

- I. THE EFFECT OF DROPLET SOLIDIFICATION
UPON TWO-PHASE FLOW IN A ROCKET NOZZLE

- II. A KINETIC THEORY INVESTIGATION OF SOME
CONDENSATION-EVAPORATION PHENOMENA
BY A MOMENT METHOD

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ABSTRACT

PART I:

The perturbation technique developed by Rannie and Marble is used to study the effect of droplet solidification upon two-phase flow in a rocket nozzle. It is shown that under certain conditions an equilibrium flow exists, where the gas and particle phases have the same velocity and temperature at each section of the nozzle. The flow is divided into three regions: the first region, where the particles are all in the form of liquid droplets; a second region, over which the droplets solidify at constant freezing temperature; and a third region, where the particles are all solid. By a perturbation about the equilibrium flow, a solution is obtained for small particle slip velocities using the Stokes drag law and the corresponding approximation for heat transfer between the particle and gas phases. Singular perturbation procedure is required to handle the problem at points where solidification first starts and where it is complete. The effects of solidification are noticeable.

PART II:

When a liquid surface, in contact with only its pure vapor, is not in thermodynamic equilibrium with it, a net condensation or evaporation of fluid occurs. This phenomenon is studied from a kinetic theory viewpoint by means of moment method developed by Lees. The evaporation - condensation rate is calculated for a spherical droplet and for a liquid sheet, when the temperatures and pressures are not too far removed from their equilibrium values. The solutions are

valid for the whole range of Knudsen numbers from the free molecule to the continuum limit. In the continuum limit, the mass flux rate is proportional to the pressure difference alone.

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PART I.
THE EFFECT OF DROPLET SOLIDIFICATION UPON
TWO-PHASE FLOW IN A ROCKET NOZZLE

1. INTRODUCTION

One-dimensional gas - particle flows in nozzles have been studied by Rannie and Marble [1, 2, 3]. The particles were treated as a continuum and linearization was then effected by assuming the particle velocity and temperature lags to be small fractions of the gas velocity and temperature. The linearization was effected about an "equilibrium flow" in which the gas and particles moved with the same velocity and temperature. This equilibrium flow was found to be equivalent to the familiar one-dimensional isentropic nozzle flow of a gas with modified thermodynamic properties. Using a well-defined velocity equilibration length, $\bar{\lambda}_v$, as perturbation parameter, the linearized solution about this flow was then obtained. These analyses have been able to describe the important features of such flows quite well.

The particles, in the above mentioned treatments, were assumed to be small solid particles. It is possible, under a combination of particular rocket propellants and chamber temperatures, for the particles to be in the liquid state over part of the flow region. For example, aluminum oxide, a typical particle constituent found in rocket exhausts, melts at 2313°K . If the chamber temperature is above 2400°K , it is to be expected that the particles will be in the liquid state over part of the flow through the nozzle. The question is, therefore, would this change of state, not taken into account in previous analyses, significantly affect flow characteristics.

The new features introduced into the problem by the existence

of the particles in the liquid state, over part of the flow field, are two fold. First, since the particles are in the form of liquid droplets initially, the phenomena of droplet growth by agglomeration, and of the deformation and breakup of droplets could be of importance. These effects will not be considered in this treatment and will be assumed to be unimportant, at least as far as the gross features are concerned. Second, the phase transition from liquid to solid, occurring over part of the flow, has to be taken into account. It is this feature that is analyzed in detail in this study.

The particles start off initially as liquid droplets, move down the nozzle until their temperature falls to the freezing point, and then begin to solidify. The solidification takes place over some distance of the flow, the particle temperature remaining at the freezing point. Once solidification is complete, the normal gas-particle flow in a nozzle takes place to the exhaust. As far as the change in gas and particle exit conditions are concerned, two factors come into play: the isothermal flow of the particles during freezing, and the latent heat of fusion given up by the particles to the gas. Now the latent heat given up to the gas tends to keep the flow in equilibrium, but without calculation it is not possible to predict how this energy is distributed between the kinetic and internal energies of the gas. That is, the mean temperature and velocity of the flow increase in some unknown fashion.

It is possible, however, to make a very rough estimate of exit temperature difference induced by the phase change. If all the latent heat goes into the heating up of the flow, the change in exit temperature

ΔT_E is roughly given by:

$$\Delta T_E = \frac{\kappa h_e}{(1+\kappa) \bar{C}_P}$$

For aluminum oxide particles and a typical gas component, this temperature change is of the order of 200°K. Clearly, this is an over-estimate, yet this indicates that in the calculation of the particle exit temperature, for example, this effect could be of importance.

2. ASSUMPTIONS AND GOVERNING EQUATIONS

Most of the assumptions made in the following analysis have been discussed in detail by Marble [2, 3]. The flow is assumed to be one-dimensional with the gas perfect and inviscid, except for its interaction with the particles. The particles will be treated as a continuum interacting with the gas but not interacting among themselves. The particle Reynolds number and the molecular mean free path are assumed small enough to permit the use of the Stokes drag law and the corresponding approximation for heat transfer between particle and gas. The particles will be assumed to be of a single size, spherical in shape, size and shape remaining unaltered through the flow. The particle properties will be assumed to be constant over the whole flow region.

The equations governing the flow are, with the exception of the energy equation, identical to those derived by Marble. For one-dimensional motion, the equations of continuity for each phase are:

$$\rho u A = \dot{m} \quad (1)$$

$$\rho_p u_p A = \kappa \dot{m} \quad (2)$$

The corresponding momentum equations follow directly:

$$\rho u \frac{du}{dx} = - \frac{dp}{dx} + F_p \quad (3)$$

$$\rho_p u_p \frac{du_p}{dx} = - F_p \quad (4)$$

Here, F_p is the force exerted by the particles on a unit volume of gas. From the Stokes drag law assumption:

$$F_p = m_p \cdot 6\pi\sigma\mu (u_p - u) = \frac{\rho_p a}{\bar{\lambda}_v} (u_p - u) \quad (5)$$

where

$$\bar{\lambda}_v = \frac{m a}{6\pi\sigma\mu} \quad (6)$$

$\bar{\lambda}_v$ is the velocity equilibration length of a single particle based on the frozen speed of sound:

$$a^2 = \bar{\gamma} \bar{R} T \quad (7)$$

The physical significance of $\bar{\lambda}_v$ has been discussed in detail by Marble. It is the distance traveled by a particle in a gas moving at sonic speed, while reducing its relative velocity to e^{-1} of the initial value. If the gas viscosity is assumed to vary as the square root of the temperature, $\bar{\lambda}_v$ is a constant. This assumption shall be made.

The energy equation for the gas can be written:

$$\rho u c_p \frac{dT}{dx} = u \frac{dp}{dx} + (u_p - u) F_p + Q_p \quad (8)$$

The term $(u_p - u) F_p$ is the work done on the gas by the particles and Q_p is the heat transfer rate from the particles given, to the same approximation as Stokes law, by

$$Q_p = m_p \left(\frac{k}{\sigma}\right) 4\pi\sigma^2 (T_p - T) = \rho_p \bar{c}_p a \left[(T_p - T) / \bar{\lambda}_T \right] \quad (9)$$

$$\bar{\lambda}_T = \bar{c}_p m a / 4\pi\sigma k = \frac{3}{2} \bar{P}_T \bar{\lambda}_v \quad (10)$$

$\bar{\lambda}_T$ and \bar{P}_T are the temperature equilibration length and the Prandtl number, respectively, based on the effective specific heat \bar{c}_p and the equilibrium sound speed. $\bar{\lambda}_T$ and \bar{P}_T are assumed to be constant with $\bar{\lambda}_v$ of the same order of magnitude as $\bar{\lambda}_T$.

For the particle flow, the first law can be written:

$$\frac{d}{dx} \left\{ \rho_p u_p \left(e_p + \frac{1}{2} u_p^2 \right) \right\} = - F_p u_p - Q_p \quad (11)$$

Adding the momentum equation (4), this reduces to:

$$\frac{d}{dx} \left\{ \rho_p u_p e_p \right\} = - Q_p \quad (12)$$

Now the particle flow is made up of a liquid fraction K_e and a solid fraction $(1 - K_e)$. Before freezing, $K_e = 1$; after freezing, $K_e = 0$, and takes values between 1 and 0 during freezing. Thus:

$$e_p = K_e e_l + (1 - K_e) e_s \quad (13)$$

We assume that the solid phase and the liquid phase have the same specific heat, C . If freezing takes place at the temperature T^* and h_e is the latent heat of fusion:

$$e_l(T^*) - e_s(T^*) = h_e \quad (14)$$

Using (13) and (14), the particle energy equation takes the form:

$$\rho_p u_p C \frac{dT_p}{dx} = - \rho_p \bar{c}_p a \frac{(T_p - T)}{\bar{\lambda}_T} - \rho_p u_p h_e \frac{dK_e}{dx} \quad (15)$$

For the gas we have the equation of state:

$$p = \rho R T \quad (16)$$

The above equations are a complete analytical description of one-dimensional gas - particle flow with solidification.

3. EQUILIBRIUM FLOW

The term "equilibrium flow" designates the circumstance where the velocities and temperatures of the two phases remain exactly equal throughout the flow, i. e., there is no velocity or temperature lag anywhere. It is to be shown that such a flow process is possible.

When there are no lags, the equations describing the flow can be rederived:

$$(1+\kappa) \rho u \frac{du}{dx} + \frac{dp}{dx} = 0 \quad (17)$$

$$\bar{C}_p T + \frac{1}{2} u^2 + \frac{\kappa}{\kappa+1} K_2 h_2 = \text{CONSTANT} \quad (18)$$

$$(1+\kappa) \rho u A = (1+\kappa) \dot{m} \quad (19)$$

$$p = \rho R T \quad (20)$$

where:

$$\bar{C}_p = \frac{C_p + \kappa C}{(1+\kappa)}$$

It is of interest to note that the above equations can be obtained from the general equations of the previous section by letting $\bar{\lambda}_v$ and $\bar{\lambda}_T$ tend to zero such that the terms $\frac{u_p - u}{\bar{\lambda}_v}$ and $\frac{T_p - T}{\bar{\lambda}_T}$ remain finite for $u_p \rightarrow u$ and $T_p \rightarrow T$. This just means that when the equilibration lengths tend to zero the flow moves towards equilibrium flow.

The flow in the nozzle can be divided into three regions of interest (Fig. 1). In region 1, the particles are all in the form of liquid droplets; and region 3, all in the form of solid particles. Solidification takes place in region 2, station 1 designating the begin-

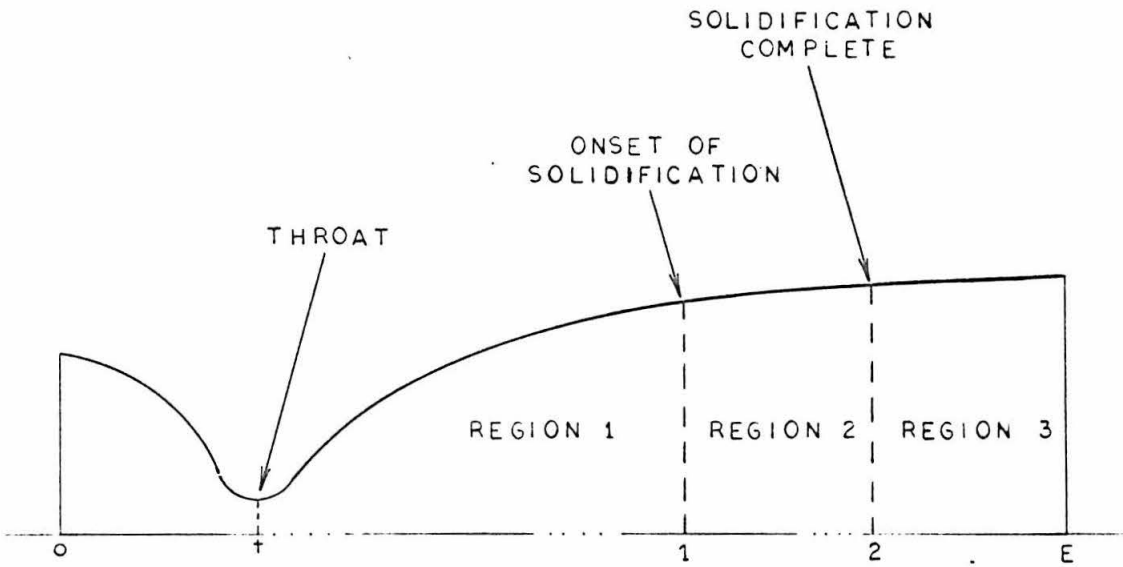


FIGURE 1. THE ROCKET NOZZLE

ning of freezing and station 2 designating the completion of solidification. We assume station 1 to be downstream of the throat.

The equilibrium flow has to be considered separately in each region.

(i) Region 1.

Here, $K_e = 1$, $\frac{dK_e}{dx} = 0$. The solution here is identical to the solution obtained by Marble. The flow is equivalent to the isentropic flow of a gas with effective properties:

$$\begin{aligned} \bar{C}_p &= \frac{C_p + Kc}{1 + K} , \quad \bar{\gamma} = \frac{C_p + Kc}{C_v + Kc} , \quad \bar{\rho} = (1 + K)\rho \\ \bar{R} &= \frac{C_p - C_v}{1 + K} = \frac{\bar{\gamma} - 1}{\bar{\gamma}} \bar{C}_p \end{aligned} \quad (21)$$

The expansion takes place according to the isentropic law:

$$\frac{T}{T_o} = \frac{T_p}{T_o} = \left(\frac{P}{P_o} \right)^{\frac{\bar{\gamma} - 1}{\bar{\gamma}}} \quad (22)$$

where the subscript 'o' refers to chamber conditions.

The complete solution in this region is written down for future reference:

$$u^2 = 2 \bar{C}_p T_o \left[1 - \left(\frac{P}{P_o} \right)^{\frac{\bar{\gamma} - 1}{\bar{\gamma}}} \right] \quad (23)$$

$$A = \frac{m}{\rho_o \sqrt{2 \bar{C}_p T_o}} \frac{1}{\left(\frac{P}{P_o} \right)^{\frac{1}{\bar{\gamma}}} \sqrt{1 - \left(\frac{P}{P_o} \right)^{\frac{\bar{\gamma} - 1}{\bar{\gamma}}}} \quad (24)$$

$$\left(\frac{P_i}{P_o} \right) = \left(\frac{T_i^*}{T_o} \right)^{\frac{\bar{\gamma}}{\bar{\gamma} - 1}} \quad (25)$$

$$u_1^2 = 2 \bar{C}_p T_0 \left[1 - \frac{T^*}{T_0} \right] \quad (26)$$

$$A_1 = \frac{m}{\rho_0 \sqrt{2 \bar{C}_p T_0}} \frac{1}{\left(\frac{T^*}{T_0} \right)^{\frac{1}{\delta}-1} \sqrt{1 - \frac{T^*}{T_0}}} \quad (27)$$

(ii) Region 2.

In this region, solidification takes place at the freezing temperature, T^* . The variable here is the liquid fraction, K_L .

$$T = T_P = T^*$$

From (17) and (18), the equation for the liquid fraction is:

$$K h_2 \frac{dK_L}{dx} = \frac{1}{\rho} \frac{d\rho}{dx} = \frac{RT^*}{\rho} \frac{d\rho}{dx} \quad (28)$$

or

$$K_L = 1 + \frac{RT^*}{K h_2} \log \frac{\rho}{\rho_1} \quad (29)$$

Using the values at station 1 determined by the solution in region 1, the solution in region 2 is given by:

$$\begin{aligned} u^2 &= 2 \left[\bar{C}_p (T_0 - T^*) - \frac{RT^*}{(1+K)} \log \frac{\rho}{\rho_1} \right] \\ &= 2 \left[\bar{C}_p (T_0 - T^*) - \frac{RT^*}{(1+K)} \log \frac{\rho}{\rho_0} \left(\frac{T_0}{T^*} \right)^{\frac{\delta}{\delta-1}} \right] \end{aligned} \quad (30)$$

$$A = \frac{m}{\rho u} = \frac{m R T^*}{\rho \sqrt{2 \left[\bar{C}_p (T_0 - T^*) - \frac{RT^*}{(1+K)} \log \frac{\rho}{\rho_1} \right]}} \quad (31)$$

At station 2, $K_e = 0$:

$$\frac{P_2}{P_1} = e^{-\frac{\kappa h_e}{RT^*}} \quad \text{or} \quad \frac{P_2}{P_0} = e^{-\frac{\kappa h_e}{RT^*} \left(\frac{T}{T_0}\right)^{\frac{\gamma}{\gamma-1}}} \quad (32)$$

$$u_2 = \sqrt{2 \left[\bar{c}_p (T_0 - T^*) + \frac{\kappa}{\kappa+1} h_e \right]} \quad (33)$$

$$T_2 = T^* , \quad K_2 = 0 \quad (34)$$

The equilibrium isothermal flow in region 2 can be understood as follows. As the flow process is an expansion, the tendency is for the gas temperature to fall. But solidification requires the particles to give up their latent heat, at constant temperature. The flow adjusts itself such that the latent heat is distributed between the internal and kinetic energies of the gas in such a fashion as to maintain 'equilibrium.' We note that at the completion of solidification, the contribution to the kinetic energy per unit mass is $\frac{\kappa}{\kappa+1} h_e$ and the contribution to the internal energy is $\frac{\kappa^2}{\kappa+1} h_e$. Since κ is usually less than 1, more latent heat goes into speeding the flow than in raising the temperature.

(iii) Region 3.

Here, $K_e = 0$, $\frac{dK_e}{dx} = 0$. Here the situation is the same as in region 1 except that the energy is increased by a constant factor. The solution is:

$$\frac{T}{T_0} = \frac{T}{T_2} \cdot \frac{T_2}{T_0} = \frac{T^*}{T_0} \left(\frac{p}{p_2} \right)^{\frac{\bar{\gamma}-1}{\bar{\gamma}}} = \left[\frac{p}{p_0} e^{\frac{\kappa h_2}{RT^*}} \right]^{\frac{\bar{\gamma}-1}{\bar{\gamma}}} \quad (35)$$

$$u = \left[2 \bar{c}_p T_0 \left\{ 1 - \left(\frac{p}{p_0} e^{\frac{\kappa h_2}{RT^*}} \right)^{\frac{\bar{\gamma}-1}{\bar{\gamma}}} \right\} + \frac{2 \kappa h_2}{(\kappa+1)} \right]^{\frac{1}{2}} \quad (36)$$

$$A = \frac{m R T_0 \left[\frac{p}{p_0} e^{\frac{\kappa h_2}{RT^*}} \right]^{\frac{\bar{\gamma}-1}{\bar{\gamma}}}}{p \left[2 \bar{c}_p T_0 \left\{ 1 - \left(\frac{p}{p_0} e^{\frac{\kappa h_2}{RT^*}} \right)^{\frac{\bar{\gamma}-1}{\bar{\gamma}}} \right\} + \frac{2 \kappa h_2}{(1+\kappa)} \right]^{\frac{1}{2}}} \quad (37)$$

Equations (22) to (37) completely describe an equilibrium flow through the nozzle.

4. PERTURBATION EQUATIONS FOR SMALL SLIP

It was shown in the last section that an equilibrium flow did exist and that this corresponded to $\bar{\lambda}_v$ tending to zero. When the equilibration length is negligible compared to the length of the nozzle, the gas and the particles are able to achieve a local equilibrium state while traversing a minute distance of the nozzle. It is natural then to seek a perturbation about this equilibrium flow for small slip. From the knowledge that the flow tends to equilibrium for $\bar{\lambda}_v \rightarrow 0$, $\bar{\lambda}_v$ seems to be the natural perturbation parameter.

Before proceeding with the perturbation analysis, the governing equations will be rewritten, following Marble, in terms of the slip quantities. As suggested by the analyses of Rannie and Marble, the pressure will be used as the nozzle variable in order to avoid the singularity at the throat.

The slip quantities u_s, p_s, T_s are defined by:

$$\begin{aligned} u_s &= u - u_p \\ p_s &= 1 - (p_p / k_p) \\ T_s &= T - T_p \end{aligned} \tag{38}$$

Let ξ be a dimensionless distance variable defined by:

$$\xi = x/L \tag{39}$$

where x is the distance along the nozzle and L is the nozzle length.

We now rewrite the equations in a form convenient for the perturbation analysis. First, eliminating between the momentum equa-

tions (3) and (4) the force F_p :

$$(1+\kappa) \rho u \frac{du}{d\xi} + \frac{dp}{d\xi} = \kappa \rho u \frac{du_s}{d\xi} \quad (40)$$

From the momentum equation for the particle:

$$u \frac{du}{d\xi} - \frac{a}{\bar{\lambda}_v/L} u_s = u \frac{du_s}{d\xi} - \frac{\rho_s a}{\bar{\lambda}_v/L} u_s \quad (41)$$

Eliminating the heat exchange between the two phases by combining (8) and (15) and using (3) and (4) yields an elementary energy integral:

$$\bar{c}_p (T - T_0) + \frac{1}{2} u^2 - \frac{\kappa}{\kappa+1} h_e (1 - \kappa_e) = \frac{\kappa}{\kappa+1} \left[c T_s + u u_s - \frac{1}{2} u_s^2 \right] \quad (42)$$

Again, the use of the two energy equations leads to the following two equations:

$$(T/T_0) / (P/P_0)^{\frac{\bar{\gamma}-1}{\bar{\gamma}}} = \exp \left[\frac{\kappa}{\kappa+1} \int_0^\xi \frac{1}{\bar{c}_p T} \left\{ \frac{a}{(u-u_s)} \frac{u_s^2}{\bar{\lambda}_v/L} + c \frac{dT_s}{d\xi'} - h_e \frac{d\kappa_e}{d\xi'} \right\} d\xi' \right] \quad (43)$$

$$u c \frac{dT}{d\xi} + h_e u \frac{d\kappa_e}{d\xi} - \frac{\bar{c}_p a}{\bar{\lambda}_v/L} T_s = u c \frac{dT_s}{d\xi} - \frac{\bar{c}_p a}{\bar{\lambda}_v/L} \rho_s T_s \quad (44)$$

With the gas pressure, P , as the independent variable, the following set of equations completely describes the flow, in a manner suitable for a perturbation treatment:

$$\rho u A = \dot{m} \quad (45)$$

$$u_s + \rho_s u = \rho_s u_s \quad (46)$$

$$\bar{c}_p (T - T_0) + \frac{1}{2} u^2 - \frac{\kappa}{\kappa + 1} h_e (1 - \kappa_e) = \frac{\kappa}{\kappa + 1} \left[c T_s + u u_s - \frac{1}{2} u_s^2 \right] \quad (47)$$

$$u \frac{du}{dp} \left(\frac{dp}{d\xi} \right) - a u_s / \bar{\lambda}_v / L = u \left(\frac{du_s}{dp} \right) \left(\frac{dp}{d\xi} \right) - \frac{a \bar{c}_p u_s}{\bar{\lambda}_v / L} \quad (48)$$

$$\left(T / T_0 \right) / \left(p / p_0 \right)^{\frac{\bar{\gamma} - 1}{\bar{\gamma}}} = \exp \left[\frac{\kappa}{\kappa + 1} \int_{p_0}^p \left\{ \frac{c}{\bar{c}_p T} \frac{dT_s}{dp'} + \frac{u_s}{\bar{c}_p T} \frac{d(u - u_s)}{dp'} - \frac{h_e}{\bar{c}_p T} \frac{d\kappa_e}{dp'} \right\} dp' \right] \quad (49)$$

$$u c \frac{dT}{dp} \left(\frac{dp}{d\xi} \right) + h_e u \frac{d\kappa_e}{dp} \left(\frac{dp}{d\xi} \right) - \frac{\bar{c}_p a}{\bar{\lambda}_v / L} T_s = \left\{ u c \frac{dT_s}{dp} \left(\frac{dp}{d\xi} \right) - \frac{\bar{c}_p a}{\bar{\lambda}_v / L} \bar{c}_p T_s \right\} \quad (50)$$

$$p = \bar{\gamma} R T \quad (51)$$

Seeking a perturbation solution, we assume that the gas properties and the liquid fraction may be written as:

$$u = u^{(0)} + (\bar{\lambda}_v / L) u^{(1)} + (\bar{\lambda}_v / L)^2 u^{(2)} + \dots$$

$$\bar{\gamma} = \bar{\gamma}^{(0)} + (\bar{\lambda}_v / L) \bar{\gamma}^{(1)} + (\bar{\lambda}_v / L)^2 \bar{\gamma}^{(2)} + \dots \quad (52)$$

$$T = T^{(0)} + (\bar{\lambda}_v / L) T^{(1)} + (\bar{\lambda}_v / L)^2 T^{(2)} + \dots$$

$$\kappa_e = \kappa_e^{(0)} + (\bar{\lambda}_v / L) \kappa_e^{(1)} + (\bar{\lambda}_v / L)^2 \kappa_e^{(2)} + \dots$$

Each variable has a non-vanishing 0th order part and all the coefficients in each of the expansions are of order unity. The slip quantities, on the other hand, have leading terms of the first order, since the perturbation is about equilibrium flow:

$$\begin{aligned}
 u_s &= (\bar{\lambda}_v/L) u_s^{(1)} + (\bar{\lambda}_v/L)^2 u_s^{(2)} + \dots \\
 T_s &= (\bar{\lambda}_v/L) T_s^{(1)} + (\bar{\lambda}_v/L)^2 T_s^{(2)} + \dots \\
 \rho_s &= (\bar{\lambda}_v/L) \rho_s^{(1)} + (\bar{\lambda}_v/L)^2 \rho_s^{(2)} + \dots
 \end{aligned} \tag{53}$$

Substitution of the expressions in (52) and (53) for each variable, into the equations (45) to (51) and separation of each equation according to powers of $(\bar{\lambda}_v/L)$ lead to equations giving the terms of the perturbation expansions.

The 0th order parts of equations (45) to (51) are exactly equations (17) to (20) describing the equilibrium flow. Clearly, this is a check on the consistency of the perturbation scheme, and on the interpretation given to the parameter $\bar{\lambda}_v$. The 0th order quantities $u^{(0)}$, $T^{(0)}$, $\rho^{(0)}$, etc. are then given in the three flow regions by the expressions worked out in the section on equilibrium flow [equations (22) to (37)].

The continuity equations (45) and (46) lead to the 1st order equations:

$$\frac{\rho_s^{(1)}}{\rho_s^{(0)}} + \frac{u_s^{(1)}}{u_s^{(0)}} + \frac{A^{(1)}}{A^{(0)}} = \frac{\dot{m}^{(1)}}{\dot{m}^{(0)}} \tag{54}$$

$$u_s^{(1)} + \rho_s^{(1)} u_s^{(0)} = 0 \tag{55}$$

We note that $\dot{m}^{(1)}$ must be independent of the pressure.

Equations (48) and (50) determine $u_s^{(1)}$ and $T_s^{(1)}$ algebraically, indicating that the perturbation is a singular one:

$$u_s^{(1)} = \frac{u^{(0)}}{\alpha^{(0)}} \frac{du^{(0)}}{dp} \left(\frac{dp}{d\xi} \right) \quad (56)$$

$$T_s^{(1)} = \frac{c}{\bar{c}_p} \left(\frac{\bar{\lambda}_T}{\bar{\lambda}_v} \right) \frac{u^{(0)}}{\alpha^{(0)}} \frac{dT^{(0)}}{dp} \left(\frac{dp}{d\xi} \right) + \frac{h_e}{\bar{c}_p} \left(\frac{\bar{\lambda}_T}{\bar{\lambda}_v} \right) \frac{u^{(0)}}{\alpha^{(0)}} \frac{dK_e^{(0)}}{dp} \left(\frac{dp}{d\xi} \right) \quad (57)$$

Equations (47) and (51) lead to the algebraic relations:

$$\bar{c}_p T^{(1)} + u^{(0)} u^{(1)} + \frac{\kappa}{\kappa+1} h_e K_e^{(1)} = \frac{\kappa}{\kappa+1} \left[c T_s^{(1)} + u^{(0)} u_s^{(1)} \right] \quad (58)$$

$$\frac{S^{(1)}}{p^{(0)}} + \frac{T^{(1)}}{T^{(0)}} = 0 \quad (59)$$

Equation (49) is slightly more difficult to handle:

$$\text{L.H.S.} = \frac{T/T_0}{(p/p_0)^{\frac{\bar{\gamma}-1}{\bar{\gamma}}}} = \frac{1}{(p/p_0)^{\frac{\bar{\gamma}-1}{\bar{\gamma}}}} \left[\frac{T^{(0)}}{T_0} + \frac{\bar{\lambda}_v}{L} \frac{T^{(1)}}{T_0} + \dots \right]$$

$$\begin{aligned} \frac{1}{T} \frac{dK_e}{dp} &= \frac{1}{(T^{(0)} + \frac{\bar{\lambda}_v}{L} T^{(1)} + \dots)} \frac{d}{dp} \left(K_e^{(0)} + \frac{\bar{\lambda}_v}{L} K_e^{(1)} + \dots \right) \\ &= \frac{1}{T^{(0)}} \frac{dK_e^{(0)}}{dp} + \frac{\bar{\lambda}_v}{L} \left\{ \frac{1}{T^{(0)}} \frac{dK_e^{(1)}}{dp} - \frac{T^{(1)}}{T^{(0)2}} \frac{dK_e^{(0)}}{dp} \right\} + \dots \end{aligned}$$

Expanding the exponential on the right hand side:

$$\begin{aligned}
 \text{R.H.S.} = & \left\{ \exp\left(-\frac{\kappa}{\kappa+1} \int_{p_0}^p \frac{h_e}{\bar{c}_p T^{(0)}} \frac{dke^{(0)}}{dp'} dp'\right) \right\} \times \\
 & \times \left[1 + \frac{\bar{\lambda}_v}{L} \frac{\kappa}{\kappa+1} \int_{p_0}^p \left\{ \frac{c}{\bar{c}_p T^{(0)}} \frac{dT_s^{(1)}}{dp'} + \frac{u_s^{(1)}}{\bar{c}_p T^{(0)}} \frac{du^{(0)}}{dp'} + \right. \right. \\
 & \left. \left. + \frac{h_e T^{(1)}}{\bar{c}_p T^{(0)2}} \frac{dke^{(0)}}{dp'} - \frac{h_e}{\bar{c}_p T^{(0)}} \frac{dke^{(1)}}{dp'} \right\} dp' \right]
 \end{aligned}$$

From the knowledge of the 0th order solution, the above lead to the first order equation for the gas temperature:

$$\frac{T^{(1)}}{T^{(0)}} = \frac{\kappa}{\kappa+1} \int_{p_0}^p \left\{ \frac{c}{\bar{c}_p T^{(0)}} \frac{dT_s^{(1)}}{dp'} + \frac{u_s^{(1)}}{\bar{c}_p T^{(0)}} \frac{du^{(0)}}{dp'} + \frac{h_e T^{(1)}}{\bar{c}_p T^{(0)2}} \frac{dke^{(0)}}{dp'} - \frac{h_e}{\bar{c}_p T^{(0)}} \frac{dke^{(1)}}{dp'} \right\} dp' \quad (60)$$

Equations (54) to (60) now give the 1st order corrections to the equilibrium flow.

5. THE SMALL SLIP SOLUTIONS

The first order perturbations to the gas and slip quantities have now to be worked out in turn, in regions 1, 2, and 3. We shall assume that solidification starts only after the throat, i. e., the throat is in region 1. All the problems associated with the singularity at the throat can be considered, then, with only the liquid and gaseous phases present.

Region 1.

The 0th order solution is given by the equilibrium solution worked out in Section 3. $K_\ell^{(0)} = 1$ and since only the liquid phase is present:

$$K_\ell^{(1)} = 0 \quad (61)$$

The first order solutions in this region are then identical to those of Marble. Equations (56) and (57) directly yield the first order velocity and temperature slips:

$$u_s^{(1)} = \frac{u^{(0)}}{a^{(0)}} \frac{du^{(0)}}{dp} \left(\frac{dp}{d\xi} \right) = - \frac{a^{(0)}}{\bar{\gamma}} \left(\frac{1}{p} \frac{dp}{d\xi} \right) \quad (62)$$

$$T_s^{(1)} = \frac{1}{\bar{c}_p} \frac{c}{\bar{c}_p} \left(\frac{\bar{\lambda}_T}{\bar{\lambda}_V} \right) \frac{a^{(0)}}{\bar{\gamma}} u^{(0)} \left(\frac{1}{p} \frac{dp}{d\xi} \right) = - \frac{1}{\bar{c}_p} \frac{c}{\bar{c}_p} \left(\frac{\bar{\lambda}_T}{\bar{\lambda}_V} \right) u^{(0)} u_s^{(1)} \quad (63)$$

Substitution of (62) into (55) yields the density slip:

$$\rho_s^{(1)} = \frac{1}{\bar{\gamma}} \frac{1}{M^{(0)}} \left(\frac{1}{p} \frac{dp}{d\xi} \right) \quad (64)$$

The gas temperature is obtained by substituting (62) and (63) into (60):

$$\begin{aligned}
 \frac{T^{(1)}}{T^{(0)}} &= \frac{\kappa}{\kappa+1} \int_{P_0}^P \frac{1}{\bar{c}_p T^{(0)}} \left[c \frac{dT_s^{(1)}}{dp'} + u_s^{(1)} \frac{du^{(0)}}{dp'} \right] dp' \\
 &= \frac{\kappa}{\kappa+1} \int_{P_0}^P \frac{1}{\bar{c}_p T^{(0)}} \left[(1-\underline{m}) u_s^{(1)} \frac{du^{(0)}}{dp'} - \underline{m} u^{(0)} \frac{du_s^{(1)}}{dp'} \right] dp' \\
 &= \frac{\kappa}{\kappa+1} G\left(\frac{P}{P_0}\right) \tag{65}
 \end{aligned}$$

where:

$$\underline{m} = \left(\frac{c}{\bar{c}_p} \right)^2 \frac{\bar{\lambda}_T}{\bar{\lambda}_V} \tag{66}$$

$$G\left(\frac{P}{P_0}\right) = \int_{P_0}^P \frac{1}{\bar{c}_p T^{(0)}} \left[(1-\underline{m}) u_s^{(1)} \frac{du^{(0)}}{dp'} - \underline{m} u^{(0)} \frac{du_s^{(1)}}{dp'} \right] dp' \tag{67}$$

From (59):

$$\frac{P^{(1)}}{P^{(0)}} = - \frac{\kappa}{\kappa+1} G\left(\frac{P}{P_0}\right) \tag{68}$$

Equations (65), (62), and (63) substituted into (58) yield the gas slip velocity:

$$\frac{u^{(1)}}{u^{(0)}} = \frac{\kappa}{\kappa+1} \frac{1}{(\bar{\gamma}-1) M^{(0)2}} \left[F\left(\frac{P}{P_0}\right) - G\left(\frac{P}{P_0}\right) \right] \tag{69}$$

where:

$$F\left(\frac{P}{P_0}\right) = (\underline{m}-1) \left(\frac{\bar{\gamma}-1}{\bar{\gamma}}\right) M^{(0)} \left(\frac{1}{P} \frac{dP}{d\xi}\right) \quad (70)$$

Equations (61) to (70) give the 1st order corrections to all the gas and slip quantities in region 1. One quantity to be determined yet is the correction to the mass flow. From equation (54):

$$\begin{aligned} \frac{\dot{m}^{(1)}}{\dot{m}^{(0)}} - \frac{A^{(1)}}{A^{(0)}} &= \frac{S^{(1)}}{P^{(0)}} + \frac{u^{(1)}}{u^{(0)}} \\ &= \frac{K}{K+1} \left[\frac{(\underline{m}-1)}{\bar{\gamma} M^{(0)}} \left(\frac{1}{P} \frac{dP}{d\xi}\right) - \frac{1 + (\bar{\gamma}-1) M^{(0)2}}{(\bar{\gamma}-1) M^{(0)2}} G\left(\frac{P}{P_0}\right) \right] \end{aligned}$$

Partial integration of (67) gives the following expression for

$$G\left(\frac{P}{P_0}\right) : \quad G\left(\frac{P}{P_0}\right) = \frac{\bar{\gamma}-1}{\bar{\gamma}} \underline{m} M^{(0)} \frac{1}{P} \frac{dP}{d\xi} + \frac{\bar{\gamma}-1}{\bar{\gamma}^2} \int_1^{P/P_0} \left\{ \frac{1 + (\bar{\gamma}-1) \underline{m} M^{(0)2}}{M^{(0)}} \right\} \frac{1}{\alpha^2} \left(\frac{d\alpha}{d\xi}\right) d\alpha \quad (71)$$

$$\alpha = P/P_0$$

Substituting we get:

$$\begin{aligned} \frac{\dot{m}^{(1)}}{\dot{m}^{(0)}} - \frac{A^{(1)}}{A^{(0)}} &= -\frac{K}{K+1} \cdot \frac{1}{\bar{\gamma} M^{(0)}} \left[\left\{ 1 + \underline{m} (\bar{\gamma}-1) M^{(0)2} \right\} \frac{1}{P} \frac{dP}{d\xi} + \right. \\ &\quad \left. + \frac{1 + (\bar{\gamma}-1) M^{(0)2}}{\bar{\gamma}^2 M^{(0)2}} \int_1^{P/P_0} \frac{1 + (\bar{\gamma}-1) \underline{m} M^{(0)2}}{M^{(0)}} \frac{1}{\alpha^2} \left(\frac{d\alpha}{d\xi}\right) d\alpha \right] \quad (72) \end{aligned}$$

If P is given as a function of ξ or the inverse, both $\dot{m}^{(1)}$

and $A^{(n)}$ are unknowns in the above equation. The key to the problem lies in the fact that $\dot{m}^{(1)}$ is independent of pressure from continuity. Then, clearly, equation (72) implies that $A^{(1)}$ at the throat is zero, giving $\dot{m}^{(1)}$. Let $p = p_t$ be the throat pressure. Then:

$$p_t = p_0 \left(\frac{2}{\bar{\gamma} + 1} \right)^{\frac{\bar{\gamma}}{\bar{\gamma} - 1}}, \quad M_t^{(0)} = 1 \quad (73)$$

$$\frac{\dot{m}^{(1)}}{\dot{m}^{(0)}} = \text{constant} = -\frac{k}{k+1} \frac{1}{\bar{\gamma}} \left[\left\{ 1 + \frac{M_t^{(0)2}}{\bar{\gamma}} \right\} \frac{1}{p_t} \left(\frac{dp}{d\xi} \right)_t + \right. \\ \left. + \frac{1 + (\bar{\gamma} - 1)}{\bar{\gamma}^2} \int_1^{p_t/p_0} \frac{1 + (\bar{\gamma} - 1) \frac{M^{(0)2}}{\bar{\gamma}}}{M^{(0)}} \frac{1}{\alpha^2} \left(\frac{d\alpha}{d\xi} \right) d\alpha \right] \quad (74)$$

Substitution of (74) into (72) then determines the first order perturbation to the area.

