# THE RESOLUTION OF THE THERMODYNAMIC PARADOX

# AND

# THE THEORY OF GUIDED WAVE PROPAGATION IN ANISOTROPIC MEDIA

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Adolph Vincent Mrstik, Jr.

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## ABSTRACT

The resolution of the so-called thermodynamic paradox is presented in this paper. It is shown, in direct contradiction to the results of several previously published papers, that the cutoff modes (evanescent modes having complex propagation constants) can carry power in a waveguide containing ferrite. The errors in all previous "proofs" which purport to show that the cutoff modes cannot carry power are uncovered. The boundary value problem underlying the paradox is studied in detail; it is shown that, although the solution is somewhat complicated, there is nothing paradoxical about it.

The general problem of electromagnetic wave propagation through rectangular guides filled inhomogeneously in cross-section with transversely magnetized ferrite is also studied. The problem is split into TE and TM parts and scalarized. Application of the standard waveguide techniques reduces the TM part to the well-known self-adjoint Sturm Liouville eigenvalue equation. The TE part, however, leads in general to a non-self-adjoint eigenvalue equation. This equation and the associated expansion problem are studied in detail. Expansion coefficients and actual fields are determined for a particular problem.

# TABLE OF CONTENTS

# GENERAL INTRODUCTION

The original objective of this research was to resolve the "thermodynamic paradox," an apparent inconsistency associated with the propagation of microwaves in guides containing magnetized ferrite slabs. It soon became apparent that the study of propagation in such guides leads to mathematical forms quite different from those obtained in the usual waveguide problems. Although many investigators have determined the modes that may exist in ferrite loaded guides, no one has been successful in finding linear combinations of these modes that would solve any fundamental problem. For example, given the electric and magnetic fields at one cross section, no one to date has mathematically determined the resulting fields at some other cross section.

As the research progressed it became apparent that the nature of wave propagation in ferrite loaded guides is not completely understood. If such propagation were completely understood, there would be no paradox. For this reason, it was decided to first study the general theory of wave propagation in ferrite loaded guides. It was felt that the development of such a theory would provide a firm foundation with which to attack the thermodynamic paradox, as well as to help clear up many of the associated misunderstandings.

This paper is divided into two parts. Part I develops the general theory of wave propagation in guides filled inhomogeneously in cross section with transversely magnetized ferrite material. Part II considers the particular case of the thermodynamic paradox.

Only original work is included; references are given for previously obtained results necessary for the development.

# PART I - THEORY OF GUIDED WAVE PROPAGATION IN ANISOTROPIC MEDIA

## 0.0 Introduction

During the past fifteen years the electrical engineering journals have been flooded with papers considering the propagation of waves through guides containing anisotropic materials. Two most pertinent papers are those by Suhl and Walker (1) and VanTrier (2). A more complete listing of papers is given in the bibliography. In this paper we will not be concerned with the many details presented in this great mass of literature. We will simply summarize the past research by stating that most of it is concerned only with determining the modes of propagation. In general, no attempt has been made to find what linear combinations of the modes yield electric and magnetic fields satisfying any waveguide boundary value problem. In some cases the determination of the required linear combination is trivial. However, when the anisotropic material fills the guide inhomogeneously, the determination is very difficult. Except for a few special cases, no one to date has determined the linear combinations of modes satisfying any given boundary conditions for guides which are inhomogeneously filled. In this part we will study the mathematical peculiarities of these problems which make them so much more difficult than the standard waveguide problems. Our objective will be to provide the necessary mathematics to understand problems of this nature and to provide a foundation with which to attack the problem of the thermodynamic paradox.

In the study of wave propagation through materials made anisotropic by the application of a biasing field, the biasing field is usually considered either parallel or perpendicular to the direction of propagation. We will consider the application of a biasing field perpendicular to the direction of propagation of a wave traveling between two parallel plates. In order to keep the mathematics as simple as possible without eliminating any of the peculiar phemomena which we wish to observe, we will also assume that the constitutive parameters do not vary along the direction of the applied biasing field. We have chosen this problem for the following reasons:

1. As may be seen from the literature, this problem is of wide interest. In particular, the thermodynamic paradox is an excellent example of the interest in this problem.
2. The mathematical formulas parallel those for cylindrical guides biased longitudinally with parameter variations only in the radial direction. Thus, the results may be applied to such cylindrical guides as well.
3. Little could be gained at the present time by considering more complicated problems.

Our analysis will assume a scalar electric permittivity and a tensor magnetic permeability. It should be noted, however, that the results can very easily be extended to problems in which the magnetic permeability is a scalar and the electric permittivity is a tensor (magnetically biased plasma).

This part begins with a mathematical formulation of the problem to be considered, i.e., Maxwell's equations and the appropriate boundary conditions. It is shown that the propagation may be split into TE and TM modes. The TM modes do not lead to interesting results. An investigation of the TE modes, however, leads to the study of a non-self-adjoint eigenvalue equation. The eigenfunctions (the waveguide modes) of this equation are not orthogonal and hence the usual waveguide techniques, which may be used on the TM modes, break down completely. A major portion of this section is devoted to finding new methods which can be applied to the TE modes.

## 1.0 Statement of the Problem

The analysis presented in this paper will be based on Maxwell's equations in a source free region,

$$\nabla \times \underline{E} = i\omega \underline{B} \quad (1.1)$$

$$\nabla \times \underline{H} = - i\omega \underline{D} \quad (1.2)$$

where the time dependence $e^{-i\omega t}$ is assumed and the vector fields $\underline{D}$ and $\underline{B}$ are related to $\underline{E}$ and $\underline{H}$ respectively through the constitutive parameters

$$\underline{D} = \varepsilon \underline{E} \quad (1.3)$$

$$\underline{B} = \underline{\underline{\mu}} \cdot \underline{H} \ . \quad (1.4)$$

The region of interest will be that confined by two perfectly conducting parallel plates of infinite extent. The x axis will be chosen perpendicular to these plates, and the problem will be assumed normalized such that the plates intersect the x axis at 0 and 1. See figure 1.1. The region between the parallel plates will contain a ferrite material biased by an external magnetic field, $H_0$, applied in the y direction such that the permeability, $\underline{\underline{\mu}}$, will be a tensor of rank two. The elements of the permeability tensor will be permitted to be functions of the x coordinate. The permittivity, $\varepsilon$ , will be assumed a scalar constant. In order to make the problem two dimensional, no variations in the y coordinate

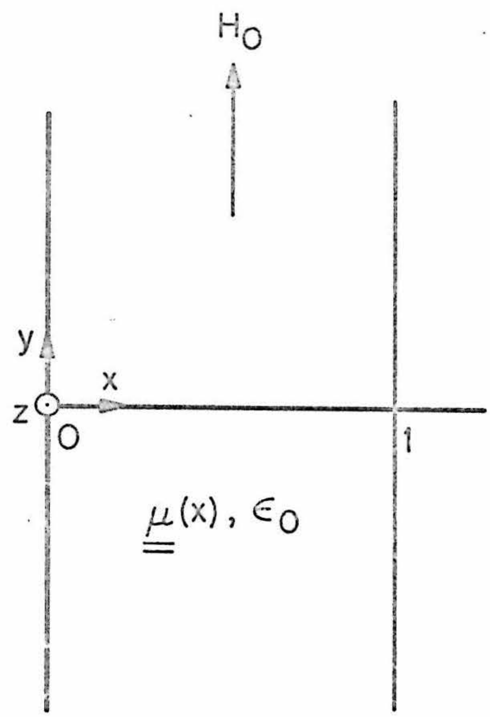


Figure 1.1

will be permitted, $\frac{\partial}{\partial y} = 0$ . Under these restrictions, Maxwell's equations become

$$\nabla \times \underline{E}\,(x,z) = i\omega \underline{\underline{\mu}}(x) \cdot \underline{H}\,(x,z) \tag{1.5}$$

$$\nabla \times \underline{H}\,(x,z) = -\,i\omega\varepsilon\,\underline{E}\,(x,z) \tag{1.6}$$

where the permeability $\underline{\underline{\mu}}(x)$ is given by

$$\underline{\underline{\mu}}(x)^{-1} = \underline{\underline{\nu}}(x) = \begin{bmatrix} \nu_1(x) & 0 & i\nu_2(x) \\ 0 & \nu_3(x) & 0 \\ -i\nu_2(x) & 0 & \nu_1(x) \end{bmatrix} \tag{1.7}$$

where -1 denotes the matrix inverse and where $\nu_1(x)$ , $\nu_2(x)$, and $\nu_3(x)$ are given real functions whose values depend on the properties of the ferromagnetic material, the operating frequency, and the amplitude of the external biasing field, $H_o$.

The objective of this paper is to solve equations 1.5 and 1.6 subject to the following boundary conditions:

$$\left.\begin{aligned} E_j(0,z) &= 0 \\ E_j(1,z) &= 0 \end{aligned}\right\} \quad j = y,z \tag{1.8}$$

where

$$E_j(x,z) = \underline{E}(x,z) \cdot \hat{e}_j$$

and $\hat{e}_j$ $(j = x,y,z)$ is the unit vector in the j direction. In addition to conditions 1.8 there must be boundary conditions in z indicating the means of excitation. These additional conditions will be discussed later at a more appropriate time.

## 2.0 Scalarization - Splitting into TE and TM Waves

The coupled vector differential equations 1.5-1.6 subject to the boundary conditions 1.8 cannot be solved directly. It is first necessary to find some means of scalarizing these equations such that they may be reduced to one or more scalar differential equations. We will now show that the coupled vector equations may be reduced to two independent scalar equations in which the unknown scalar functions are $E_y$ and $H_y$.

In component form equation 1.5 becomes

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \frac{1}{i\omega} \begin{bmatrix} -\nu_1 \dfrac{\partial E_y}{\partial z} + i\,\nu_2 \dfrac{\partial E_y}{\partial x} \\ \nu_3 \dfrac{\partial E_x}{\partial z} - \nu_3 \dfrac{\partial E_z}{\partial x} \\ i\,\nu_2 \dfrac{\partial E_y}{\partial z} + \nu_1 \dfrac{\partial E_y}{\partial x} \end{bmatrix} . \qquad (2.1)$$

Equation 1.6 becomes

$$\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \frac{-1}{i\omega\varepsilon} \begin{bmatrix} -\dfrac{\partial H_y}{\partial z} \\ \dfrac{\partial H_x}{\partial z} - \dfrac{\partial H_z}{\partial x} \\ \dfrac{\partial H_y}{\partial x} \end{bmatrix} . \qquad (2.2)$$

Substituting equation 2.2 into equation 2.1 yields

$$\begin{bmatrix} H_x \\ \\ H_y \\ \\ H_z \end{bmatrix} = \frac{1}{\omega^2\varepsilon} \begin{bmatrix} -\nu_1\left(\frac{\partial^2 H_x}{\partial z^2} - \frac{\partial^2 H_z}{\partial x \partial z}\right) + i\nu_2\left(\frac{\partial^2 H_x}{\partial x \partial z} - \frac{\partial^2 H_z}{\partial x^2}\right) \\ \\ -\nu_3\left(\frac{\partial^2 H_y}{\partial z^2} + \frac{\partial^2 H_y}{\partial x^2}\right) \\ \\ i\nu_2\left(\frac{\partial^2 H_x}{\partial z^2} - \frac{\partial^2 H_z}{\partial x \partial z}\right) + \nu_1\left(\frac{\partial^2 H_x}{\partial x \partial z} - \frac{\partial^2 H_z}{\partial x^2}\right) \end{bmatrix} . \tag{2.3}$$

Substituting equation 2.1 into equation 2.2

$$\begin{bmatrix} E_x \\ \\ E_y \\ \\ E_z \end{bmatrix} = \frac{1}{\omega^2\varepsilon} \begin{bmatrix} +\nu_3\left(\frac{\partial^2 E_z}{\partial x \partial z} - \frac{\partial^2 E_x}{\partial z^2}\right) \\ \\ -\frac{\partial}{\partial z}\nu_1\frac{\partial E_y}{\partial z} + i\frac{\partial}{\partial z}\nu_2\frac{\partial E_y}{\partial x} - i\frac{\partial}{\partial x}\nu_2\frac{\partial E_y}{\partial z} - \frac{\partial}{\partial x}\nu_1\frac{\partial E_y}{\partial x} \\ \\ \frac{\partial}{\partial x}\nu_3\frac{\partial E_x}{\partial z} - \frac{\partial}{\partial x}\nu_3\frac{\partial E_z}{\partial x} \end{bmatrix} . \tag{2.4}$$

Consider waves which have no z component of electric field, transverse electric (TE). Substituting $E_z \equiv 0$ in equation 2.4 yields

$$E_x = -\frac{\nu_3(x)}{\omega^2\varepsilon}\frac{\partial^2 E_x}{\partial z^2} \tag{2.5}$$

$$0 = \frac{\partial}{\partial x}\nu_3(x)\frac{\partial E_x}{\partial z} . \tag{2.6}$$

Equations 2.5 and 2.6 are consistent only if $E_x \equiv 0$ for the following reason.

