STEADY SURFACE WAVE PATTERN IN A SHEAR FLOW

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Abstract

The subject under investigation concerns the steady surface wave patterns created by small concentrated disturbances acting on a non-uniform flow of a heavy fluid. The initial value problem of a point disturbance in a primary flow having an arbitrary velocity distribution $(U(y), 0, 0)$ in a direction parallel to the undisturbed free surface is formulated. A geometric optics method and the classical integral transform method are employed as two different methods of solution for this problem. Whenever necessary, the special case of linear shear (i.e. $U(y) = U_o(1+\epsilon y)$) is chosen for the purpose of facilitating the final integration of the solution.

The asymptotic form of the solution obtained by the method of integral transforms agrees with the leading terms of the solution obtained by geometric optics when the latter is expanded in powers of small ϵr.

The overall effect of the shear is to confine the wave field on the downstream side of the disturbance to a region which is smaller than the wave region in the case of uniform flows. If $U(y)$ vanishes, and changes sign at a critical plane $y = y_{cr}$ (e.g. $\epsilon y_{cr} = -1$ for the case of linear shear), then the boundary of this asymmetric wave field approaches this critical vertical plane. On this boundary the wave crests are all perpendicular to the x-axis, indicating that waves are reflected at this boundary.

Inside the wave field, as in the case of a point disturbance in a uniform primary flow, there exist two wave systems. The loci of constant phases (such as the crests or troughs) of these wave systems are not symmetric with respect to the x-axis. The geometric optics method and the integral transform method yield the same result of these loci for the special case of $U(y) = U_o(1 + \epsilon y)$ and for large κr ($\epsilon r \ll 1 \ll \kappa r$).

An expression for the variation of the amplitude of the waves in the wave field is obtained by the integral transform method. This is in the form of an expansion in small ϵr. The zeroth order is identical to the expression for the uniform stream case and is thus not applicable near the boundary of the wave region because it becomes infinite in that neighborhood. Throughout this investigation the viscous terms in the equations of motion are neglected, a reasonable assumption which can be justified when the wavelengths of the resulting waves are sufficiently large.

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I. INTRODUCTION

In oceanography, the effects of strongly sheared ocean currents on the propagation of ocean waves present problems of considerable interest. Another example of water waves in shear flow is the ship waves propagating in the wake and near the stern of a ship. A steady wave pattern of surface wave is produced by a concentrated stationary disturbance located either on the surface or submerged within a steady primary shear flow. This class of problems is of basic academic interest as well as of great importance in ship hydrodynamics because it can predict the main features of the system of waves accompanying a ship moving through a sufficiently deep water.

There exists an extensive literature concerning the special case when the primary flow is uniform. The classical method of solution is discussed in Lamb's Hydrodynamics (Ref. 7) where application is made to gravity waves (Kelvin's ship wave-pattern) in §256 and to capillary and combined capillary-gravity waves in §272. The extension of this classical method to disturbances of variable or pulsating strength and following an arbitrary path in a uniform primary flow may be found in Wehausen and Laitone (Ref. 18) and Stoker (Ref. 14).

In contrast, the problem is much more difficult when the primary flow is non-uniform. Several papers treating the problems of disturbances in a rotational flow are limited to two dimensional disturbances in parallel shear flows without free surfaces. The vortex lines in these flows are straight, remain parallel to one another, and are not stretched during the motion. Hence they are relatively easy to treat.

When the disturbance is three-dimensional, even though the undisturbed flow may still be unidirectional and does not possess a free surface, the stretching and bending of vortex lines must play an important role. Certain outstanding papers on the theory of these flows have appeared.

A basic theorem that two-dimensional disturbances become unstable at a lower critical Reynolds number than three-dimensional disturbances has been given by Squire (Ref. 12).

Squire and Winter (Ref. 13) have investigated steady three-dimensional disturbances to a parallel shear flow with no free surface by the so-called "secondary flow" method. In this treatment, no restriction is placed on the disturbances but there is an assumption that the undisturbed stream is weakly sheared. The shear is usually taken to be linear though this is not an essential limitation to the method. A difficulty arising in the application of this theory is that the secondary flow disturbance due to the presence of an obstacle falls off more slowly with distance than does the primary flow disturbance. This limits the validity of the solution to the region near the obstacle.

In an effort to clear up this difficulty Lighthill (Ref. 9) has studied the fundamental solution of a small steady three-dimensional disturbance in a two-dimensional parallel shear flow without a free surface. Denoting the velocity field by $(U(y) + u, v, w)$, he has shown that the small perturbation theory based on neglecting the squares of the perturbation velocities u, v and w is valid far from the obstacle and overlaps the region where the secondary flow solution is valid. The asymptotic behavior of this solution for large r shows that a source in a shear layer produces in a region of uniform flow outside the shear layer a disturbance equivalent to a source of different strength at a different position. The strength of the equivalent source can be predicted by an image method in which the shear layer is regarded as a superposition of layers of piecewise uniform flows. The displacement in effective position is of the order of the width of the shear layer. It is expected that such modification of source strength and position will no longer be so simple in the presence of a free surface.

In fact, in the presence of a free surface, the difficulties of treating the stretching and bending of the vortex lines are further enhanced. If the undisturbed flow contains a vortex sheet i.e. a discontinuity in velocity, perpendicular to the free surface, any disturbance may produce wave motions both in the free surface and in the vortex sheet. Development of a theory describing the interaction between the waves and these free surfaces is a difficult mathematical task. For primary flows having a continuous horizontal velocity distribution, the free surface waves produced by an obstacle will definitely be affected by the vorticity of the primary flow.

In spite of the difficulties described above, Ursell (Ref. 17) has investigated the problem of steady wave patterns on a non-uniform steady fluid flow. By assuming that the primary non-uniform flow is irrotational and does not vary rapidly with distance, he has developed a theory, for the steady wave pattern, based on the following assumptions:

(i) The streaming velocity component normal to a wave crest is equal to the phase velocity based on the local wave length;

(ii) the separation between consecutive crests is equal to the local wave length.

The purpose of the present study is to develop systematically a theory for steady surface wave patterns due to a small concentrated disturbance in a primary parallel rotational flow. In the construction of the phase curves by Ursell (reference 17) it was assumed that the phase velocity relative to a slightly non-uniform stream of variable depth can be adequately approximated by the phase velocity obtained from constant depth theory. The validity of this implicit assumption of adopting the original dispersion relationship for uniform flows of constant depth for rotational non-uniform primary flows is not immediately obvious. Indeed, our theory shows that if terms other than the two lowest orders are kept, this is no longer valid.

By discarding the squares of the perturbation velocities u, v and w of the velocity field $(U(y) + u, v, w)$, we shall first formulate the problem for an arbitrary primary parallel shear flow $U(y)$ with an undisturbed free surface at $z = 0$, in Chapter II.

A method based on the notion of group velocity and geometric optics argument, as developed by Landau and Lifshitz (reference 8), by Keller (reference 6) and by others, is applied to the problem of a small stationary concentrated surface disturbance on a primary parallel shear flow $U(y)$ in Chapter III. In section 1, $U(y)$ is kept arbitrary but in sections 2 and 3, solution for the special case of $U(y) = U_o(1 + \epsilon y)$ is carried out while no restriction is placed on ϵ. It is believed that the true behavior of the flow far from the disturbance must be obtained from the full equations with the squares of the disturbance neglected. Such an approach results in a "small-perturbation" solution. The region of validity of this result is at large distances from the disturbance and overlaps the region near the disturbance where a "small-shear" solution is valid. This overlapping holds in the sense that the asymptotic behavior of the "small shear" solution is similar to the asymptotic behavior of a perturbation expansion of the "small-perturbation" solution for small ϵr. This is, indeed, indicated by comparison of the results of Chapter III, section 3, and Chapter IV.

The geometric optics method of Chapter III fails at the boundary of the wave region. For the special case of uniform primary flow, Ursell (reference 16) has determined the behavior of the waves near the boundary of the wave region by a modification of the principle of stationary phase. Thus showing that for dispersive waves, the integral transform method is the fundamental one. Therefore, in Chapter IV, we shall modify the classical integral transform method to treat a weak submerged source of constant strength in a primary linear shear flow, i.e. $U = U_o(1 + \epsilon y)$. By following the "secondary flow" method, we take the "primary flow" to coincide with that in which the undisturbed stream is uniform and the "secondary flow" to be a perturbation of the primary flow by allowing a small shear (ϵ small) in the undisturbed stream. The resulting "small shear" solution, obtained by applying the method of stationary phase to evaluate the inverse transform, is valid for large $r\ (< \frac{1}{\epsilon})$. However, a more elaborate asymptotic technique will have to be applied in order to deduce a solution valid near the boundary of the wave region.

Although the velocity distribution $U_o(1 + \epsilon y)$ considered in Chapter III and Chapter IV may seem somewhat artificial, it nevertheless provides some insight to the problem of arbitrary $U(y)$ as well as a first approximation to the horizontal velocity distribution of the ocean near the shore-line. In the present treatment, surface tension and viscous terms in the equations are neglected. This is known to be permissible except near the critical region where $U(y)$ is zero. Accordingly the theory developed here is limited to a region in which $U(y)$ is nowhere zero.

II. GENERAL FORMULATION

1. Co-ordinate system and convention

The problem in question concerns the propagation of surface waves in a body of heavy fluid, of infinite depth, initially having a prescribed non-uniform horizontal velocity distribution, due to a stationary disturbance. The fluid is taken to be inviscid and incompressible, of constant density ρ. The resulting flow will be generally rotational; the disturbance is however assumed to be so small that a linear theory can be applied. Consider a right-handed rectilinear co-ordinate system such that (Fig. 1)

(i) the z-axis points vertically upward, with $z = 0$ coinciding with the undisturbed free surface, and the gravity acting in the negative z-direction;

(ii) the x-axis is in the direction of the undisturbed stream which is taken to be a function of y only; and

(iii) the y-axis completes the system.

The x, y, z-components of the velocity vector $\vec{q}$ are denoted by u, v, w, the pressure by p, the density by ρ and the gravitational acceleration by g. Other notations will be defined in the text as needed.

2. Equations of motion

The primary flow is unidirectional and has a prescribed shear given by

$$\vec{q}_o = (U(y),\ 0, 0) \qquad (2.1)$$
$$p_o = -\rho g z$$

where $U(y)$ is an arbitrary function of y and is assumed to be at least twice differentiable. With $\vec{q}_*$ and p_* representing the total velocity vector and pressure respectively, the momentum equation for an inviscid, incompressible flow is

$$\frac{\partial \vec{q}_*}{\partial t} + (\vec{q}_* \cdot \nabla)\vec{q}_* = -\frac{\nabla p_*}{\rho} - \nabla g z \quad . \tag{2.2}$$

The continuity equation is

$$\nabla \cdot \vec{q}_* = Q(\vec{x}, t), \tag{2.3}$$

where $Q(\vec{x}, t)$ represents a source in the fluid.

3. Boundary conditions

The boundary conditions on the free surface of the heavy fluid can be stated as follows. Let the free surface be S_f: $z - \zeta(x, y, t) = 0$ where $\zeta(x, y, t)$ is the vertical displacement of the free surface measured from $z = 0$. The dynamic condition of constant pressure at the free surface required

$$\frac{dp_*}{dt} = 0 \qquad \text{on} \qquad z = \zeta(x, y, t) \quad . \tag{2.4}$$

The kinematic condition for the particles on the free surface is

$$-\frac{\partial \zeta}{\partial t} = \vec{q}_* \cdot \left(\frac{\partial \zeta}{\partial x}, \frac{\partial \zeta}{\partial y}, -1 \right) \quad \text{on} \quad z = \zeta(x, y, t) \quad . \tag{2.5}$$

On solid surfaces with specified shapes $S(x, y, z, t) = 0$ the boundary condition is

$$\frac{\partial S}{\partial t} + \vec{q}_* \cdot \nabla S = 0 \qquad \text{on} \qquad S(x, y, z, t) = 0 \quad . \tag{2.6}$$

4. Linearization

The flow field maybe considered as a combination of the primary shear flow $(\vec{q}_o, p_o)$ and a perturbation $(\vec{q}, p)$ so that

$$\vec{q}_* = \vec{q}_o + \vec{q} = (U(y) + u, v, w), \tag{2.7}$$

$$p_* = p_o + p = -\rho g z + p \quad .$$

If the departure from the primary flow is small, then by neglecting the squares and products of u, v, w and their derivatives, Eqs. (2.2), (2.3) can be linearized to give

$$Du + U'(y)v + \frac{\partial}{\partial x}\left(\frac{p}{\rho}\right) = 0, \tag{2.9}$$

$$Dv + \frac{\partial}{\partial y}\left(\frac{p}{\rho}\right) = 0, \tag{2.10}$$

$$Dw + \frac{\partial}{\partial z}\left(\frac{p}{\rho}\right) = 0, \tag{2.11}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = Q(\vec{x}, t), \tag{2.12}$$

where $D \equiv \frac{\partial}{\partial t} + U(y)\frac{\partial}{\partial x}$, and in the sequel $D' \equiv U'(y)\frac{\partial}{\partial x}$ and $D'' \equiv U''(y)\frac{\partial}{\partial x}$. The linearized free surface conditions can be readily shown to be

$$\left.\begin{array}{l} -\rho g w + Dp = 0 \\ D\zeta = w \end{array}\right\} \text{ on } z = 0, \tag{2.13}$$

whereas the boundary condition on solid surface remains unchanged. Elimination of p by cross differentiations of (2.9), (2.10) and (2.11) gives

$$\frac{\partial}{\partial y} Du = D\frac{\partial v}{\partial x} - \frac{\partial}{\partial y}(U'(y)v) \quad , \tag{2.15}$$

$$\frac{\partial}{\partial y} Dw = D \frac{\partial v}{\partial z} \quad . \tag{2.16}$$

Although from (2.12), (2.15) and (2.16) a single equation for any component of the perturbation velocity $\vec{q} = (u, v, w)$ can be obtained, the equation with v as the dependent variable is preferred. By eliminating u and w, we obtain

$$[D\nabla^2 - D'']v = \frac{\partial}{\partial y} DQ(\vec{x}, t) \quad . \tag{2.17}$$

Equation (2.17) is the basic equation of motion. When v has been solved together with appropriate boundary conditions, u and w may be deduced from (2.16) and the continuity equation. To obtain w, the lower limit in the y-integral may be taken as either $+\infty$ or $-\infty$ since the disturbances are assumed to vanish at both limits. The result obtained using either limit is the same. In integrating the continuity equation with respect to x to obtain u, however, the lower limit must be $-\infty$ because it is possible that the fluid at $+\infty$ in which region the vorticity may be permanently changed after the fluid has passed the disturbance, may not have zero disturbance velocity i.e. we shall only assume all disturbances vanish farup stream (at $x = -\infty$).

The appropriate condition on the free surface for v can be derived by differentiating (2.13) giving

$$-g \frac{\partial}{\partial y} Dw + \frac{\partial}{\partial y} D^2\left(\frac{p}{\rho}\right) = 0 \quad . \tag{2.18}$$

Eliminating w from (2.18) and (2.16), we obtain after some manipulation

$$- g \frac{\partial}{\partial z} Dv + D^2 \frac{\partial}{\partial y}\left(\frac{p}{\rho}\right) + 2DD'\left(\frac{p}{\rho}\right) = 0 \quad .$$

By using (2.10) in the second term and integrating the resulting equation once, we have

$$- g \frac{\partial v}{\partial z} - D^2 v + 2D'\left(\frac{p}{\rho}\right) = 0 \quad . \tag{2.19}$$

Recalling $D' = U'(y) \frac{\partial}{\partial x}$, Eq. (2.9) can be used to eliminate $\left(\frac{p}{\rho}\right)$, yielding

$$+ g \frac{\partial v}{\partial z} + D^2 v + 2U'(y)(Du + U'(y)v) = 0 \quad . \tag{2.20}$$

Finally, dividing (2.20) by $U'(y)$, differentiating it with respect to y, then using (2.15) to eliminate u, we obtain the free surface boundary condition as

$$\frac{\partial}{\partial y}\left\{\frac{D^2 v + g \frac{\partial v}{\partial z}}{U'(y)}\right\} = - 2 \frac{\partial}{\partial x} Dv \qquad \text{on} \qquad z = 0 \; . \tag{2.21}$$

With appropriate initial and boundary conditions, v can be solved from (2.17) and (2.21). So far no restriction has been placed on $U(y)$. They are linear partial differential equations with variable coefficients which depend on y in such a complicated manner that it is quite difficult to obtain an analytical solution for the general case of arbitrary $U(y)$. In Chapter III a geometric optics method will be developed for the case when the variation of $U(y)$, otherwise arbitrary, is assumed small. In Chapter IV an integral transform method will be applied to solve (2.17) and (2.21) for v for a particular, although somewhat artificial $U(y)$, namely that of linear shear,

$$U(y) = U_o(1 + \epsilon y) \quad . \tag{2.22}$$

It is obvious that this type of undisturbed parallel velocity distribution, if exists at all, is rare. Nevertheless the solution may give some insight to the much harder problem of arbitrary $U(y)$.

5. Review of steady wave pattern in a uniform stream

The treatment of a stationary point source of strength m, submerged in a uniform primary flow, by double Fourier transform and asymptotic evaluation of certain integrals based on the principle of stationary phase illustrates the characteristic difficulties involved (Ref. 18). We shall review this method briefly to fix ideas as well as for discussion and comparison of the results with the case in which the primary flow is non-uniform.