Digressing slightly, one may inquire about the procedure to follow if the area $A = A(x)$ is given rather than $p = p(\xi)$. Then, the location along the nozzle would also have to be expanded in a perturbation series:

$$x(p) = x^{(0)}(p) + \frac{\bar{\lambda}_v}{L} x^{(1)}(p) + \dots \quad (75)$$

$$A(x) = A[x(p)] = A^{(0)}(p) + \frac{\bar{\lambda}_v}{L} A^{(1)}(p) + \dots \\ = A(x^{(0)}) + \left. \frac{dA}{dx} \right|_{x^{(0)}} \cdot \frac{\bar{\lambda}_v}{L} x^{(1)}(p) + \dots \quad (76)$$

$$\therefore \frac{A'''}{A^{(0)}} = \frac{1}{A(x^{(0)})} \left(\frac{dA}{dx} \right)_{x^{(0)}} \cdot x'''(p) \quad (77)$$

Substituting into (72):

$$\frac{\dot{m}'''}{\dot{m}^{(0)}} - \frac{A'''}{A^{(0)}} = \frac{\dot{m}'''}{\dot{m}^{(0)}} - \frac{1}{A(x^{(0)})} \left(\frac{dA}{dx} \right)_{x^{(0)}} \cdot x'''(p) \quad (78)$$

At the throat, $\frac{dA}{dx} = 0$, and \dot{m}''' is again given by equation (74).

Equation (78) then determines the axial location perturbation as:

$$\begin{aligned} x'''(p) = & \frac{A(x^{(0)})}{\left(\frac{dA}{dx} \right)_{x^{(0)}}} \left[\frac{k}{k+1} \frac{1}{\bar{\gamma} M^{(0)}} \left\{ \left(1 + \underline{\eta}(\bar{\gamma}-1) M^{(0)2} \right) \frac{1}{P} \frac{dp}{d\xi^{(0)}} + \right. \right. \\ & \left. \left. + \frac{1 + (\bar{\gamma}-1) M^{(0)2}}{\bar{\gamma}^2 M^{(0)2}} \int_1^{P/P_0} \frac{1 + (\bar{\gamma}-1) \underline{\eta} M^{(0)2}}{M^{(0)}} \frac{1}{\alpha^2} \frac{d\alpha}{d\xi^{(0)}} \cdot d\alpha \right\} + \frac{\dot{m}'''}{\dot{m}^{(0)}} \right] \quad (79) \end{aligned}$$

This expression gives $x'''(p)$ everywhere except at $P = P_t$ since the denominator vanishes at the throat. At the throat:

$$x'''(P_t) = \lim_{\substack{P \rightarrow P_t \\ x^{(0)} \rightarrow x_t}} \left\{ \frac{\frac{\dot{m}'''}{\dot{m}^{(0)}} - \frac{S'''}{S^{(0)}} - \frac{u'''}{u^{(0)}}}{\frac{1}{A(x^{(0)})} \left(\frac{dA}{dx} \right)_{x^{(0)}}}} \right\}$$

After some algebra, assuming that $\frac{d^2 A}{dx^2}$ exists at the throat, we have:

$$\frac{1}{A_t} \left(\frac{d^2 A}{dx^2} \right)_{x_t} \cdot x^{(n)}(p_t) = \frac{K}{K+1} \frac{1}{\bar{\delta} L} \left[\frac{(1 + \underline{m}(\bar{\delta}-1))}{p_t} \left(\frac{d^2 p}{d\xi^{(n)2}} \right)_{p_t} + \right. \\ \left. + \frac{1}{p_t^2} \left(\frac{dp}{d\xi^{(n)}} \right)_{p_t}^2 \frac{\{3 + \underline{m}(4\bar{\delta} - 3\bar{\delta}^2 - 1) - \bar{\delta}\}}{2\bar{\delta}} + \right. \\ \left. + \frac{(\bar{\delta}+1)(\bar{\delta}+2)}{2\bar{\delta}^3 p_t} \left(\frac{dp}{d\xi^{(n)}} \right)_{p_t} \right] \left(\frac{1 + (\bar{\delta}-1)\underline{m} M^{(n)2}}{M^{(n)}} \frac{1}{\alpha^2} \left(\frac{d\alpha}{d\xi} \right) d\xi \right)_{p_t/p_0} \quad (80)$$

This demonstrates that the throat presents no special problems.

Region 2

In region 2, the liquid fraction is not constant, but the particle temperature, $T_p = T - T_s$, remains constant at the freezing temperature T^* . Before proceeding with the first order calculation, we note that an extra effect has to be taken into account. The first order correction to the particle temperature in region 1 causes the particle temperature at station 1 to be higher than the freezing temperature T^* by exactly the amount $T_{P1}^{(1)} = T_1^{(1)} - T_{S1}^{(1)}$, to first order (see

Figure 2). The freezing of the droplet is therefore delayed, forcing station 1 to be shifted downstream. Thus, the 0th order solution has to be corrected, the correction being of the order $\bar{\lambda}/L$. Let $\frac{\bar{\lambda}_v}{L} \tilde{T}_1$ be the correction to the 0th order temperature at station 1 in region 1:

$$T_{P1} = T_1^{(0)} + \frac{\bar{\lambda}_v}{L} \tilde{T}_1 + \frac{\bar{\lambda}_v}{L} (T_1^{(1)} - T_{S1}^{(1)}) + \dots = T^* \quad (81a)$$

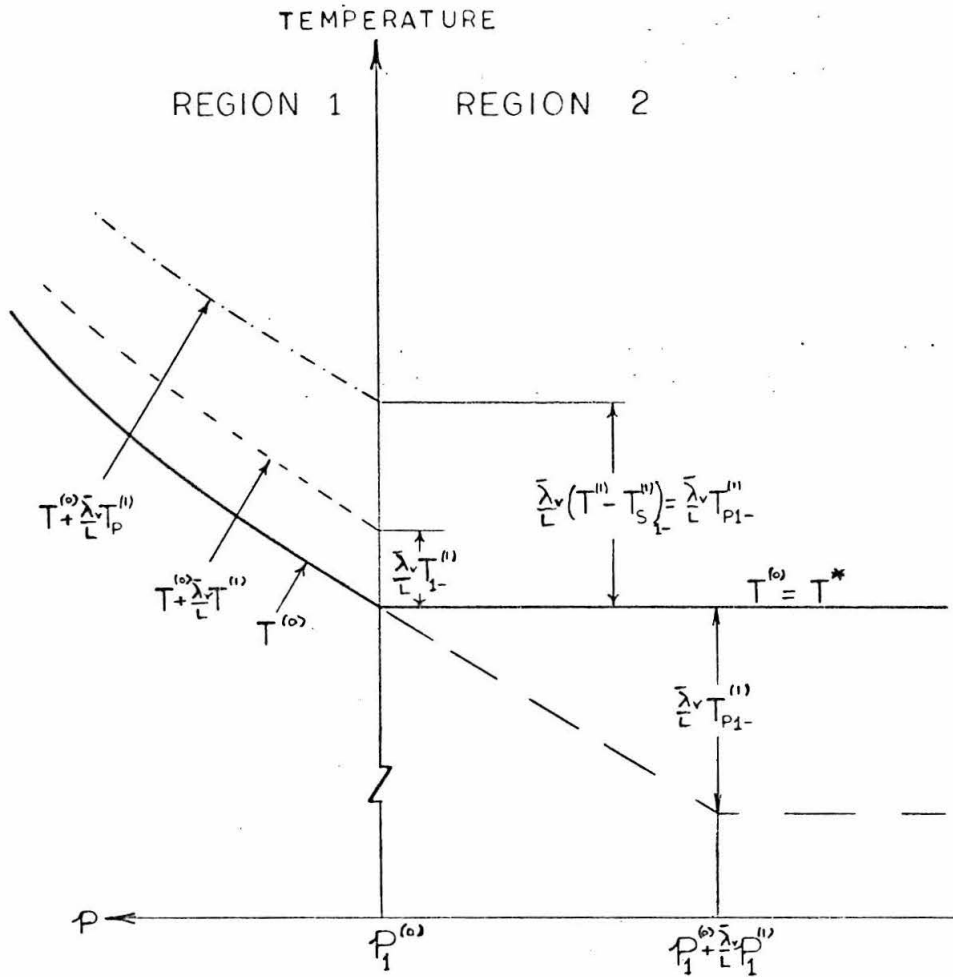


FIGURE 2. ILLUSTRATING THE SHIFT OF STATION 1

$$\therefore \tilde{T}_1 = - \left(T^{(0)} - T_S^{(0)} \right)_{1-} \quad (81b)$$

where 1- implies that the quantities are evaluated at station 1 in region 1. Therefore, in region 2, constant particle temperature implies

$$T_p = T - T_S = T^{(0)} + \frac{\bar{\lambda}_v}{L} \tilde{T} + \frac{\bar{\lambda}_v}{L} (T^{(1)} - T_S^{(1)}) + \dots = T^*$$

$$\therefore T^{(1)} - T_S^{(1)} = - \tilde{T} = - \tilde{T}_1 \quad (82)$$

The shift in station 1 and the correction to the 0th order solution cause first order corrections $\frac{\bar{\lambda}_v}{L} \tilde{u}$ and $\frac{\bar{\lambda}_v}{L} \tilde{k}_e$ in the gas velocity and the liquid fraction. These contributions have to be included in the 1st order solutions. Define:

$$\tilde{T}^{(1)} = T^{(1)} + \tilde{T} \quad (83a)$$

$$\tilde{u}^{(1)} = u^{(1)} + \tilde{u} \quad (83b)$$

$$\tilde{k}_e^{(1)} = k_e^{(1)} + \tilde{k}_e \quad (83c)$$

The corrections $\tilde{u}, \tilde{T}, \tilde{k}_e$ will be worked out in detail in the next section.

Proceeding with the first order calculation, $u_s^{(1)}$ is given by (56):

$$u_s^{(1)} = \frac{u^{(0)}}{a^{(0)}} \frac{du^{(0)}}{dp} \left(\frac{dp}{d\xi} \right) = \frac{1}{a^{(0)}} \left\{ - \frac{\bar{R} T^{(0)}}{p} \right\} \frac{dp}{d\xi}$$

$$= - \frac{a^{(0)}}{\bar{\delta}} \left(\frac{1}{p} \frac{dp}{d\xi} \right) \quad (84)$$

$$a^{(0)} = \sqrt{\bar{\delta} \bar{R} T^{(0)}} = \sqrt{\bar{\delta} \bar{R} T^*} \quad (85)$$

Equation (57) gives the first order correction to the slip temperature:

$$T_s^{(1)} = \frac{h_e}{\bar{c}_p} \left(\frac{\bar{\lambda}_T}{\bar{\lambda}_V} \right) \frac{u^{(0)}}{a^{(0)}} \frac{dke^{(0)}}{dp} \left(\frac{dp}{d\xi} \right)$$

From (29) of the 0th order solution:

$$\frac{dke^{(0)}}{dp} = \frac{RT^*}{k h_e} \frac{1}{p} = \frac{RT^{(0)}}{k h_e} \frac{1}{p} = \frac{(1+\kappa)}{\kappa} \frac{a^{(0)2}}{\bar{\delta} h_e} \frac{1}{p} \quad (86)$$

$$\begin{aligned} T_s^{(1)} &= \frac{h_e}{\bar{c}_p} \left(\frac{\bar{\lambda}_T}{\bar{\lambda}_V} \right) \frac{u^{(0)}}{a^{(0)}} \frac{(1+\kappa)}{\kappa} \frac{a^{(0)2}}{\bar{\delta} h_e} \frac{1}{p} \left(\frac{dp}{d\xi} \right) \\ &= \frac{(1+\kappa)}{\kappa \bar{c}_p} \left(\frac{\bar{\lambda}_T}{\bar{\lambda}_V} \right) \frac{a^{(0)}}{\bar{\delta}} u^{(0)} \left(\frac{1}{p} \frac{dp}{d\xi} \right) \end{aligned} \quad (87)$$

Equations (55) and (84) lead to $\rho_s^{(1)}$:

$$\rho_s^{(1)} = - \frac{u_s^{(1)}}{u^{(0)}} = \frac{1}{\bar{\delta}} \frac{1}{M^{(0)}} \left(\frac{1}{p} \frac{dp}{d\xi} \right) \quad (88)$$

Noting the fact that $dT_s^{(1)} = dT^{(1)}$ in region 2, eqn. (60)

leads to the correction in the liquid fraction $K_L^{(1)}$:

$$\frac{\kappa+1}{\kappa} \bar{c}_p \frac{dT^{(1)}}{dp} = c \frac{dT_s^{(1)}}{dp} + u_s^{(1)} \frac{du^{(0)}}{dp} + h_e \frac{T^{(1)}}{T^{(0)}} \frac{dke^{(0)}}{dp} - h_e \frac{dke^{(1)}}{dp} \quad (89)$$

$$K_e^{(1)} = A_1' + \int_{P_1}^P \left[-\frac{C_p}{\kappa h_e} \frac{dT_s^{(1)}}{dp'} + \frac{u_s^{(1)}}{h_e} \frac{du^{(1)}}{dp'} + \frac{T^{(1)}}{T^{(1)}} \frac{dK_e^{(1)}}{dp'} \right] dp' \quad (90)$$

where A_1' is a constant and P_1 is the pressure at station 1.

Substitution of (84) - (90) into (58) yields the first order gas velocity, $u^{(1)}$:

$$u^{(1)} = \frac{1}{u^{(1)}} \left[-\frac{C_p}{(\kappa+1)} T_s^{(1)} + \frac{\kappa}{\kappa+1} u^{(1)} u_s^{(1)} - \frac{\kappa}{\kappa+1} h_e K_e^{(1)} \right] \quad (91)$$

Equations (81) to (91), now, formally determine the solutions to first order in region 2:

$$u = u^{(1)} + \frac{\bar{\lambda}_v}{L} (u^{(1)} + \tilde{u}) + O\left(\frac{\bar{\lambda}_v}{L}\right)^2 = u^{(1)} + \frac{\bar{\lambda}_v}{L} \tilde{u}^{(1)} + \dots$$

$$K_e = K_e^{(1)} + \frac{\bar{\lambda}_v}{L} (K_e^{(1)} + \tilde{K}_e) + O\left(\frac{\bar{\lambda}_v}{L}\right)^2 = K_e^{(1)} + \frac{\bar{\lambda}_v}{L} \tilde{K}_e^{(1)} + \dots$$

etc.

(92)

Some interesting features are immediately apparent from the inspection of the equations. The slip temperature $T_s^{(1)}$ and the first order gas temperature $T^{(1)}$ are discontinuous across station 1. Also, A_1' , an arbitrary constant of integration, appears in the solution for $K_e^{(1)}$. These facts confirm the singular nature of the perturbation scheme.

It was assumed in the perturbation analysis that $\bar{\lambda}_v/L$ was small. This implied that the particles had time to adjust to the local

gas motion before they moved appreciably through the region. At stations 1 and 2, this does not hold. A sudden constraint, the particle temperature having to remain constant, placed on the system necessitates a relaxation zone of order $\bar{\lambda}_v$ over which the particles adjust to the change. This 'smoothing out' is lost in our 'outer solution.' To complete the solution, 'inner solutions' valid near the stations 1 and 2 will have to be worked out and matched to the outer solutions obtained.

Region 3

In this region, the particles are all solid, i. e., $K_e = 0 = K_e^{(0)} = \bar{K}_e^{(1)}$. We note that in this region too, corrections to the 0th order solution will cause first order corrections \tilde{T} & \tilde{u} to the gas temperature and velocity:

$$T = T^{(0)} + \frac{\bar{\lambda}_v}{L} (\tilde{T} + T^{(1)}) + O\left(\frac{\bar{\lambda}_v}{L}\right)^2 = T^{(0)} + \frac{\bar{\lambda}_v}{L} \tilde{T}^{(1)} + \dots \quad (93a)$$

$$u = u^{(0)} + \frac{\bar{\lambda}_v}{L} (\tilde{u} + u^{(1)}) + O\left(\frac{\bar{\lambda}_v}{L}\right)^2 = u^{(0)} + \frac{\bar{\lambda}_v}{L} \tilde{u}^{(1)} + \dots \quad (93b)$$

The calculations are very similar to the calculations in region

1. The first order solutions are:

$$u_s^{(1)} = - \frac{a^{(0)}}{\bar{\gamma}} \left(\frac{1}{p} \frac{dp}{d\xi} \right) \quad (94a)$$

$$\tilde{T}_s^{(1)} = \frac{1}{\bar{c}_p} \left(\frac{c}{\bar{c}_p} \right) \left(\frac{\bar{\lambda}_T}{\bar{\lambda}_v} \right) \frac{a^{(0)}}{\bar{\gamma}} u^{(0)} \left(\frac{1}{p} \frac{dp}{d\xi} \right) \quad (94b)$$

$$S_s^{(1)} = \frac{1}{\bar{\gamma}} \frac{1}{M^{(0)}} \left(\frac{1}{p} \frac{dp}{d\xi} \right) \quad (94c)$$

$$\frac{T^{(1)}}{T^{(0)}} = \frac{A'_2}{T^*} + \frac{K}{K+1} G_2 \left(\frac{P}{P_2} \right) \quad (94d)$$

$$\frac{U^{(1)}}{U^{(0)}} = \frac{1}{(\bar{\gamma}-1) M^{(0)2}} \frac{K}{K+1} \left[F(P/P_0) - \frac{K+1}{K} \frac{A'_2}{T^*} - G_2 \left(\frac{P}{P_2} \right) \right] \quad (94e)$$

where:

$$G_2 \left(\frac{P}{P_2} \right) = \int_{P_2}^P \frac{1}{\bar{C}_p T^{(0)}} \left[(1-\eta) u_s^{(1)} \frac{du^{(0)}}{dp} - \eta u^{(0)} \frac{du_s^{(1)}}{dp} \right] dp \quad (95)$$

and $F(P/P_0)$ is defined by (70).

We note once again the discontinuities across station 2 and the existence of an undetermined constant, A'_2 .

The solution of the first order perturbation equations is now complete.

6. THE EFFECTS OF THE SHIFTS OF THE BOUNDARIES
OF THE ZONE OF SOLIDIFICATION

In this section, we study in detail the corrections to the 0th order solution, \tilde{T} , \tilde{u} and \tilde{K}_2 mentioned in the previous section. The first order corrections to the particle temperature and liquid fraction cause shifts of the boundaries of region 2, the zone of solidification. The shifts of stations 1 and 2 produce first order corrections to the 0th order solutions, which have to be added to the first order quantities. This 'feedback effect' is peculiar to this problem and is a direct consequence of the phase change phenomenon. We consider in turn stations 1 and 2, and the effects in the regions 1, 2, and 3.

Station 1

Station 1 is by definition the point where the particle temperature, T_p , first falls to the value T^* , the solidification temperature. From the solutions in region 1, the particle temperature at station 1, T_{p1} , to first order is:

$$\begin{aligned}
 T_{p1} &= T_1 - T_{s1} = T_1^{(0)} + \frac{\bar{\lambda}_v}{L} (T_1^{(1)} - T_{s1}^{(1)}) + \dots \\
 &= T^* + \frac{\bar{\lambda}_v}{L} \left\{ \frac{\kappa}{\kappa+1} T^* G(R/p_0) - \frac{1}{\bar{c}_p} \frac{c}{c_p} \left(\frac{\bar{\lambda}_T}{\bar{\lambda}_v} \right) \frac{a_i^{(0)} u_i^{(0)}}{\delta} \left(\frac{1}{\rho} \frac{dp}{d\xi_1} \right) \right\} \quad (96)
 \end{aligned}$$

The bracket in the above expression is in general non-zero and is normally positive. The temperature at station 1 will not be T^* unless it is shifted. The shift should be such that the correction to the 0th order term cancels out the first order particle temperature at sta-

tion 1.

Let

$$P_1 = P_1^{(0)} + \frac{\bar{\lambda}_v}{L} P_1^{(1)} + \dots \quad (97a)$$

$$T_P = T^{(0)} + \frac{\bar{\lambda}_v}{L} T_P^{(1)} + \dots \quad (97b)$$

Solidification begins when $T_P = T^*$. This corresponds to the pressure P_1 .

$$\begin{aligned} T_P(P_1) &= T_P^{(0)}(P_1) + \frac{\bar{\lambda}_v}{L} T_P^{(1)}(P_1) + \dots \\ &= T_P^{(0)}(P_1^{(0)}) + \left(\frac{dT_P^{(0)}}{dP} \right)_{P_1^{(0)}} \cdot \frac{\bar{\lambda}_v}{L} P_1^{(1)} + \frac{\bar{\lambda}_v}{L} T_P^{(1)}(P_1^{(0)}) + \dots \\ &= T^* \end{aligned} \quad (98)$$

But $T_P^{(0)}(P_1^{(0)}) = T^*$

$$\therefore P_1^{(1)} = - \frac{T_P^{(1)}(P_1^{(0)})}{\left(\frac{dT_P^{(0)}}{dP} \right)_{P_1^{(0)}}} = - \frac{[T^{(1)}(P_1^{(0)}) - T_s^{(1)}(P_1^{(0)})]}{\left(\frac{dT^{(0)}}{dP} \right)_{P_1^{(0)}}}$$

$$= - \frac{\left[\frac{\kappa}{\kappa+1} T^* G\left(\frac{P_1}{P_0}\right) - \frac{1}{\bar{c}_p} \frac{c}{\bar{c}_p} \left(\frac{\bar{\lambda}_T}{\bar{\lambda}_v} \right) \frac{\alpha_1' u_1^{(0)}}{\bar{\gamma}} \left(\frac{1}{P} \frac{dP}{d\xi_1} \right) \right]}{\left(\frac{\bar{\gamma}-1}{\bar{\gamma}} \cdot \frac{T^*}{P_1^{(0)}} \right)} \quad (99)$$

This gives the first-order correction to the pressure at station

1. Note that the shift is determined entirely by the solution in region
1. Also, the correction to the 0th order temperature at 1 is:

$$\tilde{T}_1 = - (T'' - T_s'')_1 = - T_{P1}'' \quad (100)$$

Station 2

Station 2 is defined as the location where solidification is complete, that is, where K_L first vanishes. Using an argument similar to the one used for station 1, let:

$$P_2 = P_2^{(0)} + \frac{\bar{\lambda}_v}{L} P_2'' + \dots \quad (101a)$$

$$\begin{aligned} K_L(p) &= K_L^{(0)}(p) + \frac{\bar{\lambda}_v}{L} \{ \tilde{K}_L(p) + K_L'''(p) \} + \dots \\ &= K_L^{(0)}(p) + \frac{\bar{\lambda}_v}{L} \tilde{K}_L'''(p) + \dots \end{aligned} \quad (101b)$$

Here, $\tilde{K}_L(p)$ is the first-order correction to the liquid fraction in region 2 due to the shift of station 1. This will be evaluated, explicitly, later in this section. Now P_2 is the pressure corresponding to station 2.

$$\begin{aligned} K_L(P_2) &= K_L^{(0)}(P_2) + \frac{\bar{\lambda}_v}{L} \{ \tilde{K}_L'''(P_2) \} + \dots \\ &= K_L^{(0)}(P_2^{(0)}) + \left(\frac{dK_L^{(0)}}{dP} \right)_{P_2^{(0)}} \cdot \frac{\bar{\lambda}_v}{L} P_2'' + \frac{\bar{\lambda}_v}{L} \tilde{K}_L'''(P_2^{(0)}) + O\left(\frac{\bar{\lambda}_v}{L}\right)^2 \\ &= 0 \end{aligned} \quad (102)$$

But

$$K_L^{(0)}(P_2^{(0)}) = 0$$

$$\therefore p_2^{(1)} = - \frac{\tilde{K}_L^{(1)}(p_2^{(0)})}{\left(\frac{dK_L^{(0)}}{dp}\right)_{p_2^{(0)}}} \quad (103)$$

Equation (103) now gives the first order shift to the pressure at station 2. Note that $\tilde{K}_L^{(1)}(p_2^{(0)})$ is given by (90) and $\left(\frac{dK_L^{(0)}}{dp}\right)_{p_2^{(0)}}$ by (86).

It is now possible to calculate the corrections in the 0th order flow quantities due to the shifts of stations 1 and 2.

Region 1

The shift of station 1 does not affect the region upstream of it. Corrections have to be made only at station 1. At station 1, the gas temperature is given by:

$$T = T_1^{(0)} + \frac{\bar{\lambda}_v}{L} (\tilde{T}_1 + T_1^{(1)}) + \dots \quad (104a)$$

where

$$\tilde{T}_1 = - (T_1^{(1)} - T_{S1}^{(1)}) \quad (104b)$$

The first-order correction to the 0th order gas temperature due to the shift of station 1 is given by:

$$\begin{aligned} u^{(0)}(p_1) &= u^{(0)}\left(p_1^{(0)} + \frac{\bar{\lambda}_v}{L} p_1^{(1)} + \dots\right) \\ &= u_1^{(0)} + \frac{\bar{\lambda}_v}{L} \left(\frac{du^{(0)}}{dp}\right)_{p_1^{(0)}} \cdot p_1^{(1)} + \dots \\ &= u_1^{(0)} + \frac{\bar{\lambda}_v}{L} \left\{ -\frac{\bar{\sigma}-1}{\bar{\sigma}} \frac{\bar{C}_p T^*}{u_1^{(0)}} \frac{p_1^{(1)}}{p_1^{(0)}} \right\} + \dots \\ &= u_1^{(0)} + \frac{\bar{\lambda}_v}{L} \tilde{u}_1 + \dots \end{aligned}$$

To first order, therefore, the gas velocity at station 1 is given by:

$$u(p_1) = u_1^{(0)} + \frac{\bar{\lambda}_v}{L} (\tilde{u}_1 + u_1^{(1)}) + \dots \quad (105a)$$

$$\tilde{u}_1 = - \frac{\bar{\delta}-1}{\bar{\gamma}} \frac{\bar{c}_p T^*}{u_1^{(0)}} \frac{p_1^{(1)}}{p_1^{(0)}} = \frac{\bar{c}_p T_{p1}^{(1)}}{u_1^{(0)}} \quad (105b)$$

Note that \tilde{T}_1 and \tilde{u}_1 are corrections to the 0th order solution due to the shift of station 1.

Region 2

The shift of station 1 will affect all points in this region. The shift of station 2 will be of importance only to the values at station 2. The shift of station 1 corrects the 0th order temperature, $T^{(0)} = T^*$, by exactly the amount $-T_{p1-}$, so that the particle temperature is T^* to first order:

$$T = T^{(0)} + \frac{\bar{\lambda}_v}{L} \tilde{T}^{(1)} + \dots = T^* + \frac{\bar{\lambda}_v}{L} (\tilde{T} + T^{(1)}) + \dots \quad (106a)$$

$$\tilde{T} = \text{constant} = - (T^{(1)} - T_s^{(1)})_{1-} = - T_{p1-}^{(1)} \quad (106b)$$

where $1-$ implies quantities evaluated in region 1 at station 1.

The correction to the 0th order gas velocity is given by:

$$\begin{aligned} u^{(0)} \left[p_2 \left(p_1^{(0)} + \frac{\bar{\lambda}_v}{L} p_1^{(1)} \right), \left(T^{(0)} + \frac{\bar{\lambda}_v}{L} \tilde{T} \right) \right] \\ = u^{(0)} \left[p_2, p_1^{(0)}, T^* \right] + \frac{\bar{\lambda}_v}{L} \left[\frac{\partial u^{(0)}}{\partial p_1^{(0)}} \cdot p_1^{(1)} + \frac{\partial u^{(0)}}{\partial T^{(0)}} \cdot \tilde{T} \right] + O\left(\frac{\bar{\lambda}_v}{L}\right)^2 \\ = u^{(0)} + \frac{\bar{\lambda}_v}{L} \tilde{u} + \dots \end{aligned}$$

The first order corrections \tilde{u} and \tilde{K}_e to $u^{(0)}$ and $K_e^{(0)}$ due to the shift of station 1 are then:

$$\tilde{u}(p) = \frac{\bar{R} T_{P1-}^{(1)}}{u^{(0)}(p)} \log \frac{p}{P_1^{(0)}} \quad (107a)$$

$$\tilde{K}_e(p) = \frac{R T_{P1-}^{(1)}}{\kappa h_e} \left[\frac{\bar{\gamma}-1}{\bar{\gamma}} - \log \frac{p}{P_1^{(0)}} \right] \quad (107b)$$

With these corrections, in region 2:

$$u(p) = u^{(0)} + \frac{\bar{\lambda}_v}{L} (u^{(1)} + \tilde{u}) + \dots = u^{(0)} + \frac{\bar{\lambda}_v}{L} \tilde{u}^{(1)} + \dots$$

$$K_e(p) = K_e^{(0)} + \frac{\bar{\lambda}_v}{L} (K_e^{(1)} + \tilde{K}_e) + \dots = K_e^{(0)} + \frac{\bar{\lambda}_v}{L} \tilde{K}_e^{(1)} + \dots$$

where $u^{(0)}$ and $K_e^{(0)}$ are the uncorrected 0th order solutions.

The shift of station 2, however, corrects the 0th order gas velocity further by $\frac{\bar{\lambda}_v}{L} \tilde{u}_2$ given by:

$$\tilde{u}_2 = \left(\frac{du^{(0)}}{dp} \right)_{P_2^{(0)}} \cdot P_2^{(1)} \quad (108)$$

where $P_2^{(1)}$ is given by (103).

Region 3

The details of the calculation for this region are exactly as in region 2. The solutions correct to first order are:

$$u(p) = u^{(0)} + \frac{\bar{\lambda}_v}{L} (u^{(1)} + \tilde{u}) + \dots = u^{(0)} + \frac{\bar{\lambda}_v}{L} \tilde{u}^{(1)} + \dots$$

$$T(p) = T^{(0)} + \frac{\bar{\lambda}_v}{L} (T^{(1)} + \tilde{T}) + \dots = T^{(0)} + \frac{\bar{\lambda}_v}{L} \tilde{T}^{(1)} + \dots$$

etc.

where the corrections to the 0th order solution due to shifts of station

1 and 2 are given by:

$$\tilde{u}(p) = \frac{\bar{c}_p T^{(0)}(p)}{u^{(0)}(p)} \left(\frac{\bar{\delta}-1}{\bar{\delta}} \frac{p_2'''}{p_2^{(0)}} + \frac{T_{p1}'''}{T^*} \right) \quad (109a)$$

$$\tilde{T}(p) = - T^{(0)}(p) \left(\frac{\bar{\delta}-1}{\bar{\delta}} \frac{p_2'''}{p_2^{(0)}} + \frac{T_{p1}'''}{T^*} \right) \quad (109b)$$

7. THE INNER SOLUTIONS AROUND STATIONS 1 AND 2

The small slip solutions obtained in Sections 5 and 6 do not match at stations 1 and 2. In addition, the solutions in regions 2 and 3 contain two undetermined constants A'_1 and A'_2 . As explained in Section 5, these are a consequence of the singular nature of the perturbation scheme. The assumptions involved in seeking an asymptotic solution for $\bar{\lambda}_v \rightarrow 0$ do not hold at the boundaries of the solidification region. The sudden local changes introduce a region of non-uniformity of length of order $\bar{\lambda}_v$. Following Van Dyke [4], we seek 'inner solutions' about these points that can be matched to the existing outer solutions.

Introduce a stretched inner variable η defined by:

$$\eta = \begin{cases} \frac{(x-x_1)}{\bar{\lambda}_v} & \text{for solution about } x_1 & (110a) \\ \frac{(x-x_2)}{\bar{\lambda}_v} & \text{for solution about } x_2 & (110b) \end{cases}$$

$$d\eta = \frac{dx}{\bar{\lambda}_v} = \frac{L d\xi}{\bar{\lambda}_v} \quad (110c)$$

In terms of the inner variable, the governing equations

$$\rho u A = \dot{m} \quad (111a)$$

$$u_s + \rho_s u = \rho_s u_s \quad (111b)$$

$$\bar{c}_p (T - T_0) + \frac{1}{2} u^2 - \frac{K}{k+1} h_2 (1 - \kappa_2) = \frac{K}{k+1} \left[c T_s + u u_s - \frac{1}{2} u_s^2 \right] \quad (111c)$$

$$u \frac{du}{d\eta} - a u_s = u \frac{du_s}{d\eta} - a \rho_s u_s \quad (111d)$$

$$\frac{1}{T} \frac{dT}{d\eta} - \frac{\bar{\gamma}-1}{\bar{\gamma}} \frac{1}{P} \frac{dP}{d\eta} = \frac{\kappa}{\kappa+1} \frac{1}{\bar{c}_p T} \left[\frac{a}{u_p} u_s^2 + c \frac{dT_s}{d\eta} - h_e \frac{dK_e}{d\eta} \right] \quad (111e)$$

$$u c \frac{dT}{d\eta} + h_e u \frac{dK_e}{d\eta} - \frac{\bar{\lambda}_v}{\bar{\lambda}_T} \bar{c}_p a T_s = u c \frac{dT_s}{d\eta} - \frac{\bar{\lambda}_v}{\bar{\lambda}_T} \bar{c}_p a \rho_s T_s \quad (111f)$$

$$P = \rho R T \quad (111g)$$

Try perturbation solutions valid in the inner regions of the form:

$$u(\eta) = u^{(0)}(\eta) + \frac{\bar{\lambda}_v}{L} u^{(1)}(\eta) + \left(\frac{\bar{\lambda}_v}{L}\right)^2 u^{(2)}(\eta) + \dots \quad (112a)$$

$$T(\eta) = T^{(0)}(\eta) + \frac{\bar{\lambda}_v}{L} T^{(1)}(\eta) + \left(\frac{\bar{\lambda}_v}{L}\right)^2 T^{(2)}(\eta) + \dots \quad (112b)$$

$$u_s(\eta) = \frac{\bar{\lambda}_v}{L} u_s^{(1)}(\eta) + \left(\frac{\bar{\lambda}_v}{L}\right)^2 u_s^{(2)}(\eta) + \dots \quad (112c)$$

$$T_s(\eta) = \frac{\bar{\lambda}_v}{L} T_s^{(1)}(\eta) + \left(\frac{\bar{\lambda}_v}{L}\right)^2 T_s^{(2)}(\eta) + \dots \quad (112d)$$

etc.

Substitution into equations (111) yields the following 0th order equations:

$$\rho^{(0)} u^{(0)} A^{(0)} = \dot{m}^{(0)} \quad (113a)$$

$$u^{(0)} \frac{du^{(0)}}{d\eta} = 0 \quad (113b)$$

$$\bar{c}_p (T^{(0)} - T_0) + \frac{1}{2} u^{(0)2} - \frac{\kappa}{\kappa+1} h_e (1 - K_e^{(0)}) = 0 \quad (113c)$$

$$\frac{1}{T^{(0)}} \frac{dT^{(0)}}{d\eta} - \frac{\bar{\gamma}-1}{\bar{\gamma}} \frac{1}{p^{(0)}} \frac{dp^{(0)}}{d\eta} = - \frac{\kappa}{\kappa+1} \frac{h_e}{\bar{c}_p T^{(0)}} \frac{dK_e^{(0)}}{d\eta} \quad (113d)$$

$$u^{(0)} c \cdot \frac{dT^{(0)}}{d\eta} + h_e u^{(0)} \frac{dK_e^{(0)}}{d\eta} = 0 \quad (113e)$$

$$p^{(0)} = \rho^{(0)} R T^{(0)} \quad (113f)$$

The corresponding inner first-order equations are:

$$\frac{\rho^{(1)}}{\rho^{(0)}} + \frac{u^{(1)}}{u^{(0)}} + \frac{A^{(1)}}{A^{(0)}} = \frac{\dot{m}^{(1)}}{\dot{m}^{(0)}} \quad (114a)$$

$$u_s^{(1)} + u^{(0)} \rho_s^{(1)} = 0 \quad (114b)$$

$$u^{(0)} \frac{du^{(1)}}{d\eta} + u^{(1)} \frac{du^{(0)}}{d\eta} - a^{(0)} u_s^{(1)} = u^{(0)} \frac{du_s^{(1)}}{d\eta} \quad (114c)$$

$$\bar{c}_p T^{(1)} + u^{(0)} u^{(1)} + \frac{\kappa}{\kappa+1} h_e K_e^{(1)} = \frac{\kappa}{\kappa+1} \left[c T_s^{(1)} + u^{(0)} u_s^{(1)} \right] \quad (114d)$$

$$\begin{aligned} \frac{1}{T^{(0)}} \frac{dT^{(1)}}{d\eta} - \frac{\bar{\gamma}-1}{\bar{\gamma}} \frac{1}{p^{(0)}} \frac{dp^{(1)}}{d\eta} - \frac{T^{(1)}}{T^{(0)2}} \frac{dT^{(0)}}{d\eta} + \frac{\bar{\gamma}-1}{\bar{\gamma}} \frac{p^{(1)}}{p^{(0)2}} \frac{dp^{(0)}}{d\eta} \\ = \frac{\kappa}{\kappa+1} \left[\frac{c}{\bar{c}_p T^{(0)}} \frac{dT_s^{(1)}}{d\eta} - \frac{h_e}{\bar{c}_p T^{(0)}} \frac{dK_e^{(1)}}{d\eta} + \frac{h_e T^{(1)}}{\bar{c}_p T^{(0)2}} \frac{dK_e^{(0)}}{d\eta} \right] \end{aligned} \quad (114e)$$

$$\begin{aligned} u^{(0)} c \frac{dT^{(1)}}{d\eta} + h_e u^{(0)} \frac{dK_e^{(1)}}{d\eta} + u^{(1)} c \frac{dT^{(0)}}{d\eta} + h_e u^{(1)} \frac{dK_e^{(0)}}{d\eta} \\ = \frac{\bar{\lambda}_v}{\bar{\lambda}_T} \bar{c}_p a^{(0)} T_s^{(1)} + u^{(0)} c \frac{dT_s^{(1)}}{d\eta} \end{aligned} \quad (114f)$$

$$\frac{p^{(1)}}{p^{(0)}} = \frac{\rho^{(1)}}{\rho^{(0)}} + \frac{T^{(1)}}{T^{(0)}} \quad (114g)$$

Equations (113) and (114) have now to be solved for stations 1 and 2 in turn.

