ON CONCENTRATED LOADS AND GREEN'S FUNCTIONS IN ELASTOSTATICS

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Abstract

This investigation is concerned with the notion of concentrated loads in classical elastostatics and related issues. Following a limit treatment of problems involving concentrated internal and surface loads, the orders of the ensuing displacements and stress singularities, as well as the stress resultants of the latter, are determined. These conclusions are taken as a basis for an alternative direct formulation of concentrated-load problems, the completeness of which is established through an appropriate uniqueness theorem. In addition, the present work supplies a reciprocal theorem and an integral representation-theorem applicable to singular problems of the type under consideration. Finally, in the course of the analysis presented here, the theory of Green's functions in elastostatics is extended.

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Introduction

Although the notion of a "concentrated load" is a natural ingredient of the mechanics of particle systems and rigid bodies, it is inherently alien to the mechanics of deformable continua in general and to elastostatics in particular. Indeed, the introduction of concentrated loads into the linearized equilibrium theory of elastic solids gives rise to singular solutions of the governing equations that violate the basic approximative assumptions underlying the classical theory. Further, the direct formulation of concentrated-load problems in elastostatics that has become traditional is not covered by the conventional uniqueness theorem and is incomplete in the sense of admitting a multiplicity of solutions, as was emphasized by Sternberg and Rosenthal [1] (1952).

The foregoing uniqueness issue cannot be safely dismissed with a reference to the fictitious nature of concentrated loads: the point is that the fiction is useful provided it is made meaningful. Moreover, the fact that loads of this type represent merely a convenient idealization of certain physically realistic loadings hardly justifies conceptual vagueness or outright ambiguity in their mathematical treatment.

A comprehensive study aiming at a clarification and resolution of various fundamental questions connected with concentrated loads in elastostatics, was published by Sternberg and Eubanks [2] (1955). The program pursued in [2] may briefly be outlined as follows. To begin with, the solution to a problem involving concentrated internal or surface loads is defined as the limit of a sequence of regular solutions, corresponding to distributed body-forces or surface tractions. Such a limit definition is natural on physical grounds and is suggested by Kelvin's [3] original treatment of the problem of a concentrated load at a point of an elastic medium occupying the entire space. The next objective is to demonstrate the existence of the limit solution and to represent it in a manner suited to the determination of the orders and stress resultants of its singularities at the load points. Finally, the foregoing properties of the singularities — together with the boundary conditions for the regular surface tractions — are taken as a basis for an alternative direct characterization of the solution to concentrated-load problems, the completeness of which is the object of an appropriate uniqueness theorem. Such a direct formulation of concentrated-load problems obviates the necessity for carrying out explicitly a limit process that may in particular applications be highly cumbersome.

The work in [2], which provides a conceptual guide for the present investigation, fell short of its purpose. Thus, the proofs in [2] of the theorems concerning the limit definition, representation, and properties of the solution to a problem with concentrated surface loads (Theorems 7.1, 7.2) take for granted certain properties of the

Green's functions employed in these proofs[1]. Further, the argument used in [2] to establish a uniqueness theorem (Theorem 8.1) for concentrated loads, is inconclusive[2]. A remedy of these deficiencies requires some additional hypotheses, as well as a substantially different approach to the proofs of the theorems affected.

The present study serves a dual purpose. First, it amends those results in [2] that require modification and attempts to carry out rigorously the general program of [2]. Second, the current work contains various results on Green's functions and integral representations pertaining to the second boundary-value problem in elastostatics that are of interest in themselves.

The subjects of concentrated loads and of Green's functions in linear elasticity are intimately related. In fact, the theory of Green's functions supplies a helpful means for the study of concentrated loads, while at the same time the physical interpretation of the requisite Green's functions rests on the notion of concentrated loads. It should be emphasized, however, that this interconnection does not involve us

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1 Specifically, it is assumed that $\overline{u}'_i(Q, P, P^o)$ (defined in Theorem 6.1 of [2]) coincides on its domain of definition — for fixed P^o in the interior — with a function that is jointly continuous with respect to Q <u>and</u> P for Q on the boundary and P on the closure of the region, provided $Q \neq P$. It is also assumed that $\overline{u}'_{ij}(Q, P)$ (defined in Theorem 6.1 of [2]) obeys $\overline{u}'_{ij}(Q, P) = O(r^{-2})$ as $P \to Q$, for every Q on the boundary, if r is the distance from Q to P.

2 In the derivation of Equation (8.15) of [2] it is supposed that the displacements of the "difference state" are <u>uniformly</u> continuous on the intersection of the region with a deleted neighborhood of each load point, whereas only their continuity is assured directly by the hypotheses of Theorem 8.1.

in a logical circularity since the use we make of Green's functions in the analysis of concentrated loads is entirely independent of the physical significance of these functions.

In Section 1 we dispose of various notational and geometric preliminaries, and — for later economy — introduce the definition of an "elastic state". In Section 2 we recall briefly from [2] a limit definition and certain relevant properties of the solution to Kelvin's problem. This expository material is included here because a limit treatment of Kelvin's problem provides a transparent model for the more intricate analogous issue related to concentrated surface loads. In addition, Kelvin's solution plays an important role in connection with various Green's functions introduced subsequently.

Section 3 is devoted to analytical prerequisites for a treatment of concentrated surface loads. Here we construct, for any region with a sufficiently smooth boundary, certain singular solutions to the equations of elastostatics. These solutions, which possess a prescribed singularity at a given point of the boundary, are used at the end of the section to arrive at an integral representation — in terms of the given surface tractions — for the solution to the second boundary-value problem appropriate to such a region. An essential feature of the representation obtained here is that it holds up to the boundary. The basic ideas underlying the unfortunately rather lengthy and involved developments in Section 3 are drawn primarily from Weyl [4].

In Section 4 we apply the integral representation just mentioned to a limit definition of the solution to the problem of a concentrated surface load that is balanced by regular surface tractions. Further, after establishing the existence of the limit solution, we confirm that the orders of its (surface) singularities are the same as those encountered in Kelvin's problem and that the resultant of the stress singularity coincides with the given concentrated load. All of the considerations in this section are once again confined to regions with "smooth" boundaries.

The results regarding concentrated-load singularities in Section 2 and Section 4 suggest an alternative direct formulation of problems involving both concentrated internal and concentrated surface loads. The completeness of this direct formulation is established in Section 5 through a uniqueness theorem, which — in contrast to the results of Section 3 and Section 4 — applies to a broad class of regions. The principal tool employed in the proof of this theorem is furnished by Green's functions for the displacements in the second equilibrium problem, which we introduce for this purpose and whose existence for the region at hand we postulate. For bounded regions, the Green's functions used here differ in two essential respects from the customary Green's functions used in [2]. First, the Green's functions defined in Section 5 possess only one internal singularity (of the Kelvin type), the equilibration of which is achieved by conveniently chosen regular surface tractions; second, they are symmetric. The proof of the uniqueness theorem for concentrated loads, as well as the proofs of the results given in Section 6, is greatly facilitated by a generalization of the reciprocal theorem to a class of singular elastostatic fields, which is carried out at the beginning of Section 5.

In Section 6 we apply the displacement Green's functions of Section 5 (together with their counterpart for the stresses) to the derivation of an integral representation for the solution to concentrated-load problems in the formulation supplied by the uniqueness theorem of Section 5. At the end of Section 6 we establish a connection between the Green's functions entering the preceding representation theorem and the singular elastostatic fields involved in the representation theorem of Section 3. This connection, in particular, reveals the behavior of the Green's functions at the boundary of the region.

Finally, we remark that the developments in Section 5 and Section 6 — with the exception of the last theorem in Section 6 — are essentially self-contained.

1. Notation and preliminary definitions.

Throughout this investigation lower-case Latin or Greek letters, when not underlined, stand for scalars; lower-case Latin letters underlined by a tilde denote vectors, while lower-case Greek letters underlined by a tilde designate second-order tensors. Upper-case letters are reserved for sets; in particular, upper-case script letters are used for sets of functions. We employ the letter E for the entire three-dimensional euclidean space. If A is a set in E we write $\overline{A}$ and ∂A for the closure and the boundary of A, respectively. The symbol $A_{\underset{\sim}{a}}$ represents the set obtained from A by deleting the point with the position vector $\underset{\sim}{a}$; in order to avoid cumbersome notation, we agree to write $\overline{A}_{\underset{\sim}{a}}$ and $\partial A_{\underset{\sim}{a}}$ in place of $(\overline{A})_{\underset{\sim}{a}}$ and $(\partial A)_{\underset{\sim}{a}}$. Further, we call D the diagonal set defined by

$$D = \{(\underset{\sim}{x}, \underset{\sim}{y}) \mid (\underset{\sim}{x}, \underset{\sim}{y}) \in E \times E, \underset{\sim}{x} = \underset{\sim}{y}\} . \qquad (1.1)^{1}$$

Finally, the open sphere (ball) of radius ρ centered at $\underset{\sim}{x}$ is denoted by $B_\rho(\underset{\sim}{x})$, so that

$$B_\rho(\underset{\sim}{x}) = \{\underset{\sim}{y} \mid \underset{\sim}{y} \in E, (\underset{\sim}{y} - \underset{\sim}{x})^2 < \rho^2\} \quad (\rho > 0) . \qquad (1.2)$$

Standard indicial notation is used in connection with the cartesian components of tensors of any order: Latin subscripts and superscripts, unless otherwise specified, range over the integers (1, 2, 3), Greek indices have the range (1, 2), summation over repeated indices being implied; subscripts preceded by a comma indicate partial

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[1] Here and in the sequel, we use the conventional notation for the cartesian product of two sets.

differentiation with respect to the corresponding cartesian coordinate. For functions of more than one position vector, the aforementioned differentiation will be understood to be performed with respect to the coordinates of the first position vector[1]. If $\underset{\sim}{u}$ is a vector, we mean by $\nabla\underset{\sim}{u}$ the second-order tensor with the components $u_{i,j}$; the corresponding meaning is to be attached to $\nabla\underset{\sim}{\tau}$, where $\underset{\sim}{\tau}$ is a second order tensor. As usual, δ_{ij} stands for the Kronecker delta.

If φ is a function of two position vectors, then $\varphi(\cdot, \underset{\sim}{y})$ indicates the subsidiary mapping obtained by holding $\underset{\sim}{y}$ fixed. To characterize the smoothness of functions introduced, we write $\varphi \in \mathcal{C}(A)$ if φ is defined and continuous on a subset A of euclidean n-space. Moreover, if m is a positive integer, we write $\varphi \in \mathcal{C}^m(A)$ when $\varphi \in \mathcal{C}(A)$ and its partial derivatives of order up to and including m are defined as well as continuous on the interior of A and there coincide with functions continuous on A. Finally, if A is a surface in E, the statement $\varphi \in \mathcal{H}(A)$ is to convey that φ is defined and uniformly Hölder-continuous on A, i.e. that there exist $k > 0$ and $\alpha \in (0, 1]$ such that

$$|\varphi(\underset{\sim}{x}) - \varphi(\underset{\sim}{y})| \leq k |\underset{\sim}{x} - \underset{\sim}{y}|^{\alpha} \text{ for all } (\underset{\sim}{x}, \underset{\sim}{y}) \in A \times A .$$

Analogous interpretations apply to tensor-valued functions.

In the present investigation we require two classes of regions: regular and simple regions. We say that R is a regular region if it is an open region in E and there exists $\rho_o > 0$ such that for all $\rho > \rho_o$ the boundary of $R \cap B_\rho(\underset{\sim}{0})$ consists of a finite number of non-intersecting

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[1] Thus, $\varphi_{,i}(\underset{\sim}{x}, \underset{\sim}{y}) = \partial\varphi(\underset{\sim}{x}, \underset{\sim}{y})/\partial x_i$.

closed regular surfaces, the latter term being used in the sense of Kellogg [5] (p.112). Note that a regular region, as defined here, need not be bounded and, if unbounded, need not be an exterior region since its boundary may extend to infinity. In addition, the boundary of a regular region may have edges and corners. If $\underset{\sim}{y} \in \partial R$ and ∂R has a unique tangent plane at $\underset{\sim}{y}$, we always denote by $\underset{\sim}{n}(\underset{\sim}{y})$ the unit outer normal to ∂R at $\underset{\sim}{y}$. Further, in these circumstances, we call $\Omega(\underset{\sim}{y}, \lambda)$ the intersection of ∂R with a closed circular cylinder of radius λ and height 2λ, centered at $\underset{\sim}{y}$, the axis of the cylinder being parallel to $\underset{\sim}{n}(\underset{\sim}{y})$. Also, $\Pi(\underset{\sim}{y}, \lambda)$ will always designate the intersection of this cylinder and the tangent plane of ∂R at $\underset{\sim}{y}$. Thus, choosing cartesian coordinates x_i such that the x_3-axis points in the direction of $\underset{\sim}{n}(\underset{\sim}{y})$, one has

$$\left.\begin{aligned} \Omega(\underset{\sim}{y}, \lambda) &= \{\underset{\sim}{z} \mid \underset{\sim}{z} \in \partial R,\ (z_\alpha - y_\alpha)(z_\alpha - y_\alpha) \le \lambda^2,\ |z_3 - y_3| \le \lambda\}, \\ \Pi(\underset{\sim}{y}, \lambda) &= \{\underset{\sim}{z} \mid \underset{\sim}{z} \in E,\ (z_\alpha - y_\alpha)(z_\alpha - y_\alpha) \le \lambda^2,\ z_3 - y_3 = 0\}. \end{aligned}\right\} \quad (1.3)^1$$

A point $\underset{\sim}{y}$ on the boundary of a regular region R is said to be a <u>regular boundary point</u> if:

i) ∂R has a unique tangent plane at $\underset{\sim}{y}$;

ii) there exists $\lambda > 0$ such that $\Omega(\underset{\sim}{y}, \lambda)$, when referred to a rectangular cartesian frame with the origin at $\underset{\sim}{y}$ and the x_3-axis pointing in the direction of $\underset{\sim}{n}(\underset{\sim}{y})$, is given by

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[1] Recall that Greek and Latin indices have the respective ranges (1, 2) and (1, 2, 3).

$$\Omega(\underset{\sim}{0}, \lambda)=\{\underset{\sim}{z}|\underset{\sim}{z}\in E,\ z_\alpha z_\alpha \le \lambda^2,\ z_3=\varphi(z_1, z_2)\},\ \varphi\in C^2(\Pi(\underset{\sim}{0}, \lambda))\ . \qquad (1.4)^1$$

We define next a simple region to be a bounded regular region, the boundary of which is a single surface consisting of regular boundary points exclusively. We shall make frequent use of the following two properties of a simple region R:

(a) there exists $\lambda>0$ such that $(\underset{\sim}{x}, s)\in\partial R\times(0, \lambda]$ implies $\underset{\sim}{x}+s\underset{\sim}{n}(\underset{\sim}{x})\notin\overline{R}$;

(b) there exists $k>0$ such that

$$\left.\begin{aligned} &|\underset{\sim}{n}(\underset{\sim}{x})-\underset{\sim}{n}(\underset{\sim}{y})|\le k|\underset{\sim}{x}-\underset{\sim}{y}| \quad \text{for all } (\underset{\sim}{x}, \underset{\sim}{y})\in\partial R\times\partial R\ ,\\ &|(\underset{\sim}{x}-\underset{\sim}{y})\cdot\underset{\sim}{n}(\underset{\sim}{y})|\le k(\underset{\sim}{x}-\underset{\sim}{y})^2 \quad \text{for all } (\underset{\sim}{x}, \underset{\sim}{y})\in\partial R\times\partial R\ .\end{aligned}\right\} \quad (1.5)^2$$

Property (a) assures that for some $\lambda>0$ (depending only on R) any straight line segment of length λ issuing from a point of ∂R in the direction of the outer normal does not re-enter $\overline{R}$. The existence of such a λ is a direct consequence of the present definition of R and the Heine-Borel theorem. The inequalities (1.5) follow from the assumed smoothness of ∂R; the first of (1.5) is elementary, whereas a proof of the second may be found in [5] (p. 299).

Turning to preliminaries concerning the linearized theory of homogeneous and isotropic elastic solids, we now introduce

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1 Note that $\Omega(\underset{\sim}{0}, \lambda)$ here has a higher degree of smoothness than that guaranteed by Kellogg's definition of a regular surface element ([5], p. 105).

2 The symbols "$\wedge$" and "$\cdot$" are used throughout to denote vectorial and scalar multiplication of vectors, respectively.

Definition 1.1. (State, elastic state). Let A be a region in E, i.e. an open connected set together with all, some, or none of its boundary points, and let $\mathring{A}$ be the interior of A. If $\underset{\sim}{u}$ is a vector-valued and $\underset{\sim}{\tau}$ a second-order tensor valued function defined on A, we call the ordered pair $S=[\underset{\sim}{u}, \underset{\sim}{\tau}]$ a state on A. We say that $S=[\underset{\sim}{u}, \underset{\sim}{\tau}]$ is an elastic state on A, with the displacement field $\underset{\sim}{u}$ and the stress field $\underset{\sim}{\tau}$, corresponding to the body-force $\underset{\sim}{f}$, the shear modulus μ, and Poisson's ratio σ, and write

$$S=[\underset{\sim}{u}, \underset{\sim}{\tau}]\in \mathcal{E}(\underset{\sim}{f}, \mu, \sigma; A),$$

provided:

(a) $\underset{\sim}{u}\in C^1(\mathring{A})\cap C(A)$, $\underset{\sim}{\tau}\in C^1(\mathring{A})\cap C(A)$, $\underset{\sim}{f}\in C(A)$, whereas μ and σ are constants with $\mu > 0$, $-1<\sigma<1/2$;

(b) $\underset{\sim}{u}, \underset{\sim}{\tau}, \underset{\sim}{f}, \mu$ and σ satisfy

$$\tau_{ji,j}+f_i=0, \quad \tau_{ij}=\mu\left[\frac{2\sigma}{1-2\sigma}\delta_{ij}u_{k,k}+u_{i,j}+u_{j,i}\right] \text{ on } \mathring{A}; \tag{1.6}$$

(c) if A is unbounded,

$$\underset{\sim}{u}(\underset{\sim}{x})=O(x^{-1}), \quad \underset{\sim}{\tau}(\underset{\sim}{x})=O(x^{-2}), \quad \underset{\sim}{f}(\underset{\sim}{x})=O(x^{-3}) \quad \text{as } x\to\infty. \qquad (1.7)^1$$

The first of (1.6) represents the stress equations of equilibrium — the second the stress-displacement relations. In particular, (b) ensures the symmetry of the stress tensor $\underset{\sim}{\tau}$ on $\mathring{A}$. We recall that the inequalities imposed in (a) on the elastic constants μ and σ are necessary and sufficient for the positive definiteness of the strain-

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[1] Here and in the sequel, we write x in place of $|\underset{\sim}{x}|$. The order of magnitude symbols "O" and "o" are used throughout in their standard mathematical connotation.

energy density. If A is an exterior domain and $\underset{\sim}{f}=\underset{\sim}{0}$, the order conditions at infinity (1.7) are implied by

$$\underset{\sim}{u}(\underset{\sim}{x})=o(1) \quad \text{as } x\to\infty \,. \tag{1.8}$$[1]

If $S=[\underset{\sim}{u},\underset{\sim}{\tau}]$ is a state on A and Σ is one side of a regular surface with the unit outer normal vector $\underset{\sim}{n}$, we call $\underset{\sim}{t}$ the <u>traction vector</u> of S on Σ if

$$t_i=\tau_{ij}n_j \tag{1.9}$$

at all nonsingular points of Σ. If A is a region, S is a state on A, and Σ is a regular surface contained in $A\cap\partial A$, then — unless otherwise specified — we mean by the "tractions of S on Σ" the tractions of S on the side of Σ facing the exterior (complement) of A.

Equality of states, addition and multiplication by a constant, as well as differentiation and integration, are defined as follows. Suppose $S=[\underset{\sim}{u},\underset{\sim}{\tau}]$, $S'=[\underset{\sim}{u}',\underset{\sim}{\tau}']$, $S''=[\underset{\sim}{u}'',\underset{\sim}{\tau}'']$ are states on A and let c be a constant. Then,

$$S'=S'' \text{ if } \underset{\sim}{u}'=\underset{\sim}{u}'',\ \underset{\sim}{\tau}'=\underset{\sim}{\tau}'' \text{ on } A\,,$$

$$S=S'+S'' \text{ if } \underset{\sim}{u}=\underset{\sim}{u}'+\underset{\sim}{u}'',\ \underset{\sim}{\tau}=\underset{\sim}{\tau}'+\underset{\sim}{\tau}'' \text{ on } A\,,$$

$$S=c\,S' \text{ if } \underset{\sim}{u}=c\underset{\sim}{u}',\ \underset{\sim}{\tau}=c\underset{\sim}{\tau}' \text{ on } A\,.$$

Next,

$$S'=S''_{,i} \text{ if } u_j'=u''_{j,i}\,,\ \tau'_{jk}=\tau''_{jk,i} \text{ on } A\,,$$

provided the derivatives here involved exist. Further, if $S(\cdot,\lambda)=[\underset{\sim}{u}(\cdot,\lambda),\underset{\sim}{\tau}(\cdot,\lambda)]$ is a state on A for every $\lambda\in[a,b]$, then

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[1] See Fichera [6] and Gurtin and Sternberg [7] (Theorem 5.1).

$$S' = \int_a^b S(\cdot, \lambda)d\lambda \text{ if } \underset{\sim}{u}' = \int_a^b \underset{\sim}{u}(\cdot, \lambda)d\lambda , \quad \underset{\sim}{T}' = \int_a^b \underset{\sim}{T}(\cdot, \lambda)d\lambda \quad \text{on } A ,$$

provided the preceding integrations are meaningful. Finally, we attach the obvious interpretation to the limit of a sequence of states.

2. Internal concentrated loads. Kelvin's problem.

In the present section we deal with the problem of a concentrated load applied at a point of a medium occupying the entire space E. The solution to this problem was first given by Kelvin [8]; it is derived in Kelvin and Tait's treatise [3] (p. 279) through a limit process, which is made fully explicit in [2]. The limit formulation of Kelvin's problem to be presented here follows closely that adopted by Sternberg and Al-Khozaie [9] in treating the analogous problem of the linearized theory of viscoelasticity. We first introduce

Definition 2.1. (*Sequence of body-force fields tending to a concentrated load*). *Let* $\underset{\sim}{a} \in E$ *and let* $\underset{\sim}{\ell}$ *be a vector. We say that* $\{\underset{\sim}{f}^m\}$ *is a sequence of body-force fields on* E *tending to a concentrated load* $\underset{\sim}{\ell}$ *at (the point)* $\underset{\sim}{a}$ *if:*

(a) $\underset{\sim}{f}^m \in C^2(E)$ $(m=1,2,3,\ldots)$;

(b) $\underset{\sim}{f}^m = \underset{\sim}{0}$ *on* $E - B_{\rho_m}(\underset{\sim}{a})$ $(m=1,2,3,\ldots)$, *where* $\{B_{\rho_m}(\underset{\sim}{a})\}$ *is a sequence of spheres such that* $\rho_m \to 0$ *as* $m \to \infty$;

(c) $\lim\limits_{m \to \infty} \int_E \underset{\sim}{f}^m \, dV = \underset{\sim}{\ell}$;

(d) *the sequence* $\left\{ \int_E |\underset{\sim}{f}^m| \, dV \right\}$ *is bounded.*

We cite next a theorem which supplies both a definition and a representation of the solution to the problem under present consideration.

Theorem 2.1. (Limit definition of the solution to Kelvin's problem). Let $\underset{\sim}{a} \in E$ and let $\underset{\sim}{\ell}$ be a vector. Further let $\{\underset{\sim}{f}^m\}$ be a sequence of body-force fields on E tending to a concentrated load $\underset{\sim}{\ell}$ at $\underset{\sim}{a}$. Then:

(a) there exists a unique sequence of states $\{S^m\}$ such that

$$S^m = [\underset{\sim}{u}^m, \underset{\sim}{\tau}^m] \in \mathcal{E}(\underset{\sim}{f}^m, \mu, \sigma; E) \quad (m=1,2,3,\ldots);$$

(b) $\{S^m\}$ converges to a state $S=[\underset{\sim}{u}, \underset{\sim}{\tau}]$ on $E_{\underset{\sim}{a}}$, the convergence being uniform on any closed subset of $E_{\underset{\sim}{a}}$;

(c) the limit state S is independent of the sequence $\{\underset{\sim}{f}^m\}$ and admits the representation

$$S(\underset{\sim}{x}) = S^i(\underset{\sim}{x}, \underset{\sim}{a})\ell_i \quad \text{for all } \underset{\sim}{x} \in E_{\underset{\sim}{a}}, \tag{2.1}$$

where

$$S^i(\underset{\sim}{x}, \underset{\sim}{y}) = S^i(\underset{\sim}{x} - \underset{\sim}{y}, \underset{\sim}{0}) \quad \text{for all } (\underset{\sim}{x}, \underset{\sim}{y}) \in E \times E - D, \tag{2.2}$$[1]

while the displacements and stresses of $S^i(\cdot, \underset{\sim}{0})$ are, for all $\underset{\sim}{x} \in E_{\underset{\sim}{0}}$, given by

$$\left.\begin{aligned} u^i_j(\underset{\sim}{x}, \underset{\sim}{0}) &= \frac{1}{16\pi\mu(1-\sigma)x}\left[\frac{x_i x_j}{x^2} + (3-4\sigma)\delta_{ij}\right], \\ \tau^i_{jk}(\underset{\sim}{x}, \underset{\sim}{0}) &= -\frac{1}{8\pi(1-\sigma)x^3}\left[\frac{3x_i x_j x_k}{x^2} + (1-2\sigma)(\delta_{ij}x_k + \delta_{ik}x_j - \delta_{jk}x_i)\right]. \end{aligned}\right\} \tag{2.3}$$

We call S the Kelvin state corresponding to a concentrated load $\underset{\sim}{\ell}$ at $\underset{\sim}{a}$ (and to the elastic constants μ and σ). In particular, we say that $S^i(\cdot, \underset{\sim}{y})$ is the Kelvin state corresponding to a unit concentrated load at $\underset{\sim}{y}$ in the x_i-direction.

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[1] Recall from the definition of the diagonal set D in Section 1 that $E \times E - D = \{(\underset{\sim}{x}, \underset{\sim}{y}) \mid (\underset{\sim}{x}, \underset{\sim}{y}) \in E \times E, \underset{\sim}{x} \neq \underset{\sim}{y}\}$.

This theorem is proved in [2][1]. The need for condition (d) in Definition 2.1 is also established in [2], where it is shown by means of a counterexample that conclusions (b), (c) in Theorem 2.1 become invalid if this hypothesis is omitted. The foregoing requirement is no longer necessary if $\underset{\sim}{f}^m$ is parallel and unidirectional, in which case condition (d) is implied by (c) of Definition 2.1.

We now quote from [2],

<u>Theorem</u> 2.2. <u>(Properties of the Kelvin state). The Kelvin state</u> S <u>corresponding to a concentrated load</u> $\underset{\sim}{\ell}$ <u>at</u> $\underset{\sim}{a}$ <u>has the properties</u>:

(a) $S=[\underset{\sim}{u}, \underset{\sim}{T}] \in \mathcal{E}(\underset{\sim}{0}, \mu, \sigma; E_{\underset{\sim}{a}})$;

(b) $\underset{\sim}{u}(\underset{\sim}{x}) = O(|\underset{\sim}{x}-\underset{\sim}{a}|^{-1}), \quad \underset{\sim}{T}(\underset{\sim}{x}) = O(|\underset{\sim}{x}-\underset{\sim}{a}|^{-2})$ as $\underset{\sim}{x} \to \underset{\sim}{a}$;

(c) $\displaystyle\int_{\partial B_\rho(\underset{\sim}{a})} \underset{\sim}{t}\, dA = \underset{\sim}{\ell}\,, \quad \int_{\partial B_\rho(\underset{\sim}{a})} (\underset{\sim}{x}-\underset{\sim}{a}) \wedge \underset{\sim}{t}\, dA = \underset{\sim}{0}$ for every $\rho > 0$,

<u>where</u> $\underset{\sim}{t}$ <u>is the traction vector on the side of</u> $\partial B_\rho(\underset{\sim}{a})$ <u>that faces</u> $\underset{\sim}{a}$.

As is pointed out in [2], the formulation of Kelvin's problem in terms of (a) and (c) alone, which appears to have become traditional, is incomplete in view of the existence of elastic states on $E_{\underset{\sim}{a}}$ that possess self-equilibrated singularities[2] at $\underset{\sim}{a}$. In contrast, as will be shown in Section 5 (Theorem 5.2), properties (a), (b) and the first of (c) suffice to characterize the Kelvin state uniquely.

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[1] Although the uniformity of the convergence asserted in conclusion (b) is not mentioned in [2], it is easily inferred from the argument used in [2].

[2] E.g., a center of dilatation at $\underset{\sim}{a}$.

3. Representation of elastic states corresponding to given surface tractions.

The proof in [2] of Theorem 2.1 concerning the limit definition of the Kelvin state rests on a representation of the sequence of approximating states in terms of their body-force fields. On the other hand, once the Kelvin state has been explicitly determined in this manner, the proof of Theorem 2.2, which asserts various properties of Kelvin's solution, becomes entirely elementary. For a parallel treatment of the more involved issue of concentrated surface loads (in the absence of body forces) one requires first a representation of elastic states in terms of their surface tractions. A representation of this type — valid for the interior of the region at hand — is supplied by the theory of Green's functions for the second boundary-value problem of elastostatics, an exposition of which may be found in Section 6 of [2]. This theory is conveniently modified and generalized in Sections 5, 6 of the present investigation. Unfortunately, a rigorous proof of the analogues for surface loads of Theorems 2.1, 2.2 by means of Green's functions offers considerable analytical difficulties, which stem from the elusive behavior of these functions at the boundary. For this reason we deduce in the current section an alternative representation of elastic states — confined to simple regions — which holds up to the boundary. This alternative representation is better suited to a limit treatment of concentrated surface loads, which is carried out in Section 4. At the same time, as will become apparent in Section 6 (Theorem 6.2), the representation arrived at in the present section enables one to ascertain the boundary behavior of the Green's states introduced in Sections 5, 6.