Let rectangular cartesian co-ordinates (x, y, z) be chosen with the z-axis perpendicular to the undisturbed free surface, the x-axis parallel to the stream velocity U_o, and the y-axis completing the system. The gravity acts in the negative z-direction, with gravitational acceleration g. The origin is taken so that the point of disturbance is at $(0, 0, -h)$. Polar co-ordinates (r, σ, z) are defined by $x = r\cos\sigma$, $y = r\sin\sigma$. Our problem is to find the velocity potential $\phi_o(x, y, z)$ based on the linearized theory that in the flow field $z < 0$, ϕ_o satisfies the Laplace equation

$$\nabla^2\phi_o = 0 \qquad \text{except at} \qquad (0, 0, -h) \;, \tag{2.23}$$

and the boundary condition at the free surface $z = 0$ that

$$\frac{\partial^2\phi_o}{\partial x^2} + \kappa\,\frac{\partial\phi_o}{\partial z} = 0 \quad (z = 0) \;, \tag{2.24}$$

where $\kappa = g/U_o^2$ is a characteristic wave number. We further write

$$\phi_o(x, y, z) = -\frac{m}{4\pi}\frac{1}{R_1} + \tilde{\phi}(x, y, z) \quad ,$$

where $\tilde{\phi}$ is a harmonic function, regular in the region $z < 0$, and

$$R_1^2 = x^2 + y^2 + (z+h)^2 \quad . \tag{2.25}$$

For the condition at infinity we require

$$\lim_{z \to -\infty} \operatorname{grad} \phi_o = 0 \quad , \tag{2.26}$$

$$\lim_{x \to -\infty} \operatorname{grad} \phi_o = 0 \quad . \tag{2.27}$$

Applying the double Fourier Transform, defined as

$$\mathcal{F}\{\phi_o\} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; e^{-i(\alpha x + \beta y)} \phi_o \quad , \tag{2.28}$$

to (2.23), we obtain a second order ordinary differential equation with z as the only independent variable. The coefficients in the solution of this transformed equation are determined by the transform of the boundary conditions. The details of the method of solution can be found in Havelock (Ref. 4) and elsewhere and will not be repeated here. With the notation

$$\left.\begin{aligned} \alpha &= k \cos\theta \\ \beta &= k \sin\theta \end{aligned}\right\} \quad , \tag{2.29}$$

the result is as follows:

$$\phi_o(x,y,z) = -\frac{m}{4\pi}\frac{1}{R_1} + \frac{m}{4\pi}\frac{1}{R_2}$$

$$+ \mathrm{Re}\,\frac{1}{2\pi^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{\infty} dk\, e^{ikr\cos(\theta-\sigma)} e^{k(z-h)} \frac{m\kappa \sec^2\theta}{k-\kappa\sec^2\theta} ,$$

(2.30)

where $R_2^2 = x^2 + y^2 + (z-h)^2$ and "Re" denotes "the real part of". The physical significances of the terms on the right-hand side of (2.30), are respectively, the point source, its reflection into the plane $z = 0$, and the disturbance due to the free surface effects, including the surface waves. By evaluating the k-integral along an appropriate path, the integral in (2.30), for $\cos(\theta-\sigma) > 0$, yields in the final result of ϕ_o a term of the following integral representation

$$\mathrm{Re}\,\frac{i}{\pi}\int_{\theta_1}^{\theta_2} d\theta\, e^{ir\kappa\sec^2\theta\cos(\theta-\sigma)} e^{\kappa\sec^2\theta(z-h)} m\kappa\sec^2\theta ,$$

where

$$\theta_1 = \begin{cases} -\frac{\pi}{2}+\sigma \\ \sigma \end{cases} \quad \text{and} \quad \theta_2 = \begin{cases} \sigma \\ \frac{\pi}{2}+\sigma \end{cases} \quad \text{for} \quad \sigma \gtrless 0 .$$

This integral is in suitable form for the application of the method of stationary phase. For large κr and for each particular σ, the stationary points are given by the roots of

$$\frac{d}{d\theta}[\sec^2\theta\cos(\theta-\sigma)] = 0 ,$$

or

$$2\tan\theta\cos(\theta-\sigma) - \sin(\theta-\sigma) = 0 . \qquad (2.31)$$

Explicitly, there are two roots θ_- and θ_+, given by

$$\tan\theta_\pm = \frac{-1 \pm \sqrt{1-8\tan^2\sigma}}{4\tan\sigma} \quad . \tag{2.32}$$

The asymptotic form of the wave part of ϕ_o for large κr follows from the principle of stationary phase to give

$$\begin{aligned}\phi_o \simeq & - A(\theta_+) \sin\left\{r\kappa \sec^2\theta_+ \cos(\theta_+ - \sigma) + \frac{\pi}{4}\right\} \\ & - A(\theta_-) \sin\left\{r\kappa \sec^2\theta_- \cos(\theta_- - \sigma) - \frac{\pi}{4}\right\} \quad ,\end{aligned} \tag{2.33}$$

where

$$A(\theta_\pm) = \sqrt{\frac{2}{\kappa r\pi}}\, m\kappa \sec\theta_\pm \frac{(1+4\tan^2\theta_\pm)^{1/4}}{(1-2\tan^2\theta_\pm)^{1/2}} e^{\kappa \sec^2\theta_\pm (z-h)} \quad , \tag{2.34}$$

with $\theta_\pm$ given by (2.32). The above results show that the resultant wave pattern of a submerged source is the superposition of two systems of wave which are confined in the wedge bounded by two vertical planes $\sigma = \pm\sigma_*$, $\sigma_* = 19.5°$. The general features of the curves of constant phase are shown in Fig. 5b.

The variation of the amplitude is indicated by Eq. (2.34). The above expressions (2.33) and (2.34) break down when the second derivative of the phase function $\sec^2\theta\cos(\theta-\sigma)$ with respect to θ vanishes; this occurs at the boundary $\sigma = \pm\sigma_*$ of the wave region. By a more elaborate asymptotic method, Ursell (16) has shown that the leading term of ϕ_o is of order $O(\kappa r)^{-\frac{1}{3}}$ in the region near $|\sigma| = |\sigma_*|$. In this region, the method of geometric optics to be discussed in Chapter III, too, fails. For dispersive systems, the method of stationary phase therefore seems more fundamental. But it can be seen that this integral transform method, without modifications, fails when the primary flow is non-uniform because Eqs. (2.23) and (2.24) are replaced by Eqs. (2.17) and (2.21) which have variable coefficients. We shall show in Chapter IV, how this integral transform approach can be modified in a special case to overcome this difficulty.

III. METHOD OF GEOMETRIC OPTICS

1. Unsteady problem

The method of geometric optics has been developed and applied to problems involving wave motions in anisotropic and dispersive media by Landau and Lifshitz (8), Lighthill (10), Whitham (20), Keller (6), and by others. The basic assumption necessary for successful application of this method is that the non-uniformity in the dispersive medium in which the wave propagates does not change significantly over distances comparable to the typical wave-length under consideration. In our present problem of gravity waves, in a parallel shear stream with free stream velocity $U(y)$, this assumption may be interpreted as

$$U_o^2/g\ell \ll 1$$

where U_o is a characteristic constant velocity, U_o^2/g, is the typical wave length, and ℓ is a characteristic length over which the primary parallel shear flow velocity, changes appreciably. For example, ℓ may be chosen to be U_o/(mean value of $|dU/dy|$). In this section, the problem of a point disturbance applied on the surface of the fluid at $x = 0$, $y = 0$ for $t > 0$ is first formulated. The steady wave pattern due to a stationary point disturbance on the surface is then deduced quite readily.

We consider here a general distribution of velocity $U(y)$,

$$U(y) = U_o + U_1(y) \quad , \tag{3.1}$$

subject to the restriction that $U_1(y)$ is a slowly varying function and is small compared to U_o. When the variations of $U(y)$ are smooth, we can keep a clear separation between the phase of a surface wave and its amplitude. Thus we may write

$$v(\vec{x},t) = A(\vec{x},t)e^{i\phi(\vec{x},t)} = A_o \exp\{i\phi(x,t) + L(\vec{x},t)\} \quad , \tag{3.2}$$

where

$$L(\vec{x},t) = \log[A(\vec{x},t)/A_o] \quad ,$$

and A_o is a reference amplitude, $A(\vec{x},t)$ is the amplitude function, which is assumed to be slowly varying in $\vec{x}$ and t. The function $\phi(\vec{x}\ t)$ in general may be complex, its imaginary part is determined by the differential equation governing v. To represent a surface wave, the real part of ϕ assumes the form

$$\mathrm{Re}\{\phi(\vec{x},t)\} = k_o\psi(\vec{x},t) = k_o\Psi(x,t) - \omega t \tag{3.2a}$$

where k_o is a constant reference wave number appropriate to the wave motion in question, $\omega = \omega(\vec{x},t)$ is the circular frequency, and the real function $\Psi(\vec{x},t)$ is the so-called eikonal function, which defines the surfaces of equal phase.

For the geometric wave approximation we next introduce the basic assumption that the characteristic wavelength $2\pi/k_o$ is assumed to be small compared to the distance ℓ over which $U(y)$ changes appreciably. Hence, if $\vec{x}$ is referred to ℓ, and t to $k_o\ell/\omega_o$, as the basic units, then from the above assumption it follows that the derivatives of ϕ and L will assume the order of magnitudes:

$$|\mathrm{grad}\,\phi| = O(k_o\ell) \quad , \qquad |\phi_t| = O(k_o\ell) \quad ,$$
$$|\mathrm{grad}\,L| = O(1) \quad , \qquad |L_t| = O(1) \quad , \tag{3.3}$$

and the higher order derivatives of ϕ with respect to $\vec{x}$ and t are clearly also of order $(k_o\ell)$. Since the geometric wave approximation requires $k_o\ell$ to be large, the differential equations for ϕ and L can be derived by substituting v into the basic equation of motion (2.17) and the boundary condition (2.21), then expanding the equations for small $(1/k_o\ell)$. Consequently, the terms in (2.17) have the following orders of magnitude:

$$|D(\nabla^2 v)| = O(|v|(k_o\ell)^3) \quad , \qquad |U''v_x| = O(|U''v|(k_o\ell)) \quad ,$$

where the prime denotes differentiation with respect to y, as before. Hence, up to the first two leading terms for $k_o\ell$ large, we have

$$D(\nabla^2 v) = 0 \quad ,$$

which holds true for arbitrary $U(y)$ provided the variation of $U(y)$ is smooth, a factor of error $[1+O(k_o\ell)^{-2}]$ for the left side of the above equation being understood. This equation can be integrated once, giving $\nabla^2 v = f(x - Ut)$, $f(x)$ being an arbitrary function of x; but this function must be zero since $\nabla^2 v$ is required to vanish as $y^2 + z^2 \to \infty$ in the flow field. Whence

$$\nabla^2 v = 0 \quad , \tag{3.4}$$

valid for the first two leading orders. The free surface condition (2.21) becomes

$$\frac{\partial}{\partial y}\,(D^2 v + g v_z) = -\,2\,U'\,D v_x + \frac{U''}{U'}\,(D^2 v + g\,v_z) \quad . \tag{3.5}$$

It may be noted that the two terms on the left hand side of (3.5) may be regarded to be of the same order since the parameter $g\ell/U_o^2$, which is the factor multiplying v_z in the dimensionless form of the equation, can be very large. It is also noted that the terms on the right hand side of (3.5) enter in the calculation of the second order term.

By substituting (3.2) into (3.4) and retaining only the two leading orders, we obtain

$$\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 = 0 \quad , \tag{3.6}$$

and

$$2\left(\frac{\partial\phi}{\partial x}\frac{\partial L}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial L}{\partial y} + \frac{\partial\phi}{\partial z}\frac{\partial L}{\partial z}\right) + \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad . \tag{3.7}$$

From (3.6),

$$i\,\frac{\partial\phi}{\partial z} = \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]^{\frac{1}{2}} \quad . \tag{3.8}$$

Hence we observe that if the partial derivatives of ϕ with respect to x and y are real, as is required by the solution representing a surface wave propagating in the x-y plane, then $\partial\phi/\partial z$ will be purely imaginary so that v will vary exponentially with respect to z. In (3.6), an appropriate branch of the function $(\phi_x^{\,2} + \phi_y^{\,2})^{\frac{1}{2}}$ is chosen to satisfy the boundary condition that $v \to 0$ as $z \to -\infty$. By differentiating (3.6) with respect to x, y, z, and by some substitutions, we find

$$\frac{\partial^2\phi}{\partial z^2} = \frac{-1}{\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2\frac{\partial^2\phi}{\partial x^2} + 2\,\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y} + \left(\frac{\partial\phi}{\partial y}\right)^2\frac{\partial^2\phi}{\partial y^2}\right\} \quad . \tag{3.9}$$

By making use of (3.8) and (3.9), (3.7) may be written as

$$i\,\frac{\partial L}{\partial z} = \frac{\partial L}{\partial x}\,\frac{\dfrac{\partial\phi}{\partial x}}{\left[\left(\dfrac{\partial\phi}{\partial x}\right)^2+\left(\dfrac{\partial\phi}{\partial y}\right)^2\right]^{\frac{1}{2}}} + \frac{\partial L}{\partial y}\,\frac{\dfrac{\partial\phi}{\partial y}}{\left[\left(\dfrac{\partial\phi}{\partial x}\right)^2+\left(\dfrac{\partial\phi}{\partial y}\right)^2\right]^{\frac{1}{2}}}$$

$$+\ \frac{1}{2}\,\frac{\left(\dfrac{\partial\phi}{\partial x}\right)^2\dfrac{\partial^2\phi}{\partial y^2} - 2\,\dfrac{\partial\phi}{\partial x}\,\dfrac{\partial\phi}{\partial y}\,\dfrac{\partial^2\phi}{\partial x\partial y} + \left(\dfrac{\partial\phi}{\partial y}\right)^2\dfrac{\partial^2\phi}{\partial x^2}}{\left[\left(\dfrac{\partial\phi}{\partial x}\right)^2+\left(\dfrac{\partial\phi}{\partial y}\right)^2\right]^{3/2}}\ . \qquad (3.10)$$

We also note that if ϕ_x, ϕ_y, L_x, and L_y are real, then $\partial L/\partial z$ must be purely imaginary. We shall henceforth regard $i\,\dfrac{\partial\phi}{\partial z}$ and $i\,\dfrac{\partial L}{\partial z}$ as real.

Next we substitute (3.2) in (3.5), again keeping only the two leading orders, and separate the real and imaginary parts, to obtain

$$\left(\frac{\partial\phi}{\partial t} + U\,\frac{\partial\phi}{\partial x}\right)^2 - g\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right]^{\frac{1}{2}} = 0 \quad (z = 0)\ , \qquad (3.11)$$

$$\frac{\partial L}{\partial t}\left[2\left(\frac{\partial\phi}{\partial t} + U\,\frac{\partial\phi}{\partial x}\right)\right] + \frac{\partial L}{\partial x}\left[2\left(\frac{\partial\phi}{\partial t} + U\,\frac{\partial\phi}{\partial x}\right)U - g\,\frac{\dfrac{\partial\phi}{\partial x}}{\left[\left(\dfrac{\partial\phi}{\partial x}\right)^2+\left(\dfrac{\partial\phi}{\partial y}\right)^2\right]^{\frac{1}{2}}}\right]$$

$$+\ \frac{\partial L}{\partial y}\left[-\,g\,\frac{\dfrac{\partial\phi}{\partial y}}{\left[\left(\dfrac{\partial\phi}{\partial x}\right)^2+\left(\dfrac{\partial\phi}{\partial y}\right)^2\right]^{\frac{1}{2}}}\right] + \frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial t} + U\,\frac{\partial\phi}{\partial x}\right) + U\,\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial t} + U\,\frac{\partial\phi}{\partial x}\right)$$

$$-\ \frac{g}{2}\,\frac{\partial}{\partial x}\left(\frac{\dfrac{\partial\phi}{\partial x}}{\left[\left(\dfrac{\partial\phi}{\partial x}\right)^2+\left(\dfrac{\partial\phi}{\partial y}\right)^2\right]^{\frac{1}{2}}}\right) - \frac{g}{2}\,\frac{\partial}{\partial y}\left(\frac{\dfrac{\partial\phi}{\partial y}}{\left[\left(\dfrac{\partial\phi}{\partial x}\right)^2+\left(\dfrac{\partial\phi}{\partial y}\right)^2\right]^{\frac{1}{2}}}\right)$$

$$+\ 2\left(\frac{\partial\phi}{\partial t} + U\,\frac{\partial\phi}{\partial x}\right)U'(y)\,\frac{\dfrac{\partial\phi}{\partial x}}{\dfrac{\partial\phi}{\partial y}} = 0 \qquad (z = 0)\ . \qquad (3.12)$$

In (3.11), (3.8) has been used to replace $i\,\dfrac{\partial\phi}{\partial z}$. Also (3.10) has been used in (3.12) which has been further simplified by subtraction of (3.11) and its first derivative with respect to y. The eikonal equation (3.11) provides an equation for ϕ while the transport equation (3.12) is a first order partial differential equation for L. Since (3.11) and (3.12) are deduced from the linearized free surface condition (3.2), they are valid on $z = 0$ only.

To effect the integration of (3.11) and (3.12) we introduce the frequency function $\omega(\vec{x}, t)$ and wave number $\vec{k}(\vec{x}, t) = (k_1, k_2, 0)$ by

$$\frac{\partial \phi}{\partial t} = -\omega \quad , \quad \frac{\partial \phi}{\partial x} = k_1 \quad , \quad \frac{\partial \phi}{\partial y} = k_2 \quad , \quad k = \sqrt{k_1^2 + k_2^2} \quad , \tag{3.13}$$

in terms of which (3.11) becomes

$$F(-\omega, k_1, k_2, y) = (\omega - Uk_1)^2 - g(k_1^2 + k_2^2)^{\frac{1}{2}} = 0 \quad . \tag{3.14}$$

By applying the theory of first order partial differential equations (e.g. see Courant and Hilbert, (1962), Vol. 2), the characteristic equations may be written immediately. If we introduce a parameter λ along the characteristic curves the characteristic system of equations become

$$\frac{dt}{d\lambda} = -\frac{\partial F}{\partial \omega} = -2(\omega - Uk_1) \quad , \tag{3.15}$$

$$\frac{dx}{d\lambda} = \frac{\partial F}{\partial k_1} = -2(\omega - Uk_1)U - gk_1/k \quad , \tag{3.16}$$

$$\frac{dy}{d\lambda} = \frac{\partial F}{\partial k_2} = -gk_2/k \quad , \tag{3.17}$$

$$\frac{d\phi}{d\lambda} = \omega \frac{\partial F}{\partial \omega} + k_1 \frac{\partial F}{\partial k_1} + k_2 \frac{\partial F}{\partial k_2} = (\omega - Uk_1)^2 \quad , \tag{3.18}$$

$$\frac{d\omega}{d\lambda} = \frac{\partial F}{\partial t} - \omega \frac{\partial F}{\partial \phi} = 0 \quad , \tag{3.19}$$

$$\frac{dk_1}{d\lambda} = - \frac{\partial F}{\partial x} - k_1 \frac{\partial F}{\partial \phi} = 0 \quad , \tag{3.20}$$

$$\frac{dk_2}{d\lambda} = - \frac{\partial F}{\partial y} - k_2 \frac{\partial F}{\partial \phi} = 2(\omega - Uk_1)k_1 U_1'(y) \quad . \tag{3.21}$$

Equations (3.19) and (3.20) show that ω and k_1 are constants along the characteristics or ray paths though their constant values may be different for different characteristics.

