NORMAL STRUCTURES AND AUTOMORPHISM GROUPS OF t-DESIGNS

Thesis by

Richard Neil Lane

In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1968

(Submitted March 30, 1968)

ACKNOWLEDGMENTS

This work could not have been done without the continual aid, encouragement, and guidance of Professor Marshall Hall, Jr. I owe him a great deal. I would also like to thank the U. S. Office of Education for supporting me throughout my graduate career with a National Defense Education Act Fellowship.

This thesis is dedicated to my wife Suzanne, who made it all worthwhile.

ABSTRACT

Combinatorial configurations known as t-designs are studied. These are pairs $\langle B, \Pi \rangle$, where each element of B is a k-subset of Π, and each t-design occurs in exactly λ elements of B, for some fixed integers k and λ. A theory of internal structure of t-designs is developed, and it is shown that any t-design can be decomposed in a natural fashion into a sequence of "simple" subdesigns. The theory is quite similar to the analysis of a group with respect to its normal subgroups, quotient groups, and homomorphisms. The analogous concepts of normal subdesigns, quotient designs, and design homomorphisms are all defined and used.

This structure theory is then applied to the class of t-designs whose automorphism groups are transitive on sets of t points. It is shown that if G is a permutation group transitive on sets of t letters and Φ is any set of letters, then the images of Φ under G form a t-design whose parameters may be calculated from the group G. Such groups are discussed, especially for the case $t = 2$, and the normal structure of such designs is considered. Theorem 2. 2. 12 gives necessary and sufficient conditions for a t-design to be simple, purely in terms of the automorphism group of the design. Some constructions are given.

Finally, 2-designs with $k = 3$ and $\lambda = 2$ are considered in detail. These designs are first considered in general, with examples illustrating some of the configurations which can arise. Then an attempt is made to classify all such designs with an automorphism group transitive on pairs of points. Many cases are eliminated or reduced to combinations of Steiner triple systems. In the remaining cases, the simple designs are determined to consist of one infinite class and one exceptional case.

TABLE OF CONTENTS

INTRODUCTION

Combinatorial configurations known as t-designs are currently of great interest in combinatorial analysis. Here a t-design is defined to be a pair $T = \langle B, \Pi \rangle$, where Π is any finite set, and B is a set of subsets of Π, with the two properties

1) Each element of B contains k elements of Π for some fixed integer k.

2) Each set of t elements of Π lies in λ elements of B for some fixed integer $\lambda > 0$.

The elements of Π are called "points," and the elements of B are called "blocks." For $t = 2$, a t-design is a balanced incomplete block design, the subject of much study. The most important parameters of a t-design are consistently called b, v, r, k, λ where b is the number of blocks, v is the number of points, r is the number of blocks in which each point occurs, and k and λ are the numbers referred to in 1) and 2). An automorphism of a block design is a permutation on Π and a permutation on B, which preserves incidence of points on blocks. The set of elements in the image of a block is then the set of images of the elements in the block. The set of automorphisms of a t-design clearly form a group, with permutation multiplication as operation.

Much of the research accomplished in t-designs to this date has been directed toward questions of existence and construction of t-designs with various parameters, and to a lesser degree toward the various groups which arise as automorphism groups of t-designs. Here we are concerned with the internal structures of t-designs and their relations with automorphism groups.

Chapter I outlines a theory of decompositions of t-designs into normal subdesigns and quotient designs, which is in many ways analogous to the theory of normal subgroups and quotient groups of groups. The results are developed via "regular block homomorphisms" from one t-design onto another, which are mappings of designs which preserve incidence and also have the property that the inverse image of any block in the image design is the set of blocks of a subdesign of the range design. For a fixed regular block homomorphism, the set of subdesigns which arise in this manner have disjoint block sets which exhaust the block set of the range design, and this property is used to define a set of normal subdesigns. Given a set of normal subdesigns, the point sets of the normal subdesigns form the blocks of another t-design on the same points, called the quotient design. It is then shown that every regular block homomorphic image of a t-design is the quotient design produced from the set of normal subdesigns which arise as inverse images under the homomorphism. Simple designs are defined as t-designs with no non-trivial normal subdesigns, and the results allow the construction of "composition series" for a t-design, the concepts again being entirely analogous to those of group theory. The hypotheses can be weakened to produce quasi-normal subdesigns, and there are strongly simple t-designs, which have no quasi-normal subdesigns. These also play a role in the construction of composition series for a t-design. Some examples of the application of the theory are given, including two 2-designs with identical parameters, each having a doubly transitive automorphism group (on points), yet which are non-isomorphic because one has a normal subdesign and the other is simple.

Chapter II applies this theory to the special cases where the automorphism group of a t-design is transitive on sets of t points of the design. First, general permutation groups with this property are discussed, especially in the case $t = 2$. A necessary and sufficient condition for such a group to be t-ply transitive is given. Then it is shown that any permutation group transitive on sets of t points acting upon any subset of points yields a t-design, and the parameters of that t-design are computed from the structure of the given group. Finally, the "normal structure" of t-designs admitting such groups is investigated. The result of most value is Theorem 2. 2. 12, which characterizes simple designs purely in terms of the structure of their automorphism groups. Some t-designs are constructed from known groups.

In Chapter III, the results of Chapters I and II are applied to the 2-designs with $k = 3$ and $\lambda = 2$. Such designs are studied in general, with examples showing some of the various configurations which can arise, but most attention is spent on those designs whose automorphism groups are transitive on pairs of points. Various cases are shown to be combinations of Steiner triple systems (block designs with $k = 3$ and $\lambda = 1$), and the remaining cases are analyzed in terms of their normal structure. The major result is the demonstration that any simple design not a composition of Steiner triple systems either can be constructed from a finite near-field of order p or p^2 (where $p \equiv 1$, $p \equiv 2 \pmod 3$, respectively), or is the unique design with 6 points and automorphism group isomorphic to LF(2, 5). In the latter case, the further assumption that the automorphism group is doubly primitive is required.

The notations used are essentially those of Wielandt [8]: sets are represented by capital Greek letters, functions and permutations by lower case Greek letters, points by numerals or lower case Latin letters. Structures, such as groups or designs, are represented by capital Latin letters. If G is a permutation group on the set Ω, 1, 2, ..., t points of Ω, and Γ a subset of Ω, then $G_{1,2,\ldots,t}$, G_{Γ}, and $G_{\Gamma,1,2,\ldots,t}$ are respectively the subgroups fixing the t points 1, 2, ..., t; the subgroup sending the set Γ into itself (not necessarily fixing it pointwise), and the subgroup fixing the points 1, 2, ..., t <u>and</u> sending the set Γ into itself. If Ξ is a collection of orbits of the group G, then G^{Ξ} is the permutation representation of G on Ξ. If Φ is any set or group, $|\Phi|$ means the number of elements in that set or group. If T is a t-design, we write G(T) for the automorphism group of T. We say that G is transitive, primitive, etc., if G has that property when represented as a permutation group on $\Pi(T)$.

CHAPTER I

THE NORMAL STRUCTURE OF t-DESIGNS

1. 1. Homomorphism of t-designs

Let T_1, T_2 be two t-designs.

Definition 1. 1. 1. A mapping $\alpha: \Pi(T_1) \to \Pi(T_2)$ and $B(T_1) \to B(T_2)$ of the points and blocks of a t-design T_1 into the points and blocks respectively of a t-design T_2 is called a homomorphism if whenever a is a point of the block Φ of T_1, then $\alpha(a)$ is a point of the block $\alpha(\Phi)$ of T_2.

This definition appears to be too broad to be of any practical use at this time. Of more interest are block homomorphisms and regular block homomorphisms:

Definition 1. 1. 2. A homomorphism α from T_1 into T_2 is called a block homomorphism if (1) $v(T_1) = v(T_2)$, (2) $\alpha(\Pi(T_1)) = \Pi(T_2)$, and (3) $\alpha(B(T_1)) = B(T_2)$.

In short, a homomorphism is a block homomorphism if it is 1 - 1 onto for points and onto for blocks. Under these conditions, we may rewrite the blocks of T_2 using the points of T_1, so that α is in fact the identity map on the points. From now on, we assume that this has been done, so that $\Pi(T_1) = \Pi(T_2)$ and $\alpha(a) = a$ for any point a of T_1. T_1 and T_2 are then t-designs on the same set of points, and it follows directly from Definition 1. 1. 1 that the blocks of T_2 are set unions of blocks of T_1.

Definition 1. 1. 3. A block homomorphism α from T_1 onto T_2 is called regular if for each t-set $A \subseteq \Pi(T_2)$ and each two blocks Φ_1, Φ_2 of T_2 which contain A, the sets $\alpha^{-1}(\Phi_1)$, $\alpha^{-1}(\Phi_2)$ contain the same number of blocks containing A.

An extremely important class of regular block homomorphisms is the block homomorphisms mapping t-designs onto other t-designs with $\lambda = 1$. In these cases the condition of Definition 1. 1. 3 is satisfied vacuously. However, to characterize regular block homomorphisms fully, we have

Proposition 1. 1. 4. Let $\alpha: T_1 \to T_2$ be a regular block homomorphism, and let Φ_0 be a block of T_2. Then $\langle \alpha^{-1}(\Phi_0), \Phi_0 \rangle$ is a subdesign of T_1 with parameters

$$b = b(T_1)/b(T_2) \qquad v = k(T_2) \qquad r = r(T_1)/r(T_2)$$

$$k = k(T_1) \qquad \lambda = \lambda(T_1)/\lambda(T_2).$$

Conversely, suppose $\alpha: T_1 \to T_2$ is a block homomorphism and there exists an integer $\lambda' = \lambda'(\alpha)$ such that for any block $\Phi \in B(T_2)$, $\langle \alpha^{-1}(\Phi), \Phi \rangle$ is a sub-t-design of T_1 with $\lambda = \lambda'$. Then α is regular.

Proof: Since α is a block homomorphism, each point of each block in $\alpha^{-1}(\Phi_0)$ lies on Φ_0, hence $\alpha^{-1}(\Phi_0)$ is a set of $k(T_1)$-sets from Φ_0. Let A be a t-set from Φ_0, and let $\Phi_1, \Phi_2, \ldots, \Phi_s$ (where $s = \lambda(T_2)-1$) be the other blocks of T_2 containing A. The sets

$\alpha^{-1}(\Phi_i)$ are pairwise disjoint, since α is a function; and since α is regular, each set $\alpha^{-1}(\Phi_i)$ contains equally many blocks containing A. But then this number of blocks can only be $\lambda(T_1)/\lambda(T_2)$, hence there are $\lambda(T_1)/\lambda(T_2)$ blocks in $\alpha^{-1}(\Phi_0)$ containing A, and $\langle\alpha^{-1}(\Phi_0), \Phi_0\rangle$ is a subdesign of T. Calculation of the other parameters proceeds identically.