Differentiation of equation 2.6 with respect to z yields

$$0 = \frac{\partial}{\partial x} \nu_3 \frac{\partial^2 E_x}{\partial z^2} \quad . \tag{2.7}$$

Solving equation 2.5 for $\frac{\partial^2 E_x}{\partial z^2}$ and substituting this value into equation 2.7, we obtain

$$0 = \frac{\partial}{\partial x} E_x$$

or

$$E_x = P(z)$$

where P(z) is an arbitrary function of z. Using this expression of $E_x$ in equation 2.5 and rearranging,

$$- \frac{\omega^2 \varepsilon}{\nu_3 (x)} P(z) = \frac{d^2 P(z)}{dz^2} \quad . \tag{2.8}$$

Under the assumption that $\nu_3$ is a function of x and is not a constant, equation 2.8 can be valid only if $P(z) \equiv 0$ . Hence $E_x \equiv 0$, and equation 2.4 reduces to

$$E_y = \frac{1}{\omega^2 \varepsilon} \left( - \frac{\partial}{\partial z} \nu_1 \frac{\partial E_y}{\partial z} + i \frac{\partial}{\partial z} \nu_2 \frac{\partial E_y}{\partial x} - i \frac{\partial}{\partial x} \nu_2 \frac{\partial E_y}{\partial z} - \frac{\partial}{\partial x} \nu_1 \frac{\partial E_y}{\partial x} \right) \tag{2.9}$$

TE wave propagation may thus be described by the scalar differential equation 2.9. The associated magnetic field may be derived from the scalar function $E_y$ using equation 2.1

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \frac{1}{i\omega} \begin{bmatrix} -\nu_1 \frac{\partial E_y}{\partial z} + i\nu_2 \frac{\partial E_y}{\partial x} \\ 0 \\ i\nu_2 \frac{\partial E_y}{\partial z} + \nu_1 \frac{\partial E_y}{\partial x} \end{bmatrix} \quad . \tag{2.10}$$

It may be shown in a like manner* that $H_z \equiv 0$ implies $H_x \equiv 0$ and thus wave propagation having no z component of magnetic field, transverse magnetic (TM), may be described by the following scalar differential equation

$$H_y = -\frac{\nu_3}{\omega^2\varepsilon} \left( \frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial z^2} \right) \quad . \tag{2.11}$$

The associated electric field may be derived from the scalar function $H_y$ using equation 2.2

$$\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = -\frac{1}{i\omega\varepsilon} \begin{bmatrix} -\frac{\partial H_y}{\partial z} \\ 0 \\ \frac{\partial H_y}{\partial x} \end{bmatrix} \quad . \tag{2.12}$$

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*See Appendix A

In the preceding we have shown that wave propagation having no z component of magnetic field (TM) may be described by a single scalar differential equation; wave propagation having no z component of electric field (TE) may be described by another scalar differential equation. An arbitrary wave having z components of both magnetic and electric fields may be considered as the sum of two waves, one having no z component of electric field, and the other having no z component of magnetic field. Thus, the general problem of wave propagation between our parallel plates may be divided into two separate problems: one involving the scalar function $E_y$; the other, the scalar function $H_y$. In the next two chapters we will consider each of these problems.

## 3.0 TM Wave Propagation

We have shown that the study of TM wave propagation between our parallel plates reduces to the mathematical problem of solving the scalar differential equation 2.11 subject to suitable boundary conditions. As may be seen from the form of this equation, with the exception of one minor detail ( $\nu_3$ is a function of the variable x), TM wave propagation between our parallel plates gives rise to the same equations as the classical problems concerning propagation through homogeneously filled guides. Although our objective is to present only original work in this paper, we will briefly demonstrate the application of the classical methods to the TM problems in order to (1) help define the notation which will be used throughout the remaining parts of this paper, (2) refresh the reader's memory of the classical methods so that the significance of the slight differences between the TM and the much more complicated TE problem will be immediately apparent, and (3) make the discussion of propagation between parallel plates complete.

Separation of variables suggests we seek a solution of the form

$$H_y(x,z) = f(x)\, e^{ihz} \quad . \tag{3.1}$$

Using equations 1.8, 2.12, and 3.1, the boundary conditions on f(x) become

$$\left.\frac{df}{dx}\right|_{x=0} = \left.\frac{df}{dx}\right|_{x=1} = 0 \tag{3.2}$$

and the partial differential equation 2.11 reduces to

$$\frac{d^2 f}{dx^2} + \frac{\omega^2 \varepsilon}{\nu_3(x)} f = h^2 f \quad . \tag{3.3}$$

In operator notation the problem becomes

$$\begin{gathered} Mf = \lambda f \quad , \qquad \lambda = h^2 \\ \left.\frac{df}{dx}\right|_{x=0} = \left.\frac{df}{dx}\right|_{x=1} = 0 \\ M \equiv \frac{d^2}{dx^2} + \frac{\omega^2 \varepsilon}{\nu_3(x)} \quad . \end{gathered} \tag{3.4}$$

Defining an inner product

$$(f,g) = \int_0^1 f(x)\, g(x)\, dx, \tag{3.5}$$

it is immediately apparent that problem 3.4 is the classical self-adjoint Sturm Liouville boundary value problem since

$$(Mf,g) = (f, Mg) \tag{3.6}$$

for arbitrary continuous functions f and g. As shown in the standard references on such problems,

1. There is an infinite set of square-integrable functions $\{f_n\}$ $(n = 1,2,\ldots)$ each solving problem 3.4. Associated with each of these functions (eigenfunctions) is a distinct and real value of $\lambda_n$ (eigenvalue).

2. The infinite set $\{f_n\}$ forms a complete set for the set of bounded continuous functions defined on the interval [0,1]; a set $\{a_n\}$ exists such that

$$\lim_{N \to \infty} \left| F(x) - \sum_{1}^{N} a_n f_n(x) \right| = 0$$

uniformly for any function F(x) continuous on [0,1] and satisfying the boundary conditions of problem 3.4.

3. The $f_n$ may be normalized such that they are orthonormal under the inner product 3.5,

$$(f_i, f_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} .$$

We will now apply these three properties of the Sturm Liouville problem to determine $H_y(x,z)$. In the usual manner we assume

$$H_y(x,z) = \Sigma\, a_n f_n e^{ih_n z} \tag{3.7}$$

where the summation extends over all possible $f_n$, $h_n$ and the expansion coefficients $a_n$ are to be determined such that $H_y(x,z)$ satisfies some given boundary conditions in z.

Since $\lambda_n = h_n^2$ is real (property 1), we know that each $f_n$ has two associated values of $h_n$ ($\sqrt{\lambda_n}$ and $-\sqrt{\lambda_n}$ ). Thus for each $a_\ell f_\ell e^{ih_\ell z}$ under the summation 3.7, there is also a term $a_\ell f_\ell e^{-ih_\ell z}$ . The summation 3.7 may therefore be written as

$$H_y(x,z) = \Sigma \left( a_n e^{i\sqrt{\lambda_n} z} + b_n e^{-i\sqrt{\lambda_n} z} \right) f_n(x) \qquad (3.8)$$

where the $f_n$ and $\lambda_n$ under the summation are now all distinct. With the expansion for $H_y$ "split" in this manner it is very easy to determine the expansion coefficients $a_n$, $b_n$. For example, given $H_y(x,o) = 0$, equation 3.8 would require

$$0 = \Sigma (a_n + b_n) f_n(x) \quad . \qquad (3.9)$$

Using property 3, equation 3.9 implies $a_n = -b_n$ . Equation 3.8 becomes

$$H_y(x,z) = \Sigma a_n \left( e^{i\sqrt{\lambda_n} z} - e^{-i\sqrt{\lambda_n} z} \right) f_n(x) \quad . \qquad (3.10)$$

If we were also given $H_y(x,\ell)$, equation 3.10 would require

$$H_y(x,\ell) = \Sigma a_n \left( e^{i\sqrt{\lambda_n}\ell} - e^{-i\sqrt{\lambda_n}\ell} \right) f_n(x) \ . \qquad (3.11)$$

Using property 3 again, equation 3.11 requires*

$$a_n = \frac{(H_y(x,\ell),\ f_n(x))}{e^{i\sqrt{\lambda_n}\ell} - e^{-i\sqrt{\lambda_n}\ell}} \qquad (3.12)$$

and thus the expansion coefficients $a_n$ , $b_n$ have been determined such that the summation 3.8 converges to satisfy boundary conditions at $z = 0$ and $z = \ell$ . The proposed solution must then be justified in the usual manner.

The important point to keep in mind is that the determination of the expansion coefficients was trivial for the following reasons:

1. Expansion 3.7 could be "split" into the form of equation 3.8 such that the application of the boundary conditions at one z yielded a very simple relationship between the expansion coefficients ($a_n = - b_n$).
2. The completeness of the set $\{f_n\}$ guaranteed the existence of expansion coefficients $\{a_n\}$ such that equation 3.11 could be satisfied for arbitrary continuous $H_y(x,\ell)$.
3. The fact that the $f_n$ were orthogonal made the evaluation of the expansion coefficients of equation 3.11 trivial.

In the next section TE wave propagation will be considered, and it will be shown that:

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*We assume here that $(H_y(x,\ell),\ f_n) = 0$ if $e^{i\sqrt{\lambda_n}\ell} = e^{-i\sqrt{\lambda_n}\ell}$ . If this is not the case, loss must be inserted to obtain meaningful results.

1. The expansion for the scalar function cannot, in general, be "split" into the form of equation 3.8; hence, the application of boundary conditions at one z does not yield simple relationships between the expansion coefficients.
2. The completeness of the associated eigenfunctions or modes is questionable.
3. The associated eigenfunctions or modes are not orthogonal.

Thus the preceding steps used for solving the TM problem break down completely.

# 4.0 TE Wave Propagation

## 4.1 Introduction

We have shown in chapter 2 that the study of TE wave propagation between our parallel plates reduces to the mathematical problem of determining the solutions of equation 2.9 subject to boundary conditions 1.8,

$$E_y(x,z) = \frac{1}{\omega^2 \varepsilon}\left(- \frac{\partial}{\partial z} \nu_1 \frac{\partial E_y}{\partial z} + i\frac{\partial}{\partial z} \nu_2 \frac{\partial E_y}{\partial x} - i\frac{\partial}{\partial x} \nu_2 \frac{\partial E_y}{\partial z} - \frac{\partial}{\partial x} \nu_1 \frac{\partial E_y}{\partial x}\right) \quad (4.1a)$$

$$E_y(x,z)\Big|_{x=0} = E_y(x,z)\Big|_{x=1} = 0. \quad (4.1b)$$

For separable solutions of the form*

$$E_y(x,z) = f(x)\, e^{ihz}, \quad (4.2)$$

the partial differential system 4.1 reduces to an ordinary differential system,

$$\frac{d}{dx} \nu_1 \frac{df}{dx} + \left(\omega^2 \varepsilon - h^2 \nu_1 - h\frac{d\nu_2}{dx}\right) f(x) = 0 \quad (4.3a)$$

$$f(0) = f(1) = 0 \;. \quad (4.3b)$$

*The f and h used here are not related to those of the previous section.

Our task is to study the solutions of system 4.3. Except for the presence of the factor $h\frac{d\nu_2}{dx}$, equation 4.3a does not appear to differ greatly from equation 3.3 governing TM wave propagation. In fact, if $\frac{d\nu_2}{dx}$ and $\nu_1(x)$ are linearly related, system 4.3 reduces to the classical Sturm Liouville problem.

4.2 Special Case

Before attempting to study system 4.3 in its most general form, we will pause to consider the special linear case

$$\frac{d\nu_2}{dx} = K\nu_1(x) \tag{4.4}$$

where K is a real constant (zero or non-zero). For this case, system 4.3 becomes

$$\frac{1}{\nu_1(x)}\frac{d}{dx}\nu_1\frac{df}{dx} + \frac{\omega^2\varepsilon}{\nu_1(x)} f = (h^2 + hK)\, f(x) \tag{4.5}$$

$$f(0) = f(1) = 0,$$

or in operator notation,

$$\begin{aligned} Nf &= \lambda f \quad , \quad \lambda = h^2 + hK \\ f(0) &= f(1) = 0 \end{aligned} \tag{4.6}$$

where

$$N \equiv \frac{1}{\nu_1}\frac{d}{dx}\nu_1\frac{d}{dx} + \frac{\omega^2\varepsilon}{\nu_1} \quad .$$

System 4.6 is Sturm Liouville under the inner product defined by

$$(f,g) = \int_0^1 \nu_1(x)\, f(x)\, g(x)\, dx \quad . \tag{4.7}$$

Hence, we know that the eigenfunctions $\{f_n\}$ form a complete set in [0,1] for all bounded continuous functions satisfying the boundary conditions 4.3b, the eigenvalues $\lambda_n$ are real, and the eigenfunctions are orthogonal under the inner product defined by 4.7. Thus we may write

$$E_y(x,z) = \Sigma \; a_n f_n(x)\, e^{ih_n z} \quad . \tag{4.8}$$

By definition

$$\lambda = h^2 + hK$$

or

$$h_n = - \frac{K \pm \sqrt{K^2 + 4\lambda_n}}{2} \quad . \tag{4.9}$$

Thus if $K^2 \neq -4\lambda_n$ ($n = 1,2,3,\ldots$), there are two values of $h_n$ associated with each eigenfunction $f_n$ , and the summation 4.8 may be "split" into the form

$$E_y(x,z) = \Sigma \left( a_n e^{+iR_n z} + b_n e^{-iR_n z} \right) e^{-i \frac{K}{2} z} f_n(x) \tag{4.10}$$

where

$$R_n = \sqrt{\left(\frac{K}{2}\right)^2 + \lambda_n} \quad .$$

In this form it is apparent that the coefficients $a_n$, $b_n$ may be obtained in the usual manner as described in chapter 3. By specifying boundary conditions in z such as $E_y(x,o)$, $E_y(x,\ell)$, the coefficients $a_n, b_n$ may be obtained easily using the completeness and othogonality properties of the $\{f_n\}$ . It is apparent that the application of the boundary condition at one given $z(z = \ell)$ immediately yields a simple relationship between each $a_n$ and $b_n$ . That is, if $E_y(x,\ell)$ is given, then

$$a_n e^{+i(R_n - K/2)\ell} + b_n e^{-i(R_n + K/2)\ell} = c_n \tag{4.11}$$

where

$$c_n = (E_y(x,\ell), f_n(x)) \quad .$$

There are several things here which are interesting to note. First, the propagation is reciprocal for $K = 0$ ($\nu_2$ independent of x). For $K = 0$ equation 4.10 becomes

$$E_y(x,z) = \Sigma \left( a_n e^{+i\sqrt{\lambda_n}z} + b_n e^{-i\sqrt{\lambda_n}z} \right) f_n(x) \quad . \tag{4.12}$$

Just as in the case of an empty guide or an inhomogeneously filled isotropic guide, any given mode $f_n$ has two possible propagation constants, $\sqrt{\lambda_n}$ and $-\sqrt{\lambda_n}$ .