Solution around Station 1

First consider the 0th order equations (113). They just imply:

$$u^{(0)}(\eta) = \text{constant}, \quad T^{(0)}(\eta) = \text{constant}, \quad K_e^{(0)}(\eta) = \text{constant}, \quad \text{etc.} \quad (115)$$

In fact, the 0th order solution just gives constant properties. Using this 0th order solution, the first-order inner equations (114) simplify to:

$$\frac{\rho^{(1)}}{\rho^{(0)}} + \frac{u^{(1)}}{u^{(0)}} + \frac{A^{(1)}}{A^{(0)}} = \frac{\dot{m}^{(1)}}{\dot{m}^{(0)}} \quad (116a)$$

$$u_s^{(1)} + u^{(0)} \rho_s^{(1)} = 0 \quad (116b)$$

$$u^{(0)} \frac{du^{(1)}}{d\eta} - a^{(0)} u_s^{(1)} = u^{(0)} \frac{du_s^{(1)}}{d\eta} \quad (116c)$$

$$\bar{c}_p T^{(1)} + u^{(0)} u^{(1)} + \frac{\kappa}{\kappa+1} h_e K_e^{(1)} = \frac{\kappa}{\kappa+1} \left[c T_s^{(1)} + u^{(0)} u_s^{(1)} \right] \quad (116d)$$

$$\frac{1}{T^{(0)}} \frac{dT^{(1)}}{d\eta} - \frac{\bar{\gamma}-1}{\bar{\gamma}} \frac{1}{\rho^{(0)}} \frac{d\rho^{(1)}}{d\eta} = \frac{\kappa}{\kappa+1} \left[\frac{c}{\bar{c}_p T^{(0)}} \frac{dT_s^{(1)}}{d\eta} - \frac{h_e}{\bar{c}_p T^{(0)}} \frac{dK_e^{(1)}}{d\eta} \right] \quad (116e)$$

$$u^{(0)} c \frac{dT^{(1)}}{d\eta} + h_e u^{(0)} \frac{dK_e^{(1)}}{d\eta} - \frac{\bar{\lambda}_v}{\bar{\lambda}_T} \bar{c}_p a^{(0)} T_s^{(1)} = u^{(0)} c \frac{dT_s^{(1)}}{d\eta} \quad (116f)$$

$$\frac{p^{(1)}}{p^{(0)}} = \frac{\rho^{(1)}}{\rho^{(0)}} + \frac{T^{(1)}}{T^{(0)}} \quad (116g)$$

It is of importance now to realize that something is known about

the pressure. In the outer solution it was assumed that p was given as a function of ξ or x , i. e.,: $p = p(x)$

Close to station 1, we can write:

$$\begin{aligned}
 p &= p(x) = p(x_1 + \bar{\lambda}_v \eta) \\
 &= p(x_1) + \bar{\lambda}_v \eta \left(\frac{dp}{dx} \right)_{x_1} + O(\bar{\lambda}_v)^2 \\
 &= p_1^{(0)} + \frac{\bar{\lambda}_v}{L} p_1^{(1)} + \frac{\bar{\lambda}_v}{L} \left(\frac{dp}{d\xi} \right)_{\xi_1^{(0)}} \cdot \eta + O(\bar{\lambda}_v)^2 \\
 &= p(\eta)
 \end{aligned} \tag{117}$$

Thus, to 0th order, $p^{(0)}(\eta)$ is constant (exactly as predicted by the inner solution) and $p^{(1)}(\eta)$ is given by:

$$p^{(1)}(\eta) = p_1^{(1)} + \eta \left(\frac{dp}{d\xi_1} \right) \tag{118}$$

Clearly, at $\eta = 0$, the inner solution pressure matches the corrected upstream pressure in region 1 to first order.

It is also to be noted that the region being considered is downstream of station 1. The particle temperature has to remain constant at T^* :

$$\therefore T_p(\eta) = T(\eta) - T_s(\eta) = T^{(0)} + \frac{\bar{\lambda}_v}{L} (T^{(1)} - T_s^{(1)}) + \dots = T^* \tag{119}$$

$$\therefore T^{(1)}(\eta) = T_s^{(1)}(\eta) \tag{120}$$

Substituting for $K_\rho^{(1)}$ from (116e) into (116f), we obtain an equation for the slip temperature:

$$\begin{aligned}
 u^{(0)} c \frac{dT_s^{(0)}}{d\eta} + u^{(0)} \bar{c}_p T^{(0)} \left[\frac{c}{\bar{c}_p T^{(0)}} \frac{dT_s^{(1)}}{d\eta} - \frac{(k+1)}{k} \frac{1}{T^{(0)}} \frac{dT_s^{(1)}}{d\eta} + \frac{(k+1)(\bar{\gamma}-1)}{k \bar{\gamma}} \frac{1}{p^{(0)}} \frac{dp^{(1)}}{d\eta} \right] \\
 = \frac{\bar{\lambda}_v}{\bar{\lambda}_T} \bar{c}_p a^{(0)} T_s^{(1)} + u^{(0)} c \frac{dT_s^{(1)}}{d\eta} \quad (121)
 \end{aligned}$$

Simplifying and using (118) for $p^{(1)}$:

$$\frac{dT_s^{(1)}}{d\eta} + \kappa \cdot \frac{\bar{\lambda}_v}{\bar{\lambda}_T} \cdot \frac{\bar{c}_p}{c_p} \cdot \frac{a^{(0)}}{u^{(0)}} T_s^{(1)} = \frac{\bar{\gamma}-1}{\bar{\gamma}} \frac{(k+1) \bar{c}_p T^{(0)}}{c_p p^{(0)}} \left(\frac{dp}{d\xi} \right)_1 \quad (122)$$

This integrates immediately to:

$$\begin{aligned}
 T_s^{(1)}(\eta) = \frac{(k+1)}{k} \frac{\bar{\lambda}_T}{\bar{\lambda}_v} \cdot \frac{\bar{\gamma}-1}{\bar{\gamma}} \frac{u^{(0)} T^{(0)}}{a^{(0)} p^{(0)}} \left(\frac{dp}{d\xi} \right)_1 + \\
 + B_1 \exp \left\{ -\kappa \cdot \frac{\bar{\lambda}_v}{\bar{\lambda}_T} \cdot \frac{\bar{c}_p}{c_p} \frac{a^{(0)}}{u^{(0)}} \cdot \eta \right\} \quad (123)
 \end{aligned}$$

where B_1 is a constant to be determined by matching.

Differentiate (116d) and substitute for $K_e^{(1)}$ from (116e):

$$u^{(0)} \left[\frac{du^{(1)}}{d\eta} - \frac{\kappa}{k+1} \frac{du_s^{(1)}}{d\eta} \right] = - \frac{\bar{\gamma}-1}{\bar{\gamma}} \frac{\bar{c}_p T^{(0)}}{p^{(0)}} \left(\frac{dp}{d\xi} \right)_1 \quad (124)$$

Substituting for $u^{(1)}$ from (116c), results in an equation for the slip velocity:

$$\frac{du_s^{(1)}}{d\eta} + (k+1) \frac{a^{(0)}}{u^{(0)}} u_s^{(1)} = - (1+\kappa) \frac{(\bar{\gamma}-1) \bar{c}_p T^{(0)}}{\bar{\gamma} u^{(0)} p^{(0)}} \left(\frac{dp}{d\xi} \right)_1 \quad (125)$$

This integrates to:

$$u_s^{(1)}(\eta) = - \frac{(\bar{\gamma}-1) \bar{c}_p T^{(0)}}{\bar{\gamma} a^{(0)} p^{(0)}} \left(\frac{dp}{d\xi} \right)_1 + B_2 \exp \left\{ - \frac{a^{(0)} (1+\kappa)}{u^{(0)}} \eta \right\} \quad (126)$$

B_2 is another constant. The first-order gas velocity is now obtained from (124):

$$u'''(\eta) = B_3 - \frac{\bar{\delta}-1}{\bar{\delta}} \frac{\bar{c}_p T^{(0)}}{u^{(0)} p^{(0)}} \left(\frac{dp}{d\xi_1} \right) \cdot \eta + \frac{\kappa}{\kappa+1} u_s'''(\eta) \quad (127)$$

Finally, the inner first-order liquid fraction is obtained from (116d), (123), (126), and (127):

$$\begin{aligned} K_L'''(\eta) &= \frac{1}{h_L} \left[\left(c - \frac{\kappa+1}{\kappa} \bar{c}_p \right) T_s'''(\eta) + u^{(0)} u_s'''(\eta) - \frac{(\kappa+1)}{\kappa} u^{(0)} u''(\eta) \right] \\ &= \frac{1}{h_L} \left[- \frac{c_p}{\kappa} T_s'''(\eta) + u^{(0)} u_s'''(\eta) - \frac{\kappa+1}{\kappa} u^{(0)} u''(\eta) \right] \\ &= \frac{1}{h_L} \left[\frac{(\kappa+1)}{\kappa} \cdot \frac{(\bar{\delta}-1)}{\bar{\delta}} \frac{\bar{c}_p T^{(0)}}{p^{(0)}} \left(\frac{dp}{d\xi_1} \right) \cdot \eta + \left\{ - \frac{c_p}{\kappa} T_s'''(\eta) - \frac{(\kappa+1)}{\kappa} u^{(0)} B_3 \right\} \right] \quad (128) \end{aligned}$$

Solution around Station 2

The inner variable η is now defined by:

$$\eta = \frac{(x - x_2)}{\bar{\lambda}_v} \quad (129)$$

The 0th order equations again give constant values for the gas and particle flow quantities. The first order equations then simplify to equations (16 a-g). Once again, the pressure is a known function of η :

$$\begin{aligned} p &= p(x) = p(x_2 + \bar{\lambda}_v \eta) \\ &= p_2^{(0)} + \frac{\bar{\lambda}_v}{L} p_2''' + \frac{\bar{\lambda}_v}{L} \eta \left(\frac{dp}{d\xi_2} \right) + O\left(\frac{\bar{\lambda}_v}{L} \right)^2 \\ \therefore p^{(0)}(\eta) &= p_2^{(0)} \quad ; \quad p'''(\eta) = p_2''' + \eta \left(\frac{dp}{d\xi_2} \right) \quad (130) \end{aligned}$$

The particles are all in the solid phase in the region considered:

$$K_e^{(0)}(\eta) = K_e^{(1)}(\eta) = 0 \quad (131)$$

Substitute for $T^{(1)}$ from (116e) into (116f) to obtain an equation for $T_s^{(1)}$:

$$\frac{dT_s^{(1)}}{d\eta} + \frac{(K+1)\bar{c}_p}{c_p} \frac{\bar{\lambda}_v}{\bar{\lambda}_T} \frac{\bar{c}_p}{c} \frac{a^{(0)}}{u^{(0)}} T_s^{(1)} = \frac{(K+1)\bar{c}_p}{c_p} \frac{(\bar{\gamma}-1)}{\bar{\gamma}} \frac{T^{(0)}}{p^{(0)}} \left(\frac{dp}{d\xi_2} \right)$$

which integrates to:

$$T_s^{(1)}(\eta) = \frac{c}{\bar{c}_p} \cdot \frac{\bar{\lambda}_T}{\bar{\lambda}_v} \cdot \frac{(\bar{\gamma}-1)}{\bar{\gamma}} \frac{u^{(0)} T^{(0)}}{a^{(0)} p^{(0)}} \left(\frac{dp}{d\xi_2} \right) + B_4 \exp \left\{ - \frac{(K+1)\bar{c}_p}{c_p} \cdot \frac{\bar{\lambda}_v}{\bar{\lambda}_T} \cdot \frac{\bar{c}_p}{c} \cdot \frac{a^{(0)}}{u^{(0)}} \cdot \eta \right\} \quad (132)$$

$T^{(1)}$ then follows from the integration of (116e):

$$T^{(1)}(\eta) = B_5 + \frac{(\bar{\gamma}-1)}{\bar{\gamma}} \frac{T^{(0)}}{p^{(0)}} \left(\frac{dp}{d\xi_2} \right) \cdot \eta + \frac{K}{K+1} \frac{c}{\bar{c}_p} T_s^{(1)}(\eta) \quad (133)$$

From (116c, d, e):

$$u_s^{(1)}(\eta) = B_6 \exp \left\{ - \frac{(1+K)a^{(0)}}{u^{(0)}} \eta \right\} - \frac{(\bar{\gamma}-1)}{\bar{\gamma}} \frac{\bar{c}_p T^{(0)}}{a^{(0)} p^{(0)}} \left(\frac{dp}{d\xi_2} \right) \quad (134)$$

Finally, the substitution of (132) to (134) into (116d) yields the inner gas velocity:

$$u^{(1)}(\eta) = \frac{K}{K+1} u_s^{(1)}(\eta) + \frac{K}{K+1} \frac{c}{u^{(0)}} T_s^{(1)}(\eta) - \frac{\bar{c}_p}{u^{(0)}} T^{(1)}(\eta) \quad (135)$$

The two inner solutions are now complete. It remains to be shown that the inner and outer solutions can indeed be matched with a

proper choice of the constants A'_1, A'_2 and $B_1 - B_6$.

8. THE MATCHING OF THE INNER AND OUTER SOLUTIONS

We consider here the matching of the inner solutions obtained in Section 7 to the outer solutions obtained earlier. The matching should, while providing a uniformly valid solution, determine the unknown constants A'_1 and A'_2 in the outer solution.

Station 1

The procedure to be adopted is as follows. The solution upstream of station 1 is completely defined to first order. The inner solution should match this solution at $\eta = 0$. For $\eta \rightarrow \infty$, it should match the outer solution found in region 2 for $p \rightarrow p_1$, $x \rightarrow x_1$.

The 0th order quantities of the inner solution were all found to be constants. Consider the gas velocity:

$$u^{(0)}(\eta) = \text{constant} . \quad (136)$$

Clearly, to match the upstream solution, the requirement is:

$$u^{(0)}(\eta=0) = u^{(0)}(x \rightarrow x_1) = u_1^{(0)} = u^{(0)}(\eta) \quad (137)$$

Now:

$$\lim_{\eta \rightarrow \infty} u^{(0)}(\eta) = u_1^{(0)} = \lim_{x \rightarrow x_1+} u^{(0)}(x) \quad (138)$$

Similarly, all the 0th order inner quantities match correctly if they are assigned the values at station 1 as given by the outer solution.

Thus:

$$T^{(0)}(\eta) = T_1^{(0)} = T^* \quad (139a)$$

$$P^{(0)}(\eta) = P_1^{(0)} \quad (139b)$$

$$K_e^{(0)}(\eta) = K_{e1}^{(0)} = 1 \quad (139c)$$

$$P^{(0)}(\eta) = P_1^{(0)} \quad (139d)$$

For future convenience, introduce the following notation:

$$\lim_{P \rightarrow P_1+} : \text{limit as } P \text{ tends to } P_1 \text{ through region 2} \quad (140a)$$

$$\lim_{P \rightarrow P_1-} : \text{limit as } P \text{ tends to } P_1 \text{ through region 1} \quad (140b)$$

$$\lim_{P \rightarrow P_2+} : \text{limit as } P \text{ tends to } P_2 \text{ through region 3} \quad (140c)$$

$$\lim_{P \rightarrow P_2-} : \text{limit as } P \text{ tends to } P_2 \text{ through region 2.} \quad (140d)$$

Now consider the slip velocity $u_s'''(\eta)$ given by (126). For $\eta \rightarrow 0$ it must match $u_s'''(P)$ for $P \rightarrow P_1-$, as given by equation (62):

$$\begin{aligned} \lim_{\eta \rightarrow 0} u_s'''(\eta) &= -\frac{(\bar{\gamma}-1)}{\bar{\gamma}} \frac{\bar{c}_p T_1^{(0)}}{a_1^{(0)} P_1^{(0)}} \left(\frac{dP}{d\xi_1} \right) + B_2 \\ &= -\frac{a_1^{(0)}}{\bar{\gamma}} \left(\frac{1}{P} \frac{dP}{d\xi_1} \right) + B_2 \end{aligned} \quad (141a)$$

$$\lim_{P \rightarrow P_1-} u_s'''(P) = -\frac{a_1^{(0)}}{\bar{\gamma}} \left(\frac{1}{P} \frac{dP}{d\xi_1} \right) \quad (141b)$$

$\therefore B_2 = 0$ by the matching condition. Now try to match $u_s'''(\eta)$ for $\eta \rightarrow \infty$ with the downstream solution (84):

$$\lim_{\eta \rightarrow \infty} u_s'''(\eta) = -\frac{a_1^{(0)}}{\bar{\gamma}} \left(\frac{1}{P} \frac{dP}{d\xi_1} \right) = \lim_{P \rightarrow P_1+} u_s'''(P) \quad (142)$$

The slip velocity thus matches correctly on both sides, showing that it suffers no discontinuity across station 1. Now from (123) and (63):

$$\begin{aligned} \lim_{\eta \rightarrow 0} T_s^{(1)}(\eta) &= \frac{(K+1)}{K} \cdot \frac{\bar{\lambda}_T}{\bar{\lambda}_V} \cdot \frac{\bar{\gamma}-1}{\bar{\gamma}} \frac{u_1^{(0)} T_1^{(0)}}{a_1^{(0)} P_1^{(0)}} \left(\frac{dp}{d\xi} \right)_1 + B_1 \\ &= \frac{(K+1)}{K} \cdot \frac{\bar{\lambda}_T}{\bar{\lambda}_V} \cdot \frac{a_1^{(0)}}{\bar{c}_p} \frac{u_1^{(0)}}{\bar{\gamma}} \left(\frac{1}{P} \frac{dp}{d\xi} \right)_1 + B_1 \end{aligned} \quad (143a)$$

$$\lim_{P \rightarrow P_1^-} T_s^{(1)}(P) = \frac{1}{\bar{c}_p} \cdot \frac{\bar{\lambda}_T}{\bar{\lambda}_V} \cdot \frac{c}{\bar{c}_p} \cdot \frac{a_1^{(0)}}{\bar{\gamma}} u_1^{(0)} \left(\frac{1}{P} \frac{dp}{d\xi} \right)_1 \quad (143b)$$

$$\begin{aligned} \therefore B_1 &= \left(\frac{c}{\bar{c}_p} - \frac{K+1}{K} \right) \frac{1}{\bar{c}_p} \cdot \frac{\bar{\lambda}_T}{\bar{\lambda}_V} \frac{a_1^{(0)}}{\bar{\gamma}} u_1^{(0)} \left(\frac{1}{P} \frac{dp}{d\xi} \right)_1 \\ &= - \frac{c_p}{\bar{c}_p} \cdot \frac{1}{\bar{c}_p} \left(\frac{\bar{\lambda}_T}{\bar{\lambda}_V} \right) \frac{a_1^{(0)}}{\bar{\gamma}} u_1^{(0)} \left(\frac{1}{P} \frac{dp}{d\xi} \right)_1 \end{aligned} \quad (144)$$

For $\eta \rightarrow \infty$, $T_s^{(1)}(\eta)$ should match the solution in region 2, (87):

$$\lim_{\eta \rightarrow \infty} T_s^{(1)}(\eta) = \frac{(K+1)}{K} \frac{\bar{\lambda}_T}{\bar{\lambda}_V} \frac{a_1^{(0)}}{\bar{c}_p} \frac{u_1^{(0)}}{\bar{\gamma}} \left(\frac{1}{P} \frac{dp}{d\xi} \right)_1 \quad (145a)$$

$$\lim_{P \rightarrow P_1^+} T_s^{(1)}(P) = \frac{(1+K)}{K} \cdot \frac{\bar{\lambda}_T}{\bar{\lambda}_V} \frac{a_1^{(0)}}{\bar{c}_p} \frac{u_1^{(0)}}{\bar{\gamma}} \left(\frac{1}{P} \frac{dp}{d\xi} \right)_1 \quad (145b)$$

The density slip $\rho_s^{(1)}$ matches without any difficulty.

Now we note that the inner gas temperature, gas velocity, and liquid fraction must match the corrected values $(T + \tilde{T})$, $(u + \tilde{u})$,

and $(K_e + \tilde{K}_e)$ respectively, of the outer solution.

By virtue of the matching of the slip temperature, the gas temperature matches automatically:

$$\lim_{\eta \rightarrow 0} T'''(\eta) = \lim_{\eta \rightarrow 0} T_S'''(\eta) = \lim_{p \rightarrow p_1^-} T_S'''(p) = \lim_{p \rightarrow p_1^-} \tilde{T}'''(p) \quad (146a)$$

$$\lim_{\eta \rightarrow \infty} T'''(\eta) = \lim_{\eta \rightarrow \infty} T_S'''(\eta) = \lim_{p \rightarrow p_1^+} T_S'''(p) = \lim_{p \rightarrow p_1^+} \tilde{T}'''(p) \quad (146b)$$

From (127):

$$\begin{aligned} \lim_{\eta \rightarrow 0} u'''(\eta) &= B_3 + \frac{\kappa}{\kappa+1} u_{s1}''' \\ &= \lim_{p \rightarrow p_1^-} \tilde{u}''' = u_{1-}''' + \tilde{u}_1 \\ \therefore B_3 &= u_{1-}''' + \tilde{u}_1 - \frac{\kappa}{\kappa+1} u_{s1}''' \end{aligned} \quad (147)$$

where u_{1-}''' is given by (69), \tilde{u}_1 is given by (105), and u_{s1}''' is given by (141b).

We now demonstrate that with this choice for B_3 , $K_e'''(\eta) = 0$ for $\eta = 0$:

$$\begin{aligned} \lim_{\eta \rightarrow 0} K_e'''(\eta) &= \frac{1}{h_e} \left[-\frac{c_p}{\kappa} T_S'''(\eta=0) + u_1^{(0)} u_s'''(\eta=0) - \frac{(\kappa+1)}{\kappa} u_1^{(0)} u'''(\eta=0) \right] \\ &= \frac{1}{h_e} \left[-\frac{c_p}{\kappa} T_{s1-}''' + u_1^{(0)} u_{s1-}''' - \frac{(\kappa+1)}{\kappa} u_{1-}''' u_1^{(0)} - \frac{(\kappa+1)}{\kappa} \tilde{u}_1 u_1^{(0)} \right] \\ &= \frac{1}{h_e} \left[c T_{s1-}''' + u_1^{(0)} u_{s1-}''' - \frac{(\kappa+1)}{\kappa} \bar{c}_p T_{1-}''' - \frac{(\kappa+1)}{\kappa} u_{1-}''' u_1^{(0)} \right] \\ &= 0 \text{ by (58)} \end{aligned} \quad (148)$$

Thus, the matching is satisfactory at $\eta = 0$.

It remains now only to demonstrate the matching of $K_e'''(\eta)$ for $\eta \rightarrow \infty$ and evaluate the jump A_1' in the outer solution.

The matching principle requires:

$$\lim_{\eta \rightarrow \infty} K_e(\eta) = \lim_{x \rightarrow x_1+} K_e(x) \quad (149)$$

to each order of $\bar{\lambda}_v/L$.

Consider the outer solution in region 2 for $x \rightarrow x_1+$:

$$\begin{aligned} \lim_{x \rightarrow x_1+} K_e(x) &= K_e^{(0)}(x) + \frac{\bar{\lambda}_v}{L} \tilde{K}_e'''(x) + O\left(\frac{\bar{\lambda}_v}{L}\right)^2 \\ &= K_e^{(0)}(x_1 + \bar{\lambda}_v \eta) + \frac{\bar{\lambda}_v}{L} \tilde{K}_e'''(x_1 + \bar{\lambda}_v \eta) + O\left(\frac{\bar{\lambda}_v}{L}\right)^2 \\ &= K_e^{(0)}(x_1) + \frac{\bar{\lambda}_v}{L} \left[\tilde{K}_e'''(x_1) + \left(\frac{dK_e^{(0)}}{d\xi} \right)_{x_1} \eta \right] + \dots \end{aligned} \quad (150)$$

From (86):

$$\left(\frac{dK_e^{(0)}}{d\xi} \right)_{x_1} = \frac{RT_1^{(0)}}{\kappa h_e} \left(\frac{1}{P} \frac{dP}{d\xi} \right)_{x_1} = \frac{RT^*}{\kappa h_e} \left(\frac{1}{P} \frac{dP}{d\xi} \right)_{x_1}$$

$$\begin{aligned} \lim_{x \rightarrow x_1+} K_e(x) &= K_{e1}^{(0)} + \frac{\bar{\lambda}_v}{L} \left[\frac{RT^*}{\kappa h_e} \left(\frac{1}{P} \frac{dP}{d\xi} \right)_{x_1} \cdot \eta + \right. \\ &\quad \left. + \frac{\bar{\delta}}{(\bar{\delta}-1)} \frac{RT_{Pl}'''}{\kappa h_e} + A_1' \right] + O\left(\frac{\bar{\lambda}_v}{L}\right)^2 \end{aligned} \quad (151)$$

Looking at the inner solution (128):

$$\begin{aligned} \lim_{\eta \rightarrow \infty} K_e(\eta) &= K_{e1}^{(0)} + \frac{\bar{\lambda}_v}{L} \lim_{\eta \rightarrow \infty} \frac{1}{h_e} \left[\frac{(k+1)(\bar{\gamma}-1)}{k} \frac{\bar{c}_p T^{(0)}}{\bar{\delta} P^{(0)}} \left(\frac{dP}{d\xi} \right)_{x_1} \eta + \right. \\ &\quad \left. + \left\{ -\frac{c_p}{k} T_5'''(\eta) - \frac{(k+1)U_1^{(0)}}{k} B_3 \right\} \right] + \dots \end{aligned} \quad (152)$$

Now:

$$\frac{(k+1)}{k} \frac{(\bar{\gamma}-1)}{\bar{\gamma}} \frac{\bar{C}_p T^{(0)}}{h_e} = \frac{(k+1)}{k} \frac{\bar{R} T^*}{h_e} = \frac{R T^*}{k h_e}$$

Therefore, comparing (151) and (152), the terms in η match correctly. The matching will be complete provided:

$$\begin{aligned} A_1' &= \frac{1}{h_e} \left\{ -\frac{C_p}{k} T_{s1+}^{(1)} - \frac{(k+1)}{k} u_1^{(0)} B_3 \right\} - \frac{\bar{\gamma}-1}{\bar{\gamma}} \frac{R T_{p1-}^{(1)}}{k h_e} \\ &= \frac{1}{h_e} \cdot \frac{C_p}{k} (T_{s1-}^{(1)} - T_{s1+}^{(1)}) - \frac{\bar{\gamma}-1}{\bar{\gamma}} \frac{R T_{p1-}^{(1)}}{k h_e} \\ &= -\frac{1}{h_e} \left(\frac{C_p}{k \bar{C}_p} \right)^2 \left(\frac{\bar{\lambda}_T}{\bar{\lambda}_r} \right) \frac{a_1^{(0)}}{\bar{\gamma}} u_1^{(0)} \left(\frac{1}{p} \frac{dp}{d\xi_1} \right) - \frac{\bar{\gamma}-1}{\bar{\gamma}} \frac{R T_{p1-}^{(1)}}{k h_e} \end{aligned} \quad (153)$$

Thus, the unknown constant A_1' in the outer solution is determined by the matching. Since $\left(\frac{dp}{d\xi_1} \right)$ is negative, A_1' is positive. We now check that the liquid fraction is indeed less than one in the inner region:

$$\begin{aligned} \frac{dK_e(\eta)}{d\eta} &= \frac{\bar{\lambda}_r}{L} \frac{1}{h_e} \left[-\frac{C_p}{k} \frac{dT_s^{(1)}}{d\eta} + \frac{R T^*}{k} \left(\frac{1}{p} \frac{dp}{d\xi_1} \right) \right] + \dots \\ &= \frac{\bar{\lambda}_r}{L} \left[\frac{R T^*}{k h_e} \left(\frac{1}{p} \frac{dp}{d\xi_1} \right) \left\{ 1 - \frac{C_p}{C_p + kC} \exp \left(-k \cdot \frac{\bar{\lambda}_r}{\bar{\lambda}_T} \cdot \frac{\bar{C}_p}{C_p} \frac{a_1^{(0)}}{u_1^{(0)}} \eta \right) \right\} \right] \\ &\quad + \dots \dots \dots \end{aligned} \quad (154)$$

Since the above slope is always negative, the liquid fraction falls monotonically and is less than 1 away from station 1. This and the matching procedure are illustrated in Figure 3.

For completeness, some of the solutions, uniformly valid to first order over the whole of region 2, are given below:

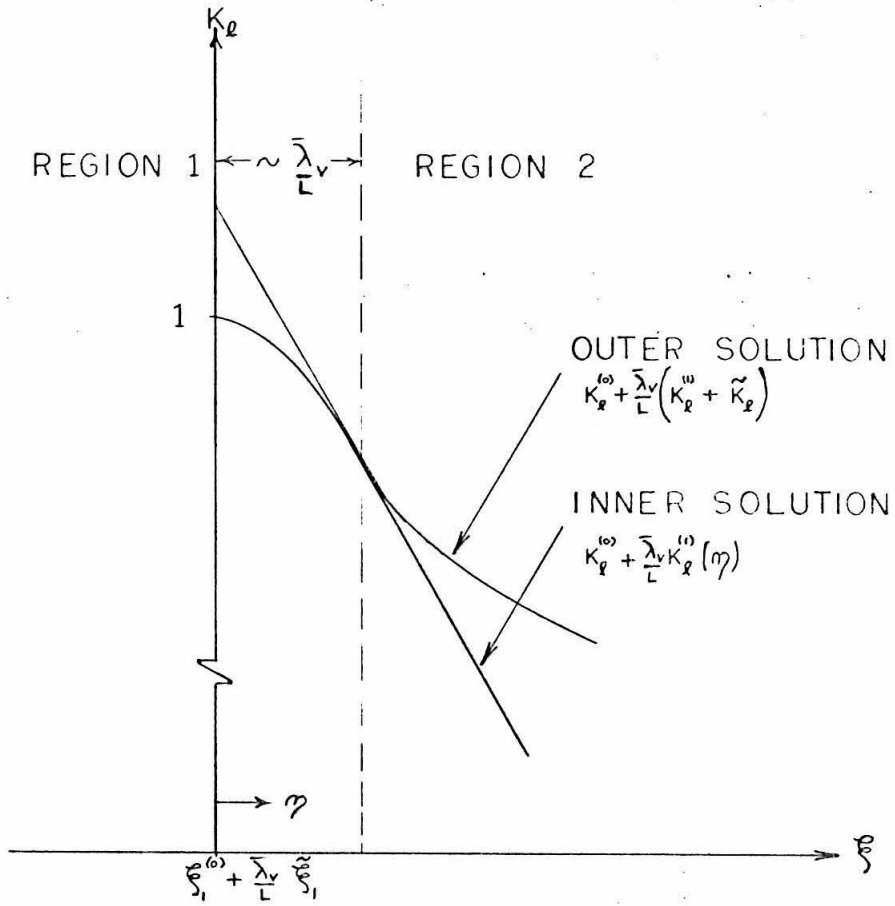


FIGURE 3. THE INNER AND OUTER SOLUTIONS FOR K_2 AT STATION 1

$$u_s = \frac{\bar{\lambda}_v}{L} \left[-\frac{a^{(0)}}{\bar{\delta}} \left(\frac{1}{P} \frac{dp}{d\xi} \right) \right] + O\left(\frac{\bar{\lambda}_v}{L}\right)^2 \quad (155a)$$

$$T_s = \frac{\bar{\lambda}_v}{L} \left[-\frac{1}{\bar{c}_p} \frac{c_p}{\bar{c}_p} \left(\frac{\bar{\lambda}_T}{\bar{\lambda}_v} \right) \frac{a_1^{(0)}}{\bar{\delta}} u_1^{(0)} \left(\frac{1}{P} \frac{dp}{d\xi} \right) \exp \left\{ -\kappa \frac{\bar{\lambda}_v}{\bar{\lambda}_T} \frac{\bar{c}_p}{c_p} \frac{a_1^{(0)}}{u_1^{(0)}} \frac{(x-x_1)}{\bar{\lambda}_v} \right\} \right. \\ \left. + \frac{1}{\bar{c}_p} \frac{(\kappa+1)}{\kappa} \cdot \frac{\bar{\lambda}_T}{\bar{\lambda}_v} \frac{a^{(0)}}{\bar{\delta}} u^{(0)} \left(\frac{1}{P} \frac{dp}{d\xi} \right) \right] + O\left(\frac{\bar{\lambda}_v}{L}\right)^2 \quad (155b)$$

$$K_e = 1 + \frac{RT^*}{\kappa h_e} \log \frac{P}{P_1^{(0)}} + \frac{\bar{\lambda}_v}{L} \left[-\frac{1}{h_e} \left(\frac{c}{\kappa \bar{c}_p} \right)^2 \left(\frac{\bar{\lambda}_T}{\bar{\lambda}_v} \right) \frac{1}{\bar{\delta}} \left(\frac{1}{P} \frac{dp}{d\xi} \right) \right. \\ \left. + \frac{RT_{P_1}^{(1)}}{\kappa h_e} \left\{ \frac{\bar{\delta}-1}{\bar{\delta}} - \log \frac{P}{P_1^{(0)}} \right\} - \frac{c_p}{\kappa h_e} B_1 \exp \left\{ -\kappa \frac{\bar{\lambda}_v}{\bar{\lambda}_T} \frac{\bar{c}_p}{c_p} \frac{a_1^{(0)}}{u_1^{(0)}} \frac{(x-x_1)}{\bar{\lambda}_v} \right\} \right. \\ \left. + \int_{P_1}^P \left\{ -\frac{c_p}{\kappa h_e} \frac{dT_s^{(1)}}{dP'} + \frac{u_s^{(1)}}{h_e} \frac{du^{(0)}}{dP'} + \frac{T^{(1)}}{T^{(0)}} \frac{dK_e^{(0)}}{dP'} \right\} dP' \right] + \dots \quad (155c)$$

Station 2

The 0th order quantities are again just equal to the quantities in the outer solution evaluated at station 2:

$$T^{(0)}(\eta) = T_2^{(0)} = T^* \quad (156a)$$

$$F^{(0)}(\eta) = F_2^{(0)} \quad (156b)$$

$$K_e^{(0)}(\eta) = K_{e2}^{(0)} = 0 \quad (156c)$$

$$P^{(0)}(\eta) = P_2^{(0)} \quad (156d)$$

Using a procedure exactly similar to that used at station 1, first consider the matching of the slip quantities. From (132) and (134):

$$\lim_{\eta \rightarrow 0} u_s'''(\eta) = B_6 - \frac{(\bar{\delta}-1) \bar{c}_p T_2^{(0)}}{\bar{\gamma} a_2^{(0)}} \left(\frac{1}{p} \frac{dp}{d\xi} \right)_2 = \lim_{p \rightarrow p_2^-} u_s'''(p) = -\frac{a_2^{(0)}}{\bar{\gamma}} \left(\frac{1}{p} \frac{dp}{d\xi} \right)_2$$

$$\therefore B_6 = 0 \quad (157a)$$

$$\lim_{\eta \rightarrow 0} T_s'''(\eta) = B_4 + \frac{(\bar{\delta}-1) \bar{\lambda}_T}{\bar{\gamma}} \frac{c}{\bar{c}_p} \frac{u_2^{(0)} T_2^{(0)}}{a_2^{(0)} p_2^{(0)}} \left(\frac{1}{p} \frac{dp}{d\xi} \right)_2$$

$$= \lim_{p \rightarrow p_2^-} T_s'''(p) = \frac{(1+\kappa)}{\kappa} \frac{1}{\bar{c}_p} \left(\frac{\bar{\lambda}_T}{\bar{\lambda}_V} \right) \frac{a_2^{(0)}}{\bar{\gamma}} u_2^{(0)} \left(\frac{1}{p} \frac{dp}{d\xi} \right)_2$$

$$\therefore B_4 = -\frac{1}{\bar{c}_p} \left(\frac{c}{\kappa \bar{c}_p} \right) \left(\frac{\bar{\lambda}_T}{\bar{\lambda}_V} \right) \frac{a_2^{(0)}}{\bar{\gamma}} u_2^{(0)} \left(\frac{1}{p} \frac{dp}{d\xi} \right)_2 \quad (157b)$$

These then match correctly with the upstream solution:

$$\lim_{\eta \rightarrow \infty} u_s'''(\eta) = -\frac{a_2^{(0)}}{\bar{\gamma}} \left(\frac{1}{p} \frac{dp}{d\xi} \right)_2 = \lim_{p \rightarrow p_2^+} u_s'''(p)$$

$$\lim_{\eta \rightarrow \infty} T_s'''(\eta) = \frac{1}{\bar{c}_p} \left(\frac{c}{\bar{c}_p} \right) \left(\frac{\bar{\lambda}_T}{\bar{\lambda}_V} \right) \frac{a_2^{(0)}}{\bar{\gamma}} u_2^{(0)} \left(\frac{1}{p} \frac{dp}{d\xi} \right)_2 = \lim_{p \rightarrow p_2^+} T_s'''(p)$$

With the slip quantities matched correctly, the matching of the gas temperature should then completely determine the solution over the whole flow. By the matching principle:

$$\lim_{\eta \rightarrow \infty} T(\eta) = \lim_{x \rightarrow x_2^+} T(x) \quad (158)$$

to each order of $\bar{\lambda}_V/L$.

For the outer solution in region 3:

$$\begin{aligned}
 \lim_{x \rightarrow x_2+} T(x) &= \lim_{x \rightarrow x_2+} \left[T^{(0)}(x) + \frac{\bar{\lambda}_v}{L} \tilde{T}^{(1)}(x) \right] + O\left(\frac{\bar{\lambda}_v}{L}\right)^2 \\
 &= T^{(0)}(x_2^{(0)}) + \frac{\bar{\lambda}_v}{L} \left[\mathcal{M} \left(\frac{dT^{(0)}}{d\xi} \right)_{x_2} + \tilde{T}^{(1)}(x_2) \right] + O\left(\frac{\bar{\lambda}_v}{L}\right)^2 \\
 &= T_2^{(0)} + \frac{\bar{\lambda}_v}{L} \left[\frac{\bar{\gamma}-1}{\bar{\gamma}} T^* \left(\frac{1}{p} \frac{dp}{d\xi} \right)_2 \cdot \mathcal{M} - \right. \\
 &\quad \left. - T^* \left\{ \frac{\bar{\gamma}-1}{\bar{\gamma}} \frac{p_2^{(0)}}{p_2^{(0)}} + \frac{T_{p1-}^{(0)}}{T^*} \right\} + A'_2 \right] + O\left(\frac{\bar{\lambda}_v}{L}\right)^2. \quad (159)
 \end{aligned}$$

From the inner solution:

$$\lim_{\eta \rightarrow 0} T^{(1)}(\eta) = B_5 + \frac{k}{k+1} \frac{c}{\bar{c}_p} T_5^{(1)}(\eta=0) = \tilde{T}_{2-}^{(1)} = T_{s2-}^{(1)}$$

$$\begin{aligned}
 \therefore B_5 &= - \frac{c_p}{(1+k)\bar{c}_p} \frac{(1+k)}{k} \frac{1}{\bar{c}_p} \left(\frac{\bar{\lambda}_T}{\bar{\lambda}_v} \right) \frac{a_2^{(0)}}{\bar{\gamma}} u_2^{(0)} \left(\frac{1}{p} \frac{dp}{d\xi} \right)_2 \\
 &= - \frac{1}{k} \frac{c_p}{\bar{c}_p} \frac{1}{\bar{c}_p} \left(\frac{\bar{\lambda}_T}{\bar{\lambda}_v} \right) \frac{a_2^{(0)}}{\bar{\gamma}} u_2^{(0)} \left(\frac{1}{p} \frac{dp}{d\xi} \right)_2 \quad (160)
 \end{aligned}$$

$$\lim_{\eta \rightarrow \infty} T^{(1)}(\eta) = B_5 + \frac{(\bar{\gamma}-1)}{\bar{\gamma}} T^* \left(\frac{1}{p} \frac{dp}{d\xi} \right)_2 \eta + \frac{k}{k+1} \frac{c}{\bar{c}_p} T_{s2+}^{(1)}$$

By the matching condition:

$$\begin{aligned}
 A'_2 &= B_5 + \frac{k}{k+1} \frac{c}{\bar{c}_p} T_{s2+}^{(1)} - \tilde{T}_{2+} \\
 &= \left[\left(\frac{k}{k+1} \frac{c^2}{\bar{c}_p} - \frac{c_p}{k} \right) \frac{1}{(\bar{c}_p)^2} \left(\frac{\bar{\lambda}_T}{\bar{\lambda}_v} \right) \frac{a_2^{(0)}}{\bar{\gamma}} u_2^{(0)} \left(\frac{1}{p} \frac{dp}{d\xi} \right)_2 - \right. \\
 &\quad \left. - \tilde{T}_{2+} \right] \quad (161)
 \end{aligned}$$

This completes the matching and the determination of the constants in the outer solution.