For the converse, let A be any t-set of $\Pi(T_2)$ and suppose $A \subseteq \Phi_0$, $A \subseteq \Phi_1$, where Φ_0 and Φ_1 are blocks of T_2. By hypothesis, $\langle\alpha^{-1}(\Phi_0), \Phi_0\rangle$ and $\langle\alpha^{-1}(\Phi_1), \Phi_1\rangle$ are subdesigns of T_1 with $\lambda = \lambda'$ for each, i.e., there are λ' blocks in $\alpha^{-1}(\Phi_0)$ which contain A, also λ' blocks in $\alpha^{-1}(\Phi_1)$ which contain A, and α is then regular.

Note that here the subdesigns of T_1 of the form $\langle\alpha^{-1}(\Phi), \Phi\rangle$ (where Φ runs over the set of blocks of T_2) comprise a set of subdesigns with the same parameters v, k, λ, whose block sets partition $B(T_1)$.

Definition 1.1.5. A subdesign T' of a t-design T is called quasi-complete if $B(T') \subseteq B(T)$ and $\Pi(T') \subseteq \Pi(T)$. Two subdesigns of a t-design are called codesigns if they have the same parameters v, k, λ.

A quasi-complete subdesign T' of T is called quasi-normal if there is a set of codesigns including T' whose block sets partition B(T), and denoted by $T' \mathrel{\tilde{\triangleleft}} T$. A quasi-complete subdesign T' of T is called complete if $\lambda(T') = \lambda(T)$. A quasi-normal subdesign T' of T is called normal if $\lambda(T') = \lambda(T)$, and denoted by $T' \triangleleft T$.

Thus Proposition 1.1.4 merely states that a block homomorphism $\alpha: T_1 \to T_2$ is regular if and only if $\langle\alpha^{-1}(\Phi), \Phi\rangle$ is a quasi-normal subdesign of T_1 for every $\Phi \in B(T_2)$. The set of co-designs $\{\langle \alpha^{-1}(\Phi), \Phi \rangle | \Phi \in B(T_2)\}$ constructed here is called the kernel of α.

Corollary 1.1.6. If $\alpha: T_1 \to T_2$ is a block homomorphism and $\lambda(T_2) = 1$, then for any block $\Phi \in B(T_2)$, $T' = \langle \alpha^{-1}(\Phi), \Phi \rangle$ is a normal subdesign of T_1.

Proof: From 1.1.4, T' is quasi-normal. Since $\lambda(T_2) = 1$, if A is any t-set of Φ, every block of T_1 which contains A must map onto Φ. This means that $\alpha^{-1}(\Phi)$ has $\lambda(T_1)$ elements which contain A, so $T' \lhd T_1$.

There are even stronger relations between homomorphisms and quasi-normal subdesigns, as will be seen next.

1.2. Factor Designs of t-designs

Proposition 1.2.1. Let T be a t-design and suppose $T' \mathrel{\tilde{\lhd}} T$ where $T_1, T_2, \ldots, T_s = T'$ are codesigns whose block sets partition B(T). Then the pair

$$T^* = \langle \{ \Pi(T_i) \}_{i=1}^{s} , \Pi(T) \rangle$$

is a t-design with parameters

$$b = s$$

$$v = v(T)$$

$$k = v(T')$$

$$\lambda = \lambda(T)/\lambda(T') \quad .$$

Proof: $\{\Pi(T_i)\}_{i=1}^{s}$ is clearly a set of s $v(T')$ - subsets of $\Pi(T)$, which has $v(T)$ points. Let A be a t-set of $\Pi(T)$. A lies in $\lambda(T)$ blocks of T, but each such block must lie in a unique codesign T_j. Since each such T_j is a t-design and a codesign of T', each such T_j has $\lambda(T')$ blocks containing A. Therefore there are $\lambda(T)/\lambda(T')$ codesigns T_j containing A, i.e., A lies in $\lambda(T)/\lambda(T')$ sets $\Pi(T_i)$.

The design T^* constructed in Proposition 1.2.1 is called the factor design T/T' by the codesigns $T_1, T_2, \ldots, T_s$. A quasi-normal subdesign can have more than one different set of such codesigns $T_1, \ldots, T_s$, so in general the factor design will depend upon the particular choice of codesigns.

Proposition 1.2.2. If T^* is the factor design T/T' by the codesigns $\{T_1, \ldots, T_s\}$, then there is a regular block homomorphism (called canonical) $\beta: T \to T^*$, with kernel $\{T_1, \ldots, T_s\}$.

Proof: Since the sets $B(T_i)$ partition B(T), each block Φ in B(T) belongs to exactly one $B(T_{i(\Phi)})$, so the map $\beta: \Phi \to \Pi(T_{i(\Phi)})$ is well-defined and easily seen to be the desired regular block homomorphism.

Theorem 1.2.3. If $\alpha: T \to T^*$ is a regular block homomorphism with kernel $\{T_1, \ldots, T_s\}$, then for any j $(1 \leq j \leq s)$, T^* is identical with T/T_j by the codesigns $\{T_1, \ldots, T_s\}$.

Proof: Let β be the canonical regular block homomorphism from T onto T/T_j constructed in Proposition 1.2.2. Then it is easily verified that for $\Phi \in B(T)$, $\beta(\Phi) = \alpha(\Phi)$, so since both β and α are onto, the blocks of T^* and T/T_j are identical. By hypothesis the point sets are identical, hence the two designs are the same.

Thus every regular block homomorphic image of a t-design T is the factor design of T by the kernel of the regular block homomorphism, and every regular block homomorphism is simply a mapping of blocks onto codesign point sets. The set of quasi-normal subdesigns of a t-design thus gives complete information about the various regular block homomorphic images possible, and conversely. For further information on these two sets, we have

Proposition 1.2.4. If $\alpha: T \to T^*$ is a regular block homomorphism, then for any quasi-complete (complete) subdesign T_1 of T^*, $\alpha^{-1}(T_1) = \langle \alpha^{-1}(B(T_1), \Pi(T_1)\rangle$ is a quasi-complete (complete) subdesign of T with $\lambda = \lambda(T) \cdot \lambda(T_1)/\lambda(T^*)$. If $T_1 \tilde{\triangleleft} T^*$ $(T_1 \triangleleft T^*)$, then $\alpha^{-1}(T_1) \tilde{\triangleleft} T$ $(\alpha^{-1}(T_1) \triangleleft T)$.

Proof: Consider $B(T^*)$ as the point sets of the codesigns in the kernel of α (Theorem 1.2.3). $B(T_1)$ is then the point sets of a subcollection of codesigns of the kernel, and $B' = \alpha^{-1}(B(T_1))$ is simply the collection of blocks in that subcollection of codesigns. Let A be a t-set of $\Pi(T_1)$. A then lies on $\lambda(T_1)$ blocks of T_1, but since α is regular, for each such block Φ of T_1, $\alpha^{-1}(\Phi)$ has $\lambda(T)/\lambda(T^*)$ blocks containing A (Proposition 1.1.4). The sets $\alpha^{-1}(\Phi)$ (where Φ runs over the blocks of T_1 containing A) are disjoint, hence A lies on $\lambda(T) \cdot \lambda(T_1)/\lambda(T^*)$ blocks of B'. $\alpha^{-1}(T_1)$ is then quasi-complete in T. If T_1 is complete in T^*, then $\lambda(T_1) = \lambda(T^*)$, so $\lambda(\alpha^{-1}(T_1)) = \lambda(T)$ and $\alpha^{-1}(T_1)$ is complete in T. If $T_1 \tilde{\triangleleft} T^*$, T_1 has a set of codesigns $\{T_1, \ldots, T_s\}$

whose block sets partition $B(T^*)$. But then the designs $\{\alpha^{-1}(T_1), \ldots, \alpha^{-1}(T_s)\}$ are a set of codesigns of $\alpha^{-1}(T_1)$ whose block sets partition $B(T)$, hence $\alpha^{-1}(T_1) \tilde{\lhd} T$. If $T_1 \lhd T^*$, then T_1 is complete in T^*, so from above $\alpha^{-1}(T_1)$ is complete in T, hence normal in T.

1.3. Composition Series and Examples

These results allow the construction of composition series for t-designs, which are helpful in constructions. We say a t-design is simple if it has no non-trivial normal subdesigns, strongly simple if it has no non-trivial quasi-normal subdesigns. For any block Φ of a t-design T, the t-design $T' = \langle \{\Phi\}, \Phi \rangle$ is, of course, always quasi-normal in T, but T/T' is simply T again by direct computation, and so such one-block designs are considered trivial. A simple design with $\lambda = 1$ is strongly simple, also, because a quasi-normal subdesign would also have $\lambda = 1$.

Given a t-design T, let $T_1 = T$, and for $i > 1$, define T_i recursively as a maximal normal subdesign of T_{i-1}, until some T_m is simple, so

$$T = T_1 \rhd T_2 \rhd \ldots \rhd T_m \quad .$$

Proposition 1.2.4 and the maximality of T_j in T_{j-1} imply the simplicity of T_j/T_{j+1} for $j = 1, 2, 3, \ldots, m-1$. From Proposition 1.2.1, $\lambda(T_j/T_{j+1}) = \lambda(T_j)/\lambda(T_{j+1}) = 1$, so from the remark above, T_j/T_{j+1} is also strongly simple. T_m is simple by construction, but not necessarily strongly simple, so we may continue, letting T_{m+i} be a maximal quasi-normal subdesign of T_{m+i-1} for $i > 0$, until some strongly simple subdesign T_{m+q} is constructed, completing a composition series for T:

$$T = T_1 \triangleright T_2 \triangleright \ldots \triangleright T_m \tilde{\triangleright} T_{m+1} \tilde{\triangleright} \ldots \tilde{\triangleright} T_{m+q} \quad .$$

Again Proposition 1.2.4 and the maximality of T_{m+i} in T_{m+i-1} can be used, here to show directly that T_{m+i-1}/T_{m+i} is strongly simple for $i = 1, 2, \ldots, q$. We have shown

1) T_i/T_{i+1} is strongly simple for $1 \leq i \leq m+q-1$,
2) T_m is a minimal normal and simple subdesign,
3) T_{m+q} is a minimal quasi-normal and strongly simple subdesign,
4) $\lambda(T_i/T_{i+1}) = 1$ for $1 \leq i \leq m-1$,
5) $1 < \lambda(T_j/T_{j+1}) | \lambda(T)$ for $m \leq j \leq m+q-1$,
6) $\lambda(T_{m+q}) | \lambda(T)$,
7) $k(T_i/T_{i+1}) = v(T_{i+1})$ for $1 \leq i \leq m+q-1$.

