.0001 | -0.9803 | 0.9901 |
| 0.92 | 0.4517 | -0.2354 | 2.6888 | -0.0000 | -0.9842 | 0.9921 |
| 0.94 | 0.4517 | -0.2457 | 3.2451 | -0.0000 | -0.9881 | 0.9941 |
| 0.96 | 0.4517 | -0.2557 | 4.0278 | -0.0000 | -0.9921 | 0.9960 |
| 0.98 | 0.4517 | -0.2652 | 4.9834 | -0.0000 | -0.9960 | 0.9980 |
| 1.00 | 0.4546 | -0.2714 | -21.2799 | 0. | -1.0000 | 1.0000 |

KNUDSEN NUMBER = 1.000 TABLE G. 3

| Z | $\tilde{N}_1$ | $\tilde{U}_1$ | $\tilde{V}_1$ | $\tilde{N}_2$ | $\tilde{U}_2$ | $\tilde{V}_2$ |
|---|---|---|---|---|---|---|
| 0. | 0.3595 | 0. | 0. | -0.0107 | -0.7403 | 0.8020 |
| 0.02 | 0.3586 | 0.0014 | 0.0009 | -0.0116 | -0.7418 | 0.8021 |
| 0.04 | 0.3582 | 0.0020 | 0.0033 | -0.0120 | -0.7427 | 0.8024 |
| 0.06 | 0.3575 | 0.0028 | 0.0072 | -0.0127 | -0.7441 | 0.8030 |
| 0.08 | 0.3570 | 0.0033 | 0.0123 | -0.0131 | -0.7454 | 0.8038 |
| 0.10 | 0.3567 | 0.0033 | 0.0186 | -0.0134 | -0.7465 | 0.8050 |
| 0.12 | 0.3566 | 0.0029 | 0.0260 | -0.0135 | -0.7475 | 0.8065 |
| 0.14 | 0.3566 | 0.0021 | 0.0343 | -0.0135 | -0.7486 | 0.8084 |
| 0.16 | 0.3567 | 0.0008 | 0.0435 | -0.0134 | -0.7496 | 0.8105 |
| 0.18 | 0.3570 | -0.0008 | 0.0535 | -0.0132 | -0.7506 | 0.8129 |
| 0.20 | 0.3573 | -0.0028 | 0.0644 | -0.0129 | -0.7517 | 0.8156 |
| 0.22 | 0.3577 | -0.0055 | 0.0759 | -0.0125 | -0.7529 | 0.8186 |
| 0.24 | 0.3581 | -0.0081 | 0.0882 | -0.0121 | -0.7543 | 0.8218 |
| 0.26 | 0.3586 | -0.0114 | 0.1012 | -0.0116 | -0.7558 | 0.8252 |
| 0.28 | 0.3591 | -0.0151 | 0.1148 | -0.0111 | -0.7575 | 0.8288 |
| 0.30 | 0.3596 | -0.0192 | 0.1291 | -0.0105 | -0.7595 | 0.8326 |
| 0.32 | 0.3602 | -0.0238 | 0.1440 | -0.0100 | -0.7616 | 0.8365 |
| 0.34 | 0.3608 | -0.0287 | 0.1596 | -0.0094 | -0.7641 | 0.8406 |
| 0.36 | 0.3613 | -0.0341 | 0.1758 | -0.0088 | -0.7668 | 0.8448 |
| 0.38 | 0.3619 | -0.0400 | 0.1927 | -0.0082 | -0.7697 | 0.8492 |
| 0.40 | 0.3625 | -0.0462 | 0.2103 | -0.0077 | -0.7730 | 0.8537 |
| 0.42 | 0.3631 | -0.0530 | 0.2286 | -0.0071 | -0.7766 | 0.8582 |
| 0.44 | 0.3636 | -0.0601 | 0.2476 | -0.0065 | -0.7805 | 0.8628 |
| 0.46 | 0.3641 | -0.0677 | 0.2674 | -0.0060 | -0.7848 | 0.8675 |
| 0.48 | 0.3647 | -0.0757 | 0.2881 | -0.0055 | -0.7893 | 0.8723 |
| 0.50 | 0.3652 | -0.0842 | 0.3096 | -0.0050 | -0.7942 | 0.8771 |
| 0.52 | 0.3657 | -0.0931 | 0.3321 | -0.0045 | -0.7994 | 0.8819 |
| 0.54 | 0.3661 | -0.1024 | 0.3556 | -0.0040 | -0.8050 | 0.8868 |
| 0.56 | 0.3665 | -0.1122 | 0.3803 | -0.0036 | -0.8108 | 0.8917 |
| 0.58 | 0.3670 | -0.1225 | 0.4062 | -0.0032 | -0.8170 | 0.8966 |
| 0.60 | 0.3673 | -0.1328 | 0.4335 | -0.0028 | -0.8235 | 0.9016 |
| 0.62 | 0.3677 | -0.1438 | 0.4623 | -0.0025 | -0.8304 | 0.9065 |
| 0.64 | 0.3680 | -0.1550 | 0.4928 | -0.0021 | -0.8375 | 0.9115 |
| 0.66 | 0.3683 | -0.1667 | 0.5253 | -0.0018 | -0.8449 | 0.9164 |
| 0.68 | 0.3686 | -0.1787 | 0.5598 | -0.0016 | -0.8526 | 0.9214 |
| 0.70 | 0.3688 | -0.1910 | 0.5968 | -0.0013 | -0.8605 | 0.9263 |
| 0.72 | 0.3691 | -0.2036 | 0.6366 | -0.0011 | -0.8687 | 0.9312 |
| 0.74 | 0.3693 | -0.2166 | 0.6796 | -0.0009 | -0.8771 | 0.9362 |
| 0.76 | 0.3694 | -0.2297 | 0.7261 | -0.0007 | -0.8858 | 0.9411 |
| 0.78 | 0.3696 | -0.2432 | 0.7769 | -0.0006 | -0.8946 | 0.9460 |
| 0.80 | 0.3697 | -0.2568 | 0.8325 | -0.0004 | -0.9036 | 0.9509 |
| 0.82 | 0.3698 | -0.2707 | 0.8938 | -0.0003 | -0.9128 | 0.9558 |
| 0.84 | 0.3699 | -0.2846 | 0.9615 | -0.0002 | -0.9222 | 0.9607 |
| 0.86 | 0.3700 | -0.2987 | 1.0368 | -0.0001 | -0.9316 | 0.9656 |
| 0.88 | 0.3701 | -0.3129 | 1.1205 | -0.0001 | -0.9412 | 0.9705 |
| 0.90 | 0.3701 | -0.3270 | 1.2137 | -0.0001 | -0.9509 | 0.9754 |
| 0.92 | 0.3701 | -0.3410 | 1.3164 | -0.0000 | -0.9606 | 0.9803 |
| 0.94 | 0.3701 | -0.3548 | 1.4259 | -0.0000 | -0.9704 | 0.9852 |
| 0.96 | 0.3701 | -0.3679 | 1.5263 | -0.0000 | -0.9803 | 0.9901 |
| 0.98 | 0.3701 | -0.3798 | 1.4448 | -0.0000 | -0.9901 | 0.9951 |
| 1.00 | 0.3713 | -0.3908 | -3.8145 | 0. | -1.0000 | 1.0000 |

KNUDSEN NUMBER = 0.500 TABLE G. 4

| Z | $\tilde{N}_1$ | $\tilde{U}_1$ | $\tilde{V}_1$ | $\tilde{N}_2$ | $\tilde{U}_2$ | $\tilde{V}_2$ |
|---|---|---|---|---|---|---|
| 0. | 0.2769 | 0. | 0. | -0.0120 | -0.5777 | 0.6739 |
| 0.02 | 0.2765 | 0.0005 | 0.0008 | -0.0123 | -0.5784 | 0.6742 |
| 0.04 | 0.2763 | 0.0007 | 0.0031 | -0.0125 | -0.5790 | 0.6750 |
| 0.06 | 0.2758 | 0.0012 | 0.0068 | -0.0130 | -0.5804 | 0.6763 |
| 0.08 | 0.2755 | 0.0013 | 0.0116 | -0.0133 | -0.5817 | 0.6782 |
| 0.10 | 0.2753 | 0.0010 | 0.0176 | -0.0135 | -0.5831 | 0.6806 |
| 0.12 | 0.2753 | 0.0002 | 0.0245 | -0.0136 | -0.5846 | 0.6836 |
| 0.14 | 0.2753 | -0.0010 | 0.0324 | -0.0135 | -0.5862 | 0.6870 |
| 0.16 | 0.2755 | -0.0026 | 0.0411 | -0.0134 | -0.5879 | 0.6909 |
| 0.18 | 0.2757 | -0.0047 | 0.0505 | -0.0131 | -0.5898 | 0.6953 |
| 0.20 | 0.2760 | -0.0072 | 0.0607 | -0.0128 | -0.5919 | 0.7001 |
| 0.22 | 0.2764 | -0.0103 | 0.0715 | -0.0124 | -0.5943 | 0.7052 |
| 0.24 | 0.2769 | -0.0139 | 0.0830 | -0.0120 | -0.5969 | 0.7107 |
| 0.26 | 0.2773 | -0.0179 | 0.0950 | -0.0115 | -0.5999 | 0.7166 |
| 0.28 | 0.2778 | -0.0225 | 0.1075 | -0.0110 | -0.6032 | 0.7227 |
| 0.30 | 0.2784 | -0.0277 | 0.1205 | -0.0104 | -0.6069 | 0.7291 |
| 0.32 | 0.2789 | -0.0333 | 0.1340 | -0.0099 | -0.6110 | 0.7357 |
| 0.34 | 0.2795 | -0.0396 | 0.1480 | -0.0093 | -0.6155 | 0.7425 |
| 0.36 | 0.2801 | -0.0464 | 0.1624 | -0.0087 | -0.6205 | 0.7496 |
| 0.38 | 0.2807 | -0.0537 | 0.1772 | -0.0082 | -0.6258 | 0.7568 |
| 0.40 | 0.2812 | -0.0617 | 0.1924 | -0.0076 | -0.6317 | 0.7641 |
| 0.42 | 0.2818 | -0.0702 | 0.2079 | -0.0070 | -0.6380 | 0.7716 |
| 0.44 | 0.2824 | -0.0793 | 0.2239 | -0.0065 | -0.6448 | 0.7792 |
| 0.46 | 0.2829 | -0.0890 | 0.2403 | -0.0059 | -0.6522 | 0.7868 |
| 0.48 | 0.2834 | -0.0992 | 0.2570 | -0.0054 | -0.6600 | 0.7946 |
| 0.50 | 0.2839 | -0.1100 | 0.2741 | -0.0049 | -0.6682 | 0.8024 |
| 0.52 | 0.2844 | -0.1214 | 0.2916 | -0.0044 | -0.6770 | 0.8103 |
| 0.54 | 0.2849 | -0.1334 | 0.3096 | -0.0039 | -0.6863 | 0.8182 |
| 0.56 | 0.2853 | -0.1459 | 0.3279 | -0.0035 | -0.6961 | 0.8261 |
| 0.58 | 0.2857 | -0.1589 | 0.3467 | -0.0031 | -0.7063 | 0.8340 |
| 0.60 | 0.2861 | -0.1725 | 0.3660 | -0.0027 | -0.7170 | 0.8420 |
| 0.62 | 0.2865 | -0.1865 | 0.3857 | -0.0023 | -0.7282 | 0.8500 |
| 0.64 | 0.2868 | -0.2011 | 0.4060 | -0.0020 | -0.7398 | 0.8580 |
| 0.66 | 0.2871 | -0.2161 | 0.4268 | -0.0017 | -0.7518 | 0.8659 |
| 0.68 | 0.2874 | -0.2315 | 0.4482 | -0.0014 | -0.7642 | 0.8739 |
| 0.70 | 0.2876 | -0.2474 | 0.4703 | -0.0012 | -0.7771 | 0.8818 |
| 0.72 | 0.2879 | -0.2637 | 0.4930 | -0.0010 | -0.7903 | 0.8898 |
| 0.74 | 0.2881 | -0.2803 | 0.5164 | -0.0008 | -0.8038 | 0.8977 |
| 0.76 | 0.2882 | -0.2972 | 0.5405 | -0.0006 | -0.8177 | 0.9056 |
| 0.78 | 0.2884 | -0.3145 | 0.5654 | -0.0005 | -0.8319 | 0.9135 |
| 0.80 | 0.2885 | -0.3320 | 0.5911 | -0.0003 | -0.8463 | 0.9214 |
| 0.82 | 0.2886 | -0.3496 | 0.6176 | -0.0002 | -0.8610 | 0.9293 |
| 0.84 | 0.2887 | -0.3675 | 0.6449 | -0.0002 | -0.8759 | 0.9371 |
| 0.86 | 0.2887 | -0.3854 | 0.6730 | -0.0001 | -0.8910 | 0.9450 |
| 0.88 | 0.2888 | -0.4034 | 0.7017 | -0.0001 | -0.9063 | 0.9529 |
| 0.90 | 0.2888 | -0.4213 | 0.7309 | -0.0000 | -0.9217 | 0.9607 |
| 0.92 | 0.2888 | -0.4390 | 0.7604 | -0.0000 | -0.9373 | 0.9686 |
| 0.94 | 0.2888 | -0.4565 | 0.7894 | -0.0000 | -0.9529 | 0.9764 |
| 0.96 | 0.2888 | -0.4735 | 0.8134 | -0.0000 | -0.9686 | 0.9843 |
| 0.98 | 0.2888 | -0.4900 | 0.7777 | -0.0000 | -0.9843 | 0.9921 |
| 1.00 | 0.2898 | -0.5070 | 1.2450 | 0. | -1.0000 | 1.0000 |