By dividing (3.16) and (3.17) by (3.15) and using (3.14), the components of the group velocity $\vec{C}g = (Cg_x, Cg_y)$ are deduced to give

$$Cg_x = \frac{dx}{dt} = \frac{\partial \omega}{\partial k_1} = U + \frac{1}{2}\sqrt{\frac{g}{k}}\,\frac{k_1}{k} \quad , \tag{3.22}$$

$$Cg_y = \frac{dy}{dt} = \frac{\partial \omega}{\partial k_2} = \frac{1}{2}\sqrt{\frac{g}{k}}\,\frac{k_2}{k} \quad . \tag{3.23}$$

The form of these group velocity components resembles those of the waves propagating on the surface of a stream with a uniform velocity except that in the present case $U = U(y)$, and k_2 is no longer constant on ray paths as can be seen by division of (3.21) by (3.15)

$$\frac{dk_2}{dt} = - k_1 U_1'(y) \tag{3.24}$$

which gives the rate of change of k_2 along a ray path. This rate is constant on each ray path for the special case of a primary parallel flow of linear shear, that is, $U'(y) = \text{const.}$, but it may differ on different rays. The ratio of (3.22) to (3.23) gives

$$\frac{dx}{dy} = \frac{U + \frac{1}{2}\sqrt{\frac{g}{k}}\frac{k_1}{k}}{\frac{1}{2}\sqrt{\frac{g}{k}}\frac{k_2}{k}} = \frac{2U(gk)^{3/2} + g^2 k_1}{g^2 k_2} \tag{3.25}$$

from which the ray paths or group lines may be deduced. Equation (3.25) may be put into a form which can be readily integrated. From (3.14),

$$\sqrt{gk} = (\omega - Uk_1) \quad , \qquad \text{or}$$

$$k_2 = \pm\sqrt{\frac{(\omega - Uk_1)^4}{g^2} - k_1^2} \quad . \tag{3.26}$$

Substitution of (3.26) in (3.25) gives

$$\frac{dx}{dy} = \frac{2U(\omega - Uk_1)^3 + g^2 k_1}{\pm g^2\sqrt{\frac{(\omega - Uk_1)^4}{g^2} - k_1^2}} \quad , \tag{3.27}$$

of which the right hand side is a function of y only because k_1 and ω are constant along each group line. For the special case of linear shear primary flow, it is readily seen that the group lines may be expressed as the elliptic functions. Integration of (3.27) for this particular case will be carried out for the steady problem in the next section.

To conclude this section we shall examine briefly the transport equation (3.12) governing the amplitude variation. In terms of the notations defined in (3.13), (3.12) becomes

$$\frac{\partial L}{\partial t} + Cg_x \frac{\partial L}{\partial x} + Cg_y \frac{\partial L}{\partial y} + \frac{1}{2(\omega - Uk_1)}\left\{\frac{\partial}{\partial t}(\omega - Uk_1) + U\frac{\partial}{\partial x}(\omega - Uk_2)\right.$$

$$\left. + \frac{g}{2}\frac{\partial}{\partial x}\left(\frac{k_1}{k}\right) + \frac{g}{2}\frac{\partial}{\partial y}\left(\frac{k_2}{k}\right)\right\} + U'(y)\frac{k_1}{k_2} = 0 \quad , \tag{3.28}$$

where

$$Cg_x = \frac{(\omega - Uk_1)U + \frac{g}{2}\frac{k_1}{k}}{(\omega - Uk_1)} \quad , \tag{3.29}$$

$$Cg_y = \frac{\frac{g}{2}\frac{k_2}{k}}{(\omega - Uk_1)} \quad . \tag{3.30}$$

These expressions for the components of the group velocity are the same as (3.22) and (3.23) by virtue of (3.14). In order to express (3.28) in a physically more meaningful form, we shall first take the partial derivatives of Cg_x and Cg_y with respect to x and y respectively, giving, after some rearrangement,

$$\frac{1}{(\omega - Uk_1)}\left\{U\frac{\partial}{\partial x}(\omega - Uk_1) + \frac{g}{2}\frac{\partial}{\partial x}\left(\frac{k_1}{k}\right)\right\} = \frac{\partial Cg_x}{\partial x} + \frac{Cg_x}{(\omega - Uk_1)}\frac{\partial}{\partial x}(\omega - Uk_1) \quad , \tag{3.31}$$

and

$$\frac{1}{(\omega - Uk_1)}\left\{\frac{g}{2}\frac{\partial}{\partial y}\left(\frac{k_2}{k}\right)\right\} = \frac{\partial Cg_y}{\partial y} + \frac{Cg_y}{(\omega - Uk_1)}\frac{\partial}{\partial y}(\omega - Uk_1) \quad . \tag{3.32}$$

We recall that the total time derivative of a function $f(x, y, t)$, which does not depend on k_2 explicitly, along the group lines is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + Cg_x \frac{\partial f}{\partial x} + Cg_y \frac{\partial f}{\partial y} \quad . \tag{3.33}$$

Hence,

$$\frac{d}{dt}(\omega - Uk_1) = -k_1 \frac{dU}{dt} = -k_1 U_1'(y)\frac{dy}{dt} = -k_1 U_1'(y) Cg_y \;, \tag{3.34}$$

since ω and k_1 are constants on each group line. But by (3.33), we can also write

$$\frac{d}{dt}(\omega - Uk_1) = \frac{\partial}{\partial t}(\omega - Uk_1) + Cg_x \frac{\partial}{\partial x}(\omega - Uk_1) + Cg_y \frac{\partial}{\partial y}(\omega - Uk_1) \;. \tag{3.35}$$

But the right hand sides of (3.34) and (3.35) must be identical, hence

$$\frac{\partial}{\partial t}(\omega - Uk_1) = -Cg_x \frac{\partial}{\partial x}(\omega - Uk_1) - Cg_y \frac{\partial}{\partial y}(\omega - Uk_1) - k_1 U_1'(y) Cg_y \;. \tag{3.36}$$

Expressions (3.31), (3.32) and (3.36) may now be used to rewrite (3.28) as

$$\frac{dL}{dt} + \frac{1}{2}\left\{\frac{\partial}{\partial x} Cg_x + \frac{\partial}{\partial y} Cg_y\right\} + U'(y)\left(\frac{k_1}{k_2} - \frac{1}{2}\frac{k_1}{\omega - Uk_1} Cg_y\right) = 0 \;. \tag{3.37}$$

To display the physical significance of (3.37), we recall $L(\vec{x}, t) = \log[A(\vec{x}, t)/A_0]$ so that in terms of A, one may obtain by multiplying (3.37) by A^2 the equation

$$\frac{d}{dt}\frac{A^2}{2} + \frac{A^2}{2}\left\{\frac{\partial}{\partial x} Cg_x + \frac{\partial}{\partial y} Cg_y\right\} = A^2 U'(y)\left\{\frac{k_1 k_2}{4k^2} - \frac{k_1}{k_2}\right\} , \tag{3.38}$$

where (3.14) and (3.23) have been used to obtain the last term of (3.38). By using the explicit form (3.33) for dA^2/dt, (3.38) may also be written as

$$\frac{\partial}{\partial t}\left(\frac{A^2}{2}\right) + \nabla_2 \cdot \left(\frac{A^2}{2}\vec{C}g\right) = -\left(\frac{A^2}{2}\right) U'(y) \frac{k_1}{k_2}\left(\frac{4k_1^2 + 3k_2^2}{2k^2}\right) , \tag{3.39}$$

where $\nabla_2 = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}$ and $U'(y) \equiv U_1'(y)$. Regarding the quantity $\frac{A^2}{2}$ as a measure of the energy density, (3.39) shows that the flux of energy along ray tubes, formed by adjacent group lines or rays, is not constant. In the region where $U'(y) > 0$, for the waves with $k_2 < 0$, (3.39) indicates that the effect of shear is to introduce a fictitious energy source while for $k_2 > 0$, a fictitious energy sink. We may conclude that the primary parallel shear flow supplies energy to the waves with $k_2 < 0$ and retrieves it from waves with $k_2 > 0$, resulting in a net transfer of energy to different regions of the flow field. It may be conjectured that the total energy of the waves are due entirely to the disturbance at the origin so that, while the parallel shear flow redistributes the energy, it does not supply its own energy to the waves.

2. Steady wave pattern

The familiar steady ship wave pattern created by a stationary point disturbance acting on the surface of a uniform stream will be modified when the primary undisturbed flow has a non-uniform horizontal velocity distribution. The steady wave pattern on the surface of a parallel linear shear flow due to a disturbance at $x = 0$, $y = 0$ furnishes at least qualitative modifications one would expect of the more general case of arbitrary shear; this special case will be considered in this section. When the wave pattern has reached a steady state, the encounter frequency ω relative to the disturbance must vanish in the wave field, hence (3.14) reduces to

$$\omega = \sqrt{gk} + (U_0 + U_1(y))k_1 = 0 \quad . \tag{3.40}$$

The result that k_1 and ω are constant along group lines must still be true so that along each group line

$$dk^2 = d(k_1^2 + k_2^2) = dk_2^2 = 2k_2 dk_2 \tag{3.41}$$

and $k_2 = k_2(y)$ on each group line.

The slope of the group lines may be deduced from (3.27) by putting $\omega = 0$, however, it can be put into a more useful form by deriving it again directly from the components of the group velocity. From (3.22) and (3.23),

$$\frac{dx}{dy} = \frac{Cg_x}{Cg_y} = \frac{k_1}{k_2} + 2\sqrt{\frac{k^3}{g}}\,\frac{U_o + U_1(y)}{k_2} \quad ,$$

which by (3.40) becomes

$$\frac{dx}{dy} = \frac{k_1}{k_2} - 2\,\frac{k^2}{k_1 k_2} = -\left(\frac{k_1}{k_2} + 2\,\frac{k_2}{k_1}\right) \quad . \tag{3.42}$$

Since the right-hand side of (3.42) is a function of y only, we let

$$\frac{dy}{dx} = m(y) \quad . \tag{3.43}$$

Upon solving for $\frac{k_2}{k_1}$ from (3.42) in terms of $m(y)$,

$$\left(\frac{k_2}{k_1}\right)_{\pm} = \frac{-1 \pm \sqrt{1 - 8m^2(y)}}{4m(y)} \quad . \tag{3.44}$$

For the waves to exist, k_1 and k_2 must be real. Equation (3.44) shows that this requirement is satisfied if and only if

$$|m| < m_* = \frac{1}{2\sqrt{2}} \quad . \tag{3.45}$$

This upper bound m_* of the slope function $m(y)$ will be used to determine the boundary of a wave region outside of which the disturbance dies down very rapidly. Introducing polar coordinates (k, θ) in the wave number plane by

$$\left.\begin{aligned} k_1 &= k\cos\theta \\ k_2 &= k\sin\theta \end{aligned}\right\} , \tag{3.46}$$

we may write (3.44) also as

$$\tan\theta_{\pm} = \frac{-1 \pm \sqrt{1 - 8m^2(y)}}{4m(y)} , \tag{3.47}$$

which will be used in the subsequent analysis. For the moment we shall simply note that for each $m(y)$ satisfying (3.45) there exist two angles of θ, denoted by θ_+ and θ_-, given by (3.47). Corresponding to $\theta_{\pm}$, the value of $k_{\pm}$ are determined by (3.40) and (3.46), to give

$$k_{\pm} = \frac{g}{U_0^2} \frac{\sec^2\theta_{\pm}}{(1+\widetilde{U}_1(y))^2} ; \tag{3.48}$$

where $\widetilde{U}_1(y) \equiv U_1(y)/U_0$, so that

$$k_{1\pm} = k_{\pm}\cos\theta_{\pm} ,$$

$$k_{2\pm} = k_{\pm}\sin\theta_{\pm} ,$$

a result which shows the presence of two systems of waves inside the wave region inferred by (3.45).

By transferring the last term of (3.40) to the right-hand side and squaring, we obtain:

$$gk = k_1^2[U_0+U_1(y)]^2 .$$

Then

$$k_2^2 = \left[\frac{k_1^2}{g^2}(U_0+U_1(y))^4 - 1\right] k_1^2 ,$$

or

$$\frac{k_2}{k_1} = \pm\sqrt{\left(\frac{k_1}{\kappa}\right)^2 (1 + \widetilde{U})^4 - 1} \quad , \tag{3.49}$$

where we have defined

$$\kappa \equiv \frac{g}{U_o^2} \quad , \tag{3.50}$$

and

$$\widetilde{U}_1 \equiv \frac{U_1(y)}{U_o} = \epsilon y \quad . \tag{3.51}$$

In the last step the linear shear primary flow has been chosen specifically for $U(y)$ for the purpose of facilitating integration of the solution. We shall now introduce a new variable

$$\xi / \xi_o = 1 + \widetilde{U}_1(y) = 1 + \epsilon y \quad , \qquad \xi_o \equiv \sqrt{\frac{k_1}{\kappa}} \quad , \tag{3.52}$$

so that (3.49) becomes

$$\frac{k_2}{k_1} = \pm\sqrt{\xi^4 - 1} \quad . \tag{3.53}$$

Here the requirement of k_1 and k_2 to be real in order for waves to exist is satisfied if and only if $\xi > 1$. On physical grounds, we expect the waves to appear in the downstream of the disturbance; this implies that $k_1 > 0$. The + or - sign in (3.49), (3.53), and in the sequel is taken according as k_2 is > or < 0. To show the equivalence of this to (3.45), we may deduce from (3.42), (3.43) and (3.53) the expression

$$m(\xi) = \mp (\xi^4 - 1)^{\frac{1}{2}} (2\xi^4 - 1)^{-1} \quad . \tag{3.54}$$

To find the maximum of $m(\xi)$ we set

$$\frac{dm}{d\xi}(\xi_*) = \frac{(3-2\xi_*^4)\,2\xi_*^3}{(2\xi_*^4-1)^2\sqrt{\xi_*^4-1}} = 0 \quad .$$

Since $\xi > 1$, the only zero is thus

$$\xi_* = \left(\frac{3}{2}\right)^{\frac{1}{4}},$$

at which the slope function m attains its maximum value

$$m(\xi_*) = \frac{1}{2\sqrt{2}} = m_* ,$$

which is identical to (3.45). The dependence of m on the parameter ξ is useful in obtaining the steady wave pattern. This is shown in Fig. 2.

Next, differentiating (3.52) we have

$$\frac{d}{d\xi}(\epsilon y) = \sqrt{\frac{\kappa}{k_1}} \quad . \tag{3.55}$$

Substitution of (3.53) and (3.55) into (3.42) then gives

$$\frac{d}{d\xi}(\epsilon x) = \mp\sqrt{\frac{\kappa}{k_1}}\,\frac{2\xi^4-1}{\sqrt{\xi^4-1}} \quad . \tag{3.56}$$

Upon integration, with the integration constants so chosen that all ray paths will pass through the origin $x = 0,\ y = 0$ or $\xi = \sqrt{k_1/\kappa}$,

$$\epsilon x = \mp\frac{1}{\xi_o}\int_{\xi_o}^{\xi}\frac{2\xi^4-1}{\sqrt{\xi^4-1}}\,d\xi \qquad \left(\xi_o = \sqrt{k_1/\kappa}\right) ,$$

or

$$\epsilon\xi_o x = \left|\frac{2}{3}\,\xi\sqrt{\xi^4-1}\,\Big|_{\xi_o}^{\xi} - \frac{1}{3}\int_{\xi_o}^{\xi}\frac{d\xi}{\sqrt{\xi^4-1}}\right| \quad . \tag{3.57}$$

The absolute value is taken because the assumption of zero upstream wave disturbance indicates only positive ϵx is required. The choice of the lower limit of integration in (3.57) arises from the assumption of a point disturbance at $x = 0$, $y = 0$ so that all the group lines pass through the origin. The integral in (3.57) can be readily expressed in terms of the elliptic integrals. Thus

$$\epsilon\xi_o x = \left| \frac{2}{3}\left[\xi\sqrt{\xi^4-1} - \xi_o\sqrt{\xi_o^4-1}\right] - \frac{1}{3\sqrt{2}}\left[F\left(\cos^{-1}\left(\frac{1}{\xi}\right),\frac{1}{\sqrt{2}}\right) - F\left(\cos^{-1}\left(\frac{1}{\xi_o}\right),\frac{1}{\sqrt{2}}\right)\right]\right|, \tag{3.58}$$

where $F(\gamma,\lambda)$ is the elliptic function of the first kind whose integral representation is

$$\int_b^{\mu} \frac{d\mu}{\sqrt{\mu^2+a^2}\sqrt{\mu^2-b^2}} = \frac{1}{\sqrt{a^2+b^2}} F(\gamma,\lambda) \tag{3.59}$$

with $\gamma = \cos^{-1}\left(\frac{b}{\mu}\right)$, $\lambda = \frac{a}{\sqrt{a^2+b^2}}$, $0 < b < \mu$ and $0 < a$. This function has been studied extensively and its values for arbitrary arguments have been tabulated (Ref. 3). Therefore, with the known properties of $F(\gamma,\lambda)$, (3.58) and (3.52) give a parametric representation of the group lines. The basic features of these two equations are shown in Fig. 3 for $k_1 > \kappa$ and for $k_1 = \kappa$. In each case, ϵy is linear with a positive slope $\frac{1}{\xi_o}$ and intersects the ξ-axis at ξ_o. The behavior of ϵx is more complicated. As ξ increases from unity, ϵx decreases from

$$\left| \frac{2}{3}\sqrt{\xi_o^4-1} - \frac{1}{3\sqrt{2}}\cdot\frac{1}{\xi_o}\cdot F\left(\cos^{-1}\left(\frac{1}{\xi_o}\right),\frac{1}{\sqrt{2}}\right)\right|$$

to zero at $\xi = \xi_o (> 1)$. Then it increases monotonically for $\xi > \xi_o$. Since from (3.55) and (3.56)

$$\left|\frac{d}{d\xi}(\epsilon x)\right| = \left|\frac{1}{\xi_o}\frac{(2\xi^4-1)}{\sqrt{\xi^4-1}}\right| > \left|\frac{d}{d\xi}(\epsilon y)\right| = \left|\frac{1}{\xi_o}\right|$$

for $\xi > 1$, the branch of ϵx for $\xi > \xi_o$ always lies above ϵy. The point $\xi_o = \sqrt{k_1/\kappa} = \sqrt{k_1 U_o^2/g}$ in Fig. 3 is particularly important. It corresponds to the point source of disturbance in the physical flow field from which all group lines originate. Therefore, it is a natural consequence from the requirement of $\xi > 1$ that the appropriate group lines of the wave region are given by $\xi_o > 1$. The limiting group line from $\xi_o = 1$ together with the line $\xi = 1$ for all k_1, $\sqrt{\frac{k_1}{\kappa}} \geqslant 1$, thus provide a bound for the wave region. This is shown in Fig. 4 in which typical group lines and lines of constant ξ are plotted inside the wave region.

The group lines are obtained from (3.58) and (3.52) with $k_1 (>\kappa)$ being kept constant on each line. In contrast to the case of stationary point disturbance on a uniform flow where the group lines are straight rays from the origin, in the present case of a linear shear stream, aside from the line $y = 0$ which remains straight, all the group lines are cubic far away from the origin. This is easily deduced from (3.55) and (3.58) since for ξ large $\epsilon x \propto \xi^3$ while $\epsilon y \propto \xi$ so that $\epsilon x \propto (\epsilon y)^3$. When the equation of the group lines are expressed in the form

$$\epsilon y = fn(\epsilon x)$$

it can be shown that for $y > 0$, they are monotonic increasing functions extending from the origin to infinity in the first quadrant of the x-y plane. For $y < 0$, they decrease monotonically, terminating with zero slope $(dy/dx = 0)$ on $\xi = 1$. This is obvious from (3.43) and (3.54).