Second, it is interesting to note that for some values of $K \neq 0$ it is possible for the two propagation constants associated with any $f_n$ to be real and have the same sign. That is, for

$$-\left(\frac{K}{2}\right)^2 < \lambda_j < 0 \qquad (4.13)$$

equation 4.10 becomes

$$E_y(x,z) = \left( a_j\, e^{iP_1 z} + b_j\, e^{iP_2 z} \right) f_j(x) + \sum_{n \neq j} \left( a_n\, e^{iR_n z} + b_n\, e^{-iR_n z} \right) e^{-i\frac{K}{2} z} f_n(x) \qquad (4.14)$$

where $P_1$ and $P_2$ are real constants having the same sign. One might think that this implies the wave $a_j\, e^{iP_1 z} f_j$ and the wave $b_j\, e^{iP_2 z} f_j$ represent two waves traveling (carrying power) in the same direction. It may be shown, however, that the direction of power flow and the sign of the propagation constant are unrelated. The direction of power flow depends on the sign of the derivative of the propagation constant with respect to $\omega$. The derivatives of $P_1$ and $P_2$ are such that $a_j\, e^{iP_1 z} f_j$ and $b_j\, e^{iP_2 z} f_j$ carry power in opposite directions.

A third interesting point to consider is the case where some eigenfunction $f_j$ has only one associated propagation constant. This occurs if

$$R_j = \sqrt{\left(\frac{K}{2}\right)^2 + \lambda_j} = 0 \qquad (4.15)$$

or

$$K = \pm\, 2\sqrt{-\lambda_j} \quad .$$

For this case, equation 4.10 would become

$$E_y(x,z) = a_j\, e^{-i\frac{K}{2}z}\, f_j(x) + \sum_{n\neq j} \left( a_n\, e^{iR_n z} + b_n\, e^{-iR_n z} \right) e^{-i\frac{K}{2}z}\, f_n(x) \quad . \tag{4.16}$$

Suppose for this special case of $K = \pm\, 2\sqrt{-\lambda_j}$ one attempted to find the coefficients $a_n$ , $b_n$ such that $E_y(x,o) = 0$ and $E_y(x,\ell)$ is a given function. Applying the properties of the Sturm Liouville functions $f_n$ , the boundary condition $E_y(x,o) = 0$ implies

$$a_j = 0 \tag{4.17}$$

$$a_n = -\, b_n \qquad n \neq j \quad .$$

Applying the boundary condition at $z = \ell$ then requires

$$E_y(x\ell) = \sum_{n\neq j} a_n \left( e^{iR_n \ell} - e^{-iR_n \ell} \right) e^{-i\frac{K}{2}\ell}\, f_n(x) \quad . \tag{4.18}$$

Clearly a set $\{a_n\}$ satisfying equation 4.18 can exist only if $E_y(x,\ell)$ is normal to $f_j$ ; that is, if

$$(E_y(x,\ell),\ f_j(x)\ ) = 0 \quad . \qquad (4.19$$

If $(E_y(x,\ell),\ f_j(x)\ ) \neq 0$, loss must be inserted into the system in order to obtain meaningful results.*

Our preceding work concerning TM wave propagation and the special cases of TE wave propagation defined by $\frac{d\nu_2}{dx} = K\nu_1(x)$ may be summarized by the following statements:

1. The coefficients $a_n$ , $b_n$ of the modal expansion may be obtained directly and easily using the standard waveguide techniques.
2. The success of these methods is based on the fact that the modes $f_n$ come from a Sturm Liouville system, and the expansion may be "split" since each mode has two associated propagation constants.

*The single mode $f_i$ in expansion 4.16 does not correspond to the unidirectional mode in the thermodynamic paradox considered later; the former does not carry power, while the latter does.

## 4.3 General Case

In order to complete the general description of TE wave propagation between our parallel plates, it remains to consider the general solution of system 4.3 with $\frac{d\nu_2}{dx} \neq K\nu_1(x)$ . For our general discussion we will write system 4.3 in operator notation

$$\begin{aligned} L_n f_n(x) &= 0 \\ f_n(o) &= f_n(1) = 0 \\ L_n &\equiv \frac{d}{dx}\nu_1(x)\frac{d}{dx} + \omega^2\varepsilon - h_n^2\nu_1 - h_n\frac{d\nu_2}{dx} \quad . \end{aligned} \tag{4.20}$$

It is immediately apparent that for $\frac{d\nu_2}{dx} \neq K\nu_1$ , the problem cannot be written in the Sturm Liouville form,

$$\begin{aligned} Of_n &= P(h_n)\, f_n \\ f_n(o) &= f_n(1) = 0 \end{aligned} \tag{4.21}$$

where $P(h_n)$ is a function of $h_n$ only and O is an operator containing only the variable x. The operator $L_n = L_n(x,h_n)$ contains the eigenvalues $h_n$ such that the problem cannot be written in the form of system 4.21. System 4.20 is non-self-adjoint; that is, an inner product cannot be found such that

$$(L\ f,\ g) = (f,\ L\ g) \tag{4.22}$$

for arbitrary bounded, continuous functions f and g in $C^2$.*

*Some authors have erroneously concluded that system 4.20 is self-adjoint by overlooking the fact that equation 4.22 must be valid for ARBITRARY f and g (27).

The properties of the self-adjoint system 4.21, namely the completeness and orthogonality properties of the eigenfunctions, may not be valid for the non-self-adjoint system 4.20. The eigenfunctions of non-self-adjoint ordinary differential systems do not always form a complete and orthogonal set; solutions cannot always be expanded in a series of the eigenfunctions.

Non-completeness of the eigenfunctions of a non-self-adjoint system is not an unusual occurence of interest only to the mathematicians. The eigenfunctions of ordinary differential systems describing very simple physical problems do not always form a complete set. An excellent example of such a problem is given in a paper by D. S. Cohen (3). Cohen considered the physical problem of diffraction by a perfectly conducting circular cylinder of radius $\alpha$ of a wave produced by a source distribution $F(r,\theta)$. The problem was to find $U(r,\theta)$ such that

$$\begin{aligned} \nabla^2 u + k^2 u &= F(r,\theta) \\ u(\alpha,\theta) &= 0 \\ \lim_{r\to\infty} \left| r^{1/2} \left(\frac{du}{dr} - iku\right) \right| &= 0 \quad \text{(Radiation Condition)} \end{aligned} \tag{4.23}$$

where k is a real non-zero number. Applying separation of variables to problem 4.23 yields

$$\frac{d^2\ \Theta}{d\ \Theta} + \lambda^2\ \Theta = 0 \tag{4.24a}$$

$$\Theta(\theta) = \Theta(\theta + 2\pi) \tag{4.24b}$$

and

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + (k^2 - \frac{\lambda^2}{r^2})R = 0 \tag{4.25a}$$

$$\lim_{r\to\infty} R^{1/2} \left( \frac{dR}{dr} - ikR \right) = 0 \tag{4.25b}$$

where

$$U(r,\theta) = R(r)\,\Theta(\theta) .$$

The usual method of solving the diffraction problem is to expand the solution in a series of eigenfunctions of system 4.24. One would expect that the solution could as well be expanded in a series of the eigenfunctions of system 4.25. That is, since both systems 4.24 and 4.25 are obtained by applying separation of variables to system 4.23, it would seem that their eigenfunctions should be equally suited for expanding the solution of system 4.23. Notice, however, that system 4.24 is self-adjoint, but system 4.25 is non-self-adjoint because of the boundary condition 4.25b. Cohen studied system 4.25 and found that there exists radial eigenfunctions $H^{(1)}_{\lambda_n}(kr)$ $(n = 1,2,\ldots)$ with complex eigenvalues $\lambda_n$. He made a detailed study of these eigenfunctions and showed that they do not form a complete set. That is, the expansion

$$f(r) = \sum_{n=1}^{\infty} a_n H_{\lambda_n}^{(1)}(kr) \tag{4.26}$$

is impossible for a large class of reasonable functions f(r), and in fact solutions of system 4.23 may not always be expanded in a series of the eigenfunctions of system 4.25.

The point which we are trying to make clear is that the eigenfunctions and eigenvalues of our non-self-adjoint system 4.20 describing TE wave propagation may differ greatly from those of the Sturm Liouville system 4.21. As demonstrated in Cohen's work, the eigenfunctions of non-self-adjoint systems associated with physical problems do not always form a complete set. Thus, the classical method of expanding the fields in a series of eigenfunctions and then determining the expansion coefficients by the boundary conditions in z, as we did for the case of TM wave propagation, can be carried out only formally, if at all, for TE wave propagation. Without first studying the eigenfunctions $f_n$, there is no guarantee that the expansion will converge to the desired boundary conditions at some given z.

Before attempting a rigorous analysis of the completeness properties of the eigenfunctions of system 4.20, let us assume that the set $\{f_n\}$ is complete and make a few simple observations. Suppose the set $\{f_n\}$ is "exactly complete." The term "exactly complete" means that the set contains as few functions as possible in the sense that no one function of the set may be expressed as a linear combination of the others. A set $\{f_n\}$ $(n = 1,2,\ldots)$ is exactly complete if $\{f_n\}$ forms a complete set,

$$\lim_{N\to\infty} \left| F(x) - \sum_{1}^{N} a_n f_n(x) \right| = 0 \qquad \text{for all } F(x) \quad c^1 \tag{4.27}$$

and no set of non-zero constants $\{c_n\}$ exists such that

$$0 = \sum_{1}^{\infty} c_n f_n(x) \quad . \tag{4.28}$$

Equation 4.28 implies that the expansion of an arbitrary function in a series of eigenfunctions $f_n$ is unique. The eigenfunctions associated with the self-adjoint Sturm Liouville systems are exactly complete.

Under the assumption that the set $f_n$ is exactly complete, consider the expansion of the desired field, $E_y(x,z)$,

$$E_y(x,z) = \sum a_n\, f_n(x)\; e^{ih_n z} \quad . \tag{4.29}$$

From the form of system 4.20 it may be seen that each $f_n$ has only one associated $h_n$.* Hence, expansion 4.29 may not be "split" as

*Suppose there were two distinct possible eigenvalues, $h_1$ and $h_2$, associated with a single eigenfunction f . Subtraction of each of the two corresponding eigenvalue equations would yield

$$\left[ (h_1^2 - h_2^2)\nu_1(x) + (h_1 - h_2)\frac{d\nu_2}{dx} \right] f(x) = 0 \quad , \quad \text{all} \quad 0< x< 1 \quad .$$

Under the assumption that f(x) is at least a piecewise continuous function which is not identically zero, this implies

$$(h_1^2 - h_2^2)\nu_1(x) + (h_1 - h_2)\frac{d\nu_2}{dx} = 0 \quad .$$

Clearly for $h_1 \neq h_2$, this is possible only if

$$\frac{d\nu_2}{dx} = K\nu_1(x) \quad .$$

were the expansions of the TM solutions and TE solutions for the special case $\frac{d\nu_2}{dx} = K\nu_1$ . Now suppose we apply the boundary condition z = 0; equation 4.29 becomes

$$E_y(x,o) = \Sigma\ a_n f_n(x) \quad . \tag{4.30}$$

Under the assumption that the set $\{f_n\}$ is exactly complete, equation 4.30 uniquely determines the $\{a_n\}$ . However, if the $\{a_n\}$ of expansion 4.29 are uniquely determined by the boundary condition at z = 0, there is no means of forcing expansion 4.29 to satisfy the necessary boundary conditions at some other z. Thus we must conclude that if solutions in the form of expansion 4.29 are to exist, the eigenfunctions $\{f_n\}$ must not form an exactly complete set. The $\{f_n\}$ must be more than complete; that is, the expansion of an arbitrary function in a series of these eigenfunctions must not be unique.

Since the eigenfunctions $\{f_n\}$ must be more than complete, it is clear that they cannot all be orthogonal.* No weighting function

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*Let $f_n (n = 1,2,...)$ denote a more than complete set. Then there exists at least one set of constants $c_n$ not all identically zero such that

$$0 = \Sigma\ c_n\ f_n(x)$$

or if $c_i \neq 0$ ,

$$c_i f_i = \sum_{n \neq i} c_n f_n \ . \tag{i}$$

Now suppose there exists a weighting factor W(x) such that

$$(f_i, f_j) = \int W\ f_i\ f_j\ dx = \delta_{ij}$$

Then

$$c_i (f_i, f_i) = \sum_{n \neq i} (f_n, f_i) \tag{ii}$$

or

$$c_i = 0 \quad .$$

It will later be clear that series (i) is uniformly convergent so that (ii) follows.

W(x) can exist such that

$$(f_i, f_j) = \int_0^1 W\, f_i\, f_j\, dx = \delta_{ij} \tag{4.31}$$

It should also be noted that, in addition to being more than complete, the set $\{f_n\}$ must be sufficiently complete for the expansion 4.29 to be capable of satisfying boundary conditions at two different values of z. That is to say, although the application of the boundary conditions at one z must not uniquely determine the expansion coefficients, the application of boundary conditions at two values of z must.

Thus it should be clear that if solutions in the form of expansion 4.29 are to exist, the eigenfunctions $\{f_n\}$ must possess some rather special properties. It should also be noted that the eigenfunctions associated with TM wave propagation and TE wave propagation for the special case $\frac{d\nu_2}{dx} = K\nu_1$ do not have to possess these special properties. For these problems, each eigenfunction has two associated propagation constants; hence, an exactly complete set of functions (Sturm Liouville) is sufficient. The expansions can be split into the form

$$\Sigma \left( a_n e^{ih_n^{(1)} z} + b_n e^{ih_n^{(2)} z} \right) f_n(x) \tag{4.32}$$

such that the application of a boundary condition at one z yields a very simple relationship between the expansion coefficients

$$a_j = \text{function } (b_j)^* \quad . \tag{4.33}$$

However, for the general case $\frac{d\nu_2}{dx} \neq K\nu_1$ , the expansion cannot be split in this manner and the eigenfunctions must therefore be more than complete. The application of boundary conditions at one z gives rise to very complicated relationships between the expansion coefficients of 4.29

$$a_j = \text{function } (a_1, a_2, \ldots a_j - 1, a_j + 1, \ldots) \quad . \tag{4.34}$$

It is interesting to note that by considering the partial differential system 4.1 (the waveguide problem), it was possible to draw conclusions about the properties of the eigenfunctions of the ordinary differential system 4.20. There appears to be an inherent relationship between the ordinary differential system and the partial differential system. We will later show a striking example of this relationship. Using the ordinary differential system, 4.20, we will derive a relationship involving the eigenfunctions. This relationship will appear to have no significance with respect to the ordinary differential system from which it was derived, yet it will provide a means of determining the expansion coefficients for solutions of the partial differential system 4.1!