Although the basic idea underlying the subsequent developments is suggested by Weyl [4][1], some of the results obtained in what follows go considerably beyond those contained in [4], while others are more closely related to the work of Kellogg [5], Giraud [11] and Pogorzelski [12] (Chapter 12). We first introduce

Definition 3.1. (Tangent states). Let R be a simple region, assume $\underset{\sim}{y} \in \partial R$ and $\lambda \in (0, \infty)$. We call

$$\overline{S}^i(\cdot, \underset{\sim}{y}, \lambda) = [\bar{\underset{\sim}{u}}^i(\cdot, \underset{\sim}{y}, \lambda),\ \overline{\underset{\sim}{T}}^i(\cdot, \underset{\sim}{y}, \lambda)]$$

the tangent state for the region R at $\underset{\sim}{y}$, corresponding to the x_i-direction, the parameter λ, and the elastic constants μ, σ if for all $\underset{\sim}{x}$ in the set

$$\overline{R} - \{\underset{\sim}{z} \,|\, \underset{\sim}{z} \in E,\ \underset{\sim}{z} = \underset{\sim}{y} + s\underset{\sim}{n}(\underset{\sim}{y}),\ s \in [0, \lambda]\},$$

$$\overline{S}^i(\underset{\sim}{x}, \underset{\sim}{y}, \lambda) = 4(1-\sigma)S^i(\underset{\sim}{x}, \underset{\sim}{y}) - 2(1-\sigma)\int_0^\lambda s S^j_{,ji}(\underset{\sim}{x}, \underset{\sim}{y} + s\underset{\sim}{n}(\underset{\sim}{y}))\,ds$$

$$-2(1-2\sigma)n_j(\underset{\sim}{y})\int_0^\lambda \left[S^i_{,j}(\underset{\sim}{x}, \underset{\sim}{y} + s\underset{\sim}{n}(\underset{\sim}{y})) - S^j_{,i}(\underset{\sim}{x}, \underset{\sim}{y} + s\underset{\sim}{n}(\underset{\sim}{y}))\right]ds\,, \qquad (3.1)^{2}$$

where $\underset{\sim}{n}$ is the unit outer normal to ∂R, while $S^i(\cdot, \underset{\sim}{y})$ is the normalized Kelvin state of Theorem 2.1. Further, we adopt the notation

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[1] See also Weyl [10] (p. 70), where an essential shortcoming of [4] is discussed.

[2] According to an agreement stated in Section 1, the differentiations in (3.1) are to be performed with respect to the first position vector.

$$\overline{S}^i(\cdot, \underset{\sim}{y}, \infty) = \lim_{\lambda \to \infty} \overline{S}^i(\cdot, \underset{\sim}{y}, \lambda) \; . \tag{3.2}$$

The state $\overline{S}^i(\cdot, \underset{\sim}{y}, \infty)$ in the preceding definition admits a simple physical interpretation. To this end hold R and $\underset{\sim}{y}$ fixed and consider the half-space

$$H = \{\underset{\sim}{z} \,|\, \underset{\sim}{z} \in E, \; (\underset{\sim}{z} - \underset{\sim}{y}) \cdot \underset{\sim}{n}(\underset{\sim}{y}) \leq 0\} \; ,$$

whose boundary coincides with the tangent plane of ∂R at $\underset{\sim}{y}$. Then $\overline{S}^i(\cdot, \underset{\sim}{y}, \infty)$ is the Boussinesq-Cerruti solution to the problem of a unit concentrated load acting at $\underset{\sim}{y}$ in the x_i-direction on an elastic body occupying H (see Love [13], p. 242 et seq.). In the present context it is essential to remark that our use of the foregoing tangent state, though motivated by, in no way depends upon, its physical significance as the solution to a particular concentrated-load problem. This physical meaning of the Boussinesq-Cerruti solution is, incidentally, readily confirmed by a limit process (see Love [13], loc. cit.) analogous to that employed in Theorem 2.1 to define the Kelvin state, but based on a sequence of distributed surface loads. The tangent state $\overline{S}^i(\cdot, \underset{\sim}{y}, \lambda)$ evidently differs from $\overline{S}^i(\cdot, \underset{\sim}{y}, \infty)$ by an elastic state regular on H.

Equations (3.1), (2.2), the first of (2.3) and the second of (1.6) yield, after some computation, that for all $\underset{\sim}{x}$ in the domain of definition of $\overline{S}^i(\cdot, \underset{\sim}{y}, \infty)$,

$$\left.\begin{aligned}
\bar{u}^i_j(\underset{\sim}{x},\underset{\sim}{y},\infty) &= \frac{1}{2\pi\mu}\left[\sigma\frac{(x_i-y_i)(x_j-y_j)}{|\underset{\sim}{x}-\underset{\sim}{y}|^3}+(1-\sigma)\frac{\delta_{ij}}{|\underset{\sim}{x}-\underset{\sim}{y}|}\right]\\
&+\frac{1-2\sigma}{4\pi\mu}\left[n_i(\underset{\sim}{y})h_{,j}(\underset{\sim}{x},\underset{\sim}{y})-n_j(\underset{\sim}{y})h_{,i}(\underset{\sim}{x},\underset{\sim}{y})+(x_p-y_p)n_p(\underset{\sim}{y})h_{,ij}(\underset{\sim}{x},\underset{\sim}{y})\right],\\
\bar{\tau}^i_{jk}(\underset{\sim}{x},\underset{\sim}{y},\infty) &= -\frac{3\sigma}{\pi}\frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|\underset{\sim}{x}-\underset{\sim}{y}|^5}-\frac{1-2\sigma}{2\pi}\frac{\delta_{ij}(x_k-y_k)+\delta_{ik}(x_j-y_j)}{|\underset{\sim}{x}-\underset{\sim}{y}|^3}\\
&+\frac{1-2\sigma}{2\pi}\left[n_i(\underset{\sim}{y})h_{,jk}(\underset{\sim}{x},\underset{\sim}{y})+(x_p-y_p)n_p(\underset{\sim}{y})h_{,ijk}(\underset{\sim}{x},\underset{\sim}{y})\right],
\end{aligned}\right\}\quad(3.3)$$

where

$$h(\underset{\sim}{x},\underset{\sim}{y})=\log\left[\,|\underset{\sim}{x}-\underset{\sim}{y}|-(\underset{\sim}{x}-\underset{\sim}{y})\cdot\underset{\sim}{n}(\underset{\sim}{y})\right]. \quad (3.4)$$

For future purposes we also note that

$$\bar{\tau}^i_{jk}(\underset{\sim}{x},\underset{\sim}{y},\infty)n_k(\underset{\sim}{y})=-\frac{3(x_i-y_i)(x_j-y_j)(x_k-y_k)n_k(\underset{\sim}{y})}{2\pi\,|\underset{\sim}{x}-\underset{\sim}{y}|^5}. \quad (3.5)$$

In connection with Definition 3.1 it is essential to recognize that the tangent state $\bar{S}^i(\cdot,\underset{\sim}{y},\infty)$ is not necessarily regular on R since the ray issuing from $\underset{\sim}{y}\in\partial R$ in the direction of $\underset{\sim}{n}(\underset{\sim}{y})$ may re-enter R unless R has certain convexity properties. Such internal singularities on R of $\bar{S}^i(\cdot,\underset{\sim}{y},\lambda)$ are precluded for sufficiently small $\lambda>0$, as is clear from

<u>Lemma</u> 3.1 (<u>Properties of the tangent states</u>). <u>Let</u> R <u>be a simple region and let</u> $\lambda>0$ <u>be such that</u>

$$\underset{\sim}{y}\in\partial R,\ \underset{\sim}{z}=\underset{\sim}{y}+s\underset{\sim}{n}(\underset{\sim}{y}),\ s\in(0,\lambda]\ \underline{\text{implies}}\ \underset{\sim}{z}\notin\bar{R}. \quad (3.6)^1$$

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[1] As was pointed out in Section 1, the existence of such a choice of λ is assured.

Further, for each $\underset{\sim}{y} \in \partial R$, *let* $\bar{S}^i(\cdot, \underset{\sim}{y}, \lambda)$ *be the tangent state for the region* R *at* $\underset{\sim}{y}$, *corresponding to the* x_i-*direction, the parameter* λ, *and the elastic constants* μ, σ. *Then:*

(a) $\bar{S}^i(\cdot, \underset{\sim}{y}, \lambda) \in \mathcal{E}(\underset{\sim}{0}, \mu, \sigma; \bar{R}_{\underset{\sim}{y}})$ *for all* $\underset{\sim}{y} \in \partial R$,

$\bar{\underset{\sim}{u}}^i(\cdot, \cdot, \lambda) \in C(\bar{R} \times \partial R - D)$, $\bar{\underset{\sim}{T}}^i(\cdot, \cdot, \lambda) \in C(\bar{R} \times \partial R - D)$,

$\nabla \bar{\underset{\sim}{u}}^i(\cdot, \cdot, \lambda) \in C(R \times \partial R)$, $\nabla \bar{\underset{\sim}{T}}^i(\cdot, \cdot, \lambda) \in C(R \times \partial R)$;

(b) *there exists* $\varkappa > 0$ (*independent of* $\underset{\sim}{y}$) *such that*

$|\bar{\underset{\sim}{u}}^i(\underset{\sim}{x}, \underset{\sim}{y}, \lambda)| < \varkappa\, |\underset{\sim}{x} - \underset{\sim}{y}|^{-1}$ *for all* $(\underset{\sim}{x}, \underset{\sim}{y}) \in \bar{R} \times \partial R - D$,

$|\bar{\underset{\sim}{T}}^i(\underset{\sim}{x}, \underset{\sim}{y}, \lambda)| < \varkappa\, |\underset{\sim}{x} - \underset{\sim}{y}|^{-2}$ *for all* $(\underset{\sim}{x}, \underset{\sim}{y}) \in \bar{R} \times \partial R - D$,

$|\bar{\underset{\sim}{t}}^i(\underset{\sim}{x}, \underset{\sim}{y}, \lambda)| < \varkappa\, |\underset{\sim}{x} - \underset{\sim}{y}|^{-1}$ *for all* $(\underset{\sim}{x}, \underset{\sim}{y}) \in \partial R \times \partial R - D$,

where $\bar{\underset{\sim}{t}}^i(\cdot, \underset{\sim}{y}, \lambda)$ *is the traction vector of* $\bar{S}^i(\cdot, \underset{\sim}{y}, \lambda)$ *on* $\partial R_{\underset{\sim}{y}}$.

Proof. Conclusion (a) is easily inferred from (3.1). To prove (b) observe on the basis of Definition 3.1 that for every $\underset{\sim}{y} \in \partial R$,

$$\left.\begin{aligned} \bar{\underset{\sim}{u}}^i(\underset{\sim}{x}, \underset{\sim}{y}, \lambda) &= \bar{\underset{\sim}{u}}^i(\underset{\sim}{x}, \underset{\sim}{y}, \infty) + O(1) \text{ as } \underset{\sim}{x} \to \underset{\sim}{y}, \\ \bar{\underset{\sim}{T}}^i(\underset{\sim}{x}, \underset{\sim}{y}, \lambda) &= \bar{\underset{\sim}{T}}^i(\underset{\sim}{x}, \underset{\sim}{y}, \infty) + O(1) \text{ as } \underset{\sim}{x} \to \underset{\sim}{y}, \end{aligned}\right\} \quad (3.7)$$

these estimates holding uniformly with respect to all $\underset{\sim}{y} \in \partial R$. The first two of conclusions (b) now follow at once from (3.7), (3.3), and (3.4). With a view toward the last of (b), note first the identities

$$\begin{aligned} \bar{t}^i_j(\underset{\sim}{x}, \underset{\sim}{y}, \lambda) &= \bar{\tau}^i_{jk}(\underset{\sim}{x}, \underset{\sim}{y}, \lambda) n_k(\underset{\sim}{x}) \\ &= \bar{\tau}^i_{jk}(\underset{\sim}{x}, \underset{\sim}{y}, \infty) n_k(\underset{\sim}{y}) + \bar{\tau}^i_{jk}(\underset{\sim}{x}, \underset{\sim}{y}, \lambda)[n_k(\underset{\sim}{x}) - n_k(\underset{\sim}{y})] \\ &\quad + [\bar{\tau}^i_{jk}(\underset{\sim}{x}, \underset{\sim}{y}, \lambda) n_k(\underset{\sim}{y}) - \bar{\tau}^i_{jk}(\underset{\sim}{x}, \underset{\sim}{y}, \infty) n_k(\underset{\sim}{y})], \end{aligned}$$

which hold for all $(\underset{\sim}{x}, \underset{\sim}{y}) \in \partial R \times \partial R - D$. Now use (3.5), (1.5) together with the estimates already confirmed to verify the traction-estimate in (b). This completes the proof.

We now turn to

Theorem 3.1. (Generation of elastic states from surface densities). Let R, λ and $\overline{S}^i(\cdot, \underset{\sim}{y}, \lambda)$ be as in Lemma 3.1. Further, let $\underset{\sim}{e} \in \mathscr{H}(\partial R)$ and define formally

$$\left.\begin{aligned} \underset{\sim}{u}(\underset{\sim}{x}) &= \int_{\partial R} \overline{\underset{\sim}{u}}^i(\underset{\sim}{x}, \underset{\sim}{y}, \lambda) e_i(\underset{\sim}{y}) dA_{\underset{\sim}{y}} \text{ for all } \underset{\sim}{x} \in \overline{R}, \\ \underset{\sim}{T}(\underset{\sim}{x}) &= \int_{\partial R} \overline{\underset{\sim}{T}}^i(\underset{\sim}{x}, \underset{\sim}{y}, \lambda) e_i(\underset{\sim}{y}) dA_{\underset{\sim}{y}} \text{ for all } \underset{\sim}{x} \in R, \\ \underset{\sim}{T}(\underset{\sim}{x}) &= \underset{\sim}{\psi}^i(\underset{\sim}{x}) e_i(\underset{\sim}{x}) + \overset{p}{\int_{\partial R}} \overline{\underset{\sim}{T}}^i(\underset{\sim}{x}, \underset{\sim}{y}, \lambda) e_i(\underset{\sim}{y}) dA_{\underset{\sim}{y}} \text{ for all } \underset{\sim}{x} \in \partial R, \end{aligned}\right\} \quad (3.8)$$

where

$$\psi^i_{jk} = \delta_{ij} n_k + \delta_{ik} n_j - n_i n_j n_k + \frac{1+2\sigma}{2} n_i (\delta_{jk} - n_j n_k) \text{ on } \partial R \quad (3.9)$$

and the last integral in (3.8) is to be interpreted as a Cauchy principal value in the sense of

$$\overset{p}{\int_{\partial R}} \overline{\underset{\sim}{T}}^i(\underset{\sim}{x}, \underset{\sim}{y}, \lambda) e_i(\underset{\sim}{y}) dA_{\underset{\sim}{y}} = \lim_{\epsilon \to 0} \int_{\partial R - \Omega(\underset{\sim}{x}, \epsilon)} \overline{\underset{\sim}{T}}^i(\underset{\sim}{x}, \underset{\sim}{y}, \lambda) e_i(\underset{\sim}{y}) dA_{\underset{\sim}{y}} . \quad (3.10)^1$$

Then:

(a) the integrals in (3.8) exist;

(b) $S = [\underset{\sim}{u}, \underset{\sim}{T}] \in \mathcal{E}(\underset{\sim}{0}, \mu, \sigma; \overline{R})$;

(c) $\underset{\sim}{t}(\underset{\sim}{x}) = \underset{\sim}{e}(\underset{\sim}{x}) + \int_{\partial R} \overline{\underset{\sim}{t}}^i(\underset{\sim}{x}, \underset{\sim}{y}, \lambda) e_i(\underset{\sim}{y}) dA_{\underset{\sim}{y}}$ for all $\underset{\sim}{x} \in \partial R$,

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[1] See (1.3) for the definition of $\Omega(\underset{\sim}{x}, \epsilon)$.

provided $\underset{\sim}{t}, \underset{\sim}{\bar{t}}^{i}(\cdot, \underset{\sim}{y}, \lambda)$ are the traction vectors of S, $\bar{S}^{i}(\cdot, \underset{\sim}{y}, \lambda)$ on ∂R and $\partial R_{\underset{\sim}{y}}$, respectively.

Proof[1]. The existence of the first two integrals in (3.8) is assured by Lemma 3.1. Note in particular that the first integral, though improper for $\underset{\sim}{x} \in \partial R$, is convergent because of (b) in this lemma. Further, (a) and (b) in the present theorem imply (c), as follows at once from the third of (3.8) together with (3.9), (1.9), and the final estimate in Lemma 3.1. Also, it is clear from the first two of (3.8), in view of (a) in the lemma, that

$$S = [\underset{\sim}{u}, \underset{\sim}{\tau}] \in \mathcal{E}(\underset{\sim}{0}, \mu, \sigma; R).$$

The preceding statement in particular guarantees the continuity of $\underset{\sim}{u}$ on R. To verify the continuity[2] of $\underset{\sim}{u}$ on $\bar{R}$, choose $\underset{\sim}{z} \in \partial R$ and $\varepsilon > 0$. Then, because of (b) in Lemma 3.1 and the boundedness of $\underset{\sim}{e}$ on ∂R, there exists $\rho > 0$ such that

$$\Big| \int_{\partial R \cap B_\rho(\underset{\sim}{z})} \underset{\sim}{\bar{u}}^{i}(\underset{\sim}{x}, \underset{\sim}{y}, \lambda) e_i(\underset{\sim}{y}) dA_{\underset{\sim}{y}} \Big| < \varepsilon \text{ for all } \underset{\sim}{x} \in \bar{R}$$

whence

$$\Big| \int_{\partial R \cap B_\rho(\underset{\sim}{z})} [\underset{\sim}{\bar{u}}^{i}(\underset{\sim}{x}, \underset{\sim}{y}, \lambda) - \underset{\sim}{\bar{u}}^{i}(\underset{\sim}{z}, \underset{\sim}{y}, \lambda)] e_i(\underset{\sim}{y}) dA_{\underset{\sim}{y}} \Big| < 2\varepsilon \text{ for all } \underset{\sim}{x} \in \bar{R} .$$

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[1] The following proof is suggested in part by Kellogg's [5] (Chapter IV, Section 5) treatment of the behavior at the boundary of the derivatives of Newtonian potentials appropriate to surface distributions.

[2] In connection with the subsequent argument see Kellogg [5] (pp. 150, 160).

On the other hand, by virtue of the continuity of $\bar{\underset{\sim}{u}}^i(\cdot,\cdot,\lambda)$ asserted in (a) of Lemma 3.1 and the boundedness of $\underset{\sim}{e}$ on ∂R, there exists $\delta>0$ such that

$$\left|\int_{\partial R - B_\rho(\underset{\sim}{z})} [\bar{\underset{\sim}{u}}^i(\underset{\sim}{x},\underset{\sim}{y},\lambda) - \bar{\underset{\sim}{u}}^i(\underset{\sim}{z},\underset{\sim}{y},\lambda)] e_i(\underset{\sim}{y}) dA_{\underset{\sim}{y}}\right| < \varepsilon \quad \text{for all } \underset{\sim}{x} \in \bar{R} \cap B_\delta(\underset{\sim}{z}) .$$

Combining the last two inequalities and using the first of (3.8), one has

$$|\underset{\sim}{u}(\underset{\sim}{x}) - \underset{\sim}{u}(\underset{\sim}{z})| < 3\varepsilon \quad \text{for all } \underset{\sim}{x} \in \bar{R} \cap B_\delta(\underset{\sim}{z}) .$$

Hence $\underset{\sim}{u}$ is continuous on $\bar{R}$.

To complete the proof it remains to be shown merely that the singular integral in the last of (3.8) is meaningful in the sense of (3.10) and that $\underset{\sim}{\tau} \in C(\bar{R})$. For this twofold purpose it is helpful to prove first that for all $\underset{\sim}{z} \in \partial R$,

$$\lim_{\varepsilon \to 0} \lim_{s \to 0+} \int_{\Omega(\underset{\sim}{z},\varepsilon)} \bar{\underset{\sim}{\tau}}^i(\underset{\sim}{z} - s\underset{\sim}{n}(\underset{\sim}{z}), \underset{\sim}{y}, \lambda) e_i(\underset{\sim}{y}) dA_{\underset{\sim}{y}} = \underset{\sim}{\psi}^i(\underset{\sim}{z}) e_i(\underset{\sim}{z}) \tag{3.11}$$

with $\underset{\sim}{\psi}^i$ given by (3.9).

We now establish (3.11). Choose $\underset{\sim}{z} \in \partial R$ and hold $\underset{\sim}{z}$ fixed. For convenience choose the coordinate frame in such a way that its origin is at $\underset{\sim}{z}$ (so that $\underset{\sim}{z} = \underset{\sim}{0}$) and the x_3-axis points in the direction opposite to $\underset{\sim}{n}(\underset{\sim}{0})$. In this frame, from (3.9),

$$\psi^i_{3j}(\underset{\sim}{0}) = \psi^i_{j3}(\underset{\sim}{0}) = -\delta_{ij} , \quad \psi^3_{\alpha\beta}(\underset{\sim}{0}) = -\frac{1+2\sigma}{2}\delta_{\alpha\beta} , \quad \psi^\alpha_{\beta\gamma}(\underset{\sim}{0}) = 0 . \tag{3.12}$$

Thus (3.11) now becomes

$$\lim_{\varepsilon \to 0} \lim_{\substack{x \to 0 \\ \underset{\sim}{x} \in L}} \int_{\Omega(\underset{\sim}{0},\varepsilon)} \bar{\underset{\sim}{\tau}}^i(\underset{\sim}{x}, \underset{\sim}{y}, \lambda) e_i(\underset{\sim}{y}) dA_{\underset{\sim}{y}} = \underset{\sim}{\psi}^i(\underset{\sim}{0}) e_i(\underset{\sim}{0}) , \tag{3.13}$$

where

$$L = \{\underset{\sim}{p} \,|\, \underset{\sim}{p} \in R\,,\ p_\alpha = 0\,,\ p_3 > 0\}\ . \tag{3.14}$$

Let $\varepsilon_o > 0$ be such that $\Omega(\underset{\sim}{0}, \varepsilon_o)$ admits the representation (1.4) and let $\bar{\underset{\sim}{y}}$ be the function defined by

$$\bar{y}_\alpha(\underset{\sim}{y}) = y_\alpha\,,\ \bar{y}_3(\underset{\sim}{y}) = -\varphi(y_1, y_2) \text{ for all } \underset{\sim}{y} \in \Pi(\underset{\sim}{0}, \varepsilon_o)\,, \tag{3.15}$$

where φ and $\Pi(\underset{\sim}{0}, \varepsilon_o)$ are as in (1.4) and (1.3), so that $\underset{\sim}{y} \in \Pi(\underset{\sim}{0}, \varepsilon_o)$ implies $\bar{\underset{\sim}{y}}(\underset{\sim}{y}) \in \Omega(\underset{\sim}{0}, \varepsilon_o)$. From (3.15) one draws, for every $\varepsilon \in (0, \varepsilon_o)$,

$$\int_{\Omega(\underset{\sim}{0}, \varepsilon_o)} \bar{\underset{\sim}{T}}^i(\underset{\sim}{x}, \underset{\sim}{y}, \lambda) e_i(\underset{\sim}{y})\, dA_{\underset{\sim}{y}} = \int_{\Pi(\underset{\sim}{0}, \varepsilon_o)} \frac{\bar{\underset{\sim}{T}}^i(\underset{\sim}{x}, \bar{\underset{\sim}{y}}(\underset{\sim}{y}), \lambda) e_i(\bar{\underset{\sim}{y}}(\underset{\sim}{y}))}{\underset{\sim}{n}(\bar{\underset{\sim}{y}}(\underset{\sim}{y})) \cdot \underset{\sim}{n}(\underset{\sim}{0})}\, dA_{\underset{\sim}{y}} \text{ for all } \underset{\sim}{x} \in L. \tag{3.16}$$

Equation (3.16) may be used to simplify (3.13). Observe first from (1.5), (3.15) that there exists $k > 0$ such that for all $\underset{\sim}{y} \in \Pi(\underset{\sim}{0}, \varepsilon_o)$,

$$|\underset{\sim}{n}(\bar{\underset{\sim}{y}}(\underset{\sim}{y})) - \underset{\sim}{n}(\underset{\sim}{0})| \le k\, |\bar{\underset{\sim}{y}}(\underset{\sim}{y})|\,,$$

$$|y_3(\bar{\underset{\sim}{y}})| = |[\bar{\underset{\sim}{y}}(\underset{\sim}{y}) - \underset{\sim}{0}] \cdot \underset{\sim}{n}(\underset{\sim}{0})| \le k[\bar{\underset{\sim}{y}}(\underset{\sim}{y})]^2.$$

Therefore, since $[\bar{\underset{\sim}{y}}(\underset{\sim}{y})]^2 = \underset{\sim}{y}^2 + [\bar{y}_3(\underset{\sim}{y})]^2$, there exists $k_1 > 0$ such that for all $\underset{\sim}{y} \in \Pi(\underset{\sim}{0}, \varepsilon_o)$,

$$|\bar{\underset{\sim}{y}}(\underset{\sim}{y})| \le k_1 |\underset{\sim}{y}|,\ |\bar{y}_3(\underset{\sim}{y})| \le k_1 \underset{\sim}{y}^2,\ |\underset{\sim}{n}(\bar{\underset{\sim}{y}}(\underset{\sim}{y})) - \underset{\sim}{n}(\underset{\sim}{0})| \le k_1 |\underset{\sim}{y}|\ . \tag{3.17}$$

Equations (3.16), (3.17), (3.7), together with the second of (b) in Lemma 3.1 and the Hölder-continuity of $\underset{\sim}{e}$ assure that (3.13) is implied by

$$\lim_{\varepsilon \to 0} \lim_{\substack{x \to 0 \\ \underset{\sim}{x} \in L}} \int_{\Pi(\underset{\sim}{0}, \varepsilon)} \bar{\tau}^i_{jk}(\underset{\sim}{x}, \bar{\underset{\sim}{y}}(\underset{\sim}{y}), \infty)\, dA_{\underset{\sim}{y}} = \psi^i_{jk}(\underset{\sim}{0})\ . \tag{3.18}$$

The verification of (3.18) involves a lengthy computation, which may be shortened by noting from (3.3) and (3.5) that

$$\left.\begin{aligned} \bar{\tau}^{i}_{j3}(\underset{\sim}{x},\underset{\sim}{y},\infty) &= \bar{\tau}^{i}_{jk}(\underset{\sim}{x},\underset{\sim}{y},\infty)[\delta_{k3}+n_k(\underset{\sim}{y})] \\ &\quad + \frac{3}{2\pi}\,\frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)n_k(\underset{\sim}{y})}{|\underset{\sim}{x}-\underset{\sim}{y}|^5}\,, \\ \bar{\tau}^{3}_{\alpha\beta}(\underset{\sim}{x},\underset{\sim}{y},\infty) &= \bar{\tau}^{\alpha}_{\beta 3}(\underset{\sim}{x},\underset{\sim}{y},\infty) + \frac{1-2\sigma}{2\pi}\,\delta_{\alpha\beta}\,\frac{x_3-y_3}{|\underset{\sim}{x}-\underset{\sim}{y}|^3} \\ &\quad + \frac{1-2\sigma}{2\pi}\left[n_3(\underset{\sim}{y})h_{,\alpha\beta}(\underset{\sim}{x},\underset{\sim}{y}) - n_\alpha(\underset{\sim}{y})h_{,3\beta}(\underset{\sim}{x},\underset{\sim}{y})\right] \end{aligned}\right\}(3.19)$$

for all $(\underset{\sim}{x},\underset{\sim}{y})\in L\times\Omega(\underset{\sim}{0},\varepsilon_o)$. By way of illustration, we confirm (3.18) for $i=j=1, k=3$. From (3.3), (3.17), since $n_k(\underset{\sim}{0})=-\delta_{k3}$, follows

$$\lim_{\varepsilon\to 0}\ \lim_{\substack{x\to 0\\ \underset{\sim}{x}\in L}}\ \int_{\Pi(\underset{\sim}{0},\varepsilon)} \bar{\tau}^{1}_{1k}(\underset{\sim}{x},\bar{\underset{\sim}{y}}(\underset{\sim}{y}),\infty)[\delta_{k3}+n_k(\bar{\underset{\sim}{y}}(\underset{\sim}{y}))]\,dA_{\underset{\sim}{y}}=0\,,$$

so that (3.19) gives

$$\lim_{\varepsilon\to 0}\ \lim_{\substack{x\to 0\\ \underset{\sim}{x}\in L}}\ \int_{\Pi(\underset{\sim}{0},\varepsilon)} \bar{\tau}^{1}_{13}(\underset{\sim}{x},\bar{\underset{\sim}{y}}(\underset{\sim}{y}),\infty)\,dA_{\underset{\sim}{y}} = \lim_{\varepsilon\to 0}\ \lim_{\substack{x\to 0\\ \underset{\sim}{x}\in L}}\ \int_{\Pi(\underset{\sim}{0},\varepsilon)} \frac{3\bar{y}_1^{\,2}(\underset{\sim}{y})[x_k-\bar{y}_k(\underset{\sim}{y})]\,n_k(\bar{\underset{\sim}{y}}(\underset{\sim}{y}))}{2\pi\,|\underset{\sim}{x}-\bar{\underset{\sim}{y}}(\underset{\sim}{y})|^5}\,dA_{\underset{\sim}{y}}\,.$$

This relation, because of (3.17), (3.15), and the inequality

$$\big|\,|\underset{\sim}{x}-\bar{\underset{\sim}{y}}(\underset{\sim}{y})| - |\underset{\sim}{x}-\underset{\sim}{y}|\,\big| \le |\bar{\underset{\sim}{y}}(\underset{\sim}{y})-\underset{\sim}{y}| = |\bar{y}_3(\underset{\sim}{y})|\,,$$

in turn yields

$$\lim_{\varepsilon\to 0}\ \lim_{\substack{x\to 0\\ \underset{\sim}{x}\in L}}\ \int_{\Pi(\underset{\sim}{0},\varepsilon)} \bar{\tau}^{1}_{13}(\underset{\sim}{x},\bar{\underset{\sim}{y}}(\underset{\sim}{y}),\infty)\,dA_{\underset{\sim}{y}} = -\frac{3}{2\pi}\lim_{\varepsilon\to 0}\ \lim_{\substack{x\to 0\\ \underset{\sim}{x}\in L}}\ \int_{\Pi(\underset{\sim}{0},\varepsilon)} \frac{y_1^2 x_3}{|\underset{\sim}{x}-\underset{\sim}{y}|^5}\,dA_{\underset{\sim}{y}}$$

$$= -\frac{3}{2}\lim_{\varepsilon\to 0}\ \lim_{x_3\to 0}\int_0^{\varepsilon}\frac{x_3\,\rho^3}{(\rho^2+x_3^2)^{5/2}}\,d\rho\,.$$

On subjecting the last integral to the change of variables $s=\rho/x_3$ one finds that

$$\lim_{\epsilon\to 0}\lim_{\substack{x\to 0\\ \underset{\sim}{x}\in L}}\int_{\Pi(\underset{\sim}{0},\epsilon)} \bar{\tau}^1_{13}(\underset{\sim}{x},\bar{\underset{\sim}{y}}(\underset{\sim}{y}),\infty)dA_{\underset{\sim}{y}} = -\int_0^\infty \frac{3s^3}{2(1+s^2)^{5/2}}ds=-1=\psi^1_{13}(\underset{\sim}{0}) .$$

The remaining limits in (3.18) may be verified in a similar manner.