From (3.58) and (3.52), keeping $\xi(\geq 1)$ constant and varying $k_1(\geq \kappa)$ we obtain the lines of constant ξ. With the exception of the line $\xi = 1$, each $\xi = \xi_c$ = constant(>1) has two branches. For $1 < \sqrt{\frac{k_1}{\kappa}} < \xi_c$, the positive branch starts from a point on the limiting group line $\sqrt{\frac{k_1}{\kappa}} = 1$ and decreases, for increasing k_1 towards the origin. For $\sqrt{\frac{k_1}{\kappa}} > \xi_c$, as k_1 increases, the negative branch which lies below $\epsilon y = 0$ decreases monotonically tending towards the asymptote $\epsilon y = -1$ which also corresponds to $\xi = 0$. The negative branch of $\xi = 1$ has the same general behavior as the negative branches of $\xi = \xi_c$ (>1), however, the positive branch is missing. This curve ($\xi = 1$) warrants special attention because, though it is not a group line, it nevertheless provides a bound for the wave region. We observe that it may be viewed as a "focal curve" or "caustic" since it is an envelope of the one parameter family of group lines. From (3.52)

$$k_1 = \frac{\kappa}{(1+\epsilon y)^2} \neq 0 \quad \text{for} \quad \xi = 1 \tag{3.60}$$

while (3.53) shows

$$\frac{k_2}{k_1} = 0 \qquad \text{for} \qquad \xi = 1 \quad . \tag{3.61}$$

Therefore, $k_2 = 0$ on $\xi = 1$, implying that all the wave crests are perpendicular to the x-axis there. Hence the net effect appears as if the waves are reflected from this curve. Beyond $\xi = 1$, it is conjectured that edge waves, with their amplitudes decreasing exponentially, may exist but we shall leave this possibility for another investigation.

To determine the traces of the wave crests, i.e. the lines of constant phase, inside the wave region, we may use the following procedure:

(i) For a given point (x_c, y_c) in the wave region we may determine the line $\xi = \xi_c$ passing through it from Fig. 4.

(ii) Either from Fig. 2, or from (3.54) we obtain

$$m_c = \mp\sqrt{\xi_c^4 - 1}\,(2\xi_c^4 - 1)^{-1} \tag{3.62}$$

which when substituted in (3.47) will give $\theta_{c\pm}$.

(iii) Then by using (3.52) in (3.48) we arrive at:

$$k_\pm = \frac{\kappa \sec^3 \theta_\pm}{\xi^2} \tag{3.63}$$

from which $k_{c\pm}$ may be deduced by evaluating the expression at $\theta_{c\pm}$ and ξ_c. Equation (3.46) now gives explicitly the components of the wave number $\vec{k}_{c\pm}$ at (x_c, y_c).

With the knowledge that the constant phase lines are orthogonal to $\vec{k}_{c\pm}$, we may graph the curves showing the traces of the wave crests. Typical traces are shown qualitatively in Fig. 5(a) for the case of a stationary point disturbance on the surface of a linear shear flow. The corresponding pattern for a point disturbance on the surface of a uniform flow is reproduced in Fig. 5(b), (Ref. 18) for comparison. Apart from the shifting of the wave crests, it is immediately obvious from Fig. 5(a) that the shear flow has the effect of suppressing the waves to a smaller region. (Dotted lines in Fig. 5(a) indicate the wave region for a uniform flow.)

In summary, we observe that while two systems of waves, one divergent and one transverse, exist inside the wave region produced by a point disturbance on the surface of a linear shear flow, the traces of the crests and the bounds to the wave region are no longer symmetric with respect to the x-axis. In particular, the wave region boundary approaches but never reaches the critical line $\epsilon y = -1$ at which $U(y) = 0$. The behavior of the waves near this critical line is not given by our analysis and will be the subject of further investigation. Finally, as a remark, it is quite obvious that essentially the same wave pattern is produced by a stationary source below the surface of a parallel linear shear stream.

3. A perturbation expansion

In order to make a comparison between the present geometric optics method and the integral transform method, which will be discussed in Chapter IV, we shall consider here a perturbation expansion in terms of a small $\widetilde{U}_1(y)$*. By regarding $|\widetilde{U}_1(y)| \ll 1$, we are actually limiting the region of validity of the resulting solution to $|y| \ll \frac{1}{\epsilon}$ in view of the definition of $\widetilde{U}_1(y)$, (3.51). For a small shear gradient (i.e. ϵ small) this may still represent an appreciably large region in the physical space.

By expanding (3.49) in powers of $\widetilde{U}_1(y)$, we have

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*It is quite obvious that the lowest order term in such an expansion represents a stationary point disturbance in a uniform stream.

$$\frac{k_2}{k_1} = \pm\sqrt{\left(\frac{k_1^2}{\kappa^2}\right)(1+\widetilde{U}_1)^4 - 1}$$

$$\approx \pm\sqrt{\left(\frac{k_1^2}{\kappa^2}\right) - 1}\left\{1 + \widetilde{U}_1 \cdot \frac{2\frac{k_1^2}{\kappa^2}}{\left(\frac{k_1^2}{\kappa^2} - 1\right)} + \widetilde{U}_1^2\left[\frac{3\frac{k_1^2}{\kappa^2}}{\left(\frac{k_1^2}{\kappa^2} - 1\right)} - \frac{2\frac{k_1^4}{\kappa^4}}{\left(\frac{k_1^2}{\kappa^2} - 1\right)^2}\right] + O(\widetilde{U}_1^3)\right\} , \tag{3.64}$$

whereas the reciprocal of (3.64) is

$$\frac{k_1}{k_2} \approx \frac{(\pm 1)}{\sqrt{\frac{k_1^2}{\kappa^2} - 1}}\left\{1 - \widetilde{U}_1 \frac{2\frac{k_1^2}{\kappa^2}}{\left(\frac{k_1^2}{\kappa^2} - 1\right)} + \widetilde{U}_1^2\left[\frac{6\frac{k_1^4}{\kappa^4}}{\left(\frac{k_1^2}{\kappa^2} - 1\right)^2} - \frac{3\frac{k_1^2}{\kappa^4}}{\left(\frac{k_1^2}{\kappa^2} - 1\right)}\right] + O(\widetilde{U}_1^3)\right\} . \tag{3.65}$$

By substituting (3.64) and (3.65) in (3.42), we obtain

$$\frac{dx}{dy} \approx \pm\left\{\frac{1}{\sqrt{\left(\frac{k_1^2}{\kappa^2} - 1\right)}} + 2\sqrt{\left(\frac{k_1^2}{\kappa^2} - 1\right)} + 2\widetilde{U}_1 \frac{k_1^2}{\kappa^2}\left[-\frac{1}{\left(\frac{k_1^2}{\kappa^2} - 1\right)^{3/2}} + \frac{2}{\sqrt{\left(\frac{k_1^2}{\kappa^2} - 1\right)}}\right]\right.$$

$$+ \widetilde{U}_1^2\left[\frac{6\frac{k_1^4}{\kappa^4}}{\left(\frac{k_1^2}{\kappa^2} - 1\right)^{5/2}} - \frac{3\frac{k_1^2}{\kappa^2}}{\left(\frac{k^2}{\kappa^2} - 1\right)^{3/2}} + \frac{6\frac{k_1^2}{\kappa^2}}{\sqrt{\left(\frac{k_1^2}{\kappa^2} - 1\right)}} - \frac{4\frac{k_1^4}{\kappa^4}}{\left(\frac{k_1^2}{\kappa^2} - 1\right)^{3/2}}\right]$$

$$\left. + O(\widetilde{U}_1^3)\right\} , \tag{3.66}$$

in which the + or - sign is taken according as y is > or < 0. Using the known result that k_1 and κ are constant on each group line and $\widetilde{U}_1(y) = \epsilon y$, we obtain by simple integration

$$\frac{x}{y} \approx \pm \left\{ \frac{1}{\sqrt{\left(\frac{k_1^2}{\kappa^2} - 1\right)}} + 2\sqrt{\left(\frac{k_1^2}{\kappa^2} - 1\right)} + \epsilon y \frac{k_1^2}{\kappa^2}\left[- \frac{1}{\left(\frac{k_1^2}{\kappa^2} - 1\right)^{3/2}} + \frac{2}{\sqrt{\frac{k_1^2}{\kappa^2} - 1}} \right] \right.$$

$$+ (\epsilon y)^2 \left[\frac{2 \frac{k_1^4}{\kappa^4}}{\left(\frac{k_1^2}{\kappa^2} - 1\right)^{5/2}} - \frac{\frac{k_1^2}{\kappa^2}}{\left(\frac{k_1^2}{\kappa^2} - 1\right)^{3/2}} + \frac{2 \frac{k_1^2}{\kappa^2}}{\sqrt{\frac{k_1^2}{\kappa^2} - 1}} - \frac{4}{3} \frac{\frac{k_1^4}{\kappa^4}}{\left(\frac{k_1^2}{\kappa^2} - 1\right)^{3/2}} \right]$$

$$\left. + O((\epsilon y)^3) \right\}, \qquad (3.67)$$

which is the required expansion of the group lines. In (3.67), the integration constant has been chosen for a point disturbance located at $x = 0$, $y = 0$, so that all the group lines pass through the origin of the x-y plane. Plotting of the group lines from (3.67) is straightforward. However, as the group lines over a much larger region have already been traced in Fig. 4, we will not repeat this operation.

We proceed to determine the boundary of wave region and the loci of constant phases (such as the wave crests) within the wave region. The extent of the wave region can be deduced from Eq. (3.67) for the ray traces. To simplify writing, we introduce the parameter

$$\beta = (k_1^2/\kappa^2 - 1)^{\frac{1}{2}} \qquad (3.68)$$

so that (3.67) becomes

$$M = \left(2\beta + \frac{1}{\beta}\right) + \epsilon y\left(2\beta + \frac{1}{\beta} - \frac{1}{\beta^3}\right) + \frac{1}{3}(\epsilon y)^2\left(2\beta + \frac{1}{\beta} + \frac{5}{\beta^3} + \frac{6}{\beta^5}\right) + O(\epsilon y)^3, \tag{3.69}$$

where

$$M \equiv x/|y| \quad , \tag{3.70}$$

which may be regarded as a function of $(\beta\,;\,\epsilon y)$, possessing the expansion for (ϵy) small. Along each ray track, k_1, and hence β remain constant. It therefore follows that at the bounding ray track, which envelopes the wave field, one must have $(\partial M/\partial\beta) = 0$, the differentiation being for fixed y. Now, by (3.69),

$$\left(\frac{\partial M}{\partial \beta}\right)_y = \left(2 - \frac{1}{\beta^2}\right) + \epsilon y\left(2 - \frac{1}{\beta^2} + \frac{3}{\beta^4}\right) + \frac{1}{3}(\epsilon y)^2\left(2 - \frac{1}{\beta^2} - \frac{15}{\beta^4} - \frac{30}{\beta^6}\right) = 0 \; . \tag{3.71}$$

The solution of (3.71), say $\beta = \beta_*(\epsilon y)$, is obtained to have the expansion

$$\frac{1}{\beta_*} = \sqrt{2}\left[1 + 3\epsilon y + \frac{7}{2}(\epsilon y)^2 + O(\epsilon y)^3\right] \quad , \tag{3.72}$$

as can readily be verified. It may be remarked here that only the positive branch of β_* is chosen since M, as defined by (3.70), is non-negative. Substituting β_* of (3.72) for β in (3.69), we obtain after some regrouping, the following result for the boundary of the wave field,

$$x_*(y) = 2\sqrt{2}\,|y|\left[1 + \frac{3}{2}(\epsilon y)^2 + O(\epsilon y)^3\right] \; . \tag{3.73}$$

It is of interest to note that the terms of order (ϵy) cancel out in the final result of $x_*(y)$; hence the boundary of wave region is symmetric with respect to the x-axis up to $O(\epsilon y)^2$. It is further noted that the stationary waves exist in the region $x > x_*(y)$, that is, on the downstream side of $x = x_*(y)$. Although the envelope of the wave field is symmetric in y, the detailed wave pattern is generally not symmetric with respect to the body trajectory $y = 0$. Since, as indicated by (3.69), β, and hence k_1, are neither even or odd in y. In fact, the value of $k_1(x_*(y), y)$, which is the x-component of the wave number at the boundary of the wave field, is different for different signs of y, as can be seen from $k_1(x_*, y) = \kappa(1+\beta_*^2)^{\frac{1}{2}}$ and (3.72). This also implies that $x = x_*(y)$ and $x = x_*(-y)$ are no longer ray tracks.

The above result may be expressed alternately in terms of cylindrical polar coordinates (r,σ) defined by

$$x = r\cos\sigma \quad , \qquad y = r\sin\sigma \quad , \tag{3.74}$$

so that at the boundary $x = x_*(y)$, the value of $\sigma = \sigma_*(r;\epsilon)$ may be expanded as

$$\sigma_*(r;\epsilon) = \sigma_0(r) + \epsilon r\sigma_1(r) + (\epsilon r)^2\sigma_2(r) + O(\epsilon r)^3 \quad . \tag{3.75}$$

Upon substitution of (3.74) and (3.75) in (3.73), together with necessary expansions, σ_0, σ_1, σ_2 are readily determined; the final result is

$$\sigma_0 = \tan^{-1}\frac{1}{2\sqrt{2}} = \sin^{-1}\left(\frac{1}{3}\right) \quad , \qquad \sigma_1 = 0 \quad ,$$

$$\sigma_2 = -3\sqrt{2}\,\sin^4\sigma_0 = -\frac{\sqrt{2}}{27} \quad . \tag{3.76}$$

Thus,

$$\sigma_*(\epsilon r) = \tan^{-1}\frac{x_*}{y} = \pm\left[\sin^{-1}(1/3) - \frac{\sqrt{2}}{27}(\epsilon r)^2 + O(\epsilon r)^3\right] \qquad (3.77)$$

in which the + or - sign is for $y >$ or < 0. This result shows that the over-all effect of the uniform shear is suppressing the wave field to a smaller region than in the uniform flow case, the deviation being of second order for (ϵr) small. These salient features are shown in Fig. 6.

The contours of constant phases (such as the wave crests) can be calculated as follows. We note first that the slope of a constant phase line, ϕ = const., at the free surface is given by

$$\left(\frac{dy}{dx}\right)_{\phi=c} = -\frac{\partial\phi/\partial x}{\partial\phi/\partial y} = -\frac{k_1}{k_2} \qquad (3.78)$$

by virtue of the characteristic equations, or more directly by the definition of the phase function. The term k_1/k_2 on the right hand side of (3.78) can be regarded as a known function of x and y in view of (3.68), (3.69) and (3.64). Consequently, the above first order differential equation can, in principle, be integrated. However, the actual calculation may prove formidable. In view of the complicated nature of the functions $k_1(x,y)$, $k_2(x,y)$, it is best to seek a parametric integration. It turns out that a convenient parameter is Θ, defined by

$$k_1/\kappa = \sec\Theta \quad , \qquad \text{or} \qquad \beta = (k_1^2/\kappa^2 - 1)^{\frac{1}{2}} = \tan\Theta \quad . \qquad (3.79)$$

Then (3.69) becomes

$$x = \pm\, y\{p(\Theta) + \epsilon y\, q(\Theta) + \epsilon^2 y^2 s(\Theta)\} \quad , \tag{3.80}$$

with

$$p(\Theta) = 2 \tan\Theta + \cot\Theta \quad , \quad q(\Theta) = p(\Theta) - \cot^3\Theta \quad ,$$

$$s(\Theta) = \frac{1}{3} q(\Theta) + 2\cot^3\Theta\csc^2\Theta \quad .$$

By making use of the expansion (3.65), (3.78) can be expressed in terms of Θ as

$$\left(\frac{dx}{dy}\right)_{\phi=c} = (\operatorname{sgn} y)\left|\frac{k_2}{k_1}\right| = (\operatorname{sgn} y)\tan\Theta\,\{1 + 2\epsilon y\csc^2\Theta + \epsilon^2 y^2 \csc^2\Theta(1 - 2\cot^2\Theta)\}. \tag{3.81}$$

We assume that the integral of (3.81) may be expressed in the following form

$$x = \sum_{n=o}^{2} \epsilon^n x_n(\Theta) + O(\epsilon^3) \quad , \quad y = \sum_{n=o}^{2} \epsilon^n y_n(\Theta) + O(\epsilon^3) \quad . \tag{3.82}$$

The two sets of functions $\{x_n(\Theta)\}$ and $\{y_n(\Theta)\}$ are not linearly independent since they are related by (3.80). In fact, upon substitution of (3.82) in (3.80), we find that for $y > 0$,

$$x_o = y_o p \quad , \quad x_1 = y_1 p + y_o^2 q \quad , \quad x_2 = y_2 p + y_1^2 q + y_o^3 s \quad , \tag{3.83}$$

whereas for $y < 0$ the signs of all the terms on the right hand sides of (3.83) are changed. Furthermore, substituting (3.82) in (3.81) yields the following sets of differential equations

$$\dot{x}_0 = \pm \dot{y}_0 \tan \Theta \ ,$$

$$\dot{x}_1 = \pm [\dot{y}_1 + 2y_0\dot{y}_0 \csc^2\Theta] \tan\Theta \ , \tag{3.84}$$

$$\dot{x}_2 = \pm \{\dot{y}_2 + 2y_0\dot{y}_1 \csc^2\Theta + \dot{y}_0[2y_1 + y_0^2(1 - 2\cot^2\Theta)]\csc^2\Theta\}\tan\Theta \ ,$$

where $\dot{x}_0$ denotes $dx_0(\Theta)/d\Theta$, etc, and the + or - sign is for $y >$ or < 0 respectively. Differentiating the first equation of (3.83), we have

$$\dot{x}_0 = \dot{y}_0 p + y_0 \dot{p} = (2\tan\Theta + \cot\Theta)\dot{y}_0 + (2\sec^2\Theta - \csc^2\Theta)y_0 \ .$$

After $\dot{x}_0$ is eliminated by making use of (3.84) we obtain the equation for y_0 as

$$\dot{y}_0 = (\cot\Theta - 2\tan\Theta)y_0 \ , \tag{3.85}$$

which has the integral

$$y_0 = A_0 \sin\Theta\cos^2\Theta \ , \tag{3.86}$$

where A_0 is an arbitrary constant of integration.

Similarly, from the second equation of (3.83) and (3.84) we derive the differential equation for y_1 as

$$\frac{d}{d\Theta}\left(\frac{y_1}{\sin\Theta\cos^2\Theta}\right) = 2A_0\dot{y}_0 - \frac{1}{\cos\Theta}\frac{d}{d\Theta}(y_0^2 q) \ . \tag{3.87}$$

This equation can be integrated explicitly, giving

$$y_1 = A_1 \sin\Theta\cos^2\Theta + A_0^2\cos^4\Theta\cos 2\Theta \ , \tag{3.88}$$

A_1 being another arbitrary constant. Integration of higher order equations becomes increasingly tedious; the integral of y_2 will not be given here.