Now that we have a basic understanding of some of the properties of these eigenfunctions $\{f_n\}$ , let us proceed to study the nature of these functions in a rigorous mathematical manner. The eigenfunctions which we wish to consider were defined by system 4.20,

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*Compare with equation 4.11.

$$\frac{d}{dx}\nu_1 \frac{df_n}{dx} + \left( \omega^2\varepsilon - h_n{}^2\nu_1 - h_n \frac{d\nu_2}{dx}\right) f_n = 0 \tag{4.35}$$

$$f_n(o) = f_n(1) = 0 \quad .$$

We would like to prove first, that the functions $\{f_n\}$ do indeed form a complete set and, second, that they are sufficiently overcomplete to expand the scalar function $E_y(x,z)$ in the form

$$E_y(x,z) = \Sigma\, a_n\, f_n(x)\, e^{ih_n z} \quad . \tag{4.36}$$

Some of the earliest rigorous mathematical work on non-selfadjoint problems in the form of system 4.35 was performed by J. Tamarkin (4). Tamarkin proved that the eigenfunctions defined by system 4.35 form a complete set for the class of continuous, bounded functions.* He obtained explicit expressions for the expansion coefficients such that the eigenfunction series converges uniformly to a function in this class. The expansion coefficients determined by Tamarkin are unique; once the function to be expanded is given, the expansion coefficients are determined. From Tamarkin's work it would appear that the eigenfunctions $\{f_n\}$ form an exactly complete set. However, we know that if the set $\{f_n\}$ is exactly complete, it is not sufficient for obtaining solutions to waveguide problems. We

*Tamarkin's work is based on the assumption that the coefficients of the differential equation ($\omega^2\varepsilon$, $\nu_1(x)$, $\frac{d\nu_2}{dx}$ ) are continuous functions of x. This assumption will apply throughout the remainder of this chapter.

would therefore expect that Tamarkin's expansion coefficients must actually be a special case of some other more general expression in which the coefficients are not unique.

Many years after Tamarkin's work appeared, the brilliant mathematician R. E. Langer studied second order ordinary differential systems more general than system 4.35. A major portion of this work concerning second order differential systems appears in a single paper entitled "The Expansion Problem in the Theory of Ordinary Linear Differential Systems of the Second Order" (5). In this paper Langer rigorously derived an expansion theorem involving the eigenfunctions of second order ordinary differential systems.

Langer's results are of more value to us than Tamarkin's because they show that the expansion coefficients are not unique; there are many possible $\{C_n\}$ such that $\Sigma\ C_n f_n(x)$ converges uniformly to a given $F(x)$ . Thus the set $\{f_n\}$ is not exactly complete. It is not surprising that Tamarkin did not realize that his expansion coefficients should not be unique and that the eigenfunctions of system 4.20 form a more than complete set. We determined that the eigenfunctions of the ordinary differential system 4.20 must form a more than complete set by considering the original partial differential system 4.1. Suppose that we were given only the ordinary differential system, as was Tamarkin, and were not told that it was obtained from a partial differential equation describing some physical problem. There would be no reason to suspect that the set $\{f_n\}$ is more than complete. Hence, there would also be no reason to question the uniqueness of the expansion coefficients. Thus it is quite understandable that Tamarkin, a mathematician studying systems of the form 4.35 without regard to what physical systems give rise to these ordinary differential systems, did not realize his evaluation of the expansion coefficients yielded just one of many possible eigenfunction expansions.

Until 1964 no one working on the problem of propagation of TE waves in ferrite filled guides realized that system 4.35 is a particular case of Langer's work. In 1964 D.S. Cohen (6) made this observation and applied Langer's results to the special case of system 4.35. Cohen was thus the first person to make rigorous mathematical statements about the completeness properties of the eigenfunctions or modes associated with TE wave propagation in ferrite filled guides. Although Cohen applied Langer's work to obtain solutions of the ordinary differential system 4.35, he did not obtain solutions of the original partial differential system 4.1. More precisely, given a function $F(x)$, Cohen showed how to determine the expansion coefficients $c_n$ such that

$$F(x) = \Sigma\, c_n\, f_n(x) \quad ; \qquad (4.37)$$

however, he did not have any means of determining the expansion coefficients $a_n$ such that

$$E_y(x,z) = \Sigma\, a_n\, f_n(x)\, e^{ih_n z} \qquad (4.38)$$

solved the original partial differential system 4.1 describing the waveguide problem.

We will now apply Cohen's results concerning the ordinary differential system 4.35 to obtain solutions of the original partial differential system; we will find for the first time what combinations of modes yield solutions to some given waveguide problems. We do not want to become involved in the details of Cohen's and Langer's works, and will therefore state only their results which we need. The details of their work may be found in the original papers (5)-(6).

The results which Cohen obtained by applying the particular form of system 4.35 to Langer's general work are as follows:*

1. Any arbitrary, bounded, continuous function $F(x)$ can be expanded in the uniformly convergent series

$$F(x) = \Sigma\, c_n\, f_n(x)\,[\nu_1(x)]^{1/2} e^{\int_0^x \frac{1}{2\,\nu_1(\xi)} \frac{d\nu_2}{d\xi} d\xi} \qquad (4.39)$$

where $f_n$ are the eigenfunctions of system 4.35.

2. The expansion coefficients $c_n$ are given as follows:

$$c_n = \frac{\int_0^1 Z_n\, R \begin{bmatrix} F(x) \\ K(x) \end{bmatrix} dx}{\int_0^1 Z_n\, R\, Y_n\, dx} \qquad (4.40)$$

*We have converted the results to our notation and simplified whereever possible.

where K(x) is an arbitrary, bounded, continuous function and

$Y_n = \begin{bmatrix} y_n^{(1)} \\ y_n^{(2)} \end{bmatrix}$ is defined by

$$Y'(x) - [-h\,R + B(x)]\,Y(x) = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} Y(0) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} Y(1) = 0 \tag{4.41}$$

and $Z_n = [\, Z_n^{(1)}, Z_n^{(2)} \,]$ is defined by the adjoint system

$$Z'(x) + Z(x)\,[-h\,R + B(x)\,] = 0$$

$$-Z(0) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + Z(1) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0 \tag{4.42}$$

and

$$R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4.43}$$

and

$$B(x) = \begin{bmatrix} 0 & e^{\int_0^x \frac{1}{\nu_1(\xi)} \frac{d\nu_2}{d\xi} d\xi} \\ q(x)\, e^{-\int_0^x \frac{1}{\nu_1(\xi)} \frac{d\nu_2}{d\xi} d\xi} & 0 \end{bmatrix} \tag{4.44}$$

and

$$q(x) = \frac{1}{2\nu_1} \left( \frac{d^2\nu_1}{dx^2} + \frac{d^2\nu_2}{dx^2} \right) - \frac{1}{4\nu_1^2} \left( \frac{d\nu_1}{dx} + \frac{d\nu_2}{dx} \right)^2 - \frac{\omega^2 \varepsilon}{\nu_1} \tag{4.45}$$

and the functions $f_n$, $y_n^{(1)}$ are related by

$$f_n(x) = \nu_1^{-1/2}(x) \quad e^{-\int_0^x \frac{1}{2\,\nu_1(\xi)} \frac{d\nu_2}{d\xi} d\xi} \; y_n^{(1)}(x) \quad . \tag{4.46}$$

Langer and Cohen obtained the expansion coefficients $c_n$ by transforming the differential system 4.35 into the matrix form 4.41 and using the adjoint system 4.42 to formally obtain expansion coefficients 4.40* Langer rigorously proved that this formal evaluation of the expansion coefficients is correct.

Notice that the expansion coefficients depend on an arbitrary function K(x) as well as the function to be expanded, F(x). By choosing different functions K(x), different expansions are obtained for the same function F(x). (Langer showed that Tamarkin's expansion could be obtained by choosing a particular K(x); hence, as was expected from our previous work, Tamarkin's expansion is a special case of a more general expansion.)

Until now no one seems to have noticed that Cohen's results concerning the expansion coefficients $c_n$ may be greatly simplified and reduced to a form appropriate for solving the original partial differential system 4.1. Since this reduction of Cohen's work is quite lengthy and involved, it has been placed in appendix B. The

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*The idea of transforming ordinary differential systems to matrix systems is often a useful method of solving non-self-adjoint problems. An excellent example of the application of the matrix approach to a much simplier problem may be found in Friedman (7).

crucial observation necessary for this reduction is that the components of the vector $Y_n$ associated with system 4.41 are related to the components of the vector $Z_n$ associated with the adjoint system 4.42. In appendix B we prove that Cohen's eight relationships 4.39-4.46 can be reduced to the following simple statement:

> Any arbitrary, bounded, continuous function F(x) can be expanded in the uniformly convergent series

$$F(x) = \Sigma\, c_n f_n(x) \tag{4.47}$$

> where $f_n(x)$ are the eigenfunctions of system 4.35 and

$$c_n = \frac{\int_0^1 (\nu_1\, h_n \quad F(x) + K(x))\, f_n(x)\, dx}{\int_0^1 (2\, h_n\, \nu_1 + \frac{d\nu_2}{dx})\, f_n{}^2\, dx} \tag{4.48}$$

> where K(x) is an arbitrary, bounded, continuous, function.

Now let us proceed to use the coefficients of 4.48 associated with the ordinary differential system to solve the original partial differential system 4.1. Assume a solution of the form

$$E_y(x,z) = \Sigma\, a_n\, f_n(x)\, e^{ih_n z} \quad . \tag{4.49}$$

Suppose we are given E (x,o). Assuming uniform convergence in z, the expansion 4.49 becomes

$$E_y(x,o) = \Sigma\, a_n\, f_n(x). \tag{4.50}$$

From 4.48 we know that this expansion is valid if the set $\{a_n\}$ is defined by

$$a_n = \frac{\int_0^1 (\nu_1 h_n E_y(x,o) + K(x))\, f_n(x)\, dx}{\int_0^1 (2\, h_n \nu_1 + \frac{d\nu_2}{dx})\, f_n^2\, dx} \qquad (4.51)$$

With $a_n = a_n\,[K(x)]$ defined by 4.51, the expansion 4.49 satisfies the boundary condition at $z = 0$. Physically, this is quite plausible. The value of $E_y(x,z)$ at one value of z does not completely determine the field; an additional boundary condition is necessary. The arbitrary function $K(x)$ must be chosen such that the additional boundary condition is satisfied. More precisely, the solution 4.49 defined by 4.51 is actually an infinite set of possible solutions which satisfy the boundary condition $E_y(x,z)|_{z=o} = E_y(x,o)$. An additional boundary condition must be given to determine which one solution of the infinite set of solutions solves the problem of interest. This additional boundary condition determines the function $K(x)$. For example, if we were also given $E_y(x,\ell)$, then we must choose $K(x)$ such that $\Sigma\, a_n\, f_n(x)\, e^{ih_n\ell}$ converges to $E_y(x,\ell)$.

Under the assumption that the series 4.49 is uniformly convergent in z, we can make the following statements:

1. Langer's work guarantees that the solution $\Sigma\, a_n\, [K(x)]\, f_n(x)\, e^{ih_n z}$ converges to $E_y(x,o)$ at $z = 0$ for any bounded, continuous function $E_y(x,o)$.

2. The problem of determining the expansion coefficients $a_n$ [K(x)] such that $\Sigma a_n f_n(x) e^{ih_n z}$ is a solution of the original partial differential system, can thus be reduced to determining the function K(x).

We must now ask if we can find a K(x) which yields expansion coefficients $a_n$ [K(x)] such that the sum $\Sigma a_n f_n e^{ih_n z}$ converges to match some additional appropriate boundary condition at some value of z. In general, determining the function K(x) is very difficult. For example, suppose we were given the value of $E_y(x,z)$ at $z = \ell$. We would then have to find a function K(x) such that

$$E_y(x,\ell) = \Sigma \frac{\int_0^1 (\nu_1 h_n E_y(\xi,o) + K(\xi)) f_n(\xi) d\xi}{\int_0^1 (2 h_n \nu_1 + \frac{d\nu_2}{d\xi}) f_n^2(\xi) d\xi} f_n(x) e^{ih_n \ell} \quad (4.52)$$

Clearly this is not a simple task. There does not appear to be any direct way of proceeding. Thus, we ask if there is any physical problem for which the function K(x) may be easily obtained. The answer to this question is yes. Suppose both the electric and magnetic fields are given for one value of z, say z = 0. According to physical considerations there is certainly a unique solution for this problem; that is, the electric and magnetic fields at all points within a guide are uniquely determined once the electric and magnetic fields at one cross section are given. We will now mathematically determine for the first time these resulting fields. We will determine a set $\{ a_n \}$ such that the expansion $\Sigma\, a_n\, f_n\, e^{ih_n z}$ satisfies Maxwell's equations, converges to a given value at z = 0, and has an associated magnetic field equal to some given value at z = 0.

First it should be noted that specifying $\underline{E}(x,o)$ and $\underline{H}(x,o)$ is equivalent to specifying $E_y(x,o)$ and $\frac{\partial}{\partial z} E_y(x,z)\Big|_{z=o} = \frac{\partial E_y(x,o)}{\partial z}$.
For the TE case which we are considering, we know from the work in chapter 2 that

$$\underline{E}(x,o) = E_y(x,o)\ \hat{e}_y$$
$$\underline{H}(x,o) = H_x(x,o)\ \hat{e}_x + H_z(x,o)\ \hat{e}_z \quad . \tag{4.53}$$

According to equation 2.10

$$H_x = \frac{1}{i\omega}\left( -\nu_1 \frac{\partial E_y}{\partial z} + i\nu_2 \frac{\partial E_y}{\partial x} \right)$$
$$H_z = \frac{1}{i\omega}\left( i\nu_2 \frac{\partial E_y}{\partial z} + \nu_1 \frac{\partial E_y}{\partial x} \right) \quad . \tag{4.54}$$

Providing $\nu_1^2 \neq \nu_2^2$, we may take the inverse of system 4.54

$$\frac{\partial E_y}{\partial z} = \frac{i\omega}{\nu_2^2 - \nu_1^2}\left(\nu_1 H_x - i\nu_2 H_z\right)$$
$$\frac{\partial E_y}{\partial x} = \frac{i\omega}{\nu_2^2 - \nu_1^2}\left( - i\nu_2 H_x - \nu_1 H_z\right) \quad . \tag{4.55}$$

Thus specifying $E_y(x,o)$ and $\frac{\partial}{\partial z} E_y(x,o)$ is equivalent to specifying

$\underline{E}(x,o)$ and $\underline{H}(x,o)$.