The existence of the singular integral in the third of (3.8) now follows easily from (3.11). Indeed, from (3.11), given $\underset{\sim}{z}\in\partial R$ and $\eta>0$, there exists $\epsilon_1>0$ such that $0<\epsilon\le\epsilon_1$ implies

$$\Big|\lim_{s\to 0+}\int_{\Omega(\underset{\sim}{z},\epsilon)} \underset{\sim}{\bar{T}}^i(\underset{\sim}{z}-s\underset{\sim}{n}(\underset{\sim}{z}),\underset{\sim}{y},\lambda)e_i(\underset{\sim}{y})dA_{\underset{\sim}{y}}-\underset{\sim}{\psi}^i(\underset{\sim}{z})e_i(\underset{\sim}{z})\Big|<\eta ,$$

so that

$$\Big|\lim_{s\to 0+}\int_{\Omega(\underset{\sim}{z},\epsilon_1)-\Omega(\underset{\sim}{z},\epsilon)} \underset{\sim}{\bar{T}}^i(\underset{\sim}{z}-s\underset{\sim}{n}(\underset{\sim}{z}),\underset{\sim}{y},\lambda)e_i(\underset{\sim}{y})dA_{\underset{\sim}{y}}\Big|<2\eta \text{ for all } \epsilon\in(0,\epsilon_1) .$$

But since $\underset{\sim}{z}$ is not in $\Omega(\underset{\sim}{z},\epsilon_1)-\Omega(\underset{\sim}{z},\epsilon)$, this inequality is equivalent to

$$\Big|\int_{\Omega(\underset{\sim}{z},\epsilon_1)-\Omega(\underset{\sim}{z},\epsilon)} \underset{\sim}{\bar{T}}^i(\underset{\sim}{z},\underset{\sim}{y},\lambda)e_i(\underset{\sim}{y})dA_{\underset{\sim}{y}}\Big|<2\eta \quad\text{for all } \epsilon\in(0,\epsilon_1)$$

and hence implies the existence of the limit in (3.10).

We turn finally to a proof of the continuity of $\underset{\sim}{T}$ on $\bar{R}$. To this end it suffices to show that

$$\lim_{\substack{\underset{\sim}{x}\to\underset{\sim}{z}\\ \underset{\sim}{x}\in R}} \underset{\sim}{T}(\underset{\sim}{x})=\underset{\sim}{T}(\underset{\sim}{z}) \text{ for all } \underset{\sim}{z}\in\partial R , \tag{3.20}$$

where $\underset{\sim}{T}(\underset{\sim}{x})$ and $\underset{\sim}{T}(\underset{\sim}{z})$ are defined by the second and third of (3.8), respectively. We prove first that (3.20) holds true if $\underset{\sim}{x}$ approaches $\underset{\sim}{z}$ along the inner normal, i.e. that

$$\lim_{s\to 0+} \underset{\sim}{T}(\underset{\sim}{z}-s\underset{\sim}{n}(\underset{\sim}{z}))=\underset{\sim}{T}(\underset{\sim}{z}) \quad \text{for all } \underset{\sim}{z}\in\partial R . \tag{3.21}$$

Choose $\underset{\sim}{z}\in\partial R$ and $\eta > 0$. Then, the existence of the limit in (3.10) now being assured, there is an $\varepsilon_1(\eta)>0$ such that

$$\left| \overset{p}{\int_{\Omega(\underset{\sim}{z},\varepsilon)}} \overline{\underset{\sim}{T}}^{i}(\underset{\sim}{z},\underset{\sim}{y},\lambda)e_i(\underset{\sim}{y})dA_{\underset{\sim}{y}} \right| < \eta \quad (0<\varepsilon<\varepsilon_1) .$$

Next, according to (3.11), there exists $\varepsilon_2\in(0,\varepsilon_1)$ and $s_1(\eta,\varepsilon_2)$ such that $0<s<s_1$ implies

$$\left| \int_{\Omega(\underset{\sim}{z},\varepsilon_2)} \overline{\underset{\sim}{T}}^{i}(\underset{\sim}{z}-s\underset{\sim}{n}(\underset{\sim}{z}),\underset{\sim}{y},\lambda)e_i(\underset{\sim}{y})dA_{\underset{\sim}{y}} - \underset{\sim}{\psi}^{i}(\underset{\sim}{z})e_i(\underset{\sim}{z}) \right| < \eta .$$

Combining these two inequalities one has

$$\left| \int_{\Omega(\underset{\sim}{z},\varepsilon_2)} \overline{\underset{\sim}{T}}^{i}(\underset{\sim}{z}-s\underset{\sim}{n}(\underset{\sim}{z}),\underset{\sim}{y},\lambda)e_i(\underset{\sim}{y})dA_{\underset{\sim}{y}} - \underset{\sim}{\psi}^{i}(\underset{\sim}{z})e_i(\underset{\sim}{z}) - \overset{p}{\int_{\Omega(\underset{\sim}{z},\varepsilon_2)}} \overline{\underset{\sim}{T}}^{i}(\underset{\sim}{z},\underset{\sim}{y},\lambda)e_i(\underset{\sim}{y})dA_{\underset{\sim}{y}} \right| < 2\eta \tag{3.22}$$

for all $s\in(0,s_1)$. On the other hand, since $\underset{\sim}{z}$ is not in $\partial R-\Omega(\underset{\sim}{z},\varepsilon_2)$, there exists $s_2(\varepsilon_2,\eta)>0$ such that $0<s<s_2$ implies

$$\left| \int_{\partial R-\Omega(\underset{\sim}{z},\varepsilon_2)} \overline{\underset{\sim}{T}}^{i}(\underset{\sim}{z}-s\underset{\sim}{n}(\underset{\sim}{z}),\underset{\sim}{y},\lambda)e_i(\underset{\sim}{y})dA_{\underset{\sim}{y}} - \int_{\partial R-\Omega(\underset{\sim}{z},\varepsilon_2)} \overline{\underset{\sim}{T}}^{i}(\underset{\sim}{z},\underset{\sim}{y},\lambda)e_i(\underset{\sim}{y})dA_{\underset{\sim}{y}} \right| < \eta . \tag{3.23}$$

Equations (3.22) and (3.23) furnish

$$\lim_{s\to 0+} \int_{\partial R} \overline{\underset{\sim}{T}}^{i}(\underset{\sim}{z}-s\underset{\sim}{n}(\underset{\sim}{z}),\underset{\sim}{y},\lambda)e_i(\underset{\sim}{y})dA_{\underset{\sim}{y}} = \underset{\sim}{\psi}^{i}(\underset{\sim}{z})e_i(\underset{\sim}{z}) + \overset{p}{\int_{\partial R}} \overline{\underset{\sim}{T}}^{i}(\underset{\sim}{z},\underset{\sim}{y},\lambda)e_i(\underset{\sim}{y})dA_{\underset{\sim}{y}} ,$$

which is equivalent to (3.21).

It is not difficult to verify that the limit in (3.21) is uniform with respect to the choice of $\underset{\sim}{z}$. In view of this uniformity one can pass from (3.21) to (3.20) by an argument analogous to that used in the proof of Theorem VII in Chapter VI of [5]. This completes the proof of Theorem 3.1 in its entirety.

Equations (3.8) may be viewed as a representation of the state $S=[\underset{\sim}{u}, \underset{\sim}{\tau}]$ in terms of a generating surface density $\underset{\sim}{e}$. What is needed for our purposes, however, is a representation of a given elastic state on $\overline{R}$ in terms of its surface tractions on ∂R. As will become apparent shortly, (3.8) remain valid if $S=[\underset{\sim}{u}, \underset{\sim}{\tau}]$ is a given elastic state and $\underset{\sim}{e}$ is replaced by the traction vector $\underset{\sim}{t}$ of S on ∂R, provided the kernel states $\overline{S}^i$ are modified suitably. Before introducing such "modified tangent states", we associate with any finite regular region a set of six fundamental vector fields that will be used repeatedly throughout the remainder of this investigation.

Definition 3.2. Let R be a bounded regular region. Let $\underset{\sim}{c}$ be the position vector of the centroid of the boundary ∂R and let $\underset{\sim}{b}^m (m=1,2,3)$ be unit base vectors of a centroidal principal frame for ∂R. Finally, let α and i_m denote respectively the area of ∂R and its (principal) moment of inertia about the (centroidal) axis determined by $\underset{\sim}{b}^m$. We then write $\underset{\sim}{q}^m (m=1,\ldots,6)$ for the six vector fields defined by

$$\underset{\sim}{q}^m(\underset{\sim}{x})=\frac{\underset{\sim}{b}^m}{\sqrt{\alpha}}, \quad \underset{\sim}{q}^{m+3}(\underset{\sim}{x})=\frac{(\underset{\sim}{x}-\underset{\sim}{c})\wedge\underset{\sim}{b}^m}{\sqrt{i_m}} \qquad (m=1,2,3), \text{ (no sum)} \quad (3.24)$$

for all $\underset{\sim}{x}\in\overline{R}$.

The dual role played by the vector fields introduced above is apparent from

Lemma 3.2. *Let R be a bounded regular region and let* $\underset{\sim}{q}^m (m=1, \ldots, 6)$ *be the vector fields defined by* (3.24).

(a) *Suppose* Σ *is a regular surface contained in* $\overline{R}$ *and* $\underset{\sim}{t}$ *is a vector field integrable on* Σ. *Then*

$$\int_{\Sigma} \underset{\sim}{t} \cdot \underset{\sim}{q}^m dA = 0 \quad (m=1, \ldots, 6)$$

if and only if

$$\int_{\Sigma} \underset{\sim}{t}\, dA = \underset{\sim}{0}, \quad \int_{\Sigma} \underset{\sim}{x} \wedge \underset{\sim}{t}\, dA = \underset{\sim}{0},$$

i.e. if and only if $\underset{\sim}{t}$ *is self-equilibrated on* Σ.

(b) *Suppose* $\underset{\sim}{u}$ *is given by*

$$\underset{\sim}{u}(\underset{\sim}{x}) = \underset{\sim}{a} + \underset{\sim}{x} \wedge \underset{\sim}{w} \quad \text{for all } \underset{\sim}{x} \in \overline{R},$$

where $\underset{\sim}{a}$ *and* $\underset{\sim}{w}$ *are constant vectors, so that* $\underset{\sim}{u}$ *is an (infinitesimal) rigid displacement field. Then*

$$\int_{\partial R} \underset{\sim}{u} \cdot \underset{\sim}{q}^m dA = 0 \quad (m=1, \ldots, 6) \text{ implies } \underset{\sim}{a} = \underset{\sim}{w} = \underset{\sim}{0}.$$

Proof. Let $\underset{\sim}{b}^m$ (m=1, 2, 3) and $\underset{\sim}{c}$ be as in Definition 3.2. Then $\underset{\sim}{t}$ is self-equilibrated on Σ if and only if

$$\int_{\Sigma} \underset{\sim}{b}^m \cdot \underset{\sim}{t}\, dA = 0, \quad \int_{\Sigma} \underset{\sim}{b}^m \cdot (\underset{\sim}{x} - \underset{\sim}{c}) \wedge \underset{\sim}{t}\, dA = 0 \quad (m=1, 2, 3),$$

and these equations, together with (3.24), establish part (a). Turning to (b), note first from Definition 3.2 the orthonormality relations

$$\int_{\partial R} \underset{\sim}{q}^m \cdot \underset{\sim}{q}^\ell dA = \begin{cases} 0 & (m \neq \ell) \\ 1 & (m = \ell) \end{cases} \quad (m, \ell = 1, \ldots, 6) . \tag{3.25}$$

Let $\underset{\sim}{c}, \alpha, \iota_m, \underset{\sim}{b}^m$ (m=1, 2, 3) have the same meaning as in Definition 3.2 and set

$$k_m = \sqrt{\alpha}\, \underset{\sim}{b}^m \cdot (\underset{\sim}{a} + \underset{\sim}{c} \wedge \underset{\sim}{w}) , \quad k_{m+3} = \sqrt{\iota_m}\, \underset{\sim}{b}^m \cdot \underset{\sim}{w} \quad (m=1, 2, 3) , \text{ (no sum)} .$$

An elementary computation then yields

$$\underset{\sim}{u} = \sum_{m=1}^{6} k_m \underset{\sim}{q}^m \text{ on } \bar{R} .$$

It thus follows from (3.25) and the assumed integral conditions on $\underset{\sim}{u}$ that $k_m = 0$ (m=1, ...6). Hence $\underset{\sim}{w}$ and $\underset{\sim}{a}$ also vanish, so that the proof is complete.

The integral conditions appearing at the end of Lemma 3.2 supply a convenient normalization of the displacement field $\underset{\sim}{u}$ appropriate to an elastic state defined as the solution of a second boundary-value problem. Such a normalization eliminates the usual arbitrary additive rigid displacement. We now return to our immediate objective.

Definition 3.3 (Modified tangent states). Let R be a simple region and let $\underset{\sim}{y} \in \partial R$. Further, let $\bar{S}^i(\cdot, \underset{\sim}{y}, \infty)$ be as in Definition 3.1. We call

$$\mathring{S}^i(\cdot, \underset{\sim}{y}) = [\mathring{\underset{\sim}{u}}^i(\cdot, \underset{\sim}{y}), \mathring{\underset{\sim}{T}}^i(\cdot, \underset{\sim}{y})]$$

the modified tangent state for the region R at $\underset{\sim}{y}$, corresponding to the x_i-direction and the elastic constants μ, σ if:

(a) $\overset{\circ}{S}^i(\cdot, \underset{\sim}{y}) \in \mathcal{E}(\underset{\sim}{0}, \mu, \sigma; \overline{R}_{\underset{\sim}{y}})$;

(b) $\overset{\circ}{\underset{\sim}{u}}{}^i(\underset{\sim}{x}, \underset{\sim}{y}) = \overline{\underset{\sim}{u}}{}^i(\underset{\sim}{x}, \underset{\sim}{y}, \infty) + O(|\underset{\sim}{x}-\underset{\sim}{y}|^{-\beta})$ <u>as</u> $\underset{\sim}{x} \to \underset{\sim}{y}$,

$\overset{\circ}{\underset{\sim}{T}}{}^i(\underset{\sim}{x}, \underset{\sim}{y}) = \overline{\underset{\sim}{T}}{}^i(\underset{\sim}{x}, \underset{\sim}{y}, \infty) + O(|\underset{\sim}{x}-\underset{\sim}{y}|^{-1-\beta})$ <u>as</u> $\underset{\sim}{x} \to \underset{\sim}{y}$ $(\beta < 1/2)$;

(c) $\overset{\circ}{t}{}^i_j(\underset{\sim}{x}, \underset{\sim}{y}) = -\sum_{m=1}^{6} q_j^m(\underset{\sim}{x}) q_i^m(\underset{\sim}{y})$ <u>for all</u> $\underset{\sim}{x} \in \partial R_{\underset{\sim}{y}}$,

<u>with</u> $\underset{\sim}{q}^m$ $(m=1, \ldots, 6)$ <u>given by Definition</u> 3.2;

(d) $\int_{\partial R} \overset{\circ}{\underset{\sim}{u}}{}^i(\underset{\sim}{x}, \underset{\sim}{y}) \cdot \underset{\sim}{q}^m(\underset{\sim}{x}) dA_{\underset{\sim}{x}} = 0 \quad (m=1, \ldots, 6).$

Note first that (c) defines the surface tractions of $\overset{\circ}{S}^i(\cdot, \underset{\sim}{y})$ uniquely on $\partial R_{\underset{\sim}{y}}$, even though the principal base vectors entering the definition of $\underset{\sim}{q}^m$ fail to be unique if the centroidal principal moments of inertia i_m $(m=1, 2, 3)$ of ∂R are not distinct. It is easily seen that conditions (a), (b), (c), (d) in the preceding definition suffice to characterize the state $\overset{\circ}{S}^i(\cdot, \underset{\sim}{y})$ <u>uniquely</u>. To confirm this claim, note with the aid of an elementary modification of the classical uniqueness proof that (a), (b), (c) determine $\overset{\circ}{\underset{\sim}{T}}{}^i(\cdot, \underset{\sim}{y})$ completely[1]. Accordingly, $\overset{\circ}{\underset{\sim}{u}}{}^i(\cdot, \underset{\sim}{y})$ is determinate except for an additive (infinitesimal) rigid displacement. This arbitrariness is removed by (d), as is clear from part (b) of Lemma 3.2.

We now state a theorem assuring the <u>existence</u> of the states $\overset{\circ}{S}^i(\cdot, \underset{\sim}{y})$ and at the same time asserting certain additional properties of these states.

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[1] In view of the limitations (b) upon the orders of the displacement and stress singularities at $\underset{\sim}{y}$, the difference of two states sharing properties (a), (b), (c) has zero total strain energy.

Theorem 3.2. (Existence and properties of the modified tangent states). Let R be a simple region. Then the modified tangent states $\mathring{S}^i(\cdot, \underset{\sim}{y})$ introduced in Definition 3.3 exist for all $\underset{\sim}{y} \in \partial R$. Moreover, these states have the properties:

(a) $\mathring{\underset{\sim}{u}}^i \in C(\overline{R} \times \partial R - D)$, $\mathring{\underset{\sim}{T}}^i \in C(\overline{R} \times \partial R - D)$,

$\nabla \mathring{\underset{\sim}{u}}^i \in C(R \times \partial R)$, $\nabla \mathring{\underset{\sim}{T}}^i \in C(R \times \partial R)$;

(b) the orders of magnitude in (b) of Definition 3.3 hold uniformly with respect to $\underset{\sim}{y}$ for all $\underset{\sim}{y} \in \partial R$.

As a prerequisite for the proof of this theorem we require some auxiliary notation as well as three additional lemmas.

Definition 3.4. (Classes of functions with surface nuclei). Let Σ be the boundary of a simple region.

(a) If $\alpha \in (0, 2)$, we write $v \in \mathcal{N}^{\alpha}(\Sigma)$ provided $v \in C(\Sigma \times \Sigma - D)$ and there exists $k > 0$ such that

$$|v(\underset{\sim}{x}, \underset{\sim}{y})| < k |\underset{\sim}{x} - \underset{\sim}{y}|^{\alpha - 2} \quad \text{for all } (\underset{\sim}{x}, \underset{\sim}{y}) \in \Sigma \times \Sigma - D.$$

If $\alpha > 2$, we write $v \in \mathcal{N}^{\alpha}(\Sigma)$ provided $v \in C(\Sigma \times \Sigma)$.

If, for each $\alpha \in (0, 2)$, $v \in \mathcal{N}^{\alpha}(\Sigma)$, we write $v \in \mathcal{N}^{2}(\Sigma)$.

(b) If $\alpha \in (0, 2]$ and $\gamma \in (0, 1]$, we write $v \in \mathcal{N}^{\alpha,\gamma}(\Sigma)$ provided $v \in \mathcal{N}^{\alpha}(\Sigma)$ and there exists $k > 0$ such that

$$|v(\underset{\sim}{x}, \underset{\sim}{y}) - v(\underset{\sim}{z}, \underset{\sim}{y})| \leq k |\underset{\sim}{x} - \underset{\sim}{z}|^{\gamma} |\underset{\sim}{x} - \underset{\sim}{y}|^{\alpha - 2 - \gamma}$$

for all $\underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z}$ on Σ subject to $2 |\underset{\sim}{x} - \underset{\sim}{z}| < |\underset{\sim}{x} - \underset{\sim}{y}|$.

If $\alpha > 2$ and $\gamma \in (0, 1]$, we write $v \in \mathcal{N}^{\alpha, \gamma}(\Sigma)$ provided $v \in \mathcal{N}^{\alpha}(\Sigma)$ and there exists $k > 0$ such that

$$|v(\underset{\sim}{x}, \underset{\sim}{y}) - v(\underset{\sim}{z}, \underset{\sim}{y})| \leq k |\underset{\sim}{x} - \underset{\sim}{z}|^{\gamma} \quad \text{for all } \underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z} \text{ on } \Sigma.$$

It is clear from the foregoing definition that $\alpha > 2$ and $v \in \mathcal{N}^{\alpha,\gamma}(\Sigma)$ implies $v(\cdot, \underset{\sim}{y}) \in \mathcal{N}(\Sigma)$ for all $\underset{\sim}{y} \in \Sigma$. Further, it is not difficult to verify that $\alpha \le \beta$ implies $\mathcal{N}^{\alpha}(\Sigma) \supset \mathcal{N}^{\beta}(\Sigma)$, while $\alpha \le \beta$, $\gamma \le \delta$ implies $\mathcal{N}^{\alpha,\gamma}(\Sigma) \supset \mathcal{N}^{\beta,\delta}(\Sigma)$.

We turn now to a lemma which is closely related to results given by Giraud [11] (p. 256).

Lemma 3.3. _(Composition of functions with surface nuclei). Let_ Σ _be as in Definition_ 3.4. _Assume_ $v_1 \in \mathcal{N}^{\alpha}(\Sigma)$, $v_2 \in \mathcal{N}^{\beta}(\Sigma)$ _and let_

$$v_3(\underset{\sim}{x}, \underset{\sim}{y}) = \int_{\Sigma} v_1(\underset{\sim}{x}, \underset{\sim}{p}) v_2(\underset{\sim}{p}, \underset{\sim}{y}) dA_{\underset{\sim}{p}}$$

for all $\underset{\sim}{x}$, $\underset{\sim}{y}$ _on_ Σ _except possibly_ $\underset{\sim}{x} = \underset{\sim}{y}$. _Then_

$$v_3 \in \mathcal{N}^{\alpha+\beta}(\Sigma) .$$

If (in addition to the original hypotheses) $v_1 \in \mathcal{N}^{\alpha,\lambda}(\Sigma)$ _while_ η _satisfies_

$$\left.\begin{array}{ll} \eta \in (0, \lambda], \ \eta < \alpha & \text{when } \alpha+\beta \le 2 , \\ \eta \in (0, \lambda), \ \eta < \alpha , \ \eta < \alpha+\beta-2 & \text{when } \alpha+\beta > 2 , \end{array}\right\} \quad (3.26)$$

then

$$v_3 \in \mathcal{N}^{\alpha+\beta,\eta}(\Sigma).$$

Proof. To establish the first part of this lemma one needs to show that (a) in Definition 3.4 holds true for $v = v_3$, provided α is replaced by $\alpha + \beta$. The required continuity of v_3 is inferred from its definition by an argument common in potential theory[1]. On the other hand, the desired order of magnitude of v_3 is a direct consequence of the known inequality

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[1] See Kellogg [5] (p. 301) and the first part of the proof of Theorem 2.1 for closely related arguments.

$$\int_{\Sigma} \frac{dA_{\underset{\sim}{p}}}{|\underset{\sim}{x}-\underset{\sim}{p}|^{2-\alpha}|\underset{\sim}{p}-\underset{\sim}{y}|^{2-\beta}} < k_1 |\underset{\sim}{x}-\underset{\sim}{y}|^{\alpha+\beta-2} \qquad (3.27)^1$$

for all $(\underset{\sim}{x},\underset{\sim}{y}) \in \Sigma \times \Sigma - D$, where k_1 is a constant and $\alpha+\beta<2$.

To confirm the second part of the lemma, choose η consistent with (3.26) and fix $\underset{\sim}{x}$, $\underset{\sim}{y}$, $\underset{\sim}{z}$ on Σ with $\underset{\sim}{x} \neq \underset{\sim}{y} \neq \underset{\sim}{z}$. Next define two complementary subsets of Σ through

$$\Sigma_1 = \left\{\underset{\sim}{p} \,|\, \underset{\sim}{p} \in \Sigma,\ |\underset{\sim}{p}-\underset{\sim}{x}| \le 2\,|\underset{\sim}{x}-\underset{\sim}{z}|\right\},\ \Sigma_2 = \Sigma - \Sigma_1 . \qquad (3.28)$$

In view of (3.28),

$$|\underset{\sim}{p}-\underset{\sim}{z}| \le |\underset{\sim}{p}-\underset{\sim}{x}| + |\underset{\sim}{x}-\underset{\sim}{z}| \le 3\,|\underset{\sim}{x}-\underset{\sim}{z}| \quad \text{for all } \underset{\sim}{p} \in \Sigma_1 .$$

Therefore, since $1 \ge \eta > 0$,

$$3\,|\underset{\sim}{x}-\underset{\sim}{z}|^{\eta} \ge |\underset{\sim}{p}-\underset{\sim}{z}|^{\eta},\ 2\,|\underset{\sim}{x}-\underset{\sim}{z}|^{\eta} \ge |\underset{\sim}{p}-\underset{\sim}{x}|^{\eta} \quad \text{for all } \underset{\sim}{p} \in \Sigma_1 . \qquad (3.29)$$

Now observe from the definition of v_3 that

$$|v_3(\underset{\sim}{x},\underset{\sim}{y}) - v_3(\underset{\sim}{z},\underset{\sim}{y})| \le \int_{\Sigma_1} |v_1(\underset{\sim}{x},\underset{\sim}{p})||v_2(\underset{\sim}{p},\underset{\sim}{y})|\,dA_{\underset{\sim}{p}} + \int_{\Sigma_1} |v_1(\underset{\sim}{z},\underset{\sim}{p})||v_2(\underset{\sim}{p},\underset{\sim}{y})|\,dA_{\underset{\sim}{p}}$$

$$+ \int_{\Sigma_2} |v_1(\underset{\sim}{x},\underset{\sim}{p}) - v_1(\underset{\sim}{z},\underset{\sim}{p})||v_2(\underset{\sim}{p},\underset{\sim}{y})|\,dA_{\underset{\sim}{p}} .$$

Hence, bearing in mind (3.28), (3.29), one has

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[1] See, for example, Pogorzelski [12] (p. 81 et seq.), where this inequality is established on the assumption that Σ is a plane; the argument used there is easily adapted to the present circumstances. Cf. also Kellogg [5], Chapter XI, Lemma II (p. 301).

$$|v_3(x,y)-v_3(z,y)| \le 2\,|x-z|^{\eta}\int_{\Sigma}|x-p|^{-\eta}|v_1(x,p)||v_2(p,y)|\,dA_p$$

$$+3\,|x-z|^{\eta}\int_{\Sigma}|z-p|^{-\eta}|v_1(z,p)||v_2(p,y)|\,dA_p$$

$$+\int_{\Sigma_2}|v_1(x,p)-v_1(z,p)||v_2(p,y)|\,dA_p\,. \qquad (3.30)$$

An application of the first part of the lemma to the pair of functions with the values

$$|x-p|^{-\eta}\,|v_1(x,p)|\,,\quad |v_2(p,y)|\quad ,$$

yields the existence of a constant k_2 (independent of the particular choice of x, y) such that[1]

$$\int_{\Sigma}|x-p|^{-\eta}|v_1(x,p)||v_2(p,y)|\,dA_p < \begin{cases} k_2\,|x-y|^{\alpha+\beta-\eta-2} & \text{if } \alpha+\beta\le 2 \\ k_2 & \text{if } \alpha+\beta>2\,. \end{cases} \qquad (3.31)$$

Further, (3.28), (3.26) and the assumed properties of v_1 entitle one to assert the existence of $k>0$ (independent of x, y, z) such that for all $p \in \Sigma_2$,

$$|v_1(x,p)-v_1(z,p)| \le \begin{cases} k\,|x-z|^{\eta}\,|x-p|^{\alpha-\eta-2} & \text{if } \alpha\le 2 \\ k\,|x-z|^{\eta} & \text{if } \alpha>2\,. \end{cases}$$

Hence, invoking once again the first part of the lemma and taking

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[1] Observe on the basis of (3.26) that $\alpha+\beta\le 2$ implies $\alpha+\beta-\eta<2$, whereas $\alpha+\beta>2$ implies $\alpha+\beta-\eta>2$.

account of the assumptions on v_2 and η, one infers the existence of $k_3 > 0$ (independent of $\underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z}$) such that

$$\int_{\Sigma_2} |v_1(\underset{\sim}{x}, \underset{\sim}{p}) - v_1(\underset{\sim}{z}, \underset{\sim}{p})| |v_2(\underset{\sim}{p}, \underset{\sim}{y})| dA_{\underset{\sim}{p}} \le \begin{cases} k_3 |\underset{\sim}{x}-\underset{\sim}{z}|^{\eta} |\underset{\sim}{x}-\underset{\sim}{y}|^{\alpha+\beta-\eta-2} & \text{if } \alpha+\beta \le 2 \\ k_3 |\underset{\sim}{x}-\underset{\sim}{z}|^{\eta} & \text{if } \alpha+\beta > 2 . \end{cases}$$

On combining this inequality with (3.31), (3.30), there follows

$$|v_3(\underset{\sim}{x}, \underset{\sim}{y}) - v_3(\underset{\sim}{z}, \underset{\sim}{y})| \le \begin{cases} |\underset{\sim}{x}-\underset{\sim}{z}|^{\eta} \left[(2k_2+k_3) |\underset{\sim}{x}-\underset{\sim}{y}|^{\alpha+\beta-\eta-2} + 3k_2 |\underset{\sim}{z}-\underset{\sim}{y}|^{\alpha+\beta-\eta-2} \right] & (\alpha+\beta \le 2) \\ |\underset{\sim}{x}-\underset{\sim}{z}|^{\eta} (5k_2+k_3) \quad (\alpha+\beta > 2) . \end{cases} \tag{3.32}$$

Finally recall that v_3 is continuous on $\Sigma \times \Sigma$ for $\alpha+\beta > 2$ and note that the assumption

$$2 |\underset{\sim}{x}-\underset{\sim}{z}| < |\underset{\sim}{x}-\underset{\sim}{y}| \quad \text{if } \alpha+\beta \le 2$$

furnishes

$$|\underset{\sim}{x}-\underset{\sim}{y}| \le 2 |\underset{\sim}{z}-\underset{\sim}{y}| + 2 |\underset{\sim}{x}-\underset{\sim}{z}| - |\underset{\sim}{x}-\underset{\sim}{y}| < 2 |\underset{\sim}{z}-\underset{\sim}{y}| \quad \text{if } \alpha+\beta \le 2 .$$

In view of these observations, the desired property of v_3 follows from (3.32) and the first part of the lemma. The proof is now complete.