Summarizing, we have determined the lines of constant phase in the parametric form

$$x = A_o\{\cos\Theta(1+\sin^2\Theta) + 2\epsilon A_o \sin\Theta\cos^5\Theta + O(\epsilon A_o)^2\} \quad , \tag{3.89}$$

$$y = A_o\{\sin\Theta\cos^2\Theta + \epsilon A_o \cos^4\Theta\cos 2\Theta + O(\epsilon A_o)^2\} \quad , \tag{3.90}$$

in which the constant A_1 has been absorbed into the higher order terms. The constant A_o can be related to the phase ϕ by noting that the phase function

$$\phi \equiv xk_1 + yk_2 = A_o\kappa\{1 + O(\epsilon A_o)\} \quad , \tag{3.91}$$

where in the last step use has been made of (3.89), (3.90) for $x(\Theta)$ and $y(\Theta)$, and of (3.81) for (k_2/k_1), (3.79) for k_1. Hence

$$A_o = \phi/\kappa = \phi U_o^2/g \quad ; \tag{3.92}$$

and from a wave crest to another, ϕ changes by $2n\pi(n=1,2,\ldots)$.

The leading terms of $x(\Theta)$ and $y(\Theta)$ are the well-known Kelvin ship wave pattern in a uniform stream. There are two wave systems within the region $|\arg\tan(y/x)| < \sin^{-1}(1/3)$, one corresponding to the range $0 < \Theta < \Theta_*$, $(\Theta_* = \tan^{-1}\beta_* = \tan^{-1}(\sqrt{2}/2))$ and the other to $\Theta_* < \Theta < \pi/2$, called respectively the diverging and transverse waves. In the present case of a uniform shear, the position and wavelength of these waves are shifted by an amount of order ϵ. From (3.90) it is noted that $|k_2|$ increases for $y > 0$ (in which region $k_2 < 0$) and decreases for $y < 0$ (where $k_2 > 0$) with increasing shear gradient ϵ. Thus the resulting configurations of constant phase lines are not symmetric with respect to the x-axis; these qualitative features are shown in Fig. 6. The perturbation expansion studied in this section shows that the resulting steady wave pattern due to a point disturbance on the surface of a linear parallel shear flow in the region $y \ll 1/\epsilon$ has essentially the same basic features as the solution of the previous section. Thus, it constitutes a good approximation. The results obtained here will be compared with that obtained by the method of integral transform to be investigated in the next chapter.

IV. INTEGRAL TRANSFORM METHOD

A combination of Fourier transform and Laplace transform has been used by several writers (see DePrima and Wu (Ref. 2) and Wu and Mei (Ref. 21)) in dealing with problems of steady and unsteady surface waves. A slight extension of the classical Fourier transform method will be made in this section to investigate the steady wave pattern in the surface of a linear parallel shear flow. A submerged source of constant strength is located at $x = 0$, $y = 0$, $z = -h$ i.e.

$$Q(\vec{x}, t) = m\delta(x)\delta(y)\delta(z+h)H(t) , \tag{4.1}$$

where $\delta(x)$ and $H(t)$ represent the Dirac delta function and the Heaviside step function respectively. Hence for $t > 0^+$ (2.17) becomes

$$D\nabla^2 v - D''v = m D_o \delta(x)\delta'(y)\delta(z+h) , \tag{4.2}$$

where $D_o = \frac{\partial}{\partial t} + U_o \frac{\partial}{\partial x}$. Although the Laplace transform may be used with respect to t and then the Tauberian theorem be applied to the resulting integrals for determining the steady state limit of the solution, it is simpler to formulate and solve the steady problem by using the artifice of Rayleigh's coefficient. We shall adopt this simpler approach.

The steady state differential equation may be deduced from (4.2) by taking $D \approx U(y)\partial/\partial x$, so that with $U(y) = U_o(1+\epsilon y)$ we have

$$U(y) \frac{\partial}{\partial x} \nabla^2 v = m U_o \frac{\partial}{\partial x} \delta(x)\delta'(y)\delta(z+h) ,$$

which after division by $U(y)$ and integration with respect to x, from $x = -\infty$, gives

$$\nabla^2 v = m \frac{U_o}{U(y)} \delta(x)\delta'(y)\delta(z+h) \quad .$$

By applying the identity

$$f(y)\delta'(y) = f(0)\delta'(y) - f'(0)\delta(y) \quad ,$$

we arrive at

$$\nabla^2 v = m[\delta'(y) + \epsilon\delta(y)]\delta(x)\delta(z+h) \quad . \tag{4.3}$$

We introduce a new function Φ by

$$v = \left(\frac{\partial}{\partial y} + \epsilon\right) \Phi \; . \tag{4.4}$$

This function Φ closely resembles the velocity potential of irrotational flow and may be called a "modified velocity potential." In fact, it becomes a velocity potential when ϵ vanishes. In this new variable, (4.3) may be written, after an integration with respect to y, as

$$\nabla^2 \Phi = \delta(x)\delta(y)\delta(z+h) \quad . \tag{4.5}$$

Derivation of the free surface condition for the case of steady state can be accomplished in a similar manner. By taking $D \approx U_o(1+\epsilon y) \frac{\partial}{\partial x}$ in (2.21) we obtain

$$\frac{\partial}{\partial y}\left[(1+\epsilon y)^2 \frac{\partial^2}{\partial x^2} + \kappa \frac{\partial}{\partial z}\right] v = -2\epsilon(1+\epsilon y) \frac{\partial^2 v}{\partial x^2} \qquad (z = 0) \quad , \tag{4.6}$$

where $\kappa = g/U_o^2$. In terms of Φ, (4.6) becomes

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} + \epsilon\right)\left[(1+\epsilon y)^2 \frac{\partial^2}{\partial x^2} + \kappa \frac{\partial}{\partial z}\right]\Phi = -2\epsilon^3 y \frac{\partial^2 \Phi}{\partial x^2} \qquad (z = 0) \; . \tag{4.7}$$

The solution of (4.5) subject to the free surface condition (4.7) may be obtained but so much extra work has to be done that it is better to keep the theory intelligible by neglecting the right-hand side of (4.7), which is of order $O(\epsilon^3)$, so that by integrating the reduced equation twice with respect to y, we finally have

$$\left[(1+\epsilon y)^2 \frac{\partial^2}{\partial x^2} + \kappa \frac{\partial}{\partial z}\right] \Phi = 0 \qquad (z = 0) \quad . \tag{4.8}$$

The resulting solution of (4.5) with condition (4.8) will be valid only when (ϵy) is small i.e. $y \ll 1/\epsilon$. Therefore the range of validity, when (4.8) is used, of the solution coincides with that of the perturbation theory of Chapter III and the results can be readily compared. Even after the free surface boundary condition is considerably simplified, its approximate form (4.8) still presents great difficulties, due mainly to the variable coefficients involving y, on an analytical approach to the problem.

We shall seek the solution of (4.5) with condition (4.8) by introducing a double Fourier transform

$$\tilde{\Phi} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\alpha x+\beta y)} \Phi \, dx \, dy. \tag{4.9}$$

In order for the Fourier transform of Φ to exist, we must further require Φ and $|\text{grad}\,\Phi|$ to be absolutely integrable with respect to x and y for every fixed $z \leqslant 0$. These conditions are associated with the physical argument that the wave velocity is finite for waves having finite wave length (see Ref. (2)) and will be assumed without proof. The transform of (4.5) then gives

$$\widetilde{\Phi}_{zz} - k^2\widetilde{\Phi} = m\,\delta(z+h), \tag{4.10}$$

where $k^2 = \alpha^2 + \beta^2$. The appropriate jumps for $\widetilde{\Phi}$ and its derivative are:

$$[\widetilde{\Phi}]_{-h_-}^{-h_+} = 0 \qquad \text{and} \qquad [\widetilde{\Phi}_z]_{-h_-}^{-h_+} = m \quad , \tag{4.11}$$

so that the solution of (4.10) has the form

$$\widetilde{\Phi} = - \frac{m}{2k}\, e^{-k|z+h|} + \widetilde{B}\, e^{k(z-h)}, \tag{4.12}$$

where k denotes the positive branch of $(\alpha^2+\beta^2)^{\frac{1}{2}}$, $\widetilde{B}(\alpha,\beta)$ represents an arbitrary function of α and β, to be determined by using the free surface condition (4.8). Hence, an integral representation of Φ for $z > -h$ is

$$\Phi = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\alpha x+\beta y)}\left[-\frac{m}{2k}\, e^{-kz} + \widetilde{B}\, e^{kz}\right] e^{-kh}\, d\alpha\, d\beta,$$

whose first partial derivative with respect to z is

$$\Phi_z = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\alpha x+\beta y)} k\left[\frac{m}{2k}\, e^{-kz} + \widetilde{B} e^{kz}\right] e^{-kh}\, d\alpha\, d\beta \quad .$$

These integral representations may now be substituted into the reduced free surface condition (4.8) to obtain

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[(1+\epsilon y)^2\alpha^2\left(\frac{m}{2k} - \widetilde{B}\right) + \kappa k\left(\frac{m}{2k} + \widetilde{B}\right)\right] e^{-kh} e^{i(\alpha x+\beta y)}\, d\alpha\, d\beta = 0\ . \tag{4.13}$$

By defining

$$\overline{B} \equiv \alpha^2\left(\widetilde{B} - \frac{m}{2k}\right) e^{-kh} \quad , \tag{4.14}$$

Eq. (4.13) may be rewritten as

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\alpha x}\left[-m\kappa e^{-kh} e^{i\beta y} + \left(1 - \frac{\kappa k}{\alpha^2}\right)\overline{B} e^{i\beta y}\right.$$

$$\left. - i2\epsilon\overline{B}\frac{\partial}{\partial\beta}\left(e^{i\beta y}\right) - \epsilon^2\overline{B}\frac{\partial^2}{\partial\beta^2}\left(e^{i\beta y}\right)\right] d\alpha d\beta = 0 \quad .$$

After integrating the above integrand by parts, with respect to β, we obtain

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\alpha x+\beta y)}\left[-m\kappa e^{-kh} + \left(1 - \frac{\kappa k}{\alpha^2}\right)\overline{B} + i2\epsilon\frac{\partial\overline{B}}{\partial\beta} - \epsilon^2\frac{\partial^2\overline{B}}{\partial\beta^2}\right] d\alpha d\beta = 0 , \tag{4.15}$$

in which the boundary terms are assumed to vanish since $e^{-kh} = 0$ at $\beta = \pm\infty$. Requiring the integrand in (4.15) to vanish identically gives the following differential equation for $\overline{B}$

$$\epsilon^2\frac{\partial^2\overline{B}}{\partial\beta^2} - i2\epsilon\frac{\partial\overline{B}}{\partial\beta} - \left(1 - \frac{\kappa k}{\alpha^2}\right)\overline{B} + m\kappa e^{-kh} = 0 \quad . \tag{4.16}$$

For ϵ small, which is the case under consideration, (4.16) is singular as $\epsilon \to 0$. However, for our purpose a straightforward series expansion in ϵ will suffice. We assume

$$\overline{B} = \overline{B}_o + \epsilon\overline{B}_1 + \epsilon^2\overline{B}_2 + O(\epsilon^3) , \tag{4.17}$$

where terms of $O(\epsilon^3)$ will be omitted. Substitution of (4.17) into (4.16) and equating like powers of ϵ, yields

$$\overline{B}_o\left(1 - \frac{\kappa k}{\alpha^2}\right) - m\kappa e^{-kh} = 0, \tag{4.18}$$

$$\overline{B}_1\left(1 - \frac{\kappa k}{\alpha^2}\right) + i2\frac{\partial\overline{B}_o}{\partial\beta} = 0, \tag{4.19}$$

$$\overline{B}_2\left(1 - \frac{\kappa k}{\alpha^2}\right) + i2\frac{\partial\overline{B}_1}{\partial\beta} - \frac{\partial^2\overline{B}_o}{\partial\beta^2} = 0 \quad . \tag{4.20}$$

The successive solutions of these algebraic equations are

$$\overline{B}_o = \frac{m\kappa\alpha^2}{\alpha^2 - \kappa k} e^{-kh},$$

$$\overline{B}_1 = -2i \frac{\alpha^2}{\alpha^2 - \kappa k} \frac{\partial}{\partial\beta} \overline{B}_o ,$$

$$\overline{B}_2 = \frac{\alpha^2}{\alpha^2 - \kappa k} \frac{\partial^2}{\partial\beta^2} \overline{B}_o - 2i \frac{\alpha^2}{\alpha^2 - \kappa k} \frac{\partial}{\partial\beta} \overline{B}_1 .$$

By carrying out the differentiation with respect to β on the right-hand side and combining (4.17) and (4.14), we have

$$\widetilde{B} = \frac{m}{2k} + B, \tag{4.21}$$

where

$$\begin{aligned} B = {} & \frac{m\kappa}{\alpha^2 - \kappa k} - i\epsilon m\kappa \left[- \frac{2h\alpha^2\beta}{k(\alpha^2 - \kappa k)^2} + \frac{2\kappa\alpha^2\beta}{k(a^2 - \kappa k)^3} \right] \\ & + \epsilon^2 m\kappa \left[\frac{h^2\alpha^2\beta^2}{k^2(\alpha^2 - \kappa k)^2} - \frac{h\alpha^4}{k^3(\alpha^2 - \kappa k)^2} - \frac{2\kappa h\alpha^2\beta^2}{k^2(\alpha^2 - \kappa k)^3} \right. \\ & - \frac{4h^2\alpha^4\beta^2}{k^2(\alpha^2 - \kappa k)^3} + \frac{4h\alpha^6}{k^3(\alpha^2 - \kappa k)^3} + \frac{\kappa\alpha^4}{k^3(\alpha^2 - \kappa k)^3} + \frac{2\kappa^2\alpha^2\beta^2}{k^2(\alpha^2 - \kappa k)^4} \\ & \left. - \frac{4\kappa\alpha^6}{k^3(\alpha^2 - \kappa k)^4} + \frac{12\kappa h\alpha^4\beta^2}{k^2(\alpha^2 - \kappa k)^4} - \frac{12\kappa^2\alpha^4\beta^2}{k^2(\alpha^2 - \kappa k)^5} \right] + O(\epsilon^3) . \end{aligned} \tag{4.22}$$

As a remark, it can be seen that only the part B of $\widetilde{B}$ will contribute to the wave motion. This will become clear shortly. For the moment, an integral representation of Φ may be written as

$$\Phi = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\alpha x + \beta y)} \left[- \frac{m}{2k} e^{-k|z+h|} + \frac{m}{2k} e^{k(z-h)} + Be^{k(z-h)} \right] d\alpha d\beta, \tag{4.23}$$

where $B = B(\alpha^2, \beta^2, k, i\beta)$ is given by (4.22). It is easily observed that Φ is real because its complex conjugate Φ^* equals Φ. Therefore, it follows that

$$\Phi = \mathrm{Re}\ \frac{1}{2\pi^2}\int_{-\infty}^{\infty} d\beta \int_{0}^{\infty} d\alpha\, e^{i(\alpha x+\beta y)}\left[-\frac{m}{2k}\, e^{-k|z+h|} + \frac{m}{2k}\, e^{k(z-h)} + B e^{k(z-h)}\right], \quad (4.24)$$

where "Re" stands for "the real part of" the integral.

Next, we introduce two sets of polar co-ordinates

$$\left.\begin{aligned} \alpha &= k\cos\theta\ , & \beta &= k\sin\theta\ , \\ x &= r\cos\sigma\ , & y &= r\sin\sigma\ , \end{aligned}\right\} \quad (4.25)$$

so that $(\alpha x+\beta y) = kr\cos(\theta-\sigma)$, and (4.24) becomes

$$\Phi = \mathrm{Re}\ \frac{1}{2\pi^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_{0}^{\infty} dk\, e^{ikr\cos(\theta-\sigma)}$$

$$\left[-\frac{m}{2}\, e^{-k|z+h|} + \frac{m}{2}\, e^{k(z-h)} + (kB)e^{k(z-h)}\right], \quad (4.26)$$

where (kB) may be deduced from (4.22) and (4.25) as

$$
\begin{aligned}
(kB) = {} & \frac{m\kappa \sec^2\theta}{k-\kappa \sec^2\theta} - i\epsilon\, m\kappa \sec^2\theta \left[-\frac{2hk \sin\theta}{(k-\kappa \sec^2\theta)^2} + \frac{2\kappa \sec^2\theta \sin\theta}{(k-\kappa \sec^2\theta)^3} \right] \\
& + \epsilon^2 m\kappa \sec^2\theta \left[- \frac{h}{\sec^2\theta(k-\kappa \sec^2\theta)^2} + \frac{h^2 k \sin^2\theta}{(k-\kappa \sec^2\theta)^2} + \frac{\kappa}{k(k-\kappa \sec^2\theta)^3} \right. \\
& - \frac{2h\kappa \sec^2\theta \sin^2\theta}{(k-\kappa \sec^2\theta)^3} + \frac{4hk}{\sec^2\theta(k-\kappa \sec^2\theta)^3} - \frac{4h^2 k^2 \sin\theta}{(k-\kappa \sec^2\theta)^3} \\
& + \frac{2\kappa^2 \sec^4\theta \sin^2\theta}{k(k-\kappa \sec^2\theta)^4} - \frac{4\kappa}{(k-\kappa \sec^2\theta)^4} + \frac{12hk\kappa \sec^2\theta \sin^2\theta}{(k-\kappa \sec^2\theta)^4} \\
& \left. - \frac{12\kappa^2 \sec^4\theta \sin^2\theta}{(k-\kappa \sec^2\theta)^5} \right] + O(\epsilon^3) \quad . \qquad (4.27)
\end{aligned}
$$

From the known integral

$$
(x^2+y^2+z^2)^{-\frac{1}{2}} = \operatorname{Re} \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^\infty dk\, e^{-k|z|}\, e^{ikr\cos(\theta-\sigma)}
$$

the first two terms of (4.26) can be integrated immediately, giving

$$
\Phi = -\frac{m}{4\pi}\frac{1}{R_1} + \frac{m}{4\pi}\frac{1}{R_2} + \Phi_w , \qquad (4.28)
$$

where

$$
R_1^2 = x^2 + y^2 + (z+h)^2,
$$

$$
R_2^2 = x^2 + y^2 + (z-h)^2,
$$

and

$$
\Phi_w = \operatorname{Re} \frac{1}{2\pi^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^\infty dk\, e^{ikr\cos(\theta-\sigma)} e^{k(z-h)} (kB) . \qquad (4.29)
$$

The first two terms of (4.28) represent the submerged source disturbance and its reflection in the plane $z = 0$, whereas Φ_w represents the waves generated and some local effects. To evaluate (4.29) one may regard k to be complex and apply the theory of residues. Equation (4.27) shows that the expansion for (kB) has two poles at $k = 0$, $\kappa \sec^2\theta$ in the k-plane. Defining

$$\kappa_o \equiv \kappa \sec^2\theta \quad , \tag{4.30}$$

an order of magnitude comparison of the various terms indicates that for $|k-\kappa_o| < \sqrt{\epsilon}$ and $|k| < \epsilon$, the expansion (4.27) becomes invalid because terms of $O(\epsilon^2)$ are greater than terms of $O(\epsilon)$ in these regions. Therefore, if the expansion (4.27) is to be used in (4.29) for Φ_w, the integration in the k-plane must be taken along a path on which (4.27) is valid i.e. $|k-\kappa_o| > \sqrt{\epsilon}$ and $|k| > \epsilon$.