The problem has now been reduced to determining a set $\{a_n\}$ such that the proposed solution $\Sigma a_n f_n e^{ih_n z}$ converges to $E_y(x,o)$ at $z = 0$ and $\frac{\partial}{\partial z} E_y(x,z)$ converges to a given arbitrary function $\frac{\partial}{\partial z} E_y(x,o)$ at $z = 0$; or assuming uniform convergence in z

$$E_y(x,o) = \Sigma\, a_n \;\; f_n(x) \tag{4.56}$$

$$\frac{\partial E_y(x,o)}{\partial z} = \Sigma\, ih_n\, a_n\, f_n(x) \quad . \tag{4.57}$$

From equations 4.47 and 4.48 we know 4.56 is satisfied by

$$a_n = \frac{\int_0^1 (\nu_1\, h_n\, E_y(x,o) + \overset{(1)}{K(x)}\,)\, f_n(x)\, dx}{\int_0^1 (2\, h_n\, \nu_1 + \frac{d\nu_2}{dx})\, f_n^2(x)\, dx} \tag{4.58}$$

where $\overset{(1)}{K(x)}$ is an arbitrary function. For the same reason, 4.57 is satisfied by

$$ih_n\, a_n = \frac{\int_0^1 (\nu_1\, h_n\, \frac{\partial E_y(x,o)}{\partial z} + \overset{(2)}{K(x)}\,)\, f_n(x)\, dx}{\int_0^1 (2\, h_n\, \nu_1 + \frac{d\nu_2}{dx})\, f_n^2(x)\, dx} \tag{4.59}$$

where $\overset{(2)}{K(x)}$ is an arbitrary function.

We must prove that a $\overset{(1)}{K(x)}$ and a $\overset{(2)}{K(x)}$ exist such that both equations 4.56 and 4.57 can be satisfied by the same set of expansion coefficients $\{a_n\}$. That is, we must show $\overset{(1)}{K}(x)$ and $\overset{(2)}{K}(x)$ may be chosen such that

$$\frac{\int_0^1 (\nu_1 h_n E_y(x,o) + \overset{(1)}{K}(x))f_n dx}{\int_0^1 (2\, h_n\, \nu_1 + \frac{d\nu_2}{dx}) f_n^2 dx} = \frac{1}{ih_n}\; \frac{\int_0^1 (\nu_1 h_n \frac{\partial}{\partial z} E_y(x,o) + \overset{(2)}{K}(x)) f_n dx}{\int_0^1 (2\, h_n\, \nu_1 + \frac{d\nu_2}{dx}) f_n^2 dx} \tag{4.60}$$

or

$$A \equiv \int_0^1 [\, \nu_1\, h_n\, E_y(x,o) + \overset{(1)}{K}(x) - \frac{1}{ih_n} (\nu_1 h_n \frac{\partial E(x,o)}{\partial z} + \overset{(2)}{K}(x))]\; f_n(x)\, dx \tag{4.61}$$

$$= 0 \quad .$$

From the original eigenvalue equation 4.35 we may write

$$\nu_1 h_n\, f_n = \frac{1}{h_n} \left( \frac{d}{dx} \nu_1 \frac{df_n}{dx} + \omega^2 \varepsilon f_n - h_n \frac{d\nu_2}{dx} f_n \right) \tag{4.62}$$

or

$$\int_0^1 \nu_1 h_n f_n E_y(x,o) dx = \frac{1}{h_n} \int_0^1 \left(\frac{d}{dx} \nu_1 \frac{df_n}{dx} + \omega^2 \varepsilon f_n - h_n \frac{d\nu_2}{dx} f_n\right) E_y(x,o) dx \; . \tag{4.63}$$

Integrating by parts and requiring the physically reasonable restriction that $E_y(x,o)|_{x=0} = E_y(x,o)|_{x=1} = 0$, we obtain

$$\int_0^1 \nu_1 h_n f_n E_y(x,o)dx = \frac{1}{h_n} \int_0^1 \Big( \frac{d}{dx} \nu_1 \frac{dE_y(x,o)}{dx} + \omega^2 \varepsilon E_y(x,o)$$

$$- h_n \frac{d\nu_2}{dx} E_y(x,o) \Big) f_n dx \quad . \tag{4.64}$$

Using equation 4.64 in equation 4.61 yields

$$A = \int_0^1 \Big( \frac{1}{h_n} \frac{d}{dx} \nu_1 \frac{dE_y(x,o)}{dx} + \frac{1}{h_n} \omega^2 \varepsilon E_y(x,o) - \frac{d\nu_2}{dx} E_y(x,o)$$

$$+ \overset{(1)}{K(x)} + i\nu_1 \frac{\partial E_y(x,o)}{\partial z} + i \frac{1}{h_n} \overset{(2)}{K(x)} \Big) f_n dx \quad . \tag{4.65}$$

Rearranging the terms into two integrals

$$A = \int_0^1 \Big( \overset{(1)}{K(x)} + i \nu_1 \frac{\partial E_y(x,o)}{\partial z} - \frac{d\nu_2}{dx} E_y(x,o) \Big) f_n dx$$

$$+ \frac{1}{h_n} \int_0^1 \Big( \frac{d}{dx} \nu_1 \frac{dE_y(x,o)}{dx} + \omega^2 \varepsilon E_y(x,o) + i \overset{(2)}{K(x)} \Big) f_n(x)\, dx \quad . \tag{4.66}$$

Clearly $A \equiv 0$ if we choose

$$K^{(1)}(x) = - i\nu_1 \frac{\partial E_y(x,o)}{\partial z} + \frac{d\nu_2}{dx} E_y(x,o) \qquad (4.67)$$

$$K^{(2)}(x) = i\frac{d}{dx}\nu_1 \frac{dE_y(x,o)}{dx} + i\omega^2\varepsilon \; E_y(x,o) \quad . \qquad (4.68)$$

Thus $K^{(1)}(x)$ and $K^{(2)}(x)$ exist such that both equation 4.56 and equation 4.57 can be satisfied by the same set of expansion coefficients $\{a_n\}$ defined by

$$a_n = \frac{\int_o^1 [ ( \nu_1 h_n + \frac{d\nu_2}{dx} ) E_y(x,o) - i\nu_1 \frac{\partial E_y(x,o)}{\partial z} ] f_n(x) \, dx}{\int_o^1 ( 2 h_n \nu_1 + \frac{d\nu_2}{dx} ) f_n^2 \, dx} \quad . \qquad (4.69)$$

Hence, assuming the necessary uniform convergence in z, we have shown in a completely rigorous manner that the proposed solution

$$E_y(x,z) = \Sigma \, a_n f_n(x) \, e^{ih_n z} \quad , \qquad (4.70)$$

where the set $\{a_n\}$ is defined by 4.69, converges to a given function $E_y(x,o)$ at $z = 0$ and $\frac{\partial E_y(x,z)}{\partial z}$ converges to another given function, $\frac{\partial E_y(x,o)}{\partial z}$, at $z = 0$.

Using equation 4.55,

$$\frac{\partial E_y}{\partial z} = \frac{i\omega}{\nu_2^2-\nu_1^2}\,[\nu_1 H_x - i\,\nu_2 H_z] \quad , \qquad (4.71)$$

we can also state that the expansion

$$E_y(x,z) = \Sigma\, a_n\, f_n(x)\, e^{ih_nz} \qquad (4.72)$$

where the set $\{a_n\}$ is defined by

$$a_n = \frac{\int_o^1 [(\nu_1 h_n + \frac{d\nu_2}{dx})E_y(x,o) + \frac{\omega\nu_1}{\nu_2^2 - \nu_1^2}(\nu_1 H_x(x,o) - i\,\nu_2 H_z(x,o))]f_n dx}{\int_o^1 (2\,h_n \nu_1 + \frac{d\nu_2}{dx})f_n^2 dx} \qquad (4.73)$$

is a solution of Maxwell's equations which converges to a given electric and magnetic field at the plane $z = 0$. Thus, given the electric and magnetic field at any cross section of the guide, we can mathematically determine the resulting electric and magnetic fields at any other cross section of the guide.

It is interesting to note that the expansion coefficients 4.69 of the rigorous solution 4.70 can be formally obtained in a much more direct manner. In order to obtain this formal expression for the expansion coefficients $\{a_n\}$ we must first derive an orthogonality relationship.

Let $f_n$ and $f_m$ be eigenfunctions of system 4.20; let $h_n$ and $h_m$

be the respective corresponding eigenvalues

$$L_n f_n = 0 \qquad f_n(o) = f_n(1) = 0 \tag{4.74}$$

$$L_m f_m = 0 \qquad f_m(o) = f_m(1) = 0 \quad . \tag{4.75}$$

We may then write

$$f_n L_m f_m = 0 \tag{4.76}$$

$$f_m L_n f_n = 0 \quad . \tag{4.77}$$

Subtracting $f_n L_m f_m$ from $f_m L_n f_n$ and integrating yields

$$\int_o^1 (f_m L_n f_n - f_n L_m f_m)\, dx = 0 \quad . \tag{4.78}$$

Substituting in the value of the operator $L_j(x)$ and integrating by parts yields

$$(h_n - h_m) \int_o^1 [(h_n + h_m)\, \nu_1 + \frac{d\nu_2}{dx}]\, f_n f_m\, dx = 0 \quad , \tag{4.79}$$

or assuming $h_n \neq h_m$ for $n \neq m$ ,

$$\int_o^1 [(h_n + h_m)\, \nu_1 + \frac{d\nu_2}{dx}]\, f_n\, f_m\, dx = 0 \quad \text{for } n \neq m.^* \tag{4.80}$$

*This relation is not new; it is a special case of a more general relationship derived by Walker (8).

Clearly there is no obvious means of using this relationship to find expansion coefficients $\{c_n\}$ such that $\Sigma\, c_n\, f_n(x)$ converges to some arbitrary given function. Except for one author who incorrectly applied this relationship to obtain some erroneous results concerning the power flow in a ferrite filled guide, no one has been able to make any use of this orthogonality relationship. Its significance has remained a mystery. We will now show that this relationship, which seems to have no significance with respect to the ordinary differential system from which it is derived (4.20), may be used to determine the expansion coefficients for solutions of the original partial differential system 4.1 (the system from which the ordinary differential system was derived)!

Assume there exists a solution to the partial differential system $E_y(x,z) = \Sigma\, a_n\, f_n(x)\, e^{ih_n z}$ uniformly convergent in z. Differentiation with respect to z yields

$$\frac{\partial E_y(x,z)}{\partial z} = \sum_n i\, h_n\, a_n\, f_n\, e^{ih_n z} \quad . \tag{4.81}$$

Thus for z = 0 we obtain

$$E_y(x,o) = \sum_n a_n\, f_n(x) \tag{4.82}$$

$$\frac{\partial E_y(x,o)}{\partial z} = \sum_n i\, h_n\, a_n\, f_n(x) \quad . \tag{4.83}$$

Multiplying both sides of equation 4.82 by $f_m h_m \nu_1$ and integrating with respect to x yields

$$\int_o^1 E_y(x,o) h_m \nu_1 f_m \, dx = \int_o^1 \sum_n a_n h_m \nu_1 f_n f_m \, dx \quad . \qquad (4.84)$$

Multiplying both sides of equation 4.82 by $\frac{d\nu_2}{dx} f_m$ and integrating with respect to x yields

$$\int_o^1 E_y(x,o) \frac{d\nu_2}{dx} f_m \, dx = \int_o^1 \sum_n a_n \frac{d\nu_2}{dx} f_n f_m \, dx \quad . \qquad (4.85)$$

Multiplying both sides of equation 4.83 by $- i \nu_1 f_m$ and integrating with respect to x yields

$$- \int_o^1 \frac{\partial E_y(x,o)}{\partial z} i\nu_1 f_m \, dx = \int_o^1 \sum_n a_n \nu_1 h_n f_n f_m \, dx \quad . \qquad (4.86)$$

Adding equations 4.84, 4.85, 4.86 and formally interchanging the summation and integration yields

$$\sum_n a_n \int_o^1 [(h_n + h_m)\nu_1 + \frac{d\nu_2}{dx}] f_n f_m dx = \int_o^1 [(h_m \nu_1 + \frac{d\nu_2}{dx}) E_y(x,o) - i\nu_1 \frac{\partial E_y(x,o)}{\partial z} ] f_m \, dx \quad . \qquad (4.87)$$

Applying the orthogonality relationship 4.80 yields

$$a_n = \frac{\int_o^1 [(h_n \nu_1 + \frac{d\nu_2}{dx}) E_y(x,o) - i \nu_1 \frac{\partial E_y(x,o)}{\partial z}] f_n dx}{\int_o^1 (2h_n \nu_1 + \frac{d\nu_2}{dx}) f_n^2 dx} \quad (4.88)$$

This is exactly the result which we obtained rigorously (Compare with equation 4.69). Thus the orthogonality relationship 4.79, which appears to have no particular significance with respect to the ordinary differential system from which it was derived, formally seems to have a great significance with respect to the partial differential system from which the ordinary system was derived. This certainly supports our previous statement that some ordinary differential systems are inherently related to particular partial differential systems.

For the benefit of readers who have studied Langer's and Cohen's works, we would like to point out one additional observation. The orthogonality relationship $(h_n - h_m)\int_o^1 Z_n RY_m dx = 0$ used by Cohen and by Langer to obtain the vector expansion theorem may be shown equivalent to the orthogonality relationship 4.79.*

---

* Appendix C contains the proof of this statement

It should be clear that the formal method of obtaining the expansion coefficients $\{ a_n\}$ of 4.88 cannot be justified without the preceding rigorous discussion of the completeness properties of the set of eigenfunctions $\{f_n\}$. Without such a discussion it is impossible to prove that $\Sigma\, a_n\, f_n(x)$ converges to $E_y(x,o)$ and $\Sigma\, a_n\, i\, h_n\, f_n(x)$ converges to $\frac{\partial}{\partial z} E_y(x,o)$.