These conditions suggest the importance of normal and quasi-normal subdesigns and imply that to construct all t-designs with a given k and λ, it is sufficient:

1) to know all simple designs with the given k and λ, and all simple designs with $\lambda = 1$ or

1') to construct all strongly simple designs with λ' dividing the given λ,

and 2) to be able to solve the extension problem for t-designs, i.e., given two t-designs T_1 and T_2 such that

$\Pi(T_1) \in B(T_2)$, construct all designs T such that $T_1 \mathrel{\tilde{\lhd}} T$ and $T/T_1 = T_2$ for some set of codesigns of T_1.

Neither of problems 1) or 1') have been solved except under very special circumstances, but 2) is solved completely by the following generalization of a result of Hanani:

Theorem 1. 3. 1. Let T_1 and T_2 be any two t-designs such $\Pi(T_1) \in B(T_2)$. Then if $B(T_2) = \{\Phi_i\}_{i=1}^{b_2}$ (say $\Pi(T_1) = \Phi_1$), $T_1' = T_1$ and for each $i > 1$ T_i' is a t-design with the same parameters v, k, λ as T_1, written on the points of Φ_i, then

$$T = \langle B(T_1') \cup B(T_2') \cup B(T_3') \cup \ldots \cup B(T_{b_2}'),\ \Pi(T_2)\rangle$$

is a t-design such that a) $T_1 \mathrel{\tilde{\lhd}} T$, and b) T/T_1 by the codesigns $\{T_i'\}$ is T_2. Conversely, any t-design T with properties a) and b) can be constructed in this way.

Proof: Let A be a t-set of $\Pi(T) = \Pi(T_2)$. A lies on $\lambda(T_2)$ blocks of T_2, and for each such block Φ_j, A lies on $\lambda(T_2)$ blocks of T_j', each of which is a distinct block of T, hence A lies on $\lambda(T_1) \cdot \lambda(T_2) = \lambda(T)$ blocks of T, hence T is a t-design. By construction, $\{T_i'\}$ is a set of codesigns whose block sets partition $B(T)$, hence $T_1 \mathrel{\tilde{\lhd}} T$. By definition (see Proposition 1. 2. 1), T/T_1 by the codesigns $\{T_i'\}_{i=1}^{b_2}$ is $\langle\{\Pi(T_i)\}_{i=1}^{b_2},\ \Pi(T)\rangle$. But $\Pi(T) = \Pi(T_2)$ by construction, also $\Pi(T_i) = \Phi_i$ by construction, hence

$$T/T_1 = \langle\{\Phi_i\}_{i=1}^{b_2},\ \Pi(T_2)\rangle \equiv T_2,$$

Conversely, suppose there is a t-design T^* such that $T_1 \tilde{\triangleleft} T^*$ and $T^*/T_1 = T_2$. Let β be the canonical regular block homomorphism from T^* onto T^*/T_1 (Proposition 1. 2. 2). Since $T^*/T_1 = T_2$, β maps T^* onto T_2. Let $B(T_2) = \{\Phi_i\}_{i=1}^{b_2}$, and set $T_i' = \langle \beta^{-1}(\Phi_i), \Phi_i \rangle$ (see Proposition 1. 1. 4). Then the construction of the first part of the argument yields T^* again.

In particular, the problem of "extending" T_1 by T_2 can always be solved in at least one way if $\Pi(T_1) \in B(T_2)$, if only by letting each T_i' be isomorphic to T_1, only written on the points of some other block of T_2. On the other hand, the T_i' need not be isomorphic to T_1, but since each T_i' is itself quasi-normal in the constructed design, this means that the composition series for t-designs are not unique. For example, if T_i' is not isomorphic to T_j' for some i and j, then the design T would have the two non-equivalent series

$$T \tilde{\triangleright} T_i'$$

$$T \tilde{\triangleright} T_j'$$

The set of composition series is, however, an isomorphic invariant, and this fact can be used to distinguish between non-isomorphic t-designs with identical parameters. For example, Figure 1a shows the blocks of a 2-design D with parameters $v = 13$, $b = 52$, $r = 12$, $k = 3$, $\lambda = 2$. The blocks on each line of the figure form a subdesign D_4 with $b = v = 4$, $k = r = 3$, $\lambda = 2$, hence a normal subdesign, since the subdesigns of the various lines are all codesigns. The quotient design is shown in Figure 1b. It is isomorphic to the projective plane of order 3, hence we have the composition series for D:

$$D \rhd D_4, \qquad D/D_4 \approx PG(2,3) \quad .$$

(Each of D_4, $PG(2,3)$ is strongly simple).

Figure 2 shows the blocks of another 2-design, D', with the same parameters. This design is simple, as will be shown in general later, but not strongly simple, as the blocks in the two columns on the left form a Steiner triple system S with parameters $b = 26$, $v = 13$, $r = 6$, $\lambda = 1$, as do the blocks in the two columns on the right. The quotient design is the 2-design T with 13 points and two identical blocks consisting of all 13 points each:

$$D' \overset{\sim}{\rhd} S \qquad D'/S = T \quad .$$

The two designs D and D' are then clearly nonisomorphic.

abd	abj	adj	bdj
bce	bck	bek	cek
cdf	cdl	cfl	dfl
deg	dem	dgm	egm
aef	aeh	afh	efh
bfg	bfi	bgi	fgi
cgh	cgj	chj	ghj
dhi	dhk	dik	hik
eij	eil	ejl	ijl
fjk	fjm	fkm	jkm
agk	agl	akl	gkl
bhl	bhm	blm	hlm
aci	acm	aim	cim

1a

abdj
bcek
cdfl
degm
aefh
aefh
cghj
dhik
eijl
fjkm
agkl
bhlm
acim

1b

Figure 1

aci	bfe	dlj	hkg
bdf	cgf	emk	ilh
cek	dhg	fal	jmi
dfl	eih	gbm	kaj
egm	fji	hca	lbk
fha	gkj	idb	mcl
gib	hlk	jec	adm
hjc	iml	kfd	bea
ikd	jam	lge	cfb
jle	kba	mhf	dgc
kmf	lcb	aig	ehd
lag	mdc	bjh	fie
mbh	aed	cki	gjf

Figure 2

CHAPTER II

t-PLY HOMOGENEOUS GROUPS AND t-DESIGNS

There is a class of permutation groups closely associated with certain t-designs. These are the permutation groups whose induced representation on unordered t-sets is transitive. If a t-design T admits such a group of automorphisms, its structure becomes much more regular, and in particular the normal subdesigns of T can be determined solely by examination of the group. Other relations exist between the design and various subgroups of the group.

2.1. t-ply Homogeneous Groups

Let G be a group of permutations of the set Ω. Then for $\Phi \subseteq \Omega$ and $\alpha \in G$, we write

$$\Phi^{\alpha} = \{x^{\alpha} \mid x \in \Phi\},$$

so G acts as a permutation group on the unordered k-sets of Ω for any $k \leq |\Omega| = n$. If some union of orbits of the representation of G on k-sets forms a t-design, then G is by definition an automorphism group of that design.

<u>Definition 2.1.1.</u> G is called <u>t-ply homogeneous</u> if for any two t-sets of Ω, say Φ_1 and Φ_2, there exists an $\alpha \in G$ such that $\Phi_1^{\alpha} = \Phi_2$. The subgroup sending Φ into itself is called G_{Φ}.

Clearly any t-ply transitive group is t-ply homogeneous, but there do exist groups which are t-ply homogeneous but not t-ply transitive, for example, the group on seven letters generated by

$$x = (a\,b\,c\,d\,e\,f\,g)$$
$$y = (a)\,(b\,c\,e)\,(d\,g\,f) \quad .$$

Such groups are called strictly t-ply homogeneous.

t-ply homogeneous groups have been studied for some time, e.g., in [2]. D. R. Hughes [7] has shown that a t-ply homogeneous group is (t-1)-ply transitive if the group is of sufficiently large degree. A relation between t-ply homogeneous groups and t-ply transitive groups is given by

Proposition 2.1.2. Let G be t-ply homogeneous on the set Ω. Then G is t-ply transitive if and only if for some t-set $B \subset \Omega$, $G_B^B \approx S^B$, the symmetric group on B.

Proof: The necessity of the condition is obvious. To show its sufficiency, let $B = \{b_i\}_{i=1}^{t}$. We shall show that for any ordered t-set $C = \{c_i\}_{i=1}^{t}$, there is an $\alpha \in G$ such that $b_i^{\alpha} = c_i$ for $1 \leq i \leq t$. Since G is t-ply homogeneous, there is a $\beta \in G$ such that $B^{\beta} = C$. Let $b_{i'} = c_i^{\alpha^{-1}}$. By hypothesis, then, there is a $\gamma \in G_B$ such that $b_i^{\alpha} = b_{i'}$ and if α is set equal to $\gamma\beta$, $b_i^{\alpha} = c_i$ for $1 \leq i \leq t$, and G is t-ply transitive.

For t = 2, the condition is readily applied:

Corollary 2.1.3. A 2-ply homogeneous group is 2-ply transitive if and only if it has even order.

Proof: If G has even order, there is an involution $\alpha \in G$:

$$\alpha = (a, b), \ldots$$

and then

$$G^{\{a,b\}}_{\{a,b\}} = \{I, (a,b)\} = S^{\{a,b\}},$$

so G is double transitive.

Conversely, if G is double transitive of degree n, the even number n(n-1) divides the order of G, since it is the index of the stabilizer of two points.

Proposition 2.1.4. If the group G of degree n is strictly 2-ply homogeneous on Ω, then

i) G has rank 3

ii) G is primitive

iii) G_1 has orbits of length 1, $\frac{n-1}{2}$ $\frac{n-1}{2}$

iv) $n \equiv 3\ (4)$

v) G is solvable

vi) n is a prime power p^r

vii) G contains a regular normal minimal elementary Abelian subgroup N.

Conversely, any rank 3 group of odd order is strictly 2-ply homogeneous.

Proof: i) If i and j are any two distinct points of Ω and a is a third point, there exists $\alpha \in G: \{a, i\}^\alpha = \{a, j\}$. If $a^\alpha = a$, $i^\alpha = j$, set $\beta = \alpha$. If $a^{\alpha} = j$, $i^{\alpha} = a$, set $\beta = \alpha^2$. Then β takes i into j and G is transitive. If $1 \in \Omega$, we compute the orbits of G_1. For $2 \in \Omega - \{1\}$, set $\Gamma = 2^{G_1}$. $\Gamma \neq \Omega - \{1\}$, because G is not doubly transitive, so we can pick a $j \in \Omega - \{1\} - \Gamma$. If $\Gamma \cup \{1\} \cup \{j\} = \Omega$, G_1 has those three sets as orbits so G is of rank 3. Otherwise, there is a further point k in $\Omega - \{1, j\} - \Gamma$. We wish to show the existence of an element fixing 1 and carrying j into k, so G_1 will have the three orbits 1, Γ, j^{G_1}.