Although (4.27), when considered as a Taylor series expansion in ϵ, can be summed to yield $(kB) = [kB(\epsilon)]$ which gives the detailed distribution of poles in the region $|k-\kappa_o| \leqslant \sqrt{\epsilon}$, it is found in Appendix A that if the path in the k-plane is taken in a region where (4.27) is valid the resulting Φ_w using (4.27) in the integrand is correct, up to $O(\epsilon^3)$. It is therefore unnecessary to consider the details of the poles in $|k-\kappa_o| \leqslant \sqrt{\epsilon}$.

Since only the real part of the integral in (4.29) is of interest, the pole at $k = 0$ does not contribute and may be neglected completely. This is shown in Appendix B.

To determine the appropriate path of integration in the k-plane, we will make use of the Rayleigh coefficient. It is shown by DePrima and Wu (Ref. 2) that this factor corresponds to a time limiting factor.

The correct use of this artifice is similar to applying a Laplace transform on t for an initial value problem and then using the Tauberian theorem to obtain the steady state solution. From the definition of $D = \frac{\partial}{\partial t} + U(y)\frac{\partial}{\partial x}$ and Eqs. (2.17) and (2.21) the method of the Rayleigh coefficient, when applied to our present problem, consists of replacing α^2 in B by $(\alpha^2 - i\mu\alpha)$ where μ is real and > 0. Furthermore, since the path of integration, when the expansion (4.27) is used, cannot be within $|k-\kappa_o| \leqslant \sqrt{\epsilon}$, we assume $\mu > \sqrt{\epsilon}$. Thus, in (4.22) for B, the factor $(\alpha^2 - \kappa k)$ becomes $(\alpha^2 - i\mu\alpha - \kappa k) =$ $k\cos^2\theta(k - i\mu\sec\theta - \kappa\sec^2\theta)$ so that the factor $(k - \kappa\sec^2\theta)$ in (4.27) is replaced by $(k - i\mu\sec\theta - \kappa\sec^2\theta)$. The only relevant pole of (kB) therefore, is at

$$k = \kappa\sec^2\theta + i\mu\sec\theta \quad ,$$

which is in the first quadrant because for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\sec\theta > 1$. Hence, the appropriate closed contour Γ for the k-integral of (4.29) is as follows: The original path along the positive real axis, broken at $k = \kappa_o$, is connected up by a small semicircle of radius $\sqrt{\epsilon} + \gamma$ $(\gamma > 0)$ in the lower half plane and is joined by a large arc of radius R in the upper (or lower) half plane for $\cos(\theta-\sigma) > 0$ (or $\cos(\theta-\sigma) < 0$), and back to $k = 0$ by the imaginary axis (see Fig. 7). The asymptotic behavior of Φ_w for large r will be considered for $\cos(\theta-\sigma) \gtrless 0$ separately.

(i) $\cos(\theta-\sigma) < 0$

In this case, the appropriate closed contour is $\Gamma = \Gamma_1 + \Gamma_4 + \Gamma_5$ (see Fig. 7). By Cauchy's integral theorem and upon passing to the limit $R \to \infty$, the k-integral of (4.29) becomes

$$-\int_{\Gamma_5} dk\, e^{ikr\cos(\theta-\sigma)} e^{k(z-h)}(kB) = -i\int_0^{\infty} d\eta\, e^{\eta r\cos(\theta-\sigma)} e^{-i\eta(z-h)}(kB),$$

where $k = -i\eta$, η real > 0, has been used in the last integral. For r large, we observe that the integrand of the above integral is exponentially small, implying that Φ_w will tend to zero as $r \to \infty$. Since for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\cos(\theta-\sigma) < 0$ and (4.25) imply $x < 0$, the natural consequence of the above is that waves do not exist upstream of the source disturbance. This is in agreement with the results of Chapter III.

(ii) $\cos(\theta-\sigma) > 0$

From $\cos(\theta-\sigma) > 0$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, we may deduce that for $0 < \sigma < \frac{\pi}{2}$ (i.e. $x > 0$, $y > 0$), $-\frac{\pi}{2} + \sigma < \theta < \sigma$ while for $-\frac{\pi}{2} < \sigma < 0$ (i.e. $x > 0$, $y < 0$), $\sigma < \theta < \frac{\pi}{2} + \sigma$. Therefore, with $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$ (see Fig. 7), application of Cauchy's integral theorem to (4.29), and passing to the limit $R \to \infty$, yields

$$\Phi_w = \mathrm{Re}\, \frac{1}{2\pi^2} \int_{-\frac{\pi}{2}+\sigma}^{\sigma} d\theta\, 2\pi i\, \mathrm{Res}\left(e^{ikr\cos(\theta-\sigma)} e^{k(z-h)} kB\right)_{k=\kappa_0} \qquad \left(0 < \sigma < \frac{\pi}{2}\right), \tag{4.31}$$

and

$$\Phi_w = \mathrm{Re}\, \frac{1}{2\pi^2} \int_{\sigma}^{\frac{\pi}{2}+\sigma} d\theta\, 2\pi i\, \mathrm{Res}\left(e^{ikr\cos(\theta-\sigma)} e^{k(z-h)} kB\right)_{k=\kappa_0} \qquad \left(-\frac{\pi}{2} < \sigma < 0\right), \tag{4.32}$$

where the integral on Γ_3 have also been discarded because by an argument similar to that given in (i) they can be shown to be small for large kr. The factor

$$\operatorname{Res}(e^{ikr\cos(\theta-\sigma)}e^{k(z-h)})_{k=\kappa_o} \equiv J \qquad (4.33)$$

in (4.31) and (4.32) denotes the residue at $k = \kappa_o$ of the quantity inside the brackets and is the only term contributing to the wave disturbance. It has been evaluated for large r in Appendix C. The final result is

$$J = A_o[1+\epsilon r A_1 + \ldots]e^{i\kappa r[\varphi_o+(\epsilon r)\varphi_1+(\epsilon r)^2\varphi_2 \ldots]}, \qquad (4.44)$$

where

$$A_o = m\kappa \sec^2\theta\, e^{\kappa\sec^2\theta(z-h)}, \qquad (4.45)$$

$$A_1 = 2(z-2h)\kappa \sec^2\theta \sin\theta\cos(\theta-\sigma), \qquad (4.46)$$

$$\varphi_o = \sec^2\theta\cos(\theta-\sigma), \qquad (4.47)$$

$$\varphi_1 = \sec^2\theta\sin\theta\cos^2(\theta-\sigma), \qquad (4.48)$$

$$\varphi_2 = \left[\frac{2}{3} - \frac{1}{3}\sec^2\theta\sin^2\theta\right]\cos^3(\theta-\sigma), \qquad (4.49)$$

so that when substituted in (4.31) and (4.32),

$$\Phi_w = \operatorname{Re}\frac{i}{\pi}\int_{\theta_1}^{\theta_2} d\theta\, A_o[1+\epsilon r A_1 \ldots]e^{i\kappa r[\varphi_o+(\epsilon r)\varphi_1+(\epsilon r)^2\varphi_2 \ldots]} \qquad (4.50)$$

where

$$\theta_1 = \begin{cases} -\frac{\pi}{2}+\sigma \\ \sigma \end{cases} \quad \text{and} \quad \theta_2 = \begin{cases} \sigma \\ \frac{\pi}{2}+\sigma \end{cases} \quad \text{for } \sigma \gtrless 0 .$$

The integral representation (4.50) is in a suitable form from which the asymptotic behavior of Φ_w may be investigated by the method of stationary phase for large values of κr provided $\epsilon r \ll 1 \ll \kappa r$.

The main idea of the principle of stationary phase is that the significant contribution of the integral of (4.50) comes from a small range of θ centered at the critical points of the phase function

$$\varphi(\theta, \sigma, \epsilon r) = \varphi_0(\theta, \sigma) + \epsilon r \varphi_1(\theta, \sigma) + (\epsilon r)^2 \varphi_2(\theta, \sigma) \quad . \tag{4.51}$$

These critical points are the points of stationary phase given by the solution of the equation

$$\frac{\partial \varphi}{\partial \theta} = \frac{\partial \varphi_0}{\partial \theta}(\theta, \sigma) + (\epsilon r)\frac{\partial \varphi_1}{\partial \theta}(\theta, \sigma) + (\epsilon r)^2 \frac{\partial \varphi_2}{\partial \theta}(\theta, \sigma) = 0 \quad . \tag{4.52}$$

Let the root of Eq. (4.52) be denoted by $\theta = \theta_c(\sigma, \epsilon r)$, which may be expanded for small ϵr as

$$\theta_c = \theta_0(\sigma) + \epsilon r\, \theta_1(\sigma) + (\epsilon r)^2 \theta_2(\sigma) + O(\epsilon r)^3 \quad . \tag{4.53}$$

By substituting (4.53) in (4.52), expanding the resulting terms for small ϵr, we obtain the following equations for θ_0, θ_1 and θ_2:

$$\frac{\partial \varphi_0}{\partial \theta}(\theta_0, \sigma) = 0 \quad , \tag{4.54}$$

$$\theta_1 \frac{\partial^2 \varphi_0}{\partial \theta^2}(\theta_0, \sigma) + \frac{\partial \varphi_1}{\partial \theta}(\theta_0, \sigma) = 0 \quad , \tag{4.55}$$

$$\theta_2 \frac{\partial^2 \varphi_0}{\partial \theta^2}(\theta_0, \sigma) + \frac{\theta_1^2}{2}\frac{\partial^3 \varphi_0}{\partial \theta^3}(\theta_0, \sigma) + \theta_1 \frac{\partial^2 \varphi_1}{\partial \theta^2}(\theta_0, \sigma) + \frac{\partial \varphi_2}{\partial \theta}(\theta_0, \sigma) = 0 \quad . \tag{4.56}$$

By making use of (4.47), (4.54) becomes

$$2\tan\theta_o\cos(\theta_o-\sigma) - \sin(\theta_o-\sigma) = 0 \quad , \tag{4.57}$$

which has two roots given by

$$\tan\theta_{o\pm} = \frac{-1 \pm \sqrt{1-8\tan^2\sigma}}{4\tan\sigma} \quad . \tag{4.58}$$

In order to have the stationary points $\theta_{o\pm}$ real, and hence to remain on the path of integration, we must have $|\sigma| < |\sigma_o| = |\sin^{-1}(1/3)|$. This, indeed, is identical to (2.32), which states that the wave region is bounded by the lines $|\sigma| = |\sigma_o|$.

From Eq. (4.55)

$$\theta_1 = -\frac{\partial\varphi_1}{\partial\theta}(\theta_{o\pm},\sigma) \Big/ \frac{\partial^2\varphi_o}{\partial\theta^2}(\theta_{o\pm},\sigma) \quad .$$

By making use of the expression for φ_o and φ_1, given by (4.47) and (4.48), we finally obtain

$$\theta_1(\theta_{o\pm},\sigma) = -\cos\theta_{o\pm}\cos(\theta_{o\pm}-\sigma) \quad , \tag{4.59}$$

where the right hand side has been simplified by using (4.58). It can be remarked that $\theta_1(\theta_{o\pm},\sigma) < 0$ for all $|\sigma| < |\sigma_o|$.

From Eq. (4.56), it follows that

$$\theta_2 = -\left\{ \frac{\frac{\theta_1^2}{2}\frac{\partial^3\varphi_o}{\partial\theta^3} + \theta_1\frac{\partial^2\varphi_1}{\partial\theta^2} + \frac{\partial\varphi_2}{\partial\theta}}{\frac{\partial^2\varphi_o}{\partial\theta^2}} \right\}_{\theta=\theta_{o\pm}} \tag{4.60}$$

The right hand side of (4.60) may be simplified by making use of the expressions for φ_0, φ_1, φ_2 and θ_1, given by (4.47) - (4.49) and (4.59), respectively, yielding

$$\theta_2(\theta_{0\pm}, \sigma) = -\frac{1}{3} \frac{\cos^2(\theta_{0\pm}-\sigma)\tan\theta_{0\pm}(1-8\tan^2\theta_{0\pm})}{\sec^2\theta_{0\pm}(1-2\tan^2\theta_{0\pm})} \quad . \tag{4.61}$$

If only the first non-zero perturbation from the uniform flow solution is required, it turns out to be unnecessary to evaluate θ_2.

Hence, for every $|\sigma| < |\sigma_0|$, (4.58) provides two roots θ_{0-} and $\theta_{0+} (> \theta_{0-})$ and (4.53) gives the corresponding stationary points $\theta_{c\pm}$ of the integrand in (4.50). The behavior of Φ_w for large κr ($\epsilon r \ll 1 \ll \kappa r$), may be calculated by application of the method of stationary phase* to the integral of (4.50) to give

$$\Phi_w \sim -A(\theta_{0+})\sin\left[\kappa r\varphi(\theta_{c+}) + \frac{\pi}{4}\right] - A(\theta_{0-})\sin\left[\kappa r\varphi(\theta_{c-}) - \frac{\pi}{4}\right], \tag{4.62}$$

where

$$A(\theta_{0\pm}) = \sqrt{\frac{2}{\kappa r \pi}} \left[\frac{A_0}{\left|\frac{\partial^2\varphi_0}{\partial\theta^2}\right|^{\frac{1}{2}}} \right]_{\theta=\theta_{0\pm}} \left\{ 1 + \epsilon r\left[\frac{\theta_1}{A_0}\frac{\partial A_0}{\partial\theta} + A_1 \right.\right.$$

$$\left.\left. - \frac{1}{2}\left(\theta_1 \frac{\partial^3\varphi_0}{\partial\theta^3} + \frac{\partial^2\varphi_1}{\partial\theta^2}\right) \Big/ \frac{\partial^2\varphi_0}{\partial\theta^2}\right]_{\theta=\theta_{0\pm}} + O(\epsilon r)^2 \right\}, \tag{4.63}$$

*For the mathematical argument of this method, see Jeffreys and Jeffreys (Ref. 5) § 17.05.

and

$$\varphi(\theta_{c\pm}) = \varphi_0(\theta_{0\pm}) + \epsilon r \varphi_1(\theta_{0\pm}) + (\epsilon r)^2 \left[\frac{\theta_1}{2} \frac{\partial \varphi_1}{\partial \theta}(\theta_{0\pm}) + \varphi_2(\theta_{0\pm}) \right] + O(\epsilon r)^3 \quad . \qquad (4.64)$$

The right hand side of Eq. (4.64) has been simplified by using (4.54) and (4.55). It is of interest to note that up to $O(\epsilon r)^2$, θ_2 does not appear in the phase function. Upon substitution of the expressions for $\varphi_0, \varphi_1, \varphi_2$ and θ_1, (4.64) becomes

$$\varphi(\theta_{c\pm}) = \frac{1}{\sqrt{1+4\tan^2\theta_{0\pm}}} \left[\sec^2\theta_{0\pm} + \epsilon r \frac{\sec\theta_{0\pm} \tan\theta_{0\pm}}{\sqrt{1+4\tan^4\theta_{0\pm}}} + \frac{(\epsilon r)^2}{6} + O(\epsilon r)^3 \right] . \qquad (4.65)$$

The form of the constant phase lines (such as the wave crests and troughs) far behind the source of the two wave systems is given approximately by

$$\kappa r \varphi(\theta_{c+}) + \frac{\pi}{4} = C_+ \quad , \qquad (4.66)$$

and

$$\kappa r \varphi(\theta_{c-}) - \frac{\pi}{4} = C_- \quad , \qquad (4.67)$$

where $\varphi(\theta_{c\pm})$ is given by (4.65) and $C_\pm$ are arbitrary constants. The qualitative pattern is shown in Fig. 6. Equation (4.66) gives the diverging waves, (4.67) the transverse waves. Their shift from the well known Kelvin's ship wave pattern is apparent by comparison of Figs. 6 and 5(b). As $\sigma \to \sigma_0$, $|\sigma| < |\sigma_0|$, Eq. (4.58) shows $\theta_{0+} = \theta_{0-}$ and the two systems of waves coalesce.

The amplitude variation of these waves is given by (4.63). By using the expressions for A_0, A_1, φ_0, φ_1 and θ_1 from Eqs. (4.45) - (4.48) and (4.59) we obtain

$$A(\theta_{o\pm}) = \sqrt{\frac{2}{\kappa r \pi}}\, m\kappa \sec\theta_{o\pm} \frac{(1+4\tan^2\theta_{o\pm})^{\frac{1}{4}}}{(1-2\tan^2\theta_{o\pm})^{\frac{1}{2}}} e^{\kappa \sec^2\theta_{o\pm}(z-h)}$$

$$\left[1 + (\epsilon r)\frac{\tan\theta_{o\pm}\cos\theta_{o\pm}}{\sqrt{1+4\tan^2\theta_{o\pm}}}\left(\frac{1}{2} + 2\kappa\sec^2\theta_{o\pm}\left[-h\sec^2\theta_{o\pm} + (z-h)\tan^2\theta_{o\pm}\right]\right) + O(\epsilon r)^2\right] . \qquad (4.68)$$

The deviation from the uniform stream case [cf. Eq. (2. 34)] appears in terms of $O(\epsilon r)$. Unfortunately, not only is this expression unduly complicated but it is also not suitable in the region near the boundaries of the wave region. As $\sigma \to \sigma_o$, $|\sigma| < |\sigma_o|$, $\tan\theta_{o\pm} \to \pm \frac{1}{\sqrt{2}}$ and the amplitude becomes infinite. A special investigation (e.g. see Ursell (16)) of this region will be necessary.