KNUDSEN NUMBER = 0.250

TABLE G.5

| Z | $\tilde{N}_1$ | $\tilde{U}_1$ | $\tilde{V}_1$ | $\tilde{N}_2$ | $\tilde{U}_2$ | $\tilde{V}_2$ |
|---|---|---|---|---|---|---|
| 0. | 0.1869 | 0. | 0. | -0.0088 | -0.3915 | 0.5219 |
| 0.02 | 0.1867 | 0.0001 | 0.0008 | -0.0090 | -0.3918 | 0.5224 |
| 0.04 | 0.1867 | 0.0001 | 0.0030 | -0.0091 | -0.3924 | 0.5237 |
| 0.06 | 0.1863 | 0.0003 | 0.0066 | -0.0094 | -0.3938 | 0.5260 |
| 0.08 | 0.1861 | 0.0001 | 0.0113 | -0.0097 | -0.3953 | 0.5291 |
| 0.10 | 0.1859 | -0.0004 | 0.0171 | -0.0098 | -0.3971 | 0.5330 |
| 0.12 | 0.1858 | -0.0013 | 0.0240 | -0.0099 | -0.3992 | 0.5377 |
| 0.14 | 0.1858 | -0.0027 | 0.0317 | -0.0099 | -0.4015 | 0.5430 |
| 0.16 | 0.1859 | -0.0046 | 0.0403 | -0.0098 | -0.4041 | 0.5491 |
| 0.18 | 0.1861 | -0.0070 | 0.0496 | -0.0097 | -0.4070 | 0.5558 |
| 0.20 | 0.1863 | -0.0100 | 0.0596 | -0.0095 | -0.4103 | 0.5631 |
| 0.22 | 0.1865 | -0.0135 | 0.0703 | -0.0092 | -0.4140 | 0.5709 |
| 0.24 | 0.1869 | -0.0175 | 0.0815 | -0.0089 | -0.4182 | 0.5792 |
| 0.26 | 0.1872 | -0.0224 | 0.0932 | -0.0085 | -0.4229 | 0.5879 |
| 0.28 | 0.1876 | -0.0278 | 0.1054 | -0.0081 | -0.4280 | 0.5970 |
| 0.30 | 0.1880 | -0.0338 | 0.1179 | -0.0077 | -0.4337 | 0.6065 |
| 0.32 | 0.1884 | -0.0406 | 0.1309 | -0.0073 | -0.4400 | 0.6164 |
| 0.34 | 0.1889 | -0.0480 | 0.1442 | -0.0069 | -0.4468 | 0.6265 |
| 0.36 | 0.1893 | -0.0561 | 0.1578 | -0.0064 | -0.4543 | 0.6368 |
| 0.38 | 0.1898 | -0.0649 | 0.1716 | -0.0060 | -0.4624 | 0.6475 |
| 0.40 | 0.1902 | -0.0745 | 0.1857 | -0.0055 | -0.4711 | 0.6582 |
| 0.42 | 0.1907 | -0.0848 | 0.2000 | -0.0051 | -0.4806 | 0.6692 |
| 0.44 | 0.1911 | -0.0958 | 0.2144 | -0.0046 | -0.4906 | 0.6803 |
| 0.46 | 0.1915 | -0.1076 | 0.2290 | -0.0042 | -0.5014 | 0.6916 |
| 0.48 | 0.1919 | -0.1201 | 0.2437 | -0.0038 | -0.5128 | 0.7029 |
| 0.50 | 0.1923 | -0.1334 | 0.2586 | -0.0034 | -0.5250 | 0.7143 |
| 0.52 | 0.1927 | -0.1473 | 0.2735 | -0.0030 | -0.5378 | 0.7258 |
| 0.54 | 0.1930 | -0.1620 | 0.2884 | -0.0027 | -0.5512 | 0.7373 |
| 0.56 | 0.1934 | -0.1774 | 0.3035 | -0.0024 | -0.5654 | 0.7489 |
| 0.58 | 0.1937 | -0.1934 | 0.3185 | -0.0021 | -0.5802 | 0.7605 |
| 0.60 | 0.1940 | -0.2101 | 0.3336 | -0.0018 | -0.5955 | 0.7721 |
| 0.62 | 0.1942 | -0.2275 | 0.3487 | -0.0015 | -0.6117 | 0.7836 |
| 0.64 | 0.1945 | -0.2455 | 0.3638 | -0.0013 | -0.6283 | 0.7952 |
| 0.66 | 0.1947 | -0.2641 | 0.3789 | -0.0011 | -0.6456 | 0.8068 |
| 0.68 | 0.1949 | -0.2832 | 0.3940 | -0.0009 | -0.6634 | 0.8183 |
| 0.70 | 0.1950 | -0.3029 | 0.4090 | -0.0007 | -0.6818 | 0.8299 |
| 0.72 | 0.1952 | -0.3231 | 0.4241 | -0.0005 | -0.7006 | 0.8414 |
| 0.74 | 0.1953 | -0.3437 | 0.4390 | -0.0004 | -0.7200 | 0.8529 |
| 0.76 | 0.1954 | -0.3648 | 0.4540 | -0.0003 | -0.7398 | 0.8643 |
| 0.78 | 0.1955 | -0.3862 | 0.4689 | -0.0002 | -0.7600 | 0.8757 |
| 0.80 | 0.1955 | -0.4080 | 0.4837 | -0.0002 | -0.7806 | 0.8871 |
| 0.82 | 0.1956 | -0.4300 | 0.4985 | -0.0001 | -0.8016 | 0.8985 |
| 0.84 | 0.1957 | -0.4523 | 0.5133 | -0.0001 | -0.8228 | 0.9098 |
| 0.86 | 0.1957 | -0.4749 | 0.5282 | -0.0000 | -0.8444 | 0.9211 |
| 0.88 | 0.1957 | -0.4975 | 0.5431 | -0.0000 | -0.8662 | 0.9324 |
| 0.90 | 0.1957 | -0.5203 | 0.5583 | -0.0000 | -0.8882 | 0.9437 |
| 0.92 | 0.1957 | -0.5431 | 0.5741 | -0.0000 | -0.9103 | 0.9550 |
| 0.94 | 0.1957 | -0.5659 | 0.5911 | -0.0000 | -0.9326 | 0.9663 |
| 0.96 | 0.1957 | -0.5887 | 0.6120 | -0.0000 | -0.9550 | 0.9775 |
| 0.98 | 0.1957 | -0.6114 | 0.6446 | -0.0000 | -0.9775 | 0.9888 |
| 1.00 | 0.1957 | -0.6340 | 2.9778 | 0. | -1.0000 | 1.0000 |

KNUDSEN NUMBER = 0.100 TABLE G. 6

| Z | $\tilde{N}_1$ | $\tilde{U}_1$ | $\tilde{V}_1$ | $\tilde{N}_2$ | $\tilde{U}_2$ | $\tilde{V}_2$ |
|---|---|---|---|---|---|---|
| 0. | 0.0913 | 0. | 0. | 0.0022 | -0.1781 | 0.3264 |
| 0.02 | 0.0912 | 0.0001 | 0.0008 | 0.0021 | -0.1784 | 0.3272 |
| 0.04 | 0.0910 | 0.0001 | 0.0032 | 0.0020 | -0.1792 | 0.3293 |
| 0.06 | 0.0906 | 0.0004 | 0.0070 | 0.0015 | -0.1808 | 0.3327 |
| 0.08 | 0.0902 | 0.0004 | 0.0122 | 0.0012 | -0.1827 | 0.3373 |
| 0.10 | 0.0898 | 0.0001 | 0.0186 | 0.0008 | -0.1851 | 0.3432 |
| 0.12 | 0.0894 | -0.0006 | 0.0261 | 0.0004 | -0.1879 | 0.3501 |
| 0.14 | 0.0891 | -0.0017 | 0.0347 | 0.0001 | -0.1911 | 0.3580 |
| 0.16 | 0.0888 | -0.0033 | 0.0442 | -0.0002 | -0.1947 | 0.3669 |
| 0.18 | 0.0886 | -0.0055 | 0.0546 | -0.0004 | -0.1989 | 0.3767 |
| 0.20 | 0.0884 | -0.0083 | 0.0658 | -0.0006 | -0.2035 | 0.3872 |
| 0.22 | 0.0883 | -0.0118 | 0.0777 | -0.0007 | -0.2087 | 0.3985 |
| 0.24 | 0.0882 | -0.0159 | 0.0903 | -0.0008 | -0.2145 | 0.4105 |
| 0.26 | 0.0881 | -0.0208 | 0.1034 | -0.0009 | -0.2209 | 0.4231 |
| 0.28 | 0.0881 | -0.0264 | 0.1171 | -0.0009 | -0.2280 | 0.4362 |
| 0.30 | 0.0881 | -0.0328 | 0.1313 | -0.0010 | -0.2358 | 0.4499 |
| 0.32 | 0.0881 | -0.0401 | 0.1459 | -0.0009 | -0.2443 | 0.4639 |
| 0.34 | 0.0881 | -0.0482 | 0.1609 | -0.0009 | -0.2536 | 0.4784 |
| 0.36 | 0.0881 | -0.0572 | 0.1761 | -0.0009 | -0.2638 | 0.4932 |
| 0.38 | 0.0882 | -0.0671 | 0.1917 | -0.0008 | -0.2747 | 0.5083 |
| 0.40 | 0.0882 | -0.0779 | 0.2075 | -0.0008 | -0.2865 | 0.5237 |
| 0.42 | 0.0883 | -0.0896 | 0.2235 | -0.0007 | -0.2991 | 0.5393 |
| 0.44 | 0.0884 | -0.1022 | 0.2397 | -0.0007 | -0.3127 | 0.5551 |
| 0.46 | 0.0884 | -0.1158 | 0.2559 | -0.0006 | -0.3271 | 0.5711 |
| 0.48 | 0.0885 | -0.1304 | 0.2723 | -0.0005 | -0.3424 | 0.5871 |
| 0.50 | 0.0886 | -0.1459 | 0.2888 | -0.0005 | -0.3586 | 0.6033 |
| 0.52 | 0.0886 | -0.1623 | 0.3053 | -0.0004 | -0.3757 | 0.6195 |
| 0.54 | 0.0887 | -0.1797 | 0.3218 | -0.0004 | -0.3938 | 0.6358 |
| 0.56 | 0.0887 | -0.1980 | 0.3383 | -0.0003 | -0.4127 | 0.6521 |
| 0.58 | 0.0888 | -0.2173 | 0.3548 | -0.0003 | -0.4325 | 0.6684 |
| 0.60 | 0.0888 | -0.2374 | 0.3713 | -0.0002 | -0.4531 | 0.6848 |
| 0.62 | 0.0888 | -0.2584 | 0.3878 | -0.0002 | -0.4746 | 0.7010 |
| 0.64 | 0.0889 | -0.2803 | 0.4042 | -0.0001 | -0.4970 | 0.7173 |
| 0.66 | 0.0889 | -0.3030 | 0.4205 | -0.0001 | -0.5201 | 0.7335 |
| 0.68 | 0.0889 | -0.3264 | 0.4368 | -0.0001 | -0.5440 | 0.7496 |
| 0.70 | 0.0890 | -0.3507 | 0.4530 | -0.0001 | -0.5686 | 0.7657 |
| 0.72 | 0.0890 | -0.3757 | 0.4691 | -0.0001 | -0.5940 | 0.7818 |
| 0.74 | 0.0890 | -0.4014 | 0.4851 | -0.0000 | -0.6200 | 0.7977 |
| 0.76 | 0.0890 | -0.4277 | 0.5011 | -0.0000 | -0.6467 | 0.8136 |
| 0.78 | 0.0890 | -0.4547 | 0.5171 | -0.0000 | -0.6740 | 0.8295 |
| 0.80 | 0.0890 | -0.4823 | 0.5330 | -0.0000 | -0.7018 | 0.8452 |
| 0.82 | 0.0890 | -0.5104 | 0.5488 | -0.0000 | -0.7301 | 0.8609 |
| 0.84 | 0.0890 | -0.5389 | 0.5647 | -0.0000 | -0.7589 | 0.8765 |
| 0.86 | 0.0890 | -0.5680 | 0.5806 | -0.0000 | -0.7881 | 0.8921 |
| 0.88 | 0.0890 | -0.5974 | 0.5966 | -0.0000 | -0.8177 | 0.9076 |
| 0.90 | 0.0890 | -0.6271 | 0.6130 | -0.0000 | -0.8476 | 0.9231 |
| 0.92 | 0.0890 | -0.6572 | 0.6302 | -0.0000 | -0.8778 | 0.9385 |
| 0.94 | 0.0890 | -0.6874 | 0.6492 | -0.0000 | -0.9081 | 0.9539 |
| 0.96 | 0.0890 | -0.7179 | 0.6746 | 0.0000 | -0.9387 | 0.9693 |
| 0.98 | 0.0890 | -0.7485 | 0.7318 | -0.0000 | -0.9693 | 0.9847 |
| 1.00 | 0.0895 | -0.7799 | 2.9652 | 0. | -1.0000 | 1.0000 |

KNUDSEN NUMBER = 0.050 TABLE G. 7

| Z | $\tilde{N}_1$ | $\tilde{U}_1$ | $\tilde{V}_1$ | $\tilde{N}_2$ | $\tilde{U}_2$ | $\tilde{V}_2$ |
|---|---|---|---|---|---|---|
| 0. | 0.0479 | 0. | 0. | 0.0090 | -0.0777 | 0.2073 |
| 0.02 | 0.0478 | 0.0001 | 0.0009 | 0.0089 | -0.0781 | 0.2082 |
| 0.04 | 0.0475 | 0.0003 | 0.0036 | 0.0087 | -0.0790 | 0.2108 |
| 0.06 | 0.0470 | 0.0007 | 0.0079 | 0.0082 | -0.0805 | 0.2149 |
| 0.08 | 0.0465 | 0.0009 | 0.0137 | 0.0077 | -0.0825 | 0.2206 |
| 0.10 | 0.0460 | 0.0010 | 0.0209 | 0.0071 | -0.0850 | 0.2276 |
| 0.12 | 0.0454 | 0.0007 | 0.0294 | 0.0065 | -0.0879 | 0.2360 |
| 0.14 | 0.0448 | 0.0000 | 0.0391 | 0.0060 | -0.0913 | 0.2456 |
| 0.16 | 0.0443 | -0.0011 | 0.0499 | 0.0054 | -0.0951 | 0.2562 |
| 0.18 | 0.0438 | -0.0028 | 0.0618 | 0.0049 | -0.0995 | 0.2680 |
| 0.20 | 0.0433 | -0.0051 | 0.0745 | 0.0045 | -0.1044 | 0.2806 |
| 0.22 | 0.0429 | -0.0081 | 0.0882 | 0.0040 | -0.1099 | 0.2942 |
| 0.24 | 0.0425 | -0.0118 | 0.1026 | 0.0036 | -0.1160 | 0.3085 |
| 0.26 | 0.0421 | -0.0163 | 0.1177 | 0.0032 | -0.1229 | 0.3236 |
| 0.28 | 0.0418 | -0.0216 | 0.1335 | 0.0029 | -0.1304 | 0.3393 |
| 0.30 | 0.0414 | -0.0278 | 0.1498 | 0.0026 | -0.1387 | 0.3556 |
| 0.32 | 0.0412 | -0.0349 | 0.1667 | 0.0023 | -0.1479 | 0.3724 |
| 0.34 | 0.0409 | -0.0430 | 0.1840 | 0.0020 | -0.1579 | 0.3897 |
| 0.36 | 0.0407 | -0.0520 | 0.2017 | 0.0018 | -0.1687 | 0.4073 |
| 0.38 | 0.0405 | -0.0621 | 0.2197 | 0.0016 | -0.1805 | 0.4253 |
| 0.40 | 0.0403 | -0.0731 | 0.2380 | 0.0014 | -0.1933 | 0.4437 |
| 0.42 | 0.0401 | -0.0853 | 0.2565 | 0.0012 | -0.2070 | 0.4622 |
| 0.44 | 0.0399 | -0.0985 | 0.2752 | 0.0011 | -0.2218 | 0.4810 |
| 0.46 | 0.0398 | -0.1129 | 0.2941 | 0.0009 | -0.2375 | 0.4999 |
| 0.48 | 0.0397 | -0.1283 | 0.3132 | 0.0008 | -0.2543 | 0.5190 |
| 0.50 | 0.0396 | -0.1449 | 0.3323 | 0.0007 | -0.2721 | 0.5381 |
| 0.52 | 0.0395 | -0.1626 | 0.3514 | 0.0006 | -0.2910 | 0.5573 |
| 0.54 | 0.0394 | -0.1813 | 0.3706 | 0.0005 | -0.3109 | 0.5766 |
| 0.56 | 0.0393 | -0.2012 | 0.3898 | 0.0004 | -0.3319 | 0.5958 |
| 0.58 | 0.0392 | -0.2222 | 0.4090 | 0.0003 | -0.3539 | 0.6151 |
| 0.60 | 0.0392 | -0.2443 | 0.4281 | 0.0003 | -0.3770 | 0.6343 |
| 0.62 | 0.0391 | -0.2675 | 0.4472 | 0.0002 | -0.4010 | 0.6534 |
| 0.64 | 0.0391 | -0.2917 | 0.4662 | 0.0002 | -0.4261 | 0.6725 |
| 0.66 | 0.0390 | -0.3169 | 0.4851 | 0.0002 | -0.4520 | 0.6915 |
| 0.68 | 0.0390 | -0.3431 | 0.5039 | 0.0001 | -0.4790 | 0.7104 |
| 0.70 | 0.0390 | -0.3702 | 0.5227 | 0.0001 | -0.5068 | 0.7292 |
| 0.72 | 0.0389 | -0.3983 | 0.5413 | 0.0001 | -0.5355 | 0.7479 |
| 0.74 | 0.0389 | -0.4272 | 0.5599 | 0.0001 | -0.5650 | 0.7664 |
| 0.76 | 0.0389 | -0.4569 | 0.5783 | 0.0000 | -0.5953 | 0.7849 |
| 0.78 | 0.0389 | -0.4875 | 0.5967 | 0.0000 | -0.6263 | 0.8033 |
| 0.80 | 0.0389 | -0.5187 | 0.6149 | 0.0000 | -0.6580 | 0.8216 |
| 0.82 | 0.0389 | -0.5507 | 0.6332 | 0.0000 | -0.6904 | 0.8397 |
| 0.84 | 0.0389 | -0.5832 | 0.6513 | 0.0000 | -0.7233 | 0.8578 |
| 0.86 | 0.0389 | -0.6163 | 0.6695 | 0.0000 | -0.7567 | 0.8758 |
| 0.88 | 0.0389 | -0.6499 | 0.6877 | 0.0000 | -0.7906 | 0.8937 |
| 0.90 | 0.0389 | -0.6840 | 0.7062 | 0.0000 | -0.8249 | 0.9115 |
| 0.92 | 0.0389 | -0.7184 | 0.7252 | 0.0000 | -0.8595 | 0.9293 |
| 0.94 | 0.0389 | -0.7532 | 0.7457 | 0.0000 | -0.8944 | 0.9470 |
| 0.96 | 0.0389 | -0.7881 | 0.7711 | 0.0000 | -0.9295 | 0.9647 |
| 0.98 | 0.0389 | -0.8233 | 0.8221 | -0.0000 | -0.9647 | 0.9824 |
| 1.00 | 0.0400 | -0.8601 | 2.3338 | 0. | -1.0000 | 1.0000 |