$k \notin \Gamma$, so there is no element of form

$$\begin{pmatrix} 1 & 2, & \ldots \\ 1 & k, & \ldots \end{pmatrix}$$

in G, hence an element taking (1, 2) into (1, k) must be of form

$$\alpha_1 = \begin{pmatrix} 1 & 2, & \ldots \\ k & 1, & \ldots \end{pmatrix} .$$

Arguing identically on j, we get the element

$$\alpha_2 = \begin{pmatrix} 1 & 2, & \ldots \\ j & 1, & \ldots \end{pmatrix}$$

in G, and

$$\alpha_2^{-1} \alpha_1 = \begin{pmatrix} 1 & j, & \ldots \\ 1 & k, & \ldots \end{pmatrix}$$

is in G_1, hence G has rank 3.

ii) A rank 3 group of odd order is primitive (Higman, [6]).

iii) 16.5 of Wielandt, [8] implies the orbits of G_1 other than $\{1\}$ have the same length.

iv) G is transitive on unordered pairs. There are $n(n-1)/2$ of these. n and this number must divide $|G|$, hence both are odd, and $n \equiv 3 \pmod 4$.

v) Feit-Thompson.

vi) 11.5 of Wielandt, [8].

vii) 11.5 of Wielandt, [8].

Conversely, suppose G is a rank 3 group of odd order. 16.5 of Wielandt implies that the lengths of the orbits of G_0 are $1, (n-1)/2$, $(n-1)/2$ (where n is the degree of G). 3.2 of Wielandt implies that the length of the orbit of $\{0,1\}$ by G is $[G:G_{\{0,1\}}]$. There is no element of G interchanging 0 and 1, so $G_{\{0,1\}} = G_{0,1}$, hence $[G:G_{\{0,1\}}]$ $= [G:G_{0,1}] = [G:G_0] \cdot [G_0:G_{0,1}] = n \cdot \frac{n-1}{2}$, whichever orbit of G_0 1 lies in. Hence G carries $\{0,1\}$ into $n(n-1)/2$ different unordered pairs. Since there are only $n(n-1)/2$ unordered pairs in all, G is 2-ply homogeneous, and strictly so because $|G|$ is odd.

There are many such groups, for example: Let G be the sharply doubly transitive group of linear substitutions $z \to az + b$ $(a \neq 0)$ in a near-field K, where K is of order $p^r \equiv 3$ (4). Then G has order $p^r(p^r - 1) \equiv 2$ (4). This is twice an odd number, so G has a subgroup G^* of index 2, and G^* is strictly (sharply) 2-ply homogeneous on the points of K.

2.2. t-designs with t-ply Homogeneous Groups

Theorem 2.2.1. Let G be a t-ply homogeneous permutation group on Ω. Then for any $\Phi \subseteq \Omega$, the pair

$$T = \langle \{\Phi^\alpha\}_{\alpha \in G}, \Omega \rangle$$

is a t-design admitting G as an automorphism group with parameters

$$v = |\Omega|, \qquad b = [G:G_\Phi], \qquad k = |\Phi|,$$

$$r = \frac{k}{v}[G:G_\Phi], \qquad \lambda = \binom{k}{t}\frac{|G_{\{1,2,\ldots,t\}}|}{|G_\Phi|}.$$

Conversely, any t-design T' admitting G as an automorphism group is a union of designs of this form.

Proof: By definition of G as a permutation group, $\{\Phi^\alpha\}$ is a set of k-subsets of Ω. $[G:G_\Phi]$ is the length of the orbit of Φ under G, hence the number of distinct sets Φ^α is $[G:G_\Phi]$.

Let A be a t-set of Ω. We wish to count the number of distinct sets Φ^α such that $A \subseteq \Phi^\alpha$. For any such set Φ^α, $(\Phi^\alpha)^\beta = \Phi^\alpha$ for any $\beta \in G_{\Phi^\alpha} = \alpha^{-1} G_\Phi \alpha$, so there are $|G_\Phi|$ group elements for each such set Φ^α.

We now count the number of group elements α such that $A \subseteq \Phi^\alpha$. But for each such α, $A^{\alpha^{-1}} \subseteq \Phi$, so we may count the number of group elements α such that $A^\alpha \subseteq \Phi$. But there are $\binom{k}{t}$ t-sets of Φ, and the image of A can be any one of them. Furthermore, for each possible image, there are $|G_A| = |G_{\{1,2,\ldots,t\}}|$ group elements sending

A into that image. Hence there are $\binom{k}{t} \cdot |G_{\{1,2,\ldots,t\}}|$ group elements α in all with $A \subseteq \Phi^{\alpha}$, or

$$\binom{k}{t} \frac{|G_{\{1,2,\ldots,t\}}|}{|G_{\Phi}|}$$

such distinct sets Φ^{α}. Thus T is a t-design, and the value for r follows by similar computation.

For the converse, suppose G has orbits $\Gamma_1, \ldots, \Gamma_u$ on the blocks of T'. Picking arbitrary blocks $\Phi_1, \Phi_2, \ldots, \Phi_u$ such that $\Phi_i \in \Gamma_i$ for $1 \leq i \leq u$, the first statement of the theorem shows that $T_i = \langle \Gamma_i, \Omega \rangle$ must be a subdesign (quasi-complete) of T', and since $\Gamma_i \cap \Gamma_j = \phi$ if $i \neq j$, $T' = \langle \bigcup_i \Gamma_i, \Omega \rangle$ by construction.

The t-design constructed in Theorem 2. 2. 1 is called the <u>action</u> of G on Φ, written Φ^G. Theorem 2. 2. 1 has many applications to t-designs admitting such groups of automorphisms. For the remainder of this chapter, let T be a t-design, and let G be a t-ply homogeneous group of automorphisms of T.

One immediate and useful result is

<u>Theorem 2. 2. 2.</u> <u>If</u> $\lambda(T) = 1$ <u>and</u> Φ <u>is any block, then</u> G_{Φ}^{Φ} <u>is t-ply homogeneous.</u>

<u>Proof:</u> Let A and B be any two t-sets of Φ. There is an $\alpha \in G$ such that $A^{\alpha} = B$. Alpha must then send all blocks containing A into blocks containing B. But since $\lambda = 1$, Φ is the only block containing A, also the only block containing B. Therefore α must send Φ into Φ, so $\alpha \in G_{\Phi}$, which is then t-ply homogeneous.

The normal structure of t-designs with such groups is readily determined. The following hold for any t-design:

Lemma 2.2.3. Let T_1 and T_2 be two complete subdesigns of T. Then $T' = \langle B(T_1) \cap B(T_2), \Pi(T_1) \cap \Pi(T_2) \rangle$ is either a trivial pair with no blocks and less than t points, or a complete subdesign of T.

Proof: If $\Pi = \Pi(T_1) \cap \Pi(T_2)$ has t or more points, there is a t-set $A \subseteq \Pi(T_1) \cap \Pi(T_2)$. Since T_1 and T_2 are both complete, the $\lambda(T)$ blocks of T containing A are all in both T_1 and T_2, so there are $\lambda(T)$ blocks in $B(T_1) \cap B(T_2) = B$. This also holds for any t-set in Π, so T' is a complete subdesign.

By simple induction and the associativity of intersections, Lemma 2.2.3 can be extended to any finite number of complete subdesigns, so for any set of complete subdesigns $T_1, T_2, \ldots, T_i$, $\bigcap_i T_i = \langle \bigcap_i B(T_i), \bigcap_i \Pi(T_i) \rangle$, is always either a complete subdesign or no blocks and less than t points.

Definition 2.2.4. For any t-set $A \subseteq \Pi(T)$, let $T_1(A)$, $T_2(A), \ldots, T_n(A)$ be the complete subdesigns of T such that $A \subseteq \Pi(T_i(A))$. Then the subdesign generated by A is written $T(A)$ and defined to be $\bigcap_i T_i(A)$.

This is, of course, always a complete subdesign, because $A \subseteq \bigcap_i \Pi(T_1(A))$, which then has at least t-points. Returning now to the special case of a t-design with t-ply homogeneous group, the following results show the importance of the subdesigns of the form T(A).

Proposition 2.2.5. For any $\alpha \in G$, $[T(A)]^{\alpha} = T(A^{\alpha})$. In particular, T(A) is isomorphic to T(B) for any t-sets $A, B \subseteq \Pi(T)$.

Proof: Alpha sends every complete subdesign containing A into one containing A^{α}, hence $[T(A)]^{\alpha} \supseteq T(A^{\alpha})$. By the same argument on $T(A^{\alpha})$, $[T(A^{\alpha})]^{\alpha-1} \supseteq T(A)$, or $T(A^{\alpha}) \supseteq [T(A)]^{\alpha}$, hence $T(A^{\alpha}) = [T(A)]^{\alpha}$. To show the isomorphism of T(A) and T(B), let β be any automorphism sending A into B. Then $[T(A)]^{\beta} = T(A^{\beta}) = T(B)$, and β is the required isomorphism.

Theorem 2.2.6. Let T_1 be any complete subdesign of T (or T itself). Then for any $A \subseteq \Pi(T_1)$, $T(A) \lhd T_1$.

Proof: We shall show that the various distinct subdesigns $T(A_i)$ (A_i a t-set in T_1) are codesigns whose block sets partition $B(T_1)$. From Proposition 2.2.4, they are all isomorphic, hence they are codesigns. Their block sets obviously exhaust $B(T_1)$. Suppose now that B(T(A)) and B(T(B)) have a block Φ in common. But then there would be a t-set $C \subseteq \Phi$, and since T(A) and T(B) are complete, by definition $T(C) \subseteq T(A) \cap T(B)$. Since T(C) is isomorphic to T(A), also to T(B), the only possible conclusion is T(B) = T(C) = T(A), and so if $T(A) \neq T(B)$, $B(T(A)) \cap B(T(B)) = \phi$, and the block sets partition $B(T_1)$, and all subdesigns T(A) are normal in T_1.

Note that if $\lambda(T) = 1$, $T(A) = \langle\{\Phi\}, \Phi\rangle$, where Φ is the single block containing A. For $\lambda(T) > 1$, T(A) must be non-trivial, and we have

Theorem 2.2.7. If $\lambda(T) > 1$, T is simple if and only if T contains no non-trivial complete subdesigns.

Proof: Every normal subdesign is complete, so the condition is sufficient. Conversely, suppose T contains a non-trivial proper complete subdesign T_1. Then for any t-set $A \subseteq \Pi(T_1)$, $T(A) \subseteq T_1 \subset T$, so by Theorem 2.2.6, T(A) is a non-trivial normal subdesign of T.

Theorem 2.2.8. Each T(A) is simple.