Lemma 3.4. (<u>Generation of elastic states from densities with surface nuclei</u>). <u>Let</u> $R, \lambda, \bar{S}^i(\cdot, \underset{\sim}{y}, \lambda)$ <u>be as in Lemma</u> 3.1. <u>Further, let</u> $\alpha > 0$, $0 < \gamma \le 1$, <u>assume</u>

$$\underset{\sim}{g} \in \mathcal{H}^{\alpha, \gamma}(\partial R) ,$$

<u>and define functions</u> $\underset{\sim}{u}, \underset{\sim}{T}$ <u>through</u>

$$\underset{\sim}{u}(\underset{\sim}{x}, \underset{\sim}{y}) = \int_{\partial R} \bar{\underset{\sim}{u}}^i(\underset{\sim}{x}, \underset{\sim}{p}, \lambda) g_i(\underset{\sim}{p}, \underset{\sim}{y}) dA_{\underset{\sim}{p}} \quad \underline{\text{for all}} \ (\underset{\sim}{x}, \underset{\sim}{y}) \in \bar{R} \times \partial R$$

except possibly for $x = y$,

$$\tau(x, y) = \int_{\partial R} \bar{\tau}^i(x, p, \lambda) g_i(p, y) dA_p \quad \text{for all } (x, y) \in R \times \partial R \; ;$$

$$\tau(x, y) = \psi^i(x) g_i(x, y) + \int_{\partial R}^{p} \bar{\tau}^i(x, p, \lambda) g_i(p, y) dA_p \quad \text{for all } (x, y) \in \partial R \times \partial R$$

except possibly for $x = y$, where ψ^i is given by (3.9) and the last integral is to be interpreted in the sense of (3.10). Then:

(a) $S(\cdot, y) = [u(\cdot, y), \tau(\cdot, y)] \in \mathcal{E}(0, \mu, \sigma; \bar{R}_y)$ for all $y \in \partial R$;

(b) $u \in C(\bar{R} \times \partial R - D)$ if $\alpha \leq 1$, $u \in C(\bar{R} \times \partial R)$ if $\alpha > 1$,

$\tau \in C(\bar{R} \times \partial R - D)$ if $\alpha \leq 2$, $\tau \in C(\bar{R} \times \partial R)$ if $\alpha > 2$,

$\nabla u \in C(R \times \partial R)$, $\nabla \tau \in C(R \times \partial R)$;

(c) $u(x, y) = O(|x - y|^{\eta - 1})$ as $x \to y$ if $\alpha \leq 1$,

$\tau(x, y) = O(|x - y|^{\eta - 2})$ as $x \to y$ if $\alpha \leq 2$,

for any fixed $\eta < \alpha$, these estimates being uniform with respect to y for all $y \in \partial R$;

(d) $t - g \in \mathcal{H}^{\alpha + 1, \nu}(\partial R)$ for some $\nu \in (0, 1)$,

provided $t(\cdot, y)$, for each $y \in \partial R$, are the tractions[1] of $S(\cdot, y)$ on ∂R_y.

Proof. Conclusions (a) and (b) are readily reached through an elementary modification of the argument employed in the proof of Theorem 3.1. Note that (b) assures the regularity of $S(\cdot, y)$ on $\bar{R}$ if $\alpha > 2$.

Turning to (c), observe first that given $\varepsilon > 0$, $\delta > 0$ with $\varepsilon + \delta > 2$, there exists $k > 0$ such that

---

[1] If $\alpha \in (1, 2]$, we define $t(x, y) - g(x, y)$ also for $x = y$ in such a way as to render $t - g$ continuous on $\partial R \times \partial R$.

$$\int_{\partial R} |\underset{\sim}{x}-\underset{\sim}{p}|^{\epsilon-2}\,|\underset{\sim}{p}-\underset{\sim}{y}|^{\delta-2}\,dA_{\underset{\sim}{p}} < k \quad \text{for all } (\underset{\sim}{x},\underset{\sim}{y})\in\overline{R}\times\overline{R}\,, \tag{3.33}$$

as is clear from the continuity of the left-hand member[1] on $\overline{R}\times\overline{R}$.

Consider first $\alpha \le 1$. Then, in view of Lemma 3.1, the present hypotheses on $\underset{\sim}{g}$, and Definition 3.4, there exists $k_1 > 0$ such that

$$|\underset{\sim}{u}(\underset{\sim}{x},\underset{\sim}{y})| < k_1 \int_{\partial R} |\underset{\sim}{x}-\underset{\sim}{p}|^{-1}\,|\underset{\sim}{p}-\underset{\sim}{y}|^{\alpha-2}\,dA_{\underset{\sim}{p}} \quad \text{for all } (\underset{\sim}{x},\underset{\sim}{y})\in\overline{R}\times\partial R-D. \tag{3.34}$$

Choose $\eta < \alpha$, $(\underset{\sim}{x},\underset{\sim}{y})\in\overline{R}\times\partial R-D$, and define

$$\partial_1 R = \left\{ \underset{\sim}{p} \,\middle|\, \underset{\sim}{p}\in\partial R,\ |\underset{\sim}{x}-\underset{\sim}{p}| > \tfrac{1}{2}|\underset{\sim}{x}-\underset{\sim}{y}| \right\},\quad \partial_2 R = \partial R - \partial_1 R\,.$$

Since

$$|\underset{\sim}{p}-\underset{\sim}{y}| \ge |\underset{\sim}{x}-\underset{\sim}{y}| - |\underset{\sim}{x}-\underset{\sim}{p}| \ge \tfrac{1}{2}\,|\underset{\sim}{x}-\underset{\sim}{y}| \quad \text{for all } \underset{\sim}{p}\in\partial_2 R\,,$$

one has

$$\int_{\partial_1 R} |\underset{\sim}{x}-\underset{\sim}{p}|^{-1}\,|\underset{\sim}{p}-\underset{\sim}{y}|^{\alpha-2}\,dA_{\underset{\sim}{p}} < 2^{1-\eta}\,|\underset{\sim}{x}-\underset{\sim}{y}|^{\eta-1}\int_{\partial R} |\underset{\sim}{x}-\underset{\sim}{p}|^{-\eta}\,|\underset{\sim}{p}-\underset{\sim}{y}|^{\alpha-2}\,dA_{\underset{\sim}{p}}\,,$$

$$\int_{\partial_2 R} |\underset{\sim}{x}-\underset{\sim}{p}|^{-1}\,|\underset{\sim}{p}-\underset{\sim}{y}|^{\alpha-2}\,dA_{\underset{\sim}{p}} \le 2^{1-\eta}\,|\underset{\sim}{x}-\underset{\sim}{y}|^{\eta-1}\int_{\partial R} |\underset{\sim}{x}-\underset{\sim}{p}|^{-1}\,|\underset{\sim}{p}-\underset{\sim}{y}|^{\alpha-\eta-1}\,dA_{\underset{\sim}{p}}\,.$$

These inequalities, together with (3.33), (3.34), yield the <u>first</u> of (c).

Next assume $\alpha < 2$. In view of conclusion (b) in the present lemma, the <u>second</u> of (c) holds true if, given $\eta < \alpha$, there exists $k_2 > 0$ such that

$$|\underset{\sim}{\tau}(\underset{\sim}{x},\underset{\sim}{y})| < k_2\,|\underset{\sim}{x}-\underset{\sim}{y}|^{\eta-2} \quad \text{for all } (\underset{\sim}{x},\underset{\sim}{y})\in R\times\partial R\,. \tag{3.35}$$

---

[1] Cf. the first footnote in the proof of Lemma 3.3.

Choose $\eta < \alpha$ and suppose without loss in generality that $\eta > \alpha - \gamma$. Further, choose $(\underset{\sim}{x}, \underset{\sim}{y}) \in R \times \partial R$ and let $\underset{\sim}{z} \in \partial R$ be such that

$$|\underset{\sim}{x}-\underset{\sim}{z}| = \min |\underset{\sim}{x}-\underset{\sim}{p}|, \quad \underset{\sim}{p} \in \partial R ,$$

whence

$$|\underset{\sim}{x}-\underset{\sim}{z}| \leq |\underset{\sim}{x}-\underset{\sim}{p}|, \quad |\underset{\sim}{p}-\underset{\sim}{z}| \leq |\underset{\sim}{p}-\underset{\sim}{x}| + |\underset{\sim}{x}-\underset{\sim}{z}| \leq 2\,|\underset{\sim}{p}-\underset{\sim}{x}| \text{ for all } \underset{\sim}{p} \in \partial R. \tag{3.36}$$

Consider first

$$|\underset{\sim}{z}-\underset{\sim}{y}| \leq |\underset{\sim}{z}-\underset{\sim}{x}| . \tag{3.37}$$

Then (3.36), (3.37) give

$$|\underset{\sim}{x}-\underset{\sim}{y}| \leq |\underset{\sim}{x}-\underset{\sim}{z}| + |\underset{\sim}{z}-\underset{\sim}{y}| \leq 2\,|\underset{\sim}{x}-\underset{\sim}{z}| \leq 2\,|\underset{\sim}{x}-\underset{\sim}{p}| \text{ for all } \underset{\sim}{p} \in \partial R . \tag{3.38}$$

In view of the properties of $\underset{\sim}{g}$ and Lemma 3.1, there exists $k_3 > 0$ (independent of the particular choice of $\underset{\sim}{x}, \underset{\sim}{y}$) such that

$$|\underset{\sim}{\tau}(\underset{\sim}{x}, \underset{\sim}{y})| < k_3 \int_{\partial R} |\underset{\sim}{x}-\underset{\sim}{p}|^{-2} |\underset{\sim}{p}-\underset{\sim}{y}|^{\alpha-2} dA_{\underset{\sim}{p}} .$$

Therefore and from (3.38),

$$|\underset{\sim}{\tau}(\underset{\sim}{x}, \underset{\sim}{y})| < 2^{2-\eta} k_3 |\underset{\sim}{x}-\underset{\sim}{y}|^{\eta-2} \int_{\partial R} |\underset{\sim}{x}-\underset{\sim}{p}|^{-\eta} |\underset{\sim}{p}-\underset{\sim}{y}|^{\alpha-2} dA_{\underset{\sim}{p}} .$$

This last inequality, because of (3.33), yields (3.35) provided (3.37) holds true.

Next, verify (3.35) for

$$|\underset{\sim}{z}-\underset{\sim}{y}| > |\underset{\sim}{z}-\underset{\sim}{x}| ,$$

in which case

$$|\underset{\sim}{x}-\underset{\sim}{y}| \leq |\underset{\sim}{z}-\underset{\sim}{x}| + |\underset{\sim}{z}-\underset{\sim}{y}| < 2\,|\underset{\sim}{z}-\underset{\sim}{y}| . \tag{3.39}$$

To this end note from the definition of $\underset{\sim}{\tau}$ that

$$|\underset{\sim}{\tau}(\underset{\sim}{x},\underset{\sim}{y})| \leq |\int_{\partial R} \overline{\underset{\sim}{\tau}}^{i}(\underset{\sim}{x},\underset{\sim}{p},\lambda)[g_i(\underset{\sim}{z},\underset{\sim}{y})-g_i(\underset{\sim}{p},\underset{\sim}{y})]dA_{\underset{\sim}{p}}| + |g_i(\underset{\sim}{z},\underset{\sim}{y})|\,|\int_{\partial R} \overline{\underset{\sim}{\tau}}^{i}(\underset{\sim}{x},\underset{\sim}{p},\lambda)dA_{\underset{\sim}{p}}| . \quad (3.40)$$

Apply Theorem 3.1 with $e_k = \delta_{ki}$ to see that

$$|\int_{\partial R} \overline{\underset{\sim}{\tau}}^{i}(\cdot,\underset{\sim}{p},\lambda)dA_{\underset{\sim}{p}}|$$

is uniformly bounded on R. Accordingly (3.39) and the present hypotheses on $\underset{\sim}{g}$ and η imply the existence of $k_4 > 0$, independent of $\underset{\sim}{x}, \underset{\sim}{y}$, such that

$$|g_i(\underset{\sim}{z},\underset{\sim}{y})|\,|\int_{\partial R} \overline{\underset{\sim}{\tau}}^{i}(\underset{\sim}{x},\underset{\sim}{p},\lambda)dA_{\underset{\sim}{p}}| < k_4 |\underset{\sim}{x}-\underset{\sim}{y}|^{\eta-2} . \quad (3.41)$$

In order to bound the leading term in (3.40) introduce

$$\partial_3 R = \left\{\underset{\sim}{p} \,\middle|\, \underset{\sim}{p} \in \partial R,\ |\underset{\sim}{z}-\underset{\sim}{p}| < \tfrac{1}{2}|\underset{\sim}{z}-\underset{\sim}{y}|\right\},\quad \partial_4 R = \partial R - \partial_3 R . \quad (3.42)$$

Then,

$$|\int_{\partial R} \overline{\underset{\sim}{\tau}}^{i}(\underset{\sim}{x},\underset{\sim}{p},\lambda)[g_i(\underset{\sim}{z},\underset{\sim}{y})-g_i(\underset{\sim}{p},\underset{\sim}{y})]dA_{\underset{\sim}{p}}| \leq \int_{\partial_3 R} |\overline{\underset{\sim}{\tau}}^{i}(\underset{\sim}{x},\underset{\sim}{p},\lambda)|\,|g_i(\underset{\sim}{z},\underset{\sim}{y})-g_i(\underset{\sim}{p},\underset{\sim}{y})|\,dA_{\underset{\sim}{p}}$$

$$+ \int_{\partial_4 R} |\overline{\underset{\sim}{\tau}}^{i}(\underset{\sim}{x},\underset{\sim}{p},\lambda)|\,|g_i(\underset{\sim}{z},\underset{\sim}{y})|\,dA_{\underset{\sim}{p}} + \int_{\partial_4 R} |\overline{\underset{\sim}{\tau}}^{i}(\underset{\sim}{x},\underset{\sim}{p},\lambda)|\,|g_i(\underset{\sim}{p},\underset{\sim}{y})|\,dA_{\underset{\sim}{p}} .$$

Thus, bearing in mind that $\gamma > \alpha - \eta$, one concludes from Lemma 3.1, (3.42), and the hypotheses on $\underset{\sim}{g}$ that there exists $k_5 > 0$, independent of $\underset{\sim}{x}, \underset{\sim}{y}$, such that

$$\left|\int_{\partial R} \underset{\sim}{T}^{i}(\underset{\sim}{x}, \underset{\sim}{p}, \lambda)[g_i(\underset{\sim}{z}, \underset{\sim}{y}) - g_i(\underset{\sim}{p}, \underset{\sim}{y})]dA_{\underset{\sim}{p}}\right| < k_5 \int_{\partial_3 R} |\underset{\sim}{x}-\underset{\sim}{p}|^{-2} |\underset{\sim}{z}-\underset{\sim}{p}|^{\alpha-\eta} |\underset{\sim}{z}-\underset{\sim}{y}|^{\eta-2} dA_{\underset{\sim}{p}}$$

$$+k_5 \int_{\partial_4 R} |\underset{\sim}{x}-\underset{\sim}{p}|^{-2} |\underset{\sim}{z}-\underset{\sim}{y}|^{\alpha-2} dA_{\underset{\sim}{p}} + k_5 \int_{\partial_4 R} |\underset{\sim}{x}-\underset{\sim}{p}|^{-2} |\underset{\sim}{p}-\underset{\sim}{y}|^{\alpha-2} dA_{\underset{\sim}{p}} . \qquad (3.43)^1$$

Further, from (3.36), (3.39),

$$\int_{\partial_3 R} |\underset{\sim}{x}-\underset{\sim}{p}|^{-2} |\underset{\sim}{z}-\underset{\sim}{p}|^{\alpha-\eta} |\underset{\sim}{z}-\underset{\sim}{y}|^{\eta-2} dA_{\underset{\sim}{p}} < 2^{2+\alpha-2\eta} |\underset{\sim}{x}-\underset{\sim}{y}|^{\eta-2} \int_{\partial_3 R} |\underset{\sim}{x}-\underset{\sim}{p}|^{-2+\alpha-\eta} dA_{\underset{\sim}{p}} . \quad (3.44)$$

In addition, (3.42), (3.36) furnish

$$|\underset{\sim}{z}-\underset{\sim}{y}| \le 2|\underset{\sim}{z}-\underset{\sim}{p}| \le 4|\underset{\sim}{p}-\underset{\sim}{x}| \quad \text{for all } \underset{\sim}{p} \in \partial_4 R ,$$

so that

$$\int_{\partial_4 R} |\underset{\sim}{x}-\underset{\sim}{p}|^{-2} |\underset{\sim}{z}-\underset{\sim}{y}|^{\alpha-2} dA_{\underset{\sim}{p}} \le 4^{\alpha-\eta} |\underset{\sim}{z}-\underset{\sim}{y}|^{\eta-2} \int_{\partial_4 R} |\underset{\sim}{x}-\underset{\sim}{p}|^{\alpha-\eta-2} dA_{\underset{\sim}{p}} ,$$

$$\int_{\partial_4 R} |\underset{\sim}{x}-\underset{\sim}{p}|^{-2} |\underset{\sim}{p}-\underset{\sim}{y}|^{\alpha-2} dA_{\underset{\sim}{p}} \le 4^{2-\eta} |\underset{\sim}{z}-\underset{\sim}{y}|^{\eta-2} \int_{\partial_4 R} |\underset{\sim}{x}-\underset{\sim}{p}|^{-\eta} |\underset{\sim}{p}-\underset{\sim}{y}|^{\alpha-2} dA_{\underset{\sim}{p}} .$$

Combining the last two inequalities with (3.33), (3.39), (3.40), (3.41), (3.43), (3.44) one obtains again (3.35).

This disposes of conclusion (c) for $\alpha < 2$. Since $\underset{\sim}{g} \in \mathcal{N}^{2,\gamma}(\partial R)$ implies $\underset{\sim}{g} \in \mathcal{N}^{\beta,\gamma}(\partial R)$ for any $\beta < 2$, the second of (c) holds also for $\alpha = 2$.

With a view toward conclusion (d), note first from the definition of $S(\cdot, \underset{\sim}{y})$ that

---

[1] Note that the assumption $\eta > \alpha - \gamma$, which ensures that $\underset{\sim}{g} \in \mathcal{N}^{\alpha, \alpha-\eta}(\partial R)$, was essential in the derivation of (3.43).

$$t(x,y)=g(x,y)+\int_{\partial R}\bar{t}^{\,i}(x,p,\lambda)g_i(p,y)dA_p \quad \text{for all } (x,y)\in\partial R\times\partial R$$

except possibly when $x=y$. Thus, according to Lemma 3.3, (d) holds true if

$$\bar{t}_j^{\,i}(\cdot,\cdot,\lambda)\in\eta^{1,1}(\partial R) . \tag{3.45}$$

On the other hand, (3.45) is implied by (a) and (b) in Lemma 3.1 if there is a constant $\kappa > 0$ such that

$$|\bar{t}_j^{\,i}(x,y,\lambda)-\bar{t}_j^{\,i}(z,y,\lambda)|\le\kappa\,|x-z|\,|x-y|^{-2} \tag{3.46}$$

for all x, y, z on ∂R subject to $|x-z|<\frac{1}{2}|x-y|$. The inequality (3.46) is confirmed through an argument strictly analogous to that employed in the first part of the proof of Lemma I, Chapter XI in [5] (p. 300) provided one establishes the existence of $\kappa_1>0$ such that

$$\left|\frac{\partial}{\partial s}\bar{t}_j^{\,i}(x,y,\lambda)\right|<\kappa_1\,|x-y|^{-2} \quad \text{for all } (x,y)\in\partial R\times\partial R-D . \tag{3.47}$$

Here, the left-hand side is the derivative of $\bar{t}_j^{\,i}(\cdot,y,\lambda)$ with respect to the distance "s" measured along any smooth arc on ∂R and evaluated at x. To see that (3.47) holds observe first that for all $(x,y)\in\partial R\times\partial R-D$,

$$\begin{aligned}\bar{t}_j^{\,i}(x,y,\lambda) &= \bar{\tau}_{jk}^{\,i}(x,y,\lambda)n_k(x)\\ &=\bar{\tau}_{jk}^{\,i}(x,y,\lambda)[n_k(x)-n_k(y)]+\bar{\tau}_{jk}^{\,i}(x,y,\infty)n_k(y)\\ &\quad+[\bar{\tau}_{jk}^{\,i}(x,y,\lambda)n_k(y)-\bar{\tau}_{jk}^{\,i}(x,y,\infty)n_k(y)] .\end{aligned} \tag{3.48}$$

Further, note from Definition 3.1 that there exists $\kappa_2>0$ such that

$$|\bar{\tau}_{jk,\ell}^{\,i}(x,y,\lambda)|<\kappa_2\,|x-y|^{-3} \quad \text{for all } (x,y)\in\bar{R}\times\partial R-D ,$$

while, as pointed out in [5] (p. 299), because of the smoothness of ∂R, there is a $\kappa_3>0$ such that

$$\left|\frac{\partial}{\partial s}\left[\frac{(x_k-y_k)n_k(\underset{\sim}{y})}{|\underset{\sim}{x}-\underset{\sim}{y}|^3}\right]\right| < \kappa_3|\underset{\sim}{x}-\underset{\sim}{y}|^{-2} \text{ for all } (\underset{\sim}{x},\underset{\sim}{y})\in\partial R\times\partial R-D .$$

By virtue of these two inequalities, (1.4), (1.5), (3.5), and conclusion (b) in Lemma 3.1, there exists $\kappa_4>0$ such that

$$\left.\begin{aligned}&\left|\frac{\partial}{\partial s}\left\{\bar{\tau}^i_{jk}(\underset{\sim}{x},\underset{\sim}{y},\lambda)[n_k(\underset{\sim}{x})-n_k(\underset{\sim}{y})]\right\}\right| < \kappa_4|\underset{\sim}{x}-\underset{\sim}{y}|^{-2},\\ &\left|\frac{\partial}{\partial s}\,\bar{\tau}^i_{jk}(\underset{\sim}{x},\underset{\sim}{y},\infty)n_k(\underset{\sim}{y})\right| < \kappa_4|\underset{\sim}{x}-\underset{\sim}{y}|^{-2},\end{aligned}\right\} \quad (3.49)$$

for all $(\underset{\sim}{x},\underset{\sim}{y})\in\partial R\times\partial R-D$. From (3.48), (3.49), upon noting that

$$\frac{\partial}{\partial s}\left[\bar{\tau}^i_{jk}(\underset{\sim}{x},\underset{\sim}{y},\lambda)n_k(\underset{\sim}{y})-\bar{\tau}^i_{jk}(\underset{\sim}{x},\underset{\sim}{y},\infty)n_k(\underset{\sim}{y})\right]$$

is uniformly bounded[1] for $(\underset{\sim}{x},\underset{\sim}{y})$ on $\partial R\times\partial R-D$, one obtains (3.47) and hence (3.45). This completes the proof.

Lemma 3.5. (A continuity property of a family of elastic states). Let R be a simple region. Suppose

$$S(\cdot,\underset{\sim}{y})=[\underset{\sim}{u}(\cdot,\underset{\sim}{y}),\underset{\sim}{\tau}(\cdot,\underset{\sim}{y})]\in\mathcal{E}(\underset{\sim}{0},\mu,\sigma;\bar{R}) \text{ for all } \underset{\sim}{y}\in\partial R ,$$

$$\int_{\partial R}\underset{\sim}{u}(\underset{\sim}{x},\underset{\sim}{y})\cdot\underset{\sim}{q}^m(\underset{\sim}{x})dA_{\underset{\sim}{x}}=0 \text{ for all } \underset{\sim}{y}\in\partial R \quad (m=1,\ldots,6) ,$$

with $\underset{\sim}{q}^m$ given by Definition 3.2. Further, assume

$$\underset{\sim}{t}\in\eta^{3,\alpha}(\partial R) \quad (0<\alpha<1) ,$$

where $\underset{\sim}{t}(\cdot,\underset{\sim}{y})$, for each $\underset{\sim}{y}\in\partial R$, are the tractions of $S(\cdot,\underset{\sim}{y})$ on ∂R. Then

$$\underset{\sim}{u}\in C(\bar{R}\times\partial R) , \quad \underset{\sim}{\tau}\in C(\bar{R}\times\partial R) ,$$

$$\nabla\underset{\sim}{u}\in C(R\times\partial R) , \quad \nabla\underset{\sim}{\tau}\in C(R\times\partial R) .$$

---

[1] By (3.5) and Definition 3.1, both $\bar{\tau}^i_{jk}(\cdot,\underset{\sim}{y},\lambda)n_k(\underset{\sim}{y})$ and $\bar{\tau}^i_{jk}(\cdot,\underset{\sim}{y},\infty)n_k(\underset{\sim}{y})$ are differentiable on $\partial R_{\underset{\sim}{y}}$, while their difference is differentiable on a neighborhood of $\underset{\sim}{y}$.

Proof. Note first from Definition 3.4 that the tractions $\underset{\sim}{t}$ of S are continuous on $\partial R \times \partial R$ and are Hölder-continuous in their first argument, uniformly with respect to the second argument.

Let λ, $\overline{S}^i(\cdot, \underset{\sim}{y}, \lambda)$ be as in Lemma 3.1 and consider the system of Fredholm integral equations of the second kind

$$\underset{\sim}{e}(\underset{\sim}{x}) = \hat{\underset{\sim}{t}}(\underset{\sim}{x}) - \int_{\partial R} \overline{\underset{\sim}{t}}^i(\underset{\sim}{x}, \underset{\sim}{y}, \lambda) e_i(\underset{\sim}{y}) dA_{\underset{\sim}{y}} \quad \text{for all } \underset{\sim}{x} \in \partial R , \tag{3.50}$$ [1]

and the adjoint homogeneous system

$$v_i(\underset{\sim}{y}) = - \int_{\partial R} \overline{\underset{\sim}{t}}(\underset{\sim}{x}, \underset{\sim}{y}, \lambda) \cdot \underset{\sim}{v}(\underset{\sim}{x}) dA_{\underset{\sim}{x}} \quad \text{for all } \underset{\sim}{y} \in \partial R . \tag{3.51}$$

We now show that the functions $\underset{\sim}{q}^m$ $(m=1, \ldots 6)$ are solutions of (3.51), i.e.,

$$q_i^m(\underset{\sim}{y}) = - \int_{\partial R} \overline{\underset{\sim}{t}}^i(\underset{\sim}{x}, \underset{\sim}{y}, \lambda) \cdot \underset{\sim}{q}^m(\underset{\sim}{x}) dA_{\underset{\sim}{x}} \quad \text{for all } \underset{\sim}{y} \in \partial R \ (m=1, \ldots 6). \tag{3.52}$$ [2]

To this end note from Theorem 3.1 that if $\underset{\sim}{e} \in \mathcal{N}(\partial R)$, the vector field defined through

$$\underset{\sim}{e}(\underset{\sim}{x}) + \int_{\partial R} \overline{\underset{\sim}{t}}^i(\underset{\sim}{x}, \underset{\sim}{y}, \lambda) e_i(\underset{\sim}{y}) dA_{\underset{\sim}{y}} \quad \text{for all } \underset{\sim}{x} \in \partial R$$

represents the surface tractions of an elastic state on $\overline{R}$ and is accordingly self-equilibrated on ∂R. Thus, in view of (a) in Lemma 3.2,

---

[1] Observe from Theorem 3.1 that if (3.50) has a Hölder-continuous solution, the latter may be used as a surface density to generate an elastic state on $\overline{R}$ whose tractions on ∂R coincide with $\hat{\underset{\sim}{t}}$.

[2] Equation (3.52) asserts that the tractions of $\overline{S}^i(\cdot, \underset{\sim}{y}, \lambda)$ on $\partial R_{\underset{\sim}{y}}$ equilibrate a unit load in the x_i-direction, applied at $\underset{\sim}{y}$.

$$\int_{\partial R} e_i(\underset{\sim}{y}) q_i^m(\underset{\sim}{y}) dA_{\underset{\sim}{y}} + \int_{\partial R}\int_{\partial R} \bar{\underset{\sim}{t}}^i(\underset{\sim}{x},\underset{\sim}{y},\lambda)\cdot \underset{\sim}{q}^m(\underset{\sim}{x}) e_i(\underset{\sim}{y}) dA_{\underset{\sim}{x}} dA_{\underset{\sim}{y}} = 0 \quad (m=1,\ldots,6).$$

Since this equation must hold true for every choice of $\underset{\sim}{e} \in \mathscr{N}(\partial R)$, (3.52) follows.