KNUDSEN NUMBER = 0.010

TABLE G. 8

| Z | $\tilde{N}_1$ | $\tilde{U}_1$ | $\tilde{V}_1$ | $\tilde{N}_2$ | $\tilde{U}_2$ | $\tilde{V}_2$ |
|---|---|---|---|---|---|---|
| 0. | 0.0096 | 0. | 0. | 0.0067 | -0.0059 | 0.0547 |
| 0.02 | 0.0096 | 0.0001 | 0.0011 | 0.0066 | -0.0060 | 0.0558 |
| 0.04 | 0.0094 | 0.0002 | 0.0043 | 0.0065 | -0.0064 | 0.0590 |
| 0.06 | 0.0092 | 0.0005 | 0.0095 | 0.0062 | -0.0072 | 0.0642 |
| 0.08 | 0.0089 | 0.0006 | 0.0165 | 0.0059 | -0.0082 | 0.0712 |
| 0.10 | 0.0085 | 0.0007 | 0.0252 | 0.0056 | -0.0095 | 0.0800 |
| 0.12 | 0.0081 | 0.0005 | 0.0356 | 0.0052 | -0.0112 | 0.0903 |
| 0.14 | 0.0077 | 0.0000 | 0.0474 | 0.0048 | -0.0132 | 0.1021 |
| 0.16 | 0.0074 | -0.0008 | 0.0606 | 0.0044 | -0.0157 | 0.1153 |
| 0.18 | 0.0070 | -0.0022 | 0.0750 | 0.0040 | -0.0186 | 0.1298 |
| 0.20 | 0.0066 | -0.0041 | 0.0906 | 0.0037 | -0.0221 | 0.1454 |
| 0.22 | 0.0062 | -0.0067 | 0.1073 | 0.0033 | -0.0262 | 0.1620 |
| 0.24 | 0.0059 | -0.0099 | 0.1249 | 0.0030 | -0.0309 | 0.1797 |
| 0.26 | 0.0056 | -0.0140 | 0.1434 | 0.0027 | -0.0364 | 0.1981 |
| 0.28 | 0.0053 | -0.0190 | 0.1627 | 0.0024 | -0.0426 | 0.2174 |
| 0.30 | 0.0050 | -0.0249 | 0.1826 | 0.0021 | -0.0498 | 0.2374 |
| 0.32 | 0.0048 | -0.0318 | 0.2031 | 0.0019 | -0.0578 | 0.2579 |
| 0.34 | 0.0046 | -0.0397 | 0.2242 | 0.0017 | -0.0669 | 0.2790 |
| 0.36 | 0.0044 | -0.0488 | 0.2458 | 0.0014 | -0.0769 | 0.3005 |
| 0.38 | 0.0042 | -0.0590 | 0.2677 | 0.0013 | -0.0881 | 0.3225 |
| 0.40 | 0.0040 | -0.0705 | 0.2899 | 0.0011 | -0.1004 | 0.3447 |
| 0.42 | 0.0039 | -0.0831 | 0.3124 | 0.0010 | -0.1140 | 0.3672 |
| 0.44 | 0.0038 | -0.0971 | 0.3351 | 0.0008 | -0.1287 | 0.3900 |
| 0.46 | 0.0036 | -0.1124 | 0.3580 | 0.0007 | -0.1447 | 0.4128 |
| 0.48 | 0.0035 | -0.1289 | 0.3810 | 0.0006 | -0.1619 | 0.4358 |
| 0.50 | 0.0034 | -0.1469 | 0.4040 | 0.0005 | -0.1804 | 0.4589 |
| 0.52 | 0.0034 | -0.1661 | 0.4271 | 0.0004 | -0.2003 | 0.4819 |
| 0.54 | 0.0033 | -0.1867 | 0.4501 | 0.0004 | -0.2214 | 0.5050 |
| 0.56 | 0.0032 | -0.2087 | 0.4731 | 0.0003 | -0.2439 | 0.5280 |
| 0.58 | 0.0032 | -0.2320 | 0.4960 | 0.0002 | -0.2677 | 0.5509 |
| 0.60 | 0.0031 | -0.2566 | 0.5189 | 0.0002 | -0.2927 | 0.5738 |
| 0.62 | 0.0031 | -0.2826 | 0.5416 | 0.0002 | -0.3191 | 0.5965 |
| 0.64 | 0.0031 | -0.3098 | 0.5642 | 0.0001 | -0.3466 | 0.6191 |
| 0.66 | 0.0030 | -0.3383 | 0.5866 | 0.0001 | -0.3754 | 0.6415 |
| 0.68 | 0.0030 | -0.3679 | 0.6089 | 0.0001 | -0.4054 | 0.6638 |
| 0.70 | 0.0030 | -0.3988 | 0.6310 | 0.0001 | -0.4365 | 0.6859 |
| 0.72 | 0.0030 | -0.4307 | 0.6529 | 0.0000 | -0.4688 | 0.7078 |
| 0.74 | 0.0030 | -0.4638 | 0.6747 | 0.0000 | -0.5020 | 0.7296 |
| 0.76 | 0.0030 | -0.4978 | 0.6963 | 0.0000 | -0.5363 | 0.7512 |
| 0.78 | 0.0030 | -0.5328 | 0.7177 | 0.0000 | -0.5715 | 0.7726 |
| 0.80 | 0.0029 | -0.5687 | 0.7390 | 0.0000 | -0.6076 | 0.7939 |
| 0.82 | 0.0029 | -0.6054 | 0.7601 | 0.0000 | -0.6445 | 0.8150 |
| 0.84 | 0.0029 | -0.6429 | 0.7811 | 0.0000 | -0.6821 | 0.8359 |
| 0.86 | 0.0029 | -0.6811 | 0.8020 | 0.0000 | -0.7203 | 0.8568 |
| 0.88 | 0.0029 | -0.7198 | 0.8229 | 0.0000 | -0.7592 | 0.8775 |
| 0.90 | 0.0029 | -0.7591 | 0.8437 | 0.0000 | -0.7986 | 0.8981 |
| 0.92 | 0.0029 | -0.7988 | 0.8646 | 0.0000 | -0.8383 | 0.9186 |
| 0.94 | 0.0029 | -0.8389 | 0.8859 | 0.0000 | -0.8785 | 0.9390 |
| 0.96 | 0.0029 | -0.8792 | 0.9086 | 0.0000 | -0.9188 | 0.9594 |
| 0.98 | 0.0029 | -0.9198 | 0.9389 | -0.0000 | -0.9594 | 0.9797 |
| 1.00 | 0.0039 | -0.9615 | 1.3695 | 0. | -1.0000 | 1.0000 |

# APPENDIX H

## NUMERICAL VALUES OF MOMENT PARAMETERS FOR ROTATING SPHERE

The ten parametric components of the rotating sphere weighting function are shown on the right hand side of Eqs. (5.122) to (5.126) along with their assumed $\theta$-dependence. A mixed analytic and numerical solution was required to determine these ten functions of R. The six parameters, $\tilde{N}_+$, $\tilde{N}_-$, $\tilde{\tau}_+$, $\tilde{\tau}_-$, $\hat{N}_+$ and $\hat{N}_-$, are given analytically by the expressions (5.90) to (5.93), (5.100) and (5.109), and numerical values of these expressions are tabulated in this Appendix.

The remaining four parameters were obtained by a Runge-Kutta-Gill integration scheme integrated from the sphere towards infinity. The boundary conditions at infinity were satisfied by matching the moment velocity vector and its slope to a continuum-like velocity field at the point, $x^2 = .95$. The numerical results are presented on the following pages for a series of Knudsen numbers representing free-molecule to continuum conditions.

TABLE H. 1

KNUDSEN NUMBER = INFINITY

| $x^2$ | $\tilde{N}_-$ | $\tilde{N}_+$ | $\tilde{T}_-$ | $\tilde{T}_+$ | $\hat{V}_-$ | $\hat{V}_+$ | $\hat{N}_-$ | $\hat{N}_+$ | $\hat{U}_-$ | $\hat{U}_+$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0. | -0. | 0. | 0. | 0. | -0.0031 | 0.0031 | -0.0009 | -0.0009 | -0.0018 | 0.0018 |
| 0.05 | 0.0042 | 0.0042 | 0. | 0. | 0.0069 | 0.0112 | -0.0009 | -0.0009 | -0.0018 | 0.0015 |
| 0.10 | 0.0088 | 0.0088 | 0. | 0. | 0.0160 | 0.0194 | -0.0009 | -0.0009 | -0.0019 | 0.0011 |
| 0.15 | 0.0140 | 0.0140 | 0. | 0. | 0.0252 | 0.0279 | -0.0009 | -0.0009 | -0.0021 | 0.0006 |
| 0.20 | 0.0199 | 0.0199 | 0. | 0. | 0.0345 | 0.0367 | -0.0009 | -0.0009 | -0.0023 | 0.0001 |
| 0.25 | 0.0265 | 0.0265 | 0. | 0. | 0.0441 | 0.0460 | -0.0009 | -0.0009 | -0.0027 | -0.0005 |
| 0.30 | 0.0341 | 0.0341 | 0. | 0. | 0.0543 | 0.0558 | -0.0009 | -0.0009 | -0.0031 | -0.0012 |
| 0.35 | 0.0428 | 0.0428 | 0. | 0. | 0.0651 | 0.0664 | -0.0009 | -0.0009 | -0.0036 | -0.0020 |
| 0.40 | 0.0531 | 0.0531 | 0. | 0. | 0.0767 | 0.0779 | -0.0009 | -0.0009 | -0.0042 | -0.0028 |
| 0.45 | 0.0651 | 0.0651 | 0. | 0. | 0.0893 | 0.0904 | -0.0009 | -0.0009 | -0.0048 | -0.0036 |
| 0.50 | 0.0796 | 0.0796 | 0. | 0. | 0.1033 | 0.1042 | -0.0009 | -0.0009 | -0.0056 | -0.0046 |
| 0.55 | 0.0973 | 0.0973 | 0. | 0. | 0.1189 | 0.1198 | -0.0009 | -0.0009 | -0.0064 | -0.0055 |
| 0.60 | 0.1194 | 0.1194 | 0. | 0. | 0.1367 | 0.1375 | -0.0009 | -0.0009 | -0.0073 | -0.0066 |
| 0.65 | 0.1478 | 0.1478 | 0. | 0. | 0.1574 | 0.1582 | -0.0009 | -0.0009 | -0.0083 | -0.0078 |
| 0.70 | 0.1857 | 0.1857 | 0. | 0. | 0.1823 | 0.1831 | -0.0009 | -0.0009 | -0.0094 | -0.0091 |
| 0.75 | 0.2387 | 0.2387 | 0. | 0. | 0.2131 | 0.2139 | -0.0009 | -0.0009 | -0.0107 | -0.0105 |
| 0.80 | 0.3183 | 0.3183 | 0. | 0. | 0.2534 | 0.2542 | -0.0009 | -0.0009 | -0.0122 | -0.0121 |
| 0.85 | 0.4509 | 0.4509 | 0. | 0. | 0.3105 | 0.3113 | -0.0009 | -0.0009 | -0.0139 | -0.0139 |
| 0.90 | 0.7162 | 0.7162 | 0. | 0. | 0.4032 | 0.4039 | -0.0009 | -0.0009 | -0.0160 | -0.0162 |
| 0.95 | 1.5120 | 1.5120 | 0. | 0. | 0.6096 | 0.6100 | -0.0009 | -0.0009 | -0.0189 | -0.0192 |

TABLE H. 2

KNUDSEN NUMBER = 3.000

| $X^2$ | $\tilde{N}_-$ | $\tilde{N}_+$ | $\tilde{T}_-$ | $\tilde{T}_+$ | $\hat{V}_-$ | $\hat{V}_+$ | $\hat{N}_-$ | $\hat{N}_+$ | $\hat{U}_-$ | $\hat{U}_+$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0. | 0.0005 | -0.0013 | -0.0011 | 0.0011 | -0.0023 | 0.0023 | -0.0013 | -0.0014 | -0.0026 | 0.0026 |
| 0.05 | 0.0046 | 0.0029 | -0.0009 | 0.0011 | 0.0075 | 0.0103 | -0.0013 | -0.0014 | -0.0026 | 0.0023 |
| 0.10 | 0.0091 | 0.0074 | -0.0008 | 0.0011 | 0.0163 | 0.0185 | -0.0013 | -0.0014 | -0.0026 | 0.0019 |
| 0.15 | 0.0142 | 0.0125 | -0.0006 | 0.0012 | 0.0251 | 0.0268 | -0.0013 | -0.0014 | -0.0028 | 0.0013 |
| 0.20 | 0.0198 | 0.0182 | -0.0005 | 0.0013 | 0.0339 | 0.0355 | -0.0013 | -0.0014 | -0.0031 | 0.0007 |
| 0.25 | 0.0262 | 0.0246 | -0.0003 | 0.0013 | 0.0430 | 0.0444 | -0.0013 | -0.0014 | -0.0034 | 0.0001 |
| 0.30 | 0.0334 | 0.0319 | -0.0002 | 0.0014 | 0.0526 | 0.0539 | -0.0013 | -0.0014 | -0.0038 | -0.0007 |
| 0.35 | 0.0418 | 0.0403 | -0.0001 | 0.0014 | 0.0627 | 0.0639 | -0.0013 | -0.0014 | -0.0043 | -0.0015 |
| 0.40 | 0.0515 | 0.0501 | 0.0001 | 0.0015 | 0.0735 | 0.0747 | -0.0013 | -0.0014 | -0.0048 | -0.0023 |
| 0.45 | 0.0629 | 0.0616 | 0.0002 | 0.0015 | 0.0853 | 0.0864 | -0.0013 | -0.0013 | -0.0054 | -0.0032 |
| 0.50 | 0.0767 | 0.0754 | 0.0003 | 0.0015 | 0.0981 | 0.0993 | -0.0013 | -0.0013 | -0.0061 | -0.0042 |
| 0.55 | 0.0934 | 0.0922 | 0.0004 | 0.0016 | 0.1124 | 0.1136 | -0.0013 | -0.0013 | -0.0069 | -0.0052 |
| 0.60 | 0.1143 | 0.1132 | 0.0005 | 0.0016 | 0.1286 | 0.1298 | -0.0013 | -0.0013 | -0.0077 | -0.0063 |
| 0.65 | 0.1412 | 0.1402 | 0.0006 | 0.0016 | 0.1474 | 0.1485 | -0.0013 | -0.0013 | -0.0087 | -0.0075 |
| 0.70 | 0.1770 | 0.1761 | 0.0007 | 0.0016 | 0.1697 | 0.1708 | -0.0013 | -0.0013 | -0.0097 | -0.0088 |
| 0.75 | 0.2272 | 0.2263 | 0.0008 | 0.0016 | 0.1970 | 0.1982 | -0.0013 | -0.0013 | -0.0109 | -0.0103 |
| 0.80 | 0.3024 | 0.3016 | 0.0008 | 0.0016 | 0.2323 | 0.2334 | -0.0013 | -0.0013 | -0.0122 | -0.0119 |
| 0.85 | 0.4278 | 0.4271 | 0.0009 | 0.0016 | 0.2811 | 0.2822 | -0.0013 | -0.0013 | -0.0138 | -0.0137 |
| 0.90 | 0.6787 | 0.6782 | 0.0010 | 0.0015 | 0.3571 | 0.3580 | -0.0013 | -0.0013 | -0.0158 | -0.0158 |
| 0.95 | 1.4317 | 1.4313 | 0.0010 | 0.0014 | 0.5052 | 0.5057 | -0.0013 | -0.0013 | -0.0183 | -0.0185 |