Proof: Suppose $T_1 \lhd T(A)$. Then $\lambda(T_1) = \lambda(T(A)) = \lambda(T)$. If B is any t-set in $\Pi(T_1)$, $T(B) \subseteq T_1$ by definition, and from Theorem 2.2.6, $T(B) \lhd T_1 \lhd T(A)$. Since T(B) is isomorphic to T(A) from Proposition 2.2.5, we can only have $T(B) = T_1 = T(A)$. Therefore every normal subdesign of T(A) is T(A), and T(A) is simple.

Theorem 2.2.9. If $\lambda(T) > 1$, every simple complete subdesign T_1 of T is a T(A) and so is normal in T. In particular, if T is simple, $T = T(A)$ for every t-set $A \subseteq \Pi(T)$.

Proof: Let A_1 be a t-set in $\Pi(T_1)$. From Theorem 2.2.6 applied to T_1, $T(A_1) \lhd T_1$, and since $\lambda(T(A_1)) = \lambda(T_1) = \lambda(T) > 1$, $T(A_1)$ is a non-trivial normal subdesign of T_1, hence the simplicity of T_1 implies that $T_1 = T(A_1)$.

Given a design T, the subdesigns of the form T(A) are relatively simple to construct: one simply takes all the blocks containing A and all the points on those blocks, then continues the same process, using new t-sets from the new blocks to produce newer blocks, until each t-set in the points produced already occurs $\lambda(T)$ times in the blocks already found. However, the subdesigns $T(A)$ can be found directly from examination of the automorphism group:

<u>Theorem 2.2.10.</u> <u>If</u> T_1 <u>is a simple complete subdesign of</u> T,

1) $G_{\Pi(T_1)}$ <u>is t-ply homogeneous on</u> $\Pi(T_1)$,

2) $G_\Phi \subseteq G_{\Pi(T_1)}$ <u>for any</u> $\Phi \in B(T_1)$, <u>and</u>

3) $G_A \subseteq G_{\Pi(T_1)}$ <u>for any t-set</u> $A \subseteq \Pi(T_1)$.

<u>Proof:</u> T_1 is equal to $T(A)$ for any t-set $A \subseteq \Pi(T_1)$. Therefore any automorphism which sends a t-set $A \subseteq \Pi(T_1)$ into another t-set $B \subseteq \Pi(T_1)$ also sends $T_1 = T(A)$ into $T(B) = T_1$, i.e., fixes T_1.

1) Let A, B be two t-sets in $\Pi(T_1)$. There is an $\alpha \in G$ such that $A^\alpha = B$. By the above argument $T_1^\alpha = T_1$, so $\alpha \in G_{\Pi(T_1)}$. Since A, B were chosen arbitrarily, $G_{\Pi(T_1)}$ must be t-ply homogeneous on $\Pi(T_1)$.

2) Let $\alpha \in G_\Phi$ be arbitrary. Alpha then sends any t-set of Φ into another t-set of Φ. Since $\Phi \subseteq \Pi(T_1)$, these two t-sets are in $\Pi(T_1)$, so the above argument applies to α here, so $\alpha \in G_{\Pi(T_1)}$. Since α was chosen arbitrarily, $G_\Phi \subseteq G_{\Pi(T_1)}$.

3) Any $\alpha \in G_A$ sends A into itself, so again by the above argument, $\alpha \in G_{\Pi(T_1)}$.

Theorem 2.2.11. Let T_1 be a simple complete subdesign of T. Then for each block Φ of T_1, $T_1 = \Phi^{G_{\Pi(T_1)}}$.

Proof: From Theorem 2.2.10, $H = G_{\Pi(T_1)}$ is t-ply homogeneous on $\Pi(T_1)$, hence by Theorem 2.2.1, $T' = \Phi^H$ is a t-design and a subdesign of T. We now calculate the parameters of T' from Theorem 2.2.1.

$v(T') = v(T_1)$, since H is transitive on $\Pi(T_1)$.

$k(T') = k(T_1)$ by definition.

$\lambda(T') = \binom{k}{t}\dfrac{|H_A|}{|H_{\Phi}|}$ where A is any t-set in $\Pi(T_1)$.

But from Theorem 2.2.10, $H_A = G_A$ and $H_{\Phi} = G_{\Phi}$, hence

$$\lambda(T') = \binom{k}{t}\frac{|G_A|}{|G_{\Phi}|} = \lambda(T) = \lambda(T_1) \quad .$$

T' is then a subdesign of T_1 with the same parameters, hence must be all of T_1.

These results lead to a converse of Theorem 2.2.10 and a characterization of simple t-designs purely in terms of their automorphism groups:

Theorem 2.2.12. Let A be any t-set of $\Pi(T)$, let Φ be any block of T, and suppose $\lambda(T) > 1$. Then T is simple if and only if G contains no proper subgroup H such that

1) $G_A \subseteq H$,

2) $G_{\bar{\Phi}} \subseteq H$,

3) H <u>is t-ply homogeneous on the points in the t-sets</u> A^H.

<u>Proof</u>: If T has a proper normal subdesign T^*, then T^* has a codesign T' containing $\bar{\Phi}$ and so A. Since T' is also normal, it is complete, therefore $T(A) \subseteq T'$ is a proper simple complete subdesign of T, and from Theorem 2.2.10, $H = G_{\Pi(T(A))}$ satisfies 1), 2), and 3). Conversely, suppose such an H exists. Then $\bar{\Phi}^H$ is a subdesign by Theorem 2.2.1.

$$\lambda(\bar{\Phi}^H) = \binom{k}{t} \frac{|H_A|}{|H_{\bar{\Phi}}|} = \binom{k}{t} \frac{|G_A|}{|H_{\bar{\Phi}}|} = \lambda(T)$$

as before, so $\bar{\Phi}^H$ is complete. But then $T(A) \subseteq \bar{\Phi}^H$ is a proper non-trivial normal subdesign.

We can also relate the quotient designs of T to the group G:

<u>Theorem 2.2.13</u>. <u>If</u> T_1 <u>is a simple normal subdesign of</u> T, <u>then</u>

$$T/T_1 = \left[\Pi(T_1)\right]^G .$$

<u>Proof</u>: $T_1 = T(A_1)$ for any t-set A_1 in $\Pi(T_1)$, furthermore, all codesigns of T_1 must also be of the form $T(A_i)$, $1 \leq i \leq m$. Therefore,

$$T/T_1 = \left\langle \{\Pi(T(A_i))\}_{i=1}^{m} , \quad \Pi(T) \right\rangle .$$

But from Proposition 2. 2. 5, for every $\alpha \in G$ and every i, $[T(A_i)]^\alpha = T(A_i^\alpha)$, so $[\Pi(T(A_i))]^\alpha = \Pi(T(A_i^\alpha))$, and G permutes the sets $\Pi(T(A_i))$. Furthermore, G is t-ply homogeneous, hence for each i there is an $\alpha_i \in G$ such that $A_1^{\alpha_i} = A_i$, so $[\Pi(T(A_1))]^{\alpha_i} = \Pi(T(A_i))$. Thus the sets $\{[\Pi(T_1)]^\alpha\}_{\alpha \in G}$ are exactly the sets $\{\Pi(T(A_i))\}_{i=1}^m$, and $T/T_1 = [\Pi(T_1)]^G$.

Theorem 2. 2. 14. *Let T_1 be a simple complete subdesign of T. Then*

$$G(T) \subseteq G(T/T_1).$$

Proof: $T_1 = T(A)$ for some t-set A. By Proposition 2. 2. 5, any automorphism of T permutes the various sets $T(A_i)$, hence from Theorem 2. 2. 13, every automorphism of T permutes the blocks of T/T_1.

Note that this inequality may indeed be strict, because a permutation on $\Pi(T)$ could permute the sets $\Pi(T(A_i))$, but not the actual blocks of the subdesigns $T(A_i)$. However, we do have that the group of T/T_1 is t-ply homogeneous, hence the previous analysis applies to T/T_1 and $G(T/T_1)$, and by continuation, the complete normal structure of T may be derived from the groups of the various normal and quotient designs.

2. 3. Some Constructions of t-designs from Known Groups

By Theorem 2. 2. 1, many t-designs can be constructed from a given t-ply homogeneous group, but many of these will be trivial or uninteresting, because the parameters b and λ are exceedingly large by comparison with v and k respectively. For example, the Mathieu group M_{12} is quintuply transitive on 12 letters, and it is not difficult to show that M_{12} is t-ply homogeneous for all t between 1 and 12, except for $t = 6$. Thus $\Phi^{M_{12}}$ will be the trivial design of all $|\Phi|$-sets, unless $|\Phi| = 6$, in which case $\Phi^{M_{12}}$ is one of the well-known few 5-designs. Inspection of the formulae for b and λ in Theorem 2.2.1 yields the information that a design of the form Φ^G will only have reasonable parameters if G_Φ is quite large in G. In the extreme case of Φ^G being a symmetric design, we must have $|G_\Phi| = |G_0|$. Such subgroups appear to be quite scarce. For example, we have

Proposition 2.3.1. Let G be a 2-ply transitive permutation group on Ω, and suppose G has a subgroup H such that H is 2-ply transitive on an orbit Γ, and if 0, 1 are any two points of Γ, $|H| = |G_0|$ and $G_{0,1} \subset H$. Then $v = |\Omega| = p^{2r} + p^r + 1$ for some prime p and some integer $r > 0$, $k = |\Gamma| = p^r + 1$, and G is isomorphic to a subgroup of $LF(3, p^r)$.

Proof: We apply Theorem 2.2.1 to Γ^G. Since $H \subseteq G_\Gamma$, $b(\Gamma^G) = [G:G_\Gamma] \leq [G:H] = v$, so $b \leq v$, also from 2.2.1,

$$\lambda = \binom{k}{2} \frac{|G_{\{0,1\}}|}{|G_\Gamma|} = \frac{k^2 - k}{2} \cdot \frac{2 \cdot |G_{0,1}|}{|G_\Gamma|} .$$

G_Γ is surely doubly transitive on Γ, so

$$\lambda = \frac{(k^2 - k)\,|G_{0,1}|}{(k^2 - k)\,|G_{0,1,\Gamma}|} \leq \frac{|H_{0,1}|}{|H_{0,1,\Gamma}|} = 1 .$$

Since $\lambda > 0$, $\lambda = 1$, $G_\Gamma = H$, $b = v$, and Γ^G is a symmetric 2-design with $\lambda = 1$, otherwise known as a projective plane. Since the collineation (automorphism) group of Γ^G contains the doubly transitive (on points) group G, Γ^G must be Desarguesian and can be coordinatized by a finite field with p^r elements for some prime p and integer $r > 0$. The group of such a plane is $LF(3, p^r)$, and so G is by construction isomorphic to a subgroup of $LF(3, p^r)$.

Some interesting 2-designs (balanced incomplete block designs), can be constructed from known groups. A doubly primitive group is one which is doubly transitive and whose subgroups fixing a point are primitive on the remaining points.