The continuity and order-of-magnitude properties of $\bar{\underset{\sim}{t}}^i$ given in Lemma 3.1 guarantee the applicability of Fredholm's theory[1] to the pair of systems (3.50), (3.51). Hence (3.51) has at most a finite number of linearly independent continuous solutions $\underset{\sim}{v}^m$ $(m=1,\ldots,k)$ which, because of (3.52) and (3.25), may be assumed to satisfy

$$\left.\begin{array}{l} \underset{\sim}{v}^m = \underset{\sim}{q}^m \text{ on } \partial R \quad (m=1,\ldots,6) \\ \displaystyle\int_{\partial R} \underset{\sim}{v}^m \cdot \underset{\sim}{v}^\ell \, dA = \begin{cases} 0 & (m\neq \ell) \\ 1 & (m=\ell) \quad (m,\ell=1,\ldots k). \end{cases} \end{array}\right\} \quad (3.53)$$

Further, (3.50) is solvable if and only if $\hat{\underset{\sim}{t}}$ is orthogonal to the k vector fields $\underset{\sim}{v}^m$ in the sense of

$$\int_{\partial R} \hat{\underset{\sim}{t}} \cdot \underset{\sim}{v}^m \, dA = 0 \quad (m=1,2,\ldots k).$$

We now define

$$\underset{\sim}{t}'(\underset{\sim}{x},\underset{\sim}{y}) = \underset{\sim}{t}(\underset{\sim}{x},\underset{\sim}{y}) - \sum_{m=7}^{k} \underset{\sim}{v}^m(\underset{\sim}{x}) \int_{\partial R} \underset{\sim}{v}^m(\underset{\sim}{p})\cdot \underset{\sim}{t}(\underset{\sim}{p},\underset{\sim}{y}) dA_{\underset{\sim}{p}} \quad \text{for all } (\underset{\sim}{x},\underset{\sim}{y}) \in \partial R \times \partial R. \quad (3.54)$$

Then (3.53), (3.54), (a) in Lemma 3.2, and the self-equilibration of $\underset{\sim}{t}(\cdot,\underset{\sim}{y})$ on ∂R, furnish

$$\int_{\partial R} \underset{\sim}{t}'(\underset{\sim}{x},\underset{\sim}{y})\cdot \underset{\sim}{v}^m(\underset{\sim}{x}) dA_{\underset{\sim}{x}} = 0 \quad (m=1,\ldots,k) \text{ for all } \underset{\sim}{y} \in \partial R .$$

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[1] See, for example, [12], Chapter III.

Thus, the system of integral equations

$$\underset{\sim}{g}(\underset{\sim}{x},\underset{\sim}{y})=\underset{\sim}{t}'(\underset{\sim}{x},\underset{\sim}{y})-\int_{\partial R}\overline{\underset{\sim}{t}}^{i}(\underset{\sim}{x},\underset{\sim}{p},\lambda)g_i(\underset{\sim}{p},\underset{\sim}{y})dA_{\underset{\sim}{p}} \text{ for all } (\underset{\sim}{x},\underset{\sim}{y})\in\partial R\times\partial R \tag{3.55}$$

which, for fixed $\underset{\sim}{y}\in\partial R$, is of the form (3.50), has a (nonunique) solution. This solution may be chosen so as to ensure that

$$\underset{\sim}{g}\in C(\partial R\times\partial R)\ . \qquad (3.56)^{1}$$

We show next that any solution $\underset{\sim}{g}$ of (3.55) that conforms to (3.56) also obeys the stronger regularity condition

$$\underset{\sim}{g}\in\mathcal{H}^{3,\alpha}(\partial R)\ . \tag{3.57}$$

For this purpose one may use an argument analogous to that employed in deducing (3.45) to show that $\overline{\overline{t}}{}^{i}_{j}$ defined by

$$\overline{\overline{t}}{}^{i}_{j}(\underset{\sim}{y},\underset{\sim}{x},\lambda)=\overline{t}{}^{j}_{i}(\underset{\sim}{x},\underset{\sim}{y},\lambda) \text{ for all } (\underset{\sim}{x},\underset{\sim}{y})\in\partial R\times\partial R-D\ ,$$

has the property

$$\overline{\overline{t}}{}^{i}_{j}(\cdot,\cdot,\lambda)\in\mathcal{H}^{1,1}(\partial R)\ .$$

Hence (3.51) and Lemma 3.3 furnish

$$\underset{\sim}{v}^{m}\in\mathcal{H}(\partial R)\quad(m=1,\ldots,k)\ , \tag{3.58}$$

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[1] This claim may be confirmed as follows. One first reduces (3.55) through the usual iteration process to an equivalent system of integral equations whose kernel is continuous on $\partial R\times\partial R$. Subsequently one constructs a resolvent of the latter system in infinite series form and deduces the continuity of the resolvent on $\partial R\times\partial R$. Finally, one verifies (3.56) by an appeal to the representation of $\underset{\sim}{g}$ in terms of the resolvent and the given (continuous) data. Cf. [12], Chapters 2, 3.

the exponent of this Hölder-condition[1] being any number in the interval (0, 1). From (3.58), (3.54) and the assumed regularity of $\underset{\sim}{t}$ follows

$$\underset{\sim}{t}' \in \mathcal{H}^{3,\alpha}(\partial R) ,$$

and this conclusion, together with (3.55), (3.56), (3.45) and Lemma 3.3, implies (3.57).

In view of (3.57) we may employ $\underset{\sim}{g}$ in conjunction with Lemma 3.4 to generate a family of states

$$S'(\cdot, \underset{\sim}{y}) = [\underset{\sim}{u}'(\cdot, \underset{\sim}{y}), \underset{\sim}{\tau}'(\cdot, \underset{\sim}{y})] \in \mathcal{E}(\underset{\sim}{0}, \mu, \sigma; \overline{R}) \text{ for all } \underset{\sim}{y} \in \partial R , \tag{3.59}$$

with

$$\left.\begin{aligned} \underset{\sim}{u}' \in \mathcal{C}(\overline{R} \times \partial R) , \quad \underset{\sim}{\tau}' \in \mathcal{C}(\overline{R} \times \partial R) , \\ \nabla \underset{\sim}{u}' \in \mathcal{C}(R \times \partial R) , \quad \nabla \underset{\sim}{\tau}' \in \mathcal{C}(R \times \partial R) , \end{aligned}\right\} \tag{3.60}$$

whose tractions on ∂R, because of (3.55), are $\underset{\sim}{t}'(\cdot, \underset{\sim}{y})$ for each $\underset{\sim}{y} \in \partial R$.

With a view towards constructing an elastic state with the surface tractions $\underset{\sim}{t}(\cdot, \underset{\sim}{y})$ we recall (3.54) and bear in mind that the fields $\underset{\sim}{v}^m$ (m=7, ...k) are self-equilibrated and Hölder-continuous on the boundary of the simple region R. The foregoing properties of R and of $\underset{\sim}{v}^m$ (m=7, ..., k) entitle us to conclude from Korn's [14][2] existence theorem for the second boundary-value problem of elastostatics the existence of elastic states S^m (m=7, ..., k) on $\overline{R}$ whose tractions on ∂R coincide with $\underset{\sim}{v}^m$ (m=7, ..., k). Thus the family of states S'' defined by

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[1] See Section 1.

[2] See also Korn [15].

$$S''(\underset{\sim}{x},\underset{\sim}{y})=S'(\underset{\sim}{x},\underset{\sim}{y})+\sum_{m=7}^{k}\left\{\int_{\partial R}\underset{\sim}{v}^{m}(\underset{\sim}{p})\cdot\underset{\sim}{t}(\underset{\sim}{p},\underset{\sim}{y})dA_{\underset{\sim}{p}}\right\}S^{m}(\underset{\sim}{x}) \text{ for all } (\underset{\sim}{x},\underset{\sim}{y})\in\overline{R}\times\partial R ,$$

because of (3.54), (3.59), (3.60), has the properties

$$\left.\begin{aligned} &S''(\cdot,\underset{\sim}{y})=[\underset{\sim}{u}''(\cdot,\underset{\sim}{y}),\underset{\sim}{\tau}''(\cdot,\underset{\sim}{y})]\in\mathcal{E}(\underset{\sim}{0},\mu,\sigma;\overline{R}) \text{ for all } \underset{\sim}{y}\in\partial R ,\\ &\underset{\sim}{u}''\in\mathcal{C}(\overline{R}\times\partial R),\ \underset{\sim}{\tau}''\in\mathcal{C}(\overline{R}\times\partial R) ,\\ &\nabla\underset{\sim}{u}''\in\mathcal{C}(R\times\partial R),\ \nabla\underset{\sim}{\tau}''\in\mathcal{C}(R\times\partial R) ,\\ &\underset{\sim}{t}''(\underset{\sim}{x},\underset{\sim}{y})=\underset{\sim}{t}(\underset{\sim}{x},\underset{\sim}{y}) \text{ for all } (\underset{\sim}{x},\underset{\sim}{y})\in\partial R\times\partial R , \end{aligned}\right\} \quad (3.61)$$

where $\underset{\sim}{t}''(\cdot,\underset{\sim}{y})$ are the surface tractions of $S''(\cdot,\underset{\sim}{y})$ for each $\underset{\sim}{y}\in\partial R$. The given state $S(\cdot,\underset{\sim}{y})$ evidently differs from $S''(\cdot,\underset{\sim}{y})$ by a rigid displacement field. By virtue of (3.25) and the assumed normalization of the displacements belonging to $S(\cdot,\underset{\sim}{y})$ we arrive at the representation

$$\underset{\sim}{u}(\underset{\sim}{x},\underset{\sim}{y})=\underset{\sim}{u}''(\underset{\sim}{x},\underset{\sim}{y})-\sum_{m=1}^{6}\underset{\sim}{q}^{m}(\underset{\sim}{x})\int_{\partial R}\underset{\sim}{u}''(\underset{\sim}{p},\underset{\sim}{y})\cdot\underset{\sim}{q}^{m}(\underset{\sim}{p})dA_{\underset{\sim}{p}} ,\ \underset{\sim}{\tau}(\underset{\sim}{x},\underset{\sim}{y})=\underset{\sim}{\tau}''(\underset{\sim}{x},\underset{\sim}{y})$$

for all $(\underset{\sim}{x},\underset{\sim}{y})\in\overline{R}\times\partial R$. This representation, together with (3.61), implies the desired continuity property of S. This completes the proof.

We are now ready to turn to the

<u>Proof of Theorem</u> 3.2. Our initial objective here is to reduce the construction of the (singular) modified tangent states to the solution of a regular boundary-value problem in elastostatics. Let λ and the tangent states $\overline{S}^{i}(\cdot,\underset{\sim}{y},\lambda)$ be as in Lemma 3.1. Define $\underset{\sim}{g}^{i}$ through

$$\underset{\sim}{g}^{i}(\underset{\sim}{x},\underset{\sim}{y},\lambda)=\overline{\underset{\sim}{t}}^{i}(\underset{\sim}{x},\underset{\sim}{y},\lambda)-\int_{\partial R}\overline{\underset{\sim}{t}}^{k}(\underset{\sim}{x},\underset{\sim}{p},\lambda)\overline{t}_{k}^{i}(\underset{\sim}{p},\underset{\sim}{y},\lambda)dA_{\underset{\sim}{p}} \text{ for all } (\underset{\sim}{x},\underset{\sim}{y})\in\partial R\times\partial R-D, \quad (3.62)$$

so that from (3.45) and Lemma 3.3,

$$\underset{\sim}{g}^{i}(\cdot,\cdot,\lambda) \in \mathcal{N}^{1,\gamma}(\partial R) \text{ for any } \gamma \in (0,1) . \tag{3.63}$$

Define $\underset{\sim}{\tilde{u}}^{i}$, $\underset{\sim}{\tilde{\tau}}^{i}$ by means of

$$\left.\begin{aligned}
&\underset{\sim}{\tilde{u}}^{i}(\underset{\sim}{x},\underset{\sim}{y},\lambda)=\underset{\sim}{\bar{u}}^{i}(\underset{\sim}{x},\underset{\sim}{y},\lambda)-\int_{\partial R}\underset{\sim}{\bar{u}}^{j}(\underset{\sim}{x},\underset{\sim}{p},\lambda)g_{j}^{i}(\underset{\sim}{p},\underset{\sim}{y},\lambda)dA_{\underset{\sim}{p}} \text{ for all } (\underset{\sim}{x},\underset{\sim}{y})\in\bar{R}\times\partial R-D ,\\
&\underset{\sim}{\tilde{\tau}}^{i}(\underset{\sim}{x},\underset{\sim}{y},\lambda)=\underset{\sim}{\bar{\tau}}^{i}(\underset{\sim}{x},\underset{\sim}{y},\lambda)-\int_{\partial R}\underset{\sim}{\bar{\tau}}^{j}(\underset{\sim}{x},\underset{\sim}{p},\lambda)g_{j}^{i}(\underset{\sim}{p},\underset{\sim}{y},\lambda)dA_{\underset{\sim}{p}} \text{ for all } (\underset{\sim}{x},\underset{\sim}{y})\in R\times\partial R ,\\
&\underset{\sim}{\tilde{\tau}}^{i}(\underset{\sim}{x},\underset{\sim}{y},\lambda)=\underset{\sim}{\bar{\tau}}^{i}(\underset{\sim}{x},\underset{\sim}{y},\lambda)-\underset{\sim}{\psi}^{j}(\underset{\sim}{x})g_{j}^{i}(\underset{\sim}{x},\underset{\sim}{y},\lambda)\\
&\qquad -\overset{p}{\int_{\partial R}}\underset{\sim}{\bar{\tau}}^{j}(\underset{\sim}{x},\underset{\sim}{p},\lambda)g_{j}^{i}(\underset{\sim}{p},\underset{\sim}{y},\lambda)dA_{\underset{\sim}{p}} \text{ for all } (\underset{\sim}{x},\underset{\sim}{y})\in\partial R\times\partial R-D ,
\end{aligned}\right\} \tag{3.64}$$

with $\underset{\sim}{\psi}^{j}$ given by (3.9). Then, in view of Lemma 3.1, (3.63), Lemma 3.4, and (3.7),

$$\left.\begin{aligned}
&\underset{\sim}{\tilde{S}}^{i}(\cdot,\underset{\sim}{y},\lambda)=[\underset{\sim}{\tilde{u}}^{i}(\cdot,\underset{\sim}{y},\lambda),\underset{\sim}{\tilde{\tau}}^{i}(\cdot,\underset{\sim}{y},\lambda)]\in\mathcal{E}(\underset{\sim}{0},\mu,\sigma;\bar{R}_{\underset{\sim}{y}} \text{ for all } \underset{\sim}{y}\in\partial R ,\\
&\quad\underset{\sim}{\tilde{u}}^{i}(\cdot,\cdot,\lambda)\in\mathcal{C}(\bar{R}\times\partial R-D), \underset{\sim}{\tilde{\tau}}^{i}(\cdot,\cdot,\lambda)\in\mathcal{C}(\bar{R}\times\partial R-D) ,\\
&\quad\nabla\underset{\sim}{\tilde{u}}^{i}(\cdot,\cdot,\lambda)\in\mathcal{C}(R\times\partial R), \nabla\underset{\sim}{\tilde{\tau}}^{i}(\cdot,\cdot,\lambda)\in\mathcal{C}(R\times\partial R) ,\\
&\quad\underset{\sim}{\tilde{u}}^{i}(\underset{\sim}{x},\underset{\sim}{y},\lambda)=\underset{\sim}{\bar{u}}^{i}(\underset{\sim}{x},\underset{\sim}{y},\infty)+O(|\underset{\sim}{x}-\underset{\sim}{y}|^{-\beta}) \text{ as } \underset{\sim}{x}\to\underset{\sim}{y} ,\\
&\quad\underset{\sim}{\tilde{\tau}}^{i}(\underset{\sim}{x},\underset{\sim}{y},\lambda)=\underset{\sim}{\bar{\tau}}^{i}(\underset{\sim}{x},\underset{\sim}{y},\infty)+O(|\underset{\sim}{x}-\underset{\sim}{y}|^{-1-\beta}) \text{ as } \underset{\sim}{x}\to\underset{\sim}{y} ,
\end{aligned}\right\} \tag{3.65}$$

for any $\beta > 0$, uniformly in $\underset{\sim}{y}$ for all $\underset{\sim}{y} \in \partial R$. From (3.64), (3.9), (3.62), after a brief computation, follows

$$\underset{\sim}{\tilde{t}}^{i}(\underset{\sim}{x},\underset{\sim}{y},\lambda)=\underset{\sim}{\hat{t}}^{i}(\underset{\sim}{x},\underset{\sim}{y},\lambda)-\sum_{m=1}^{6}\underset{\sim}{q}^{m}(\underset{\sim}{x})q_{i}^{m}(\underset{\sim}{y}) \text{ for all } (\underset{\sim}{x},\underset{\sim}{y})\in\partial R\times\partial R-D , \tag{3.66}$$

where $\underset{\sim}{\tilde{t}}^{i}(\cdot,\underset{\sim}{y},\lambda)$ are the surface tractions of $\underset{\sim}{\tilde{S}}^{i}(\cdot,\underset{\sim}{y},\lambda)$ for each

$\underset{\sim}{y} \in \partial R$, $\underset{\sim}{q}^m$ $(m=1, \ldots 6)$ is as in Definition 3.2, while

$$\hat{\underset{\sim}{t}}^i(\underset{\sim}{x}, \underset{\sim}{y}, \lambda) = \int_{\partial R}\int_{\partial R} \bar{\underset{\sim}{t}}^j(\underset{\sim}{x}, \underset{\sim}{p}, \lambda)\bar{t}_j^k(\underset{\sim}{p}, \underset{\sim}{v}, \lambda)\bar{t}_k^i(\underset{\sim}{v}, \underset{\sim}{y}, \lambda)\, dA_{\underset{\sim}{p}}\, dA_{\underset{\sim}{v}} + \sum_{m=1}^{6} \underset{\sim}{q}^m(\underset{\sim}{x}) q_i^m(\underset{\sim}{y}) \tag{3.67}$$

for all $(\underset{\sim}{x}, \underset{\sim}{y}) \in \partial R \times \partial R$.

Next, (3.67), (3.45), Lemma 3.3, (3.52) and (3.25) yield

$$\hat{\underset{\sim}{t}}^i(\cdot, \cdot, \lambda) \in \mathfrak{N}^{3,\gamma}(\partial R), \int_{\partial R} \hat{\underset{\sim}{t}}^i(\underset{\sim}{x}, \underset{\sim}{y}, \lambda) \cdot \underset{\sim}{q}^m(\underset{\sim}{x})\, dA_{\underset{\sim}{x}} = 0 \text{ for all } \underset{\sim}{y} \in \partial R \quad (m=1, \ldots 6) \tag{3.68}$$

for any $\gamma \in (0, 1)$. In particular, (3.68) implies that the fields $\hat{\underset{\sim}{t}}^i(\cdot, \underset{\sim}{y}, \lambda)$ are Hölder-continuous and self-equilibrated on ∂R for each $\underset{\sim}{y} \in \partial R$. Thus, from Korn's [14] existence theorem, (3.68), and Lemma 3.5, one infers the existence of states $\hat{S}^i$ with

$$\left.\begin{aligned}
&\hat{S}^i(\cdot, \underset{\sim}{y}, \lambda) = [\hat{\underset{\sim}{u}}^i(\cdot, \underset{\sim}{y}, \lambda), \hat{\underset{\sim}{\tau}}^i(\cdot, \underset{\sim}{y}, \lambda)] \in \mathcal{E}(\underset{\sim}{0}, \mu, \sigma; \bar{R}) \text{ for all } \underset{\sim}{y} \in \partial R, \\
&\int_{\partial R} \hat{\underset{\sim}{u}}^i(\underset{\sim}{x}, \underset{\sim}{y}) \cdot \underset{\sim}{q}^m(\underset{\sim}{x})\, dA_{\underset{\sim}{x}} = 0 \text{ for all } \underset{\sim}{y} \in \partial R \ (m=1, \ldots 6), \\
&\hat{\underset{\sim}{u}}^i(\cdot, \cdot, \lambda) \in \mathcal{C}(\bar{R} \times \partial R), \ \hat{\underset{\sim}{\tau}}^i(\cdot, \cdot, \lambda) \in \mathcal{C}(\bar{R} \times \partial R), \\
&\nabla \hat{\underset{\sim}{u}}^i(\cdot, \cdot, \lambda) \in \mathcal{C}(R \times \partial R), \ \nabla \hat{\underset{\sim}{\tau}}^i(\cdot, \cdot, \lambda) \in \mathcal{C}(R \times \partial R),
\end{aligned}\right\} \tag{3.69}$$

whose tractions on ∂R are $\hat{\underset{\sim}{t}}^i(\cdot, \underset{\sim}{y}, \lambda)$ for each $\underset{\sim}{y} \in \partial R$.

Finally, define states

$$\mathring{S}^i(\underset{\sim}{x}, \underset{\sim}{y}) = [\mathring{\underset{\sim}{u}}^i(\underset{\sim}{x}, \underset{\sim}{y}), \mathring{\underset{\sim}{\tau}}^i(\underset{\sim}{x}, \underset{\sim}{y})]$$

for all $(\underset{\sim}{x}, \underset{\sim}{y}) \in \bar{R} \times \partial R - D$ through

$$\overset{\circ}{\underset{\sim}{u}}{}^{i}(\underset{\sim}{x},\underset{\sim}{y})=\underset{\sim}{\tilde{u}}{}^{i}(\underset{\sim}{x},\underset{\sim}{y},\lambda)-\underset{\sim}{\hat{u}}{}^{i}(\underset{\sim}{x},\underset{\sim}{y},\lambda)-\sum_{m=1}^{6}\underset{\sim}{q}{}^{m}(\underset{\sim}{x})\int_{\partial R}\underset{\sim}{\tilde{u}}{}^{i}(\underset{\sim}{p},\underset{\sim}{y},\lambda)\cdot\underset{\sim}{q}{}^{m}(\underset{\sim}{p})dA_{\underset{\sim}{p}},$$

$$\overset{\circ}{\underset{\sim}{\tau}}{}^{i}(\underset{\sim}{x},\underset{\sim}{y})=\underset{\sim}{\tilde{\tau}}{}^{i}(\underset{\sim}{x},\underset{\sim}{y},\lambda)-\underset{\sim}{\hat{\tau}}{}^{i}(\underset{\sim}{x},\underset{\sim}{y},\lambda)\ .$$

It is clear from (3.65), (3.66), (3.69), (3.25) that $\overset{\circ}{S}{}^{i}$ so constructed conforms to Definition 3.3 of the modified tangent states and possesses the additional properties (a), (b), asserted in the present theorem. This completes the proof.

The preceding existence theorem for the modified tangent states, together with Theorem 3.1 on the generation of elastic states from given surface densities, enables us to establish

Theorem 3.3. (*A representation of elastic states on simple regions in terms of their surface tractions*). *Let* R *be a simple region and let*

$$S=[\underset{\sim}{u},\underset{\sim}{\tau}]\in\mathcal{E}(\underset{\sim}{0},\mu,\sigma;\overline{R})\ .$$

Assume

$$\int_{\partial R}\underset{\sim}{u}\cdot\underset{\sim}{q}{}^{m}dA=0\quad(m=1,\ldots 6),\ \underset{\sim}{t}\in\mathcal{X}(\partial R)\ ,$$

where $\underset{\sim}{q}{}^{m}$ *is given by Definition* 3.2 *and* $\underset{\sim}{t}$ *are the tractions of* S *on* ∂R. *Let* $\overset{\circ}{S}{}^{i}(\cdot,\underset{\sim}{y})$ *be the modified tangent state for the region* R *at* $\underset{\sim}{y}$ *corresponding to the* x_i *direction and the elastic constants* μ, σ, *in the sense of Definition* 3.3. *Then* S *admits the representation*

$$\underset{\sim}{u}(\underset{\sim}{x})=\int_{\partial R}\overset{\circ}{\underset{\sim}{u}}{}^{i}(\underset{\sim}{x},\underset{\sim}{y})t_i(\underset{\sim}{y})dA_{\underset{\sim}{y}}\quad\text{for all }\underset{\sim}{x}\in\overline{R}\ ,$$

$$\underset{\sim}{\tau}(\underset{\sim}{x})=\int_{\partial R}\overset{\circ}{\underset{\sim}{\tau}}{}^{i}(\underset{\sim}{x},\underset{\sim}{y})t_i(\underset{\sim}{y})dA_{\underset{\sim}{y}}\quad\text{for all }\underset{\sim}{x}\in R\ ,$$

$$\underset{\sim}{\tau}(\underset{\sim}{x})=\underset{\sim}{\psi}^{i}(\underset{\sim}{x})t_i(\underset{\sim}{x})+\int_{\partial R}^{p}\underset{\sim}{\mathring{\tau}}^{i}(\underset{\sim}{x},\underset{\sim}{y})t_i(\underset{\sim}{y})dA_{\underset{\sim}{y}} \quad \underline{\text{for}}\ \underline{\text{all}}\ \underset{\sim}{x}\in\partial R ,$$

<u>where</u> $\underset{\sim}{\psi}^{i}$ <u>is</u> <u>given</u> <u>by</u> (3.9) <u>and</u> <u>the</u> <u>last</u> <u>integral</u> <u>is</u> <u>to</u> <u>be</u> <u>interpreted</u> <u>as</u> <u>in</u> (3.10).

<u>Proof</u>. Let λ, $\overline{S}^{i}(\cdot,\underset{\sim}{y},\lambda)$ be as in Lemma 3.1 and define

$$\left.\begin{aligned}
&\underset{\sim}{u}'(\underset{\sim}{x})=\int_{\partial R}\underset{\sim}{\overline{u}}^{i}(\underset{\sim}{x},\underset{\sim}{y},\lambda)t_i(\underset{\sim}{y})dA_{\underset{\sim}{y}} \quad \text{for all } \underset{\sim}{x}\in\overline{R} ,\\
&\underset{\sim}{\tau}'(\underset{\sim}{x})=\int_{\partial R}\underset{\sim}{\overline{\tau}}^{i}(\underset{\sim}{x},\underset{\sim}{y},\lambda)t_i(\underset{\sim}{y})dA_{\underset{\sim}{y}} \quad \text{for all } \underset{\sim}{x}\in R ,\\
&\underset{\sim}{\tau}'(\underset{\sim}{x})=\underset{\sim}{\psi}^{i}(\underset{\sim}{x})t_i(\underset{\sim}{x})+\int_{\partial R}^{p}\underset{\sim}{\overline{\tau}}^{i}(\underset{\sim}{x},\underset{\sim}{y},\lambda)t_i(\underset{\sim}{y})dA_{\underset{\sim}{y}} \quad \text{for all } \underset{\sim}{x}\in\partial R .
\end{aligned}\right\} \qquad (3.70)$$

These defining equations are meaningful in view of Theorem 3.1 and the assumed regularity of $\underset{\sim}{t}$. Further, Theorem 3.1 furnishes

$$S'=[\underset{\sim}{u}',\underset{\sim}{\tau}']\in\mathcal{E}(\underset{\sim}{0},\mu,\sigma;\overline{R}),\quad \underset{\sim}{t}'=\underset{\sim}{t}+\int_{\partial R}\underset{\sim}{\overline{t}}^{i}(\cdot,\underset{\sim}{y},\lambda)t_i(\underset{\sim}{y})dA_{\underset{\sim}{y}} \quad \text{on } \partial R , \qquad (3.71)$$

where $\underset{\sim}{t}'$ and $\underset{\sim}{\overline{t}}^{i}(\cdot,\underset{\sim}{y},\lambda)$ are the respective surface tractions of S' and $\overline{S}^{i}(\cdot,\underset{\sim}{y},\lambda)$ on ∂R.

Next, introduce $\check{S}^{i}$ through

$$\check{S}^{i}(\underset{\sim}{x},\underset{\sim}{y},\lambda)=\mathring{S}^{i}(\underset{\sim}{x},\underset{\sim}{y})-\overline{S}^{i}(\underset{\sim}{x},\underset{\sim}{y},\lambda) \quad \text{for all } (\underset{\sim}{x},\underset{\sim}{y})\in\overline{R}\times\partial R-D \qquad (3.72)$$

and note on the basis of Lemma 3.1, Definition 3.3, Theorem 3.2, and (3.7) that

$$\left.\begin{aligned}
&\check{S}^i(\cdot,\underset{\sim}{y},\lambda)=[\check{\underset{\sim}{u}}^i(\cdot,\underset{\sim}{y},\lambda),\check{\underset{\sim}{\tau}}^i(\cdot,\underset{\sim}{y},\lambda)]\in\mathcal{E}(\underset{\sim}{0},\mu,\sigma;\overline{R}_{\underset{\sim}{y}}) \text{ for all } \underset{\sim}{y}\in\partial R ,\\
&\check{\underset{\sim}{u}}^i(\cdot,\cdot,\lambda)\in\mathcal{C}(\overline{R}\times\partial R-D),\ \check{\underset{\sim}{\tau}}^i(\cdot,\cdot,\lambda)\in\mathcal{C}(\overline{R}\times\partial R-D),\\
&\nabla\check{\underset{\sim}{u}}^i(\cdot,\cdot,\lambda)\in\mathcal{C}(R\times\partial R),\ \nabla\check{\underset{\sim}{\tau}}^i(\cdot,\cdot,\lambda)\in\mathcal{C}(R\times\partial R),\\
&\check{\underset{\sim}{u}}^i(\underset{\sim}{x},\underset{\sim}{y},\lambda)=O(|\underset{\sim}{x}-\underset{\sim}{y}|^{-\beta}),\ \check{\underset{\sim}{\tau}}^i(\underset{\sim}{x},\underset{\sim}{y},\lambda)=O(|\underset{\sim}{x}-\underset{\sim}{y}|^{-1-\beta}) \text{ as } \underset{\sim}{x}\to\underset{\sim}{y}\ \ (\beta<1/2),
\end{aligned}\right\}\quad(3.73)$$

these estimates being uniform with respect to $\underset{\sim}{y}\in\partial R$. Because of (3.73), the state S'' defined by

$$S''=\int_{\partial R}\check{S}^i(\cdot,\underset{\sim}{y},\lambda)t_i(\underset{\sim}{y})dA_{\underset{\sim}{y}} \text{ on } \overline{R} \qquad (3.74)$$

has the properties

$$S''=[\underset{\sim}{u}'',\underset{\sim}{\tau}'']\in\mathcal{E}(\underset{\sim}{0},\mu,\sigma;\overline{R}),\ \underset{\sim}{t}''=\int_{\partial R}\check{\underset{\sim}{t}}^i(\cdot,\underset{\sim}{y},\lambda)t_i(\underset{\sim}{y})dA_{\underset{\sim}{y}} \text{ on } \partial R . \qquad (3.75)$$

Here $\underset{\sim}{t}''$ and $\check{\underset{\sim}{t}}^i(\cdot,\underset{\sim}{y},\lambda)$ are the tractions of S'' and $\check{S}^i(\cdot,\underset{\sim}{y},\lambda)$ on ∂R.