TABLE H. 3

KNUDSEN NUMBER = 1.000

| $x^2$ | $\tilde{N}_-$ | $\tilde{N}_+$ | $\tilde{T}_-$ | $\tilde{T}_+$ | $\hat{V}_-$ | $\hat{V}_+$ | $\hat{N}_-$ | $\hat{N}_+$ | $\hat{U}_-$ | $\hat{U}_+$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0. | 0.0015 | -0.0038 | -0.0029 | 0.0029 | -0.0027 | 0.0027 | -0.0019 | -0.0022 | -0.0037 | 0.0037 |
| 0.05 | 0.0054 | 0.0003 | -0.0025 | 0.0030 | 0.0066 | 0.0109 | -0.0019 | -0.0022 | -0.0037 | 0.0033 |
| 0.10 | 0.0096 | 0.0047 | -0.0021 | 0.0032 | 0.0150 | 0.0190 | -0.0019 | -0.0021 | -0.0037 | 0.0028 |
| 0.15 | 0.0144 | 0.0096 | -0.0017 | 0.0033 | 0.0231 | 0.0272 | -0.0019 | -0.0021 | -0.0038 | 0.0022 |
| 0.20 | 0.0196 | 0.0150 | -0.0013 | 0.0035 | 0.0313 | 0.0354 | -0.0019 | -0.0021 | -0.0041 | 0.0014 |
| 0.25 | 0.0255 | 0.0211 | -0.0009 | 0.0036 | 0.0397 | 0.0439 | -0.0019 | -0.0021 | -0.0044 | 0.0006 |
| 0.30 | 0.0322 | 0.0279 | -0.0006 | 0.0038 | 0.0483 | 0.0526 | -0.0019 | -0.0021 | -0.0047 | -0.0002 |
| 0.35 | 0.0398 | 0.0357 | -0.0002 | 0.0039 | 0.0573 | 0.0617 | -0.0019 | -0.0021 | -0.0051 | -0.0012 |
| 0.40 | 0.0486 | 0.0447 | 0.0001 | 0.0040 | 0.0669 | 0.0713 | -0.0019 | -0.0021 | -0.0056 | -0.0022 |
| 0.45 | 0.0590 | 0.0553 | 0.0004 | 0.0042 | 0.0771 | 0.0815 | -0.0019 | -0.0021 | -0.0062 | -0.0032 |
| 0.50 | 0.0714 | 0.0679 | 0.0008 | 0.0043 | 0.0881 | 0.0926 | -0.0019 | -0.0021 | -0.0068 | -0.0043 |
| 0.55 | 0.0865 | 0.0832 | 0.0010 | 0.0044 | 0.1002 | 0.1046 | -0.0019 | -0.0021 | -0.0075 | -0.0055 |
| 0.60 | 0.1053 | 0.1022 | 0.0013 | 0.0044 | 0.1136 | 0.1179 | -0.0019 | -0.0020 | -0.0082 | -0.0067 |
| 0.65 | 0.1295 | 0.1265 | 0.0016 | 0.0045 | 0.1283 | 0.1329 | -0.0019 | -0.0020 | -0.0090 | -0.0080 |
| 0.70 | 0.1616 | 0.1589 | 0.0018 | 0.0045 | 0.1462 | 0.1502 | -0.0019 | -0.0020 | -0.0100 | -0.0093 |
| 0.75 | 0.2066 | 0.2042 | 0.0020 | 0.0045 | 0.1668 | 0.1704 | -0.0019 | -0.0020 | -0.0110 | -0.0108 |
| 0.80 | 0.2741 | 0.2719 | 0.0022 | 0.0044 | 0.1916 | 0.1949 | -0.0019 | -0.0020 | -0.0121 | -0.0123 |
| 0.85 | 0.3867 | 0.3848 | 0.0024 | 0.0043 | 0.2221 | 0.2249 | -0.0019 | -0.0020 | -0.0135 | -0.0140 |
| 0.90 | 0.6122 | 0.6106 | 0.0026 | 0.0041 | 0.2576 | 0.2596 | -0.0019 | -0.0019 | -0.0150 | -0.0157 |
| 0.95 | 1.2894 | 1.2883 | 0.0027 | 0.0038 | 0.2465 | 0.2475 | -0.0019 | -0.0019 | -0.0170 | -0.0176 |

TABLE H. 4

KNUDSEN NUMBER = 0.500

| $X^2$ | $\tilde{N}_-$ | $\tilde{N}_+$ | $\tilde{T}_-$ | $\tilde{T}_+$ | $\hat{V}_-$ | $\hat{V}_+$ | $\hat{N}_-$ | $\hat{N}_+$ | $\hat{U}_-$ | $\hat{U}_+$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0. | 0.0025 | -0.0075 | -0.0051 | 0.0051 | -0.0078 | 0.0078 | -0.0024 | -0.0031 | -0.0047 | 0.0047 |
| 0.05 | 0.0062 | -0.0035 | -0.0043 | 0.0053 | 0.0012 | 0.0162 | -0.0024 | -0.0031 | -0.0046 | 0.0041 |
| 0.10 | 0.0102 | 0.0008 | -0.0036 | 0.0056 | 0.0092 | 0.0244 | -0.0024 | -0.0031 | -0.0046 | 0.0033 |
| 0.15 | 0.0145 | 0.0055 | -0.0030 | 0.0058 | 0.0168 | 0.0324 | -0.0024 | -0.0031 | -0.0047 | 0.0024 |
| 0.20 | 0.0193 | 0.0105 | -0.0023 | 0.0061 | 0.0244 | 0.0403 | -0.0024 | -0.0031 | -0.0048 | 0.0014 |
| 0.25 | 0.0245 | 0.0161 | -0.0017 | 0.0064 | 0.0320 | 0.0482 | -0.0024 | -0.0030 | -0.0050 | 0.0003 |
| 0.30 | 0.0305 | 0.0224 | -0.0010 | 0.0067 | 0.0397 | 0.0562 | -0.0024 | -0.0030 | -0.0053 | -0.0009 |
| 0.35 | 0.0372 | 0.0295 | -0.0004 | 0.0069 | 0.0477 | 0.0642 | -0.0024 | -0.0030 | -0.0057 | -0.0022 |
| 0.40 | 0.0449 | 0.0376 | 0.0001 | 0.0072 | 0.0559 | 0.0724 | -0.0024 | -0.0030 | -0.0061 | -0.0035 |
| 0.45 | 0.0540 | 0.0470 | 0.0007 | 0.0074 | 0.0645 | 0.0809 | -0.0024 | -0.0029 | -0.0065 | -0.0049 |
| 0.50 | 0.0647 | 0.0581 | 0.0012 | 0.0076 | 0.0735 | 0.0896 | -0.0024 | -0.0029 | -0.0071 | -0.0063 |
| 0.55 | 0.0778 | 0.0715 | 0.0017 | 0.0078 | 0.0830 | 0.0987 | -0.0024 | -0.0029 | -0.0077 | -0.0077 |
| 0.60 | 0.0940 | 0.0881 | 0.0022 | 0.0079 | 0.0930 | 0.1082 | -0.0024 | -0.0029 | -0.0083 | -0.0092 |
| 0.65 | 0.1147 | 0.1093 | 0.0027 | 0.0080 | 0.1037 | 0.1181 | -0.0024 | -0.0028 | -0.0090 | -0.0107 |
| 0.70 | 0.1424 | 0.1373 | 0.0031 | 0.0080 | 0.1148 | 0.1283 | -0.0024 | -0.0028 | -0.0098 | -0.0122 |
| 0.75 | 0.1810 | 0.1764 | 0.0035 | 0.0080 | 0.1260 | 0.1382 | -0.0024 | -0.0028 | -0.0106 | -0.0137 |
| 0.80 | 0.2389 | 0.2348 | 0.0038 | 0.0078 | 0.1354 | 0.1461 | -0.0024 | -0.0027 | -0.0116 | -0.0151 |
| 0.85 | 0.3357 | 0.3321 | 0.0041 | 0.0076 | 0.1374 | 0.1460 | -0.0024 | -0.0027 | -0.0126 | -0.0164 |
| 0.90 | 0.5295 | 0.5267 | 0.0044 | 0.0072 | 0.1049 | 0.1103 | -0.0024 | -0.0026 | -0.0136 | -0.0173 |
| 0.95 | 1.1127 | 1.1107 | 0.0046 | 0.0066 | -0.1961 | -0.1936 | -0.0024 | -0.0025 | -0.0146 | -0.0174 |

TABLE H. 5

KNUDSEN NUMBER = 0.250

| $x^2$ | $\tilde{N}_-$ | $\tilde{N}_+$ | $\tilde{T}_-$ | $\tilde{T}_+$ | $\hat{V}_-$ | $\hat{V}_+$ | $\hat{N}_-$ | $\hat{N}_+$ | $\hat{U}_-$ | $\hat{U}_+$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0. | 0.0040 | -0.0142 | -0.0079 | 0.0079 | -0.0273 | 0.0273 | -0.0028 | -0.0047 | -0.0056 | 0.0056 |
| 0.05 | 0.0072 | -0.0103 | -0.0068 | 0.0084 | -0.0185 | 0.0372 | -0.0028 | -0.0046 | -0.0053 | 0.0043 |
| 0.10 | 0.0107 | -0.0062 | -0.0057 | 0.0089 | -0.0108 | 0.0465 | -0.0028 | -0.0046 | -0.0050 | 0.0027 |
| 0.15 | 0.0144 | -0.0019 | -0.0047 | 0.0094 | -0.0035 | 0.0552 | -0.0028 | -0.0045 | -0.0049 | 0.0009 |
| 0.20 | 0.0184 | 0.0027 | -0.0037 | 0.0099 | 0.0037 | 0.0635 | -0.0028 | -0.0045 | -0.0049 | -0.0010 |
| 0.25 | 0.0227 | 0.0077 | -0.0027 | 0.0104 | 0.0107 | 0.0714 | -0.0028 | -0.0044 | -0.0050 | -0.0031 |
| 0.30 | 0.0275 | 0.0132 | -0.0017 | 0.0109 | 0.0177 | 0.0789 | -0.0028 | -0.0044 | -0.0051 | -0.0054 |
| 0.35 | 0.0329 | 0.0192 | -0.0008 | 0.0113 | 0.0246 | 0.0861 | -0.0028 | -0.0043 | -0.0053 | -0.0077 |
| 0.40 | 0.0391 | 0.0260 | 0.0001 | 0.0117 | 0.0315 | 0.0928 | -0.0028 | -0.0042 | -0.0056 | -0.0100 |
| 0.45 | 0.0461 | 0.0337 | 0.0009 | 0.0121 | 0.0384 | 0.0990 | -0.0028 | -0.0042 | -0.0059 | -0.0125 |
| 0.50 | 0.0545 | 0.0427 | 0.0017 | 0.0124 | 0.0451 | 0.1046 | -0.0028 | -0.0041 | -0.0063 | -0.0149 |
| 0.55 | 0.0645 | 0.0534 | 0.0025 | 0.0127 | 0.0514 | 0.1094 | -0.0028 | -0.0040 | -0,0067 | -0.0173 |
| 0.60 | 0.0769 | 0.0665 | 0.0032 | 0.0128 | 0.0571 | 0.1129 | -0.0028 | -0.0040 | -0.0072 | -0.0197 |
| 0.65 | 0.0927 | 0.0831 | 0.0039 | 0.0129 | 0.0614 | 0.1144 | -0.0028 | -0.0039 | -0.0077 | -0.0220 |
| 0.70 | 0.1136 | 0.1048 | 0.0045 | 0.0129 | 0.0630 | 0.1123 | -0.0028 | -0.0038 | -0.0083 | -0.0241 |
| 0.75 | 0.1429 | 0.1349 | 0.0051 | 0.0128 | 0.0589 | 0.1034 | -0.0028 | -0.0037 | -0.0089 | -0.0259 |
| 0.80 | 0.1869 | 0.1798 | 0.0057 | 0.0126 | 0.0418 | 0.0803 | -0.0028 | -0.0036 | -0.0096 | -0.0272 |
| 0.85 | 0.2604 | 0.2543 | 0.0061 | 0.0121 | -0.0091 | 0.0215 | -0.0028 | -0.0035 | -0.0101 | -0.0275 |
| 0.90 | 0.4081 | 0.4031 | 0.0065 | 0.0115 | -0.1758 | -0.1553 | -0.0028 | -0.0034 | -0.0105 | -0.0261 |
| 0.95 | 0.8534 | 0.8499 | 0.0068 | 0.0103 | -1.0884 | -1.0805 | -0.0028 | -0.0032 | -0.0099 | -0.0207 |

# TABLE H. 6

KNUDSEN NUMBER = 0.100

| $X^2$ | $\tilde{N}_-$ | $\tilde{N}_+$ | $\tilde{T}_-$ | $\tilde{T}_+$ | $\hat{V}_-$ | $\hat{V}_+$ | $\hat{N}_-$ | $\hat{N}_+$ | $\hat{U}_-$ | $\hat{U}_+$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0. | 0.0055 | -0.0292 | -0.0109 | 0.0109 | -0.1056 | 0.1056 | -0.0027 | -0.0073 | -0.0055 | 0.0055 |
| 0.05 | 0.0079 | -0.0254 | -0.0095 | 0.0119 | -0.0960 | 0.1279 | -0.0027 | -0.0072 | -0.0043 | 0.0017 |
| 0.10 | 0.0104 | -0.0216 | -0.0080 | 0.0129 | -0.0873 | 0.1486 | -0.0027 | -0.0071 | -0.0033 | -0.0031 |
| 0.15 | 0.0130 | -0.0176 | -0.0067 | 0.0140 | -0.0789 | 0.1679 | -0.0027 | -0.0069 | -0.0025 | -0.0085 |
| 0.20 | 0.0156 | -0.0136 | -0.0053 | 0.0149 | -0.0706 | 0.1856 | -0.0027 | -0.0068 | -0.0019 | -0.0145 |
| 0.25 | 0.0184 | -0.0095 | -0.0040 | 0.0159 | -0.0624 | 0.2019 | -0.0027 | -0.0067 | -0.0014 | -0.0211 |
| 0.30 | 0.0214 | -0.0052 | -0.0027 | 0.0167 | -0.0543 | 0.2165 | -0.0027 | -0.0065 | -0.0010 | -0.0281 |
| 0.35 | 0.0247 | -0.0006 | -0.0015 | 0.0175 | -0.0464 | 0.2292 | -0.0027 | -0.0064 | -0.0008 | -0.0354 |
| 0.40 | 0.0282 | 0.0042 | -0.0004 | 0.0182 | -0.0387 | 0.2399 | -0.0027 | -0.0063 | -0.0006 | -0.0430 |
| 0.45 | 0.0322 | 0.0095 | 0.0008 | 0.0188 | -0.0315 | 0.2480 | -0.0027 | -0.0061 | -0.0006 | -0.0509 |
| 0.50 | 0.0367 | 0.0154 | 0.0018 | 0.0193 | -0.0252 | 0.2529 | -0.0027 | -0.0060 | -0.0007 | -0.0588 |
| 0.55 | 0.0421 | 0.0221 | 0.0028 | 0.0197 | -0.0204 | 0.2537 | -0.0027 | -0.0058 | -0.0008 | -0.0668 |
| 0.60 | 0.0486 | 0.0300 | 0.0038 | 0.0199 | -0.0181 | 0.2487 | -0.0027 | -0.0056 | -0.0011 | -0.0745 |
| 0.65 | 0.0568 | 0.0396 | 0.0047 | 0.0200 | -0.0205 | 0.2352 | -0.0027 | -0.0054 | -0.0014 | -0.0818 |
| 0.70 | 0.0676 | 0.0519 | 0.0056 | 0.0199 | -0.0314 | 0.2085 | -0.0027 | -0.0052 | -0.0018 | -0.0881 |
| 0.75 | 0.0827 | 0.0684 | 0.0063 | 0.0196 | -0.0591 | 0.1592 | -0.0027 | -0.0050 | -0.0022 | -0.0929 |
| 0.80 | 0.1053 | 0.0927 | 0.0070 | 0.0190 | -0.1229 | 0.0662 | -0.0027 | -0.0048 | -0.0025 | -0.0951 |
| 0.85 | 0.1433 | 0.1325 | 0.0077 | 0.0181 | -0.2777 | -0.1274 | -0.0027 | -0.0045 | -0.0025 | -0.0928 |
| 0.90 | 0.2202 | 0.2114 | 0.0082 | 0.0168 | -0.7327 | -0.6338 | -0.0027 | -0.0042 | -0.0018 | -0.0820 |
| 0.95 | 0.4539 | 0.4478 | 0.0086 | 0.0147 | -3.0665 | -3.0321 | -0.0027 | -0.0038 | 0.0015 | -0.0530 |