9. CALCULATIONS FOR AN EXPONENTIAL NOZZLE

In order to illustrate the application of the foregoing analysis, a numerical calculation is considered here. It is assumed that the pressure distribution is known as a function of distance and is given by:

$$\left(\frac{1}{P} \frac{dP}{d\xi} \right) = \text{constant} = -1$$

As discussed by Marble in [2], this distribution approximates the pressure distribution in a nozzle optimized for minimum particle lag loss. The gaseous component will be assumed to be the same as in [1], with the following properties:

$$C_p = 0.5 \text{ cal/gm}^\circ\text{K}, \quad R = 0.11 \text{ cal/gm}^\circ\text{K}, \quad P_r = 0.74.$$

The chamber and exit conditions are

$$T_0 = 3500^\circ\text{K}, \quad \frac{P_0}{P_E} = 66.69.$$

The particles are to be aluminum oxide (Al_2O_3) with the following properties [5, 6]: $h_L = 26000 \text{ cal/mole} = 255 \text{ cal/gm}$, $\kappa = 0.667$, $T^* = 2313^\circ\text{K}$, $C = 0.47 \text{ cal/gm}^\circ\text{K}$. With this choice of characteristics, $\bar{C}_p = 0.489 \text{ cal/gm}^\circ\text{K}$, $\bar{R} = 0.066 \text{ cal/gm}^\circ\text{K}$, $\bar{\gamma} = 1.155$, $\bar{m} = 1.00$, $\frac{\bar{\lambda}_T}{\bar{\lambda}_V} = 1.11$.

First, the equilibrium flow is calculated from the results of Section 3:

$$\frac{P_0}{P_E^{(0)}} = 1.738; \quad \frac{P_0}{P_1^{(0)}} = 21.8; \quad \frac{P_0}{P_2^{(0)}} = 42.5; \quad \frac{P_0}{P_E} = 66.69$$

$$U_1^{(0)} = 34.1; \quad U_2^{(0)} = 36.9; \quad U_E^{(0)} = 38.68$$

$$T_1^{(0)} = 2313^\circ\text{K}; \quad T_2^{(0)} = 2313^\circ\text{K}; \quad T_E^{(0)} = 2180^\circ\text{K}$$

Note that the velocity is in C. G. S. heat units. For the exponential pressure distribution, the integrals involved in the first order calculation can be evaluated in closed form. The first order results at station 1 are:

$$U_{S1-}^{(1)} = 11.5 ; T_{S1-}^{(1)} = -855^{\circ}K ; T_{S1+}^{(1)} = -2220^{\circ}K$$

$$U_{I-}^{(1)} = 0.426 ; T_{I-}^{(1)} = 42.6^{\circ}K ; \frac{P_1^{(1)}}{P_1^{(0)}} = -0.0513$$

$$A_1' = 3.92$$

At station 2:

$$U_{S2-}^{(1)} = 11.5 ; T_{S2-} = -2400^{\circ}K$$

$$K_{L2-}^{(1)} = 5.18 ; \tilde{K}_{L2} = 0.43 ; \frac{P_2^{(1)}}{P_2^{(0)}} = -3.74$$

Since to first order $P_2 = P_2^{(0)} + \frac{\bar{\lambda}_v}{L} P_2^{(1)}$, station 2 will be shifted to the nozzle exit for $\frac{\bar{\lambda}_v}{L} \geq 0.1$, and a pressure ratio $\frac{P_0}{P_E} = 66.69$. In region 3, assuming complete solidification,

$$\tilde{T}_{2+} = 275^{\circ}K ; A_2' = 785^{\circ}K ; T_E^{(1)} = 740^{\circ}K$$

$$T_{SE}^{(1)} = 940^{\circ}K ; \tilde{T}_E = 260^{\circ}K$$

$$U_{SE}^{(1)} = 11.2 , U_E^{(1)} = -3.78 , \tilde{U}_E = -3.3$$

Thus, at nozzle exit, taking into account the phase change, to first order:

$$U_E = 38.68 - \frac{\bar{\lambda}_v}{L} \cdot 7.08$$

$$U_{PE} = 38.68 - \frac{\bar{\lambda}_v}{L} 17.28$$

$$T_E = 2180^\circ K + \frac{\bar{\lambda}_v}{L} 1000^\circ K$$

$$T_{PE} = 2180^\circ K + \frac{\bar{\lambda}_v}{L} 1900^\circ K$$

If the phase change is neglected, the calculations for region 1 apply to the whole flow; the results to first order are then:

$$U_E = 38.33 - \frac{\bar{\lambda}_v}{L} 0.76$$

$$U_{PE} = 38.33 - \frac{\bar{\lambda}_v}{L} 11.46$$

$$T_E = 2000^\circ K + \frac{\bar{\lambda}_v}{L} 59^\circ K$$

$$T_{PE} = 2000^\circ K + \frac{\bar{\lambda}_v}{L} 1010^\circ K$$

10. SUMMARY AND DISCUSSION

To recapitulate, the major steps in the analysis are:

- (1) the demonstration of the existence of an equilibrium flow for $\bar{\lambda}_v \rightarrow 0$, where the particles and the gas move together without velocity and temperature lags;
- (2) the calculation of the first order corrections to the equilibrium flow for $\bar{\lambda}_v \neq 0$;
- (3) the correction of the equilibrium solution to take into account the shifts in the boundaries of the zone of solidification;
- (4) the matching of the solutions at the two stations, 1 and 2.

The effects of the phase change are quite noticeable in the calculation of the last section. For example, the error in the particle exit temperature, caused by neglecting the phase change, is

$(180 + \frac{\bar{\lambda}_v}{L} 900)^\circ \text{K}$ or 225°K for $\frac{\bar{\lambda}_v}{L} = 0.05$. Comparing only the 0th order solutions at the exit, it is apparent that 88 per cent of the latent heat released goes into the heating up of the flow, while only 12 per cent increases the kinetic energy.

Looking at the whole flow field, the most striking effects are the large discontinuities across the boundaries of the zone of phase change. The temperature slip in the zone of phase change is almost $2\frac{1}{2}$ times its value close to the boundaries in regions 1 or 3. The reason for this is clear: the heat released during solidification has to be transported by conduction, and so larger temperature differences are required. The particle temperature T_P is, of course, greater than the gas temperature (negative slip) so that energy is carried away from the droplets. The magnitudes of the constants A_1' and A_2'

emphasize the importance of correct matching at the boundaries. In this particular case, the constant A_1' , found by the matching procedure, accounts for about 70 per cent of the first-order liquid fraction at station 2. The constant A_2' accounts for about 80 per cent of the first-order gas temperature throughout region 2.

It is clear from the analysis that the first order corrections to the equilibrium flow can shift station 2 to the exit. That is, depending on $\frac{\bar{\lambda}_v}{L}$ and the pressure ratio across the nozzle, the particles can exit with a fraction in the liquid state. In that case, region 3 will not exist, and the analysis can then predict the liquid fraction. It is to be noted that in any case, the analysis will tend to be inaccurate for short nozzles and large particle sizes (greater than 2 microns), when the slip velocities may be large enough to invalidate the Stokes drag law used. Corrections for Reynolds numbers much in excess of unity can be made as suggested by Marble [7].

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LIST OF SYMBOLS USED IN PART I

a	equilibrium speed of sound
A	nozzle area
c	specific heat of particles
c_p, c_v	specific heats of gas
e	internal energy of gas per unit mass
F_p	force exerted upon gas by particles, per unit volume
h_e	latent heat of fusion
k	thermal conductivity of gas
K_e	liquid fraction
L	length of nozzle
m	particle mass
\dot{m}	mass flow rate
M	Mach number based on a
n	number density of particles
p	pressure
\bar{Pr}	mixture Prandtl number = $\frac{\mu \bar{c}_p}{k}$
Q_p	heat transferred from particles to gas, per unit volume
R	gas constant
T	temperature of gas
T^*	droplet freezing temperature
u	velocity of gas
x	axial distance along nozzle
γ	ratio of specific heats for gas
η	inner variable; equal to = $\frac{x-x_1}{\bar{\lambda}_v}$ or $\frac{x-x_2}{\bar{\lambda}_v}$

$K = \rho_p / \rho$	particle density ratio
$\bar{\lambda}_v$	velocity equilibration length
$\bar{\lambda}_T$	thermal equilibration length
μ	viscosity of gas
ξ	dimensionless length variable = x/L
ρ	density of gaseous phase
σ	diameter of particles

Superscripts

$(0), (1), (2)$	0^{th} order, 1^{st} order, 2^{nd} order, etc.
$-$	values for equilibrium mixture
\sim	1^{st} order correction to 0^{th} order solution
$\sim(1)$	total first order quantity

Subscripts

E	nozzle exit
P	particle
S	slip
t	throat
O	chamber
1	station 1
1-, 1+	approaching station 1 from upstream or downstream
2	station 2
2-, 2+	approaching station 2 from upstream or downstream

II. A KINETIC THEORY INVESTIGATION OF
SOME CONDENSATION - EVAPORATION PHENOMENA
BY A MOMENT METHOD

1.1 Introduction

Under many circumstances of technological interest, flow fields involving a vapor carrying a small volume fraction of its liquid can be greatly affected by the processes of phase change. The flow of wet vapors in power cycles utilizing alkali metals as working fluids provides examples in both the turbine and condenser flow processes.

When the time scale of the flow is short, the dynamics of the phase change assumes considerable importance; the vapor-pressure equilibrium condition is not satisfied over the scale of droplet spacing. Frequently, the question is not one of droplet nucleation but one of increase or decrease in the sizes of droplets that generally retain their identity throughout the flow. The radii of these droplets may range from much smaller to much larger than the local mean free path of vapor molecules.

The mechanism and rate of transfer of vapor to and from liquid droplets differs completely, depending upon whether the medium is a chemically pure substance or contains an inert diluent gas. In the latter case, only a rather small mass fraction of the diluent gas is required to make the diffusion mechanism rate controlling. Then the vapor molecules condensing on the liquid surface must diffuse through the inert gaseous component and the rate is low in comparison with molecular velocities. For such flux rates, the effects of phase change are of similar magnitude as thermal conductivity and viscosity and, as a consequence, do not add novel features. For a pure substance, however, the flux rates of condensation can take place at nearly molecular velocities; the effects become much larger and take place

much more rapidly than those associated with diffusive mechanisms. The new phenomena that enter the problem are associated with this extremely short relaxation time.

At the present time, the formulation of this vapor transfer process in a pure substance is incomplete. Figure 4 shows the surface of a liquid at temperature T_L in contact with its pure vapor, at ambient conditions P_0, T_0 . At the liquid surface, molecules are incident (P_i, T_i) , reflected (P_r, T_r) and emitted from the liquid (P_L, T_L) . Now, on the one hand there exists the classical formula of Hertz [1] and Knudsen [2] which is valid strictly in the free-molecule regime. It is based upon the assumption that molecular flux from the liquid may be computed from a Maxwellian distribution at liquid temperature T_L and number density corresponding to the saturation pressure P_L . Similarly, the flux of molecules from the surrounding vapor to the surface is computed from a Maxwellian distribution at the assumed vapor state P_0, T_0 . Now since the Maxwellian mass-flux rate [1] is $P/\sqrt{2\pi RT}$, the net condensation rate per unit area is

$$\frac{P_0}{\sqrt{2\pi RT_0}} - \frac{P_L}{\sqrt{2\pi RT_L}} \approx \frac{1}{2\pi} P_0 \sqrt{2\pi RT_0} \left\{ \frac{(P_0 - P_L)}{P_0} - \frac{1}{2} \frac{(T_0 - T_L)}{T_0} \right\}$$

the approximation on the right holding for small differences between vapor and saturation states. The free-molecule restriction of this result is clearly seen in the facts that the source of vapor molecules is independent of the molecular flux from the surface and that no molecules leaving the liquid are scattered back into the liquid.

PURE VAPOR
 P_0, T_0

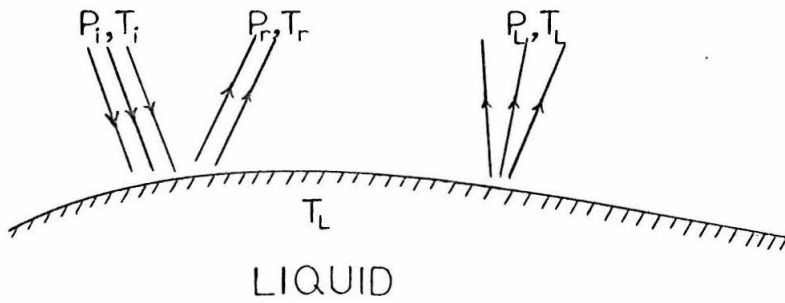


FIGURE 4. THE SURFACE OF A LIQUID IN CONTACT WITH ITS VAPOR

On the other hand, Marble [3] has proposed a rough formulation for the condensation from continuum flow. The Hertz-Knudsen exchange is retained in the immediate vicinity of the liquid surface, but the state from which the vapor molecules flow is not the continuum state of the vapor many mean free paths from the surface. Rather this unknown state (P_i, T_i) is related to the continuum state (P_c, T_c) by conservation of mass, momentum, and energy. This leads to the three equations:

$$\frac{P_i}{\sqrt{2\pi RT}} - \frac{P_L}{\sqrt{2\pi RT_L}} = -\rho_c u_c$$

$$\frac{1}{2} P_i + \frac{1}{2} P_L = P_c + \rho_c u_c^2$$

$$\frac{1}{\pi} \sqrt{2\pi RT_i} P_i - \frac{1}{\pi} \sqrt{2\pi RT_L} P_L = -\rho_c u_c \left(c_p T_c + \frac{1}{2} u_c^2 \right)$$

which, under the assumption that the differences among vapor states are not large, gives the mass condensation rate as

$$-\rho_c u_c = \frac{1}{2\pi} \left(\frac{\delta}{2 + \frac{\gamma}{\delta-1}} \right) \rho_c \sqrt{2\pi RT_c} \left(\frac{P_c - P_L}{P_c} \right)$$

The state P_c, T_c , however, is not the state of the remote vapor, since the former is moving toward the surface and the latter is stationary. As a consequence, the state P_c, T_c is connected to the remote vapor state through an isentropic expansion fan, giving the pressure difference

$$\frac{P_c - P_0}{P_0} \approx \gamma \frac{\rho_c u_c}{\rho_0 a_0}$$

where $a_0 = \sqrt{\gamma R T_0}$ is the sound speed in the remote vapor.

Then, eliminating the state p_c, T_c under the assumption of small state changes, the final form of the condensation mass flow is

$$\dot{m} = \frac{1}{2\pi} \left(\frac{\gamma}{2 + \frac{\gamma}{\gamma-1}} \right) \left(\frac{1}{1 + \sqrt{\frac{\gamma}{2\pi} \left(\frac{\gamma}{2 + \frac{\gamma}{\gamma-1}} \right)}} \right) p_0 \sqrt{2\pi R T_0} \left(\frac{p_0 - p_L}{p_0} \right)$$

For $\gamma = 5/3$,

$$\dot{m} = 0.909 \frac{1}{2\pi} p_0 \sqrt{2\pi R T_0} \left(\frac{p_0 - p_L}{p_0} \right)$$

Among several facts concerning this result, two are of special interest. First, the condensation rate depends upon the difference between the pressure and the saturation pressure, not upon the corresponding temperature difference; this is in distinct contrast with the Hertz-Knudsen formula. Secondly, and somewhat more surprising, the coefficient relating the mass condensation rate to the pressure difference is within 10 percent of that which occurs in the Hertz-Knudsen formula. These two observations suggest that the transition from free-molecule to continuum range has mainly the effect of suppressing flux caused by temperature difference.

Such conclusions at this state of development may, however, be somewhat dangerous, not only because theoretical investigations of this transition regime have not been done, but because the continuum calculation of Marble is based upon a physical model rather than deduced directly from kinetic theory. There are questions, therefore, about the nature of intermediate vapor states near the wall and of the process of "patching" the expansion fan to the quasi-steady continuum solution. Furthermore, the extension from a plane

surface to a spherical droplet is not at all obvious. Clearly, the expansion wave plays a different role, if any, since the corresponding wave would attenuate at least as the reciprocal radius from the sphere.

It is the aim of the present work to analyze this problem of condensation from the kinetic theory fundamentals, as nearly as is possible, over the range from free-molecule to continuum flow. The moment method of Maxwell will be employed. Even with the recognized limitation of the technique, it offers the most tractable approach to a problem of this degree of complexity. Furthermore, the method permits a general treatment for all Knudsen numbers and thus may be checked against existing results in the asymptotic limits. Moreover, the resulting condensation rates may be computed for different numbers of moments, the relative insensitivity to these moments providing further confidence in its accuracy.

1.2 Some General Assumptions

(i) Regarding the governing equations. - We shall be concerned only with the vapor, the liquid making itself felt only through the boundary conditions. It shall be assumed that the governing equation is the Maxwell-Boltzmann equation for the one-particle distribution function, f :

$$\frac{\partial f}{\partial t} + \underline{\xi} \cdot \nabla f + \frac{\underline{F}}{m} \cdot \nabla_{\underline{\xi}} f = \left(\frac{\partial f}{\partial t} \right)_{\text{COLL}} \quad (1)$$

where $\underline{\xi}$ is the vector in the molecular velocity space, \underline{F} is the

external force field, and $(\partial f / \partial t)_{\text{COLL}}$ represents the collision integral (!). For the assumptions involved in the derivation of the equation, its range of validity, etc., we refer the reader to Grad's [4] article. This equation will be solved approximately by the Maxwell moment method using Lees' [5] approach. This will be described in the next section.

(ii) Regarding the nature of the vapor. - It shall be convenient to treat the vapor as a pure monatomic gas, with equation of state $p = \rho RT$. Further, in view of the immense simplification of the analysis, an inverse fifth power law force between the molecules will be assumed. Since at moderate temperatures the rotational and vibrational energy exchanges between molecules will be small, the results should not be too bad for non-monatomic gases at moderate temperatures away from the critical point.

(iii) Regarding the boundary conditions. - In the present analysis, the accommodation coefficient will be taken to be unity. This means that every molecule hitting the surface will be absorbed by it; it will be assumed, further, that the molecules are re-emitted with a Maxwellian distribution corresponding to the liquid temperature with zero mean velocity. Since very little is actually known about the complex interactions which take place at the liquid surface, these restrictions seem justified for a first step in understanding the phenomena.

1.3 A Brief Description of the Maxwell Moment Method

The kinetic theory treatment of the vapor phase involves the solution of the Maxwell-Boltzmann equation. Unfortunately, the difficulty in solving this integro-differential equation forces us to consider approximate indirect approaches, e.g., assume approximate forms for the collision integral, or solve the equations of transfer rather than the Maxwell-Boltzmann equation itself, etc. Here the Maxwell moment method as developed by Lees will be used. A brief description of the method follows; the reader is referred to Lees' [5, 6] original papers for details.

Often, in fluid mechanics, one is not particularly interested in the velocity distribution function itself, but in certain lower moments of it, such as mean velocity, mean temperature, etc. In such cases, instead of seeking exact solutions of the Maxwell-Boltzmann equation, one can solve the transport equation for the lower moments. In this way one satisfies the Maxwell-Boltzmann equation in some average sense.

The general transport equation can be derived in a number of ways, either directly or from the Boltzmann equation. Let $Q(\xi_i)$ be any function of the velocity components of a particle. The general equation for the rate of transport of Q can be obtained directly by considering the various sources of change in the amount of Q in a fixed volume. The rate of increase of Q per unit volume, $\frac{\partial}{\partial t} \int Q(\xi_i) f(\underline{r}, \underline{\xi}, t) d\underline{\xi}$ must equal the sum of:

- (i) the rate of change due to the flux of particles crossing the

bounding surface of the volume, $-\nabla_{\underline{r}} \cdot \left[\int f \underline{\xi} Q d\underline{\xi} \right]$

(ii) the effect of external forces and curvature of the coordinate system on particle acceleration, given by

$$\int f \left\{ \frac{\underline{F}}{m} - (\underline{\Omega} \wedge \underline{\xi}) \right\} \cdot \nabla_{\underline{\xi}} Q d\underline{\xi}$$

(iii) the effect of collisions given by:

$$\Delta Q = \int (\underline{Q}' - Q) f f' | \underline{\xi}_1 - \underline{\xi} | d\underline{\xi}_1 d\underline{\xi}_2 b db d\epsilon \quad (163)$$

The resulting equation is Maxwell's integral equation of transfer:

$$\begin{aligned} \frac{\partial}{\partial t} \int f Q d\underline{\xi} + \nabla_{\underline{r}} \cdot \int f \underline{\xi} Q d\underline{\xi} \\ = \int f \left\{ \frac{\underline{F}}{m} - (\underline{\Omega} \wedge \underline{\xi}) \right\} \cdot \nabla_{\underline{\xi}} Q d\underline{\xi} + \Delta Q \end{aligned} \quad (164)$$

This is the basic equation for the present investigation. We note that the gas dynamic conservation equations are obtained by setting Q to be successively the collisional invariants of mass ($Q = m$), momentum ($Q = m \xi_i$), and energy ($Q = m \xi^2/2$), for which $\Delta Q = 0$.

The procedure then is as follows:

(i) Represent the distribution function $f(\underline{r}, \underline{\xi}, t)$ by M

arbitrary functions $F_M(\underline{r}, t)$ of space and time, selected in such a way that the essential physical features of the problem are preserved.

(ii) Take M appropriate choices of Q in equation (164) to derive M moment equations.

(iii) Substitute $f = f\{F_M(\underline{r}, t), \underline{\xi}\}$ and solve the M first order partial differential equations for the unknown functions $F_M(\underline{r}, t)$

(iv) Once the boundary and/or initial conditions are satisfied one can hope that $f\{F_M(\underline{r}, t), \underline{\xi}\}$ represents $f(\underline{r}, \underline{\xi}, t)$ in some fashion; more important that the lower moments such as $\rho, \underline{u}, \underline{P}$ derived from $f\{F_M, \underline{\xi}\}$ are approximations to those derived from the correct distribution function, f .

It is clear that this procedure is similar to the integral methods used for boundary layers. Rather than satisfying the Maxwell-Boltzmann equation point by point, one satisfies it in 'some' undefined average sense. Though it is hard to make any definite statement in advance, it would seem that 'better' solutions would be obtained by using more moments (larger M 's).

The choice of the arbitrary functions F_M is not rigid. One is guided by the basic requirements for the distribution function to be employed in the moment method: (i) it must have the 'two sided' character essential to rarefied flows, (ii) its form must be such that the boundary conditions are easily applicable, (iii) it must be capable of providing a smooth transition from rarefied flow to the Navier-

Stokes regime.

The above requirements are well satisfied, for the present investigation, by Lees' 'two-stream' Maxwellian (Fig. 5). In body coordinates all outwardly directed particle velocity vectors lying within the "cone of influence" (region 1 in Fig. 5) are described by $f = f_1$ where:

$$f_1 = \frac{m_1(\underline{r}, t)}{[2\pi RT_1(\underline{r}, t)]^{3/2}} \exp \left\{ - \frac{[\underline{\xi} - \underline{u}_1(\underline{r}, t)]^2}{2RT_1(\underline{r}, t)} \right\} \quad (165a)$$

and region 2 (all other $\underline{\xi}$) by:

$$f_2 = \frac{m_2(\underline{r}, t)}{[2\pi RT_2(\underline{r}, t)]^{3/2}} \exp \left\{ - \frac{[\underline{\xi} - \underline{u}_2(\underline{r}, t)]^2}{2RT_2(\underline{r}, t)} \right\} \quad (165b)$$

where $m_1, m_2, T_1, T_2, \underline{u}_1$ and \underline{u}_2 are ten initially undetermined functions. (It is not necessary to use all ten functions.) With the correct choice of these functions the free molecule limit can be described exactly for diffuse re-emission. The boundary condition and transition requirement are also satisfied.

2. The Steady State Spherical Droplet

2.1 Formulation of the Problem

Consider a spherical liquid droplet of radius r_0 , temperature T_0 in equilibrium with its pure vapor. Then, neglecting surface energy corrections, the vapor would be at a temperature T_0 and

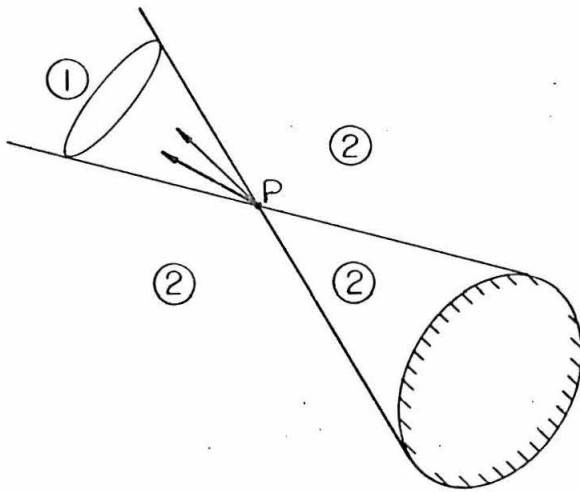


FIGURE 5 a. CONE OF INFLUENCE FOR TWO STREAM MAXWELLIAN

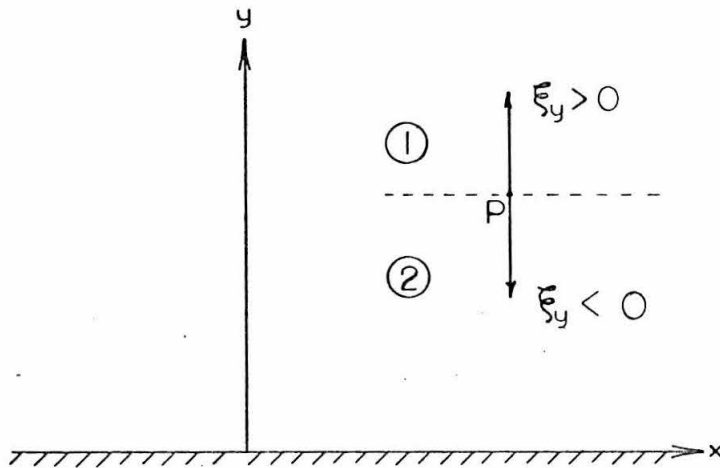


FIGURE 5 b. CONFIGURATION FOR ONE DIMENSIONAL PROBLEM

pressure P_0 , where P_0 is the saturation vapor pressure corresponding to T_0 . Now suppose, for instance, that the temperature of the droplet is suddenly changed to $T_L \neq T_0$. Now we ask, is there a steady state condition with net condensation on to or evaporation from the droplet? We shall neglect the growth of the droplet and the latent heat effects, i. e., the mathematical problem will be a droplet with a straw to suck out the condensing fluid. This model will be a good one if the calculated growth rate is small compared to the radius r_0 .

In order to keep the formulation as simple as possible, we choose to describe the distribution function f for the vapor molecules by a two stream Maxwellian with four unknown functions. Referring to Fig. 6a:

$$f(r, \underline{\xi}) = \begin{cases} f_1 = \frac{m_1(r)}{[2\pi R T_1(r)]} \exp\left[-\frac{\{\xi_r^2 + \xi_\theta^2 + \xi_\phi^2\}}{2 R T_1(r)}\right] & \text{in 1} \\ f_2 = \frac{m_2(r)}{[2\pi R T_2(r)]} \exp\left[-\frac{\{\xi_r^2 + \xi_\theta^2 + \xi_\phi^2\}}{2 R T_2(r)}\right] & \text{in 2} \end{cases} \quad (166)$$

The four unknown functions m_1, T_1, m_2, T_2 are functions of r only because of spherical symmetry and the assumed steady state.

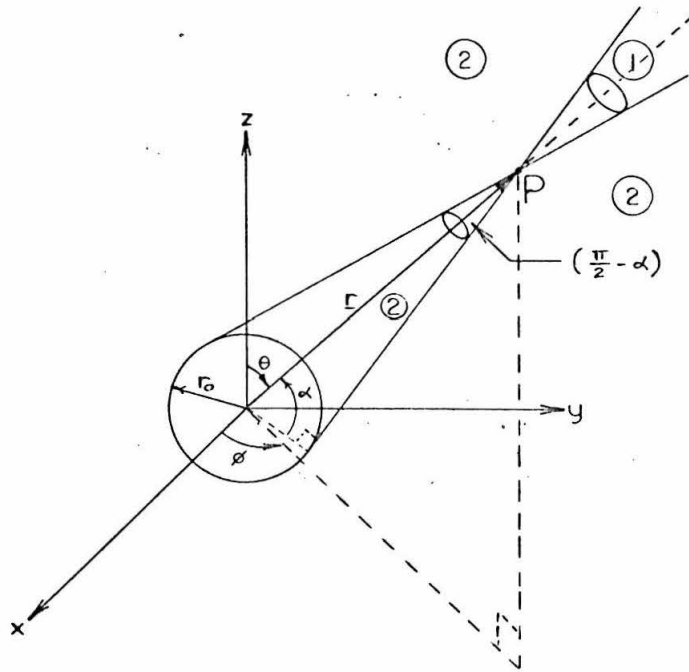


FIGURE 6a. GEOMETRY FOR THE DROPLET

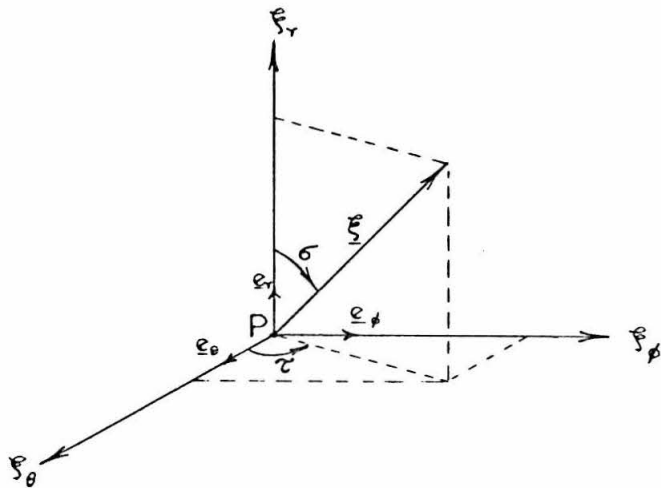


FIGURE 6b. VELOCITY SPACE AT P

The "cone of influence" and regions 1 and 2 are defined precisely through Figs. 6a and 6b. At the physical point P , position vector \underline{r} , the molecular velocity vector $\underline{\xi}$ has components $(\xi_r, \xi_\theta, \xi_\phi)$ where:

$$\begin{aligned}\xi_r &= \xi \cos \sigma \\ \xi_\theta &= \xi \sin \sigma \cos \tau \\ \xi_\phi &= \xi \sin \sigma \sin \tau\end{aligned}\tag{167}$$

When the direction of $\underline{\xi}$ is in the outward cone it is in region 1. That is, if $(\frac{\pi}{2} - \alpha)$ is the angle between the radius vector \underline{r} to P and the rays defining the cone at P :

$$0 < \sigma < \frac{\pi}{2} - \alpha \quad \text{defines region 1}\tag{168}$$

$$\frac{\pi}{2} - \alpha < \sigma < \pi \quad \text{defines region 2}$$

and $\cos \alpha = r_0/r$ (169)

Equation (166) can then be rewritten:

$$f(r, \underline{\xi}) = \begin{cases} f_1(r, \underline{\xi}) & 0 < \sigma < \frac{\pi}{2} - \alpha \\ f_2(r, \underline{\xi}) & \frac{\pi}{2} - \alpha < \sigma < \pi \end{cases}\tag{170}$$

With the distribution function defined in this fashion the various mean quantities like density, temperature, stress, etc., can be evaluated in terms of the four unknown functions. One notes that the volume element in velocity space can be written:

$$\begin{aligned}
 d\underline{\xi} &= d\xi_r d\xi_\theta d\xi_\phi \rightarrow d\underline{\xi} (\xi d\sigma) (\xi \sin\sigma d\tau) \\
 &\rightarrow \xi^2 \sin\sigma d\xi d\sigma d\tau \quad (171)
 \end{aligned}$$

Then if $Q(\xi_i)$ is any function of the velocity components, the mean value, $\langle Q \rangle(\tau)$, can be written:

$$\begin{aligned}
 \langle Q \rangle(\tau) &= \int Q f d\underline{\xi} \\
 &= \int_0^\infty \int_0^{(\frac{\pi}{2}-\alpha)} \int_0^{2\pi} Q f_1 \xi^2 \sin\sigma d\xi d\sigma d\tau + \\
 &\quad + \int_0^\infty \int_0^\pi \int_0^{2\pi} Q f_2 \xi^2 \sin\sigma d\xi d\sigma d\tau \quad (172)
 \end{aligned}$$

For example from the kinetic theory definition (Appendix

I) of density, ρ :

$$\begin{aligned}
 \rho(\gamma) &= \int m f d\underline{\xi} = \\
 &= \int_0^{\infty} \int_0^{(\frac{\pi}{2}-\alpha)} \int_0^{2\pi} \frac{m m_1(\gamma)}{[2\pi R T_1(\gamma)]} \exp\left[-\frac{\xi^2}{2R T_1(\gamma)}\right] \xi^2 \sin\sigma (d\xi \times \\
 &\quad \times d\sigma d\tau) \\
 &+ \int_0^{\infty} \int_0^{\pi} \int_0^{2\pi} \frac{m m_2(\gamma)}{[2\pi R T_2(\gamma)]} \exp\left[-\frac{\xi^2}{2R T_2(\gamma)}\right] \xi^2 \sin\sigma \times \\
 &\quad \times (d\xi d\sigma d\tau) \quad (173)
 \end{aligned}$$

$$\rho(\gamma) = \frac{m}{2} \left[m_1(\gamma)(1 - \sin\alpha) + m_2(\gamma)(1 + \sin\alpha) \right] \quad (174)$$

For future convenience define:

$$\sin\alpha = \sqrt{1 - \frac{\gamma_0^2}{\gamma^2}} = x(\gamma) = x \quad (175)$$

Thus:

$$\rho(\gamma) = \frac{m}{2} \left[m_1(\gamma)(1 - x) + m_2(\gamma)(1 + x) \right] \quad (176)$$

All other flow quantities can be similarly evaluated in terms of m_1, m_2, T_1, T_2 .

2.2 The Moment Equations

In the absence of external forces the time independent moment equation with spherical symmetry takes the form:

$$\begin{aligned} & \frac{1}{r^2} \frac{d}{dr} \int r^2 f \xi_r Q d\underline{\xi} + \frac{1}{r} \frac{\cos \theta}{\sin \theta} \int f \xi_\theta Q d\underline{\xi} - \\ & - \frac{1}{r} \int f \left[(\xi_\theta^2 + \xi_\phi^2) \frac{\partial Q}{\partial \xi_r} + \left(\frac{\cos \theta}{\sin \theta} \xi_\phi^2 - \xi_\theta \xi_r \right) \frac{\partial Q}{\partial \xi_\theta} - \right. \\ & \quad \left. - \left(\xi_\phi \xi_r + \frac{\cos \theta}{\sin \theta} \xi_\phi \xi_\theta \right) \frac{\partial Q}{\partial \xi_\phi} \right] d\underline{\xi} \\ & = \Delta Q \end{aligned} \tag{177}$$

where Q is any function of the components of the particle velocity and ΔQ is the rate of change of Q produced by particle collisions. For Maxwell particles it can be shown [5] that:

$$\Delta Q = 0 \quad \text{for } Q = m, \xi_i, \frac{1}{2} m \xi^2 \tag{178}$$

$$\Delta Q = \frac{\rho}{\mu} P_{jk} \quad \text{for } Q = m \xi_j \xi_k$$

$$\Delta Q = \frac{\rho}{\mu} \left[-\frac{2}{3} q_j + \sum_k P_{jk} u_k \right] \quad \text{for } Q = m \xi_j \xi^2 / 2$$

where $P_{jk} = P_{jk} + \mathcal{P} \delta_{jk}$. P, μ, q_j are the thermodynamic pressure, viscosity coefficient and the heat flux vector respectively.

Since we have four arbitrary functions $m_1(r), T_1(r), m_2(r)$ and $T_2(r)$ defining the distribution function we need to use four moment equations: the three conservation equations plus one higher moment. In this case it seems appropriate to use the moment corresponding to the radial heat flux. Thus, setting $Q = m, m \xi_r, m \xi^2/2$ and $m \xi_r \xi^2/2$ respectively the following four moment equations are obtained:

$$(a) \quad \frac{d}{dr} \int r^2 f m \xi_r d\underline{\xi} = 0$$

$$(b) \quad \frac{1}{r^2} \frac{d}{dr} \int r^2 f \xi_r^2 m d\underline{\xi} - \frac{1}{r} \int f [\xi_\theta^2 + \xi_\phi^2] m d\underline{\xi} = 0 \quad (179)$$

$$(c) \quad \frac{d}{dr} \int r^2 f m \xi_r \xi^2/2 d\underline{\xi} = 0$$

$$(d) \quad \frac{1}{r^2} \frac{d}{dr} \int r^2 f m \xi_r \frac{\xi^2}{2} d\underline{\xi} - \frac{1}{r} \int f \left[m \frac{\xi^2}{2} (\xi_\theta^2 + \xi_\phi^2) \right] \times d\underline{\xi} \\ = \frac{P}{\mu} \left[-\frac{2}{3} q_r + \nu P_{rr} \right]$$

where V is the mean radial velocity. It is easy to show from the kinetic theory definitions (App. I) that (179a, b, c) are just the equations of mass, radial momentum, and energy conservation:

$$(a) \quad \frac{d}{dr} (\gamma^2 \rho V) = 0$$

$$(b) \quad \rho V \frac{dV}{dr} = \frac{d}{dr} P_{rr} + \frac{1}{r} (2 P_{rr} - P_{\theta\theta} - P_{\phi\phi}) \quad (180)$$

$$(c) \quad \frac{d}{dr} \left[\gamma^2 \left(q_r + \frac{3}{2} \rho V - V P_{rr} + \frac{1}{2} \rho V^3 \right) \right] = 0$$

Equations (179a, b, c, d) are then our required moment equations.