By virtue of (3.70), (3.72), (3.74) the proof will be complete if we show that

$$\underset{\sim}{u}=\underset{\sim}{u}'+\underset{\sim}{u}'',\quad \underset{\sim}{\tau}=\underset{\sim}{\tau}'+\underset{\sim}{\tau}'' \text{ on } \overline{R}. \qquad (3.76)$$

On the other hand (3.76) is implied by the first of (3.71), the first of (3.75), the present hypotheses on S, (b) in Lemma 3.2, and the uniqueness theorem for the second boundary-value problem of elastostatics, provided

$$\underset{\sim}{t}=\underset{\sim}{t}'+\underset{\sim}{t}'' \text{ on } \partial R,\ \int_{\partial R}(\underset{\sim}{u}'+\underset{\sim}{u}'')\cdot\underset{\sim}{q}^m dA=0\ \ (m=1,\ldots 6). \qquad (3.77)$$

To confirm (3.77) observe from (3.70), (3.71), (3.72), (3.74), (3.75) that

$$\left.\begin{aligned} \underset{\sim}{u}'(\underset{\sim}{x})+\underset{\sim}{u}''(\underset{\sim}{x}) &= \int_{\partial R} \overset{\circ}{\underset{\sim}{u}}{}^{i}(\underset{\sim}{x},\underset{\sim}{y})\, t_i(\underset{\sim}{y})\, dA_{\underset{\sim}{y}} \quad \text{for all } \underset{\sim}{x}\in\overline{R}\,, \\ \underset{\sim}{t}'(\underset{\sim}{x})+\underset{\sim}{t}''(\underset{\sim}{x}) &= \underset{\sim}{t}(\underset{\sim}{x}) + \int_{\partial R} \overset{\circ}{\underset{\sim}{t}}{}^{i}(\underset{\sim}{x},\underset{\sim}{y})\, t_i(\underset{\sim}{y})\, dA_{\underset{\sim}{y}} \quad \text{for all } \underset{\sim}{x}\in\partial R\,. \end{aligned}\right\} \quad (3.78)$$

Finally use (c), (d) in Definition 3.3, together with (a) in Lemma 3.2 and the self-equilibration of the tractions $\underset{\sim}{t}$ of S on ∂R, to see that (3.78) implies (3.77).

The preceding theorem, which constituted the main objective of this section, will be used in what follows for a limit treatment of concentrated surface loads. In view of the rather elaborate developments that were required to arrive at this theorem, it should be emphasized once more that the integral representation of elastic states deduced here — though confined to simple regions — is essentially stronger than the representation in terms of Green's states[1], which is not applicable to points on the boundary of the region. The usefulness of the present representation, which is free from this deficiency, transcends the particular purpose for which it was derived. Thus, for example, Theorem 3.3 supplies also a convenient tool for the study of singularities induced by discontinuous surface loads, which are beyond the scope of this investigation.

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[1] Cf. Theorems 6.1, 6.2 in [2], as well as our Section 6.

4. Limit treatment of concentrated surface loads.

The present section contains a counterpart for concentrated surface loads of the limit treatment of Kelvin's problem in Section 2. Thus, we first define the solution to a problem involving concentrated surface loads through an appropriate limit process and subsequently examine the nature of the singularities inherent in the solution so defined. In carrying out this task we shall confine our attention exclusively to simple regions and to a single concentrated load that is equilibrated by regular surface tractions, in the absence of body forces. The generalization of what follows to any finite number of concentrated surface loads and to non-vanishing body forces is entirely elementary. Further, the extension of most of the results deduced in this section to the broader class of regular regions presents no essential difficulties, provided the point of application of the given concentrated surface load lies within a sufficiently smooth subset of the boundary.

With a view toward our present objective we first introduce

Definition 4.1. (Sequence of traction fields tending to a concentrated surface load and to regular surface tractions). Let R be a simple region and $\underset{\sim}{a} \in \partial R$. Let $\underset{\sim}{\ell} \neq \underset{\sim}{0}$ be a vector and $\overset{*}{\underset{\sim}{t}} \in \mathscr{C}(\partial R)$. We say that $\{\underset{\sim}{t}^m\}$ is a sequence of traction fields on ∂R tending to a concentrated load $\underset{\sim}{\ell}$ at (the point) $\underset{\sim}{a}$ and tractions $\overset{*}{\underset{\sim}{t}}$ on ∂R if:

(a) $\underset{\sim}{t}^m = \overset{*}{\underset{\sim}{t}} + \hat{\underset{\sim}{t}}^m$ on ∂R, $\hat{\underset{\sim}{t}}^m \in \mathscr{C}(\partial R)$ $(m=1,2,3,\ldots)$;

(b) $\hat{\underset{\sim}{t}}^m = \underset{\sim}{0}$ on $\partial R - B_{\rho_m}(\underset{\sim}{a})$ $(m=1,2,3,\ldots)$, where $\{B_{\rho_m}(\underset{\sim}{a})\}$ is a sequence of spheres such that $\rho_m \to 0$ as $m \to \infty$;

(c) $\lim_{m\to\infty} \int_{\partial R} \hat{\underset{\sim}{t}}^m dA = \underset{\sim}{\ell}$;

(d) the sequence $\left\{ \int_{\partial R} |\hat{\underset{\sim}{t}}^m| dA \right\}$ is bounded ;

(e) $\int_{\partial R} \underset{\sim}{t}^m dA = \int_{\partial R} \underset{\sim}{x} \wedge \underset{\sim}{t}^m dA = \underset{\sim}{0}$ $(m=1, 2, 3, \dots)$.

In connection with the foregoing definition, which is an analogue of Definition 2.1, it is natural to ask whether an approximating sequence of traction fields $\{\underset{\sim}{t}^m\}$ exists for given $\underset{\sim}{t}$ and $\underset{\sim}{\ell}$. This question is answered by

Theorem 4.1. (Existence of a sequence of traction fields tending to a concentrated surface load and to regular surface tractions). Let R, $\underset{\sim}{a}$, $\underset{\sim}{\ell}$ and $\overset{*}{\underset{\sim}{t}}$ be as in Definition 4.1. Then necessary and sufficient for the existence of a sequence of traction fields on ∂R tending to a concentrated load $\underset{\sim}{\ell}$ at $\underset{\sim}{a}$ and tractions $\overset{*}{\underset{\sim}{t}}$ on ∂R is that the entire given loading be self-equilibrated, i.e.,

$$\int_{\partial R} \overset{*}{\underset{\sim}{t}}\, dA + \underset{\sim}{\ell} = \underset{\sim}{0}\,, \int_{\partial R} \underset{\sim}{x} \wedge \overset{*}{\underset{\sim}{t}}\, dA + \underset{\sim}{a} \wedge \underset{\sim}{\ell} = \underset{\sim}{0}\,. \tag{4.1}$$

Proof. To confirm the necessity of (4.1), observe first from (a) and (e) in Definition 4.1 that

$$\int_{\partial R} \overset{*}{\underset{\sim}{t}}\, dA + \int_{\partial R} \hat{\underset{\sim}{t}}^m dA = \underset{\sim}{0}\,, \quad \int_{\partial R} \underset{\sim}{x} \wedge \overset{*}{\underset{\sim}{t}}\, dA + \underset{\sim}{a} \wedge \int_{\partial R} \hat{\underset{\sim}{t}}^m dA + \int_{\partial R} (\underset{\sim}{x} - \underset{\sim}{a}) \wedge \hat{\underset{\sim}{t}}^m dA = \underset{\sim}{0} \quad (m=1, 2, 3, \dots)\,. \tag{4.2}$$

Now let $m \to \infty$, use (c) in Definition 4.1, and note that because of (b) and (d), the last integral in (4.2) tends to zero. Thus, (4.1) holds.

To establish the sufficiency of (4.1), one merely needs to exhibit a sequence $\{\hat{\underset{\sim}{t}}^m\}$ such that

$$\hat{\underset{\sim}{t}}^m \in \mathscr{K}(\partial R),\ \hat{\underset{\sim}{t}}^m = \underset{\sim}{0} \text{ on } \partial R - B_{\rho_m}(\underset{\sim}{a}) \quad (m=1,2,3,\ldots), \tag{4.3}$$

with $\{\rho_m\}$ a null sequence, and

$$\int_{\partial R} \hat{\underset{\sim}{t}}^m dA = \underset{\sim}{\ell},\ \int_{\partial R} (\underset{\sim}{x} - \underset{\sim}{a}) \wedge \hat{\underset{\sim}{t}}^m dA = \underset{\sim}{0},\ \int_{\partial R} |\hat{\underset{\sim}{t}}^m| dA < k \quad (m=1,2,3,\ldots), \tag{4.4}$$

where k is a constant.

Without loss of generality, assume henceforth that $\underset{\sim}{\ell}$ is a <u>unit</u> vector. Suppose first that $\underset{\sim}{\ell}$ is not tangential to ∂R at $\underset{\sim}{a}$, so that

$$\underset{\sim}{\ell} \cdot \underset{\sim}{n}(\underset{\sim}{a}) \neq 0 , \tag{4.5}$$

where $\underset{\sim}{n}(\underset{\sim}{a})$ is the unit outer normal to ∂R at $\underset{\sim}{a}$. Choose a rectangular cartesian frame with the origin at $\underset{\sim}{a}$, such that the x_3-axis points in the direction of $\underset{\sim}{n}(\underset{\sim}{0})$ while the x_1-axis is perpendicular to $\underset{\sim}{\ell}$. Further, consider the cylinder

$$F(\delta) = \{\underset{\sim}{x} \mid \underset{\sim}{x} \in E,\ x_\alpha x_\alpha \leq \delta^2,\ |x_3| \leq \delta\} .$$

It then follows from (1.3), (1.4) and the present hypotheses on R that for some $\lambda > 0$,

$$\Omega(\underset{\sim}{0}, \lambda) = F(\lambda) \cap \partial R = \{\underset{\sim}{x} \mid \underset{\sim}{x} \in F(\lambda),\ \theta(\underset{\sim}{x}) = 0\} , \tag{4.6}$$

where

$$\left.\begin{aligned} &\theta(\underset{\sim}{x}) = x_3 - \varphi(x_1, x_2) \text{ for all } \underset{\sim}{x} \in F(\lambda) , \\ &\varphi \in C^2(\Pi(\underset{\sim}{0}, \lambda)),\ \varphi(0,0) = \varphi_{,\alpha}(0,0) = 0 . \end{aligned}\right\} \tag{4.7}$$

Next, introduce cartesian coordinates x_i' through a rotation about the x_1-axis that brings the x_3'-axis to coincidence with $\underset{\sim}{\ell}$, i.e.,

$$x_1'=x_1\ ,\ x_2'=\ell_3x_2-\ell_2x_3\ ,\ x_3'=\ell_2x_2+\ell_3x_3\ . \tag{4.8}$$

If ℓ_i' denotes the components of $\underset{\sim}{\ell}$ in this new frame, evidently

$$\ell_1'=\ell_2'=0\ ,\ \ \ell_3'=1\ . \tag{4.9}$$

Equations (4.7), (4.8) now yield

$$\theta(\underset{\sim}{x})=\theta'(x_1', x_2', x_3')=-\ell_2x_2'+\ell_3x_3'-\varphi(x_1', \ell_3x_2'+\ell_2x_3') \text{ for all } \underset{\sim}{x}\in F(\lambda), \tag{4.10}$$

whence

$$\frac{\partial\theta'}{\partial x_3'}\Bigg|_{\underset{\sim}{x}=\underset{\sim}{0}}=\ell_3=\underset{\sim}{\ell}\cdot\underset{\sim}{n}(\underset{\sim}{0})\neq 0\ , \tag{4.11}$$

since $\underset{\sim}{\ell}$ is at present non-tangential to ∂R. Thus, adopting the notation

$$\left.\begin{aligned} &F'(\delta, h)=\{\underset{\sim}{x}\,|\,\underset{\sim}{x}\in E,\ x_\alpha'x_\alpha'\leq\delta^2,\ |x_3'|\leq h\}\ ,\\ &\Pi'(\underset{\sim}{0}, \delta)=\{\underset{\sim}{x}\,|\,\underset{\sim}{x}\in E,\ x_\alpha'x_\alpha'\leq\delta^2,\ x_3'=0\}\ , \end{aligned}\right\} \tag{4.12}$$

one concludes from (4.6), (4.7), (4.10, (4.11) and the implicit-function theorem the existence of $\nu>0$ and of $\varphi'\in C^2(\Pi'(\underset{\sim}{0}, \lambda))$ such that

$$F'(\nu, \nu)\cap\partial R=\{\underset{\sim}{x}\,|\,\underset{\sim}{x}\in F'(\nu, \nu)\ ,\ x_3'=\varphi'(x_1', x_2')\}\ . \tag{4.13}$$

Now define

$$\underset{\sim}{\hat{t}}^m(\underset{\sim}{x})=\begin{cases}\dfrac{3m^6}{\pi\nu^6}\left(\dfrac{\nu^2}{m^2}-x_1'^2-x_2'^2\right)^2|n_3'(\underset{\sim}{x})|\underset{\sim}{\ell} \text{ for all } \underset{\sim}{x}\in\partial R\cap F'(\frac{\nu}{m}, \nu)\\[2ex] \underset{\sim}{0} \text{ for all } \underset{\sim}{x}\in\partial R-F'(\frac{\nu}{m}, \nu)\quad (m=1, 2, 3, \ldots)\ .\end{cases} \tag{4.14}$$

The sequence $\{\underset{\sim}{\hat{t}}^m\}$ so constructed clearly conforms to (4.3). Further, because of (4.13), (4.14),

$$\int_{\partial R} \hat{\underset{\sim}{t}}^m dA = \frac{3m^6}{\pi \nu^6} \underset{\sim}{\ell} \int_{\Pi'(\underset{\sim}{0}, \frac{\nu}{m})} \left(\frac{\nu^2}{m^2} - x_1'^2 - x_2'^2 \right)^2 dA = \underset{\sim}{\ell} \quad (m=1,2,3,\ldots),$$

and similarly from (4.9), (4.13), (4.14),

$$\int_{\partial R} (\underset{\sim}{x} - \underset{\sim}{a}) \wedge \hat{\underset{\sim}{t}}^m dA = \underset{\sim}{0}, \quad \int_{\partial R} |\hat{\underset{\sim}{t}}^m| dA = |\underset{\sim}{\ell}| = 1 \quad (m=1,2,3,\ldots).$$

This completes the proof provided (4.5) holds. If, finally, $\underset{\sim}{\ell}$ is tangential to ∂R at $\underset{\sim}{a}$, so that $\underset{\sim}{\ell} \cdot \underset{\sim}{n}(\underset{\sim}{a}) = 0$, consider the unit vectors

$$\underset{\sim}{\ell}_1 = \underset{\sim}{n}(\underset{\sim}{a}), \quad \underset{\sim}{\ell}_2 = \frac{\underset{\sim}{\ell} - \underset{\sim}{n}(\underset{\sim}{a})}{|\underset{\sim}{\ell} - \underset{\sim}{n}(\underset{\sim}{a})|},$$

which are not tangential to ∂R. Then, there are sequences $\{\hat{\underset{\sim}{t}}_1^m\}$, $\{\hat{\underset{\sim}{t}}_2^m\}$ satisfying (4.3), (4.4) with $\underset{\sim}{\ell}$ replaced by $\underset{\sim}{\ell}_1$, $\underset{\sim}{\ell}_2$, respectively. Hence, the sequence $\{\hat{\underset{\sim}{t}}^m\}$ defined by

$$\hat{\underset{\sim}{t}}^m = \hat{\underset{\sim}{t}}_1^m + |\underset{\sim}{\ell} - \underset{\sim}{n}(\underset{\sim}{a})| \hat{\underset{\sim}{t}}_2^m \text{ on } \partial R \quad (m=1,2,3,\ldots)$$

fulfills requirements (4.3) and (4.4). The proof is now complete.

It is clear from the foregoing theorem that Definition 4.1 is empty unless the self-equilibration relations (4.1) hold true. On the other hand, (4.1) in conjunction with (a) and (e) in Definition 4.1 imply

$$\int_{\partial R} \hat{\underset{\sim}{t}}^m dA = \underset{\sim}{\ell} \quad (m=1,2,3,\ldots), \tag{4.15}$$

which is stronger than (c) in this definition.

The following theorem supplies a definition through a limit process, and at the same time a representation in terms of the load data, of the solution to a problem corresponding to a given concentrated load that is equilibrated by preassigned regular surface tractions. In analogy with Theorem 2.1 one has

Theorem 4.2. (Limit definition of the solution to a problem involving a concentrated surface load). Let R be a simple region and $\underset{\sim}{a}\in\partial R$. Let $\underset{\sim}{\ell}\neq\underset{\sim}{0}$ be a vector and $\overset{*}{\underset{\sim}{t}}\in\mathcal{X}(\partial R)$. Assume

$$\int_{\partial R}\overset{*}{\underset{\sim}{t}}\,dA+\underset{\sim}{\ell}=\underset{\sim}{0}\,,\quad \int_{\partial R}\underset{\sim}{x}\wedge\overset{*}{\underset{\sim}{t}}\,dA+\underset{\sim}{a}\wedge\underset{\sim}{\ell}=\underset{\sim}{0}\,.$$

Further, let $\{\underset{\sim}{t}^m\}$ be a sequence of traction fields on ∂R tending to a concentrated load $\underset{\sim}{\ell}$ at $\underset{\sim}{a}$ and tractions $\overset{*}{\underset{\sim}{t}}$ on ∂R. Then:

(a) there exists a unique sequence of states $\{S^m\}$ such that $S^m=[\underset{\sim}{u}^m,\underset{\sim}{\tau}^m]\in\mathcal{E}(\underset{\sim}{0},\mu,\sigma;\overline{R})$, $\tau^m_{ij}n_j=t^m_i$ on ∂R $(m=1,2,3,\ldots)$,

$$\int_{\partial R}\underset{\sim}{u}^m\cdot\underset{\sim}{q}^p\,dA=0\quad(m=1,2,3,\ldots,\ p=1,\ldots,6)\,,$$

with $\underset{\sim}{q}^p$ given by Definition 3.2;

(b) $\{S^m\}$ converges to a state $S=[\underset{\sim}{u},\underset{\sim}{\tau}]$ on $\overline{R}_{\underset{\sim}{a}}$, the convergence being uniform on any closed subset of $\overline{R}_{\underset{\sim}{a}}$;

(c) the limit state S is independent of the sequence $\{\underset{\sim}{t}^m\}$ and admits the representation

$$\left.\begin{aligned}
\underset{\sim}{u}(\underset{\sim}{x})&=\overset{\circ}{\underset{\sim}{u}}{}^i(\underset{\sim}{x},\underset{\sim}{a})\ell_i+\int_{\partial R}\overset{\circ}{\underset{\sim}{u}}{}^i(\underset{\sim}{x},\underset{\sim}{y})\overset{*}{t}_i(\underset{\sim}{y})\,dA_{\underset{\sim}{y}}\quad\text{for all } \underset{\sim}{x}\in\overline{R}_{\underset{\sim}{a}}\,,\\
\underset{\sim}{\tau}(\underset{\sim}{x})&=\overset{\circ}{\underset{\sim}{\tau}}{}^i(\underset{\sim}{x},\underset{\sim}{a})\ell_i+\int_{\partial R}\overset{\circ}{\underset{\sim}{\tau}}{}^i(\underset{\sim}{x},\underset{\sim}{y})\overset{*}{t}_i(\underset{\sim}{y})\,dA_{\underset{\sim}{y}}\quad\text{for all } \underset{\sim}{x}\in R\,,
\end{aligned}\right\}\quad(4.16)$$

$$\tau(x)=\overset{\circ}{\tau}{}^i(x,a)\ell_i+\psi^i(x)\overset{*}{t}_i(x)+\overset{p}{\int_{\partial R}}\overset{\circ}{\tau}{}^i(x,y)\overset{*}{t}_i(y)dA_y \quad \underline{\text{for all}}\ x\in\partial R_a\ , \qquad \begin{matrix}(4.16)\\ (\text{Cont.})\end{matrix}$$

where $\overset{\circ}{S}{}^i(\cdot,y)$ is the modified tangent state for the region R at y corresponding to the x_i-direction and the elastic constants μ, σ, in the sense of Definition 3.3, while ψ^i is given by (3.9) and the last integral in (4.16) is to be interpreted as in (3.10).

We call S the state corresponding to a concentrated surface load ℓ at a and tractions $\overset{*}{t}$ on ∂R (as well as to the elastic constants μ, σ).

Proof. Conclusion (a) follows immediately from the present hypotheses, in view of (a), (e) in Definition 4.1, with the aid of Korn's [14] existence theorem, (b) in Lemma 3.2, and the uniqueness theorem[1] for the second boundary-value problem of elastostatics.

To reach the remaining conclusions, note first from Theorem 3.3 that S^m $(m=1,2,3,\ldots)$ admits the representation

$$\left.\begin{aligned}
u^m(x)&=\int_{\partial R}\overset{\circ}{u}{}^i(x,y)t_i^m(y)dA_y \quad \text{for all } x\in\overline{R}\ ,\\
\tau^m(x)&=\int_{\partial R}\overset{\circ}{\tau}{}^i(x,y)t_i^m(y)dA_y \quad \text{for all } x\in R\ ,\\
\tau^m(x)&=\psi^i(x)t_i^m(x)+\overset{p}{\int_{\partial R}}\overset{\circ}{\tau}{}^i(x,y)t_i^m(y)dA_y \quad \text{for all } x\in\partial R\ .
\end{aligned}\right\} \qquad (4.17)$$

Now define u, τ through (4.16). Then, by virtue of (4.17) and (a) in

[1] Recall that u^m has been normalized so as to preclude an arbitrary additive rigid displacement field.

Definition 4.1,

$$\left.\begin{aligned}
&\underset{\sim}{u}^{m}(\underset{\sim}{x})-\underset{\sim}{u}(\underset{\sim}{x})=\int_{\partial R}\overset{\circ}{\underset{\sim}{u}}{}^{i}(\underset{\sim}{x},\underset{\sim}{y})\hat{t}_{i}^{m}(\underset{\sim}{y})dA_{\underset{\sim}{y}}-\overset{\circ}{\underset{\sim}{u}}{}^{i}(\underset{\sim}{x},\underset{\sim}{a})\ell_{i} \text{ for all } \underset{\sim}{x}\in\overline{R}_{\underset{\sim}{a}},\\
&\underset{\sim}{\tau}^{m}(\underset{\sim}{x})-\underset{\sim}{\tau}(\underset{\sim}{x})=\int_{\partial R}\overset{\circ}{\underset{\sim}{\tau}}{}^{i}(\underset{\sim}{x},\underset{\sim}{y})\hat{t}_{i}^{m}(\underset{\sim}{y})dA_{\underset{\sim}{y}}-\overset{\circ}{\underset{\sim}{\tau}}{}^{i}(\underset{\sim}{x},\underset{\sim}{a})\ell_{i} \text{ for all } \underset{\sim}{x}\in R,\\
&\underset{\sim}{\tau}^{m}(\underset{\sim}{x})-\underset{\sim}{\tau}(\underset{\sim}{x})=\underset{\sim}{\psi}^{i}(\underset{\sim}{x})\hat{t}_{i}^{m}(\underset{\sim}{x})+\overset{p}{\int_{\partial R}}\overset{\circ}{\underset{\sim}{\tau}}{}^{i}(\underset{\sim}{x},\underset{\sim}{y})\hat{t}_{i}^{m}(\underset{\sim}{y})dA_{\underset{\sim}{y}}-\overset{\circ}{\underset{\sim}{\tau}}{}^{i}(\underset{\sim}{x},\underset{\sim}{a})\ell_{i} \text{ for all } \underset{\sim}{x}\in\partial R_{\underset{\sim}{a}}.
\end{aligned}\right\}\quad(4.18)$$

To complete the argument it remains to be shown that the left-hand members in (4.18) tend to zero uniformly on any closed subset of $\overline{R}_{\underset{\sim}{a}}$. Let G be such a set and let $\{\rho_m\}$ be the null sequence of radii associated with $\{\underset{\sim}{t}^m\}$ in the sense of (b) in Definition 4.1. Further, let m_o be such that $\overline{B}_{\rho_m}(\underset{\sim}{a})\cap G$ is empty whenever $m>m_o$. Then, because of (4.18), (4.15) and (b) in Definition 4.1,

$$\underset{\sim}{u}^{m}(\underset{\sim}{x})-\underset{\sim}{u}(\underset{\sim}{x})=\int_{\partial R\cap B_{\rho_m}(\underset{\sim}{a})}[\overset{\circ}{\underset{\sim}{u}}{}^{i}(\underset{\sim}{x},\underset{\sim}{y})-\overset{\circ}{\underset{\sim}{u}}{}^{i}(\underset{\sim}{x},\underset{\sim}{a})]\,\hat{t}_{i}^{m}(\underset{\sim}{y})dA_{\underset{\sim}{y}},$$

$$\underset{\sim}{\tau}^{m}(\underset{\sim}{x})-\underset{\sim}{\tau}(\underset{\sim}{x})=\int_{\partial R\cap B_{\rho_m}(\underset{\sim}{a})}[\overset{\circ}{\underset{\sim}{\tau}}{}^{i}(\underset{\sim}{x},\underset{\sim}{y})-\overset{\circ}{\underset{\sim}{\tau}}{}^{i}(\underset{\sim}{x},\underset{\sim}{a})]\,\hat{t}_{i}^{m}(\underset{\sim}{y})dA_{\underset{\sim}{y}},$$

for all $\underset{\sim}{x}\in G$ and every $m>m_o$. Hence,

$$\left.\begin{aligned}
&|\underset{\sim}{u}^{m}(\underset{\sim}{x})-\underset{\sim}{u}(\underset{\sim}{x})|\leq k_{i}^{m}\int_{\partial R}|\hat{t}_{i}^{m}|dA \text{ for all } \underset{\sim}{x}\in G,\ m>m_o,\\
&|\underset{\sim}{\tau}^{m}(\underset{\sim}{x})-\underset{\sim}{\tau}(\underset{\sim}{x})|\leq \varkappa_{i}^{m}\int_{\partial R}|\hat{t}_{i}^{m}|dA \text{ for all } \underset{\sim}{x}\in G,\ m>m_o,
\end{aligned}\right\}\quad(4.19)$$

where

$$k_i^m = \max|\overset{\circ}{u}^i(x, y) - \overset{\circ}{u}^i(x, a)|, \quad x \in G, \ y \in \partial R \cap \overline{B}_{\rho_m}(a),$$

$$\varkappa_i^m = \max|\overset{\circ}{\tau}^i(x, y) - \overset{\circ}{\tau}^i(x, a)|, \quad x \in G, \ y \in \partial R \cap \overline{B}_{\rho_m}(a).$$

Observe that the existence of these maxima is assured by the first continuity assertion in (a) of Theorem 3.2; for the same reason and since $\{\rho_m\}$ is a null sequence, k_i^m and $\varkappa_i^m$ tend to zero as $m \to \infty$. The desired conclusion thus follows at once from (4.19) and (d) in Definition 4.1. This completes the proof.

The next theorem is an analogue for the problem under present consideration of Theorem 2.2 on properties of the Kelvin state.

Theorem 4.3. (*Properties of the state corresponding to a concentrated surface load and to regular surface tractions*). *Let* R, a, ℓ, $\overset{*}{t}$ *and* S *be as in Theorem* 4.2. *Then* S *has the properties:*

(a) $S = [u, \tau] \in \mathcal{E}(0, \mu, \sigma; \overline{R}_a)$;

(b) $u(x) = O(|x - a|^{-1})$, $\tau(x) = O(|x - a|^{-2})$ *as* $x \to a$;

(c) *the tractions of* S *on* ∂R_a *coincide with* $\overset{*}{t}$;

(d) $\lim_{\rho \to 0} \int_{R \cap \partial B_\rho(a)} t \, dA = \ell$, $\lim_{\rho \to 0} \int_{R \cap \partial B_\rho(a)} (x - a) \wedge t \, dA = 0$,

where t *are the tractions of* S *on the side of* $R \cap \partial B_\rho(a)$ *that faces* a.

Proof. Define u' and τ' on $\overline{R}$ through

$$\left.\begin{aligned}
&\underset{\sim}{u}'(\underset{\sim}{x})=\int_{\partial R}\overset{\circ}{\underset{\sim}{u}}{}^{i}(\underset{\sim}{x},\underset{\sim}{y})\overset{*}{t}_i(\underset{\sim}{y})dA_{\underset{\sim}{y}} \quad \text{for all } \underset{\sim}{x}\in\bar{R}\ ,\\
&\underset{\sim}{\tau}'(\underset{\sim}{x})=\int_{\partial R}\overset{\circ}{\underset{\sim}{\tau}}{}^{i}(\underset{\sim}{x},\underset{\sim}{y})\overset{*}{t}_i(\underset{\sim}{y})dA_{\underset{\sim}{y}} \quad \text{for all } \underset{\sim}{x}\in R\ ,\\
&\underset{\sim}{\tau}'(\underset{\sim}{x})=\underset{\sim}{\psi}^{i}(\underset{\sim}{x})\overset{*}{t}_i(\underset{\sim}{x})+\int_{\partial R}^{p}\overset{\circ}{\underset{\sim}{\tau}}{}^{i}(\underset{\sim}{x},\underset{\sim}{y})\overset{*}{t}_i(\underset{\sim}{y})dA_{\underset{\sim}{y}} \quad \text{for all } \underset{\sim}{x}\in\partial R\ ,
\end{aligned}\right\} \tag{4.20}$$

where $\overset{\circ}{S}{}^{i}(\cdot,\underset{\sim}{y})$ is once again the modified tangent state of Definition 3.3 and $\underset{\sim}{\psi}^{i}$ is given by (3.9). An elementary modification of the argument employed in the proof of Theorem 3.3 yields

$$S'=[\underset{\sim}{u}',\underset{\sim}{\tau}']\in\mathcal{E}(\underset{\sim}{0},\mu,\sigma;\bar{R})\ . \tag{4.21}$$

Further, from (4.16), (4.20),

$$S=S'+\overset{\circ}{S}{}^{i}(\cdot,\underset{\sim}{a})\ell_i \text{ on } \bar{R}_{\underset{\sim}{a}}\ . \tag{4.22}$$

Conclusions (a) and (b) now follow directly from (4.22), (4.21), (a) and (b) in Definition 3.3, (3.7), and (b) in Lemma 3.1.