TABLE H. 7

KNUDSEN NUMBER = 0.050

| $x^2$ | $\tilde{N}_-$ | $\tilde{N}_+$ | $\tilde{T}_-$ | $\tilde{T}_+$ | $\hat{V}_-$ | $\hat{V}_+$ | $\hat{N}_-$ | $\hat{N}_+$ | $\hat{U}_-$ | $\hat{U}_+$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0. | 0.0054 | -0.0438 | -0.0107 | 0.0107 | -0.2093 | 0.2093 | -0.0021 | -0.0092 | -0.0042 | 0.0042 |
| 0.05 | 0.0071 | -0.0399 | -0.0093 | 0.0123 | -0.1980 | 0.2710 | -0.0021 | -0.0090 | -0.0020 | -0.0031 |
| 0.10 | 0.0088 | -0.0360 | -0.0080 | 0.0139 | -0.1870 | 0.3293 | -0.0021 | -0.0088 | -0.0003 | -0.0128 |
| 0.15 | 0.0105 | -0.0323 | -0.0067 | 0.0154 | -0.1759 | 0.3841 | -0.0021 | -0.0087 | 0.0017 | -0.0245 |
| 0.20 | 0.0122 | -0.0285 | -0.0054 | 0.0168 | -0.1646 | 0.4354 | -0.0021 | -0.0085 | 0.0031 | -0.0379 |
| 0.25 | 0.0139 | -0.0248 | -0.0042 | 0.0181 | -0.1531 | 0.4828 | -0.0021 | -0.0083 | 0.0044 | -0.0529 |
| 0.30 | 0.0156 | -0.0210 | -0.0030 | 0.0193 | -0.1413 | 0.5260 | -0.0021 | -0.0080 | 0.0053 | -0.0693 |
| 0.35 | 0.0175 | -0.0172 | -0.0018 | 0.0204 | -0.1294 | 0.5643 | -0.0021 | -0.0078 | 0.0061 | -0.0871 |
| 0.40 | 0.0194 | -0.0133 | -0.0007 | 0.0213 | -0.1175 | 0.5971 | -0.0021 | -0.0076 | 0.0067 | -0.1061 |
| 0.45 | 0.0214 | -0.0093 | 0.0003 | 0.0221 | -0.1059 | 0.6233 | -0.0021 | -0.0074 | 0.0071 | -0.1260 |
| 0.50 | 0.0237 | -0.0051 | 0.0013 | 0.0227 | -0.0952 | 0.6415 | -0.0021 | -0.0071 | 0.0074 | -0.1467 |
| 0.55 | 0.0263 | -0.0006 | 0.0023 | 0.0231 | -0.0862 | 0.6496 | -0.0021 | -0.0069 | 0.0074 | -0.1677 |
| 0.60 | 0.0293 | 0.0044 | 0.0032 | 0.0233 | -0.0804 | 0.6444 | -0.0021 | -0.0066 | 0.0073 | -0.1886 |
| 0.65 | 0.0331 | 0.0101 | 0.0041 | 0.0233 | -0.0810 | 0.6211 | -0.0021 | -0.0063 | 0.0071 | -0.2086 |
| 0.70 | 0.0380 | 0.0170 | 0.0049 | 0.0231 | -0.0935 | 0.5712 | -0.0021 | -0.0060 | 0.0067 | -0.2266 |
| 0.75 | 0.0448 | 0.0259 | 0.0056 | 0.0226 | -0.1303 | 0.4788 | -0.0021 | -0.0057 | 0.0063 | -0.2408 |
| 0.80 | 0.0550 | 0.0383 | 0.0063 | 0.0217 | -0.2207 | 0.3098 | -0.0021 | -0.0053 | 0.0059 | -0.2482 |
| 0.85 | 0.0723 | 0.0580 | 0.0069 | 0.0204 | -0.4477 | -0.0256 | -0.0021 | -0.0049 | 0.0058 | -0.2434 |
| 0.90 | 0.1077 | 0.0961 | 0.0074 | 0.0186 | -1.1316 | -0.8557 | -0.0021 | -0.0044 | 0.0069 | -0.2157 |
| 0.95 | 0.2167 | 0.2086 | 0.0077 | 0.0156 | -4.7047 | -4.6134 | -0.0021 | -0.0037 | 0.0117 | -0.1396 |