2.3 Boundary Conditions

For the steady state problem that we are considering, the gas far from the sphere should be uniform with:

$$\left. \begin{array}{l} T \rightarrow T_0 \\ \rho \rightarrow \rho_0 \end{array} \right\} \text{ for } r \rightarrow \infty \quad (181)$$

Under the assumptions made regarding the accommodation

coefficient, surface energy, etc., all that is known at the droplet surface is the outward part of the distribution function. The molecules from the droplet are at the temperature of the droplet T_L ; their number density is assumed to be equal to the number density m_L corresponding to the equilibrium condition at T_L . We note that m_L and T_L do, indeed, constitute two boundary conditions for the vapor; for a given T_L the value of m_L will depend on the liquid concerned. This boundary condition is particularly easily represented by the assumed form of the distribution function:

$$\left. \begin{aligned} T_i(r) &= T_L \\ m_i(r) &= m_L \end{aligned} \right\} \text{at } r = r_0 \quad (182)$$

2.4 The Linearized Problem

Equations (179a, b, c, d) together with the boundary conditions (181) and (182) will provide the solution to the problem. However, in order to get an analytically tractable solution for small deviations from equilibrium, a perturbation solution is attempted. We can write:

$$\begin{aligned}
 m_1(\tau) &= m_0 \left[1 + \lambda N_1^{(1)}(\tau) + \lambda^2 N_1^{(2)}(\tau) + \dots \right] \\
 T_1(\tau) &= T_0 \left[1 + \lambda t_1^{(1)}(\tau) + \lambda^2 t_1^{(2)}(\tau) + \dots \right] \\
 m_2(\tau) &= m_0 \left[1 + \lambda N_2^{(1)}(\tau) + \lambda^2 N_2^{(2)}(\tau) + \dots \right] \\
 T_2(\tau) &= T_0 \left[1 + \lambda t_2^{(1)}(\tau) + \lambda^2 t_2^{(2)}(\tau) + \dots \right]
 \end{aligned} \tag{183}$$

Here λ is a perturbation parameter to indicate non-equilibrium: when $\lambda = 0$, $m_1 = m_2 = m_0$ and $T_1 = T_2 = T_0$ to give the required steady state Maxwellian.

The mean flow quantities can now be evaluated to all orders of λ .

It is convenient to introduce the following notation:

$$\begin{aligned}
 N_+^{(1)}(\tau) &= N_1^{(1)} + N_2^{(1)} \quad ; \quad N_-^{(1)} = N_1^{(1)} - N_2^{(1)} \\
 t_+^{(1)}(\tau) &= t_1^{(1)} + t_2^{(1)} \quad ; \quad t_-^{(1)}(\tau) = t_1^{(1)} - t_2^{(1)}
 \end{aligned} \tag{184}$$

and similarly for the second order quantities. For brevity, the complete list of flow quantities to second order has been left to an appendix, with only a few presented here:

$$\begin{aligned}
 \text{(a) } \rho &= \rho_0 \left[1 + \lambda \left\{ \frac{1}{2} N_+^{(1)} - \frac{x}{2} N_-^{(1)} \right\} + \right. \\
 &\quad \left. + \lambda^2 \left\{ \frac{1}{2} N_+^{(2)} - \frac{x}{2} N_-^{(2)} \right\} + \dots \right]
 \end{aligned}$$

$$\begin{aligned}
 \text{(b) } V = & \frac{\beta(1-x^2)}{2\pi} \left[\lambda \left\{ N_-^{(1)} + \frac{1}{2} t_-^{(1)} \right\} + \lambda^2 \left\{ -\frac{(t_1^{(12)} - t_2^{(12)})}{8} + \right. \right. \\
 & + \left. \left(N_-^{(1)} + \frac{1}{2} t_-^{(1)} \right) \left(\frac{1}{2} N_+^{(1)} - \frac{x}{2} N_-^{(1)} \right) + \right. \\
 & \left. \left. + \frac{1}{2} (N_1^{(1)} t_1^{(1)} - N_2^{(1)} t_2^{(1)}) + N_-^{(2)} + \frac{1}{2} t_-^{(2)} \right\} + \dots \right] \quad (185)
 \end{aligned}$$

$$\begin{aligned}
 \text{(c) } p = & \int RT \\
 = & p_0 \left[1 + \lambda \left\{ \frac{1}{2} N_+^{(1)} - \frac{x}{2} N_-^{(1)} + \frac{1}{2} t_+^{(1)} - \frac{x}{2} t_-^{(1)} \right\} + \right. \\
 & + \lambda^2 \left\{ -\frac{(1-x^2)^2}{6\pi} (N_-^{(1)} + \frac{1}{2} t_-^{(1)})^2 + \frac{(1-x)}{2} N_1^{(1)} t_1^{(1)} + \right. \\
 & + \frac{(1+x)}{2} N_2^{(1)} t_2^{(1)} + \frac{1}{2} N_+^{(2)} - \frac{x}{2} N_-^{(2)} + \\
 & \left. \left. + \frac{1}{2} t_+^{(2)} - \frac{x}{2} t_-^{(2)} \right\} + \dots \right]
 \end{aligned}$$

where
$$\beta = \sqrt{2\pi RT_0} \quad (186)$$

We note that when $\lambda = 0$, $V = 0$, $p = p_0$, $P = p_0$, etc. The scheme is therefore consistent.

2.5 First Order Solution

Using the perturbation scheme of the previous section, the moment equations (179a, b, c, d) to order λ reduce to:

$$(a) \quad N_-^{(1)} + \frac{1}{2} t_-^{(1)} = A^{(1)}$$

$$(b) \quad N_+^{(1)} + t_+^{(1)} = C^{(1)}$$

$$(c) \quad 7t_-^{(1)} - 2N_-^{(1)} = B^{(1)}$$

$$(d) \quad r \frac{d}{dr} \left(\frac{5}{2} t_+^{(1)} \right) = - \frac{\gamma_0}{3} \frac{\rho_0}{\beta \mu_0} \cdot \frac{\gamma_0}{r} \cdot B^{(1)}$$

(187)

These equations are identical to those obtained in [6].

The boundary conditions (181) and (182) to first order become:

$$(a) \quad \text{For } r \rightarrow \infty : t_2^{(1)} \rightarrow 0, N_2^{(1)} \rightarrow 0$$

(188)

$$(b) \quad \text{For } r = r_0 : t_1^{(1)} = \frac{T_L - T_0}{\lambda T_0} = \frac{\Delta T}{\lambda T_0}$$

$$N_1^{(1)} = \frac{m_L - m_0}{\lambda m_0} = \frac{\Delta m}{\lambda m_0}$$

We note that:

$$\frac{\gamma_0 \rho_0}{\beta \mu_0} = \frac{\gamma_0 \rho_0 R T_0}{\sqrt{2\pi R T_0} \rho_0 l_0 \left(\frac{2 R T_0}{\pi} \right)^{1/2}} = \frac{\gamma_0}{2 l_0} = \frac{1}{4 Km} \quad (189)$$

where l_0 is the undisturbed mean free path and Km is the Knudsen number based on the droplet diameter. Integrating

equations (187) and applying the boundary conditions (188) we have:

$$(a) \quad t_1'''(\tau) = \left(\frac{1}{4} A^{(1)} + \frac{1}{8} B^{(1)} \right) + \frac{1}{30} \frac{\tau_0}{l_0} B^{(1)} \cdot \frac{\tau_0}{\tau}$$

$$(b) \quad t_2'''(\tau) = \frac{1}{30} \frac{\tau_0}{l_0} \cdot B^{(1)} \cdot \frac{\tau_0}{\tau}$$

$$(c) \quad N_1'''(\tau) = \left(\frac{7}{8} A^{(1)} - \frac{1}{16} B^{(1)} \right) - \frac{1}{30} \frac{\tau_0}{l_0} B^{(1)} \cdot \frac{\tau_0}{\tau}$$

$$(d) \quad N_2'''(\tau) = - \frac{1}{30} \frac{\tau_0}{l_0} B^{(1)} \cdot \frac{\tau_0}{\tau}$$

(190)

$$A^{(1)} = \frac{\left(\frac{1}{4} + \frac{2}{15} \frac{\tau_0}{l_0} \right) \frac{\Delta T}{\lambda T_0} + \left(\frac{1}{2} + \frac{2}{15} \frac{\tau_0}{l_0} \right) \frac{\Delta m}{\lambda m_0}}{\left(\frac{1}{2} + \frac{3}{20} \frac{\tau_0}{l_0} \right)} \quad (191)$$

$$B^{(1)} = \frac{7}{\left(2 + \frac{3}{5} \frac{\tau_0}{l_0} \right)} \frac{\Delta P}{P_0} \quad (192)$$

NOTE : $\frac{\Delta P}{P_0} = \frac{\Delta T}{T_0} + \frac{\Delta m}{m_0}$

From (185b) the evaporating mass flux is given to first order by:

$$\begin{aligned} \rho V &= \frac{\rho_0 \beta}{2\pi} (1-x^2) \cdot \lambda \left(N_-'' + \frac{1}{2} t_-''' \right) \\ &= \frac{\rho_0 \beta}{2\pi} \cdot \lambda A''' \left(\frac{\gamma_0}{\gamma} \right)^2 \end{aligned}$$

Thus at the surface of the droplet to first order:

$$\begin{aligned} \dot{m} &= \rho V (\gamma = \gamma_0) \\ &= \frac{\rho_0 \sqrt{2\pi R T_0}}{2\pi} \frac{\left(\frac{1}{4} + \frac{2}{15} \frac{\gamma_0}{l_0} \right) \frac{\Delta T}{T_0} + \left(\frac{1}{2} + \frac{2}{15} \frac{\gamma_0}{l_0} \right) \frac{\Delta m}{m_0}}{\left(\frac{1}{2} + \frac{3}{20} \frac{\gamma_0}{l_0} \right)} \\ &= \frac{\rho_0 \sqrt{2\pi R T_0}}{2\pi} \frac{\left(\frac{1}{2} + \frac{2}{15} \frac{\gamma_0}{l_0} \right) \frac{\Delta p}{p_0} - \frac{1}{4} \frac{\Delta T}{T_0}}{\left(\frac{1}{2} + \frac{3}{20} \frac{\gamma_0}{l_0} \right)} \quad (193) \end{aligned}$$

This is precisely the result that was sought: given T_0 , p_0 and ΔT and Δp the mass evaporation or condensation rate is determined for all Knudsen numbers. We note that all the formulas seem to hold for both positive and negative Δp , ΔT , i. e., the problem seems to be well formulated for both evaporation and condensation.

In Fig. 7 the dependence of the coefficients of $\Delta p/p_0$ and $\Delta T/T_0$ on the ratio γ_0/l_0 is shown. The coefficient

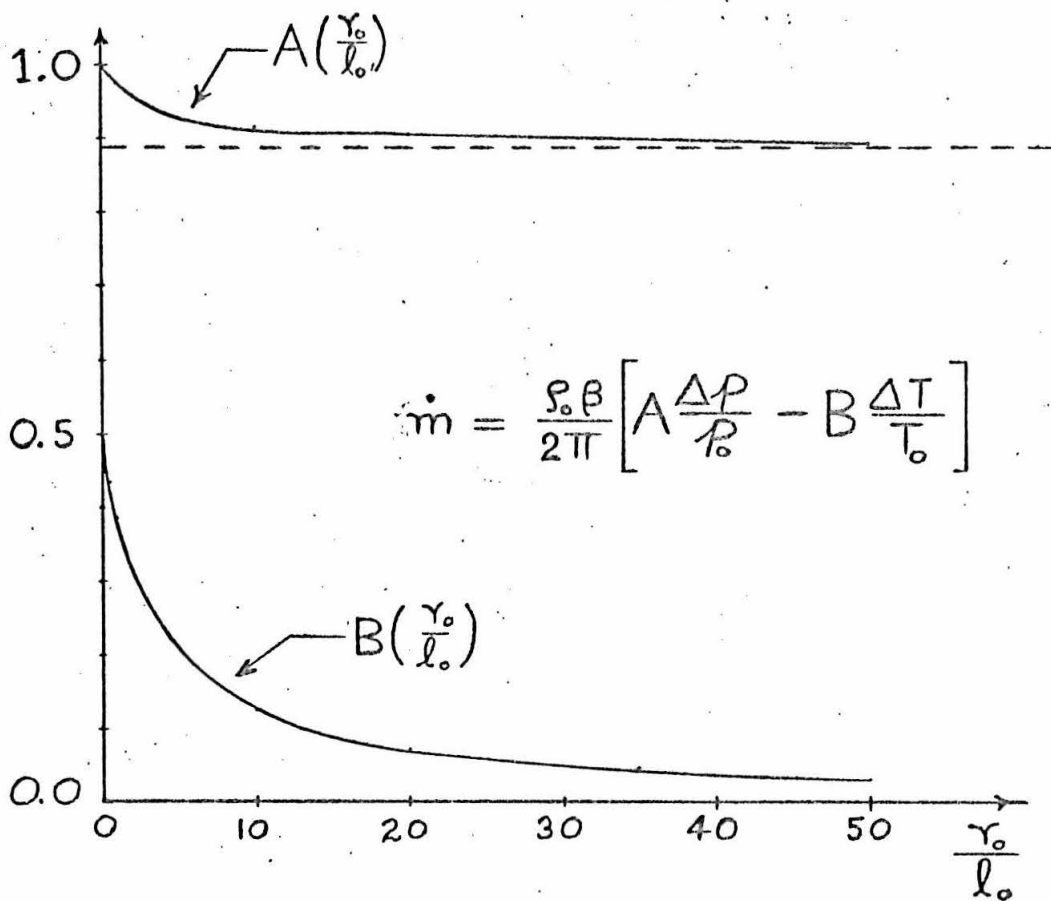


FIGURE 7. THE DEPENDENCE OF \dot{m} ON γ_0/l_0 .

of $\Delta P/P_0$ changes little with r_0/l_0 but the coefficient of $\Delta T/T_0$ vanishes for $(r_0/l_0) \rightarrow \infty$. The two limiting cases are to be noted:

(i) Free molecule limit ($K_M = \frac{l_0}{2r_0} \rightarrow \infty$)

$$\rho V(r=r_0) \rightarrow \frac{P_0 \beta}{2\pi} \cdot \frac{1}{2} \left(2 \frac{\Delta P}{P_0} - \frac{\Delta T}{T_0} \right) \quad (194)$$

This agrees exactly with the Hertz-Knudsen formula.

(ii) Continuum limit ($K_M = \frac{l_0}{2r_0} \rightarrow 0$)

$$\rho V(r=r_0) \rightarrow \frac{P_0 \beta}{2\pi} \cdot \frac{8}{9} \frac{\Delta P}{P_0} = 0.142 P_0 \sqrt{2\pi RT_0} \frac{\Delta P}{P_0} \quad (195)$$

Thus in the continuum limit the mass flux is proportional to the difference in pressure alone!

These two cases will be discussed in detail later.

2.6 The Solution to Second Order

The first order solution of the previous section gives us most of the information that we seek. The reasons for carrying out the

calculation to higher order are twofold: (i) to demonstrate the consistency of the perturbation scheme and (ii) to see how the mass flux depends on products like $(\Delta p)^2$, $\Delta p \Delta T$, etc. This will enable us to see the difference in behavior between evaporation and condensation.

From equations (179a, b, c, d) the moment equations for the second order quantities are:

$$(a) \left(N_-^{(2)} + \frac{1}{2} t_-^{(2)} \right) + \frac{1}{2} (N_1^{(1)} t_1^{(1)} - N_2^{(1)} t_2^{(1)}) - \frac{1}{8} (t_1^{(1)2} - t_2^{(1)2}) = A^{(2)}$$

$$(b) \left(N_-^{(2)} + \frac{3}{2} t_-^{(2)} \right) + \frac{3}{2} (N_1^{(1)} t_1^{(1)} - N_2^{(1)} t_2^{(1)}) + \frac{3}{8} (t_1^{(1)2} - t_2^{(1)2}) = B^{(2)}$$

(196)

$$(c) \frac{d}{dr} \left[N_+^{(2)} - x^3 N_-^{(2)} + t_+^{(2)} - x^3 t_-^{(2)} + (1-x^3) N_1^{(1)} t_1^{(1)} + (1+x^3) N_2^{(1)} t_2^{(1)} \right] + \frac{2}{r} \left[\frac{3}{2} x(1-x^2) (N_-^{(2)} + t_-^{(2)} + N_1^{(1)} t_1^{(1)} - N_2^{(1)} t_2^{(1)}) \right] = 0$$

$$(d) \frac{5}{8\pi} P_0 \beta^2 \frac{d}{dr} \left[N_+^{(2)} - x^3 N_-^{(2)} + 2 t_+^{(2)} - 2 x^3 t_-^{(2)} + (1-x^3) (2 N_1^{(1)} t_1^{(1)} + t_1^{(1)2}) + (1+x^3) (2 N_2^{(1)} t_2^{(1)} + t_2^{(1)2}) \right] +$$

$$\begin{aligned}
 & + \frac{5}{4\pi} \frac{\rho_0 \beta^2}{\gamma} x(1-x^2) \left[\frac{3}{2} N_-^{(2)} + 3 t_-^{(2)} + 3 N_1^{(1)} t_1^{(1)} + 3 N_2^{(1)} t_2^{(1)} + \right. \\
 & \qquad \qquad \qquad \left. + \frac{3}{2} t_1^{(1)2} - \frac{3}{2} t_2^{(1)2} \right] \\
 & = \frac{\rho_0}{\mu_0} \left[-\frac{2}{3} \left(\frac{1}{2} N_+^{(1)} - \frac{x}{2} N_-^{(1)} \right) \cdot \frac{\rho_0 \beta}{8\pi} (1-x^2) (7 t_-^{(1)} - 2 N_-^{(1)}) - \right. \\
 & \quad - \frac{2}{3} \frac{\rho_0 \beta}{2\pi} (1-x^2) \left\{ \frac{7}{4} (N_1^{(1)} t_1^{(1)} - N_2^{(1)} t_2^{(1)}) + \frac{17}{16} (t_1^{(1)2} - t_2^{(1)2}) + \right. \\
 & \quad + \left(N_-^{(1)} + \frac{1}{2} t_-^{(1)} \right) \left(\frac{x}{2} (x^2-1) N_-^{(1)} - \frac{5}{4} t_+^{(1)} + \frac{1}{2} \left(\frac{3}{2} x + x^3 \right) t_-^{(1)} \right) + \\
 & \quad \left. + \frac{1}{4} (7 t_-^{(2)} - 2 N_-^{(2)}) \right\} \\
 & \quad \left. + \frac{\rho_0 \beta}{2\pi} (1-x^2) \left\{ \left(N_-^{(1)} + \frac{1}{2} t_-^{(1)} \right) \left(\frac{x}{2} (x^2-1) N_-^{(1)} + \frac{(x^2-1)}{2} t_-^{(1)} \right) \right\} \right]
 \end{aligned}$$

In the above equations the four unknowns are $N_1^{(2)}$, $t_1^{(2)}$, $N_2^{(2)}$, and $t_2^{(2)}$. The first order quantities $N_1^{(1)}$, $t_1^{(1)}$, etc. are known functions of Υ from (190). It is to be

noted that the change of viscosity with temperature was taken into account in deriving (196d). The boundary conditions for the second order quantities are:

$$(a) \text{ For } r \rightarrow \infty : t_2^{(2)} \rightarrow 0, N_2^{(2)} \rightarrow 0$$

$$(b) \text{ At } r = r_0 : t_1^{(2)} = 0, N_1^{(2)} = 0$$
(197)

The details of the simple but tedious calculation are given in Appendix III. For the present purposes the only important result is the value of the constant $A^{(2)}$:

$$\begin{aligned} \left(4 + \frac{6}{5} \frac{\gamma_0}{\ell_0}\right) A^{(2)} = & \left[\frac{15}{32} + \frac{1}{15} \frac{\gamma_0}{\ell_0} \left(\frac{41\pi}{128} + \frac{53}{16} \right) \right] A^{(1)2} + \\ & + \left[\frac{7}{32} + \frac{1}{15} \frac{\gamma_0}{\ell_0} \left(\frac{7\pi}{128} + \frac{43}{16} \right) + \frac{8}{225} \frac{\gamma_0^2}{\ell_0^2} \left(\frac{11}{32} + \right. \right. \\ & \left. \left. + \frac{3\pi}{512} \right) \right] A^{(1)} B^{(1)} \\ & + \left[-\frac{1}{128} + \frac{1}{15} \frac{\gamma_0}{\ell_0} \left(\frac{13\pi}{512} - \frac{15}{64} \right) + \right. \\ & \left. + \frac{8}{225} \frac{\gamma_0^2}{\ell_0^2} \left(\frac{3\pi}{1024} - \frac{21}{64} \right) - \frac{4}{3375} \frac{\gamma_0^3}{\ell_0^3} \right] B^{(1)2} \end{aligned}$$
(198)

where $A^{(1)}$ and $B^{(1)}$ are defined by (191, 192). We note that the

solution is again good for the whole range of Knudsen number. To second order the radial mass flux is:

$$\dot{m} = \frac{\rho_0 \beta}{2\pi} (1-x^2) \left[\lambda A'' + \lambda^2 A^{(2)} + \dots \right] \quad (199)$$

From (198) and (191), in the continuum limit $\left(\frac{\gamma_0}{l_0} \rightarrow \infty \right)$:

$$A'' \rightarrow \frac{8}{9} \frac{\Delta p}{\lambda \rho_0}, \quad B'' \rightarrow 0 \quad (200)$$

$$A^{(2)} \rightarrow \frac{5}{6} \left[\frac{1}{15} \left(\frac{41\pi}{128} + \frac{53}{16} \right) \left(\frac{8}{9} \right)^2 + \frac{8}{225} \left(\frac{3\pi}{512} + \frac{11}{32} \right) \frac{280}{27} - \frac{4}{3375} \left(\frac{35}{3} \right)^2 \right] \left(\frac{\Delta p}{\lambda \rho_0} \right)^2 \quad (201)$$

From (199), (200) and (201) the mass flux at the droplet surface, in the continuum limit, to second order is:

$$\dot{m} = \dot{m}(\gamma = \gamma_0, \frac{\gamma_0}{l_0} \rightarrow \infty) = \frac{\rho_0 \beta}{2\pi} \left[\frac{8}{9} \frac{\Delta p}{\rho_0} + 0.167 \left(\frac{\Delta p}{\rho_0} \right)^2 \right] \quad (202)$$

Thus the mass flux in the continuum limit depends only on Δp , to second order. Also the asymmetry in Δp says that for a given $|\Delta p|$ more fluid will evaporate than condense. Thus the difference between evaporation and condensation arises in the second order.

3. The One Dimensional Transient Problem

3.1 Formulation

Every steady state phenomenon is, in actual fact, the asymptotic limit, for large time, of some transient phenomenon. In the steady state solution obtained for the droplet, we implied uniform conditions far from the droplet. In actual fact pressure (and other) waves generated by sudden changes at the surface travel outward. But in a spherical geometry these waves die off at least as fast as r^{-1} , and become infinitesimally weak far away. In order to understand this better, it is of interest to consider a transient problem. In order to avoid the mathematical complexity of the spherical geometry, the plane one dimensional problem will be considered.

Consider an infinite, plane sheet of liquid at temperature T_0 , in equilibrium with its pure vapor at temperature T_0 and pressure p_0 , the equilibrium vapor pressure at T_0 . At some instant the temperature of the liquid is suddenly changed to $T_L \neq T_0$. This is the transient problem to be studied.

Let the distribution function $f(y, \xi, t)$ of the vapor molecules be defined (Fig. 5b) by the two stream Maxwellian:

$$f(y, \xi, t) = \begin{cases} f_1 = \frac{m_1(y, t)}{[2\pi RT_1(y, t)]^{3/2}} \exp\left[-\frac{\xi^2}{2RT_1(y, t)}\right] \dots \xi_y > 0 \\ f_2 = \frac{m_2(y, t)}{[2\pi RT_2(y, t)]^{3/2}} \exp\left[-\frac{\xi^2}{2RT_2(y, t)}\right] \dots \xi_y < 0 \end{cases} \quad (203)$$

where m_1, T_1, m_2, T_2 are four unknown functions of y and t .

The mean flow quantities are now given by:

$$\begin{aligned} \langle Q(\underline{\xi}) \rangle (y, t) &= \int Q f d\underline{\xi} \\ &= \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} Q f_1 d\xi_x d\xi_y d\xi_z + \\ &\quad + \int_{-\infty}^{\infty} \int_{-\infty}^0 \int_{-\infty}^{\infty} Q f_2 d\xi_x d\xi_y d\xi_z \quad (204) \end{aligned}$$

For example:

$$\begin{aligned} \rho V &= \int m \xi_y f d\underline{\xi} \\ &= \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} m \xi_y \frac{m_1}{[2\pi RT_1]^{3/2}} \exp\left[-\frac{\xi^2}{2RT_1}\right] d\xi_x d\xi_y d\xi_z + \\ &\quad + \int_{-\infty}^{\infty} \int_{-\infty}^0 \int_{-\infty}^{\infty} m \xi_y \frac{m_2}{[2\pi RT_2]^{3/2}} \exp\left[-\frac{\xi^2}{2RT_2}\right] d\xi_x d\xi_y d\xi_z \end{aligned}$$

$$= \sqrt{\frac{R}{2\pi}} m \left[m_1(y, t) \sqrt{T_1(y, t)} - m_2(y, t) \sqrt{T_2(y, t)} \right] \quad (205)$$

3.2 The Exact and Linearized Equations and Boundary Conditions

In the absence of external forces, the general moment equation (164) for the present geometry takes the form

$$\frac{\partial}{\partial t} \int f Q d\underline{\xi} + \frac{\partial}{\partial y} \int f \xi_y Q d\underline{\xi} = \Delta Q \quad (206)$$

Setting Q to be equal to the three collisional invariants m , $m \xi_y$ and $m \xi^2/2$ and equal to $m \xi_y \xi^2/2$ for the heat flux moment, we get the equations:

$$(a) \quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial y} (\rho v) = 0$$

$$(b) \quad \rho \frac{\partial v}{\partial t} + \rho v \frac{\partial v}{\partial y} - \frac{\partial P_{yy}}{\partial y} = 0$$

(207)

$$(c) \quad \frac{\partial p}{\partial t} + \frac{\partial}{\partial y} (p v) + \frac{2}{3} \frac{\partial q_y}{\partial y} - \frac{2}{3} P_{yy} \frac{\partial v}{\partial y} = 0$$

$$(d) \quad \frac{\partial}{\partial t} \int m \xi_y \frac{\xi^2}{2} f d\underline{\xi} + \frac{\partial}{\partial y} \int m \xi_y \frac{\xi^2}{2} f d\underline{\xi} \\ = \frac{p}{\mu} \left[-\frac{2}{3} q_y + v P_{yy} \right]$$

The first three are again just the familiar continuum conservation equations.

The initial and boundary conditions are:

- (a) For $y \geq 0, t \leq 0$; $T = T_0, p = p_0, v = 0$
- (b) For $y \rightarrow \infty, \text{all } t$: $T \rightarrow T_0, p \rightarrow p_0, v \rightarrow 0$ (208)
- (c) For $y = 0, t > 0$: $T_1(y, t) = T_L, m_1(y, t) = m_L$

The second condition states that all flow quantities approach the ambient state far from the liquid surface. The third condition implies diffuse re-emission from the liquid at a known temperature and number density (or pressure).

For small changes in conditions, we seek a perturbation solution:

- (a) $m_1(y, t) = m_0 [1 + \lambda n_1(y, t) + O(\lambda^2)]$
- (b) $m_2(y, t) = m_0 [1 + \lambda n_2(y, t) + O(\lambda^2)]$
- (c) $T_1(y, t) = T_0 [1 + \lambda t_1(y, t) + O(\lambda^2)]$
- (d) $T_2(y, t) = T_0 [1 + \lambda t_2(y, t) + O(\lambda^2)]$
- (209)

where λ is again a perturbation parameter to specify non equilibrium. Only first order quantities will be considered, with the superscripts left out. The mean flow quantities (Appendix IV) can

then be evaluated; for example:

$$(a) \quad P = P_0 \left[1 + \lambda \cdot \frac{1}{2} N_+ + \dots \right]$$

$$(b) \quad P V = \frac{P_0 \beta}{2\pi} \left[\lambda \left\{ N_- + \frac{1}{2} t_- \right\} + \dots \right]$$

(210)

$$(c) \quad P = P R T = P_0 \left[1 + \lambda \left\{ \frac{1}{2} N_+ + \frac{1}{2} t_+ \right\} + \dots \right]$$

$$(d) \quad P_y = \frac{P_0 \beta}{2\pi} \left[\lambda \left\{ 7t_- - 2N_- \right\} + \dots \right]$$

Clearly for $\lambda = 0$ we get equilibrium as desired.

Substituting these into the moment equations (207):

$$(a) \quad \frac{\partial}{\partial t} [N_+] + \frac{\beta}{2\pi} \frac{\partial}{\partial y} [2N_- + t_-] = 0$$

$$(b) \quad \frac{\partial}{\partial t} [2N_- + t_-] + \frac{2\pi P_0}{P_0 \beta} \frac{\partial}{\partial y} [N_+ + t_+] = 0$$

(211)

$$(c) \quad \frac{\partial}{\partial t} [3N_+ + 3t_+] + \frac{\beta}{\pi} \frac{\partial}{\partial y} [4N_- + 6t_-] = 0$$

$$(d) \frac{\partial}{\partial t} [4N_- + 6t_-] + \frac{\beta}{2} \frac{\partial}{\partial y} [5N_+ + 10t_+] = - \frac{\rho_0}{3\mu_0} [7t_- - 2N_-]$$

The above four moment equations are identical to those obtained in [7] . The boundary conditions (208b, c) are now:

$$(a) \text{ For } y \rightarrow \infty : N_+(y, t) \rightarrow 0, t_+(y, t) \rightarrow 0 \quad (212)$$

$$(b) \text{ At } y = 0, t > 0 : t_1 = \frac{T_L - T_0}{\lambda T_0} = \frac{\Delta T}{\lambda T_0}, N_1 = \frac{m_L - m_0}{\lambda m_0} = \frac{\Delta m}{\lambda m_0}$$

For future convenience we non-dimensionalize the variables:

$$\underline{t} = t/\tau_f, \quad \underline{y} = \frac{y}{\beta \tau_f}, \quad \tau_f = \frac{\pi}{4} \frac{\mu_0}{\rho_0} \quad (213)$$

where τ_f is the mean free time.

In dimensionless form the pertinent moment equations are:

$$(a) \frac{\partial}{\partial \underline{t}} [N_+] + \frac{1}{2\pi} \frac{\partial}{\partial \underline{y}} [2N_- + t_-] = 0$$

$$(b) \frac{\partial}{\partial \underline{t}} [2N_- + t_-] + \frac{\partial}{\partial \underline{y}} [N_+ + t_+] = 0$$

(214)

$$(c) \frac{\partial}{\partial \underline{t}} [3N_+ + 3t_+] + \frac{1}{\pi} \frac{\partial}{\partial \underline{y}} [4N_- + 6t_-] = 0$$

$$(d) \frac{\partial}{\partial \underline{t}} [4N_- + 6t_-] + \frac{1}{2} \frac{\partial}{\partial \underline{y}} [5N_+ + 10t_+] + \frac{\pi}{12} [7t_- - 2N_-] = 0$$

3.3 A Solution by the Method of Laplace Transforms

The Laplace transform [8] technique is particularly suitable for the solution of the system of linear, first order partial differential equations (214), with boundary conditions (212). Define the Laplace transform with zero initial conditions:

$$(a) \mathcal{L}\{\varphi\} = \tilde{\varphi} = \int_0^{\infty} e^{-s\underline{t}} \varphi(\underline{y}, \underline{t}) d\underline{t}$$

$$(b) \mathcal{L}\left\{\left(\frac{\partial^m \tilde{\varphi}}{\partial \underline{t}^m}\right)\right\} = s^m \tilde{\varphi} \quad (215)$$

$$(c) \varphi(\underline{y}, \underline{t}) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{s\underline{t}} \tilde{\varphi}(\underline{y}, s) ds$$

where γ is the largest real part of all singularities.

Applying the transform to equations (214), the transformed equations are:

$$(a) \frac{d}{d\underline{y}} \left[2 \tilde{N}_- + \tilde{t}_- \right] + 2\pi s \tilde{N}_+ = 0$$

$$(b) \frac{d}{d\underline{y}} \left[\tilde{N}_+ + \tilde{t}_+ \right] + s \left[2 \tilde{N}_- + \tilde{t}_- \right] = 0$$

(216)

$$(c) \frac{d}{d\underline{y}} \left[4 \tilde{N}_- + 6 \tilde{t}_- \right] + 3\pi s \left[\tilde{N}_+ + \tilde{t}_+ \right] = 0$$

$$(d) \frac{d}{d\underline{y}} \left[5 \tilde{N}_+ + 10 \tilde{t}_+ \right] + \left(8s - \frac{\pi}{3} \right) \tilde{N}_- + \left(12s + \frac{7\pi}{6} \right) \tilde{t}_- = 0$$

Assume solutions to these linear equations in \underline{y} of the form:

$$\tilde{N}_+(y, s) = A(s, \omega) e^{\omega y}; \quad \tilde{N}_- = B(s, \omega) e^{\omega y}$$

(217)

$$\tilde{t}_+(y, s) = C(s, \omega) e^{\omega y}; \quad \tilde{t}_-(y, s) = D(s, \omega) e^{\omega y}$$

Substituting (217) into (216):

$$\begin{pmatrix} 2\pi s & 2\omega & 0 & \omega \\ \omega & 2s & \omega & s \\ 2\pi s & 4\omega & 3\pi s & 6\omega \\ 5\omega & (8s - \frac{\pi}{3}) & 10\omega & (12s + \frac{7\pi}{6}) \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = 0 \quad (218)$$

The determinant for ω can be rewritten:

$$\begin{vmatrix} 2\pi s & 0 & 0 & \omega \\ 0 & 0 & \omega & s \\ 0 & -8\omega & 3\pi s & 0 \\ -5\omega & (-16s - \frac{8\pi}{3}) & 0 & (-10s - \frac{5\pi}{6}) \end{vmatrix} = 0 \quad (219)$$

Or:

$$\omega^4 - 2\pi s (2s + \frac{\pi}{6}) \omega^2 + \frac{3\pi^2}{5} s^3 (4s + \frac{2\pi}{3}) = 0 \quad (220)$$

The quadratic in ω^2 is easily solved to give four roots:

$$\omega_{1,2} = - \left[\pi s \left(\frac{\pi}{6} + 2s \right) \pm \frac{\pi s}{2} \left(\frac{\pi^2}{9} + \frac{16\pi}{15} s + \frac{32}{5} s^2 \right)^{1/2} \right]^{1/2} \quad (221)$$

$$\omega_{3,4} = + \left[\pi s \left(\frac{\pi}{6} + 2s \right) \pm \frac{\pi s}{2} \left(\frac{\pi^2}{9} + \frac{16\pi}{15} s + \frac{32}{5} s^2 \right)^{1/2} \right]^{1/2} \quad (222)$$

The positive roots ω_3 and ω_4 are rejected by the boundedness condition (212a), for $y \rightarrow \infty$.

For each ω equations (218) can be solved to give:

$$(a) B(s) = b(s) \cdot A(s) = - \left[\frac{5\omega^2 + 2\pi s \left(2s + \frac{7\pi}{6} \right)}{\omega \left(16s + \frac{8\pi}{3} \right)} \right] A(s)$$

$$(b) C(s) = c(s) A(s) = \left[3 + \frac{8\omega}{3\pi s} b(s) \right] A(s) \quad (223)$$

$$(c) D(s) = d(s) A(s) = - \left[\frac{2\pi s}{\omega} + 2 b(s) \right] A(s)$$

From (217) and (223) the transformed variables are:

$$\begin{aligned}
 \text{(a)} \quad \tilde{N}_+(y, s) &= A_1 e^{\omega_1 y} + A_2 e^{\omega_2 y} \\
 \text{(b)} \quad \tilde{N}_-(y, s) &= b_1 A_1 e^{\omega_1 y} + b_2 A_2 e^{\omega_2 y} \\
 \text{(c)} \quad \tilde{t}_+(y, s) &= c_1 A_1 e^{\omega_1 y} + c_2 A_2 e^{\omega_2 y} \\
 \text{(d)} \quad \tilde{t}_-(y, s) &= d_1 A_1 e^{\omega_1 y} + d_2 A_2 e^{\omega_2 y}
 \end{aligned}
 \tag{224}$$

We need to solve, now, only for $A_1(s)$ and $A_2(s)$.

From the boundary conditions (212b) at the liquid surface ($y=0$):

$$\begin{aligned}
 \text{(a)} \quad 2 \tilde{t}_l(y=0) &= \tilde{t}_+(y=0) + \tilde{t}_-(y=0) \\
 &= (c_1 + d_1) A_1 + (c_2 + d_2) A_2 = \frac{2 t_{lL}}{s}
 \end{aligned}
 \tag{225}$$

$$\begin{aligned}
 \text{(b)} \quad 2 \tilde{N}_l(y=0) &= \tilde{N}_+(y=0) + \tilde{N}_-(y=0) \\
 &= (1 + b_1) A_1 + (1 + b_2) A_2 = \frac{2 N_{lL}}{s}
 \end{aligned}$$

where:

$$t_{lL} = \frac{\Delta T}{\lambda T_0} \quad ; \quad N_{lL} = \frac{\Delta m}{\lambda m_0}
 \tag{226}$$

Solving for $A_1(s)$ and $A_2(s)$:

$$(a) A_1(s) = \frac{\frac{2}{s} \left[t_{1L} (1 + b_2) - N_{1L} (c_2 + d_2) \right]}{(c_1 + d_1)(1 + b_2) - (c_2 + d_2)(1 + b_1)} \quad (227)$$

$$(b) A_2(s) = \frac{\frac{2}{s} \left[t_{1L} (1 + b_1) - N_{1L} (c_1 + d_1) \right]}{(c_2 + d_2)(1 + b_1) - (c_1 + d_1)(1 + b_2)}$$

By substituting (221) and (223) into (227), the transformed variables (224) can be written down simply as functions of S , \underline{y} , t_{1L} and N_{1L} . In principle then these can be inverted to give the solutions for all time. However, our interest lies mainly in the wave pattern and mass flux rates for large time. We therefore seek solutions in the large time limit.

By inspection the origin is the singularity with the largest real part. For fixed \underline{y} , large time corresponds to S small [8] ; one therefore expands the transformed variables in power series in S . Keeping only highest order terms:

$$(a) \omega_1(s) = -\frac{\pi}{\sqrt{3}} \sqrt{s} + o(\sqrt{s}).$$

$$(b) \omega_2(s) = -\sqrt{\frac{6\pi}{5}} S + o(s)$$

(228)

Since $e^{\omega_1 y} \rightarrow e^{-\frac{\pi}{\sqrt{3}} \sqrt{s} y}$ this transform leads to a solution that is diffusive in character. But

$e^{\omega_2 y} \rightarrow e^{-\sqrt{\frac{6\pi}{5}} s y}$ gives a shift of the form:

$$\begin{aligned} \left[t - \sqrt{\frac{6\pi}{5}} y \right] &= \frac{1}{\tau_f} \left[t - \sqrt{\frac{6\pi}{5}} \frac{y}{\sqrt{2\pi RT_0}} \right] \\ &= \frac{1}{\tau_f} \left[t - \frac{y}{\sqrt{\frac{5}{3} RT_0}} \right] = \frac{1}{\tau_f} \left[t - \frac{y}{a_0} \right] \end{aligned} \quad (229)$$

where $a_0 = \sqrt{\frac{5}{3} RT_0}$ is the isentropic speed of sound. This is exactly as in [7].