Turning to (c), note first from (4.22), the last of (4.20), and (3.9), (1.9) that

$$\underset{\sim}{t}=\overset{*}{\underset{\sim}{t}}+\int_{\partial R}\overset{\circ}{\underset{\sim}{t}}{}^{i}(\cdot,\underset{\sim}{y})\overset{*}{t}_i(\underset{\sim}{y})dA_{\underset{\sim}{y}}+\overset{\circ}{\underset{\sim}{t}}{}^{i}(\cdot,\underset{\sim}{a})\ell_i \text{ on } \partial R_{\underset{\sim}{a}}\ , \tag{4.23}$$

where $\underset{\sim}{t}$ is the traction vector of S on $\partial R_{\underset{\sim}{a}}$ and $\overset{\circ}{\underset{\sim}{t}}{}^{i}$ is given by (c) in Definition 3.3. Observe that the integral in (4.23) is proper. Next, (4.1) and (a) in Lemma 3.2 furnish

$$\int_{\partial R}\underset{\sim}{q}^{m}(\underset{\sim}{y})\cdot\overset{*}{\underset{\sim}{t}}(\underset{\sim}{y})dA_{\underset{\sim}{y}}+\underset{\sim}{q}^{m}(\underset{\sim}{a})\cdot\underset{\sim}{\ell}=0 \quad (m=1,\ldots,6)\ . \tag{4.24}$$

Equations (4.23) and (4.24), together with (c) in Definition 3.3, imply conclusion (c).

Finally, turn to conclusions (d). As a consequence of conclusion (a), the tractions $\underset{\sim}{t}$ of S are self-equilibrated on the boundary of the region $R - B_\rho(\underset{\sim}{a})$ for all sufficiently small $\rho > 0$. Thus,

$$\lim_{\rho \to 0} \int_{R \cap \partial B_\rho(\underset{\sim}{a})} \underset{\sim}{t} dA + \int_{\partial R} \underset{\sim}{t} dA = \underset{\sim}{0} . \tag{4.25}$$

Also, because of (4.1) and conclusion (c),

$$\int_{\partial R} \underset{\sim}{t} dA + \underset{\sim}{\ell} = \underset{\sim}{0} . \tag{4.26}$$

Combining (4.25) with (4.26) one obtains the first of (d). The second of conclusions (d) is immediate from the second of (b), so that the argument is complete.

It will become apparent through specialization of a general uniqueness theorem for concentrated-load problems established in the next section that properties (a), (b), (c) together with the first of (d) suffice to characterize the limit state S uniquely (except for an additive rigid displacement) and therefore furnish an alternative definition of S.

The orders of the displacement and stress singularities at the point of application of the concentrated surface load are given by (b) in Theorem 4.3 regardless of the particular shape of the boundary. We emphasize, however, that the detailed structure of these singularities depends upon the specific geometry of the boundary in a neighborhood of the load point, as is apparent from the results in [1], [16], [17].

5. Alternative characterization and uniqueness of solutions to concentrated load problems.

We have so far defined the solution to a problem involving concentrated loads through a limit process applied to a sequence of solutions corresponding to regular loadings. We then examined the solution generated by means of the foregoing limit process and in particular determined the orders of the ensuing displacement and stress singularities, as well as the stress resultants of the latter. This program was carried out for internal concentrated loads in connection with Kelvin's problem in Section 2 (Theorems 2.1, 2.2); the analogous results for concentrated surface loads were obtained in Section 4 (Theorems 4.2, 4.3) with limitation to simple regions.

The results to which we have just referred provide the motivation for an alternative formulation of concentrated-load problems. This direct formulation rests on the a priori specification of the concentrated-load singularities as to their orders and stress resultants, in addition to the assignment of the regular body forces and surface tractions.

In this section we seek to establish the completeness of such an alternative formulation of concentrated-load problems through an appropriate uniqueness theorem. For this purpose we first extend Betti's reciprocal theorem to elastic states with singularities of the type arising in the limit treatment of concentrated loads. The generalized reciprocal theorem, which is of interest on its own merits, greatly facilitates the proof of the uniqueness theorem that is our primary objective.

All of the results given in the present section are applicable to arbitrary regular regions, in contrast to those in Section 4, which are confined to simple regions.

Theorem 5.1. (Generalization of the reciprocal theorem to a class of singular elastic states). Let R be a regular region. Let

$$P' = \{\underset{\sim}{a}'_1, \ldots, \underset{\sim}{a}'_{k'}\}, \; P'' = \{\underset{\sim}{a}''_1, \ldots, \underset{\sim}{a}''_{k''}\},$$

be two sets of points in $\overline{R}$ which have no elements in common and consist of k' and k'' distinct (interior or boundary) points. Further, let S', S'' be two states with the properties:

(a) $S' = [\underset{\sim}{u}', \underset{\sim}{\tau}'] \in \mathcal{E}(\underset{\sim}{f}', \mu, \sigma; \overline{R} - P')$, $\underset{\sim}{f}' \in \mathcal{C}(\overline{R})$,

$S'' = [\underset{\sim}{u}'', \underset{\sim}{\tau}''] \in \mathcal{E}(\underset{\sim}{f}'', \mu, \sigma; \overline{R} - P'')$, $\underset{\sim}{f}'' \in \mathcal{C}(\overline{R})$;

(b) $\underset{\sim}{u}'(\underset{\sim}{x}) = O(|\underset{\sim}{x} - \underset{\sim}{a}'_m|^{-1})$, $\underset{\sim}{\tau}'(\underset{\sim}{x}) = O(|\underset{\sim}{x} - \underset{\sim}{a}'_m|^{-2})$ as $\underset{\sim}{x} \to \underset{\sim}{a}'_m$ $(m = 1, \ldots, k')$,

$\underset{\sim}{u}''(\underset{\sim}{x}) = O(|\underset{\sim}{x} - \underset{\sim}{a}''_m|^{-1})$, $\underset{\sim}{\tau}''(\underset{\sim}{x}) = O(|\underset{\sim}{x} - \underset{\sim}{a}''_m|^{-2})$ as $\underset{\sim}{x} \to \underset{\sim}{a}''_m$ $(m = 1, \ldots, k'')$;

(c) $\lim_{\rho \to 0} \int_{\Lambda'_m(\rho)} \underset{\sim}{t}' \, dA = \underset{\sim}{\ell}'_m$ $(m = 1, \ldots, k')$, $\lim_{\rho \to 0} \int_{\Lambda''_m(\rho)} \underset{\sim}{t}'' \, dA = \underset{\sim}{\ell}''_m$ $(m = 1, \ldots, k'')$,

where

$$\Lambda'_m(\rho) = R \cap \partial B_\rho(\underset{\sim}{a}'_m) \; (m = 1, \ldots k'), \quad \Lambda''_m(\rho) = R \cap \partial B_\rho(\underset{\sim}{a}''_m) \; (m = 1, \ldots, k''),$$

while $\underset{\sim}{t}'$ are the tractions of S' on the side of $\Lambda'_m(\rho)$ that faces $\underset{\sim}{a}'_m$, and $\underset{\sim}{t}''$ is defined analogously;

(d) the tractions of S' and S'' on ∂R are integrable.

Then

$$\sum_{m=1}^{k'} \underset{\sim}{\ell}'_m \cdot \underset{\sim}{u}''(\underset{\sim}{a}'_m) + \int_{\partial R} \underset{\sim}{t}' \cdot \underset{\sim}{u}'' dA + \int_R \underset{\sim}{f}' \cdot \underset{\sim}{u}'' dV$$

$$= \sum_{m=1}^{k''} \underset{\sim}{\ell}''_m \cdot \underset{\sim}{u}'(\underset{\sim}{a}''_m) + \int_{\partial R} \underset{\sim}{t}'' \cdot \underset{\sim}{u}' dA + \int_R \underset{\sim}{f}'' \cdot \underset{\sim}{u}' dV , \tag{5.1}$$

__if__ $\underset{\sim}{t}'$, $\underset{\sim}{t}''$ __here denote the tractions of__ S', S'' __on__ ∂R.

__Proof.__ Let $\rho_o > 0$ be such that any two spheres (balls) of radius ρ_o centered at points of $P' \cup P''$ are disjoint. Define

$$R(\rho) = R - \bigcup_{m=1}^{k'} \overline{B}_\rho(\underset{\sim}{a}'_m) - \bigcup_{m=1}^{k''} \overline{B}_\rho(\underset{\sim}{a}''_m) \quad (0<\rho<\rho_o) , \tag{5.2}$$

and let $\rho_1 \in (0, \rho_o)$ be such that $R(\rho)$ is a regular region whenever $0<\rho<\rho_1$. Applying Betti's reciprocal theorem to the pair of elastic states S', S'' on $R(\rho)$ $(0<\rho<\rho_1)$ one has[1]

$$\int_{\partial R(\rho)} \underset{\sim}{t}' \cdot \underset{\sim}{u}'' dA + \int_{R(\rho)} \underset{\sim}{f}' \cdot \underset{\sim}{u}'' dV = \int_{\partial R(\rho)} \underset{\sim}{t}'' \cdot \underset{\sim}{u}' dA + \int_{R(\rho)} \underset{\sim}{f}'' \cdot \underset{\sim}{u}' dV \quad (0<\rho<\rho_1) . \tag{5.3}$$

Next, hypothesis (c) implies

$$\int_{\Lambda'_m(\rho)} \underset{\sim}{t}'(\underset{\sim}{x}) \cdot \underset{\sim}{u}''(\underset{\sim}{x}) dA_{\underset{\sim}{x}} = \int_{\Lambda'_m(\rho)} \underset{\sim}{t}'(\underset{\sim}{x}) \cdot [\underset{\sim}{u}''(\underset{\sim}{x}) - \underset{\sim}{u}''(\underset{\sim}{a}'_m)] dA_{\underset{\sim}{x}} + \underset{\sim}{\ell}'_m \cdot \underset{\sim}{u}''(\underset{\sim}{a}'_m) + o(1) \text{ as } \rho \to 0 \tag{5.4}$$

for $m=1, \ldots, k'$, and

$$\int_{\Lambda''_m(\rho)} \underset{\sim}{t}''(\underset{\sim}{x}) \cdot \underset{\sim}{u}'(\underset{\sim}{x}) dA_{\underset{\sim}{x}} = \int_{\Lambda''_m(\rho)} \underset{\sim}{t}''(\underset{\sim}{x}) \cdot [\underset{\sim}{u}'(\underset{\sim}{x}) - \underset{\sim}{u}'(\underset{\sim}{a}''_m)] dA_{\underset{\sim}{x}} + \underset{\sim}{\ell}''_m \cdot \underset{\sim}{u}'(\underset{\sim}{a}''_m) + o(1) \text{ as } \rho \to 0 \tag{5.5}$$

[1] Observe from (1.7) that the reciprocal theorem holds also if R is unbounded.

for $m=1,\ldots,k''$. Since S' is regular on the intersection of $\overline{R}$ with a neighborhood of P'', and S'' is regular on the intersection of $\overline{R}$ with a neighborhood of P'', equations (5.4), (5.5) and hypothesis (b) furnish

$$\left.\begin{aligned}&\int_{\Lambda'_m(\rho)} \underset{\sim}{t}'\cdot\underset{\sim}{u}''dA=\underset{\sim}{\ell}'_m\cdot\underset{\sim}{u}''(\underset{\sim}{a}'_m)+o(1),\quad \int_{\Lambda'_m(\rho)} \underset{\sim}{t}''\cdot\underset{\sim}{u}'dA=o(1) \text{ as } \rho\to 0\ (m=1,\ldots k'),\\ &\int_{\Lambda''_m(\rho)} \underset{\sim}{t}''\cdot\underset{\sim}{u}'dA=\underset{\sim}{\ell}''_m\cdot\underset{\sim}{u}'(\underset{\sim}{a}''_m)+o(1),\quad \int_{\Lambda''_m(\rho)} \underset{\sim}{t}'\cdot\underset{\sim}{u}''dA=o(1) \text{ as } \rho\to 0\ (m=1,\ldots k'').\end{aligned}\right\}\quad(5.6)$$

Now proceed to the limit as $\rho\to 0$ in (5.3), using (5.2), (5.6) and bearing in mind hypothesis (d), as well as the continuity of $\underset{\sim}{f}'$, $\underset{\sim}{f}''$ on $\overline{R}$, to obtain the desired identity (5.1).

It is not difficult to see from the foregoing proof that the conclusion in Theorem 5.1 continues to hold if hypothesis (d) is omitted provided the surface integrals in (5.1) are interpreted as suitable principal values. Note also that Theorem 5.1 reduces to Betti's reciprocal theorem if S' and S'' are regular on $\overline{R}$. Finally, the generalization of the preceding theorem to inhomogeneous and anisotropic linearly elastic materials is elementary.

As a further preliminary to the uniqueness theorem at which we are aiming we require

Definition 5.1. (Green's states for the displacements in the second boundary-value problem). Let R be a regular region and $\underset{\sim}{y}\in R$. We call

$$\hat{S}^i(\cdot,\underset{\sim}{y})=[\hat{\underset{\sim}{u}}^i(\cdot,\underset{\sim}{y}),\hat{\underset{\sim}{\tau}}^i(\cdot,\underset{\sim}{y})]$$

the displacement Green's states for the region R at $\underset{\sim}{y}$, corresponding to the elastic constants μ and σ, provided:

(a) $\hat{S}^i(\cdot,\underset{\sim}{y}) = S^i(\cdot,\underset{\sim}{y}) + \tilde{S}^i(\cdot,\underset{\sim}{y})$ on $\overline{R}_{\underset{\sim}{y}}$,

where $S^i(\cdot,\underset{\sim}{y})$ is the normalized Kelvin state introduced in Theorem 2.1;

(b) $\tilde{S}^i(\cdot,\underset{\sim}{y}) = [\tilde{\underset{\sim}{u}}^i(\cdot,\underset{\sim}{y}), \tilde{\underset{\sim}{\tau}}^i(\cdot,\underset{\sim}{y})] \in \mathcal{E}(\underset{\sim}{0}, \mu, \sigma; \overline{R})$;

$$\text{(c)}\quad \tilde{\underset{\sim}{t}}^i(\cdot,\underset{\sim}{y}) = \begin{cases} -\underset{\sim}{t}^i(\cdot,\underset{\sim}{y}) - \sum_{m=1}^{6} \underset{\sim}{q}^m(\cdot) q_i^m(\underset{\sim}{y}) \text{ on } \partial R \text{ if } R \text{ is bounded} \\ -\underset{\sim}{t}^i(\cdot,\underset{\sim}{y}) \text{ on } \partial R \text{ if } R \text{ is unbounded,} \end{cases}$$

where $\tilde{\underset{\sim}{t}}^i(\cdot,\underset{\sim}{y})$, $\underset{\sim}{t}^i(\cdot,\underset{\sim}{y})$ are the surface tractions of $\tilde{S}^i(\cdot,\underset{\sim}{y})$, $S^i(\cdot,\underset{\sim}{y})$, while $\underset{\sim}{q}^m$ $(m=1,\ldots,6)$ is given by Definition 3.2;

$$\text{(d)}\quad \int_{\partial R} \hat{\underset{\sim}{u}}^i(\underset{\sim}{x},\underset{\sim}{y}) \cdot \underset{\sim}{q}^m(\underset{\sim}{x})\, dA_{\underset{\sim}{x}} = 0 \quad (m=1,\ldots,6) \text{ if } R \text{ is bounded.}$$

The regular part $\tilde{S}^i(\cdot,\underset{\sim}{y})$ of the Green's state $\hat{S}^i(\cdot,\underset{\sim}{y})$ is defined through (b), (c) as the solution of a second boundary-value problem for the region R. Thus requirements (b), (c), because of (1.7), determine $\tilde{S}^i(\cdot,\underset{\sim}{y})$ uniquely[1] if R is unbounded but, if R is bounded, leave this state determinate merely within an additive rigid displacement field. Accordingly, $\hat{S}^i(\cdot,\underset{\sim}{y})$ is defined by (a), (b), (c) to the same degree of indeterminacy. This indeterminacy is eliminated by the normalization condition (d), as is clear from (b) in Lemma 3.2.

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[1] Cf. the remark immediately following Definition 3.3.

The surface tractions $\tilde{\underset{\sim}{t}}^{i}(\cdot, \underset{\sim}{y})$, defined by (c), are self-equilibrated on ∂R if R is bounded, as is easily verified with the aid of (c) in Theorem 2.2, together with Definition 3.2 and (3.25); further, they evidently possess the same smoothness on the boundary as does the unit normal vector of ∂R. Consequently the <u>existence</u> of of $\tilde{S}^{i}(\cdot, \underset{\sim}{y})$ — and hence of the Green's states $\hat{S}^{i}(\cdot, \underset{\sim}{y})$— is assured for <u>simple</u> regions by Korn's [14] existence theorem. The existence of $\hat{S}^{i}(\cdot, \underset{\sim}{y})$ for the broader class of <u>regular</u> regions hinges on the solvability of the second boundary-value problem for such regions in the presence of surface tractions with the degree of smoothness of the unit normal vector of ∂R. Note also that $\hat{S}^{i}(\cdot, \underset{\sim}{y})$ is known explicitly (in elementary form) for the special cases of the entire space and the half space: in the former instance it coincides with the Kelvin state $S^{i}(\cdot, \underset{\sim}{y})$, while in the latter it is furnished by Mindlin's [18] solution to the problem of a half-space under an internal concentrated load.

It is worth mentioning that the Green's states in Definition 5.1, which are related to those used by Bergman and Schiffer in [19] (p. 223), differ from the analogous traditional Green's states employed in [2][1] (Theorem 6.1). There, the equilibration of the concentrated load at $\underset{\sim}{y}$ induced by the Kelvin state $S^{i}(\cdot, \underset{\sim}{y})$ is effected through the introduction of a second internal singularity; further, the normalization of the displacement field is achieved by requiring the displacements and rotations to vanish at the location of this supplemental singularity.

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[1] See also [9] (Definition 3.2).

The Green's states of Definition 5.1, in contrast to their counterpart in [2], are symmetric in the sense of

$$\hat{u}^{i}_{j}(\underset{\sim}{x},\underset{\sim}{y})=\hat{u}^{j}_{i}(\underset{\sim}{y},\underset{\sim}{x}) \text{ for all } (\underset{\sim}{x},\underset{\sim}{y})\in R \times R - D \,. \tag{5.7}$$

These symmetry relations follow at once from an application of Theorem 5.1 to the pair of states $\hat{S}^{i}(\cdot,\underset{\sim}{y})$, $\hat{S}^{j}(\cdot,\underset{\sim}{x})$. We shall show later on that there is a simple connection between the Green's states $\hat{S}^{i}(\cdot,\underset{\sim}{y})$ and the modified tangent states defined in Section 3 (Definition 3.3).

Definition 5.1 will be used in Section 6 to deduce an integral-representation theorem for solutions to problems involving concentrated internal and surface loads. We now turn directly to

Theorem 5.2. (A uniqueness theorem for problems involving concentrated internal and surface loads). Let R be a regular region and assume the displacement Green's states for the region R at $\underset{\sim}{y}$, introduced in Definition 5.1, exist for all $\underset{\sim}{y}\in R$. Let

$$P=\{\underset{\sim}{a}_{1},\ldots\underset{\sim}{a}_{k}\}$$

be a set consisting of k distinct (interior or boundary) points in $\overline{R}$. Further, let S', S'' be two states with the properties:

(a) $S'=[\underset{\sim}{u}',\underset{\sim}{\tau}']\in\mathcal{E}(\underset{\sim}{f},\mu,\sigma;\overline{R}-P)$,

$S''=[\underset{\sim}{u}'',\underset{\sim}{\tau}'']\in\mathcal{E}(\underset{\sim}{f},\mu,\sigma;\overline{R}-P)$;

(b) $\underset{\sim}{u}'(\underset{\sim}{x})=O(|\underset{\sim}{x}-\underset{\sim}{a}_{m}|^{-1})$, $\underset{\sim}{\tau}'(\underset{\sim}{x})=O(|\underset{\sim}{x}-\underset{\sim}{a}_{m}|^{-2})$ *as* $\underset{\sim}{x}\to\underset{\sim}{a}_{m}$ $(m=1,\ldots,k)$,

$\underset{\sim}{u}''(\underset{\sim}{x})=O(|\underset{\sim}{x}-\underset{\sim}{a}_{m}|^{-1})$, $\underset{\sim}{\tau}''(\underset{\sim}{x})=O(|\underset{\sim}{x}-\underset{\sim}{a}_{m}|^{-2})$ *as* $\underset{\sim}{x}\to\underset{\sim}{a}_{m}$ $(m=1,\ldots,k)$;

(c) $\lim_{\rho\to 0} \int_{\Lambda_m(\rho)} \underset{\sim}{t}' dA = \lim_{\rho\to 0} \int_{\Lambda_m(\rho)} \underset{\sim}{t}'' dA \quad (m=1,\ldots,k)$,

<u>where</u>

$$\Lambda_m(\rho)=R\cap\partial B_\rho(\underset{\sim}{a}_m) \quad (m=1,\ldots,k) , \tag{5.8}$$

<u>while</u> $\underset{\sim}{t}', \underset{\sim}{t}''$ <u>are the tractions of</u> S', S'' <u>on the side of</u> $\Lambda_m(\rho)$ <u>that faces</u> $\underset{\sim}{a}_m$;

(d) $\underset{\sim}{t}'=\underset{\sim}{t}''$ <u>on</u> $\partial R - P$,

<u>if</u> $\underset{\sim}{t}', \underset{\sim}{t}''$ <u>here denote the surface tractions of</u> S', S'';

(e) $\int_{\partial R} \underset{\sim}{u}'\cdot\underset{\sim}{q}^m dA=0$, $\int_{\partial R} \underset{\sim}{u}''\cdot\underset{\sim}{q}^m dA=0 \quad (m=1,\ldots,6)$ <u>if</u> R <u>is bounded</u>,

<u>with</u> $\underset{\sim}{q}^m$ <u>given by Definition</u> 3.2.

<u>Then</u>

$$S'=S'' \text{ on } \overline{R}-P .$$

<u>Proof</u>. Choose $\underset{\sim}{y}\in R-P$ and hold $\underset{\sim}{y}$ fixed. Let $\hat{S}^i(\cdot,\underset{\sim}{y})$ be the displacement Green's states for the region R at $\underset{\sim}{y}$, corresponding to the elastic constants μ, σ. Then, in view of (a), (b) in Definition 5.1 and (a), (b), (c) in Theorem 2.2, one has

$$\left.\begin{aligned} &\hat{S}^i(\cdot,\underset{\sim}{y})=[\hat{\underset{\sim}{u}}^i(\cdot,\underset{\sim}{y}), \hat{\underset{\sim}{T}}^i(\cdot,\underset{\sim}{y})]\in\mathcal{E}(\underset{\sim}{0},\mu,\sigma;\overline{R}_{\underset{\sim}{y}}) ,\\ &\hat{\underset{\sim}{u}}^i(\underset{\sim}{x},\underset{\sim}{y})=O(|\underset{\sim}{x}-\underset{\sim}{y}|^{-1}),\ \hat{\underset{\sim}{T}}^i(\underset{\sim}{x},\underset{\sim}{y})=O(|\underset{\sim}{x}-\underset{\sim}{y}|^{-2}) \text{ as } \underset{\sim}{x}\to\underset{\sim}{y} ,\\ &\lim_{\rho\to 0}\int_{\partial B_\rho(\underset{\sim}{y})} \hat{\underset{\sim}{t}}^i(\underset{\sim}{x},\underset{\sim}{y})dA_{\underset{\sim}{x}}=\underset{\sim}{e}^i , \end{aligned}\right\} \tag{5.9}$$

where $\hat{\underset{\sim}{t}}^i(\cdot, \underset{\sim}{y})$ are the tractions of $\hat{S}^i(\cdot, \underset{\sim}{y})$ on the side of $\partial B_\rho(\underset{\sim}{y})$ that faces $\underset{\sim}{y}$, while $\underset{\sim}{e}^i$ is a unit vector in the x_i-direction. Further, from (a) and (c) in Definition 5.1,

$$\hat{\underset{\sim}{t}}^i(\cdot, \underset{\sim}{y}) = \begin{cases} -\sum_{m=1}^{6} \underset{\sim}{q}^m(\cdot) q_i^m(\underset{\sim}{y}) \text{ on } \partial R \text{ if R is bounded} \\ \underset{\sim}{0} \text{ on } \partial R \text{ if R is unbounded ,} \end{cases} \tag{5.10}$$

provided $\hat{\underset{\sim}{t}}^i(\cdot, \underset{\sim}{y})$ here are the tractions of $\hat{S}^i(\cdot, \underset{\sim}{y})$ on ∂R.

Next, define the state $S = [\underset{\sim}{u}, \underset{\sim}{\tau}]$ through

$$S = S' - S'' \text{ on } \overline{R} - P , \tag{5.11}$$

so that by (a), (b), (c), (d),

$$\left.\begin{array}{c} S = [\underset{\sim}{u}, \underset{\sim}{\tau}] \in \mathcal{E}(\underset{\sim}{0}, \mu, \sigma; \overline{R} - P) , \\ \underset{\sim}{u}(\underset{\sim}{x}) = O(|\underset{\sim}{x} - \underset{\sim}{a}_m|^{-1}) , \ \underset{\sim}{\tau}(\underset{\sim}{x}) = O(|\underset{\sim}{x} - \underset{\sim}{a}_m|^{-2}) \text{ as } \underset{\sim}{x} \to \underset{\sim}{a}_m \ (m=1, \ldots, k) , \\ \lim_{\rho \to 0} \int_{\Lambda_m(\rho)} \underset{\sim}{t} dA = \underset{\sim}{0} \ (m=1, \ldots, k) , \\ \underset{\sim}{t} = \underset{\sim}{0} \text{ on } \partial R - P \end{array}\right\} \tag{5.12}$$

where $\underset{\sim}{t}$ are the surface tractions of S and $\Lambda_m(\rho)$ is given by (5.8).

Taking account of (5.9), (5.10), (5.12) and applying the generalized reciprocal theorem (Theorem 5.1) to the pair of states S, $\hat{S}^i(\cdot, \underset{\sim}{y})$, one draws

$$u_i(\underset{\sim}{y}) + \int_{\partial R} \hat{\underset{\sim}{t}}^i(\underset{\sim}{x}, \underset{\sim}{y}) \cdot \underset{\sim}{u}(\underset{\sim}{x}) dA_{\underset{\sim}{x}} = 0 . \tag{5.13}$$

The integral in (5.13) vanishes if R is unbounded because of (5.10). On the other hand, if R is bounded, (5.10), (5.11), and hypothesis (e) yield

$$\int_{\partial R} \hat{\underset{\sim}{t}}^{i}(\underset{\sim}{x}, \underset{\sim}{y}) \cdot \underset{\sim}{u}(\underset{\sim}{x}) dA_{\underset{\sim}{x}} = - \sum_{m=1}^{6} q_i^m(\underset{\sim}{y}) \int_{\partial R} \underset{\sim}{q}^m(\underset{\sim}{x}) \cdot \underset{\sim}{u}(\underset{\sim}{x}) dA_{\underset{\sim}{x}} = 0 \; .$$

Thus, (5.13) implies $u_i(\underset{\sim}{y})=0$. Therefore, since $\underset{\sim}{y}$ was chosen arbitrarily in R-P,

$$\underset{\sim}{u}=\underset{\sim}{0} \text{ on } R-P \; . \tag{5.14}$$

Equations (5.14), (1.6) assure that $\underset{\sim}{\tau}$ also vanishes on R-P. The desired conclusion now follows immediately from (5.11) together with the continuity of $\underset{\sim}{u}$ and $\underset{\sim}{\tau}$ on $\overline{R}$-P.

The hypotheses in Theorem 5.2 may be weakened in several respects. First, as may be shown by considerations strictly analogous to those employed in the proofs of Theorems 5.1, 5.2 in [7], if $\underset{\sim}{a}_m$ is an interior point, then

$\underset{\sim}{u}'(\underset{\sim}{x})=O(|\underset{\sim}{x}-\underset{\sim}{a}_m|^{-1})$ if and only if $\underset{\sim}{\tau}'(\underset{\sim}{x})=O(|\underset{\sim}{x}-\underset{\sim}{a}_m|^{-2})$ as $\underset{\sim}{x}\rightarrow\underset{\sim}{a}_m$,

$\underset{\sim}{u}''(\underset{\sim}{x})=O(|\underset{\sim}{x}-\underset{\sim}{a}_m|^{-1})$ if and only if $\underset{\sim}{\tau}''(\underset{\sim}{x})=O(|\underset{\sim}{x}-\underset{\sim}{a}_m|^{-2})$ as $\underset{\sim}{x}\rightarrow\underset{\sim}{a}_m$.

Hence for internal singularities, hypotheses (b) — though mutually consistent — are redundant. Second, note that the regularity conditions on $\underset{\sim}{f}$ at infinity, implied by hypothesis (a) and the last of (c) in Definition 1.1 if R is unbounded, were not used in the preceding uniqueness proof. Suppose, in particular, R is an exterior region and all of conditions (c) in Definition 1.1 are replaced by the weaker requirement

$$\underset{\sim}{\tau}(\underset{\sim}{x})=o(1) \text{ as } x\rightarrow\infty \; .$$

Then hypotheses (a), (b), (c), (d) ensure that the states S' and S'' can differ only by a rigid displacement field, as is easily seen from Theorem 5.2 in [7]. Further, in view of the remarks made in connection with Definition 5.1, the hypothesis concerning the existence of the Green's states becomes superfluous if R is a simple region, a half-space, or the entire space. Next, it is worth mentioning that the positive-definiteness of the strain-energy density assumed in (a) of Definition 1.1 nowhere entered the proof of Theorem 5.2; consequently, uniqueness prevails for all values of μ and σ for which the requisite Green's states exist. Finally, it would appear that a generalization of Theorem 5.2 to anisotropic elastic solids can be carried out with the aid of Fredholm's [20] work on basic singular solutions in the linearized equilibrium theory for such media.