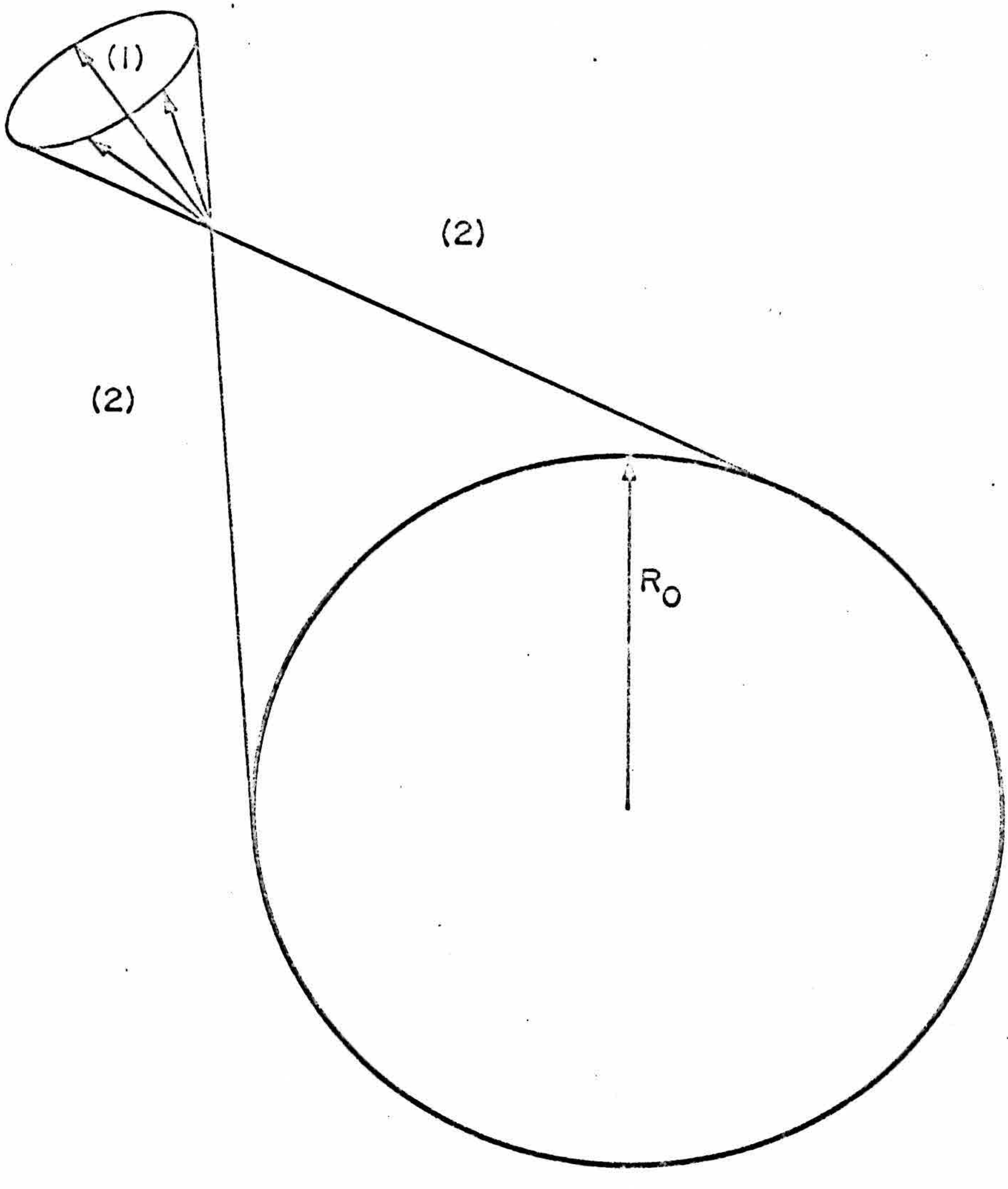


FIG. I GEOMETRY FOR TWO-SIDED MAXWELLIAN

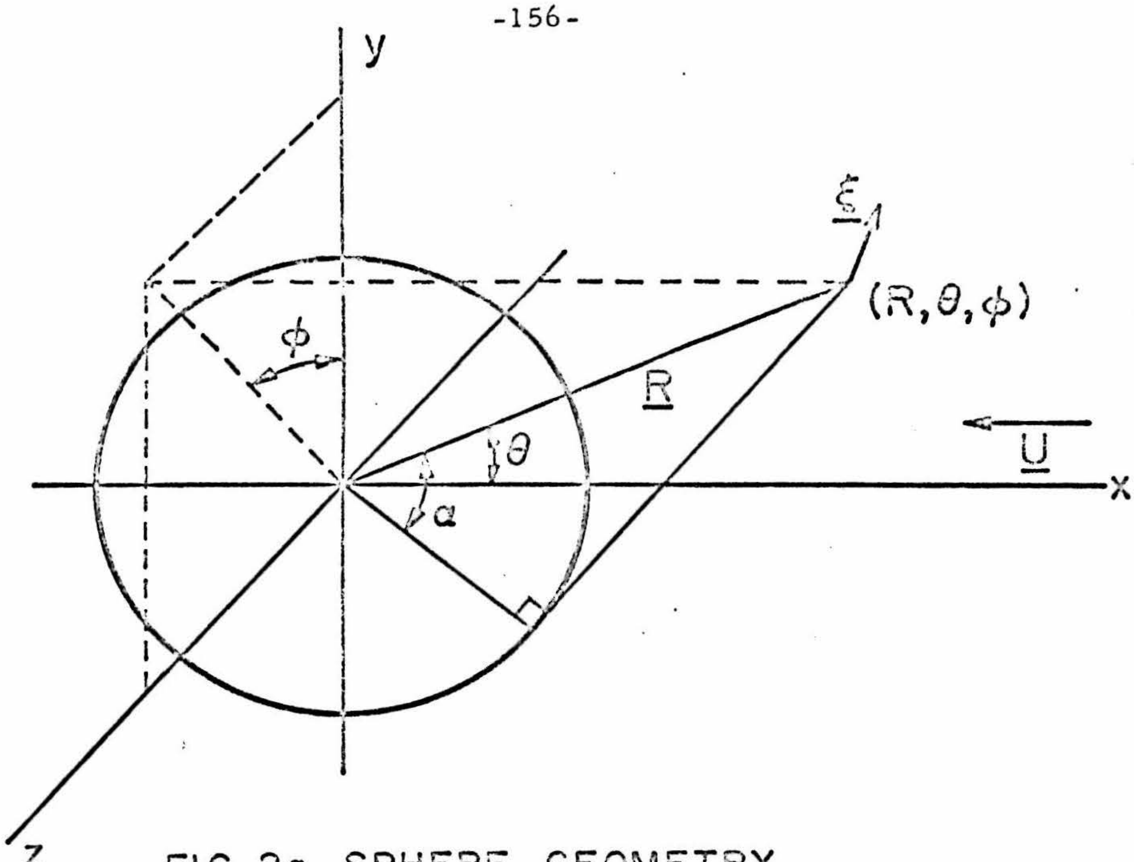


FIG. 2a SPHERE GEOMETRY

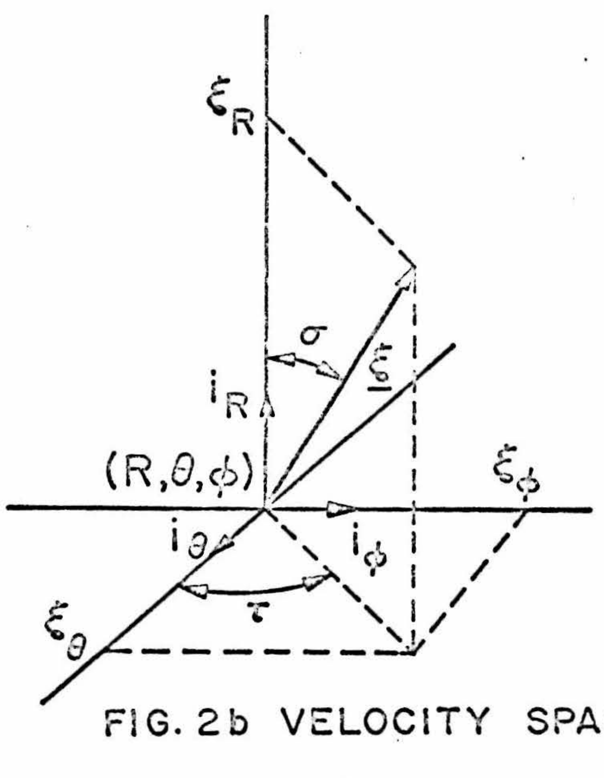


FIG. 2b VELOCITY SPACE

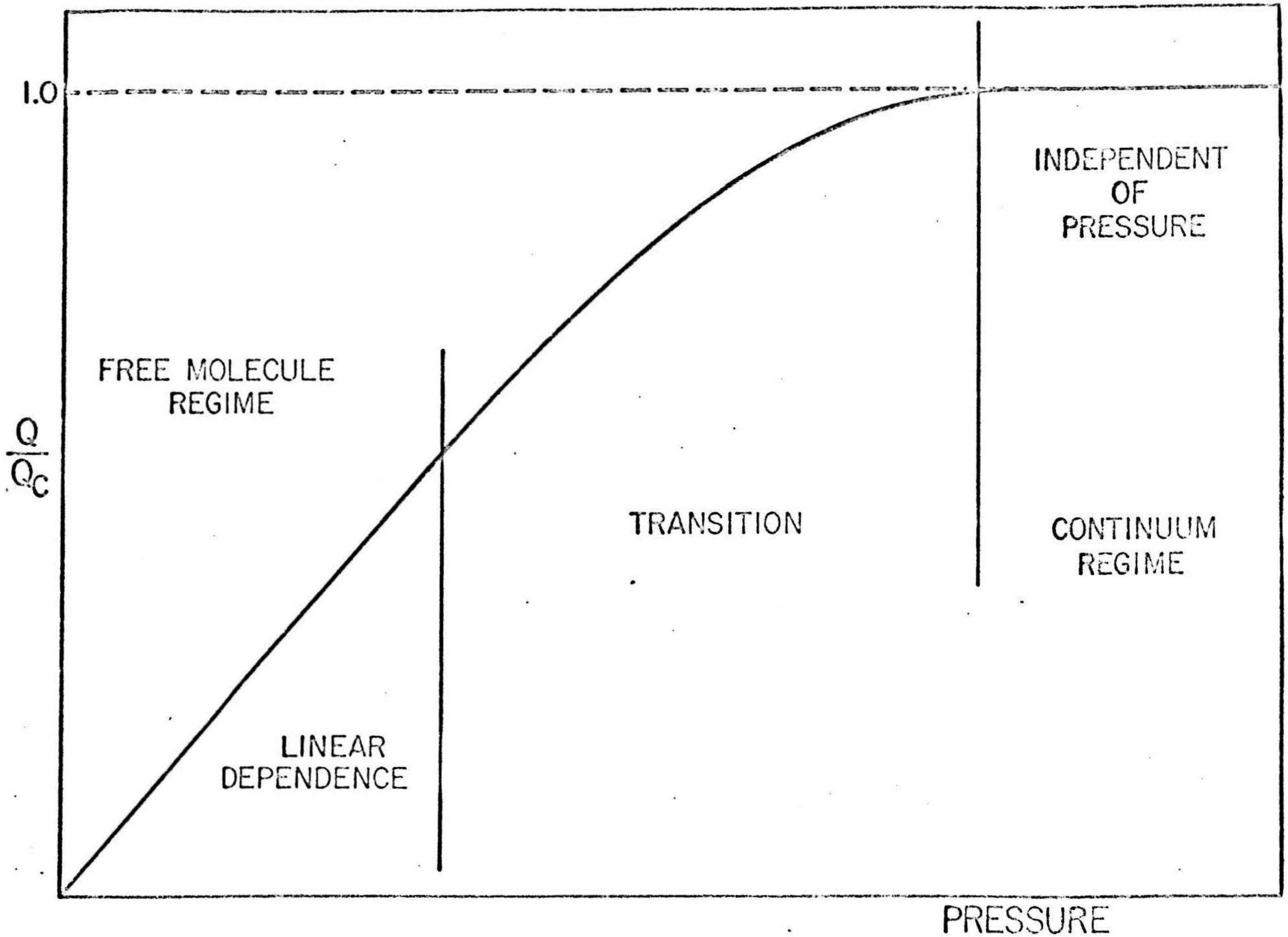


FIG. 3 VARIATION OF HEAT TRANSFER WITH GAS PRESSURE

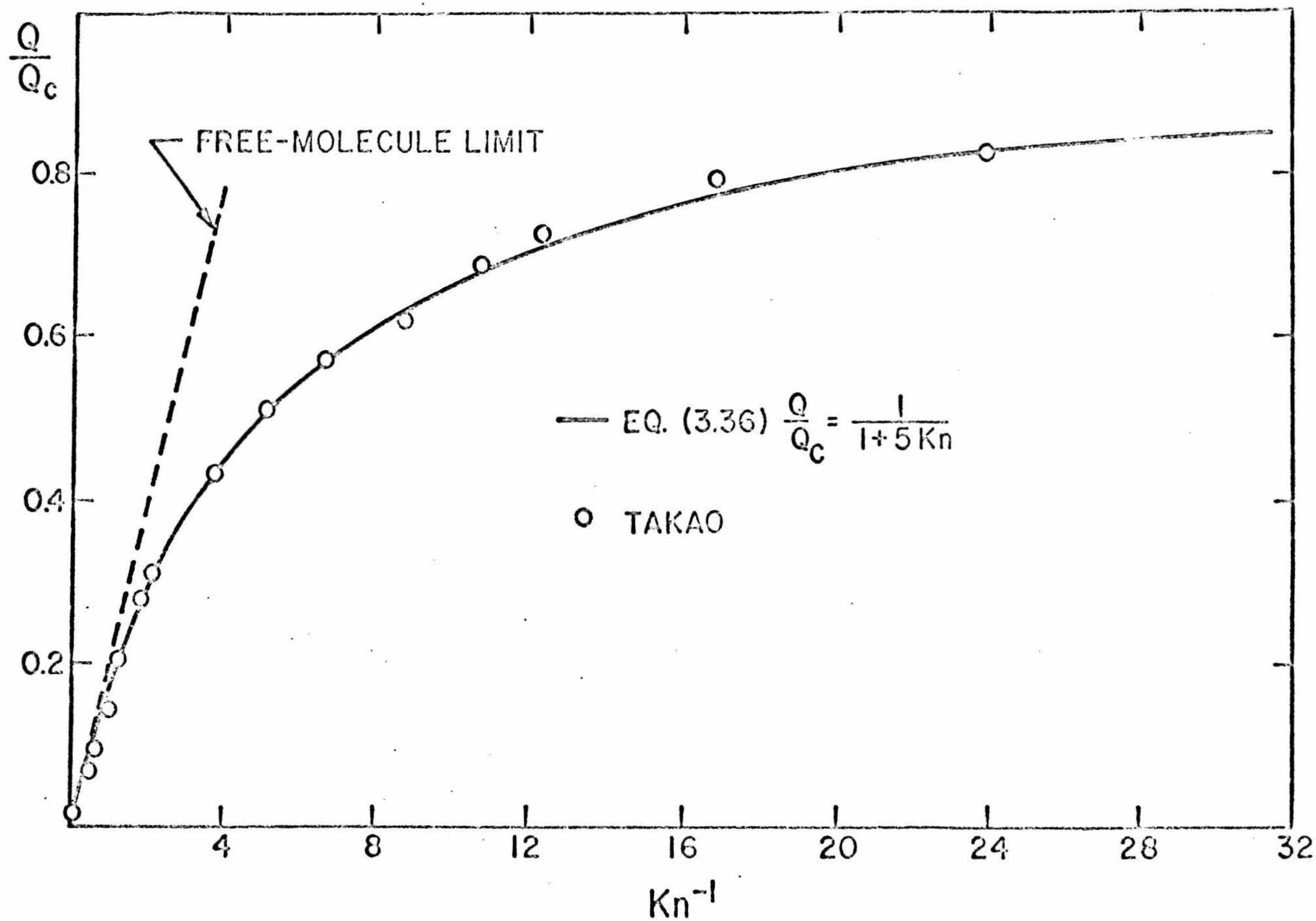


FIG. 4 HEAT TRANSFER FROM A SPHERE

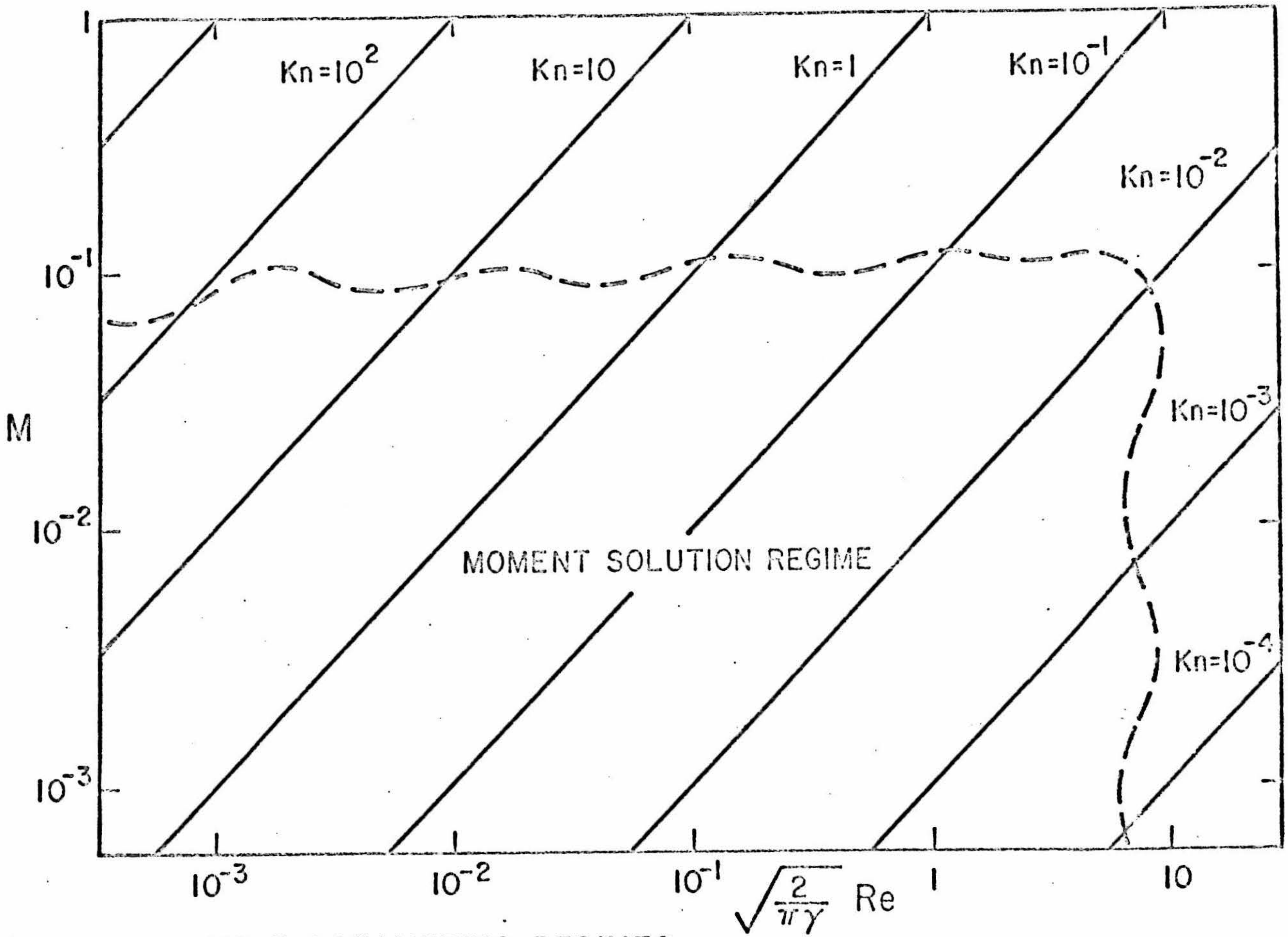


FIG. 5 PARAMETRIC REGIMES