Theorem 2.3.2. Let G be doubly transitive but not doubly primitive, and let Γ be a non-trivial block of G_0. Then Γ^G is a block design with $\lambda | k-1$.

Proof: Let 1 be a point of Γ. It is well known (see, e.g., [5], pp. 64-65) that $G_{0,\Gamma}$ is transitive on Γ, and that $G_{0,1} \subseteq G_{1,\Gamma}$. Hence the group $G_{0,\Gamma}$ of order $k \cdot |G_{0,1}|$ is a subgroup of G_Γ, G_Γ is transitive on Γ, and from 2.2.1,

$$\lambda = \binom{k}{2} \frac{|G_{\{0,1\}}|}{|G_\Gamma|} = \frac{k(k-1)|G_{0,1}|}{|G_\Gamma|} = \frac{(k-1)|G_{0,1}|}{|G_{\Gamma,1}|}$$

$$= \frac{k-1}{\left(\frac{G_{\Gamma,1}}{G_{0,1}}\right)}.$$

Since $G_{0,1} \subset G_{\Gamma,1}$, the denominator is an integer, and $\lambda | k-1$.

CHAPTER III

BLOCK DESIGNS WITH $k = 3$ AND $\lambda = 2$

In this chapter we consider 2-designs (balanced incomplete block designs) with $k = 3$, $\lambda = 2$, especially with regard to their normal structures and automorphism groups. Let D be such a design, and let G be its group of automorphisms. General equations on parameters of block designs yield

$$r = v - 1,$$

$$b = \frac{v(v-1)}{3} .$$

Thus necessarily $v \not\equiv 2 \pmod 3$. Bhattacharya [1] has shown that this condition is sufficient for the existence of D with v points by constructing such designs for $v = 6t + 4$ and $v = 6t$ for all t. Designs with $v = 6t + 1$ and $v = 6t + 3$ can readily be constructed by taking each block of a Steiner triple system on v points twice. Steiner triple systems with $6t + 1$ and $6t + 3$ points exist for all t (see Hall [4], pp. 237-241). Thus designs exist for all $v \not\equiv 2 \pmod 3$. However, with a few exceptions, the automorphism groups of the known designs are relatively small.

3.1. The Operator τ

Definition 3.1.1. The function τ, from the set of unordered pairs of points of D into the set of pairs of points of D is defined for a pair (a, b) $(a \neq b)$ as follows:

Let the two blocks of D containing a and b be

$$\begin{matrix} a & b & c \\ a & b & d \end{matrix} \quad .$$

Then $\tau(a, b) = (a, b)^{\tau} = (c, d)$. If $c = d$, we still consider (c, c) as a "pair." Tau is undefined on such pairs.

Theorem 3. 1. 2. A permutation α of the points of D is an automorphism of D if and only if α commutes with τ on the unordered pairs of points.

Proof: Let (a, b) be an arbitrary pair, lying on the two blocks

$$\begin{matrix} a & b & c \\ a & b & d \end{matrix} \quad .$$

If α is an automorphism of D, we have the two blocks

$$\begin{matrix} a^{\alpha} & b^{\alpha} & c^{\alpha} \\ a^{\alpha} & b^{\alpha} & d^{\alpha} \end{matrix} \quad ,$$

hence $(a, b)^{\tau\alpha} = (c, d)^{\alpha} = (c^{\alpha}, d^{\alpha}) = (a^{\alpha}, b^{\alpha})^{\tau} = (a, b)^{\alpha\tau}$ and α commutes with τ on the unordered pairs.

Conversely, suppose $\alpha\tau = \tau\alpha$ on unordered pairs. We need to show that if (a b c) is any block of D, then $a^{\alpha}\ b^{\alpha}\ c^{\alpha}$ is also a block of D. Let (a b d) be the other block of D containing a and b, and let

$$\begin{matrix} a^{\alpha} & b^{\alpha} & i \\ a^{\alpha} & b^{\alpha} & j \end{matrix}$$

be the two blocks of D containing a^{α} and b^{α}. Then

$$(i,j) = (a^{\alpha}, b^{\alpha})^{\tau} = (a,b)^{\alpha\tau} = (a,b)^{\tau\alpha} = (c,d)^{\alpha} = (c^{\alpha}, d^{\alpha}) \quad ,$$

hence either $c^{\alpha} = i$ or $c^{\alpha} = j$, but in either case $a^{\alpha}\, b^{\alpha}\, c^{\alpha}$ is a block of D.

Thus the automorphism group of D can be computed directly from τ. Tau has a further property of general interest: repeated applications of τ to pairs from a set Σ will yield the smallest complete subdesign containing Σ:

Theorem 3. 1. 3. Let $\Sigma \subseteq \Pi(D)$ be a set of at least two points with the following property: If a pair (a,b) of distinct points is in Σ, then $(a,b)^{\tau}$ is also in Σ. Then Σ is the set of points of a complete subdesign.

Proof: Let B_1 be the set of blocks of D whose points are all in Σ. We need to show that $\langle B_1, \Sigma \rangle$ is a complete subdesign of D, i. e., for any pair $x, y \in \Sigma$, $x \neq y$, there are 2 blocks of B_1 containing x and y. But by hypothesis, $(x,y)^{\tau} = (a,b)$ is in Σ, so the blocks

x y a

x y b

lie in B_1. Since $|\Sigma| \geq 2$, B_1 is non-void, and $\langle B_1, \Sigma \rangle$ is indeed a complete subdesign of D.

If S satisfies the hypothesis of 3. 1. 3, Σ is said to have "the τ-property."

3.2. Designs with 2-ply Homogeneous Groups

In this section, we assume throughout that G, the automorphism group of D, is 2-ply homogeneous on $\Pi(D)$. We wish to classify and analyze all such designs. In some cases, the problem is reduced to a similar problem for Steiner triple systems which has largely been solved (see M. Hall, [4]). (In part 4, the simple designs in the remaining cases are determined, with one possible exception.) Under the assumption of a 2-ply homogeneous group, the τ-function is much better behaved, and we can readily dispose of two important cases:

Theorem 3.2.1. If any block of D is repeated, then D is two copies of a Steiner triple system S. We have $S \precsim D$, and D/S is the trivial b=2, v=v(D), k=v, r=2, λ=2 design consisting of two blocks, each containing all points.

Proof: Suppose we have the two distinct blocks

$$x,\ y,\ z$$
$$x,\ y,\ z$$

Let (a, b, c) be any other block. We need to show that there are two blocks (a, b, c), (a, b, c). Let (a, b, d) be the other block of D containing a and b. By hypothesis there is an automorphism α carrying (x, y) into (a, b). Alpha must then carry

$$\begin{matrix} x,\ y,\ z \\ x,\ y,\ z \end{matrix} \quad \text{into} \quad \begin{matrix} a,\ b,\ c \\ a,\ b,\ d \end{matrix}\ ,$$

so $(z,z)^{\alpha} = (x,y)^{\tau\alpha} = (x,y)^{\alpha\tau} = (a,b)^{\tau} = (c,d)$, thus $c = z^{\alpha} = d$, and the block a,b,c is repeated, so all are. Trivially the set B' of blocks consisting of one block from each such pair of equal blocks contains each pair of points just once, hence $S_1 = \langle B', \Pi(D) \rangle$ is a Steiner triple system. $S_2 = \langle B'', \Pi(D) \rangle$ (where B'' is the set of the other blocks of the pairs of equal blocks) is another Steiner triple system, a codesign to S_1. Hence D can be divided into two copies of S_1, so $S_1 \tilde{\trianglelefteq} D$. The quotient design $D/S_1 = \langle \{\Pi(S_1), \Pi(S_2)\}, \Pi(D) \rangle$ $= \langle \{\Pi(D), \Pi(D)\}, \Pi(D) \rangle$ obviously is the design stated in the theorem.

Thus under these conditions the construction and structure of such designs rest upon the same problems for Steiner triple systems. If D is the Steiner triple system S doubled, then any non-normal, quasi-normal subdesign of D is a normal subdesign of S, and any normal subdesign of D is a normal subdesign of S doubled.

We now assume that no block is repeated. A further possibility is disposed of by

Theorem 3.2.2. If G is transitive on B(D), then G is doubly transitive. If G is intransitive on B(D), it has two orbits: Γ, $\Psi : \Gamma \cup \Psi = B(D)$, and $\langle \Gamma, \Pi(D) \rangle$ and $\langle \Psi, p(D) \rangle$ are quasi-normal Steiner triple subdesigns of D.

Proof: If G is transitive on B(D), $y^G = D$ for any block Φ. By Theorem 2.2.1, $\lambda(D) = \lambda(\Phi^G) =$

$$2 = \binom{3}{2} \frac{|G_{\{0,1\}}|}{|G_y|} = \frac{3 \cdot |G_{\{0,1\}}|}{|G_y|} ,$$

hence necessarily $2 \,\big|\, |G_{\{0,1\}}|$, so $2 \,\big|\, |G|$ and by Corollary 2.1.3, G is doubly transitive.

On the other hand, suppose G is intransitive on B(D). Then by the converse statement of Theorem 2.2.1, D is the union of designs of the form Φ^G. Furthermore, if $D = \bigcup_i \Phi_i^G$, $2 = \lambda(D) = \sum_i \lambda(\Phi_i^G)$, so the only possibility is that $D = \Phi_1^G \cup \Phi_2^G$, and $\lambda(\Phi_1^G) = \lambda(\Phi_2^G) = 1$, hence each Φ_i^G is a Steiner triple system, quasi-normal in D. Setting $\Gamma = B(\Phi_1^G)$ and $\Psi = B(\Phi_2^G)$ completes the proof.

For further study we can now assume that G is doubly transitive on $\Pi(D)$ and transitive on B(D) and that no block of D is repeated. Under this last condition, $(0,1)^\tau$ is always an unordered pair of distinct points, so that the repeated function τ^n is defined for all $n \geq 0$.

Proposition 3.2.3. Tau is a permutation on the unordered pairs of points, consisting of disjoint t-cycles for a fixed integer $t = t(D)$.

Proof: Consider the sequence

$$(x,y) \xrightarrow{\tau} (x,y)^\tau \xrightarrow{\tau} (x,y)^{\tau^2} \xrightarrow{\tau}, \ldots \quad .$$

Since the set of pairs of points is finite, this sequence must repeat itself eventually:

$$(x,y)^{\tau^m} = (x,y)^{\tau^n} = (w,z) \qquad (m > n) \quad .$$

Setting $t_1 = m-n$, $\qquad (w,z)^{\tau^{t_1}} = (w,z) \quad .$

If (a, b) is any other pair, let α be an automorphism with (a, b) $= (w, z)^{\alpha}$. Then $(a, b)^{\tau^{t_1}} = [(w, z)^{\alpha}]^{\tau^{t_1}} = (w, z)^{\alpha\tau^{t_1}} = (w, z)^{\tau^{t_1}\alpha} = (w, z)^{\alpha}$ (a, b), so $(a, b)^{\tau^{t_1}} = (a, b)$ for all pairs (a, b).