It is an immediate consequence of Theorem 5.2 that the properties of the Kelvin state listed in Theorem 2.2 uniquely characterize that state. Similarly, Theorem 5.2 guarantees that the solution to the problem of a concentrated surface load balanced by regular tractions on the boundary of a simple region defined in Theorem 4.2 through a limit process, is uniquely characterized by the properties listed in Theorem 4.3 — provided the displacements are suitably normalized.

We emphasize that the conclusion in Theorem 5.2 no longer follows if hypotheses (b) are omitted, i.e. if the orders of the singularities at the load points are not preassigned. This lack of uniqueness is due to the existence of elastic states with higher-order self-equilibrated point singularities.[1] In particular, [1] contains

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[1] Cf. the remark at the end of Section 2.

examples of "pseudo-solutions" to concentrated-load problems for the half-space and the sphere that possess singularities with the requisite stress-resultants at the load points and that satisfy the prescribed boundary conditions for the regular surface tractions, but fail to coincide with the corresponding limit solutions. These examples demonstrate the incompleteness of the commonly employed direct formulation of concentrated-load problems, in which the orders of the singularities are not specified.

The usefulness of Theorem 5.2 stems from the fact that the direct formulation of concentrated-load problems furnished by this theorem enables one to validate the solution to such problems without carrying out a possibly cumbersome limit process. In conclusion we observe that Theorem 5.1 may now be viewed as a generalization of the classical reciprocal theorem to problems involving concentrated loads.

6. Integral representations of solutions to concentrated-load problems. Behavior of Green's states at the boundary.

This section aims primarily at integral representations for solutions to concentrated-load problems in the direct formulation supplied by the uniqueness theorem (Theorem 5.2) of the preceding section. In particular we show that the displacement fields of such solutions may be represented in integral form with the aid of the Green's states $\hat{\mathcal{S}}^i$ of Definition 5.1. To obtain an analogous representation for the associated fields of stress we require

Definition 6.1. (Green's states for the stresses in the second boundary-value problem). Let R be a regular region and $\underset{\sim}{y}\in R$. We call

$$\hat{\mathcal{S}}^{ij}(\cdot,\underset{\sim}{y})=[\hat{\underset{\sim}{u}}^{ij}(\cdot,\underset{\sim}{y}),\hat{\underset{\sim}{\tau}}^{ij}(\cdot,\underset{\sim}{y})]$$

the stress Green's states for the region R at $\underset{\sim}{y}$, corresponding to the elastic constants μ and σ, provided:

(a) $\hat{\mathcal{S}}^{ij}(\cdot,\underset{\sim}{y})=\mathcal{S}^{ij}(\cdot,\underset{\sim}{y})+\tilde{\mathcal{S}}^{ij}(\cdot,\underset{\sim}{y})$ on $\overline{R}_{\underset{\sim}{y}}$,

where

$$\mathcal{S}^{ij}(\cdot,\underset{\sim}{y})=-\mu\left[\frac{2\sigma}{1-2\sigma}\delta_{ij}\mathcal{S}^{k}_{,k}(\cdot,\underset{\sim}{y})+\mathcal{S}^{i}_{,j}(\cdot,\underset{\sim}{y})+\mathcal{S}^{j}_{,i}(\cdot,\underset{\sim}{y})\right] \text{ on } \overline{R}_{\underset{\sim}{y}}, \qquad (6.1)$$

while $\mathcal{S}^i(\cdot,\underset{\sim}{y})$ are the normalized Kelvin states introduced in Theorem 2.1;

(b) $\tilde{\mathcal{S}}^{ij}(\cdot,\underset{\sim}{y})=[\tilde{\underset{\sim}{u}}^{ij}(\cdot,\underset{\sim}{y}),\tilde{\underset{\sim}{\tau}}^{ij}(\cdot,\underset{\sim}{y})]\in\mathcal{E}(\underset{\sim}{0},\mu,\sigma;\overline{R})$;

(c) $\tilde{\underset{\sim}{t}}^{ij}(\cdot,\underset{\sim}{y})=-\underset{\sim}{t}^{ij}(\cdot,\underset{\sim}{y})$ on ∂R,

where $\tilde{\underset{\sim}{t}}^{ij}(\cdot,\underset{\sim}{y})$, $\underset{\sim}{t}^{ij}(\cdot,\underset{\sim}{y})$ are the surface tractions of $\tilde{\mathcal{S}}^{ij}(\cdot,\underset{\sim}{y})$, $\mathcal{S}^{ij}(\cdot,\underset{\sim}{y})$;

(d) $\int_{\partial R} \hat{u}^{ij}(x, y) \cdot q^m(x) dA_x = 0$ $(m=1, \ldots, 6)$ if R is bounded,

with q^m given by Definition 3.2.

Note that the state $S^{ij}(\cdot, y)$ defined through (6.1) is a linear combination of a center of dilatation and of two force doublets with equal and opposite moments, and hence has a self-equilibrated singularity[1] at y. Thus, from (c), the tractions of $\tilde{S}^{ij}(\cdot, y)$ are self-equilibrated on ∂R.

The remarks made in connection with Definition 5.1 that concern the existence and uniqueness of the displacement Green's states $\hat{S}^i$, are equally applicable to the stress Green's states $\hat{S}^{ij}$ of Definition 6.1. In particular, the existence of the latter states is assured when R is a simple region, a half-space, or the entire space. A connection between the Green's states and the modified tangent states of Definition 3.3 will be established later on. We now proceed to

Theorem 6.1. (Integral representation of solutions to concentrated-load problems). Let R be a regular region. Assume the displacement Green's states $\hat{S}^i(\cdot, y)$ of Definition 5.1 and the stress Green's states $\hat{S}^{ij}(\cdot, y)$ of Definition 6.1, for the region R at y, exist for all $y \in R$. Let

$$P = \{a_1 \ldots, a_k\}$$

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[1] See [2] (Theorem 5.2).

be a set in $\overline{R}$, which consists of k distinct (interior or boundary) points. Further, let S be a state with the properties:

(a) $S=[\underset{\sim}{u}, \underset{\sim}{\tau}]\in\mathcal{E}(\underset{\sim}{f}, \mu, \sigma; \overline{R}-P)$, $\underset{\sim}{f}\in\mathcal{C}(\overline{R})$;

(b) $\underset{\sim}{u}(\underset{\sim}{x})=O(|\underset{\sim}{x}-\underset{\sim}{a}_m|^{-1})$, $\underset{\sim}{\tau}(\underset{\sim}{x})=O(|\underset{\sim}{x}-\underset{\sim}{a}_m|^{-2})$ as $\underset{\sim}{x}\to\underset{\sim}{a}_m$ $(m=1, \ldots, k)$;

(c) $\lim\limits_{\rho\to 0}\int_{\Lambda_m(\rho)} \underset{\sim}{t}\, dA=\underset{\sim}{\ell}_m \quad (m=1, \ldots, k)$,

where

$$\Lambda_m(\rho)=R\cap\partial B_\rho(\underset{\sim}{a}_m) \quad (m=1, \ldots, k) ,$$

while $\underset{\sim}{t}$ are the tractions of S on the side of $\Lambda_m(\rho)$ that faces $\underset{\sim}{a}_m$;

(d) the tractions of S on ∂R are integrable;

(e) $\int_{\partial R} \underset{\sim}{u}\cdot\underset{\sim}{q}^m dA=0 \quad (m=1, \ldots, 6)$ if R is bounded,

with $\underset{\sim}{q}^m$ given by Definition 3.2.

Then S on R-P admits the representation

$$u_i(\underset{\sim}{y})=\int_{\partial R}\underset{\sim}{\hat{u}}^i(\underset{\sim}{x}, \underset{\sim}{y})\cdot\underset{\sim}{t}(\underset{\sim}{x})dA_{\underset{\sim}{x}}+\int_R \underset{\sim}{\hat{u}}^i(\underset{\sim}{x}, \underset{\sim}{y})\cdot\underset{\sim}{f}(\underset{\sim}{x})dV_{\underset{\sim}{x}}$$

$$+\sum_{m=1}^{k}\underset{\sim}{\hat{u}}^i(\underset{\sim}{a}_m, \underset{\sim}{y})\cdot\underset{\sim}{\ell}_m \text{ for all } \underset{\sim}{y}\in R-P , \tag{6.2}$$

$$\tau_{ij}(\underset{\sim}{y})=\int_{\partial R}\underset{\sim}{\hat{u}}^{ij}(\underset{\sim}{x}, \underset{\sim}{y})\cdot\underset{\sim}{t}(\underset{\sim}{x})dA_{\underset{\sim}{x}}+\int_R \underset{\sim}{\hat{u}}^{ij}(\underset{\sim}{x}, \underset{\sim}{y})\cdot\underset{\sim}{f}(\underset{\sim}{x})dV_{\underset{\sim}{x}}$$

$$+\sum_{m=1}^{k}\underset{\sim}{\hat{u}}^{ij}(\underset{\sim}{a}_m, \underset{\sim}{y})\cdot\underset{\sim}{\ell}_m \text{ for all } \underset{\sim}{y}\in R-P , \tag{6.3}$$

if $\underset{\sim}{t}$ here denotes the tractions of S on ∂R.

Proof. Let $\underset{\sim}{y} \in R-P$. Then, bearing in mind (a), (b), (c), (d) and (5.9), and applying Theorem 5.1 to the pair of states S, $\hat{S}^i(\cdot, \underset{\sim}{y})$, one obtains

$$\int_{\partial R} \hat{\underset{\sim}{t}}^i(\underset{\sim}{x}, \underset{\sim}{y}) \cdot \underset{\sim}{u}(\underset{\sim}{x}) dA_{\underset{\sim}{x}} + u_i(\underset{\sim}{y}) = \int_{\partial R} \hat{\underset{\sim}{u}}^i(\underset{\sim}{x}, \underset{\sim}{y}) \cdot \underset{\sim}{t}(\underset{\sim}{x}) dA_{\underset{\sim}{x}}$$

$$+ \int_R \hat{\underset{\sim}{u}}^i(\underset{\sim}{x}, \underset{\sim}{y}) \cdot \underset{\sim}{f}(\underset{\sim}{x}) dV_{\underset{\sim}{x}} + \sum_{k=1}^{m} \hat{\underset{\sim}{u}}^i(\underset{\sim}{a}_m, \underset{\sim}{y}) \cdot \underset{\sim}{\ell}_m \,. \tag{6.4}$$

It follows from (5.10) and hypothesis (e) that the integral in the left-hand member of (6.4) vanishes. Hence (6.4) implies (6.2). Turning to the proof of (6.3), we recall first from Theorem 2.2 that

$$\left.\begin{aligned} &S^i(\cdot, \underset{\sim}{y}) = [\underset{\sim}{u}^i(\cdot, \underset{\sim}{y}), \underset{\sim}{\tau}^i(\cdot, \underset{\sim}{y})] \in \mathcal{E}(\underset{\sim}{0}, \mu, \sigma; \overline{R}_{\underset{\sim}{y}}) \,, \\ &\underset{\sim}{u}^i(\underset{\sim}{x}, \underset{\sim}{y}) = O(|\underset{\sim}{x} - \underset{\sim}{y}|^{-1}), \ \underset{\sim}{\tau}^i(\underset{\sim}{x}, \underset{\sim}{y}) = O(|\underset{\sim}{x} - \underset{\sim}{y}|^{-2}) \text{ as } \underset{\sim}{x} \to \underset{\sim}{y} \,, \\ &\lim_{\rho \to 0} \int_{\partial B_\rho(\underset{\sim}{y})} \underset{\sim}{t}^i(\underset{\sim}{x}, \underset{\sim}{y}) dA_{\underset{\sim}{x}} = \underset{\sim}{e}^i \,, \end{aligned}\right\} \tag{6.5}$$

where $\underset{\sim}{t}^i(\cdot, \underset{\sim}{y})$ are the tractions of $S^i(\cdot, \underset{\sim}{y})$ on the side of $\partial B_\rho(\underset{\sim}{y})$ that faces $\underset{\sim}{y}$, while $\underset{\sim}{e}^i$ is a unit base-vector in the x_i-direction. Hence, an application of Theorem 5.1 to the pair of states S, $S^i(\cdot, \underset{\sim}{y})$ yields

$$u_i(\underset{\sim}{y}) = \int_{\partial R} [\underset{\sim}{u}^i(\underset{\sim}{x}, \underset{\sim}{y}) \cdot \underset{\sim}{t}(\underset{\sim}{x}) - \underset{\sim}{t}^i(\underset{\sim}{x}, \underset{\sim}{y}) \cdot \underset{\sim}{u}(\underset{\sim}{x})] dA_{\underset{\sim}{x}}$$

$$+ \int_R \underset{\sim}{u}^i(\underset{\sim}{x}, \underset{\sim}{y}) \cdot \underset{\sim}{f}(\underset{\sim}{x}) dV_{\underset{\sim}{x}} + \sum_{m=1}^{k} \underset{\sim}{u}^i(\underset{\sim}{a}_m, \underset{\sim}{y}) \cdot \underset{\sim}{\ell}_m \,. \tag{6.6}$$

Since $\underset{\sim}{y}$ was chosen arbitrarily in R-P, (6.6) holds for all $\underset{\sim}{y} \in R-P$.
From (6.6), (2.3), (2.2) follows

$$u_{i,j}(\underset{\sim}{y}) = \int_{\partial R} \left[\frac{\partial}{\partial y_j} \underset{\sim}{u}^i(\underset{\sim}{x}, \underset{\sim}{y}) \cdot \underset{\sim}{t}(\underset{\sim}{x}) - \frac{\partial}{\partial y_j} \underset{\sim}{t}^i(\underset{\sim}{x}, \underset{\sim}{y}) \cdot \underset{\sim}{u}(\underset{\sim}{x}) \right] dA_{\underset{\sim}{x}}$$

$$+ \int_R \frac{\partial}{\partial y_j} \underset{\sim}{u}^i(\underset{\sim}{x}, \underset{\sim}{y}) \cdot \underset{\sim}{f}(\underset{\sim}{x}) dV_{\underset{\sim}{x}} + \sum_{m=1}^{k} \frac{\partial}{\partial y_j} \underset{\sim}{u}^i(\underset{\sim}{a}_m, \underset{\sim}{y}) \cdot \underset{\sim}{\ell}_m \text{ for all } \underset{\sim}{y} \in R-P. \quad (6.7)^1$$

According to (2.2), (2.3), $S^i(\underset{\sim}{x}, \underset{\sim}{y})$ is differentiable with respect to both of its arguments, provided $\underset{\sim}{x} \neq \underset{\sim}{y}$. Further, (2.2) gives

$$\frac{\partial}{\partial y_j} \underset{\sim}{u}^i(\underset{\sim}{x}, \underset{\sim}{y}) = -\frac{\partial}{\partial x_j} \underset{\sim}{u}^i(\underset{\sim}{x}, \underset{\sim}{y}), \quad \frac{\partial}{\partial y_j} \underset{\sim}{\tau}^i(\underset{\sim}{x}, \underset{\sim}{y}) = -\frac{\partial}{\partial x_j} \underset{\sim}{\tau}^i(\underset{\sim}{x}, \underset{\sim}{y}),$$

so that (6.7), (6.1), and (1.6) imply

$$\tau_{ij}(\underset{\sim}{y}) = \int_{\partial R} [\underset{\sim}{u}^{ij}(\underset{\sim}{x}, \underset{\sim}{y}) \cdot \underset{\sim}{t}(\underset{\sim}{x}) - \underset{\sim}{t}^{ij}(\underset{\sim}{x}, \underset{\sim}{y}) \cdot \underset{\sim}{u}(\underset{\sim}{x})] \, dA_{\underset{\sim}{x}}$$

$$+ \int_R \underset{\sim}{u}^{ij}(\underset{\sim}{x}, \underset{\sim}{y}) \cdot \underset{\sim}{f}(\underset{\sim}{x}) dV_{\underset{\sim}{x}} + \sum_{m=1}^{k} \underset{\sim}{u}^{ij}(\underset{\sim}{a}_m, \underset{\sim}{y}) \cdot \underset{\sim}{\ell}_m \text{ for all } \underset{\sim}{y} \in R-P. \quad (6.8)$$

Next, for each $\underset{\sim}{y} \in R-P$, let $\tilde{S}^{ij}(\cdot, \underset{\sim}{y})$ be the "regular part" of the stress Green's state $\hat{S}^{ij}(\cdot, \underset{\sim}{y})$ in Definition 6.1. Then (b) in this definition, together with the present hypotheses on S and Theorem 5.1 applied to S, $\tilde{S}^{ij}(\cdot, \underset{\sim}{y})$, furnish

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[1] The differentiation under the integral sign of the improper volume integral in (6.6) is easily justified with the aid of (2.3), (2.2). Cf. the proof for the differentiability under the integral sign of Newtonian potentials of volume distributions in Kellogg [5] (p. 151).

$$0=\int_{\partial R}[\tilde{\mathbf{u}}^{ij}(\mathbf{x},\mathbf{y})\cdot\mathbf{t}(\mathbf{x})-\tilde{\boldsymbol{\tau}}^{ij}(\mathbf{x},\mathbf{y})\cdot\mathbf{u}(\mathbf{x})]\,dA_{\mathbf{x}}$$

$$+\int_{R}\tilde{\mathbf{u}}^{ij}(\mathbf{x},\mathbf{y})\cdot\mathbf{f}(\mathbf{x})\,dV_{\mathbf{x}}+\sum_{m=1}^{k}\tilde{\mathbf{u}}^{ij}(\mathbf{a}_m,\mathbf{y})\cdot\boldsymbol{\ell}_m \quad \text{for all } \mathbf{y}\in R-P\,. \qquad (6.9)$$

Finally, add (6.8) and (6.9), and use (a), (c) in Definition 6.1 to obtain (6.3). This completes the proof.

We now establish a connection between the Green's states of Definitions 5.1, 6.1, and the modified tangent states of Definition 3.3.

Theorem 6.2. (*A connection between the Green's states and the modified tangent states*). *Let* R *be a simple region. Let* $\hat{S}^i$, $\hat{S}^{ij}$, $\mathring{S}^i$ *respectively denote the displacement Green's states of Definition* 5.1, *the stress Green's states of Definition* 6.1, *and the modified tangent states of Definition* 3.3, *for the region* R. *Then:*

(a) $\hat{u}^i_k(\mathbf{x},\mathbf{y})=\mathring{u}^k_i(\mathbf{y},\mathbf{x})$, $\hat{u}^{ij}_k(\mathbf{x},\mathbf{y})=\mathring{\tau}^k_{ij}(\mathbf{y},\mathbf{x})$ *for all* $(\mathbf{x},\mathbf{y})\in\partial R\times R$;

(b) $\lim_{\mathbf{z}\to\mathbf{y}}\hat{u}^i_j(\mathbf{x},\mathbf{z})=\mathring{u}^i_j(\mathbf{x},\mathbf{y})$ *for all* $(\mathbf{x},\mathbf{y})\in\bar{R}\times\partial R-D$.

Proof. Let $\mathbf{x}\in\partial R$ and let $\mathring{S}^k(\cdot,\mathbf{x})$ be the modified tangent state for the region R at $\mathbf{x}$ corresponding to the x_k-direction[1]. Let $\mathbf{q}^m$ (m=1, ..., 6) be given by Definition 3.2. Then (a) in Definition 3.3 and (a) in Lemma 3.2 imply

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[1] Recall that the existence of $\breve{S}^k(\cdot,\mathbf{x})$ is assured by Theorem 3.3.

$$\int_{\partial[R-B_\rho(x)]} \overset{\circ}{t}{}^k(p,x)\cdot q^m(p)\,dA_p = 0 \quad (m=1,\ldots,6)\ ,$$

for all sufficiently small $\rho>0$. Thus, from (c) in Definition 3.3 and (3.25),

$$\lim_{\rho\to 0}\int_{\Lambda(\rho)} \overset{\circ}{t}{}^k(p,x)\cdot q^m(p)\,dA_p = \sum_{\ell=1}^{6} q_k^{\ell}(x)\int_{\partial R} q^{\ell}(p)\cdot q^m(p)\,dA_p$$

$$= q_k^m(x) \quad (m=1,\ldots,6)\ , \tag{6.10}$$

where

$$\Lambda(\rho) = R\cap\partial B_\rho(x)\ ,$$

while $\overset{\circ}{t}{}^k(\cdot,x)$ in (6.10) are the tractions of $\overset{\circ}{S}{}^k(\cdot,x)$ on the side of $\Lambda(\rho)$ that faces x. For $m=1,2,3$, Equations (6.10) in conjunction with Definition 3.2 yield

$$\lim_{\rho\to 0}\int_{\Lambda(\rho)} \overset{\circ}{t}{}^k(p,x)\,dA_p = e^k\ , \tag{6.11}$$

where e^k is a unit vector in the x_k-direction.

Next, (b) in Definition 3.3, (3.7), and (b) in Lemma 3.1 imply

$$\overset{\circ}{u}{}^k(p,x) = O(|p-x|^{-1}),\quad \overset{\circ}{T}{}^k(p,x) = O(|p-x|^{-2}) \text{ as } p\to x\ . \tag{6.12}$$

In view of (6.11), (6.12) and (a), (c), (d) in Definition 3.3, it follows from Theorem 6.1 that $\overset{\circ}{S}{}^k(\cdot,x)$ admits the representation

$$\left.\begin{aligned} \overset{\circ}{u}{}^{k}_{i}(\underset{\sim}{y},\underset{\sim}{x}) &= \int_{\partial R} \hat{\underset{\sim}{u}}{}^{i}(\underset{\sim}{p},\underset{\sim}{y})\cdot \overset{\circ}{\underset{\sim}{t}}{}^{k}(\underset{\sim}{p},\underset{\sim}{x})\,dA_{\underset{\sim}{p}} + \hat{u}^{i}_{k}(\underset{\sim}{x},\underset{\sim}{y}) \quad \text{for all } \underset{\sim}{y}\in R\,, \\ \overset{\circ}{\tau}{}^{k}_{ij}(\underset{\sim}{y},\underset{\sim}{x}) &= \int_{\partial R} \hat{\underset{\sim}{u}}{}^{ij}(\underset{\sim}{p},\underset{\sim}{y})\cdot \overset{\circ}{\underset{\sim}{t}}{}^{k}(\underset{\sim}{p},\underset{\sim}{x})\,dA_{\underset{\sim}{p}} + \hat{u}^{ij}_{k}(\underset{\sim}{x},\underset{\sim}{y}) \quad \text{for all } \underset{\sim}{y}\in R\,. \end{aligned}\right\} \tag{6.13}$$

The integrals appearing in (6.13) vanish because of (d) in Definition 5.1, (d) in Definition 6.1, and (c) in Definition 3.3. Thus, since $\underset{\sim}{x}$ was chosen arbitrarily on ∂R, (6.13) implies conclusion (a).

We turn next to the verification of conclusion (b). To this end first choose $(\underset{\sim}{x},\underset{\sim}{y})$ on $\partial R\times\partial R - D$ and observe that (6.11), (6.12) together with (a), (c) in Definition 3.3 guarantee the applicability of Theorem 5.1 to the pair of states $\overset{\circ}{S}{}^{i}(\cdot\,,\underset{\sim}{y})$, $\overset{\circ}{S}{}^{j}(\cdot\,,\underset{\sim}{x})$. Indeed, one obtains in this manner

$$\overset{\circ}{u}{}^{i}_{j}(\underset{\sim}{x},\underset{\sim}{y}) + \int_{\partial R} \overset{\circ}{\underset{\sim}{t}}{}^{j}(\underset{\sim}{p},\underset{\sim}{x})\cdot \overset{\circ}{\underset{\sim}{u}}{}^{i}(\underset{\sim}{p},\underset{\sim}{y})\,dA_{\underset{\sim}{p}} = \overset{\circ}{u}{}^{j}_{i}(\underset{\sim}{y},\underset{\sim}{x}) + \int_{\partial R} \overset{\circ}{\underset{\sim}{t}}{}^{i}(\underset{\sim}{p},\underset{\sim}{y})\cdot \overset{\circ}{\underset{\sim}{u}}{}^{j}(\underset{\sim}{p},\underset{\sim}{x})\,dA_{\underset{\sim}{p}} \tag{6.14}$$

for all $(\underset{\sim}{x},\underset{\sim}{y})\in\partial R\times\partial R - D$. Hence, invoking (c), (d) of Definition 3.3, one sees that each of the two integrals in (6.14) vanishes, and arrives at the symmetry relations

$$\overset{\circ}{u}{}^{i}_{j}(\underset{\sim}{x},\underset{\sim}{y}) = \overset{\circ}{u}{}^{j}_{i}(\underset{\sim}{y},\underset{\sim}{x}) \quad \text{for all } (\underset{\sim}{x},\underset{\sim}{y})\in\partial R\times\partial R - D\,. \tag{6.15}$$

In addition, recall from (5.7) that

$$\hat{u}^{i}_{j}(\underset{\sim}{x},\underset{\sim}{y}) = \hat{u}^{j}_{i}(\underset{\sim}{y},\underset{\sim}{x}) \quad \text{for all } (\underset{\sim}{x},\underset{\sim}{y})\in R\times R - D\,. \tag{6.16}$$

From (a) in Definition 3.3, (a) in Definition 5.1, conclusion (a) in the present theorem, (6.15), (6.16), it follows that

$$\overset{\circ}{u}{}^i_j(\underset{\sim}{x}, \underset{\sim}{y}) = \hat{u}^j_i(\underset{\sim}{y}, \underset{\sim}{x}) = \lim_{\underset{\sim}{z} \to \underset{\sim}{y}} \hat{u}^j_i(\underset{\sim}{z}, \underset{\sim}{x})$$

$$= \lim_{\underset{\sim}{z} \to \underset{\sim}{y}} \hat{u}^i_j(\underset{\sim}{x}, \underset{\sim}{z}) \text{ for all } (\underset{\sim}{x}, \underset{\sim}{y}) \in R \times \partial R \;, \qquad (6.17)$$

and

$$\overset{\circ}{u}{}^i_j(\underset{\sim}{x}, \underset{\sim}{y}) = \overset{\circ}{u}{}^j_i(\underset{\sim}{y}, \underset{\sim}{x}) = \lim_{\underset{\sim}{z} \to \underset{\sim}{y}} \overset{\circ}{u}{}^j_i(\underset{\sim}{z}, \underset{\sim}{x})$$

$$= \lim_{\underset{\sim}{z} \to \underset{\sim}{y}} \hat{u}^i_j(\underset{\sim}{x}, \underset{\sim}{z}) \text{ for all } (\underset{\sim}{x}, \underset{\sim}{y}) \in \partial R \times \partial R - D \;. \qquad (6.18)$$

Relations (6.17), (6.18) imply conclusion (b), so that the proof is complete.

It is apparent from (6.2), (6.3) in Theorem 6.1 that one requires merely a knowledge of the Green's displacements $\hat{\underset{\sim}{u}}{}^i(\cdot, \underset{\sim}{y})$, $\hat{\underset{\sim}{u}}{}^{ij}(\cdot, \underset{\sim}{y})$ <u>on the boundary</u> ∂R in order to arrive at an integral representation of $\underset{\sim}{u}(\underset{\sim}{y})$, $\underset{\sim}{\tau}(\underset{\sim}{y})$, in the absence of body forces and internal concentrated loads. Conclusion (a) in Theorem 6.2 now reveals that this limited information concerning the two types of Green's displacements is supplied completely by the displacements and stresses of the modified tangent states, if the region is simple. This observation is apt to be of practical interest in connection with the actual construction of the general solution to the second boundary-value problem for such regions. Beyond this, the theoretical significance of Theorem 6.2 stems from the fact that it reveals the behavior <u>at</u> the boundary of the Green's states $\hat{S}^i$, $\hat{S}^{ij}$ since the corresponding behavior of the modified tangent states $\overset{\circ}{S}{}^i$ is known a priori from Definition 3.3.

In connection with the original definition of the modified tangent states we note from (6.11), (6.12), in conjunction with the uniqueness theorem for concentrated-load problems (Theorem 5.2), that (b) in Definition 3.3 may now be replaced by:

$$\text{(b}'\text{)} \quad \overset{\circ}{\underset{\sim}{u}}{}^i(\underset{\sim}{x},\underset{\sim}{y}) = O(|\underset{\sim}{x}-\underset{\sim}{y}|^{-1}), \quad \overset{\circ}{\underset{\sim}{\tau}}{}^i(\underset{\sim}{x},\underset{\sim}{y}) = O(|\underset{\sim}{x}-\underset{\sim}{y}|^{-2}) \text{ as } \underset{\sim}{x}\to\underset{\sim}{y},$$

$$\lim_{\rho\to 0} \int_{\Lambda_\rho(\underset{\sim}{y})} \overset{\circ}{\underset{\sim}{t}}{}^i(\underset{\sim}{x},\underset{\sim}{y})\, dA_{\underset{\sim}{x}} = \underset{\sim}{e}^i, \quad \Lambda_\rho(\underset{\sim}{y}) = R\cap\partial B_\rho(\underset{\sim}{y}),$$

where $\overset{\circ}{\underset{\sim}{t}}{}^i(\cdot,\underset{\sim}{y})$ are the tractions of $\overset{\circ}{S}{}^i(\cdot,\underset{\sim}{y})$ on the side of $\Lambda_\rho(\underset{\sim}{y})$ that faces $\underset{\sim}{y}$, while $\underset{\sim}{e}^i$ is a unit vector in the x_i-direction. This alternative and more transparent characterization of the modified tangent state $\overset{\circ}{S}(\cdot,\underset{\sim}{y})$ identifies the latter as the solution to a problem corresponding to a unit concentrated load at the boundary point $\underset{\sim}{y}$ together with the equilibrating regular surface tractions

$$\overset{\circ}{\underset{\sim}{t}}{}^i(\cdot,\underset{\sim}{y}) = -\sum_{m=1}^{6} \underset{\sim}{q}^m(\cdot)\, q_i^m(\underset{\sim}{y}) \quad \text{on } \partial R_{\underset{\sim}{y}}$$

specified in (c) of Definition 3.3.

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