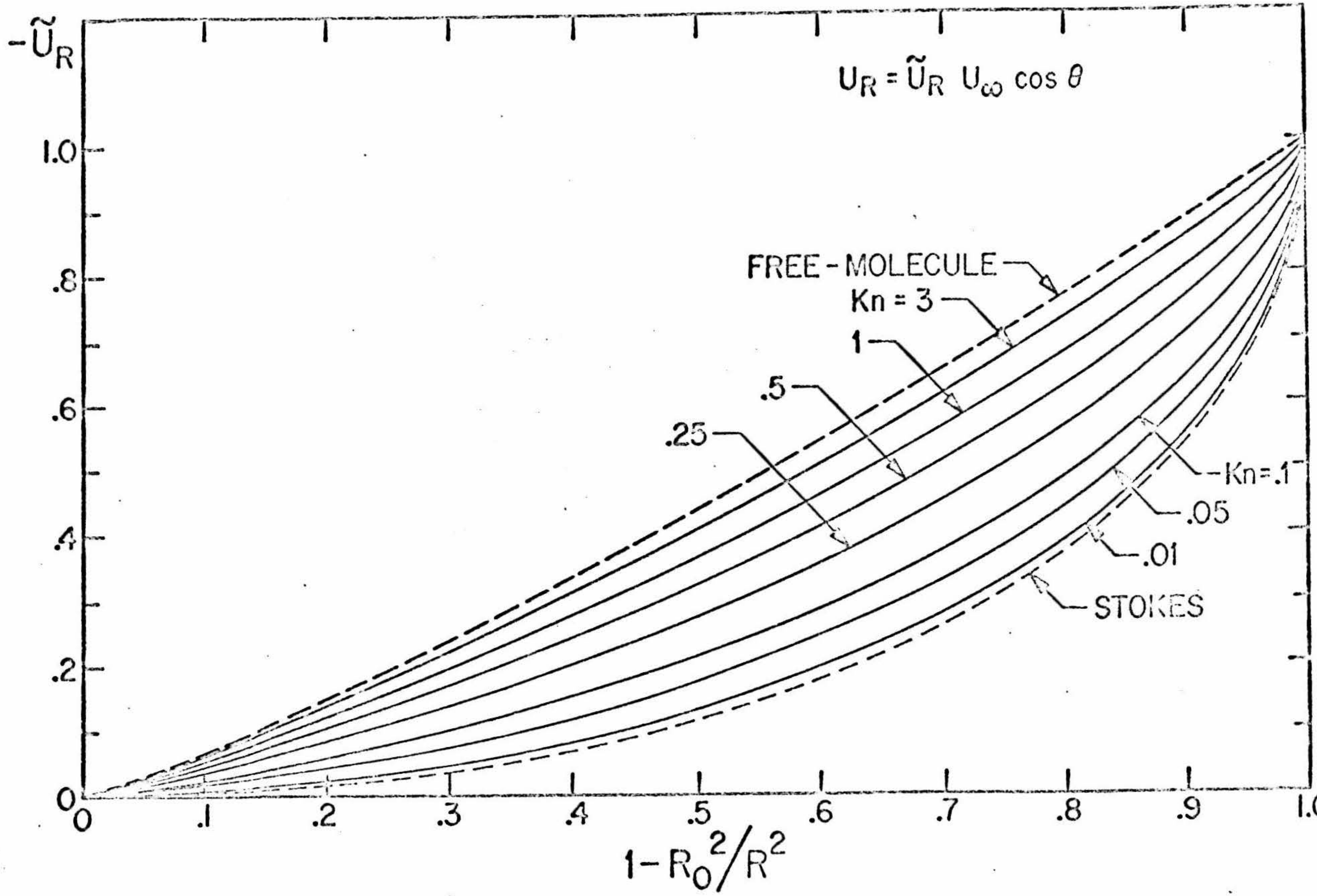


FIG. 6 RADIAL VELOCITY FIELD

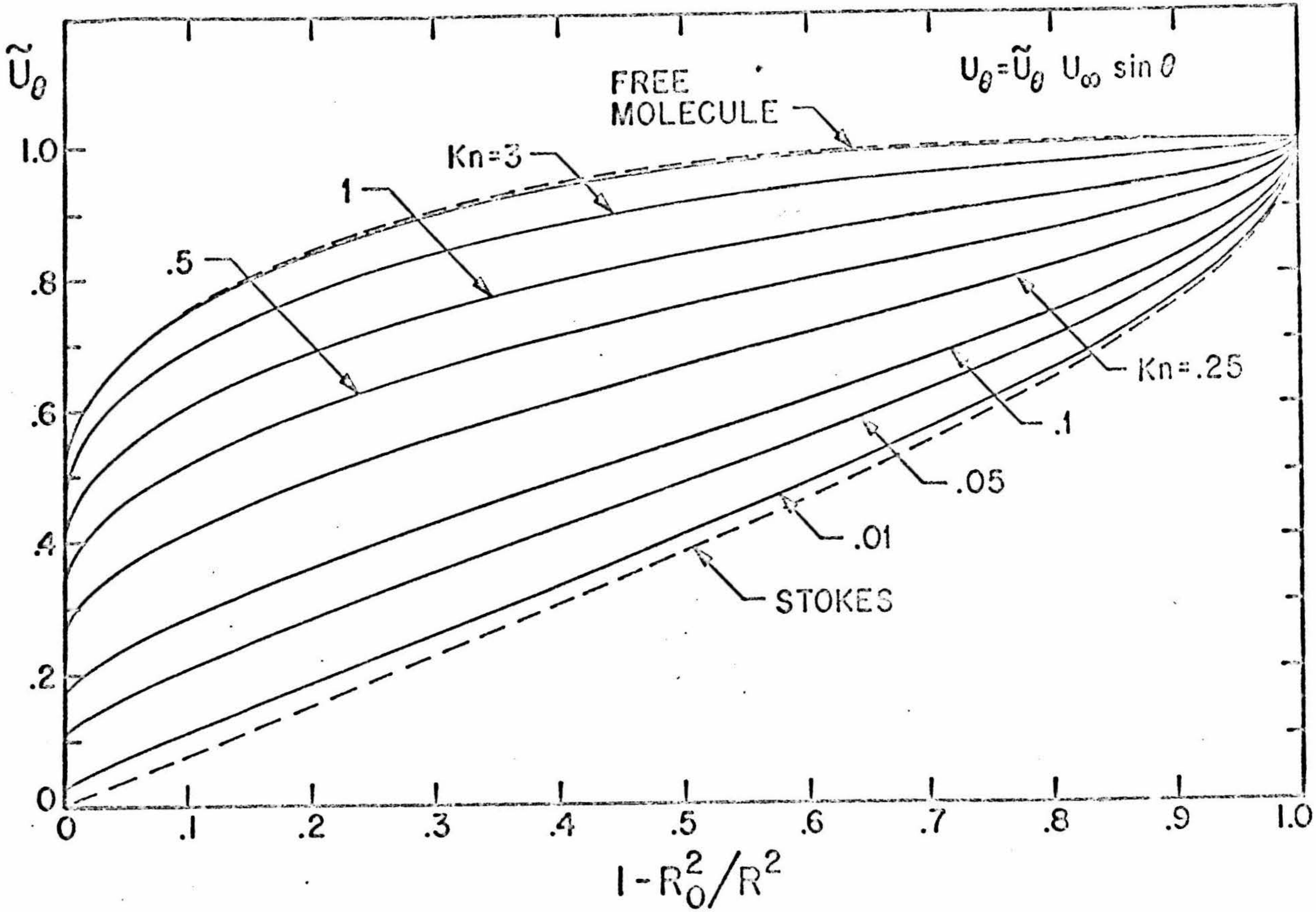


FIG. 7 TANGENTIAL VELOCITY FIELD

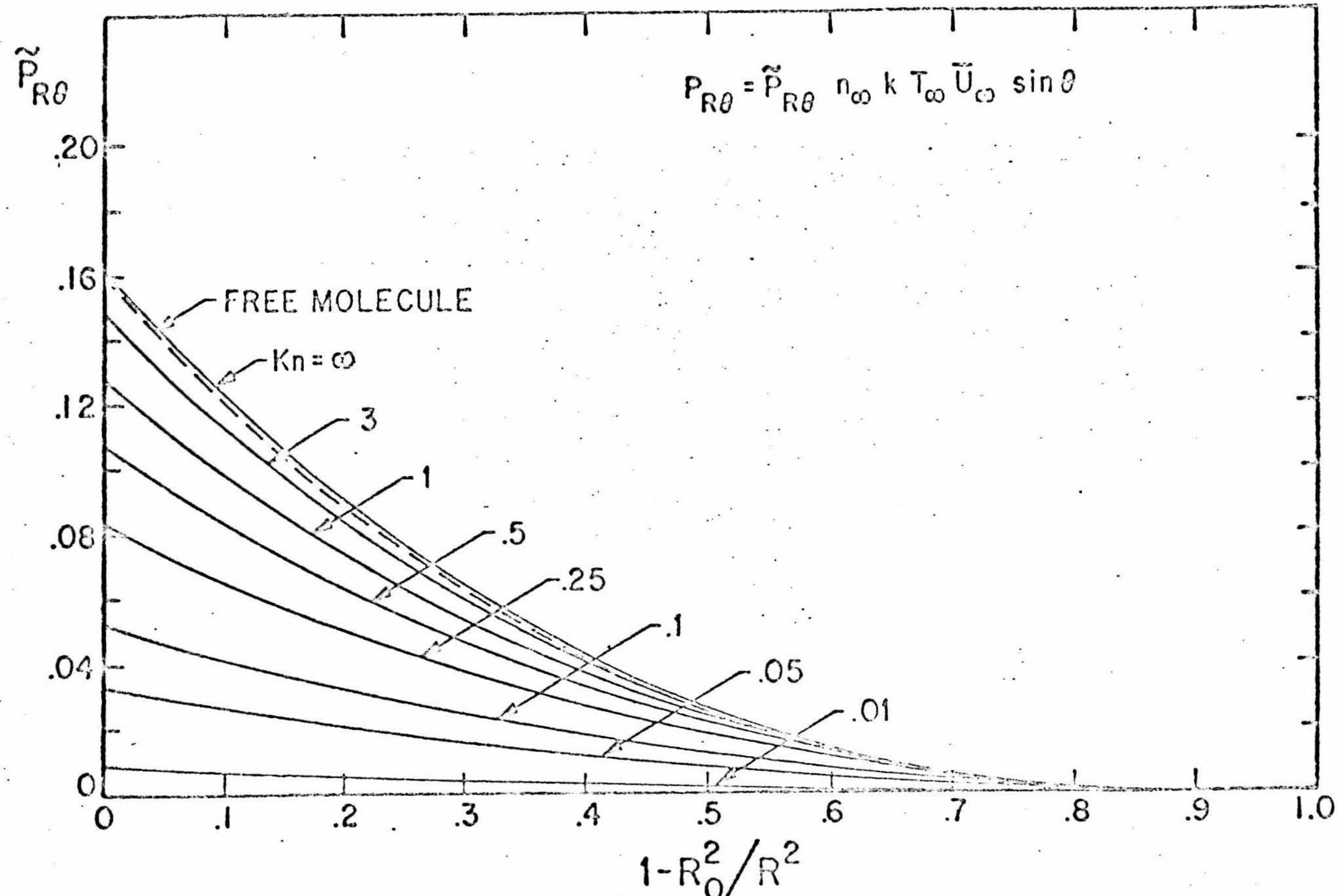


FIG. 8 SHEAR STRESS DISTRIBUTION

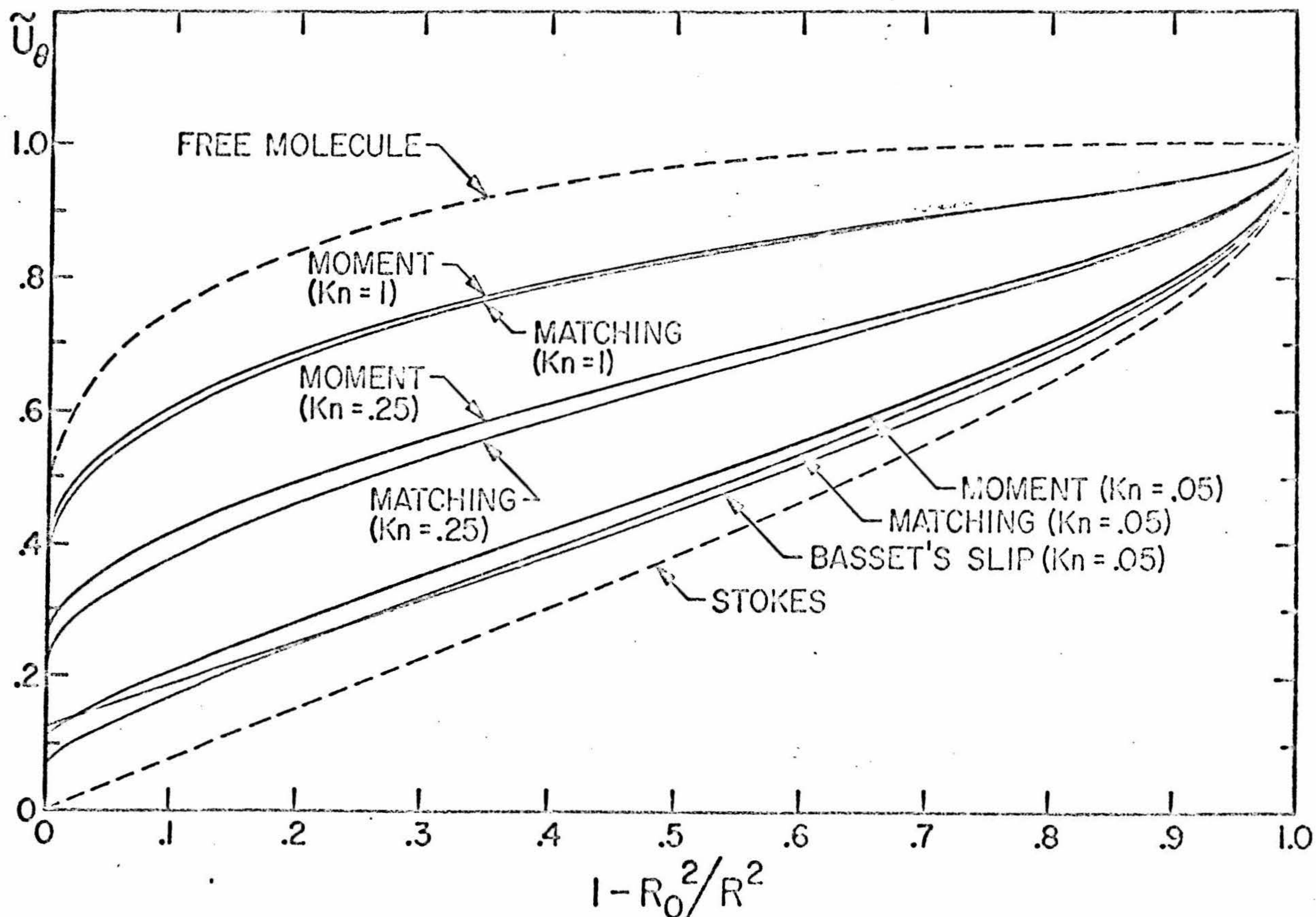


FIG. 9 COMPARISON OF MOMENT, MATCHING AND SLIP CALCULATIONS OF VELOCITY

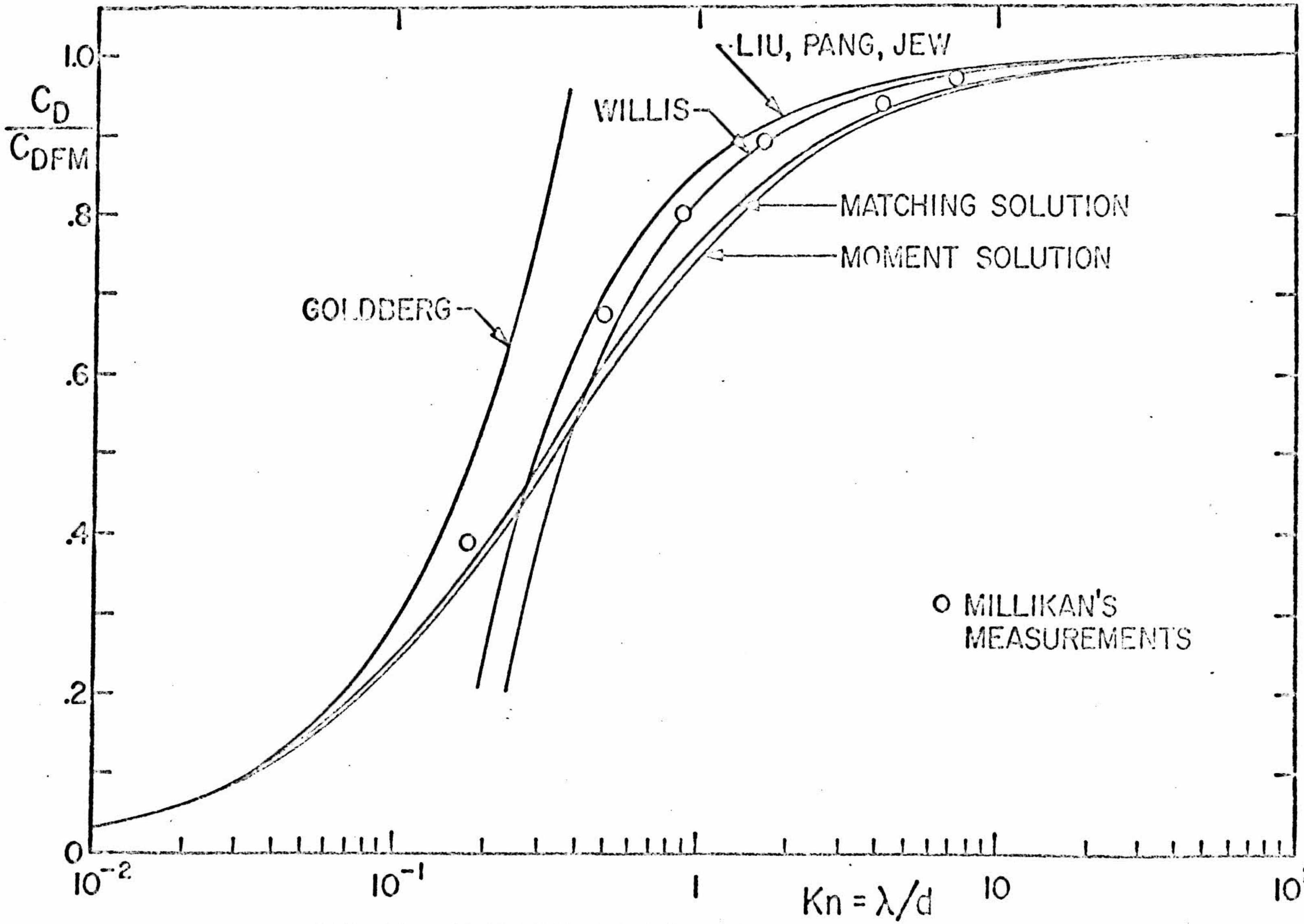


FIG. 10 SPHERE DRAG RATIO

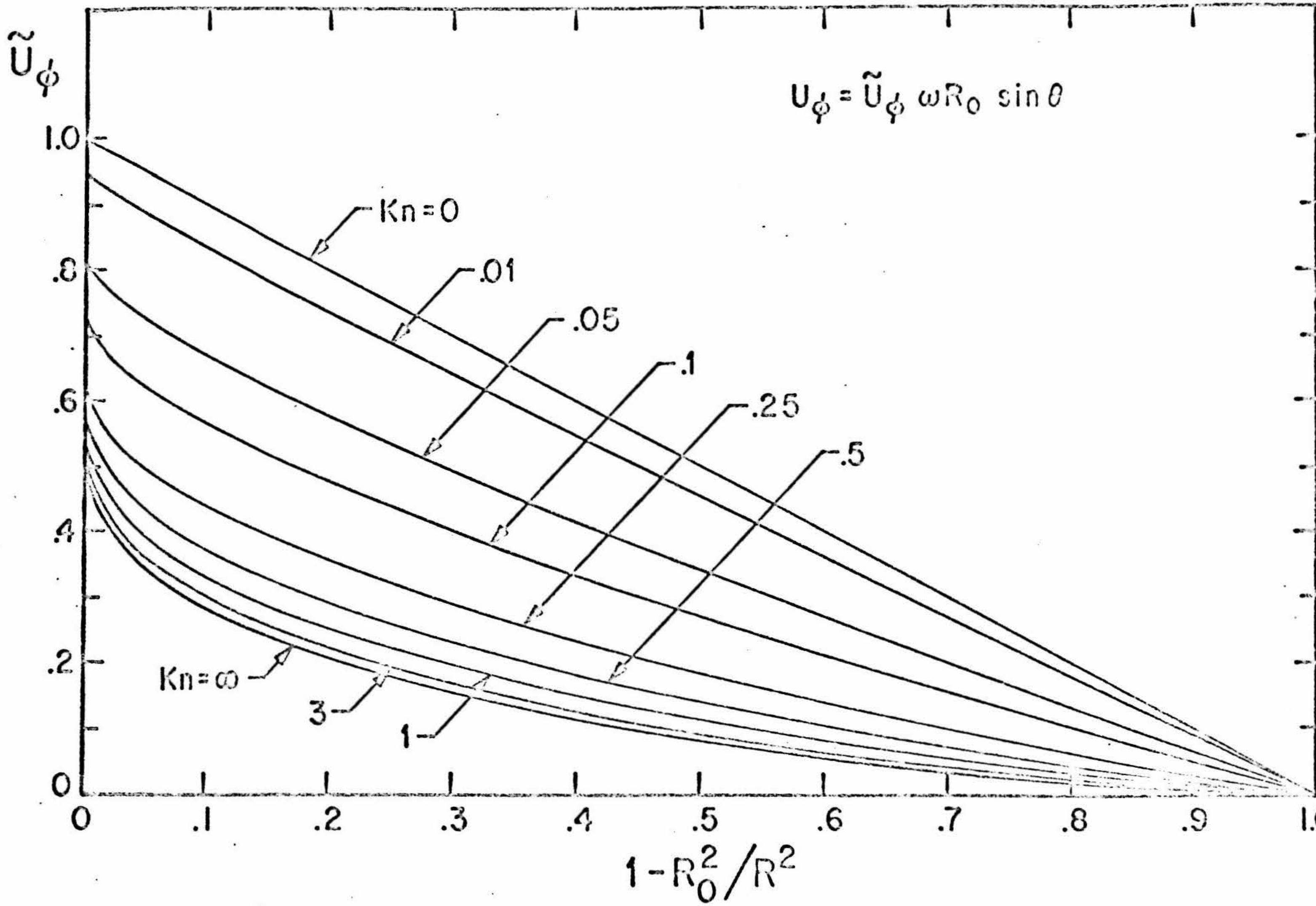