Now that we have developed a means of determining the electric and magnetic fields at any point in a ferrite loaded guide from a knowledge of the fields at any one cross-section, let us make a few statements concerning other possible waveguide boundary value problems. Suppose the given boundary conditions are $E_y(x,o) = 0$ and $E_y(x,z)\big|_{z=\ell} = E_y(x,\ell)$. Assuming a solution of the form

$$E_y(x,z) = \Sigma\; a_n\, f_n(x)\, e^{ih_n z} \quad , \qquad (4.89)$$

the problem reduces to finding a set $\{ a_n \}$ such that

$$o = \Sigma\;\; a_n\, f_n(x) \qquad (4.90a)$$

$$E_y(x,\ell) = \Sigma\; a_n\;\; e^{ih_n\ell}\, f_n(x) \quad . \qquad (4.90b)$$

From equations 4.47-4.48 we know that equation 4.90a is satisfied by

$$a_n = \frac{\int_o^1 K^{(1)}_{(x)}\, f_n(x)\, dx}{\int_o^1 (2\, h_n\, \nu_1 + \frac{d\nu_2}{dx})\, f_n^2(x)\, dx} \quad ; \qquad (9.91a)$$

and 4.90b, by

$$e^{ih_n \ell} \quad a_n = \frac{\int_0^1 (\nu_1 \; h_n \; E_y(x,\ell) + K^{(2)}_{(x)} ) \; f_n(x) \; dx}{\int_0^1 (2 \; h_n \; \nu_1 + \frac{d\nu_2}{dx} ) \; f_n^2 \; dx} \tag{4.91b}$$

where $K^{(1)}_{(x)}$ and $K^{(2)}_{(x)}$ are arbitrary, bounded, continuous functions. In order to obtain a solution, we would have to prove the existence of and then determine $K^{(1)}$, $K^{(2)}$ such that equations 4.90a and 4.90b would be satisfied by the same set of expansion coefficients $\{ a_n \}$. In analog with equation 4.61 we would have to find $K^{(1)}$ and $K^{(2)}$ such that $A_n \equiv 0$ for all n where

$$A_n = \int_0^1 (\nu_1 \; h_n \; E_y(x,\ell) + K^{(2)}_{(x)} - e^{ih_n \ell} \; K^{(1)}_{(x)} ) \; dx \quad . \tag{4.92}$$

It is clear that the dependence on n cannot be easily isolated to yield solutions for $K^{(1)}$ and $K^{(2)}$ . Herein lies the general difficulty. In general, ferrite loaded waveguide boundary value problems lead to infinite sets of integral equations which cannot be solved rigorously. The set of integral equations 4.92 is a case in point.

The fact the rigorous solutions cannot in general be obtained should not surprise the reader. Even for empty waveguides, rigorous solutions cannot usually be obtained and it is necessary to resort to approximate methods. For empty waveguides it has been possible to apply the general theoretical work to develop reasonably successful approximate methods (Schwinger's variational approach, etc.). Until now, such general theoretical work for ferrite loaded guides was not available. The theoretical work which we have presented may be used to provide a firm foundation with which to develop sensible methods of approximation for ferrite loaded waveguide problems.

# PART II - THE THERMODYNAMIC PARADOX

## 5.0 Introduction - Statement of the Paradox

In this part we will study the problem of the thermodynamic paradox. We will state the paradox, review past significant research, provide important corrections to this research, and present the resolution. We begin by describing the problem which gives rise to the thermodynamic paradox.

In 1956 K. J. Button and B. Lax (9) investigated the propagation of electromagnetic energy in an infinitely long rectangular waveguide partially filled with a transversely magnetized ferrite slab. See figure 5.1. They restricted their study of the propagation of $TE_{no}$ modes ( $\frac{\partial}{\partial y} = 0$) and hence essentially considered the propagation of electromagnetic energy in a ferrite region between two parallel plates. They also assumed that the transverse biasing field was uniform and sufficiently strong to saturate the ferrite and that the region of operation did not include ferromagnetic resonance.

Using the notation of Part I, Maxwell's equations describing wave propagation in such guides may be written as

$$\nabla \times \underline{E}(x,z) = i\omega\mu_o \, \underline{\underline{\nu}}^{-1}(x) \cdot \underline{H}(x,z) \tag{5.1a}$$

$$\nabla \times \underline{H}(x,z) = - i\omega\varepsilon\underline{E}(x,z) \tag{5.1b}$$

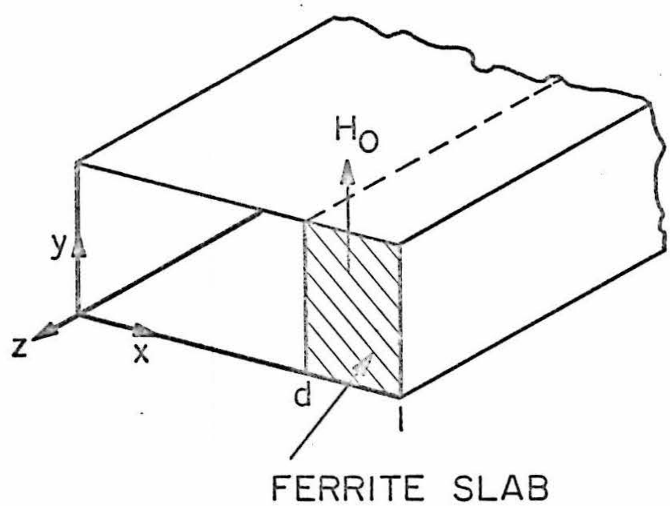


Figure 5.1

where

$$\underline{\nu}(x) = \begin{bmatrix} \nu_1(x) & 0 & i\nu_2(x) \\ 0 & \nu_3(x) & 0 \\ -i\nu_2(x) & 0 & \nu_1(x) \end{bmatrix}$$

where for $0 < x < d$

$$\nu_1(x) = 1$$

$$\nu_2(x) = 0$$

$$\nu_3(x) = 1$$

and for $d < x < 1$

$$\nu_1(x) = A$$

$$\nu_2(x) = B$$

$$\nu_3(x) = C$$

where A, B, C are real constants whose values depend on the properties of ferrite, the amplitude of the biasing magnetic field, and the frequency of operation.

Using the Heavyside step function notation we may thus write for $0 < x < 1$

$$\begin{aligned} \nu_1(x) &= 1 + (A - 1)\, H(x - d) \\ \nu_2(x) &= B\, H\,(x - d) \\ \nu_3(x) &= 1 + (C - 1)\, H(x - d) \end{aligned} \tag{5.2}$$

where

$$H(x - d) = \begin{cases} 0 & x < d \\ 1 & x > d \end{cases} .$$

From the work in Part I we know that only the y component of the electric field can be non-zero for $TE_{no}$ propagation. Assuming this component may be written as

$$E_y(x,z) = f(x)\, e^{ihz} ,$$

Maxwell's equations may be reduced to the following system

$$\frac{d}{dx}\nu_1(x) \frac{df}{dx} + (\omega^2 \varepsilon\mu_o - h^2 \nu_1(x) - h \frac{d\nu_2}{dx})f = 0$$

$$f(0) = f(1) = 0 \tag{5.3}$$

where the associated magnetic field is given in terms of f(x) by

$$\underline{H}(x,z) = (\nu_2 \frac{df}{dx} - \nu_1 hf)\frac{e^{ihz}}{\omega\mu_o} \hat{e}_x$$

$$+ (\nu_1 \frac{df}{dx} - \nu_2 hf)\frac{e^{ihz}}{i\omega\mu_o} \hat{e}_z . \tag{5.4}$$

Lax and Button studied system 5.3 for each of the two regions. For $0 < x < d$ system 5.3 becomes

$$\frac{d^2 f}{dx^2} + ( \omega^2 \varepsilon \mu_o - h^2 ) f = 0$$

$$f(0) = 0 \quad ; \qquad (5.5)$$

for $d < x < 1$ ,

$$\frac{d^2 f}{dx} + ( \frac{\omega^2 \varepsilon \mu_o}{A} - h^2 ) f = 0$$

$$f(1) = 0 \quad . \qquad (5.6)$$

Solutions of 5.5 are

$$f(x) = (\text{const.}) \cdot \sinh k^{(1)} x \qquad (5.7)$$

where $k^{(1)} = \sqrt{h^2 - \omega^2 \varepsilon \mu_o}$ .

Solutions of 5.6 are

$$f(x) = (\text{const.}) \cdot \sinh k^{(2)} (x-1) \qquad (5.8)$$

where $k^{(2)} = \sqrt{h^2 - \frac{\omega^2 \varepsilon \mu_o}{A}}$ .

Matching the tangential components of the electric and magnetic fields at $x = d$ in the usual manner, Lax and Button obtained the following transcendental equation:

$$T_{(h)} = k^{(1)} \coth k^{(1)} d - A k^{(2)} \coth k^{(2)} (d-1) + hB = 0 \qquad (5.9)$$

Clearly this equation cannot be solved explicitly for the propagation constants h. Lax and Button studied equation 5.9 and found that for a given range of frequencies it is possible to choose the ferrite parameters A, B, and slab thickness (l-d) such that 5.9 admits the possibility of a single propagating mode (a mode having a pure real propagation constant). Their now famous plot of possible real propagation constants versus ferrite slab thickness appears in figure 5.2. Notice that there is only one propagating mode for a sufficiently large slab thickness. Lax and Button concluded that this single propagating mode implied the existence of a lossless unidirectional transmission system. Such a system would constitute a clear violation of the basic laws of thermodynamics, hence the so-called thermodynamic paradox.

In order to make the meaning of the paradox more clear, let us consider a finite section of such a ferrite filled guide reactively terminated at one end, say $z = \ell$, and connected to an empty guide at the other end, say $z = 0$. See figure 5.3. Now suppose the empty guide is fed from the left with energy such that the unidirectional mode is excited in the ferrite guide. This unidirectional mode will carry power to the right toward the reactive termination at $z = \ell$. Lax and Button would reason that since there are no propagating modes carrying power to the left away from the reactive termination, the input power is being continually fed into a lossless system, clearly an inconsistency.

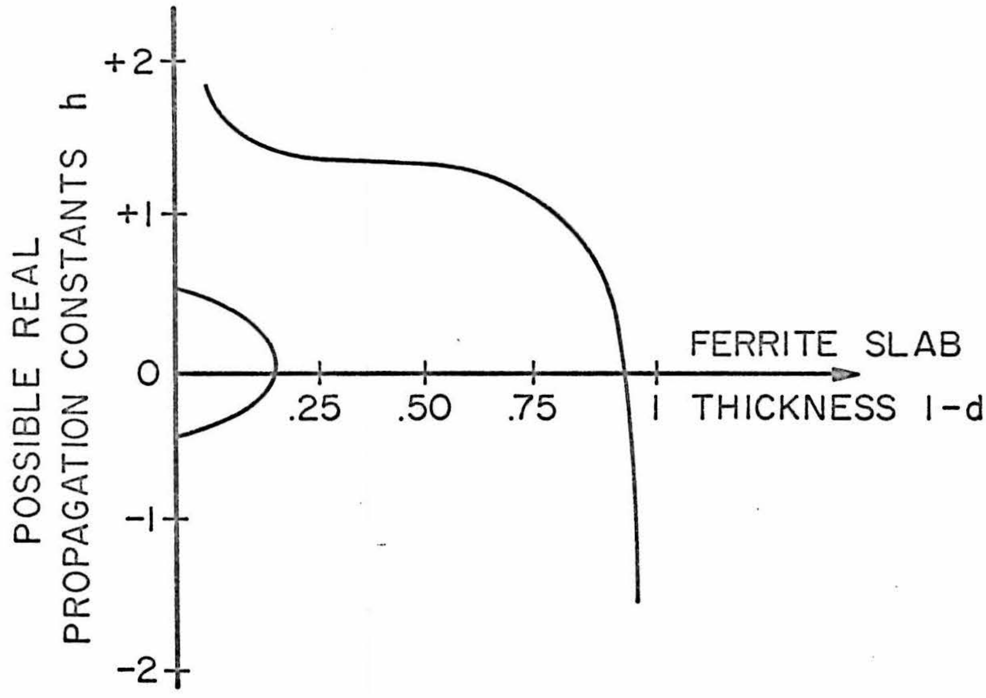


Figure 5.2

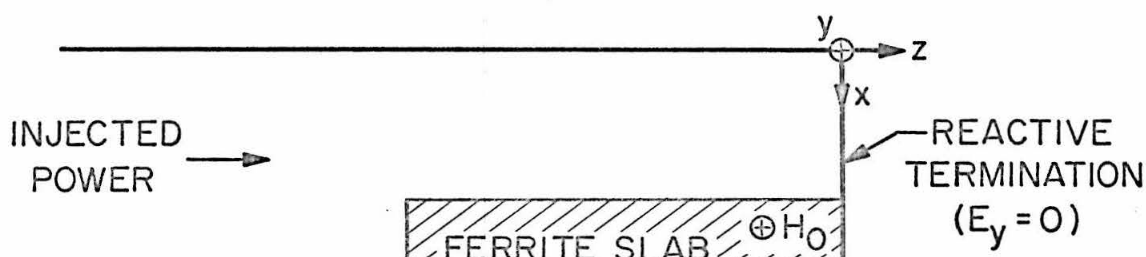


Figure 5.3

It should be noted that no one has ever solved the boundary value problem posed in figure 5.3; that is, no one has successfully determined the proper linear combination of modes satisfying the boundary conditions of figure 5.3. The existence of the paradox has only been implied by the reasoning in the preceding paragraph based on the assumption that the fields in the finite section of the ferrite filled guide can be expressed as a linear combination of the modes of the infinite guide.

As the research progressed, it became apparent that no one had been able to determine the linear combination of modes solving any practical ferrite guide problem.* This realization led to the general work presented in Part I.

Since Lax and Button's work in 1956 there have been many papers published concerning the paradox. In the next chapter we will give a very brief chronological summary of the previous significant research.

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*That is, no one has ever rigorously solved any ferrite guide problem in which the ferrite fills the guide inhomogeneously. As previously shown in Part I, solutions for homogeneously filled ferrite guides are trivial.