Similarly by neglecting higher order terms:

$$\begin{aligned} \text{(a)} \quad b_1(s) &= \frac{3\sqrt{3}}{2} \sqrt{s} + \dots \\ \text{(b)} \quad c_1(s) &= -1 + \dots \\ \text{(c)} \quad d_1(s) &= -\sqrt{3} \sqrt{s} + \dots \\ \text{(d)} \quad b_2(s) &= \frac{7\pi}{8} \sqrt{\frac{s}{6\pi}} + \dots \\ \text{(e)} \quad c_2(s) &= \frac{2}{3} + \dots \end{aligned} \quad (230)$$

$$(f) \quad d_2(s) = \frac{\pi}{4} \sqrt{\frac{s}{6\pi}} + \dots$$

$$(g) \quad A_1(s) = \frac{2}{s} \frac{\left[\left(1 + \frac{7\pi}{8} \sqrt{\frac{s}{6\pi}}\right) t_{IL} - \left(\frac{2}{3} + \frac{\pi}{4} \sqrt{\frac{s}{6\pi}}\right) N_{IL} \right]}{\left[-\left(1 + \frac{7\pi}{8} \sqrt{\frac{s}{6\pi}}\right) - \left(\frac{2}{3} + \frac{\pi}{4} \sqrt{\frac{s}{6\pi}}\right) \right]} + \dots$$

$$(h) \quad A_2(s) = \frac{2}{s} \frac{\left[t_{IL} + N_{IL} \right]}{\left[\left(\frac{2}{3} + \frac{\pi}{4} \sqrt{\frac{s}{6\pi}}\right) + \left(1 + \frac{7\pi}{8} \sqrt{\frac{s}{6\pi}}\right) \right]} + \dots$$

From equation (210b, c) we have to first order:

$$\begin{aligned} \tilde{p}(s) &= p_0 \left[\frac{1}{s} + \lambda \left\{ \frac{1}{2} \tilde{N}_+ + \frac{1}{2} \tilde{t}_+ \right\} + \dots \right] \\ &= p_0 \left[\frac{1}{s} + \lambda \left\{ \frac{(1+c_1)}{2} A_1 e^{\omega_1 y} + \frac{(1+c_2)}{2} A_2 e^{\omega_2 y} \right\} \right] \quad (231) \end{aligned}$$

$$\begin{aligned} \tilde{v}(s) &= \frac{\lambda \beta}{2\pi} \left[\tilde{N}_- + \frac{1}{2} \tilde{t}_- \right] \\ &= \frac{\lambda \beta}{2\pi} \left[(b_1 + \frac{1}{2} d_1) A_1 e^{\omega_1 y} + (b_2 + \frac{1}{2} d_2) A_2 e^{\omega_2 y} \right] \quad (232) \end{aligned}$$

The relevant inversions are:

$$\frac{1}{\sqrt{s}} e^{-a\sqrt{s}} \longrightarrow \frac{e^{-\frac{a^2}{4\underline{t}}}}{(\pi \underline{t})^{1/2}} \quad (233)$$

$$\frac{e^{-as}}{s} \longrightarrow \begin{cases} 0 & \text{for } 0 < \underline{t} < a \\ 1 & \text{for } \underline{t} > a \end{cases} \quad (234)$$

Thus for large time the solutions are:

$$p(y, \underline{t}) = \begin{cases} p_0 & \text{for } 0 < \underline{t} < \sqrt{\frac{6\pi}{5}} \underline{y} \\ p_0 [1 + \delta_1] & \text{for } \underline{t} > \sqrt{\frac{6\pi}{5}} \underline{y} \end{cases} \quad (235)$$

$$v(y, \underline{t}) = \begin{cases} \frac{\delta_2}{(\pi \underline{t})^{1/2}} \exp \left\{ -\frac{\pi^2 \underline{y}^2}{12 \underline{t}} \right\} & 0 < \underline{t} < \sqrt{\frac{6\pi}{5}} \underline{y} \\ \frac{\delta_2}{(\pi \underline{t})^{1/2}} \exp \left\{ -\frac{\pi^2 \underline{y}^2}{12 \underline{t}} \right\} + \delta_3 & \underline{t} > \sqrt{\frac{6\pi}{5}} \underline{y} \end{cases} \quad (236)$$

where

$$\delta_1 = \frac{1}{2} \cdot \frac{5}{3} \frac{2}{\left(\frac{5}{3} + \frac{9\pi}{8}\sqrt{\frac{5}{6\pi}}\right)} \frac{\Delta P}{P_0} \quad (237)$$

$$\delta_2 = \frac{\beta}{2\pi} \cdot \frac{\sqrt{3}}{2} \cdot \frac{2 \left[\left(\frac{2}{3} + \frac{\pi}{4}\sqrt{\frac{5}{6\pi}} \right) \frac{\Delta m}{m_0} - \left(1 + \frac{7\pi}{8}\sqrt{\frac{5}{6\pi}} \right) \frac{\Delta T}{T_0} \right]}{\left(\frac{5}{3} + \frac{9\pi}{8}\sqrt{\frac{5}{6\pi}} \right)} \quad (238)$$

$$\delta_3 = \frac{\beta}{2\pi} \cdot \pi \sqrt{\frac{5}{6\pi}} \cdot \frac{2}{\left(\frac{5}{3} + \frac{9\pi}{8}\sqrt{\frac{5}{6\pi}}\right)} \frac{\Delta P}{P_0} \quad (239)$$

It is clear from the above equations that the large time solution is composed to two parts - a diffusive part and a wave part. The diffusive part vanishes rapidly away from the plate, leaving a purely inviscid solution. On the other hand, the pure wave

travelling at the isentropic sound speed $a_0 = \sqrt{\frac{5}{3}RT_0}$ never dies out. From (235), it is seen that the pressure is composed only of the wave part. For $t \rightarrow \infty$, the evaporation-condensation rate at the liquid surface is given by:

$$\begin{aligned}
 \dot{m}(y=0) &= \rho V(y=0) \\
 &= \rho_0 \sqrt{2\pi RT_0} \frac{\sqrt{\frac{5}{6\pi}}}{\left(\frac{5}{3} + \frac{9\pi}{8}\sqrt{\frac{5}{6\pi}}\right)} \frac{\Delta P}{P_0} \\
 &= 0.148 \rho_0 \sqrt{2\pi RT_0} \frac{\Delta P}{P_0} \tag{240}
 \end{aligned}$$

Comparing with (195) for the spherical droplet, we see that in the continuum limit the mass fluxes are almost the same both in form ($\sim \Delta P/P_0$) and magnitude (0.148 compared with 0.142). Detailed discussion will be left to the last section.

The large time solutions obtained here are valid only when \underline{y} is held fixed. For solutions near the wave front one would have to use the method of steepest descent or the method due to Whitham [1959, Comm. Pure Appl. Math., 12]. This is not necessary for our purposes. In the next section we examine a 'quasi-steady state solution'.

4. One Dimensional Quasi-Steady State Solution

4.1 Formulation

We saw in the last section that the large time solution, for the one dimensional problem, was essentially composed of a uniform steady state and a pure inviscid wave that never really died out. We ask ourselves, can we solve the steady state moment

equations, patch on a wave and get the same result as the transient solution? It is not really obvious that this can be done, since we do not know the details of the transient solution or of the wave.

To bring up a point that will be discussed later, one cannot be sure that this can be done for both evaporation and condensation.

In order to investigate this possibility the following programme is suggested:

(i) Solve the 'steady state' moment equations with known boundary conditions at the liquid surface but with uniform (pressure = P_c , temperature = T_c) but unknown conditions far away.

(ii) Patch on a pure wave to connect the 'steady solution' with the known uniform conditions, P_0 and T_0 at infinity. Thus P_c , T_c , ΔP , ΔT and the evaporation rate will be connected to P_0 and T_0 through the wave.

If the above scheme yields the answer for large time of the transient problem, we will have gained a better understanding of the phenomenon.

As in the previous section, the two stream Maxwellian defined by (203) with four initially unknown functions $m_1(y)$, $T_1(y)$, $m_2(y)$, $T_2(y)$ will be used. Dropping out the time dependence from equations (207) we have the required moment equations:

$$(a) \quad \frac{d}{dy} (\rho v) = 0$$

$$(b) \quad \rho v \frac{dv}{dy} - \frac{dP_{yy}}{dy} = 0$$

(241)

$$(c) \quad \frac{d}{dy} (\rho v) + \frac{2}{3} \frac{dq_y}{dy} - \frac{2}{3} P_{yy} \frac{dv}{dy} = 0$$

$$(d) \quad \frac{d}{dy} \int m \xi_y^2 \frac{\xi^2}{2} f d\xi = \frac{\rho}{\mu} \left[-\frac{2}{3} q_y + v P_{yy} \right]$$

The first three of the above equations can be integrated immediately to give the usual conservation equations:

$$(a) \quad \rho v = \text{constant}$$

$$(b) \quad \rho v^2 - P_{yy} = \text{constant}$$

(242)

$$(c) \quad \rho v (3\rho v + 2q_y) - P_{yy}^2 = \text{constant}$$

Equations (241a, b, c, d) can therefore be reduced to a single first order non-linear equation for one of the functions $m_i(y)$, $T_i(y)$, etc. However, we shall again seek a perturbation solution

for small disturbances from equilibrium.

We can write:

$$\begin{aligned}
 \text{(a)} \quad \rho(y) &= \rho_0 (1 + \lambda \rho^{(1)} + \lambda^2 \rho^{(2)} + \dots) \\
 \text{(b)} \quad p(y) &= p_0 (1 + \lambda p^{(1)} + \lambda^2 p^{(2)} + \dots) \\
 \text{(c)} \quad v(y) &= \beta (\lambda v^{(1)} + \lambda^2 v^{(2)} + \dots)
 \end{aligned}
 \tag{243}$$

etc.

where the superscripts refer to the orders of the quantities. Of course $\rho^{(1)}$, $p^{(1)}$, etc. are functions of $N_1^{(1)}(y)$, $t_1^{(1)}(y)$, etc. as given in Appendix IV or equations (210), e.g.:

$$\begin{aligned}
 \text{(a)} \quad \rho^{(1)} &= \frac{1}{2} N_+^{(1)} = \frac{1}{2} (N_1^{(1)} + N_2^{(1)}) \\
 \text{(b)} \quad p^{(1)} &= \frac{1}{2} (N_+^{(1)} + t_+^{(1)}) = \frac{1}{2} (N_1^{(1)} + N_2^{(1)} + t_1^{(1)} + t_2^{(1)})
 \end{aligned}
 \tag{244}$$

Now by direct substitution of (243) into the conservation equations (242), it is easy to show (App. V) that:

$$q_y^{(1)} = 0 \quad ; \quad q_y^{(2)} = -\frac{1}{2} \beta p_0 v^{(1)} [3p^{(1)} - 5\rho^{(1)}]
 \tag{245}$$

i. e., the heat flux is zero to first order and to second order is as given above. This result will be used later.

The boundary conditions, for the steady state solution are:

$$(a) \text{ At } y=0 : t_1''' = \frac{T_{1L} - T_0}{\lambda T_0} = \frac{\Delta T}{\lambda T_0}, N_1''' = \frac{m_{1L} - m_0}{\lambda m_0} = \frac{\Delta m}{\lambda m_0}$$

(246)

$$(b) \text{ For } y \rightarrow \infty : p \rightarrow p_c, V \rightarrow V_c$$

We note that p_c and V_c are the uniform pressure and velocity behind the wave.

The linearized moment equations for the problem, derived from (241), are to first order:

$$(a) \frac{d}{dy} [2N_- + t_-] = 0$$

$$(b) \frac{d}{dy} [N_+ + t_+] = 0$$

(247)

$$(c) \frac{d}{dy} \left[-N_- + \frac{7}{2} t_- \right] = 0$$

$$(d) \frac{d}{dy} [5N_+ + 10t_+] = -\frac{2}{3} \frac{p_0}{\beta \mu_0} [7t_- - 2N_-]$$

The first three equations state that to first order the velocity, pressure, and heat flux are constant. These can be integrated immediately to give:

$$\begin{aligned}
 \text{(a)} \quad 2 N_- + t_- &= B_1 \\
 \text{(b)} \quad N_+ + t_+ &= B_2 \\
 \text{(c)} \quad -N_- + \frac{7}{2} t_- &= B_3 \\
 \text{(d)} \quad 5 N_+ + 10 t_+ &= B_4 - \frac{4}{3} \frac{\rho_0}{\beta \mu_0} B_3 y
 \end{aligned}
 \tag{248}$$

From (248d) we note that $B_3 = 0$ for finiteness, i. e., the heat flux must be 0 to first order. This leaves three constants B_1, B_2, B_4 to satisfy four conditions (246), i. e., the problem seems overspecified.

At this juncture it is instructive to examine the simple one dimensional heat convection problem. This is treated in detail in Appendix VI. It can be seen there that ∞ is a singular point for the equations and that a perturbation solution will break down. The singular perturbation can be treated by the method of inner and outer expansions [9].

The equations (241) that we are dealing with here are very similar to the continuum heat convection equations. It seems reasonable to expect that, as in the continuum case, an 'inner-outer' expansion technique will overcome the difficulties encountered.

Equations (247a, b, c, d) are then taken to be the inner equations.

Equations (248) are then the inner solution:

$$\begin{aligned} \text{(a)} \quad & 2 N_-(y) + t_-(y) = B_1 \\ \text{(b)} \quad & N_+(y) + t_+(y) = B_2 \\ \text{(c)} \quad & -N_-(y) + \frac{7}{2} t_-(y) = 0 \\ \text{(d)} \quad & 5 N_+(y) + 10 t_+(y) = B_4 \end{aligned} \tag{249}$$

4.2 Steady State Outer Solution and Matching

Let η be the outer variable defined by:

$$\eta = \lambda y \quad , \quad \frac{d}{dy} = \lambda \frac{d}{d\eta} \tag{250}$$

Assume, again, perturbation expansions of the form:

$$\begin{aligned} m_1(\eta) &= m_0 \left(1 + \lambda N_1^{(1)}(\eta) + \lambda^2 N_1^{(2)}(\eta) + \dots \right) \\ T_1(\eta) &= T_0 \left(1 + \lambda t_1^{(1)}(\eta) + \lambda^2 t_1^{(2)}(\eta) + \dots \right) \end{aligned} \tag{251}$$

etc.

Noting that the algebraic equations should be identical to the inner equations by the matching condition, the outer equations are:

$$(a) \quad 2 N_-(\eta) + t_-(\eta) = B_1$$

$$(b) \quad N_+(\eta) + t_+(\eta) = B_2$$

(252)

$$(c) \quad -N_-(\eta) + \frac{7}{2} t_-(\eta) = 0$$

$$(d) \quad \frac{1}{2} \frac{\rho_0 \beta^4}{8\pi^2} \frac{d}{d\eta} [5 N_+(\eta) + 10 t_+(\eta)] = -\frac{2}{3} q_y^{(2)} \frac{\rho_0}{\mu_0}$$

But from (245)

$$\begin{aligned} q_y^{(2)} &= -\frac{1}{2} \beta \rho_0 v^{(1)} [3 p^{(1)} - 5 f^{(1)}] \\ &= -\frac{1}{2} \beta \rho_0 v^{(1)} \left[\frac{3}{2} t_+ - N_+ \right] \end{aligned} \quad (253)$$

where $v^{(1)} = \frac{\beta}{4\pi} \cdot B_1$, is the velocity perturbation. Thus (252b, d) and (253) yield:

$$\frac{dt_+}{d\eta} = \frac{4\pi}{3} \frac{\rho_0 v^{(1)}}{\mu_0 \beta} \left(t_+ - \frac{2}{5} B_2 \right) \quad (254)$$

Integrating we have the outer solutions:

$$(a) \quad t_+(\eta) = \frac{2}{5} B_2 + C \exp \left\{ \frac{4\pi}{3} \frac{\rho_0 v^{(1)}}{\mu_0 \beta} \eta \right\}$$

$$(b) \quad N_+(\eta) = \frac{3}{5} B_2 - C \exp \left\{ \frac{4\pi}{3} \frac{\rho_0 v^{(1)}}{\mu_0 \beta} \eta \right\}$$

$$(c) \quad t_-(\eta) = \frac{1}{8} B_1 \quad (255)$$

$$(d) \quad N_-(\eta) = \frac{7}{16} B_1$$

To the above outer solution we apply the boundary conditions (246b):

$$\text{For } \eta \rightarrow \infty : \quad p \rightarrow p_c, \quad v \rightarrow v_c$$

$$\therefore B_2 = \frac{2(p_c - p_0)}{\lambda p_0} \quad (256)$$

$$B_1 = \frac{4\pi v_c}{\lambda \beta} \quad (257)$$

We also note from (255a) that for the finiteness of the temperature field we require:

$$v^{(1)} < 0 \quad (258)$$

This means that the 'steady state' problem is properly posed only for condensation. The solution obtained above is now clearly like the convection solution (Appendix VI). The velocity has to oppose the temperature gradient to maintain a finite temperature infinitely far away.

To proceed with the solution, the inner and outer solutions

will match completely $\left(\lim_{y \rightarrow \infty} \varphi(y) = \lim_{\eta \rightarrow 0} \varphi(\eta) \right)$ provided:

$$C = \frac{1}{5} B_4 - \frac{7}{5} B_2 \quad (259)$$

The boundary conditions (246a) can be applied to the inner solution. At $y = 0$:

$$(a) \quad 2 t_1'''(y=0) = \frac{1}{8} B_1 + \frac{1}{5} B_4 - B_2 = \frac{2 \Delta T}{\lambda T_0} \quad (260)$$

$$(b) \quad 2 N_1'''(y=0) = \frac{7}{16} B_1 - \frac{1}{5} B_4 + 2 B_2 = \frac{2 \Delta m}{\lambda m_0}$$

Thus:

$$\frac{9}{16} B_1 = \frac{2 \Delta p}{\lambda p_0} - B_2 \quad (261)$$

Or using (95) and (96):

$$\frac{9}{16} \frac{4\pi v_c}{\lambda \beta} = \frac{2(p_L - p_0)}{\lambda p_0} - \frac{2(p_L - p_0)}{\lambda p_0} = \frac{2(p_L - p_c)}{\lambda p_0}$$

$$\therefore v_c = \beta \cdot \frac{8}{9\pi} \frac{(p_L - p_c)}{p_0} \quad (262)$$

This is the constant mean velocity to first order in the steady solution. Now v_c and p_c are the uniform velocity and pressure behind the wave.

4.3 Completion of the Solution with the Wave

The transient solution showed that for large time the solution was composed of a steady part and a pure undiminished wave. The 'steady' solution of the last subsection, showed by comparison with the convection problem that to first order, viscous effects would not be important. Further the steady solution was seen to be valid only for condensation.

One can now conjecture that the transient condensation problem is similar to the piston-tube problem in gas dynamics [10]. Lowering the liquid temperature and vapor pressure is like pulling a piston away from stationary gas in a tube. The kinetic effects take place close to the liquid surface but a pure expansion fan propagates into the undisturbed medium. In the analogy the piston speed would be equal to the velocity of condensation. Assuming such a picture to be true, the relationship between P_c , V_c and P_0 would be given by the corresponding relationships in a pure expansion fan [10, pg. 79] :

$$\frac{P_c}{P_0} = \left(1 + \frac{\gamma-1}{2} \frac{V_c}{a_0}\right)^{\frac{2\gamma}{\gamma-1}} \quad (263)$$

Linearizing and noting that $\gamma = \frac{5}{3}$:

$$P_c = P_0 \left(1 + \frac{5}{3} \frac{V_c}{\sqrt{\frac{5}{3} RT_0}}\right) = P_0 \left(1 + \sqrt{\frac{10\pi}{3}} \frac{V_c}{\beta}\right) \quad (264)$$

But from equation (262)

$$V_c = \frac{8}{9\pi} \beta \frac{(P_L - P_c)}{P_0} = \frac{8}{9\pi} \left[\beta \cdot \frac{\Delta P}{P_0} - \sqrt{\frac{10\pi}{3}} V_c \right]$$

$$\therefore V_c = \beta \cdot \frac{\frac{8}{9\pi}}{\left[1 + \frac{8}{9\pi} \sqrt{\frac{10\pi}{3}} \right]} \quad (265)$$

Thus to first order the condensation mass flux rate predicted by this 'quasi-steady state' approach is:

$$\begin{aligned} \dot{m}(y=0) &= \rho V = \rho_0 \beta \frac{\frac{8}{9\pi}}{\left[1 + \frac{8}{9\pi} \sqrt{\frac{10\pi}{3}} \right]} \frac{\Delta P}{P_0} \\ &= \rho_0 \beta \frac{\sqrt{\frac{5}{6\pi}}}{\left[\frac{9\pi}{8} \sqrt{\frac{5}{6\pi}} + \sqrt{\frac{10\pi}{3}} \sqrt{\frac{5}{6\pi}} \right]} \frac{\Delta P}{P_0} \\ &= \rho_0 \beta \frac{\sqrt{\frac{5}{6\pi}}}{\left[\frac{5}{3} + \frac{9\pi}{8} \sqrt{\frac{5}{6\pi}} \right]} \frac{\Delta P}{P_0} \quad (266) \end{aligned}$$

This is identical to the result (240) obtained in the transient problem. One is therefore encouraged to have some belief in the 'quasi-steady' picture for condensation. This approach does not work for evaporation; this point will be discussed later.

We now note that the expansion fan will become infinitesimally

weak in the case of the droplet. This is because in a spherical geometry the inviscid waves go like $f(r-ct)/r$ and die out for large τ . The solution obtained for the droplet is therefore the correct solution for large time.

5. The Effect of Higher Moments

So far, all the calculations have been made using four initially unknown functions to represent the distribution function and then using four moment equations. In an integral method such as the present one, one's confidence is greatly increased if it can be shown that the gross features are changed little, by the use of higher moments. In an attempt to see the effect of higher moments, we seek a six moment solution for the one dimensional problem.

Let the distribution function for the one dimensional case (Fig. 5b) be represented by the two stream Maxwellian:

$$f(y, \underline{\xi}, t) = \begin{cases} f_1 = \frac{m_1(y, t)}{[2\pi RT_1(y, t)]^{3/2}} \exp \left\{ -\frac{\xi_x^2 + \xi_z^2 + [\xi_y - V_1(y, t)]^2}{2RT_1(y, t)} \right\} \\ f_2 = \frac{m_2(y, t)}{[2\pi RT_2(y, t)]^{3/2}} \exp \left\{ -\frac{\xi_x^2 + \xi_z^2 + [\xi_y - V_2(y, t)]^2}{2RT_2(y, t)} \right\} \end{cases} \quad (267)$$

where $m_{1,2}(y, t)$, $T_{1,2}(y, t)$, $V_{1,2}(y, t)$ are six initially undetermined functions. This is the most obvious extension of the four-function approach, since it is the y velocity

that we are trying to calculate. The reason for not using five moments is that it is not clear which five functions would be most suitable.

The mean flow quantities can now be determined in terms of the six unknown functions $m_{1,2}$, $T_{1,2}$, $V_{1,2}$. The integrations are now complicated by the presence of V_1 and V_2 in the exponentials and lead to error functions, exponentials, etc. However, we are only going to solve the linearized problem. It therefore proves advantageous to evaluate the mean flow quantities after linearizing m_1 , m_2 , etc.:

$$\begin{aligned} m_1 &= m_0 [1 + \lambda N_1(y,t) + \dots] \\ T_1 &= T_0 [1 + \lambda t_1(y,t) + \dots] \\ V_1 &= \beta [\lambda \nu_1(y,t) + \dots] \\ &\text{etc.} \end{aligned} \tag{268}$$

V_1 and V_2 have no 0th order parts since they vanish at equilibrium. Now the exponentials in (267) can be linearized before integrating, e.g.:

$$\begin{aligned} \exp \left\{ - \frac{\xi_x^2 + \xi_z^2 + [\xi_y - \lambda \beta \nu_1 + \dots]^2}{2RT_0 [1 + \lambda t_1 + \dots]} \right\} \\ = \left[1 + \lambda \cdot \frac{\beta \nu_1}{RT_0} \xi_y + o(\lambda^2) \right] e^{-\frac{\xi^2}{2RT_0}} \end{aligned} \tag{269}$$

Then, for example, the density is given to first order by:

$$\begin{aligned} \rho(y, t) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m f d\xi_x d\xi_y d\xi_z \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m f_1 d\xi_x d\xi_y d\xi_z + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m f_2 d\xi_x d\xi_y d\xi_z \quad (270) \\ &= \rho_0 \left[1 + \lambda \left\{ \frac{1}{2} N_+ + \nu_- \right\} + O(\lambda^2) \right] \end{aligned}$$

where: $N_+ = N_1 + N_2$, $\nu_- = \nu_1 - \nu_2$, etc. (271)

Appendix IV lists all the necessary mean flow quantities to first order; a few are given here:

$$(a) \quad \rho = \rho_0 \left[1 + \lambda \left\{ \frac{1}{2} N_+ + \nu_- \right\} + \dots \right]$$

$$(b) \quad \rho V = \frac{\rho_0 \beta}{2\pi} \left[\lambda \left\{ N_- + \frac{1}{2} t_- + \pi \nu_+ \right\} + \dots \right]$$

$$(c) \quad p = p_0 \left[1 + \lambda \left\{ \frac{1}{2} N_+ + \frac{1}{2} t_+ + \frac{4}{3} \nu_- \right\} + \dots \right] \quad (272)$$

$$(d) \quad P_{yy} = -p_0 \left[1 + \lambda \left\{ \frac{1}{2} N_+ + \frac{1}{2} t_+ + 2 \nu_- \right\} + \dots \right]$$

$$(e) \quad q_y = \frac{\rho_0 \beta}{8\pi} \left[\lambda \left\{ 7 t_- - 2 N_- \right\} + \dots \right]$$

We note that the introduction of V_1 and V_2 permit the stress tensor to be non-isotropic ($P \neq -P_{yy}$); also when $\lambda = 0$ equilibrium is obtained.

The six moment equations to be used, corresponding to $Q = m, m \xi_y, m \xi^2/2, m \xi_x^2, \frac{1}{2} m \xi_y \xi^2$ and $m \xi_y^3$, are from equation (206):

$$(a) \quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial y} (\rho v) = 0$$

$$(b) \quad \rho \frac{\partial v}{\partial t} + \rho v \frac{\partial v}{\partial y} - \frac{\partial P_{yy}}{\partial y} = 0$$

$$(c) \quad \frac{\partial p}{\partial t} + \frac{\partial}{\partial y} (\rho v) + \frac{2}{3} \frac{\partial q_y}{\partial y} - \frac{2}{3} P_{yy} \frac{\partial v}{\partial y} = 0$$

(273)

$$(d) \quad \frac{\partial}{\partial t} \int m \xi_x^2 f d \underline{\xi} + \frac{\partial}{\partial y} \int m \xi_y \xi_x^2 f d \underline{\xi} = \frac{P}{\mu} p_{xx}$$

$$(e) \quad \frac{\partial}{\partial t} \int m \xi_y \frac{\xi^2}{2} f d \underline{\xi} + \frac{\partial}{\partial y} \int m \xi_y^2 \frac{\xi^2}{2} f d \underline{\xi} = \frac{P}{\mu} \left[-\frac{2}{3} q_y + v P_{yy} \right]$$

$$(f) \quad \frac{\partial}{\partial t} \int m \xi_y^3 f d \underline{\xi} + \frac{\partial}{\partial y} \int m \xi_y^4 f d \underline{\xi} \\ = \frac{P}{\mu} \left[q_y + 3v P_{yy} - \frac{3}{2} \int m (\xi_y - v)^3 f d \underline{\xi} \right]$$

Equations (273a, b, c, e) are of course identical to those used in the four moment problem. Since ΔQ for $Q = m \xi y^3$ is not given in [5] , this is derived here in Appendix VII.

The boundary conditions for the transient problem are:

$$\begin{aligned} \text{(a) For } y \rightarrow \infty : T \rightarrow T_0, P \rightarrow P_0, V \rightarrow 0 \\ \text{(b) At } y = 0, t > 0 : m_1 = m_L, T_1 = T_L, V_1 = 0 \end{aligned} \quad (274)$$

One can now solve the linearized problem, by Laplace transforms exactly as in section 3, to obtain solutions for large time. Since the details are complicated, but not instructive, they are given in Appendix VIII.

The solution turns out to be composed of a diffusive part and a wave-like part, as before, plus a wave damped exponentially in y . For $t \rightarrow \infty$, the mass flux at the liquid surface is given by:

$$\begin{aligned} \dot{m} = \rho V(y=0) &= \rho_0 \beta \frac{1}{\left[\sqrt{\frac{10\pi}{3}} + \frac{9\pi + 12\sqrt{\frac{5\pi}{2}}}{8 + 3\sqrt{\frac{5\pi}{2}}} \right]} \frac{\Delta P}{P_0} \\ &= 0.143 \rho_0 \beta \frac{\Delta P}{P_0} \end{aligned} \quad (275)$$

Comparing with (240) it seems that the four moment solution is remarkably good considering its crudeness. It seems as though the extra moments only give greater detail but do not change the magnitudes of the mean quantities appreciably. This result, again, gives one confidence in the present approach.

The 'quasi-steady' approach to the condensation problem again gives exactly the same result (275) for condensation (Appendix VIII).

6. Discussion

Summary of the Results

For ease in comparison, we present, here, in brief, the main results of the calculations. If the ambient vapor temperature and pressure are T_0 and P_0 , and the liquid temperature is T_L , then the net evaporating or condensing mass flux \dot{m} , at the liquid surface, is given to first order by:

(i) For a Spherical Droplet (4 Moments):

$$\dot{m} = \frac{\rho_0 \sqrt{2\pi RT_0}}{2\pi} \frac{\left(\frac{1}{4} + \frac{2}{15} \frac{\gamma_0}{l_0}\right) \frac{\Delta T}{T_0} + \left(\frac{1}{2} + \frac{2}{15} \frac{\gamma_0}{l_0}\right) \frac{\Delta m}{m_0}}{\left(\frac{1}{2} + \frac{3}{20} \frac{\gamma_0}{l_0}\right)}$$

(ii) For an Infinite Plane Liquid Sheet (4 Moments):

$$\dot{m} = \rho_0 \sqrt{2\pi RT_0} \frac{\sqrt{\frac{5}{6\pi}}}{\left(\frac{5}{3} + \frac{9\pi}{8} \sqrt{\frac{5}{6\pi}}\right)} \frac{\Delta P}{P_0} = 0.148 \rho_0 \beta \frac{\Delta P}{P_0}$$

(iii) For an Infinite Plane Liquid Sheet (6 Moments):

$$\dot{m} = \rho_0 \sqrt{2\pi RT_0} \frac{1}{\left[\sqrt{\frac{10\pi}{3}} + \frac{9\pi + 12\sqrt{\frac{5\pi}{2}}}{8 + 3\sqrt{\frac{5\pi}{2}}} \right]} \frac{\Delta P}{\rho_0} = 0.143 \rho_0 \beta \frac{\Delta P}{\rho_0}$$

where:

r_0 is the radius of the droplet

l_0 is the mean free path at ρ_0, T_0

$$\Delta T = T_L - T_0 ; \Delta P = P_L - P_0 ; \Delta m = m_L - m_0$$

$$\frac{\Delta P}{\rho_0} = \frac{\Delta T}{T_0} + \frac{\Delta m}{m_0}$$

$$\beta = \sqrt{2\pi RT_0}$$

We note first of all that the one dimensional problem has no natural length scale or Knudsen number associated with it. In this case "free molecular flow" takes place for small time and continuum flow for large time. The droplet, on the other hand, has a natural length, its radius, r_0 . One would expect that in the continuum limit, $(r_0/l_0) \rightarrow \infty$, the Knudsen layer would be so thin that the evaporation or condensation rate would be close to the value for the one dimensional problem. This is so: comparing the two four moment calculations:

$$\dot{m} \quad (\text{sphere, } \frac{r_0}{l_0} \rightarrow \infty) = 0.142 \rho_0 \sqrt{2\pi RT_0} \frac{\Delta P}{\rho_0}$$

$$\dot{m} \text{ (one dimensional)} = 0.148 \rho_0 \sqrt{2\pi RT_0} \frac{\Delta P}{P_0}$$

The slight difference in the answers comes about from the continuum part of the flow field, and not from the Knudsen layer.

(One cannot obtain one dimensional heat convection by letting $\tau_0 \rightarrow \infty$ in three dimensional heat convection.)

By letting $(\tau_0/l_0) \rightarrow 0$ or ∞ , the limiting solutions for the droplet are:

$$\text{Free molecule } \left(\frac{\tau_0}{l_0} \rightarrow 0 \right): \dot{m} = \frac{\rho_0 \beta}{2\pi} \cdot \frac{1}{2} \left(2 \frac{\Delta P}{P_0} - \frac{\Delta T}{T_0} \right)$$

$$\text{Continuum } \left(\frac{\tau_0}{l_0} \rightarrow \infty \right): \dot{m} = \frac{\rho_0 \beta}{2\pi} \cdot \frac{8}{9} \frac{\Delta P}{P_0}$$

where $\beta = \sqrt{2\pi RT_0}$. Thus in one limit the mass flux depends only on ΔP and in the other limit the ratio of the coefficients of ΔP to ΔT is 2:1.

Referring to Fig. 4 this can be understood in the following fashion (with no reflected molecules in our case). From elementary kinetic theory:

$$\dot{m} \propto \left(\frac{P_L}{\sqrt{T_L}} - \frac{P_i}{\sqrt{T_i}} \right)$$

In the free molecule limit the pressure and temperature of the incoming molecules is exactly P_0 , T_0 the pressure and temperature far away, i.e. $P_i = P_0$, $T_i = T_0$. And for small temperature, pressure differences:

$$\frac{P_L}{\sqrt{T_L}} = \frac{P_0}{\sqrt{T_0}} \left(1 + P_2^{(1)} - \frac{1}{2} T_L^{(1)} + \dots \right)$$

$$\therefore \dot{m} \propto \frac{P_0}{\sqrt{T_0}} \left(2 \frac{\Delta P}{P_0} - \frac{\Delta T}{T_0} \right)$$

On the other hand, in the continuum limit the pressure is approximately constant up to the Knudsen layer, i.e. $P_i \approx P_0$. But now the temperature jump across the thin Knudsen layer is negligible, i.e. $T_i \approx T_L$.

$$\therefore \dot{m} \propto \left(\frac{P_L}{\sqrt{T_L}} - \frac{P_0}{\sqrt{T_L}} \right)$$

$$\propto \frac{P_0}{\sqrt{T_0}} \cdot \frac{\Delta P}{P_0} \quad \text{to first order.}$$

This explains the form of the two limiting cases. One notes that actual formula is valid over the whole range of Knudsen numbers.

In the one dimensional problem for small time, $P_i = P_0$, $T_i = T_0$, i.e., the molecules coming in have not yet known the change in the liquid temperature. Thus for small time ($t \rightarrow 0$), the flow is "free molecule", $\dot{m} \propto (2\Delta P - \Delta T)$. For large time the flow becomes "continuum" and $\dot{m} \propto \Delta P$.

From the general second order solution for the droplet, it was shown that in the continuum limit

$$\dot{m} = \frac{P_0 \beta}{2\pi} \left[\frac{8}{9} \frac{\Delta P}{P_0} + 0.167 \left(\frac{\Delta P}{P_0} \right)^2 + \dots \right]$$

Since the sign of the second term is always positive, for a given $|\Delta p|$, \dot{m} is greater for evaporation than condensation. Therefore the difference between condensation and evaporation arises in the second order.

One of the interesting consequences of the study of the one dimensional problem is that the transient condensation problem can be treated by a quasi-steady approach. The effect of a mass flow towards the liquid surface, by say lowering its temperature, is comparable to pulling a piston in a tube away from the stationary gas. Kinetic effects take place close to the liquid surface, a spatially increasing steady region is formed away from the surface and an inviscid expansion fan propagates into the vapor. On the other hand evaporation cannot be treated by such a simple quasi-steady picture. Raising the liquid's temperature, say, causes a 'weak shock' to propagate into the gas; however, even for large time the flow behind the wave cannot be treated as steady. In order to say anything more definite one would have to study the transient solution close to the wave front; perhaps, another nonsteady region is needed for the evaporation case. We note once again that in the spherical geometry the propagating waves decay inversely as the radius. The 'steady' solution obtained is therefore a valid one in that sense.

It is remarkable that the crude four function splitting of the distribution function leads to solutions that show the transition from free molecular to continuum flow regimes. The correction in the mass flux due to the six moment formulation is just about 2 percent in this case. Such a small correction increases one's confidence in the present approach.

In conclusion we note some of the limitations of the present approach. The solutions obtained are strictly valid for a monatomic vapor with an inverse fifth power repulsion between the molecules. Still the results should not be too bad for non monatomic gases, not too close to their critical point and at moderate temperatures. A possible way of correcting for non monatomic gases is suggested in [6] . In the boundary conditions, an accommodation coefficient of unity was assumed in order to simplify the analysis. This restriction could be lifted.

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LIST OF SYMBOLS USED IN PART II

a_0	isentropic speed of sound
\underline{c}	relative molecular velocity, $\underline{\xi} - \underline{u}$
f	molecular velocity distribution function
\underline{F}	external force on a single particle
k	Boltzmann constant
Km	Knudsen number
l	mean free path
m	mass of molecule
\dot{m}	net evaporation or condensation rate
n	particle number density, per unit volume
Δm	$m_L - m_0$
p	thermodynamic pressure = mkT
Δp	$p_L - p_0$
\underline{P}	stress tensor
P_{ij}	equals P_{ij} for $i \neq j$; equals $P_{ii} + p$ for $i = j$
\underline{q}	heat flux vector
\underline{Q}	arbitrary function of molecular velocity components
ΔQ	change in Q produced by collisions
$\underline{r}(x, y, z)$	} position vector
$\underline{r}(r, \theta, \phi)$	
r_0	radius of liquid drop
R	gas constant of vapor = $\frac{k}{m}$
s	Laplace transform variable
t	time

T	absolute temperature
ΔT	$T_L - T_0$
\underline{u}	mean velocity vector
v	x or y component of mean velocity
\underline{y}	$y/\beta\tau_f$
$x = x(r)$	equals $\sin \alpha = \sqrt{1 - (\tau_0^2/r^2)}$
α, β, γ	angles defined by Fig. 6a, b
β	$= \sqrt{2\pi RT_0}$
γ	ratio of the specific heats
η	outer variable = λy
κ	$= \tau_0 P_0 / \beta \mu_0 = 1/4 \text{ Km}$
λ	perturbation parameter
μ	viscosity coefficient
ξ_a	accommodation coefficient
$\underline{\xi}$	vector in molecular velocity space
ρ	density
τ_f	mean free time

Superscripts

$(0), (1), (2)$ 0^{th} order, 1^{st} order, 2^{nd} order, etc.