We shall proceed to determine the extent of the wave region. As in Eq. (3.75), we may expand the value of $\sigma(r) = \sigma_*(r)$ at the boundary as

$$\sigma_*(r,\epsilon) = \sigma_o(r) + \epsilon r\sigma_1(r) + (\epsilon r)^2\sigma_2(r) + O(\epsilon r)^3 \quad . \qquad (4.69)$$

Upon substitution of (4.69) and (4.53) into (4.52), together with the necessary expansions, we obtain the following equations for σ_o, σ_1, and σ_2

$$\frac{\partial\varphi_o}{\partial\theta}(\theta_o^*,\sigma_o) = 0 \quad , \qquad (4.70)$$

$$\sigma_1 \frac{\partial^2\varphi_o}{\partial\theta\partial\sigma}(\theta_o^*,\sigma_o) + \theta_1^* \frac{\partial^2\varphi_o}{\partial\theta^2}(\theta_o^*,\sigma_o) + \frac{\partial\varphi_1}{\partial\theta}(\theta_o^*,\sigma_o) = 0 \quad , \qquad (4.71)$$

$$\sigma_2 \frac{\partial^2\varphi_o}{\partial\theta\partial\sigma}(\theta_o^*,\sigma_o) + \frac{\sigma_1^2}{2}\frac{\partial^3\varphi_o}{\partial\theta\partial\sigma^2}(\theta_o^*,\sigma_o) + \sigma_1\theta_1^* \frac{\partial^2\varphi_o}{\partial\theta^2\partial\sigma}(\theta_o^*,\sigma_o)$$

$$+ \sigma_1 \frac{\partial^2\varphi_1}{\partial\theta\partial\sigma}(\theta_o^*,\sigma_o) + \theta_2^* \frac{\partial^2\varphi_o}{\partial\theta^2}(\theta_o^*,\sigma_o) + \frac{\theta_1^{*2}}{2}\frac{\partial^3\varphi_o}{\partial\theta^3}(\theta_o^*,\sigma_o)$$

$$+ \theta_1^* \frac{\partial^2\varphi_1}{\partial\theta^2}(\theta_o^*,\sigma_o) + \frac{\partial\varphi_2}{\partial\theta}(\theta_o^*,\sigma_o) = 0 \quad . \tag{4.72}$$

The solution of Eq. (4.70) with φ_o given by (4.47) is obtained as

$$\tan\theta_{o\pm}^* = \frac{-1 \pm \sqrt{1-8\tan^2\sigma_o}}{4\tan\sigma_o} \quad . \tag{4.73}$$

Since at the boundary the two systems of waves coalesce (i.e. $\theta_{o+}^* = \theta_{o-}^*$), Eq. (4.73) implies $\sigma_o = \pm \sin^{-1}(1/3)$ and $\tan\theta_o^* = \pm 1/\sqrt{2}$. Then from (4.70) and (4.48) we may deduce

$$\frac{\partial^2\varphi_o}{\partial\theta^2}(\theta_o^*,\sigma_o) = 0 \quad \text{and} \quad \frac{\partial\varphi_1}{\partial\theta}(\theta_o^*,\sigma_o) = 0 \quad .$$

Hence, Eq. (4.71) shows $\sigma_1 \equiv 0$. By the above result, Eq. (4.72) becomes

$$\sigma_2 \frac{\partial^2\varphi_o}{\partial\theta\partial\sigma}(\theta_o^*,\sigma_o) = -\left[\frac{\theta_1^{*2}}{2}\frac{\partial^3\varphi_o}{\partial\theta^3}(\theta_o^*,\sigma_o) + \theta_1^* \frac{\partial^2\varphi_1}{\partial\theta^2}(\theta_o^*,\sigma_o) + \frac{\partial\varphi_2}{\partial\theta}(\theta_o^*,\sigma_o)\right] . \tag{4.74}$$

Upon substitution of the expressions for $\varphi_o, \varphi_1, \varphi_2$ and θ_1 given by (4.47) - (4.49) and (4.59), σ_2 is readily determined; the final result is

$$\sigma_2 = -\frac{1}{3}\frac{\tan\theta_o^*(1-8\tan^2\theta_o^*)}{\sec^2\theta_o^*(1+4\tan^2\theta_o^*)^2} = -\frac{\sqrt{2}}{27} \quad . \tag{4.75}$$

Thus,

$$\sigma_{*}(r,\epsilon) = \pm\left[\sin^{-1}(1/3) - \frac{\sqrt{2}}{27}(\epsilon r)^2 + O(\epsilon r)^3\right] , \qquad (4.76)$$

in which the + or - sign is for $y >$ or < 0. This result shows that the wave field is suppressed by the linear shear to a smaller region than in the uniform stream case, the deviation being of $O(\epsilon r)^2$ (see Fig. 6).

The Eq. (4.76) for the boundary of the wave region is identical to Eq. (3.77) obtained in Chapter III. Also Eqs. (4.65) - (4.67) and (3.89) - (3.90) are different parametric representations of the loci of constant phases within the wave region. Hence, for large $r(< 1/\epsilon)$, the integral transform method of the present chapter yields the same result as the geometric optics method of Chapter III.

V. DISCUSSION AND CONCLUSION

In the treatment of a steady point disturbance given in Chapter III, it is necessary to assume that all the group lines pass through the location of the point disturbance. It may be remarked that with this assumption, the theory of geometric optics can adequately handle all point disturbances including submerged and periodic types.

Similarly, the method of integral transform of Chapter IV may be adopted to problems of surface and periodic point disturbances.

Of the results of this investigation three main features may be recapitulated. First, by comparison with the steady wave pattern generated by a point disturbance moving with constant velocity, the parallel shear flow has the effect of compressing the region of wave disturbance. In particular, a boundary of this resulting asymmetric wave region approaches but never attains the critical horizontal line $\epsilon y = -1$ (i.e. where $U(y) = 0$). A physical interpretation of this result is that this line effectively reflects the surface waves. The asymmetry of the wave pattern is also apparent from the traces of constant phases.

Another result of general interest is concerned with the behavior of the solution for large $r \left(< \frac{1}{\epsilon}\right)$: both the small perturbation expansion of the geometric optics solution and the solution by the integral transform method result in a symmetric wave region with asymmetric traces of wave crests, indicating that the effect of the shear is not fully developed in the region $r < 1/\epsilon$. The similarity of the wave patterns generated by a surface and a submerged disturbance is also demonstrated by these solutions.

Finally, the integral transform method provides an expression showing the corrections to the amplitude (for large $r < \frac{1}{\epsilon}$) in comparison with that of the flow without shear. However, this expression becomes infinite at the boundary of the wave field and is thus not applicable in that neighborhood. For the geometric optics method, due to the fact that energy is not constant in each part of the frequency spectrum, the amplitude cannot be inferred by the usual argument of energy flux being constant along adjacent group lines and the relationship between the energy density and amplitude.

The above results are valid at a distance of many wave lengths downstream of the disturbance. Near the disturbance, the boundaries of the wave region and near the critical line where $U(y) = 0$, the present theory is not applicable. These, indeed, are areas where further study is necessary.

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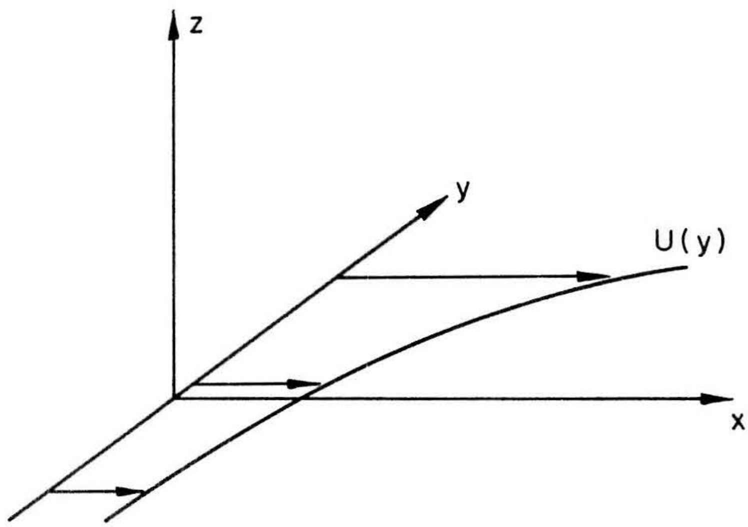


Fig. 1 - The co-ordinate system.

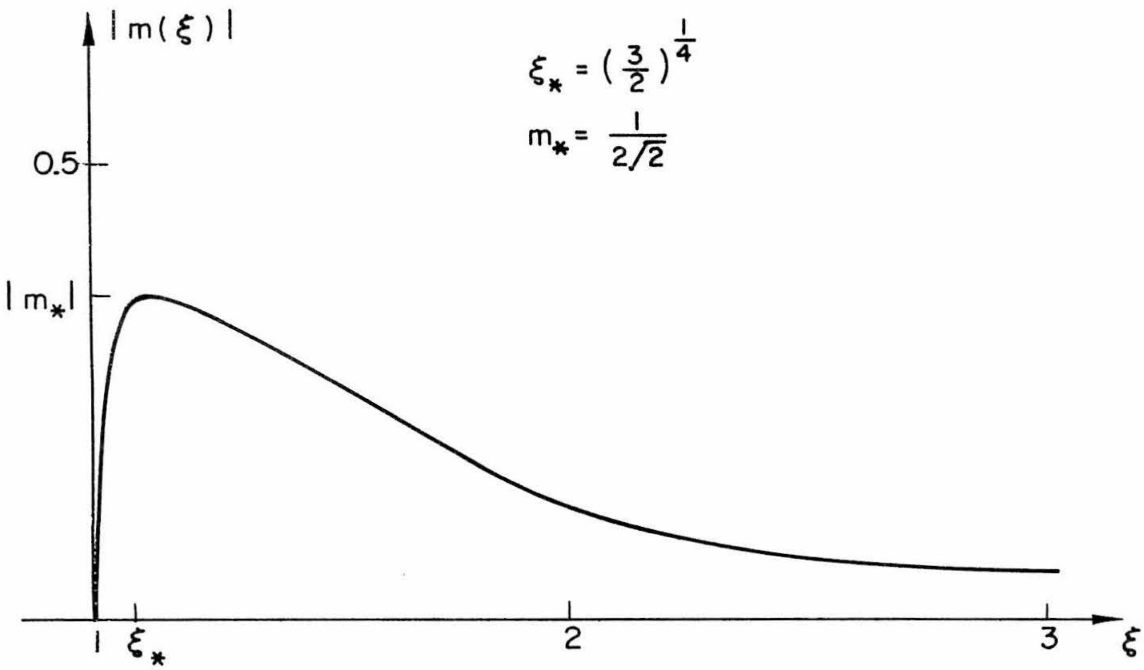


Fig. 2 - The dependence of the slope function m on ξ.

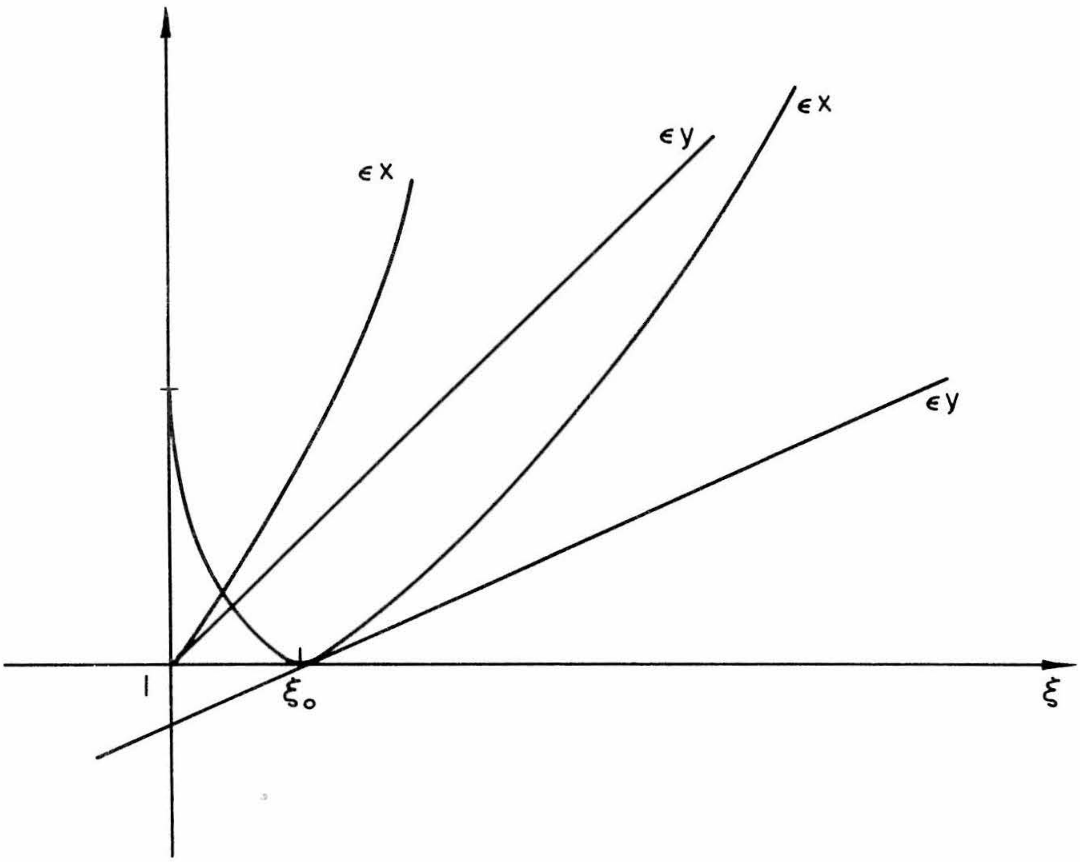


Fig. 3 - The parametric representation of the group lines for the special case $U(y) = U_o(1+\epsilon y)$.

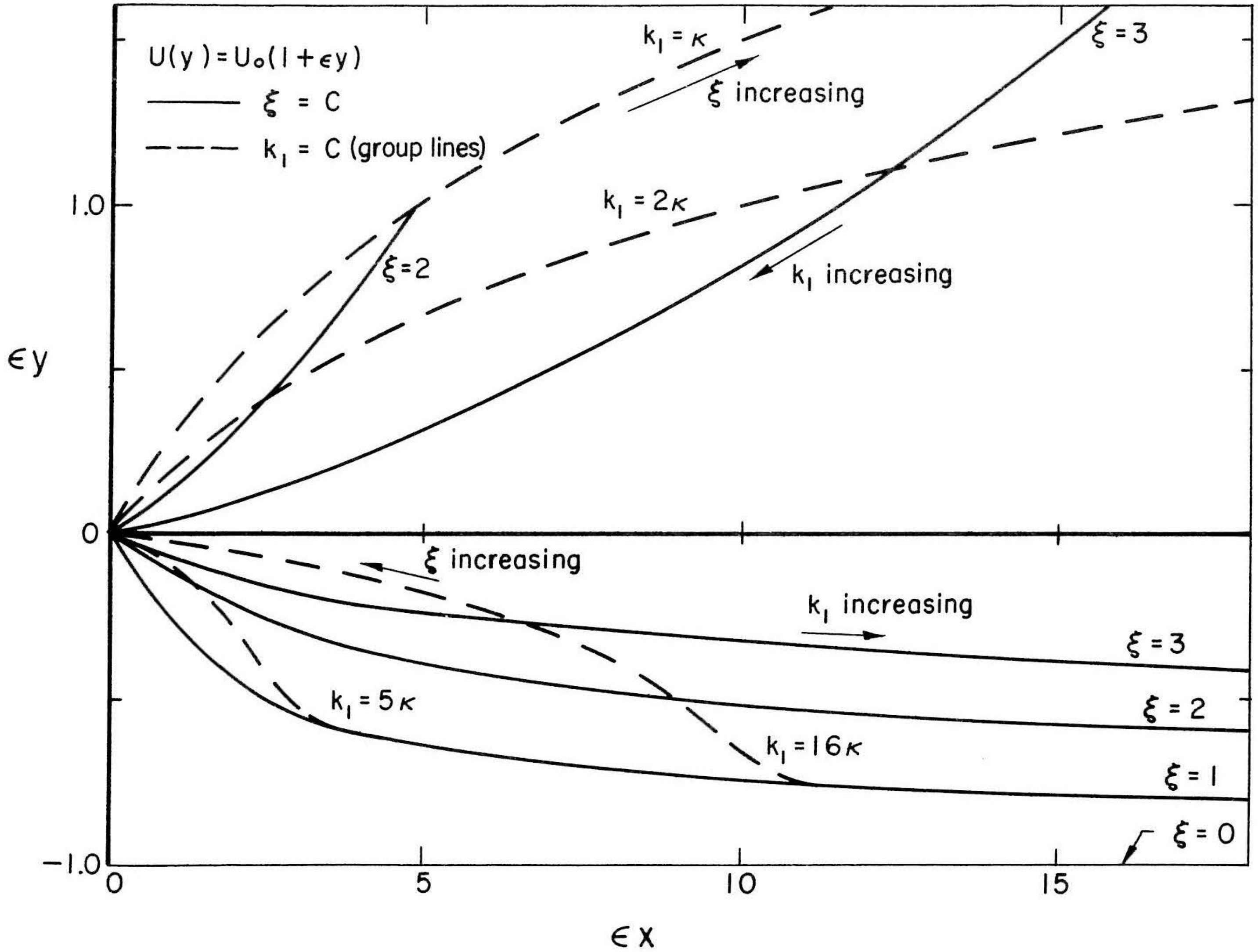
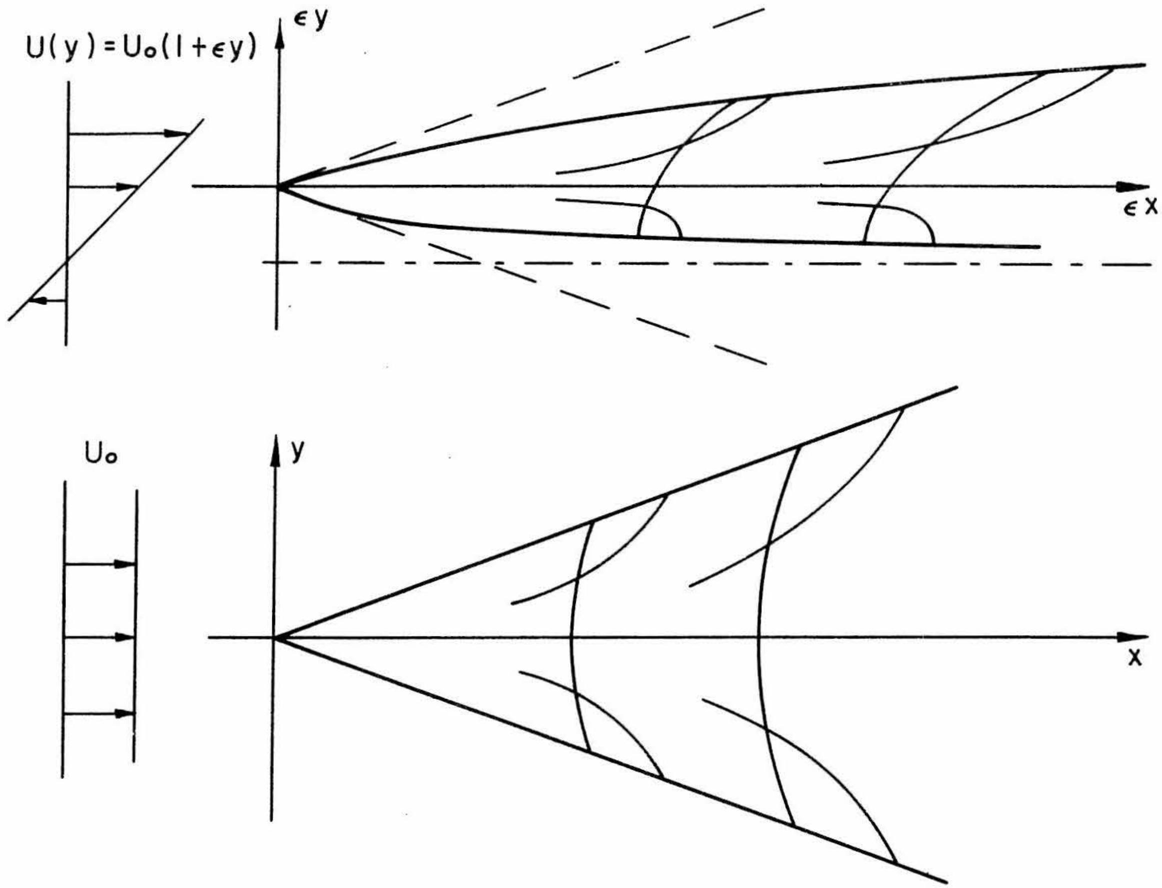


Fig. 4 - The traces of the group lines and ξ = const. in the wave field.