## 6.0 Review of Previous Research

In this chapter we will give a very brief review of the past significant research concerning the thermodynamic paradox. Only the conclusions of the past research will be presented; the reader will be referred to the original papers for details. Some of these conclusions which have been accepted to date are incorrect. In the next section the errors in these incorrect conclusions will be presented; only details which are essential to correct these errors will be given.

The first published statement concerning Lax and Button's thermodynamic paradox was made in 1956 by M. L. Kales. Kales made the following statement: (10)

> "Even in the case of a conventional waveguide at a frequency for which all modes are cut off, it is possible to transmit energy through a finite length. This requires only that two properly phased modes of the same kind, and attenuated in reverse direction, be present. It therefore does not seem unreasonable to expect that propagation through the finite ferrite section is possible for either direction of propagation, when modes belonging to both directions are present simultaneously."

Kales did not prove that the power carried to the right by the propagating mode in figure 5.3 returned via the cutoff modes, he merely stated that it seemed reasonable that it could.

In 1957 H. Seidel (11) studied the atomic model of ferrite materials and concluded that lossless ferrite materials cannot exist.

According to Seidel, all ferrites have an "intrinsic" loss which prevents the transmission of energy without decay; thus, the propagating unidirectional mode of the thermodynamic paradox cannot exist.

Two years later H. Seidel and R. C. Fletcher investigated the propagation of higher order modes ($TE_{nm}$, $m > 0$ $\frac{\partial}{\partial y} \neq 0$ ) in rectangular guides containing ferrite slabs and found that these higher order (gyromagnetic) modes can carry power. They concluded that the power carried by the single propagating mode in the problem of the thermodynamic paradox returned via these gyromagnetic modes. They rejected Kales' explanation that the power returned via the cutoff modes with the following statement: (12)

> "...Our reason for favoring the gyromagnetic mode resolution rather than the cutoff modes is that we have experimental evidence for the coupling to the gyromagnetic modes..."

It should be immediately clear to the reader that the gyromagnetic modes ( $\frac{\partial}{\partial y} \neq 0$) cannot possibly resolve the thermodynamic paradox. Although Seidel and Fletcher interpreted their experimental results concerning rectangular guides as showing that there is coupling to the gyromagnetic modes, it is clear that such coupling cannot explain the paradox for propagation between parallel plates. The problem of the paradox for propagation between parallel plates may be reduced to a two dimensional problem in which the gyromagnetic modes have no role.

In 1960 A. D. Bresler (13) studied the problem of the thermodynamic paradox in great detail. Bresler came to the following conclusions:

1. It is impossible for power to return via the cutoff modes.
2. Seidel's "intrinsic" loss resolution (11) is not acceptable because it does not resolve the paradox within the framework in which the problem was posed.
3. If the problem of propagation in guides containing a ferrite slab against one side wall as in figure 5.1 is solved by first considering the slab to be a distance $\varepsilon$ away from the wall and then taking the limit $\varepsilon \to 0$, no unidirectional propagation will be found. Thus "the two idealizations $\varepsilon = 0$ and $\varepsilon \to 0$ lead to distinctly different solutions. Without asking why this difference arises, we are justified in choosing between them on the basis that the idealization which leads to a thermodynamic paradox must be discarded." (14)

During the same year that Bresler published his paper concerning the resolution of the thermodynamic paradox, C. T. Tai (15) considered the propagation of the cutoff modes in rectangular guides inhomogeneously filled with transversely magnetized ferrite. Although he did not offer a resolution of the thermodynamic paradox, he did give an independent proof that the propagating power in the problem of the paradox cannot return via the cutoff modes.

Thus by 1960, three investigators had independently rejected

Kales' cutoff mode resolution. Since 1960 all papers published on the thermodynamic paradox have accepted and referred to these previous proofs in rejecting the cutoff mode resolution. In the next section we will show that each of these proofs is incorrect.

Two years after Bresler's paper appeared, Lax and Button published a book on ferrites and ferrimagnetics (16). An entire chapter of this book is devoted to the propagation of waves in rectangular guides containing ferrite slabs. In this chapter, the possible higher order $TE_{nm}$ modes are studied in detail. It is shown that for particular ferrite parameters, even these higher order modes cannot resolve the paradox as Seidel and Fletcher had proposed. According to Lax and Button (17):

> "...the thermodynamic paradox associated with unidirectional waveguide propagation still exists despite the discovery of this new class of anomalous (higher order) modes. It is unlikely that this paradox will be completely resolved, at least until the solutions are investigated with methods similar to those of C. T. Tai."*

It is clear that Lax and Button became so involved in the details of these higher order modes that they overlooked the previously presented simple argument which makes it apparent that the resolution of the paradox cannot be found in these higher order modes.

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*Tai's methods did not lead to a resolution of the paradox.

In 1962 Akira Ishimaru (18) studied the problem of the thermodynamic paradox and arrived at the following conclusions:

1. Seidel's (12) and Bresler's (13) proofs that the power cannot return via the cutoff modes are valid.
2. Seidel's (11) intrinsic loss idea does not solve the difficulty within the framework of Maxwell's equations; it should be possible to resolve the paradox purely on the basis of mathematical arguments without employing the atomic model or the idea of intrinsic loss.
3. Bresler's resolution is not valid for the general case of a single unidirectional propagating mode. There exists a unidirectional mode in a lossless medium.

Ishimaru considered a unidirectional mode existing at a ferrite-metal interface and showed that the solution of Maxwell's equations for such a structure is discontinuous as the conductivity g of the conductor approaches $\infty$ . He concluded (19)

> "... the problem of solving Maxwell's equations for a purely lossless medium constitutes an 'Improperly-posed problem,' which simply does not correspond to physical reality."

In 1966 G. Barzilai and G. Gerosa published a paper on the thermodynamic paradox (20). Although they did not resolve the paradox, they did take a very important step forward by showing that Bresler's and Ishimaru's resolutions did not resolve the paradox. They proved that

1. "...the structure with a vanishingly small vacuum gap and capable of carrying back a finite power, used by Bresler to resolve the paradox, is not the same structure as the one originally considered by Lax and Button. In fact, the former structure allows the electric field to undergo a finite discontinuity on the ferrite-wall interface, while the latter assumes a zero-continuous electric field on the ferrite-wall interface." (21)

2. "...the ferrite-metal interface structure suggested by Ishimaru to resolve the paradox, in the limit $g \rightarrow \infty$, is not the same structure as the interface between ferrite and a medium on which the tangential electric field is assumed to be zero, since the first can carry a surface wave with a nonvanishing tangential electric field, while the second cannot. These conclusions can be extended to a rectangular guide loaded with a slab of transversely magnetized ferrite." (22)

Barzilai and Gerosa thus proved that the limit structures proposed by Bresler and Ishimaru do not approach the original problem of the thermodynamic paradox, and hence they do not help to resolve the apparent inconsistency.

The most recent paper on the thermodynamic paradox appeared in September 1967 and was written by F. E. Gardiol (23). According to Gardiol, the lossless ferrite model used in the problem of the paradox is not physically realizable; it violates the principle of causality, and therefore should not be used in general theoretical developments.

The past research to date can be summarized as follows:

1. Barzilai and Gerosa (20) have conclusively proved that

Bresler's (13) and Ishimaru's (18) resolutions do not apply to the problem of the paradox.

2. The higher order modes proposed by Seidel and Fletcher (12) cannot possibly resolve the paradox.
3. Seidel's (11) intrinsic loss approach does not resolve the problem within the framework of Maxwell's equations.
4. Kales' proposed resolution, that the power returns via the cutoff modes, has been rejected due to the independent works of Seidel and Fletcher, Tai, and Bresler.
5. Gardiol's resolution has remained unchallenged.

## 7.0 Corrections to Previous Research and the Resolution of the Thermodynamic Paradox

In this section we will show that:

1. The proofs offered by Seidel and Fletcher, Tai, and Bresler to show that power cannot return via the cutoff modes are incorrect.
2. Gardiol's proof that the lossless ferrite model used in the problem of the paradox violates causality is incorrect.

We begin with Gariol's work (23). In the frequency domain, $\underline{B}$ and $\underline{H}$ are related by

$$\underline{B}(\omega) = \underline{\underline{\mu}}(\omega) \cdot \underline{H}(\omega) . \quad (7.1)$$

Application of the convolution integral to equation 7.1 yields the time domain relationship

$$\underline{B}(t) = \int_{-\infty}^{\infty} \underline{\underline{g}}(\tau) \cdot \underline{H}(t-\tau)\, d\tau \quad (7.2)$$

where

$$\underline{\underline{g}}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{\underline{\mu}}(\omega)\, e^{i\omega t}\, d\omega . \quad (7.3)$$

It is clear, as Gardiol states, that if the principle of causality is to be satisfied, all components of $\underline{\underline{g}}(t)$ must be zero for $t < 0$, or if $g_{\ell m}$ denotes the component in the $\ell^{th}$ row and $m^{th}$ column of the tensor $\underline{\underline{g}}$, then

$$g_{\ell m}(t) = \frac{1}{2\pi} \int_{\infty}^{\infty} \mu_{\ell m}(\omega)\, e^{i\omega t}\, d\omega \qquad (7.4)$$

must be zero for all t less than zero and for all $\ell$, m.

Gardiol points out that $\mu_{12}(\omega)$ for a lossless ferrite model (such as used in the problem of the paradox) is given in the form

$$\mu_{12}(\omega) = \frac{ic\omega}{\omega_o^{\,2} - \omega^2} \qquad (7.5)$$

where c and $\omega_o$ are positive real constants. Thus for a lossless ferrite

$$g_{12}(t) = \frac{ic}{2\pi} \int_{\infty}^{\infty} \frac{\omega}{\omega_o^{\,2} - \omega^2}\, e^{i\omega t}\, d\omega \quad . \qquad (7.6)$$

According to Gardiol, the integration of equation 7.6 yields

$$g_{12}(t) = \begin{cases} -\dfrac{c}{2}\cos\ \omega_o t & t < 0 \\ 0 & t = 0 \\ \dfrac{c}{2}\cos\ \omega_o t & t > 0 \end{cases} \qquad (7.7)$$

and hence, since $g_{12}(t) \neq 0$ for $t < 0$, causality is violated by this lossless ferrite. He then concludes that lossless ferrite models are not physically realizable and "may lead to physical absurdities, of which the so-called 'thermodynamic paradox' is an example." (24)

We cannot accept Gardiol's results on the grounds that he has not properly interpreted the integral represented in equation 7.6. Clearly the integration is undefined because the integrand has poles on the real axis along which the integration is to be performed. Integrals whose paths of integration pass over poles are, however, not uncommon to mathematical idealizations of physical problems. Such integrals can usually be defined by:

1. Redefining the path of integration around the troublesome poles.
2. Using the Cauchy principle value. The principle value of the integral 7.6 is

$$\lim_{\varepsilon \to 0} \left[ \int_{-\infty}^{-\omega_0-\varepsilon} + \int_{-\omega_0+\varepsilon}^{+\omega_0-\varepsilon} + \int_{+\omega_0+\varepsilon}^{+\infty} \right] \frac{ic\omega}{2\pi(\omega_0^2-\omega^2)} e^{i\omega t} \, d\omega \ .$$

Gardiol obtained the results 7.7 by using the Cauchy principle value. It is our contention that this is an unreasonable interpretation of the integral 7.6, and the application of such an interpretation to standard classical guide problems yields equally ridiculous results.

By definition the solution $E_y(x,z)$ of a zero-loss idealization of a physical problem is

$$E_y(x,z) = \lim_{\Delta\varepsilon \to 0} E_y(x,z,\Delta\varepsilon) \tag{7.8}$$

where $E_y(x,z\ \Delta\varepsilon)$ represents the solution in the presence of a small loss $\Delta\varepsilon$ . For example, if the lossy solution is given by

$$E_y(x,z,\Delta\varepsilon\ ) = \int_{\infty}^{\infty} G(x,z,\xi,\Delta\varepsilon)\ f(\xi)\ d\xi \tag{7.9}$$

where $f(\xi)$ is some source distribution and $G(x,z,\xi,\Delta\varepsilon)$ is the Green's function for the lossy system, then the lossless solution, if it exists, is given by

$$E_y(x,z) = \lim_{\Delta\varepsilon \to 0} \int_{-\infty}^{\infty} G(x,z,\xi,\Delta\varepsilon)\ f(\xi)\ d\xi \quad . \tag{7.10}$$

Assuming that $G(x,z,\xi,\Delta\varepsilon)$ is Fourier transformable,

$$G(x,z,\xi,\Delta\varepsilon) = \int_{-\infty}^{\infty} g(x,h,\xi,\Delta\varepsilon)\ e^{ihz}\ dh, \tag{7.11}$$

equation 7.10 may be written

$$E_y(x,z) = \lim_{\Delta\varepsilon \to 0} \int_{-\infty}^{\infty} f(\xi) \int_{-\infty}^{\infty} g(x,h,\xi,\Delta\varepsilon)\ e^{ihz}\ dh\ d\xi \quad . \tag{7.12}$$

In general, it is far more difficult to obtain the lossy transformed Green's function $g(x,h,\xi,\Delta\varepsilon)$ than it is to find the zero-loss transformed Green's function $g(x,h,\xi)$ and it is tempting to move the $\lim_{\Delta\varepsilon\to 0}$ under the integrals

$$E_y(x,z) = \int_{-\infty}^{\infty} f(\xi) \lim_{\Delta\varepsilon\to 0} \int_{-\infty}^{\infty} g(x,h,\xi,\Delta\varepsilon)\, e^{ihz}\, dh d\xi \tag{7.13a}$$

$$= \int_{-\infty}^{\infty} f(\xi) \int_{-\infty}^{\infty} \lim_{\Delta\varepsilon\to 0} g(x,h,\xi,\Delta\varepsilon)\, e^{ihz}\, dh\, d\xi \tag{7.13b}$$

$$= \int_{-\infty}^{\infty} f(\xi) \int_{-\infty}^{\infty} g(x,h,\xi) e^{ihz}\, dh\, d\xi \quad . \tag{7.13c}$$