Subscripts

c	steady condition behind wave
i	incident on liquid surface
L	corresponding to Maxwellian from liquid
r	reflected from liquid surface

- ambient condition
- 1,2 corresponding to the two parts of the two-stream Maxwellian
- + , - sum or negation of corresponding parts of two-stream Maxwellian, e.g.,
$$t_+ = t_1 + t_2$$
$$t_- = t_1 - t_2$$

APPENDIX I. Kinetic Theory Definitions

The following is a partial list of useful kinetic theory definitions:

1. $f(\underline{r}, \underline{\xi}, t)$, the molecular velocity distribution function, is so defined that:

$$f(\underline{r}, \underline{\xi}, t) d\underline{r}, d\underline{\xi}$$

is the number of molecules which, at time t , have positions lying within a volume element $d\underline{r}$ about \underline{r} and velocities lying within a velocity space element $d\underline{\xi}$ about $\underline{\xi}$.

2. $\rho(\underline{r}, t)$, the mass density, is given in terms of the distribution function f by:

$$\rho(\underline{r}, t) = \int m f(\underline{r}, \underline{\xi}, t) d\underline{\xi}$$

where m is the mass of a molecule.

3. $\underline{u}(\underline{r}, t)$, the mean velocity, is given by:

$$\underline{u}(\underline{r}, t) = \frac{\int m f \underline{\xi} d\underline{\xi}}{\int m f d\underline{\xi}} = \frac{\int m f \underline{\xi} d\underline{\xi}}{\rho(\underline{r}, t)}$$

4. $\underline{c}(\underline{r}, \underline{\xi}, t)$, the intrinsic, or relative particle velocity is defined:

$$\underline{c} = \underline{\xi} - \underline{u}$$

5. $\underline{P}(\underline{r}, t)$, the stress tensor in any locally orthogonal coordinate system, is defined:

$$P_{ij} = - \int m c_i c_j f d\underline{\xi}$$

One can also write:

$$\underline{P} = -p \underline{I} + \underline{P}$$

where $p = \rho RT$ and:

$$P_{ii} = -\frac{m}{3} \int (2c_i^2 - c_j^2 - c_k^2) f d\underline{c}$$

$$P_{ij} = -m \int c_i c_j f d\underline{c}, \quad i \neq j$$

6. $T(\underline{r}, t)$, the mean temperature, is defined:

$$T(\underline{r}, t) = \frac{m \int \frac{c^2}{2} f d\underline{c}}{\frac{3}{2} R \rho(\underline{r}, t)}$$

where R is the gas constant.

7. $\underline{q}(\underline{r}, t)$, the heat flux vector, is defined:

$$\underline{q}(\underline{r}, t) = m \int \underline{c} \cdot \frac{c^2}{2} f d\underline{c}$$

APPENDIX II. The Linearised Flow Quantities For A Spherically Symmetric Geometry.

Let the distribution function f for a spherically symmetric geometry be represented by the two stream Maxwellian:

$$f = \begin{cases} f_1 = \frac{m_1(r)}{[2\pi R T_1(r)]^{3/2}} \exp\left\{-\frac{\xi^2}{2RT_1(r)}\right\} & 0 < \sigma < \frac{\pi}{2} - \alpha \\ f_2 = \frac{m_2(r)}{[2\pi R T_2(r)]^{3/2}} \exp\left\{-\frac{\xi^2}{2RT_2(r)}\right\} & \frac{\pi}{2} - \alpha < \sigma < \pi \end{cases} \quad (2-1)$$

Linearise the four arbitrary functions:

$$m_1 = m_0 (1 + \lambda N_1^{(1)} + \lambda^2 N_1^{(2)} + \dots) \quad (2-2 a)$$

$$T_1 = T_0 (1 + \lambda t_1^{(1)} + \lambda^2 t_1^{(2)} + \dots) \quad (2-2 b)$$

$$m_2 = m_0 (1 + \lambda N_2^{(1)} + \lambda^2 N_2^{(2)} + \dots) \quad (2-2 c)$$

$$T_2 = T_0 (1 + \lambda t_2^{(1)} + \lambda^2 t_2^{(2)} + \dots) \quad (2-2 d)$$

Define:

$$N_+^{(1)} = N_1^{(1)} + N_2^{(1)}, \quad N_-^{(1)} = N_1^{(1)} - N_2^{(1)} \quad (2-3)$$

$$N_+^{(2)} = N_1^{(2)} + N_2^{(2)}, \quad N_-^{(2)} = N_1^{(2)} - N_2^{(2)}$$

etc.

Then to 2nd order the mean quantities are:

$$\rho = \frac{\rho_0}{2} \left[2 + \lambda \left\{ N_+^{(1)} - x N_-^{(1)} \right\} + \lambda^2 \left\{ N_+^{(2)} - x N_-^{(2)} \right\} \dots \right] \quad (2-4 a)$$

$$V = \frac{\beta(1-x^2)}{2\pi} \left[\lambda \left\{ N_-^{(1)} + \frac{1}{2} t_-^{(1)} \right\} + \lambda^2 \left\{ \frac{1}{8} (t_2^{(1)2} - t_1^{(1)2}) + \right. \right. \\ \left. \left. + \frac{1}{2} (N_1^{(1)} t_1^{(1)} - N_2^{(1)} t_2^{(1)}) - \frac{1}{2} (N_-^{(1)} + \frac{1}{2} t_-^{(1)}) (N_+^{(1)} - x N_-^{(1)}) + \right. \right. \\ \left. \left. + N_-^{(2)} + \frac{1}{2} t_-^{(2)} \right\} \dots \dots \dots \right] \quad (2-4 b)$$

$$\int m f \xi_y^2 d\xi = \frac{\rho_0}{2} \left[2 + \lambda \left\{ N_+^{(1)} - x^3 N_-^{(1)} + t_+^{(1)} - x^3 t_-^{(1)} \right\} \right. \\ \left. + \lambda^2 \left\{ (1-x^3) N_1^{(1)} t_1^{(1)} + (1+x^3) N_2^{(1)} t_2^{(1)} \right. \right. \\ \left. \left. + N_+^{(2)} - x^3 N_-^{(2)} + t_+^{(2)} - x^3 t_-^{(2)} \right\} + \dots \right] \quad (2-4 c)$$

$$\int m f \xi_0^2 d\xi = \frac{\rho_0}{2} \left[2 + \lambda \left\{ N_+^{(1)} + \left(\frac{1}{2} x^3 - \frac{3}{2} x \right) N_-^{(1)} + t_+^{(1)} + \right. \right. \\ \left. \left. + \left(\frac{1}{2} x^3 - \frac{3}{2} x \right) t_-^{(1)} \right\} + \right. \\ \left. + \lambda^2 \left\{ \left(1 - \frac{3}{2} x + \frac{1}{2} x^3 \right) N_1^{(1)} t_1^{(1)} + \right. \right. \\ \left. \left. + \left(1 + \frac{3}{2} x - \frac{1}{2} x^3 \right) N_2^{(1)} t_2^{(1)} + N_+^{(2)} + \right. \right. \\ \left. \left. + \left(\frac{1}{2} x^3 - \frac{3}{2} x \right) N_-^{(2)} + t_+^{(2)} + \right. \right. \\ \left. \left. + \left(\frac{1}{2} x^3 - \frac{3}{2} x \right) t_-^{(2)} \right\} + \dots \dots \dots \right] \quad (2-4 d)$$

$$P_{rr} = - \frac{p_0}{2} \left[2 + \lambda \left\{ N_+^{(1)} - x^3 N_-^{(1)} + t_+^{(1)} - x^3 t_-^{(1)} \right\} + \right. \\ \left. + \lambda^2 \left\{ (1-x^3) N_1^{(1)} t_1^{(1)} + (1+x^3) N_2^{(1)} t_2^{(1)} - \frac{(1-x^2)^2}{\pi} \left(N_-^{(1)} + \frac{1}{2} t_-^{(1)} \right) + \right. \right. \\ \left. \left. + N_+^{(2)} - x^3 N_-^{(2)} + t_+^{(2)} - x^3 t_-^{(2)} \right\} + \dots \right] \quad (2-4 e)$$

$$p = p_0 \left[1 + \lambda \left\{ \frac{1}{2} N_+^{(1)} - \frac{1}{2} x N_-^{(1)} + \frac{1}{2} t_+^{(1)} - \frac{x}{2} t_-^{(1)} \right\} + \right. \\ \left. + \lambda^2 \left\{ \left(\frac{1}{2} - \frac{x}{2} \right) N_1^{(1)} t_1^{(1)} + \left(\frac{1}{2} + \frac{x}{2} \right) N_2^{(1)} t_2^{(1)} - \frac{(1-x^2)^2}{6\pi} \left(N_-^{(1)} + \frac{1}{2} t_-^{(1)} \right)^2 \right. \right. \\ \left. \left. + \frac{1}{2} N_+^{(2)} - \frac{x}{2} N_-^{(2)} + \frac{1}{2} t_+^{(2)} - \frac{x}{2} t_-^{(2)} \right\} + \dots \right] \quad (2-4 f)$$

$$P_{rr} = p_0 \left[\lambda \left\{ \frac{1}{2} x (x^2-1) N_-^{(1)} + \frac{1}{2} x (x^2-1) t_-^{(1)} \right\} + \right. \\ \left. + \lambda^2 \left\{ \frac{(1-x^2)}{3\pi} \left(N_-^{(1)} + \frac{1}{2} t_-^{(1)} \right)^2 + \frac{1}{2} x (x^2-1) N_1^{(1)} t_1^{(1)} - \right. \right. \\ \left. \left. - \frac{x}{2} (x^2-1) N_2^{(1)} t_2^{(1)} + \frac{x}{2} (x^2-1) N_-^{(2)} + \frac{x}{2} (x^2-1) t_-^{(2)} \right\} + \dots \right] \quad (2-4 g)$$

$$T = T_0 \left[1 + \lambda \left\{ \frac{1}{2} t_+^{(1)} - \frac{x}{2} t_-^{(1)} \right\} + \dots \right] \quad (2-4 h)$$

$$\int m \xi_r \xi^2 f d\xi = \frac{\rho_0 \beta^3}{\pi^2} (1-x^2) \left[\lambda \left\{ N_-^{(1)} + \frac{3}{2} t_-^{(1)} \right\} + \right. \\ \left. + \lambda^2 \left\{ \frac{3}{2} (N_1^{(1)} t_1^{(1)} - N_2^{(1)} t_2^{(1)}) + \frac{3}{8} (t_1^{(12)} - t_2^{(12)}) + \right. \right. \\ \left. \left. + N_-^{(2)} + \frac{3}{2} t_-^{(2)} \right\} + \dots \right] \quad (2-4 i)$$

$$\int m f \xi_r \xi^2 d\xi = \frac{5}{4\pi} \rho_0 \beta^2 \left[2 + \lambda \left\{ N_+^{(1)} - x^3 N_-^{(1)} + 2 t_+^{(1)} - 2 x^3 t_-^{(1)} \right\} + \right. \\ \left. + \lambda^2 \left\{ (1-x^3) (2 N_1^{(1)} t_1^{(1)} + t_1^{(12)}) + (1+x^3) (2 N_2^{(1)} t_2^{(1)} + t_2^{(12)}) + \right. \right. \\ \left. \left. + N_+^{(2)} - x^3 N_-^{(2)} + 2 t_+^{(2)} - 2 x^3 t_-^{(2)} \right\} + \dots \right] \quad (2-4 j)$$

$$\int m f \xi_\theta \xi^2 d\xi = \frac{5}{4\pi} \rho_0 \beta^2 \left[2 + \lambda \left\{ N_+^{(1)} + 2 t_+^{(1)} + (x^3 - 3x) \left(\frac{1}{2} N_-^{(1)} + t_-^{(1)} \right) \right\} \right. \\ \left. + \lambda^2 \left\{ \left(1 - \frac{3}{2} x + \frac{1}{2} x^3 \right) (2 N_1^{(1)} t_1^{(1)} + t_1^{(12)}) + \right. \right. \\ \left. \left. + \left(1 + \frac{3}{2} x - \frac{1}{2} x^3 \right) (2 N_2^{(1)} t_2^{(1)} + t_2^{(12)}) + \right. \right. \\ \left. \left. + N_+^{(2)} + 2 t_+^{(2)} + (x^3 - 3x) \left(\frac{1}{2} N_-^{(2)} + t_-^{(2)} \right) \right\} + \right. \\ \left. + \dots \right] \quad (2-4 k)$$

$$\begin{aligned}
 q_{\gamma} = & \frac{\rho_0 \beta (1-x^2)}{2\pi} \left[\lambda \left\{ \frac{7}{4} t_-^{(1)} - \frac{1}{2} N_-^{(1)} \right\} + \right. \\
 & + \lambda^2 \left\{ \frac{7}{4} (N_1^{(1)} t_1^{(1)} - N_2^{(1)} t_2^{(1)}) + \frac{17}{16} (t_1^{(1)2} - t_2^{(1)2}) + \right. \\
 & + (N_-^{(1)} + \frac{1}{2} t_-^{(1)}) \left(\frac{1}{2} x(x^2-1) N_-^{(1)} - \frac{5}{4} t_+^{(1)} + \right. \\
 & \left. \left. + \frac{1}{2} \left(\frac{3}{2} x + x^3 \right) t_-^{(1)} \right) + \right. \\
 & \left. \left. + \frac{1}{4} \left(7 t_-^{(2)} - 2 N_-^{(2)} \right) \right\} + \dots \right] \quad (2-41)
 \end{aligned}$$

Note that:

$$x = \sin \alpha = \sqrt{1 - \frac{\gamma_0^2}{\gamma^2}} \quad ; \quad \beta = \sqrt{2\pi R T_0} \quad (2-5)$$

APPENDIX III. Some Details Of The 2nd Order Calculation
For The Droplet.

From the moment equations, the four 2nd order equations when simplified are:

$$N_-^{(2)} + \frac{1}{2} t_-^{(2)} = A^{(2)} + \frac{1}{8} (t_1^{(12)} - t_2^{(12)}) - \frac{1}{2} (N_1^{(1)} t_1^{(1)} - N_2^{(1)} t_2^{(1)}) \quad (3-1 a)$$

$$N_-^{(2)} + \frac{3}{2} t_-^{(2)} = B^{(2)} - \frac{3}{8} (t_1^{(12)} - t_2^{(12)}) - \frac{3}{2} (N_1^{(1)} t_1^{(1)} - N_2^{(1)} t_2^{(1)}) \quad (3-1 b)$$

$$\begin{aligned} \frac{d}{dr} [N_+^{(2)} + t_+^{(2)}] &= x^3 \frac{dN_-^{(2)}}{dr} + x^3 \frac{dt_-^{(2)}}{dr} - (1-x^3) \frac{d}{dr} (N_1^{(1)} t_1^{(1)}) \\ &\quad + (1+x^3) \frac{d}{dr} (N_2^{(1)} t_2^{(1)}) \quad (3-1 c) \end{aligned}$$

$$\begin{aligned} \frac{d}{dr} [N_+^{(2)} + 2t_+^{(2)}] &= - \frac{d}{dr} \left[2(N_1^{(1)} t_1^{(1)} + N_2^{(1)} t_2^{(1)}) + (t_1^{(12)} + t_2^{(12)}) \right] + \\ &\quad + \frac{(\gamma^2 - \gamma_0^2)^{3/2}}{\gamma^3} \frac{d}{dr} \left[N_-^{(2)} + 2t_-^{(2)} + (t_1^{(12)} - t_2^{(12)}) + \right. \\ &\quad \left. + 2(N_1^{(1)} t_1^{(1)} - N_2^{(1)} t_2^{(1)}) \right] - \\ &\quad - \frac{1}{15} \frac{\gamma_0}{l_0} \cdot \frac{\gamma_0}{\gamma^2} \left[B^{(1)} \left(\frac{1}{2} N_+^{(1)} - \frac{x}{2} N_-^{(1)} \right) + \left\{ 7(N_1^{(1)} t_1^{(1)} - N_2^{(1)} t_2^{(1)}) + \right. \right. \\ &\quad \left. \left. + \frac{17}{4} (t_1^{(12)} - t_2^{(12)}) + (7t_-^{(2)} - 2N_-^{(2)}) \right\} + A^{(1)} \left\{ -5t_+^{(1)} + \right. \right. \\ &\quad \left. \left. + \frac{\gamma_0^2}{\gamma^2} \frac{(\gamma^2 - \gamma_0^2)^{3/2}}{\gamma} N_-^{(2)} + (5x - x^3) t_-^{(1)} \right\} \right] \quad (3-1 d) \end{aligned}$$

In the above equations $A^{(2)}$ and $B^{(2)}$ are as yet undetermined constants, while $A^{(1)}$, $B^{(1)}$ and all the first order functions are known. Since in (3-1 c, d,) the right hand sides are known functions of γ , the problem then reduces to two quadratures, which are straight forward but tedious.

From the two algebraic equations (3-1 a, b):

$$t_-^{(2)} = B^{(2)} - A^{(2)} - \frac{1}{2} (t_1^{(1)2} - t_2^{(1)2}) - (N_1^{(1)} t_1^{(1)} - N_2^{(1)} t_2^{(1)}) \quad (3-2 a)$$

$$N_-^{(2)} = \frac{3}{2} A^{(2)} - \frac{1}{2} B^{(2)} + \frac{3}{8} (t_1^{(1)2} - t_2^{(1)2}) \quad (3-2 b)$$

(3-1 c) integrates to:

$$N_+^{(2)} + t_+^{(2)} = C^{(2)} - (N_1^{(1)} t_1^{(1)} + N_2^{(1)} t_2^{(1)}) + \frac{G_8}{8} \left[-\frac{(\gamma^2 - \gamma_0^2)^{3/2}}{4\gamma^4} + \frac{3}{8} \frac{(\gamma^2 - \gamma_0^2)^{3/2} - \gamma^2 (\gamma^2 - \gamma_0^2)^{1/2}}{\gamma_0^2 \gamma^2} + \frac{3}{8\gamma_0} \cos^{-1} \frac{\gamma_0}{\gamma} \right] \quad (3-3)$$

where $C^{(2)}$ is a constant and

$$G_8 = \frac{1}{15} \frac{\gamma_0^2}{l_0} \left(\frac{1}{4} A^{(1)} B^{(1)} + \frac{1}{8} B^{(1)2} \right) \quad (3-4)$$

Next equation (3-1 d) integrates to:

$$N_+^{(2)} + 2t_+^{(2)} = D^{(2)} + C^{(2)} - \left\{ 2(N_1^{(1)} t_1^{(1)} + N_2^{(1)} t_2^{(1)}) + (t_1^{(1)2} + t_2^{(1)2}) \right\} - \frac{3}{8} G_8 \left\{ -\frac{(\gamma^2 - \gamma_0^2)^{3/2}}{4\gamma^4} + \frac{3}{8} \frac{(\gamma^2 - \gamma_0^2)^{3/2} - \gamma^2 (\gamma^2 - \gamma_0^2)^{1/2}}{\gamma_0^2 \gamma^2} + \frac{3}{8\gamma_0} \cos^{-1} \frac{\gamma_0}{\gamma} \right\} -$$

$$\begin{aligned}
 & -\frac{1}{15} \frac{\gamma_0^2}{l_0} \left[-\frac{H_1}{\gamma} - \frac{H_2}{2\gamma^2} - \frac{B^{(1)} N_-^{(1)}}{2} \left\{ -\frac{\sqrt{\gamma^2 \gamma_0^2}}{2\gamma^2} + \frac{1}{2\gamma_0} \cos^{-1} \frac{\gamma_0}{\gamma} \right\} + \right. \\
 & \quad + A^{(1)} (N_-^{(1)} + 5t_-^{(1)}) \left\{ -\frac{\sqrt{\gamma^2 \gamma_0^2}}{4\gamma^4} + \frac{\sqrt{\gamma^2 \gamma_0^2}}{8\gamma_0^2 \gamma^2} + \right. \\
 & \quad \quad \quad \left. + \frac{1}{8\gamma_0^3} \cos^{-1} \frac{\gamma_0}{\gamma} \right\} + \\
 & \quad + 4A^{(1)} t_-^{(1)} \left\{ -\frac{(\gamma^2 - \gamma_0^2)^{3/2}}{4\gamma^4} + \frac{3}{8\gamma_0} \cos^{-1} \frac{\gamma_0}{\gamma} + \right. \\
 & \quad \quad \quad \left. + \frac{3}{8} \frac{(\gamma^2 - \gamma_0^2)^{3/2} - \gamma^2 (\gamma^2 - \gamma_0^2)^{1/2}}{\gamma_0^2 \gamma^2} \right\} \left. \right] \quad (3-5)
 \end{aligned}$$

where $D^{(2)}$ is the fourth arbitrary constant and H_1 and H_2 are defined by:

$$\begin{aligned}
 H_1 &= \left[\frac{35}{64} A^{(1)2} + \frac{47}{64} A^{(1)} B^{(1)} - \frac{5}{256} B^{(1)2} \right] \\
 H_2 &= \frac{\gamma_0^2}{30 l_0} \left[-10 A^{(1)} B^{(1)} - B^{(1)2} \right] \quad (3-6)
 \end{aligned}$$

Equations (3-2), (3-3) and (3-5) for $N_-^{(2)}$, $t_-^{(2)}$, $N_+^{(2)}$, $t_+^{(2)}$ involve four arbitrary constants $A^{(2)}$, $B^{(2)}$, $C^{(2)}$, $D^{(2)}$.

The four boundary conditions are:

$$\text{At } \gamma = \gamma_0 : t_+^{(2)} + t_-^{(2)} = 0, \quad N_+^{(2)} + N_-^{(2)} = 0 \quad (3-7 a)$$

$$\text{For } \gamma = \infty : t_+^{(2)} - t_-^{(2)} = 0, \quad N_+^{(2)} - N_-^{(2)} = 0 \quad (3-7 b)$$

Applying these 4 conditions the 2nd order mass flux $A^{(2)}$ can be obtained:

$$\begin{aligned}
 \left(4 + \frac{6}{5} \frac{\gamma_0}{\ell_0}\right) A^{(2)} &= \left[\frac{15}{32} + \frac{1}{15} \frac{\gamma_0}{\ell_0} \left(\frac{41\pi}{128} + \frac{53}{16} \right) \right] A^{(1)2} + \\
 &+ \left[\frac{7}{32} + \frac{1}{15} \frac{\gamma_0}{\ell_0} \left(\frac{7\pi}{128} + \frac{43}{16} \right) + \frac{8}{225} \frac{\gamma_0^2}{\ell_0^2} \left(\frac{3\pi}{512} + \frac{11}{32} \right) \right] \times \\
 &\quad \times A^{(1)} B^{(1)} \\
 &+ \left[-\frac{1}{128} + \frac{1}{15} \frac{\gamma_0}{\ell_0} \left(\frac{13\pi}{512} - \frac{15}{64} \right) + \frac{8}{225} \frac{\gamma_0^2}{\ell_0^2} \left(\frac{3\pi}{1024} - \frac{21}{64} \right) - \right. \\
 &\quad \left. - \frac{4}{3375} \frac{\gamma_0^3}{\ell_0^3} \right] B^{(1)2} \tag{3-8}
 \end{aligned}$$

Where $A^{(1)}$ and $B^{(1)}$ are given by the first order solution in terms of $\frac{\Delta m}{m_0}$ and $\frac{\Delta T}{T_0}$.

APPENDIX IV. The Linearised Mean Quantities For The
One Dimensional Problem

Let the distribution function for the one dimensional problem be represented by:

$$f = \begin{cases} f_1 = \frac{m_1(y,t)}{[2\pi RT_1(y,t)]^{3/2}} \exp\left\{-\frac{\xi_x^2 + \xi_z^2 + (\xi_y - v_1(y,t))^2}{2RT_1(y,t)}\right\} & \xi_y > 0 \\ f = \frac{m_2(y,t)}{[2\pi RT_2(y,t)]^{3/2}} \exp\left\{-\frac{\xi_x^2 + \xi_z^2 + (\xi_y - v_2(y,t))^2}{2RT_2(y,t)}\right\} & \xi_y < 0 \end{cases}$$

Linearise the six arbitrary functions $m_{1,2}$, $T_{1,2}$, and

$v_{1,2}$ as follows:

$$m_{1,2} = m_0 \left[1 + \lambda N_{1,2}^{(1)}(y,t) + \lambda^2 N_{1,2}^{(2)}(y,t) + \dots \right]$$

$$T_{1,2} = T_0 \left[1 + \lambda t_{1,2}^{(1)}(y,t) + \lambda^2 t_{1,2}^{(2)}(y,t) + \dots \right] \quad (4-1)$$

$$v_{1,2} = \beta \left[\lambda v_{1,2}^{(1)}(y,t) + \lambda^2 v_{1,2}^{(2)}(y,t) + \dots \right]$$

where $\beta = \sqrt{2\pi RT_0}$. Define:

$$N_+^{(1)} = N_1^{(1)} + N_2^{(1)}, \quad N_-^{(1)} = N_1^{(1)} - N_2^{(1)}$$

$$v_+^{(1)} = v_1^{(1)} + v_2^{(1)}, \quad v_-^{(1)} = v_1^{(1)} - v_2^{(1)}$$

(4-2)

etc.

Then, leaving out the superscripts to denote first order quantities, the mean flow quantities to first order are given by:

$$\rho = \rho_0 \left[1 + \lambda \left\{ \frac{1}{2} N_+ + \nu_- \right\} + \dots \right] \quad (4-3 a)$$

$$\rho v = \frac{\rho_0 \beta}{2\pi} \left[\lambda \left\{ N_- + \frac{1}{2} t_- + \pi \nu_+ \right\} + \dots \right] \quad (4-3 b)$$

$$P_{xx} = -\rho_0 \left[1 + \lambda \left\{ \frac{1}{2} N_+ + \frac{1}{2} t_+ + \nu_- \right\} + \dots \right] \quad (4-3 c)$$

$$P_{yy} = -\rho_0 \left[1 + \lambda \left\{ \frac{1}{2} N_+ + \frac{1}{2} t_+ + 2\nu_- \right\} + \dots \right] \quad (4-3 d)$$

$$P = \rho_0 \left[1 + \lambda \left\{ \frac{1}{2} N_+ + \frac{1}{2} t_+ + \frac{4}{3} \nu_- \right\} + \dots \right] \quad (4-3 e)$$

$$T = T_0 \left[1 + \lambda \left\{ \frac{1}{2} t_+ + \frac{1}{3} \nu_- \right\} + \dots \right] \quad (4-3 f)$$

$$q_y = \frac{\rho_0 \beta}{8\pi} \left[\lambda \left\{ 7 t_- - 2 N_- \right\} + \dots \right] \quad (4-3 g)$$

$$\int m \xi_x^2 f d \underline{\xi} = \rho_0 \left[1 + \lambda \left\{ \frac{1}{2} N_+ + \frac{1}{2} t_+ + \nu_- \right\} + \dots \right] \quad (4-3 h)$$

$$\int m \xi_y^2 f d \underline{\xi} = \frac{\rho_0 \beta}{2\pi} \left[\lambda \left\{ 2 N_- + 3 t_- + 3\pi \nu_+ \right\} + \dots \right] \quad (4-3 i)$$

$$\int m \xi_y \frac{\xi^2}{2} f d \underline{\xi} = \frac{\rho_0 \beta}{4\pi} \left[\lambda \left\{ 4 N_- + 6 t_- + 5\pi \nu_+ \right\} + \dots \right] \quad (4-3 j)$$

$$\int m \xi_y \xi^2 f d \underline{\xi} = \frac{\rho_0 \beta}{2\pi} \left[\lambda \left\{ N_- + \frac{3}{2} t_- + \pi \nu_+ \right\} + \dots \right] \quad (4-3 k)$$

$$\int m \xi_y^4 f d\xi = \frac{\rho_0 \beta^2}{4\pi} \left[6 + \lambda \{ 3N_+ + 6t_+ + 16v_- \} + \dots \right] \quad (4-3 l)$$

$$\int m \xi_y^2 \xi_x^2 f d\xi = \frac{\rho_0 \beta^2}{4\pi} \left[2 + \lambda \{ N_+ + 2t_+ + 4v_- \} + \dots \right] \quad (4-3 m)$$

$$\int m \xi_y^2 \xi^2 f d\xi = \frac{\rho_0 \beta^2}{4\pi} \left[10 + \lambda \{ 5N_+ + 10t_+ + 24v_- \} + \dots \right] \quad (4-3 n)$$

$$\int m (\xi_y - v)^3 f d\xi = \frac{\rho_0 \beta}{4\pi} \left[\lambda \{ -2N_- + 3t_- \} + \dots \right] \quad (4-3 o)$$

APPENDIX V. The Perturbation Of The Conservation Equations

The quasi-steady state one dimensional problem involves a singular perturbation. The 'outer solution' it turns out requires a knowledge of the 2nd order heat flux, $q_y^{(2)}$. The purpose of this appendix is to show that the second order heat flux is very simply related to the first order quantities. This relationship comes about from the conservation equations.

The first three moment equations, which are just the equations of conservation of mass, y momentum and energy, are:

$$\rho V = \text{constant} \quad (5-1 \text{ a})$$

$$\rho V^2 - P_{yy} = \text{constant} \quad (5-1 \text{ b})$$

$$\rho V (3\rho V + 2 q_y) - P_{yy} = \text{constant} \quad (5-1 \text{ c})$$

The flow quantities ρ, V etc. can be linearised as follows:

$$\rho = \rho_0 (1 + \lambda \rho^{(1)} + \lambda^2 \rho^{(2)} + \dots) \quad (5-2 \text{ a})$$

$$P = P_0 (1 + \lambda P^{(1)} + \lambda^2 P^{(2)} + \dots) \quad (5-2 \text{ b})$$

$$V = V_0 (\lambda V^{(1)} + \lambda^2 V^{(2)} + \dots) \quad (5-2 \text{ c})$$

$$P_{yy} = p_0 (-1 + \lambda P_{yy}^{(1)} + \lambda^2 P_{yy}^{(2)} + \dots) \quad (5-2 d)$$

$$q_y = (\lambda q_y^{(1)} + \lambda^2 q_y^{(2)} + \dots) \quad (5-2 e)$$

Substituting the above into (5-1):

$$\rho v = \rho_0 \beta [\lambda v^{(1)} + \lambda^2 (v^{(2)} + \rho^{(1)} v^{(1)}) + \dots] = \text{constant} \quad (5-3 a)$$

$$\begin{aligned} \rho v^2 - P_{yy} &= \rho_0 \beta^2 [\lambda^2 v^{(1)2} + \lambda^3 (\rho^{(1)} v^{(1)2} + 2 v^{(1)} v^{(2)}) + \dots] \\ &\quad - p_0 [-1 + \lambda P_{yy}^{(1)} + \lambda^2 P_{yy}^{(2)} + \lambda^3 P_{yy}^{(3)} + \dots] \\ &= \text{constant} \end{aligned} \quad (5-3 b)$$

$$\begin{aligned} \rho v (3\rho v + 2q_y) - P_{yy}^2 &= 3\rho_0 \rho_0 \beta^2 [\lambda^2 v^{(1)2} + \lambda^3 (2v^{(1)} v^{(2)} + \rho^{(1)} v^{(1)2} + \rho^{(1)} v^{(1)2}) + \dots] + \\ &\quad + 2\rho_0 \beta [\lambda^2 q_y^{(1)} v^{(1)} + \lambda^3 (q_y^{(1)} v^{(1)} \rho^{(1)} + v^{(1)} q_y^{(2)} + v^{(2)} q_y^{(1)}) + \dots] - \\ &\quad - p_0^2 [1 - 2\lambda P_{yy}^{(1)} + \lambda^2 (P_{yy}^{(1)} - 2P_{yy}^{(2)}) + \lambda^3 (2P_{yy}^{(1)} P_{yy}^{(2)} - 2P_{yy}^{(3)}) + \dots] \\ &= \text{constant} \end{aligned} \quad (5-3 c)$$

From (5-3) it follows that:

$$v^{(1)} = \text{constant} \quad (5-4 a)$$

$$V^{(2)} + f^{(1)} v^{(1)} = \text{constant} \quad (5-4 \text{ b})$$

$$P_{yy}^{(1)} = \text{constant} ; P_{yy}^{(2)} = \text{constant} \quad (5-4 \text{ c})$$

$$P_{yy}^{(3)} - \frac{\rho_0 \beta^2}{\rho_0} (f^{(1)} v^{(1)2} + 2 v^{(1)} V^{(2)}) = \text{constant} \quad (5-4 \text{ d})$$

$$q_y^{(1)} = \text{constant} \quad (5-4 \text{ e})$$

$$3\rho_0 \rho_0 \beta^2 (2 v^{(1)} V^{(2)} + f^{(1)} v^{(1)2} + p^{(1)} v^{(1)2}) \\ + v^{(1)} q_y^{(2)} \cdot 2\rho_0 \beta + 2\rho_0^2 P_{yy}^{(3)} = \text{constant} \quad (5-4 \text{ f})$$

Algebraic manipulation of the above leads to the result:

$$q_y^{(2)} = -\frac{1}{2} \rho_0 \beta v^{(1)} [3p^{(1)} - 5f^{(1)}] + \text{constant} \quad (5-5)$$

APPENDIX VI. The Perturbation Solution Of The Continuum
Heat Convection Equations

In order to explain the nature of the singular perturbation arising in the one dimensional problem, we consider here the perturbation solution of the heat convection equations. Specifically the problem considered is that of an infinite porous plate at a temperature different from the ambient temperature. The governing continuum (Navier-Stokes) equations are:

$$\frac{d}{dy} (\rho v) = 0 \quad (6-1 a)$$

$$\rho v \frac{dv}{dy} = - \frac{dp}{dy} + \mu \frac{d^2 v}{dy^2} \quad (6-1 b)$$

$$\rho v c_p \frac{dT}{dy} - v \frac{dp}{dy} = k \frac{d^2 T}{dy^2} + \frac{4}{3} \mu \left(\frac{dv}{dy} \right)^2 \quad (6-1 c)$$

If the $\frac{dp}{dy}$ and dissipation terms are neglected i.e. for small velocities, it is easy to show that:

$$T = T_0 + (T_p - T_0) \exp \left\{ \frac{\dot{m} c_p}{k} y \right\} \quad (6-2)$$

where T_p is the temperature of the plate and $\dot{m} = \rho v$ is the constant mass flux. This result implies that the mass flux has to be towards the plate, for there to be finite temperature far from it. If the temperature gradient and the mass flux are in the same

direction, an infinite amount of energy will be carried to infinity, causing infinite temperatures there. Equation (6-2) also shows that if linearisation is effected for $\dot{m} \rightarrow 0$, the linear temperature distribution will not be able to satisfy the ambient conditions for $y \rightarrow \infty$. This is the nature of the singular perturbation.

In order to attempt a systematic perturbation solution for small temperature differences and small velocities, let:

$$V = \lambda v^{(1)} + O(\lambda^2) \quad (6-3 \text{ a})$$

$$\rho = \rho_0 (1 + \lambda \rho^{(1)} + \lambda^2 \rho^{(2)} + \dots) \quad (6-3 \text{ b})$$

$$p = p_0 (1 + \lambda p^{(1)} + \lambda^2 p^{(2)} + \dots) \quad (6-3 \text{ c})$$

$$T = T_0 (1 + \lambda T^{(1)} + \lambda^2 T^{(2)} + \dots) \quad (6-3 \text{ d})$$

Substitution into (6-1) leads to the first order solutions:

$$v^{(1)} = \text{constant} = A_1 \quad (6-4 \text{ a})$$

$$p^{(1)} = \text{constant} = A_2 \quad (6-4 \text{ b})$$

$$T''' = A_3 + A_4 y \quad (6-4 c)$$

We note that the linearised boundary conditions are:

$$\text{At } y = 0 : T''' = \frac{T_P - T_0}{\lambda T_0} ; v''' = \frac{\dot{m}}{\rho_0 \lambda} \quad (6-5 a)$$

$$\text{For } y \rightarrow \infty : T''' = 0 , p''' = 0 \quad (6-5 b)$$

It is clear that the above boundary conditions cannot be satisfied by (6-4).

Define an 'outer variable' η by:

$$\eta = \lambda y , \quad d\eta = \lambda dy \quad (6-6)$$

In terms of the outer variable the first order equations are:

$$v'''(\eta) = \text{constant} = B_1 \quad (6-7 a)$$

$$p'''(\eta) = \text{constant} = B_2 \quad (6-7 b)$$

$$P^{(0)} v''' c_p \frac{dT'''(\eta)}{d\eta} = k \frac{d^2 T'''(\eta)}{d\eta^2} \quad (6-7 c)$$

(6-7 c) integrates to:

$$T'''(\eta) = B_3 + B_4 \exp\left\{\frac{\rho_0 B_1 c_p}{k} \cdot \eta\right\} \quad (6-8)$$

From the boundary conditions at infinity:

$$B_1 < 0, \quad B_3 = 0, \quad B_2 = 0 \quad (6-9)$$

Again since $B_1 < 0$, solution is valid only for suction at the plate. From the matching condition:

$$\lim_{y \rightarrow \infty} \varphi(y) = \lim_{\eta \rightarrow 0} \varphi(\eta)$$

to all order of λ , it is clear that:

$$A_1 = B_1; \quad A_2 = B_2 = 0; \quad A_4 = 0; \quad A_3 = B_4 \quad (6-10)$$

From the boundary conditions at the plate:

$$B_1 = \frac{\dot{m}}{\rho_0 \lambda}; \quad A_3 = B_4 = \frac{T_p - T_0}{\lambda T_0} \quad (6-11)$$

Thus, to first order, the uniformly valid solution for the temperature distribution is:

$$T = T_0 \left\{ 1 + \lambda \cdot \frac{(T_p - T_0)}{\lambda T_0} \exp \left(\frac{\rho_0 c_p}{k} \cdot \frac{\dot{m}}{\rho_0 \lambda} \cdot r \right) \right\} \quad (6-12)$$
$$= T_0 + (T_p - T_0) \exp \left(\frac{\dot{m} c_p}{k} y \right)$$

This is identical to the result (6-12). The conditions at infinity require that the 2nd order term $\rho v \frac{dT}{dy}$ be kept in the equations. This is precisely what the outer equations do.

In the spherical geometry of course, infinity is an ordinary point, and the linearised equations are sufficient.

APPENDIX VII. Collision Integral for $Q = m \xi_x^3$

Ref: GALCIT, HRP Memo. 51, Lester Lees, pp. 27-31.