5 (a)

5 (b)

Fig. 5 - Steady surface wave patterns due to point disturbances in (a) linear shear flow and (b) uniform flow.

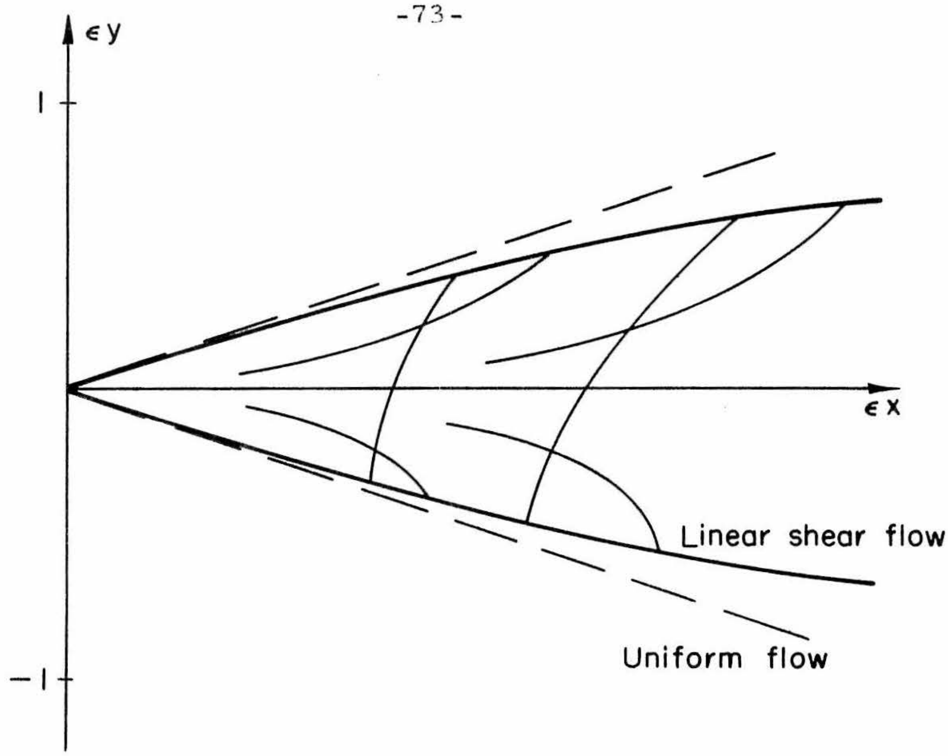


Fig. 6 - The boundary of the wave field and traces of constant phase produced by a point disturbance in a linear shear flow.

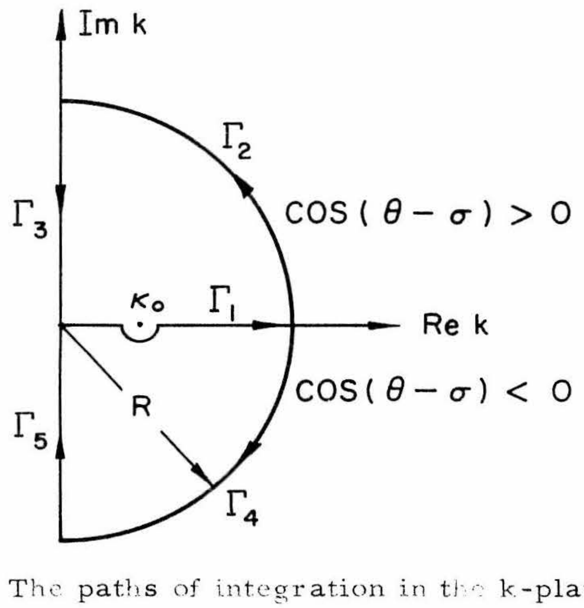


Fig. 7 - The paths of integration in the k-plane.

APPENDIX A

To illustrate the approach that will be used in this appendix we consider the integral

$$I = \frac{1}{2\pi i} \oint_C \frac{g(z)}{z-\epsilon} \, dz \quad , \tag{A.1}$$

where $g(z)$ is analytic everywhere within and on the closed contour C. If ϵ is any point interior to C, then Cauchy's integral theorem implies

$$I = g(\epsilon) \quad . \tag{A.2}$$

But for $|z| > \epsilon$, a straightforward expansion gives

$$\frac{1}{z-\epsilon} = \frac{1}{z} + \frac{\epsilon}{z^2} + \frac{\epsilon^2}{z^3} + O(\epsilon^3) \quad , \tag{A.3}$$

so that (A.1) may be rewritten as

$$I = \frac{1}{2\pi i} \oint_C g(z) \left[\frac{1}{z} + \frac{\epsilon}{z^2} + \frac{\epsilon^2}{z^3} + O(\epsilon^3) \right] \quad . \tag{A.4}$$

With $z = 0$ and ϵ interior to C, the application of Cauchy's integral theorem to (A.4) results in

$$I = g(0) + \epsilon g'(0) + \frac{\epsilon^2}{2} g''(0) + O(\epsilon^3), \tag{A.5}$$

which is seen to be a Taylor series expansion of (A.2) in ϵ. Therefore, up to $O(\epsilon^3)$, (A.4) gives exactly the same result as (A.1) provided the path of integration encloses the pole under consideration i.e. C is in the region $|z| > \epsilon$ where (A.3) is valid.

We shall apply these ideas to the expression for (kB).

Consider (4.22) as a Taylor series expansion of B about $\epsilon = 0$ and write

$$B(\epsilon) = B(\epsilon = 0) + \epsilon \frac{\partial B}{\partial \epsilon}(\epsilon = 0) + \frac{\epsilon^2}{2}\frac{\partial^2 B}{\partial \epsilon^2}(\epsilon = 0) + O(\epsilon^3) \quad , \qquad (A.6)$$

so that by direct identification with (4.22) all the coefficients of (A.6) may be determined as

$$B(\epsilon=0) = \frac{m\kappa}{\alpha^2 - \kappa k} \quad ,$$

$$\frac{\partial B}{\partial \epsilon}(\epsilon=0) = -\, im\kappa\left[-\frac{2h\alpha^2\beta}{k(\alpha^2-\kappa k)^2} + \frac{2\kappa\alpha^2\beta}{k(\alpha^2-\kappa k)^3}\right] \quad ,$$

and so on. Also, up to $O(\epsilon^3)$, we assume B has the form:

$$B(\epsilon) = \frac{m\kappa}{\alpha^2 - \kappa k + \epsilon B_1 + \epsilon^2 B_2} + O(\epsilon^3) \quad , \qquad (A.7)$$

which for small ϵ may be expanded to give

$$B(\epsilon) = \frac{m\kappa}{\alpha^2-\kappa k}\left[1 - \epsilon\frac{B_1}{\alpha^2-\kappa k} + \epsilon^2\left(\frac{B_1^2}{(\alpha^2-\kappa k)^2} - \frac{B_2}{(\alpha^2-\kappa k)}\right) + O(\epsilon^3)\right] \quad .(A.8)$$

Hence, it is obvious that B_1 and B_2 can be determined by equating terms of like powers of ϵ of (A.6) and (A.8). For our discussion, the precise forms of $B_1(\alpha, \beta, \kappa, h)$ and $B_2(\alpha, \beta, \kappa, h)$ will not be necessary. It is sufficient to note

$$B_1(\alpha, \beta, \kappa, h) = \frac{i}{\alpha^2 - \kappa k}\,\overline{P}_1(\alpha, \beta, \kappa, h) \quad ,$$

and

$$B_2(\alpha, \beta, \kappa, h) = \frac{1}{(\alpha^2-\kappa k)^2}\,\overline{P}_2(\alpha, \beta, \kappa, h) \quad ,$$

where $\overline{P}_1$ and $\overline{P}_2$ are analytic functions, so that by (4.25) and (A.7) we may deduce

$$(kB) = \frac{m\kappa \sec^2\theta}{k-\kappa \sec^2\theta + \dfrac{i\epsilon P_1(k,\theta)}{k-\kappa \sec^2\theta} + \dfrac{\epsilon^2 P_2(k,\theta)}{(k-\kappa \sec^2\theta)^3}} + O(\epsilon^3) \quad , \tag{A.9}$$

where P_1 and P_2 are again analytic.

The poles of (kB) are the zeros of

$$(k-\kappa \sec^2\theta)^4 + i\epsilon P_1(k,\theta)(k-\kappa \sec^2\theta)^2 + \epsilon^2 P_2(k,\theta) = 0 \quad . \tag{A.10}$$

The solution of (A. 10) gives the detailed distribution of poles in the neighborhood of $k = \kappa \sec^2\theta$. It can be clearly seen that all the poles in this neighborhood lie in the region $|k-\kappa \sec^2\theta| \leqslant \sqrt{\epsilon}$. Hence by analogy to the situation of (A. 1) - (A. 5), the integral with respect to k of (4. 29)

$$\oint_\Gamma dk\, e^{ikr\cos(\theta-\sigma)} e^{k(z-h)} (kB) \quad ,$$

will yield, up to $O(\epsilon^3)$, the same result no matter whether (kB) is given by (A. 9) or by (4. 27) provided the path Γ stays in the region $|k-\kappa \sec^2\theta| > \sqrt{\epsilon}$ i. e. the region in which the expansion (4. 27) is valid.

APPENDIX B

We shall show that the pole of (kB) at $k = 0$ does not contribute to Φ_w. Denoting by $(kB)_*$ the terms in (kB) that have this pole, we may write from (4.27)

$$(kB)_* = \epsilon^2 m\kappa \sec^2\theta \left[\frac{\kappa}{k(k-\kappa\sec^2\theta)^3} + \frac{2\kappa^2\sec^4\theta\sin^2\theta}{k(k-\kappa\sec^2\theta)^4} \right] . \qquad (B.1)$$

The corresponding $(\Phi_w)_*$ arising from $(kB)_*$ is

$$(\Phi_w)_* = \text{Re}\ \frac{1}{2\pi^2} \int_{\theta_1}^{\theta_2} d\theta \oint_C dk\, e^{ikr\cos(\theta-\sigma)} e^{k(z-h)} (kB)_* . \qquad (B.2)$$

If $k = 0$ is interior to C, by using (B.1) in (B.2), it follows from Cauchy's integral theorem that

$$(\Phi_w)_* = \text{Re}\ \frac{1}{2\pi^2} \int_{\theta_1}^{\theta_2} d\theta\, 2\pi i \epsilon^2 \frac{m}{\kappa} \left[-\cos^4\theta + 2\cos^2\theta\sin^2\theta \right] ,$$

which can be integrated once more to yield

$$(\Phi_w)_* = \text{Re}\ \frac{i}{\pi} \frac{\epsilon^2 m}{\kappa} \left[\left(-\frac{3\theta}{8} + \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32} \right) \Bigg|_{\theta_1}^{\theta_2} + \frac{1}{4}\left(\theta - \frac{\sin 4\theta}{4} \right) \Bigg|_{\theta_1}^{\theta_2} \right] . \qquad (B.3)$$

Thus for any θ_1 and θ_2, the right-hand side of (B.3) vanishes because the real part of a purely imaginary number is zero. Therefore, the pole $k = 0$ does not contribute towards Φ_w and the path of integration in the k-plane in the neighborhood of $k = 0$ is irrelevant. It may be taken to either include or exclude $k = 0$.

APPENDIX C

We shall evaluate for large κr the following integral

$$J = \mathrm{Res}\left(e^{ikr\cos(\theta-\sigma)} e^{k(z-h)} kB\right)_{k=\kappa_o} = \frac{1}{2\pi i}\oint_C dk\, e^{ikr\cos(\theta-\sigma)} e^{k(z-h)} kB, \tag{C.1}$$

where $\kappa_o \equiv \kappa \sec^2\theta$ is interior to C and (kB) is given by (4.27). In terms of the variable

$$\eta = k - \kappa_o \quad , \qquad 1 \gg \eta > \epsilon \quad , \tag{C.2}$$

(C.1) becomes

$$J = \frac{1}{2\pi i}\, e^{\lambda \kappa_o} \oint_C d\eta\, e^{\lambda\eta} (kB) \quad , \tag{C.3}$$

where

$$\lambda \equiv [\, ir\cos(\theta-\sigma) + (z-h)] \quad . \tag{C.4}$$

By using (4.27) the integrand of (C.3) may be written in terms of η as

$$\begin{aligned} e^{\lambda\eta}(kB) = {} & \frac{m\kappa_o}{\eta} e^{\lambda\eta} - i\epsilon m\kappa_o \left[-\frac{2h\kappa_o \sin\theta}{\eta^2}\left(1+\frac{\eta}{\kappa_o}\right) + \frac{2\kappa_o\sin\theta}{\eta^3}\right] e^{\lambda\eta} \\ & + \epsilon^2 m\kappa \Bigg[-\frac{h}{\eta^2} + \frac{h^2\sec^2\theta\sin^2\theta}{\eta^2}\left(1+\frac{\eta}{\kappa_o}\right)\kappa_o + \frac{1}{\eta^3}\left(1 - \frac{\eta}{\kappa_o} + \frac{\eta^2}{\kappa_o^2} \;\ldots\right) \\ & - \frac{2h\kappa_o\sec^2\theta\sin^2\theta}{\eta^3} + \frac{4h\kappa_o}{\eta^3}\left(1+\frac{\eta}{\kappa_o}\right) - \frac{4h\sec^2\theta\sin^2\theta}{\eta^3}\kappa_o^2\left(1+\frac{\eta}{\kappa_o}\right)^2 \\ & + \frac{2\sec^2\theta\sin^2\theta}{\eta^4}\kappa_o\left(1 - \frac{\eta}{\kappa_o} + \frac{\eta^2}{\kappa_o^2} - \frac{\eta^3}{\kappa_o^3}\ldots\right) - \frac{4\kappa_o}{\eta^4} \\ & + \frac{12h\sec^2\theta\sin^2\theta\kappa_o^2}{\eta^4}\left(1+\frac{\eta}{\kappa_o}\right) - \frac{12\sec^2\theta\sin^2\theta\kappa_o^2}{\eta^5}\Bigg] e^{\lambda\eta} + O(\epsilon^3) \quad . \end{aligned} \tag{C.5}$$

For large r, we shall consider $(\lambda\eta)$ small (i.e. $r\eta \ll 1$) and expand $e^{\lambda\eta}$ in (C.5) in a power series of $(\lambda\eta)$. The residue at the pole may then be evaluated by using the well-known theorem that if z_o is interior to C and g(z) analytic

$$\frac{1}{2\pi i}\oint_C dz\ \frac{g(z)}{(z-z_o)^n} = \frac{g^{n-1}(z_o)}{(n-1)!} \quad .$$

Hence, from (C.3) and (C.5) we obtain

$$\begin{aligned} J = e^{\lambda\kappa_o}\Big[& m\kappa_o - i\epsilon m\kappa_o\Big[-2h\sin\theta\,\kappa_o\Big(\lambda+\frac{1}{\kappa_o}\Big)+\sin\theta\,\kappa_o\lambda^2\Big] \\ & +\epsilon^2 m\kappa\Big[-h\lambda + h^2\sec^2\theta\sin^2\theta\,\kappa_o\Big(\lambda+\frac{1}{\kappa_o}\Big)+4h\kappa_o\Big(\frac{\lambda^2}{2}+\frac{\lambda}{\kappa_o}\Big) \\ & +\Big(\frac{\lambda^2}{2}-\frac{\lambda}{\kappa_o}+\frac{1}{\kappa_o^2}\Big) - h\sec^2\theta\sin^2\theta\,\kappa_o\lambda^2 - 4h^2\sec^2\theta\sin^2\theta\,\kappa_o^2\Big(\frac{\lambda^2}{2}+\frac{2\lambda}{\kappa_o}+\frac{1}{\kappa_o^2}\Big) \\ & +2\sec^2\theta\sin^2\theta\,\kappa_o\Big(\frac{\lambda^3}{3!}-\frac{\lambda^2}{2\kappa_o}+\frac{\lambda}{\kappa_o^2}-\frac{1}{\kappa_o^3}\Big)-\frac{4\kappa_o\lambda^3}{3!} \\ & +12h\sec^2\theta\sin^2\theta\,\kappa_o^2\Big(\frac{\lambda^3}{3!}+\frac{\lambda^2}{2\kappa_o}\Big) - 12\sec^2\theta\sin^2\theta\,\frac{\kappa_o^2\lambda^4}{4!}\Big] \\ & + O(\epsilon^3)\Big] \end{aligned} \qquad (C.6)$$

By substituting (C.4) into (C.6) and separating real and imaginary parts we may write

$$J = m\kappa_o e^{i\kappa_o r\cos(\theta-\sigma)}\, e^{\kappa_o(z-h)}\{1+\epsilon H_1+\epsilon^2 H_2+i\epsilon I_1+i\epsilon^2 I_2+O(\epsilon^3)\} \quad , \qquad (C.7)$$

where

$$H_1 = 2r(z-2h)\kappa_o\sin\theta\cos(\theta-\sigma) \quad , \qquad (C.8)$$

$$H_2 = -\frac{r^4}{2}\,\kappa_o^2\sin^2\theta\cos^4(\theta-\sigma) + O(r^2,(z-h)^4) \quad , \qquad (C.9)$$

$$I_1 = r^2\kappa_0 \sin\theta \cos^2(\theta-\sigma) + O(\ (z-h)^2\ ) \quad , \tag{C.10}$$

$$I_2 = \frac{r^3}{3}\ [\ 2\kappa - \kappa_0 \sin^2\theta]\cos^3(\theta-\sigma)$$
$$+ 2r^3(z-2h)\kappa_0^2 \sin^2\theta \cos^3(\theta-\sigma) + O(r) \quad . \tag{C.11}$$

In (C.8) to (C.11), we have retained only the terms of highest order in r in each case. This is justified because the asymptotic behavior of J for large r is of interest here. A physically more significant form of the above may be easily deduced by rewriting (C.7) to (C.11) in the following form:

$$J = A_0(1+\epsilon r A_1 + \ .\ .\ .\ )\ e^{i\kappa r[\ \varphi_0 + (\epsilon r)\varphi_1 + (\epsilon r)^2\varphi_2\ .\ .\ .\ ]} \quad , \tag{C.12}$$

where

$$A_0 = m\kappa_0 e^{\kappa_0(z-h)} \quad , \tag{C.13}$$

$$A_1 = 2(z-2h)\kappa_0 \sin\theta \cos^2(\theta-\sigma) \quad , \tag{C.14}$$

$$.\ .\ .\ ;$$

$$\varphi_0 = \sec^2\theta \cos(\theta-\sigma) \quad , \tag{C.15}$$

$$\varphi_1 = \sec^2\theta \sin\theta \cos^2(\theta-\sigma), \tag{C.16}$$

$$\varphi_2 = \left[\frac{2}{3} - \frac{1}{3}\ \sec^2\theta \sin^2\theta\right] \cos^3(\theta-\sigma) \quad . \tag{C.17}$$

The derivation of (C.13) to (C.17) from (C.8) to (C.11) is straightforward and will be left out. The meaning of (C.12) to (C.17) is explained in the text.