Let t be the smallest integer > 0 such that $(a, b)^{\tau^t} = (a, b)$ for some pair (a, b). The above argument shows that $(x, y)^{\tau^t} = (x. y)$ for all pairs (x, y), so τ can be written as disjoint t-cycles of pairs. Since τ is so defined on all unordered pairs, it is then a permutation on them.

In general, if D does not have a doubly transitive group of automorphisms, τ is not 1 - 1, τ^t is not the identity function for any $t > 0$, and τ is not a permutation. For example, the designs of Bhattacharya for $v = 6t+4$ each fail all these tests (example 3. 3. 4).

<u>Corollary 3. 2. 4.</u> Let t be the order of τ as a permutation. Then $t \mid \frac{v^2-v}{2}$.

<u>Proof:</u> Tau consists of disjoint t-cycles on the $\frac{v^2-v}{2}$ unordered pairs, so of $\frac{v^2-v}{2t}$ t-cycles, hence a priori $t \mid \frac{v^2-v}{2}$.

<u>Corollary 3. 2. 5.</u> If the automorphism α sends the pair (x_1, x_2) into itself, then α also fixes all pairs of the form $(x_1, x_2)^{\tau^s}$.

<u>Proof:</u> $((x_1, x_2)^{\tau^s})^{\alpha} = (x_1, x_2)^{\tau^s\alpha} = (x_1, x_2)^{\alpha\tau^s} = ((x_1, x_2)^{\alpha})^{\tau^s}$ $= (x_1, x_2)^{\tau^s}$.

<u>Corollary 3. 2. 6.</u> $t = 0(\tau) \leq v/2$.

Proof: Suppose $t > v/2$. Since there are $2t$ points represented in the sequence

$$(1) \qquad (x_1, x_2) \xrightarrow{\tau} (x_3, x_4) \xrightarrow{\tau} \ldots \xrightarrow{\tau} (x_{2t-1}, x_{2t}) \xrightarrow{\tau} (x_1, x_2),$$

at least one point must be repeated. Without loss of generality, we may assume that x_1 appears in two pairs (x_1, x_2) and (x_1, x_s). Since G is assumed doubly transitive, there is an involutory automorphism α which interchanges x_1 and x_2:

$$\alpha = (x_1, x_2)(\ldots.), \ldots$$

From Corollary 3. 2. 5, α then also fixes all pairs in the sequence (1). In particular, it sends the pair (x_1, x_s) into itself, i. e., $x_1^\alpha = x_1$, or $x_1^\alpha = x_s$, each of which contradicts $x_1^\alpha = x_2$.

In some cases, simply the order of τ gives a great deal of information about D:

Theorem 3. 2. 7. If $O(\tau) = 2$, then every pair of points generates a normal subdesign D' with four points. If G' is the automorphism group of D/D', and Φ is a block of D/D', then $G_{\Phi}^{\Phi} \supseteq A_4$, the alternating group on 4 letters.

Proof: Let (a, b) be any pair. From the sequence $(a, b) \xrightarrow{\tau} (c, d) \xrightarrow{\tau} (a, b)$, we obtain the blocks

a b c

a b d

a c d

b c d ,

which form a design, which in terms of Chapter 2, is $T(\{a,b\})$, hence is normal from Theorem 2.2.5. From Theorem 2.2.9, $G^{\Phi}_{\Phi} \cong G^{\{a,b,c,d\}}_{\{a,b,c,d\}}$ is t-ply homogeneous on Φ. It must be doubly transitive, from Corollary 2.1.3 and Proposition 2.1.4, so must contain A_4.

Theorem 3.2.8. If $0(\tau) = \frac{v-1}{2}$, then G is sharply doubly transitive on $\Pi(D)$.

Proof: Pick $\alpha \in G_{a,b}$. We wish to show $\alpha = 1$. Since α fixes the pair (a,b), it must either interchange the points or fix pointwise each pair in the sequence

$$(a,b) \xrightarrow{\tau} \dots \xrightarrow{\tau} \dots \xrightarrow{\tau} (a,b) \tag{1}$$

(Corollary 3.2.5). The proof of Corollary 3.2.6 implies that all the points appearing in the sequence (1) are distinct. This accounts for the images of $2t = v-1$ points, hence the one remaining point must be fixed, and so $\alpha^2 = 1$. We now need

Lemma 3.2.9. If β is an automorphism of D, $\beta^2 = 1$ and β fixes u points, then there is an integer $s \geq u-1$ such that

$$u^2 - 2u + v = 2s \cdot 0(\tau) .$$

Proof: For each unordered pair (x,y) that β fixes, β also fixes the sequence $(x,y) \xrightarrow{\tau} (\ .\ .\ ) \longrightarrow \dots \xrightarrow{\tau} (x,y)$ pair by pair. Therefore, the fixed pairs of β fall into s disjoint sets of t each, namely the sets of fixed pairs in the τ-sequences fixed by β. Now β fixes

$\frac{u^2-u}{2}$ pairs pointwise, and since $\beta^2 = 1$, β has $\frac{v-u}{2}$ transpositions. Thus β fixes $\frac{u^2-u}{2} + \frac{v-u}{2}$ pairs in all, so

$$\frac{u^2-u}{2} + \frac{v-u}{2} = s \cdot 0(\tau) ,$$

by counting the pairs fixed by β in two different ways. Since the points in a τ-sequence are all distinct, if β fixes 1, 2, . . . , u, then the u-1 sequences

$$(1,2) \xrightarrow{\tau} \ldots$$
$$(1,3) \xrightarrow{\tau} \ldots$$
$$\vdots$$
$$(1,u) \xrightarrow{\tau} \ldots$$

are all distinct and fixed by β. Hence $s \geq u - 1$.

We now apply the lemma to α. Knowing $0(\tau) = \frac{v-1}{2}$, we get

$$u^2 - 2u + v = 2s(\tfrac{v-1}{2}) , \quad \text{so}$$

$$u^2 - 2u + (v-sv+s) = 0$$

and $\quad u = \dfrac{2 \pm \sqrt{4 - 4(v-sv+s)}}{2} = 1 \pm \sqrt{1-v+sv-s} = 1 \pm \sqrt{(1-v)(1-s)}$.

$\alpha \in G_{0,1}$, so $u \geq 2$, hence $u = 1 + \sqrt{(v-1)(s-1)}$.

$$s \geq u-1, \quad \text{so} \quad s-1 \geq u-2, \quad \text{and we get}$$

$$u \geq 1 + \sqrt{(u-2)(v-1)} , \quad \text{or}$$

$$u-1 \geq \sqrt{(u-2)(v-1)}$$

$$(u-1)^2 \geq (u-2)(v-1)$$

(2) $$u^2 - (v+1)\,u + 2v-1 \geq 0\,.$$

Now we claim that if u satisfies (2) and $2 \leq u \leq v$, then $u = v$. The equation $x^2 - (v+1)\,x + 2v-1 = 0$ has solutions

$$\frac{1}{2}\,[v+1 \pm \sqrt{(v-5)(v-1)}\,].$$

$0(\tau) \geq 2$ implies $v \geq 5$, but v cannot be 5 because there is no design with $v = 5$, $k = 3$, $\lambda = 2$. Thus $v \geq 6$ and $x^2 - (v+1)\,x + 2v-1 = 0$ has two distinct real roots x_1 and x_2, where $x_2 > x_1$

Here we must have $u \leq x_1$ or $u \geq x_2$. But

$$3^2 - 3(v+1) + 2v-1 = 5 - v < 0\,, \quad \text{and}$$

$$(v-2)^2 - (v-2)(v+1) + 2v-1 = 5 - v < 0\,,$$

so we certainly must have $u < 3$ or $u > v-2$. But $v = 2 \cdot 0(\tau) + 1$ is an odd number, therefore if $\alpha \neq 1$, u must be odd also, as α cannot fix an even number of points and be an involution on an odd number of points. Therefore $u \leq 1$ or $u \geq v$. Since $\alpha \in G_{a,b}$, $u \geq 2$, so $u = v$, and $\alpha = 1$, i.e., $G_{a,b} = \{1\}$ and G is sharply doubly transitive.

Remark 3.2.10. Hans Zassenhaus has shown [9] that a doubly transitive group G can be faithfully represented as the set of linear substitutions in a near-field K: $G = \{\alpha : K \to K \mid \alpha(z) = a\,z + b;$ $a, b \in K,\ a \neq 0\}$. K has order p^r for some prime p and integer $r > 0$, so G has order $p^r(p^r - 1)$. The Sylow p-subgroup of order p^r is isomorphic with the elementary abelian additive group of K. The subgroup G_0 is isomorphic to the multiplicative group of K. Since

$$2 = \lambda = \binom{3}{2} \frac{|G_{\{0,1\}}|}{|G_{\bar{\Phi}}|} = \frac{3 \cdot 2}{|G_{\bar{\Phi}}|} ,$$

$|G_{\bar{\Phi}}| = 3$ for any block $\bar{\Phi}$. There are two cases: either $p = 3$ and $3 \nmid 1^r - 1$, or $p \neq 3$ and $3 \mid p^r - 1$. In the first case, $G_{\bar{\Phi}}$ is in the "additive" subgroup and is conjugate to the subgroup $\{1, \alpha, \alpha^2 \mid \alpha(z) = z + 1\}$, and $\bar{\Phi}$ can be identified with the points 0, 1, 2 in K. Then $\bar{\Phi}$ is a sub-near field of K, and $G_{\bar{\Phi}}$ includes the linear substitutions β, γ, and δ, where $\beta(z) = 2z$, $\gamma(z) = 2z+1$, $\delta(z) = 2z+2$, hence $G_{\bar{\Phi}}$ is of order 6 and $\lambda = 1$, contrary to assumption. In the second case, $G_{\bar{\Phi}}$ is conjugate to a subgroup of order 3 in the multiplicative group, and $\bar{\Phi}$ can be identified with the points of K 1, x, x^2 where $x^3 = 1$ in K. In this case, $\bar{\Phi}^G$ is a genuine $k = 3$, $\lambda = 2$ design.

3.3. Examples

<u>3.3.1.</u>	B(D) =	012	023	The trivial 4-point design D_4.
		013	123	$G(D) \approx S_4$, $0(\tau) = 2$. Strongly simple.