We need to evaluate:

$$\Delta Q = \sqrt{2mK} \iint f f_1 J d\xi d\xi_1$$

where

$$J = \int_0^\infty \int_0^{2\pi} (Q'_1 - Q) \alpha d\alpha d\epsilon$$

From the Memo:

$$\xi'_x = \xi_x + (\xi_1 - \xi)_x \cos^2(\theta'/2) + \frac{1}{2} \sqrt{V^2 - (\xi_1 - \xi)_x^2} \sin \theta' \cos \epsilon$$

$$V^2 = (\xi_1 - \xi)_x^2 + (\xi_1 - \xi)_y^2 + (\xi_1 - \xi)_z^2$$

In the integral for J , terms in the expression $(Q'_1 - Q)$ proportional to $\cos \epsilon$ and $\cos^3 \epsilon$ will vanish. With these terms discarded:

$$\begin{aligned} \frac{Q'_1 - Q}{m} &= \xi_x'^3 - \xi_x^3 \\ &= (\xi_1 - \xi)_x^3 \cos^6(\theta'/2) + 3 \left[\xi_x^2 (\xi_1 - \xi)_x \cos^2(\theta'/2) + \right. \\ &\quad \left. + \xi_x (\xi_1 - \xi)_x^2 \cos^4(\theta'/2) + \frac{1}{4} \{ V^2 - (\xi_1 - \xi)_x^2 \} \sin^2 \theta' \right. \\ &\quad \left. \times \cos^2 \epsilon \left\{ \xi_x + (\xi_1 - \xi)_x \cos^2(\theta'/2) \right\} \right] \end{aligned}$$

$$\int_0^{2\pi} d\epsilon = 2\pi, \quad \int_0^{2\pi} \cos^2 \epsilon d\epsilon = \pi$$

$$J = \pi m \int_0^{\infty} \left[2(\xi_{1x} - \xi_x)^3 \cos^6(\theta'/2) + 6\xi_x^2(\xi_{1x} - \xi_x) \cos^2(\theta'/2) + \frac{3}{4} \left\{ V^2 (\xi_{1x} - \xi_x)^2 \right\} \left\{ \xi_x + (\xi_{1x} - \xi_x) \cos^2(\theta'/2) \right\} \sin^2 \frac{\theta'}{2} + 6\xi_x (\xi_{1x} - \xi_x)^2 \cos^4(\theta'/2) \right] \alpha d\alpha$$

Now

$$\cos^4(\theta'/2) = \cos^2(\theta'/2) - \frac{1}{4} \sin^2 \theta'$$

The expression for J can now be rewritten:

$$\begin{aligned} \frac{1}{\pi m} J &= 2(\xi_{1x} - \xi_x)^3 \int_0^{\infty} \cos^6(\theta'/2) \alpha d\alpha + \\ &+ 6(\xi_x \xi_{1x}^2 - \xi_{1x} \xi_x^2) \int_0^{\infty} \cos^2(\theta'/2) \alpha d\alpha + \\ &+ \frac{3}{4} (\xi_{1x} - \xi_x) \left\{ (\xi_{1y} - \xi_y)^2 + (\xi_{1z} - \xi_z)^2 \right\} \int_0^{\infty} \cos^2(\theta'/2) \sin^2 \theta' \alpha d\alpha + \\ &+ \left[-\frac{6}{4} \xi_x^2 (\xi_{1x} - \xi_x)^2 + \frac{3}{4} \xi_x \left\{ (\xi_{1y} - \xi_y)^2 + (\xi_{1z} - \xi_z)^2 \right\} \right] \times \\ &\quad \times \int_0^{\infty} \sin^2 \theta' \alpha d\alpha \end{aligned}$$

The first two lines in the above expression do not contribute to ΔQ

because:

$$\begin{aligned} &\iint f f_1 (\xi_{1x} - \xi_x)^3 d\xi_2 d\xi_1 \\ &= \iint f f_1 \left[(\xi_{1x}^3 - \xi_x^3) + 3(\xi_x^2 \xi_{1x} - \xi_x \xi_{1x}^2) \right] d\xi_2 d\xi_1 \\ &= 0 \end{aligned}$$

$$\iint f f_1 (\xi_x \xi_{1x}^2 - \xi_{1x} \xi_x^2) d\xi d\xi_1 = 0$$

$$\begin{aligned} & \iint (\xi_{1x} - \xi_x) \left\{ (\xi_{1y} - \xi_y)^2 + (\xi_{1z} - \xi_z)^2 \right\} f f_1 d\xi_1 d\xi \\ &= \iint \left[\xi_{1x} (\xi_{1y}^2 + \xi_{1z}^2) - \xi_x (\xi_y^2 + \xi_z^2) + 2(\xi_x \xi_y \xi_{1y} - \xi_{1x} \xi_y \xi_y) + \right. \\ & \quad \left. + 2(\xi_x \xi_z \xi_{1z} - \xi_{1x} \xi_z \xi_z) \right] f f_1 d\xi d\xi_1 \\ &= 0 \end{aligned}$$

Now:

$$A_z = \pi \int_0^{\infty} \sin^2 \theta' \cdot \alpha d\alpha$$

$$\frac{3}{2} A_z \sqrt{2mK} = \frac{kT}{\mu} = \frac{P}{\mu m}$$

Substituting into the expression for ΔQ :

$$\Delta Q = -\frac{1}{2} \frac{mP}{\mu m} \iint \left[\xi_x \left\{ 2(\xi_{1x} - \xi_x)^2 - (\xi_{1y} - \xi_y)^2 - (\xi_{1z} - \xi_z)^2 \right\} \right] f f_1 d\xi d\xi_1$$

Now if

$$\underline{c} = \underline{\xi} - \underline{u} \quad ; \quad \xi_{ij} - \xi_j = c_{ij} - c_j$$

$$\therefore \Delta Q = -\frac{1}{2} \frac{mP}{\mu m} \iint (c_x + u_x) \left\{ 2(c_{1x} - c_x)^2 - (c_{1y} - c_y)^2 - (c_{1z} - c_z)^2 \right\} f f_1 d\xi d\xi_1$$

Note that: $\int c_i f d\underline{\xi} = 0$

$$\begin{aligned} \therefore \Delta Q &= -\frac{1}{2} \frac{m\rho}{\mu m} \left[2m u_x \int [2c_x^2 - c_y^2 - c_z^2] f d\underline{\xi} + \right. \\ &\quad \left. + m \int c_x [2c_x^2 - c_y^2 - c_z^2] f d\underline{\xi} \right] \\ &= \frac{P}{\mu} \left[3u_x p_{xx} + q_x - \frac{3}{2} \int m c_x^3 f d\underline{\xi} \right] \end{aligned}$$

APPENDIX VIII. Some Details of the Six-Moment
One-Dimensional Calculation

The Transient Problem

The six moment equations are:

$$(a) \quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial y} (\rho v) = 0$$

$$(b) \quad \rho \frac{\partial v}{\partial t} + \rho v \frac{\partial v}{\partial y} = \frac{\partial P_{yy}}{\partial y}$$

$$(c) \quad \frac{\partial p}{\partial t} + \frac{\partial}{\partial y} (pv) = -\frac{2}{3} \frac{\partial q_y}{\partial y} + \frac{2}{3} P_{yy} \frac{\partial v}{\partial y}$$

(8-1)

$$(d) \quad \frac{\partial}{\partial t} \int m \xi_x^2 f d\xi + \frac{\partial}{\partial y} \int m \xi_y \xi_x^2 f d\xi = \frac{\rho}{\mu} p_{xx}$$

$$(e) \quad \frac{\partial}{\partial t} \int m \xi_y \frac{\xi^2}{2} f d\xi + \frac{\partial}{\partial y} \int m \xi_y^2 \frac{\xi^2}{2} f d\xi = \frac{\rho}{\mu} \left[-\frac{2}{3} q_y + v p_{yy} \right]$$

$$(f) \quad \frac{\partial}{\partial t} \int m \xi_y^3 f d\xi + \frac{\partial}{\partial y} \int m \xi_y^4 f d\xi = \frac{\rho}{\mu} \left[q_y + 3v p_{yy} - \frac{3}{2} m \overline{C_y^3} \right]$$

The corresponding linearized equations to first order are:

$$(a) \quad \frac{\partial}{\partial t} [N_+ + 2v_-] + \frac{\beta}{2\pi} \frac{\partial}{\partial y} [2N_- + t_- + 2\pi v_+] = 0$$

(8-2)

$$(b) \quad \frac{\partial}{\partial t} [2N_- + t_- + 2\pi v_+] + \frac{2\pi \rho_0}{\rho \beta} \frac{\partial}{\partial y} [N_+ + t_+ + 4v_-] = 0$$

$$(c) \frac{\partial}{\partial t} [3N_+ + 3t_+ + 8v_-] + \frac{\beta}{\pi} \frac{\partial}{\partial y} [4N_- + 6t_- + 5\pi v_+] = 0$$

$$(d) \frac{\partial}{\partial t} [N_+ + t_+ + 2v_-] + \frac{\beta}{2\pi} \frac{\partial}{\partial y} [2N_- + 3t_- + 2\pi v_+] - \frac{2\rho_0}{3\mu_0} v_- = 0$$

$$(e) \frac{\partial}{\partial t} [4N_- + 6t_- + 5\pi v_+] + \frac{\beta}{2} \frac{\partial}{\partial y} [5N_+ + 10t_+ + 24v_-] + \frac{\rho_0}{\mu_0} \frac{1}{3} [7t_- - 2N_-] = 0 \quad (8-2)$$

$$(f) \frac{\partial}{\partial t} [2N_- + 3t_- + 3\pi v_+] + \frac{\beta}{2} \frac{\partial}{\partial y} [3N_+ + 6t_+ + 16v_-] + \frac{\rho_0}{\mu_0} \frac{1}{2} [t_- - 2N_-] = 0$$

Define:

$$\underline{t} = t/\tau_f, \quad \underline{y} = \frac{y}{\beta\tau_f}, \quad \tau_f = \frac{\pi}{4} \frac{\mu_0}{\rho_0} \quad (8-3)$$

After some algebraic manipulation, the equations (8-2) can be simplified to:

$$(a) \frac{\partial}{\partial \underline{t}} [N_+ + 2v_-] + \frac{1}{2\pi} \frac{\partial}{\partial \underline{y}} [2N_- + \underline{t}_- + 2\pi v_+] = 0 \quad (8-4)$$

$$(b) \frac{\partial}{\partial \underline{t}} [2N_- + \underline{t}_- + 2\pi \underline{v}_+] + \frac{\partial}{\partial \underline{y}} [N_+ + \underline{t}_+ + 4\underline{v}_-] = 0$$

$$(c) 2 \frac{\partial}{\partial \underline{t}} (\underline{v}_-) + \frac{1}{\pi} \frac{\partial}{\partial \underline{y}} [N_- + \frac{3}{2} \underline{t}_- + 2\pi \underline{v}_+] + \frac{\pi}{2} \underline{v}_- = 0$$

$$(d) \frac{\partial}{\partial \underline{t}} (\underline{t}_+) + \frac{1}{\pi} \frac{\partial}{\partial \underline{y}} (\underline{t}_-) - \frac{\pi}{6} \underline{v}_- = 0$$

(8-4)

$$(e) \frac{\partial}{\partial \underline{t}} [4N_- + 3\pi \underline{v}_+] + \frac{\partial}{\partial \underline{y}} [\frac{3}{2} N_+ + 4\underline{v}_-] + \frac{\pi}{8} [2N_- - \underline{t}_-] = 0$$

$$(f) \frac{\partial}{\partial \underline{t}} [2N_- + 3\underline{t}_-] - 4 \frac{\partial}{\partial \underline{y}} (\underline{v}_-) + \frac{\pi}{8} [6N_- + 9\underline{t}_-] = 0$$

The Laplace transformed equations are:

$$(a) \frac{d}{d\underline{y}} [2\tilde{N}_- + \tilde{t}_- + 2\pi \tilde{v}_+] + 2\pi s [\tilde{N}_+ + 2\tilde{v}_-] = 0$$

$$(b) \frac{d}{d\underline{y}} [\tilde{N}_+ + \tilde{t}_+ + 4\tilde{v}_-] + s [2\tilde{N}_- + \tilde{t}_- + 2\pi \tilde{v}_+] = 0 \quad (8-5)$$

$$(c) \frac{d}{d\underline{y}} [\tilde{N}_- + \frac{3}{2} \tilde{t}_- + 2\pi \tilde{v}_+] + [2\pi s + \frac{\pi^2}{2}] \tilde{v}_- = 0$$

$$(d) \frac{d}{dy} [\tilde{t}_-] + \pi s \tilde{t}_+ - \frac{\pi^2}{6} \tilde{v}_- = 0$$

$$(e) \frac{d}{dy} \left[\frac{3}{2} \tilde{N}_+ + 4\tilde{v}_- \right] + s \left[4\tilde{N}_- + 3\pi \tilde{v}_+ \right] + \frac{\pi}{8} \left[2\tilde{N}_- - \tilde{t}_- \right] = 0 \quad (8-5)$$

$$(f) 4 \frac{d}{dy} [\tilde{v}_-] - s \left[2\tilde{N}_- + 3\tilde{t}_- \right] - \frac{\pi}{8} \left[6\tilde{N}_- + 9\tilde{t}_- \right] = 0$$

Now assume solutions of the form

$$\begin{aligned} \tilde{N}_+ &= A e^{\omega y} & \tilde{N}_- &= B e^{\omega y} \\ \tilde{t}_+ &= C e^{\omega y} & \tilde{t}_- &= D e^{\omega y} \\ \tilde{v}_+ &= E e^{\omega y} & \tilde{v}_- &= F e^{\omega y} \end{aligned} \quad (8-6)$$

Substituting into (8-5), one obtains the following relations between the coefficients:

$$(a) B = -\frac{3}{2} D + \frac{4\omega}{(2s + \frac{3\pi}{4})} F$$

$$(b) E = -\frac{1}{2\pi} \left[\frac{4\omega}{(2s + \frac{3\pi}{4})} + \frac{\pi(2s + \frac{\pi}{2})}{\omega} \right] F \quad (8-7)$$

$$(c) C = \frac{1}{\pi s} \left[\frac{\pi^2}{6} F - \omega D \right]$$

$$(d) A = -\frac{2}{3\omega} \left[3\pi s E + (4s + \frac{\pi}{4}) B - \frac{\pi}{8} D + 4\omega F \right]$$

$$(e) D = \frac{12\omega^4 \left(60\pi s^2 + \frac{29\pi^2}{2}s + \frac{9\pi^3}{8} \right) \omega^2 + 6\pi^2 s^2 \left(4s^2 + \frac{5\pi}{2}s + \frac{3\pi^2}{8} \right)}{6\omega \left(2s + \frac{3\pi}{4} \right) \left[\omega^2 - \pi s \left(4s + \frac{\pi}{3} \right) \right]} F \quad (8-7)$$

Thus, the coefficients A, B, C, D and E can be expressed in terms of the coefficient F .

The determinant for ω is:

$$\begin{vmatrix} 2\pi s & 2\omega & 0 & \omega & 2\pi\omega & 2\pi s \\ \omega & 2s & \omega & s & 2\pi s & 4\omega \\ 0 & \omega & 0 & \frac{3}{2}\omega & 2\pi\omega & \left(2\pi s + \frac{\pi^2}{2} \right) \\ 0 & 0 & \pi s & \omega & 0 & -\frac{\pi^2}{6} \\ \frac{3}{2}\omega & \left(2s + \frac{\pi}{4} \right) & 0 & -\frac{\pi}{8} & 3\pi s & 4\omega \\ 0 & -\left(2s + \frac{3\pi}{4} \right) & 0 & -\left(3s + \frac{9\pi}{8} \right) & 0 & 4\omega \end{vmatrix} = 0 \quad (8-8)$$

Adding and subtracting columns and rows, the above determinant reduces to:

$$\begin{vmatrix}
 2\omega & 0 & 0 & 0 & 2\pi s & -\frac{4\pi}{3}s \\
 2s & \omega & 0 & 0 & 0 & 0 \\
 0 & 0 & \omega & 0 & 0 & (2\pi s + \frac{\pi^2}{2}) \\
 0 & \pi s & 0 & \omega & -\pi s & -(\frac{4\pi s}{3} + \frac{\pi^2}{6}) \\
 (5s + \frac{\pi}{2}) & 0 & -(s + \frac{\pi}{4}) & -s & \frac{3}{2}\omega & 0 \\
 -(4s + \frac{3\pi}{2}) & 0 & (2s + \frac{3\pi}{4}) & -(4s + \frac{3\pi}{2}) & 0 & 4\omega
 \end{vmatrix} = 0 \quad (8-9)$$

Expanding the determinant, the following cubic in ω^2 is obtained:

$$12\omega^6 + a(s)\omega^4 + b(s)\omega^2 + c(s) = 0 \quad (8-10)$$

where

$$(a) \quad a(s) = -\frac{15\pi^3}{8} - \frac{45\pi^2}{2}s - 84\pi s^2$$

$$(b) \quad b(s) = \frac{5\pi^5}{8}s + \frac{41\pi^4}{3}s^2 + 80\pi^3s^3 + 144\pi^2s^4 \quad (8-11)$$

$$(c) \quad c(s) = -\frac{3\pi^6}{4}s^3 - \frac{19\pi^5}{2}s^4 - 38\pi^4s^5 - 48\pi^3s^6$$

Examining (8-10) for $s \rightarrow 0$, it is possible to show that the roots of the equation to leading orders are:

$$\begin{aligned}
 \text{(a)} \quad \omega_{1,4} &= \mp \frac{\pi}{4} \sqrt{\frac{5\pi}{2}} \left[1 + \frac{74}{15\pi} s + \dots \right] \\
 \text{(b)} \quad \omega_{2,5} &= \mp \frac{\pi}{\sqrt{3}} \sqrt{s} \left[1 + \frac{21}{5\pi} s + \dots \right] \\
 \text{(c)} \quad \omega_{3,6} &= \mp \sqrt{\frac{6\pi}{5}} s \left[1 - \frac{14}{5\pi} s + \dots \right]
 \end{aligned} \tag{8-12}$$

Retaining only the negative roots for finiteness for $y \rightarrow \infty$ the transform for $s \rightarrow 0$ can be written:

$$\begin{aligned}
 \tilde{N}_+ &= A_1 e^{\omega_1 y} + A_2 e^{\omega_2 y} + A_3 e^{\omega_3 y} \\
 \tilde{N}_- &= B_1 e^{\omega_1 y} + B_2 e^{\omega_2 y} + B_3 e^{\omega_3 y}
 \end{aligned} \tag{8-13}$$

etc.

Now all the coefficients $A-E$ can be written in terms of F ; define:

$$\begin{aligned}
 A_1 &= a_1(s, \omega_1) F_1; \quad A_2 = a_2(s, \omega_2) F_2; \quad A_3 = a_3(s, \omega_3) F_3 \\
 B_1 &= b_1(s, \omega_1) F_1; \quad B_2 = b_2(s, \omega_2) F_2; \quad B_3 = b_3(s, \omega_3) F_3
 \end{aligned} \tag{8-14}$$

etc.

From (8-14), (8-12), and (8-7), for $s \rightarrow 0$:

$$\begin{aligned}
 \text{(a)} \quad \alpha_1 &= -\frac{2\pi}{3} \sqrt{\frac{2}{5\pi}} + O(s) \\
 \text{(b)} \quad b_1 &= -\frac{7\pi}{3} \sqrt{\frac{2}{5\pi}} + O(s)
 \end{aligned} \tag{8-15}$$

$$(c) e_1 = \frac{8}{3} \sqrt{\frac{2}{5\pi}} + O(s)$$

$$(d) c_1 = -\frac{4}{5} + O(s)$$

$$(e) a_1 = -\frac{16}{5} + O(s)$$

$$(f) d_2 = -\frac{5\pi^2\sqrt{3}}{72} \frac{1}{s^{3/2}} + O(s^{-1/2})$$

$$(g) b_2 = \frac{5\pi^2\sqrt{3}}{48} \frac{1}{s^{3/2}} + O(s^{-1/2})$$

$$(h) e_2 = \frac{\sqrt{3}}{4} \frac{1}{\sqrt{s}} + O(\sqrt{s})$$

$$(i) c_2 = -\frac{5\pi^2}{72} \frac{1}{s^2} + O(s^{-1})$$

(8-15)

$$(j) a_2 = \frac{5\pi^2}{72} \frac{1}{s^2} + O(s^{-1})$$

$$(k) d_3 = \frac{3\pi}{5} \sqrt{\frac{5}{6\pi}} + O(s)$$

$$(l) b_3 = -\frac{9\pi}{10} \sqrt{\frac{5}{6\pi}} + O(s)$$

$$(m) e_3 = \frac{\pi}{4} \sqrt{\frac{5}{6\pi}} \frac{1}{s} + O(1)$$

$$(n) c_3 = \frac{\pi}{6} \frac{1}{s} + O(1)$$

$$(o) a_3 = \frac{\pi}{4} \frac{1}{s} + O(1)$$

From the boundary conditions at the liquid surface ($y = 0$):

$$(a) \quad 2\tilde{v}_1 = \tilde{v}_+ + \tilde{v}_- = (e_1+1)F_1 + (e_2+1)F_2 + (e_3+1)F_3 = 0 \quad (8-16)$$

$$(b) \quad 2\tilde{t}_1 = \tilde{t}_+ + \tilde{t}_- = (c_1+d_1)F_1 + (c_2+d_2)F_2 + (c_3+d_3)F_3 \\ = \frac{2\Delta T}{\lambda T_0} \cdot \frac{1}{S}$$

$$(c) \quad 2\tilde{N}_1 = \tilde{N}_+ + \tilde{N}_- = (a_1+b_1)F_1 + (a_2+b_2)F_2 + (a_3+b_3)F_3 \\ = \frac{2\Delta m}{\lambda m_0} \cdot \frac{1}{S}$$

Define:

$$t_{1L} = \frac{\Delta T}{\lambda T_0}, \quad N_{1L} = \frac{\Delta m}{\lambda m_0} \quad (8-17)$$

Then solving (8-16) for F_1, F_2 and F_3 :

$$(a) \quad F_1 = \frac{2t_{1L}}{S} \left[\frac{(e_3+1)(a_2+b_2) - (e_2+1)(a_3+b_3)}{\Delta} \right] + \\ + \frac{2N_{1L}}{S} \left[\frac{(e_2+1)(c_3+d_3) - (e_3+1)(c_2+d_2)}{\Delta} \right] \quad (8-18)$$

$$(b) \quad F_2 = \frac{2t_{1L}}{S} \left[\frac{(e_1+1)(a_3+b_3) - (e_3+1)(a_1+b_1)}{\Delta} \right] + \\ + \frac{2N_{1L}}{S} \left[\frac{(e_3+1)(c_1+d_1) - (e_1+1)(c_3+d_3)}{\Delta} \right]$$

$$(c) F_3 = \frac{2t_{IL}}{S} \left[\frac{(e_2+1)(a_1+b_1) - (e_1+1)(a_2+b_2)}{\Delta} \right] + \\ + \frac{2N_{IL}}{S} \left[\frac{(e_1+1)(c_2+d_2) - (e_2+1)(c_1+d_1)}{\Delta} \right]$$

where:

$$\Delta = (e_1+1) \{ (c_2+d_2)(a_3+b_3) - (a_2+b_2)(c_3+d_3) \} + \quad (8-19) \\ + (e_2+1) \{ (c_3+d_3)(a_1+b_1) - (a_3+b_3)(c_1+d_1) \} + \\ + (e_3+1) \{ (c_1+d_1)(a_2+b_2) - (a_1+b_1)(c_2+d_2) \}$$

Substituting (8-12) and (8-15) into (8-18), we have for $S \rightarrow 0$:

$$(a) F_1 = - \frac{\frac{5\pi^3 \sqrt{S}}{144 \sqrt{6\pi}}}{S \{ K_1 + K_2 \sqrt{S} + O(S) \}} \left[t_{IL} \left(1 + \frac{3}{2} \sqrt{3} \sqrt{S} + O(S) \right) + \right. \\ \left. + N_{IL} (1 + \sqrt{3} \sqrt{S} + O(S)) \right] \quad (8-20)$$

$$(b) F_2 = - \frac{\frac{\pi}{2} \cdot S}{S \{ K_1 + K_2 \sqrt{S} + O(S) \}} \left[t_{IL} \left\{ \left(1 + \frac{8}{3} \sqrt{\frac{2}{5\pi}} \right) + \sqrt{\frac{5}{6\pi}} \left(\frac{16}{5} + \frac{7\pi \sqrt{2}}{3 \sqrt{5\pi}} \right) \right\} + \right. \\ \left. + N_{IL} \left\{ \frac{2}{3} \left(1 + \frac{8}{3} \sqrt{\frac{2}{5\pi}} \right) + \sqrt{\frac{5}{6\pi}} \left(\frac{4}{5} + \frac{2\pi \sqrt{2}}{3 \sqrt{5\pi}} \right) \right\} + O(S) \right]$$

$$(c) \quad F_3 = \frac{\frac{5\pi^2}{36} \left(1 + \frac{8}{3} \sqrt{\frac{2}{5\pi}}\right)}{\{K_1 + K_2 \sqrt{S} + O(S)\}} \left[t_{IL} \left\{1 + \frac{3}{2} \sqrt{3} \sqrt{S} + O(S)\right\} + N_{IL} \left\{1 + \sqrt{3} \sqrt{S} + O(S)\right\} \right]$$

where:

$$(a) \quad K_1 = \frac{5\pi^3}{288} \left[\frac{5}{3} \left\{1 + \frac{8}{3} \sqrt{\frac{2}{5\pi}}\right\} + \sqrt{\frac{5}{6\pi}} \left\{4 + 3\pi \sqrt{\frac{2}{5\pi}}\right\} \right] \quad (8-21)$$

$$(b) \quad K_2 = \frac{5\pi^3}{288} \sqrt{3} \left[2 \left\{1 + \frac{8}{3} \sqrt{\frac{2}{5\pi}}\right\} + \sqrt{\frac{5}{6\pi}} \left\{\frac{22}{5} + \frac{8\pi}{3} \sqrt{\frac{2}{5\pi}}\right\} \right]$$

Now, the gas velocity is given to first order by:

$$V = \frac{\beta}{2\pi} \left[\lambda \left\{ N_- + \frac{1}{2} t_- + \pi v_+ \right\} + O(\lambda^2) \right]$$

$$= \beta \left[\lambda v + O(\lambda^2) \right] \quad (8-22)$$

or:

$$\tilde{v} = \frac{1}{4\pi} \left[2 \tilde{N}_- + \tilde{t}_- + 2\pi \tilde{v}_+ \right] \quad (8-23)$$

For $S \rightarrow 0$:

$$\begin{aligned} \tilde{v} &= \frac{1}{4\pi} \left\{ 2b_1 + d_1 + 2\pi e_1 \right\} F_1 e^{\omega_1 \underline{y}} + \\ &+ \frac{1}{4\pi} \left\{ 2b_2 + d_2 + 2\pi e_2 \right\} F_2 e^{\omega_2 \underline{y}} + \\ &+ \frac{1}{4\pi} \left\{ 2b_3 + d_3 + 2\pi e_3 \right\} F_3 e^{\omega_3 \underline{y}} \end{aligned} \quad (8-24)$$

From (8-12), (8-20), and (8-24), for $s \rightarrow 0$:

$$\begin{aligned} &\tilde{v}(s, \underline{y}) \\ &= \left\{ \frac{O(1)}{4\pi \{k_1 + k_2 \sqrt{s} + O(s)\}} \cdot e^{-\frac{\pi}{4} \sqrt{\frac{s\pi}{2}} \left[1 + \frac{74}{15\pi} s + O(s^2) \right] \underline{y}} \right\} - \\ &- \left\{ \frac{5\pi^2 \sqrt{3}}{288} \left[t_{IL} \left\{ \left(1 + \frac{8}{3} \sqrt{\frac{2}{5\pi}} \right) + \sqrt{\frac{s}{6\pi}} \left(\frac{16}{5} + \frac{7\pi}{3} \sqrt{\frac{2}{5\pi}} \right) \right\} - N_{IL} \left\{ \frac{2}{3} \left(1 + \frac{8}{3} \sqrt{\frac{2}{5\pi}} \right) + \sqrt{\frac{s}{6\pi}} \left(\frac{4}{5} + \frac{2\pi}{3} \sqrt{\frac{2}{5\pi}} \right) \right\} + O(s) \right] \right\} \times \\ &\times \left\{ \frac{\exp \left[-\frac{\pi}{\sqrt{3}} \sqrt{s} \left(1 + \frac{21}{8\pi} s + O(s) \right) \underline{y} \right]}{s^{1/2} \left(k_1 + k_2 \sqrt{s} + O(s) \right)} \right\} + \quad (8-25) \\ &+ \left\{ \frac{\frac{5\pi^3}{288} \sqrt{\frac{s}{6\pi}} \left(1 + \frac{8}{3} \sqrt{\frac{2}{5\pi}} \right) \left[\left(t_{IL} + N_{IL} \right) + \sqrt{3} \sqrt{s} \left(\frac{3}{2} t_{IL} + N_{IL} \right) + O(s) \right]}{s \left(k_1 + k_2 \sqrt{s} + O(s) \right)} \right\} \times \\ &\times \left\{ \exp \left[-\sqrt{\frac{6\pi}{s}} s \left(1 - \frac{14}{5\pi} s + O(s^2) \right) \underline{y} \right] \right\} \end{aligned}$$

Note that for large time the solution exponentially damped in y will not contribute to the velocity. Thus, for large time ($t \gg \frac{y}{a_0}$):

$$v \rightarrow C_1 e^{\frac{K_1}{K_2} \frac{\pi y}{\sqrt{3}} + \frac{K_1^2}{K_2^2} t} \cdot \operatorname{erfc} \left\{ \frac{1}{2} \frac{\pi y}{\sqrt{3t}} + \frac{K_1}{K_2} \sqrt{t} \right\} + \quad (8-26)$$

$$+ C_2 \left[1 + C_3 \frac{1}{\left(t - \sqrt{\frac{6\pi}{5}} y \right)^{1/2}} \right]$$

where:

$$(a) C_1 = - \frac{5\pi^2 \sqrt{3}}{288 K_2} \left[t_{IL} \left\{ \left(1 + \frac{8}{3} \sqrt{\frac{2}{5\pi}} \right) + \sqrt{\frac{5}{6\pi}} \left(\frac{16}{5} + \frac{7\pi}{3} \sqrt{\frac{2}{5\pi}} \right) \right\} - \right. \\ \left. - N_{IL} \left\{ \frac{2}{3} \left(1 + \frac{8}{3} \sqrt{\frac{2}{5\pi}} \right) + \sqrt{\frac{5}{6\pi}} \left(\frac{4}{5} + \frac{2\pi}{3} \sqrt{\frac{2}{5\pi}} \right) \right\} \right] \quad (8-27)$$

$$(b) C_2 = \frac{5\pi^3}{288} \sqrt{\frac{5}{6\pi}} \left(1 + \frac{8}{3} \sqrt{\frac{2}{5\pi}} \right) \frac{[t_{IL} + N_{IL}]}{K_1}$$

$$(c) C_3 = \left[\sqrt{3} \left(\frac{3}{2} t_{IL} + N_{IL} \right) - \frac{K_2}{K_1} (t_{IL} + N_{IL}) \right]$$

For large time:

$$v \longrightarrow \text{constant} = C_2$$

The evaporation - condensation rate at the liquid surface is therefore given to first order by:

$$\begin{aligned}
 \dot{m} &= \lambda \beta \rho_0 c_2 \\
 &= \beta \rho_0 \frac{\left(1 + \frac{8}{3} \sqrt{\frac{2}{5\pi}}\right) \sqrt{\frac{5}{6\pi}}}{\left[\frac{5}{3} \left(1 + \frac{8}{3} \sqrt{\frac{2}{5\pi}}\right) + \sqrt{\frac{5}{6\pi}} \left(4 + 3\pi \sqrt{\frac{2}{5\pi}}\right)\right]} \frac{\Delta p}{p_0} \\
 &= \beta \rho_0 \frac{1}{\left[\sqrt{\frac{10\pi}{3}} + \frac{9\pi + 12 \sqrt{\frac{5\pi}{2}}}{8 + 3 \sqrt{\frac{5\pi}{2}}}\right]} \frac{\Delta p}{p_0} \quad (8-28)
 \end{aligned}$$

The Quasi-Steady State Solution for Condensation

The six linearized, steady moment equations to first order are:

- (a) $2N_- + t_- + 2\pi v_+ = B_1$
- (b) $N_+ + t_+ + 4v_- = B_2$
- (c) $7t_- - 2N_- = 2B_3$ (8-29)
- (d) $\frac{d}{dy} [2N_- + 3t_- + 2\pi v_+] = \frac{\pi^2}{3} v_-$
- (e) $\frac{d}{dy} [5N_+ + 10t_+ + 24v_-] = \frac{\pi}{6} [2N_- - 7t_-]$
- (f) $\frac{d}{dy} [3N_+ + 6t_+ + 16v_-] = \frac{\pi}{4} [2N_- - t_-]$

The boundary conditions are:

$$\begin{aligned}
 & \text{(a) } P \rightarrow P_c, \quad V \rightarrow V_c \quad \text{for } \underline{y} \rightarrow \infty \\
 & \left. \begin{aligned}
 & \text{(b) } t_1^{(1)} = \frac{T_{1L} - T_0}{\lambda T_0} = \frac{\Delta T}{\lambda T_0} \\
 & \text{(c) } N_1^{(1)} = \frac{m_{1L} - m_0}{\lambda m_0} = \frac{\Delta m}{\lambda m_0} \\
 & \text{(d) } \nu_1^{(1)} = 0
 \end{aligned} \right\} \text{ at } \underline{y} = 0
 \end{aligned}
 \tag{8-30}$$

Note that P_c , T_c , and V_c are the steady conditions behind the wave.

Equations (8-29) can be integrated immediately to give:

$$\begin{aligned}
 & \text{(a) } t_- = \frac{1}{5} B_3 + B_4 \exp\left(-\frac{\pi}{4} \sqrt{\frac{S\pi}{2}} \underline{y}\right) + B_5 \exp\left(\frac{\pi}{4} \sqrt{\frac{S\pi}{2}} \underline{y}\right) \\
 & \text{(b) } \nu_- = \frac{3}{2\pi} \sqrt{\frac{S\pi}{2}} \left[-B_4 \exp\left(-\frac{\pi}{4} \sqrt{\frac{S\pi}{2}} \underline{y}\right) + B_5 \exp\left(+\frac{\pi}{4} \sqrt{\frac{S\pi}{2}} \underline{y}\right) \right] \\
 & \text{(c) } N_- = \frac{7}{2} t_- - B_3 \\
 & \text{(d) } t_+ = -B_2 + \frac{1}{5} \left[-\frac{\pi}{3} B_3 \underline{y} + B_6 \right] - \frac{4}{5} \nu_- \\
 & \text{(e) } N_+ = B_2 - t_+ - 4 \nu_- \\
 & \text{(f) } \nu_+ = \frac{1}{2\pi} \left[B_1 - t_- - 2 N_- \right]
 \end{aligned}
 \tag{8-31}$$

where B_4 , B_5 , and B_6 are constants. Now if one tries to apply the boundary conditions (8-30), one finds that there are an insufficient number of constants. The perturbation is again singular.

Define the outer variable :

$$\eta = \lambda y, \quad \lambda \frac{d}{d\eta} = \frac{d}{dy} \quad (8-32)$$

The first order equations in terms of the outer variable are:

$$\begin{aligned} (a) \quad & 2 N_-(\eta) + t_-(\eta) + 2\pi \nu_+(\eta) = A_1 \\ (b) \quad & N_+(\eta) + t_+(\eta) + 4 \nu_-(\eta) = A_2 \\ (c) \quad & 7 t_-(\eta) - 2 N_-(\eta) = 2 A_3 \\ (d) \quad & \nu_-(\eta) = 0 \\ (e) \quad & \frac{d}{d\eta} [5 N_+(\eta) + 10 t_+(\eta) + 24 \nu_-(\eta)] = -\frac{\pi}{6} \frac{8\pi}{\rho\beta} q_y^{(2)} \\ (f) \quad & 2 N_-(\eta) - t_-(\eta) = 0 \end{aligned} \quad (8-33)$$

From Appendix V:

$$\begin{aligned} q_y^{(2)} &= -\frac{1}{2} \rho\beta v^{(1)} [3\rho^{(1)} - 5f^{(1)}] \\ &= -\frac{1}{2} \rho\beta v_c^{(1)} \left[\frac{3}{2} t_+ - N_+ - \nu_- \right] \end{aligned} \quad (8-34)$$

Thus, (8-33 b, e, d) lead to:

$$5 \frac{dt_+}{d\eta} = \frac{2\pi^2}{3} \left[\frac{5}{2} t_+ - A_2 \right] v_c^{(1)} \quad (8-35)$$

or:

$$t_+(\eta) = \frac{2}{5} A_2 + A_4 \exp \left\{ \frac{\pi^2}{3} v_c^{(1)} \eta \right\} \quad (8-36)$$

Note that $v_c^{(1)}$ has to be negative, i. e., solution is valid only for condensation.

Matching the inner and outer solutions:

$$A_1 = B_1 ; A_2 = B_2 ; A_3 = B_3 = 0 ; B_5 = 0 \quad (8-37)$$

$$-B_2 + \frac{1}{5} B_6 = \frac{2}{5} A_2 + A_4$$

From the boundary conditions at infinity:

$$A_2 = 2 \frac{(P_c - P_0)}{\lambda P_0} ; A_1 = \frac{4\pi V_c}{\beta \lambda} \quad (8-38)$$

Now satisfying the boundary conditions at the liquid surface:

$$B_1 = \frac{(8 + 3\sqrt{\frac{5\pi}{2}})}{(\frac{9}{2} + \frac{6}{\pi}\sqrt{\frac{5\pi}{2}})} \left[\frac{2\Delta P}{\lambda P_0} - B_2 \right] \quad (8-39)$$

or:

$$\frac{4\pi V_c}{\beta \lambda} = \frac{(8 + 3\sqrt{\frac{5\pi}{2}})}{(\frac{9}{2} + \frac{6}{\pi}\sqrt{\frac{5\pi}{2}})} \left[\frac{2\Delta P}{\lambda P_0} - \frac{2(P_c - P_0)}{\lambda P_0} \right] \quad (8-40)$$

Now, across the expansion fan we have, to first order:

$$P_c = P_0 \left(1 + \frac{\sqrt{2\pi\gamma}}{\beta} V_c \right) = P_0 \left(1 + \sqrt{\frac{10\pi}{3}} \frac{V_c}{\beta} \right) \quad (8-41)$$

Substituting into (8-40), the condensation velocity is given by:

$$V_c = \beta \frac{1}{\left[\sqrt{\frac{10\pi}{3}} + \frac{9\pi + 12\sqrt{\frac{5\pi}{2}}}{8 + 3\sqrt{\frac{5\pi}{2}}} \right]} \frac{\Delta P}{P_0} \quad (8-42)$$

This is identical to (8-28) of the transient solution.