FIG. 11 FIRST-ORDER TANGENTIAL VELOCITY FIELD FOR ROTATING SPHERE

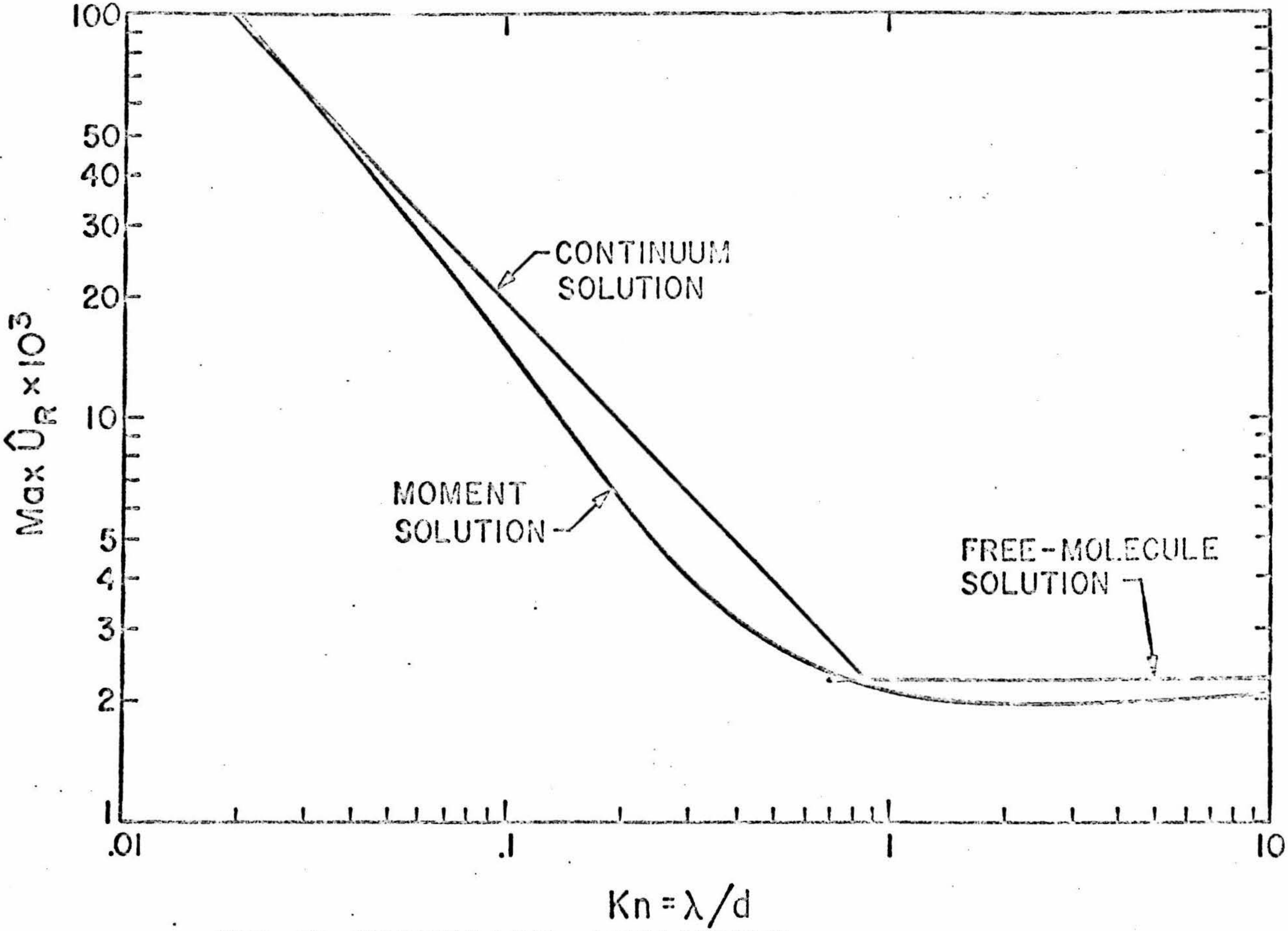


FIG. 12 SECONDARY AMPLITUDE

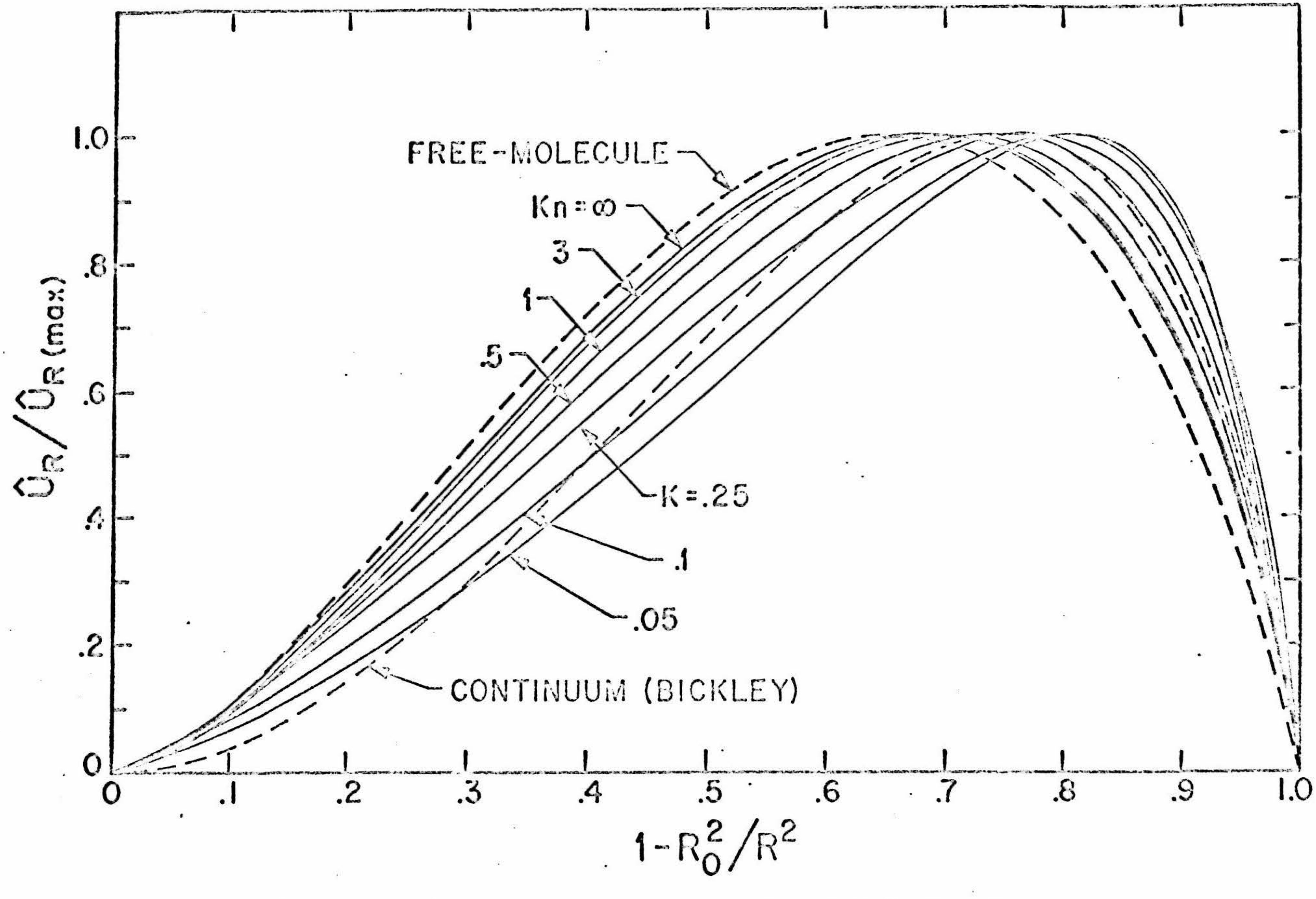


FIG. 13 SECONDARY RADIAL VELOCITY FIELD

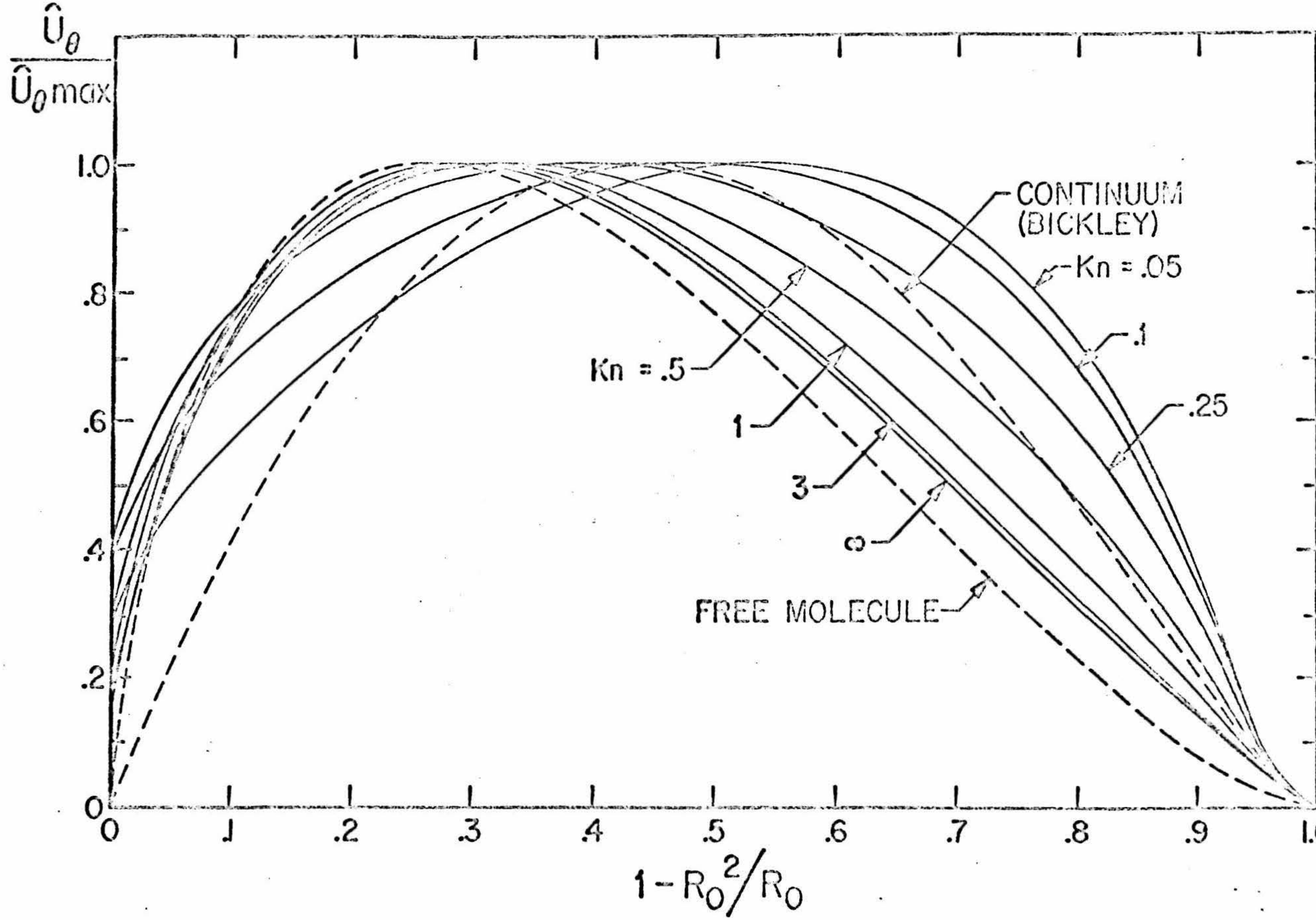


FIG. 14 SECONDARY TANGENTIAL VELOCITY FIELD

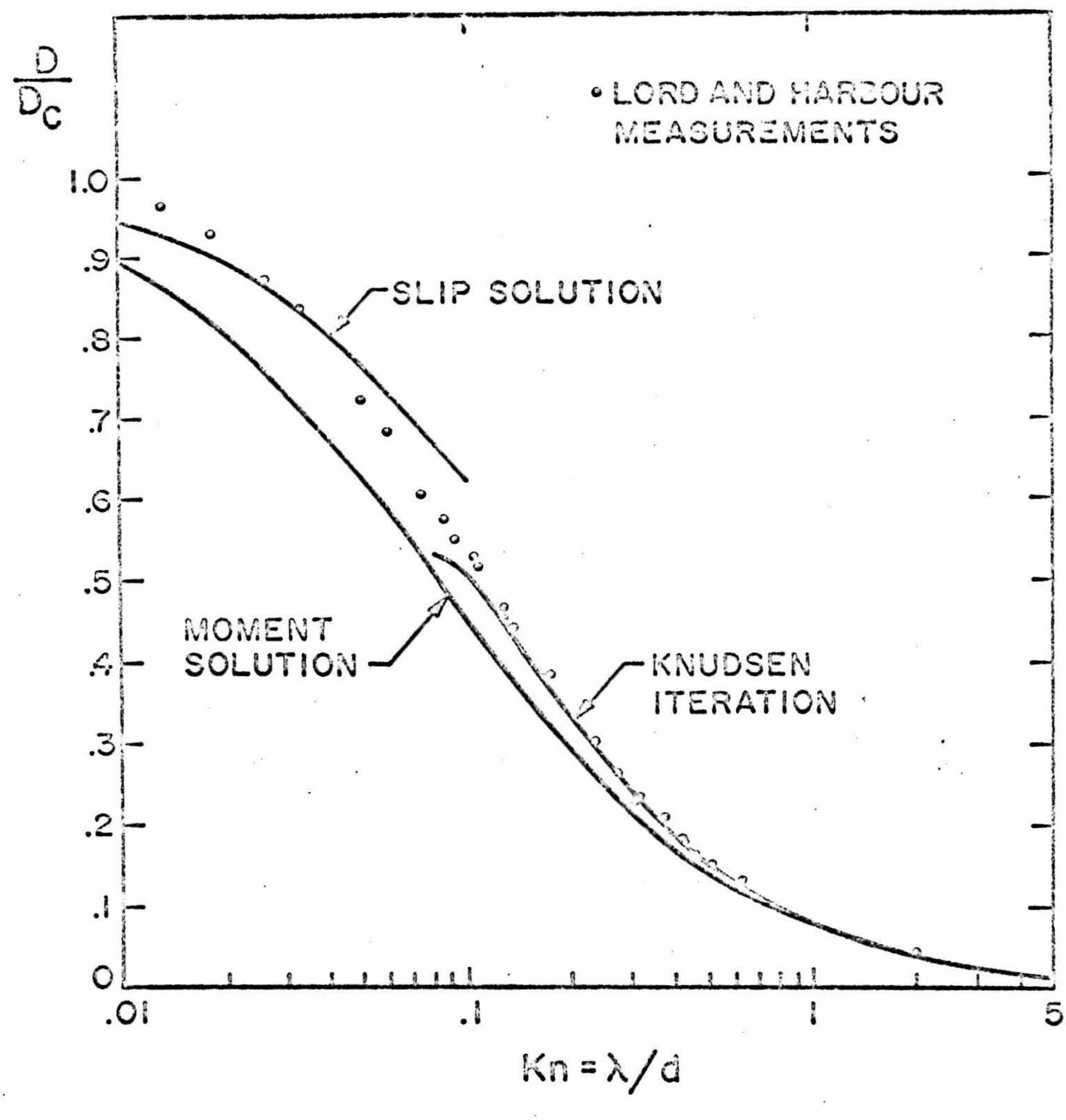


FIG. 15 DRAG TORQUE COMPARISON

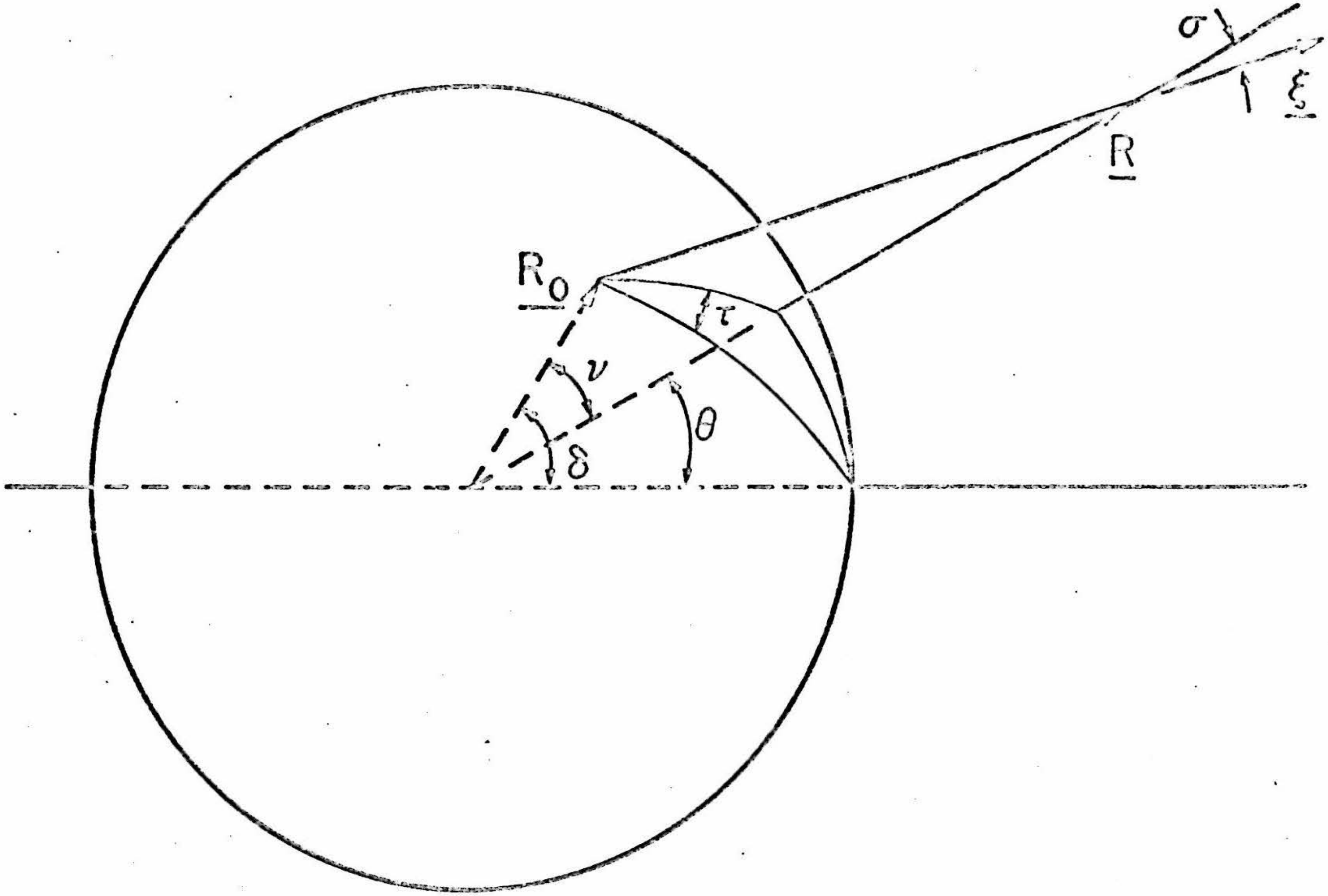


FIG. 16 GEOMETRY FOR FREE-MOLECULE INTEGRATIONS