<u>3.3.2.</u>	B(D) =	012	025	The 6-point design D_6.
		013	034	$G(D) \approx LF(2, 5)$. $0(\tau) = 3 = v/2$.
		234	135	Strongly simple.
		235	124	
		450		
		451		

3.3.3. $B(D) =$

013	023	One 7-point design. G(D) is Frobenius group of order $7 \cdot 6$ [Linear substitutions $x \to a \cdot x + b$ in GF(7), $a \neq 0$]. $0(\tau) = 3 = (v-1)/2$. Simple, but not strongly simple. Each column of blocks forms a Steiner triple system $S \cong\!\!\!\lhd D$.
124	134	
235	245	
346	356	
450	460	
561	501	
602	612	

3.3.4. The designs of Bhattacharya for $v = 6t+4$. These exist for all $t \geq 0$ and are generated by the base blocks

$$A = (\infty, 0, 3t+1)$$

$$B = (0, 2t+1, 4t+2)$$

$$C_i = (0, i, 2t+1-i) \qquad i = 1, 2, \ldots, t$$

$$D_i = (0, 2i, 3t+1+i) \qquad i = 1, 2, \ldots, t$$

The sets $\{A+j, \; C_i+j, \; D_i+j\}$ for $j = 0, 1, 2, \ldots, 6t+3$

$\{B+j\}$ for $j = 0, 1, \ldots, 2t$

(All elements taken modulo $6t+3$, ∞ fixed by all such translations) form a block design with parameters $b = (2t+1)(6t+3)$, $v = 6t+4$, $r = 6t+3$, $k = 3$, $\lambda = 2$.

However, for each $t \geq 0$, we have the blocks

$$A: \; (\infty, 0, 3t+1)$$

$$A+3t+2: \; (\infty, 0, 3t+2)$$

$$B+t: \; (t, 3t+1, 5t+2)$$

$$D_t+3t+2: \; (3t+2, 5t+2, t),$$

so by inspection, $(0, \infty)^\tau = (3t+1, 3t+2) = (t, 5t+2)^\tau$, hence τ is not 1 - 1 and G(D) is not doubly transitive for any of these designs.

3.3.5. The design of Figure 1a. $G(D) \approx LF(3,3)$.

3.3.6. The design of Figure 2. $G(D)$ is the sharply doubly transitive group of linear substitutions in $GF(13)$ of order 156. As discussed at the end of section 1.3, this design has the same parameters as the design of Figure 1a, but the two are not isomorphic. However, each has a doubly transitive group.

3.4. The Simple Designs Remaining

We are still left with the problem of classifying the designs with $2 < 0(\tau) < \frac{v-1}{2}$ or $0(\tau) = \frac{v}{2}$, with doubly transitive group transitive on the blocks. This problem remains unsolved, in general, but here the simple designs are discussed. From Chapter 2, any $k = 3$, $\lambda = 2$ design with doubly transitive group must contain a simple normal subdesign D_1, and other relations on the automorphism groups of D, D_1, and D/D_1 must be satisfied (see 2.2.13). Thus the determination of the simple designs is a step toward the complete classification.

From now on, we assume that D is a simple design with doubly transitive group. From Corollary 2.2.8, $D = T(A)$ for every pair of points A, so every pair generates the entire design. Furthermore, from Theorem 3.1.2, $\Pi(D)$ is the only set of at least two points which has the τ-property. The following lemmas are necessary:

Lemma 3.4.1. $G_{a,b}$ is a 2-group for any $a, b \in \Pi(D)$.

Proof: Suppose an odd prime p_1 divides the order of $G_{a,b}$. Let I be the set of fixed points of α, an element of $G_{a,b}$ of order p_1.

We have the blocks abc, abd which, since α fixes a and b, must be fixed pointwise, or interchanged. If they are interchanged, α contains the transposition (c, d): $\alpha = (a)(b)(c,d), \ldots$, so α is of even order, contrary to assumption. Therefore, α must fix c and d. Applying the same argument to all other pairs of points in I, we see that I has the τ-property, hence $I = \Pi(D)$ by 3. 1. 2, so α fixes all points and is the identity. Therefore, there can be no element of order p_1, and p_1 cannot divide the order of $G_{a,b}$.

Lemma 3. 4. 2. Either G is a Frobenius group or $G_{a,b}$ has an orbit of length 2.

Proof: Let I be the set of fixed points of $G_{a,b}$. For x, y, any two points of I, we have two blocks of D:

$$x\ y\ w$$
$$x\ y\ z,$$

where case 1) w and z are both fixed by $G_{a,b}$, or case 2) (w, z) are interchanged by an element of $G_{a,b}$. If case 1) applies for every pair of points of I, then I has the τ-property, so $G_{a,b} = 1$ and G is a Frobenius group. If case 2) applies for some pair x, y of I, then (w, z) is an orbit of length 2 of $G_{a,b}$.

We can now prove the characterization:

Theorem 3. 4. 3. There are two main types for D:

1) G is a Frobenius group. $\Pi(D)$ can then be identified with the point set of a near field k of order p or p^2 (as $p \equiv 1$ or $p \equiv 2$ (mod 3) respectively). B(D) is the action of the linear substitutions in

K <u>upon the points of a subgroup of order</u> 3 <u>in the multiplicative group of</u> K.

2) $|G_{a,b}| = 2^a > 1$. <u>Here</u> D <u>can only be the four-point design</u> D_4 <u>or the six-point design</u> D_6, <u>if we further assume that</u> G_a <u>is primitive on the points it moves.</u>

<u>Proof</u>: 1) From 3. 2. 10, G is the group of linear substitutions in K, where $|K| = p^r$, and it only remains to show that the order of K must be p if $p \equiv 1 \pmod 3$ and the order of K must be p^2 if $p \equiv 2 \pmod 3$. Theorem 2. 2. 12 applied here states that there can be no proper subgroup H doubly transitive on the orbit Γ containing the block Φ if $G_\Phi \subset H$. But if $p \equiv 1 \pmod 3$ and $r > 1$, then k has a sub-near-field K^* of order p whose multiplicative group of order $p - 1$ contains an element of order 3 fixing a block Φ'. Hence the subgroup of linear substitutions $\{z \to a, z+b \mid a, b \in K^*\} = H$ satisfies the hypotheses of the converse of 2. 2. 12, and Φ^H is a normal subdesign. If $p \equiv 2 \pmod 3$, and $r > 2$, then for p^r to be congruent to $1 \pmod 3$, r must be even. But then K contains a sub-near-field of order p^2 and the above argument again produces a normal subdesign. On the other hand, if $|K| = p$ or p^2 respectively, there are no subgroups satisfying 2. 2. 12, and D is simple.

2) It remains to show that if G_a is primitive on the remaining points, then D is the six-point design D_6 (3. 3. 2) or the four-point design D_4 (3. 3. 1). From Lemma 3. 4. 2, we may assume that $G_{a,b}$ has an orbit of length 2. But then 18. 7 of Wielandt [8] applied to G_a implies that G_a is a Frobenius group and has a regular normal

subgroup of index 2, hence $|G_a| = (v-1) \cdot 2$. Furthermore, 18.8 of Wielandt [8] implies that the regular normal subgroup of G_a has prime order p. Thus $v-1 = p$, $|G_a| = 2p$, and since G_a is not cyclic, it must be isomorphic to the dihedral group of degree p. Then $|G| = (p+1) \cdot p \cdot 2$, and since G_a is a Frobenius group, G has the property that only the identity fixes three letters.

To finish the proof, we need the following theorem of Walter Feit [3]:

Theorem 1 (Feit). Let G be a doubly transitive permutation group on v letters of order $v(v-1)q$ in which no non-trivial permutation leaves three letters fixed. Then either G contains a normal subgroup of order v, or $v-1 = p_1^e$ for some prime p_1. In the latter case, $[S_{p_1}:S'_{p_1}] < 4q^2$, where S_p is the Sylow p-group of G, and if $S'_{p_1} = \{1\}$, there exists an exactly triply transitive permutation group G_0 containing G such that $[G_0:G] \leq 2$.

Since our group G satisfies the hypotheses of Feit's Theorem with $q = |G_{a,b}| = 2$, we have the conclusion. If G contains a regular normal subgroup, then $v = p_1^e$ for some prime p_1 and $s > 0$. But $v-1 = p$ is a prime, hence either $v = 2^s$ and $v-1$ is a Mersenne prime, or $v = 3$ and $v-1 = 2$. The second case yields the trivial 3 point design with blocks

a b c

a b c

which was considered in Theorem 3.2.1. For the first case, since G is transitive on B(D), $D = \Phi^G$ for any block Φ, thus

$$2 = \lambda = \frac{2 \cdot 3 \cdot |G_{a,b}|}{|G_{\bar{\Phi}}|} = \frac{2 \cdot 3 \cdot 2}{|G_{\bar{\Phi}}|}$$

so necessarily $3 \mid |G_{\bar{\Phi}}|$, so $3 \mid |G|$. If $v = 2^s$, and $|G_{a,b}| = 2^t$ (Lemma 3.4.1), $3 \mid |G|$ implies $3 \mid v-1$. $v-1$ is a prime p, hence $v-1 = 3$, $v = 4$, and D is the four-point design.

The case $v-1 = p_1^e$ remains. We already know that $v-1 = p'$, however, and so an S_p must have order p. Then $S_p' = 1$, and Feit's Theorem implies that there exists a triply transitive group G_0 containing G with $[G_0:G] \leq 2$. If $G = G_0$, D would consist of all triples of points from $\Pi(D)$, i.e., D is the complete balanced block design with $d = 3$ and $\lambda = 2$. This is none other than the four-point design again. If $[G_0:G] = 2$, then G_0 is a sharply triply transitive group on v letters of order $2 \cdot v \cdot (v-1) \cdot 2 = v(v-1) \cdot 4$, hence $v-2 = 4$, $v = 6$, and the design must be the six-point design of example 3.3.2.

It should be noted that no other simple designs with $k = 3$, $\lambda = 2$ are known to the writer, and it is conjectured that the simplicity of the design in some way forces the group G_a to be primitive on $\Pi(D) - \{a\}$.

BIBLIOGRAPHY

1. Bhattacharya, K. N. "A Note on Two-fold Triple Systems," Sankhya, 6 (1943), pp. 313-314.

2. Beaumont, R. A., and Peterson, R. P. "Set Transitive Permutation Groups," Can. J. of Math., 7 (1955), pp. 35-42.

3. Feit, Walter. "On a Class of Doubly Transitive Permutation Groups," Ill. J. of Math., 4 (1960), pp. 170-186.

4. Hall, Marshall, Jr. Combinatorial Theory. Waltham, Mass.: Blaisdell, 1967.

5. Hall, Marshall, jr. Group Theory. New York: Macmillan Company, 1959.

6. Higman, Donald G. "Finite Permutation Groups of Rank 3," Math. Zeitschr., 86 (1964), pp. 145-156.

7. Hughes, D. R. "On k-homogeneous Groups," Arch. Math., 15 (1964), pp. 401-402.

8. Wielandt, H. Finite Permutation Groups. New York: Academic Press, 1964.

9. Zassenhaus, H. "Uber Endliche Fastkorper," Abh. Math. Sem. Hamburg, 11 (1936), pp. 187-220.