However, it is often the case that the function $g(x,h,\xi)$ has poles on the real axis which means that 7.13b does not follow from 7.13a because the integral

$$\int_{-\infty}^{\infty} g(x,h,\xi)\, e^{ihz}\, dh \tag{7.14}$$

is undefined, while the function

$$\lim_{\Delta\varepsilon\to 0} \int_{-\infty}^{\infty} g(x,h,\xi,\Delta\varepsilon) e^{ihz}\, dh \tag{7.15}$$

does, however, exist and is well-defined. Thus the zero-loss transformed Green's function is of no use in obtaining the lossless solution unless the integral 7.14 may be interpreted such that it is defined and equal to integral 7.15. This may be done in general, not by taking the Cauchy principle value of integral 7.14, but by choosing a new path of integration P such that

$$\int_P g(x,h,\xi)\, e^{ihz}\, dh \tag{7.16}$$

is well-defined and EQUAL to integral 7.15. In general, the function $g(x,h,\xi,\Delta\varepsilon)$ for loss $\Delta\varepsilon$ will not have poles on the real h axis. If the path P is chosen such that it lies below all poles in the half plane $\mathrm{Im}\ h > 0$ for $\Delta\varepsilon > 0$, and over all poles in the half plane $\mathrm{Im}\ h < 0$ for $\Delta\varepsilon \geq 0$, then integral 7.16 will give the same result as integral 7.15 and the lossless transformed Green's function may be used.*

As an example, consider the well-known transformed Green's function associated with propagation in the z direction in a free space region between two parallel plates separated by unit distance. For $x < \xi$

$$g(x,h,\xi,\Delta\varepsilon) = \frac{\sin \gamma x \sin \gamma\ (\xi-1)}{\gamma \sin \gamma} \tag{7.17}$$

where

$$\gamma = + \sqrt{\omega^2 \mu_o\, \varepsilon_o\, (1+ i\Delta\varepsilon) - h^2} \tag{7.18}$$

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* This is true only if the real axis poles of $g(x,h,\xi)$ are simple poles. If these poles are not simple, the function defined by equation 7.15 does not exist.

and $\Delta\varepsilon$ is a small loss term introduced in the dielectric constant for the free space between the parallel plates. The function

$$\int_{-\infty}^{\infty} \lim_{\Delta\varepsilon \to 0} g(x,h,\xi,\Delta\varepsilon)\, e^{ihz}\, dh \qquad (7.19)$$

clearly does not exist; however, the quantity of interest

$$\lim_{\Delta\varepsilon \to 0} \int_{-\infty}^{\infty} g(x,h,\xi,\Delta\varepsilon)\, e^{ihz}\, dh \qquad (7.20)$$

does exist and is equal to

$$2\pi i \sum_{n=1}^{\infty} \frac{\sin n\pi x \sin n\pi(\xi-1)}{n\pi}\, e^{\pm i \sqrt{\omega^2 \varepsilon_o \mu_o - (n\pi)^2}\, z} \quad \text{for } z \gtrless 0. \qquad (7.21)$$

It is clear that the poles of $g(x,h,\xi,\Delta\varepsilon)$ which are on the positive real h axis for $\Delta\varepsilon = 0$ move above the real axis for $\Delta\varepsilon > 0$; those on the negative real axis, below. Thus, by defining the contour P as in figure 7.1

$$\int_P g(x,h,\xi)e^{ihz}\, dh \equiv \lim_{\Delta\varepsilon \to 0} \int_{-\infty}^{\infty} g(x,h,\xi,\Delta\varepsilon)e^{ihz}\, dh \qquad (7.22)$$

and the lossless transformed Green's function $g(x,h,\xi)$ may be used to obtain the desired result 7.21. Notice, however, that the Cauchy principle value of 7.19 is not equal to 7.20. The Cauchy principle value has no physical significance.

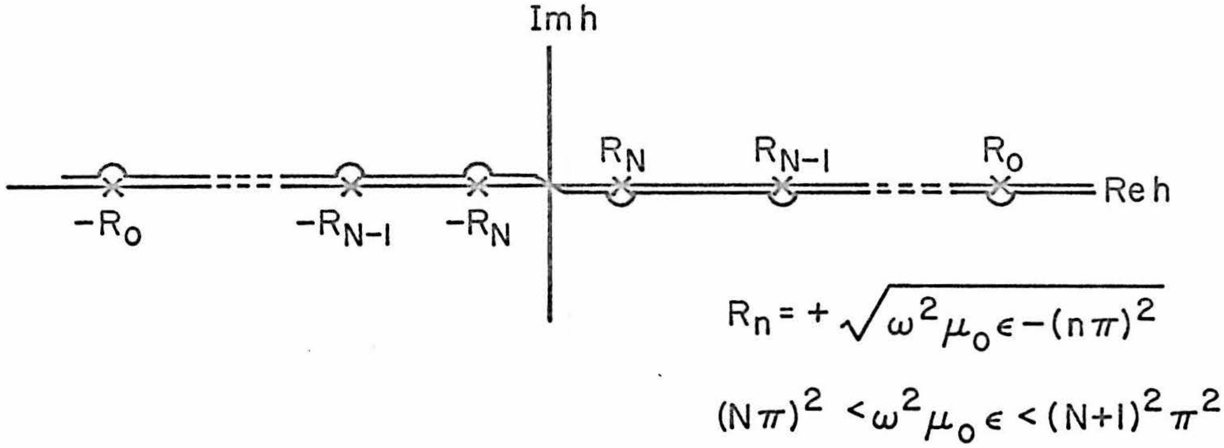


Figure 7.1

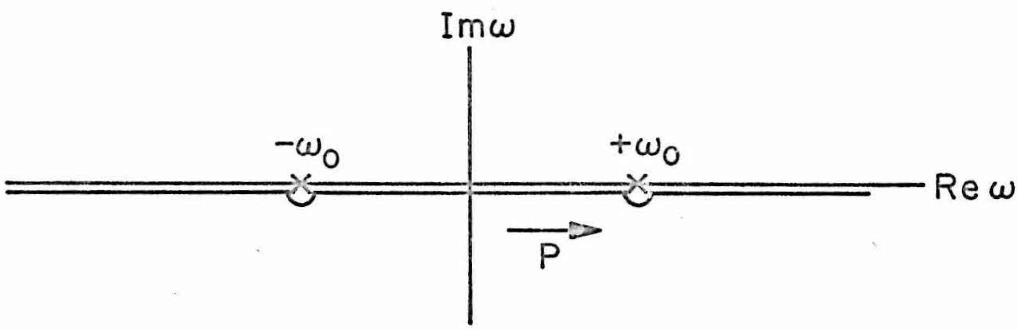


Figure 7.2

Now consider the integral 7.6 presented by Gardiol. It should now be clear from the preceding that the undefined integral 7.6 should be replaced by

$$\frac{1}{2\pi} \lim_{\Delta\varepsilon \to 0} \int_{-\infty}^{\infty} \mu_{12} (\omega, \Delta\varepsilon)\, e^{i\omega t}\, d\omega \tag{7.23}$$

where $\Delta\varepsilon$ represents a small loss term. According to Lax and Button (25) $\mu_{12}$ $(\omega, \Delta\varepsilon)$ is given in the form

$$\mu_{12}(\omega, \Delta\varepsilon) = \frac{ic\omega}{(\omega_o + j\omega\Delta\varepsilon)^2 - \omega^2} \tag{7.24}$$

where c and $\omega_o$ are real constants. For small $\Delta\varepsilon$ the poles of $\mu_{12}(\omega, \Delta\varepsilon)$ all lie above the real $\omega$ axis and thus the integral 7.23 may be replaced by

$$g_{12}(t) = \frac{ic}{2\pi} \int_P \frac{\omega}{\omega_o^2 - \omega^2}\, e^{i\omega t}\, d\omega \tag{7.25}$$

where the path P is defined in figure 7.2. By closing the contour in the upper half plane and performing the integration,

$$g_{12}(t) = \begin{cases} 0 & t < 0 \\ c \cos \omega_o t & t > 0 \end{cases} \tag{7.26}$$

and causality is not violated. It is thus clear that Gardiol's resolution of the paradox must be rejected on the grounds that he has not properly interpreted the meaning of integral 7.6. The application of Gardiol's interpretation to integrals obtained in classical guide problems yield equally ridiculous results.

We will now demonstrate that the proofs offered by Seidel and Fletcher, Tai, and Bresler to show that power cannot return via the cutoff modes are incorrect. We begin with Bresler's proof since it seems to be the most commonly accepted work.

Bresler's proof is rather long and somewhat difficult to follow (perhaps this is why it has remained unquestioned for so many years) and therefore we will not try to reproduce it here. The reader is referred to Bresler's original paper for the details of his work (26). Our objective here is not to give a presentation of Bresler's proof, but only to point out why we cannot accept it.

Bresler considers the junction of an empty guide with a reactively terminated partially filled ferrite guide. See figure 5.3 He assumes, without loss of generality, that the single unidirectional mode which can exist in the ferrite section carries power to the right, toward the reactive termination. A careful analysis of Bresler's proof reveals that it rests on the crucial assumption that if the ferrite section is very long, the cutoff modes decaying to the right are negligible near $z = \ell$ as compared to the single unidirectional propagating mode which does not decay. This assumption is unjustified. To make this clear, consider the modal expansion of the fields in the ferrite guide

$$E_y(x,z) = a_o\, f_o(x)\, e^{ih_o z} + \Sigma\ a_n\, f_n(x)\, e^{ih_n z} \tag{7.27}$$

where $f_o(x)$ is the unidirectional mode associated with the real propagation constant $h_o$, and $f_n(x)$ $(n = 1,2...)$ are the cutoff modes associated with complex propagation constants $h_n = \beta_n + i\alpha_n$ ($\alpha_n, \beta_n$ real). A cutoff mode decaying to the right would have the form

$$a_n\, f_n(x)\, e^{(-\alpha_n + i\,\beta_n)z} \qquad \text{where} \quad \alpha_n > 0\ . \tag{7.28}$$

According to Bresler, at $z = \ell$ these cutoff modes will be decayed by the factor $e^{-\alpha_n \ell}$ and hence for large enough $\ell$ they may be neglected as compared with the unidirectional propagating mode, $a_o\, f_o\, e^{ih_o \ell}$. Bresler has overlooked the fact that the expansion coefficients of the cutoff modes are functions of the guide length $\ell$. That is, $a_n = a_n(\ell)$ and hence it does not follow that $a_n\, f_n\, e^{(-\alpha_n + i\beta_n)\ell} \to 0$ for large $\ell$. For example, if $a_n(\ell) = \text{const.}\ e^{+\alpha_n \ell}$, then $a_n\, f_n\, e^{(-\alpha_n + i\beta_n \ell)} = \text{const.}\ f_n\, e^{i\beta_n \ell}$ which certainly does not become small as $\ell$ becomes large. It should thus be clear that Bresler's proof is invalid because he is unjustified in assuming that if the ferrite section is very long, the cutoff modes decaying to the right are negligible near $z = \ell$.

Now let us consider the work by Seidel and Fletcher (12). Seidel and Fletcher rejected the cutoff mode resolution in favor of the gyromagnetic mode resolution on the grounds of some experimental measurements. According to their measurements, the gyromagnetic modes carried the return power; however we know, from the simple reasoning already presented in the preceding chapter, that the gyromagnetic modes cannot be generated in the parallel plate problem of the paradox. The fact that their experiment generated the gyromagnetic modes clearly means their experiment did not duplicate the problem of the paradox, and hence their results cannot be used to reject the cutoff mode resolution.

We now come to perhaps the least known but most convincing proof that the cutoff modes cannot carry power. C. T. Tai's approach (15) seems to be the most direct and reasonable manner of studying the power flow in ferrite loaded guides, simply calculate the Poynting vector. Using the notation in Part I of this paper, propagation of fields such that $\frac{\partial}{\partial y} = 0$ in an inhomogeneously filled trans-magnetized guide may be described by the expansion

$$E_y(x,z) = \Sigma\, a_n\, f_n(x)\, e^{ih_n z} \quad . \tag{7.29}$$

The power flowing in the positive z direction in the guide is then given by

$$P(z) = \frac{1}{2} \operatorname{Re} \int_0^1 \underline{E} \times \overline{\underline{H}} \cdot \hat{e}_z \, dx$$

$$= -\frac{1}{4} \int_0^1 (E_y \overline{H}_x + \overline{E}_y H_x) \, dx \quad . \tag{7.30}$$

Using 7.29 to write $E_y$ in terms of $f_n(x)$ and $h_n$ ; and 5.4 to write $H_x$ in terms of $f_n(x)$, $h_n$ , $\nu_1(x)$, $\nu_2(x)$ leads to*

$$P(z) = \frac{1}{4\omega\mu_o} \sum_n \sum_m a_n \overline{a}_m e^{i(h_n - \overline{h}_m)z} \int_0^1 [\nu_1 (h_n + \overline{h}_m) + \frac{d\nu_2}{dx} ] f_n \overline{f}_m \, dx \quad . \tag{7.31}$$

In a manner analogous to the derivation of equation 4.79, it may be shown that

$$(h_n - \overline{h}_m) \int_0^1 [(h_n + \overline{h}_m) \nu_1 + \frac{d\nu_2}{dx} ] f_n \overline{f}_m \, dx = 0 \quad . \tag{7.32}$$

Tai assumed that 7.32 implied

$$\int_0^1 [(h_n + \overline{h}_m)\nu_1 + \frac{d\nu_2}{dx} ] f_n \overline{f}_m \, dx = 0 \tag{7.33}$$

---

* The steps leading to equation 7.31 consist of an integration by parts, a formal interchange of summations and integrations, and use of the fact that $f_n(0) = f_n(1) = 0$ .

for $n \neq m$ and reduced 7.31 to

$$P(z) = \frac{1}{4\omega\mu_o} \sum_n |a_n|^2 e^{i(h_n - \bar{h}_n)z} \int_o^1 [\nu_1 (h_n + \bar{h}_n) + \frac{d\nu_2}{dx}] \, |f_n|^2 dx \, . \qquad (7.34)$$

For n = m equation 7.32 implies

$$\int_o^1 [(h_n + \bar{h}_n) \, \nu_1 + \frac{d\nu_2}{dx}] \, |f_n|^2 dx = 0 \qquad \text{for } \operatorname{Im} h_n \neq 0 \qquad (7.35)$$

and thus 7.34 further reduces to

$$P(x) = \frac{1}{4\omega\mu_o} \sum_{\substack{n \\ \text{Real } h_n}} |a_n|^2 \int_o^1 [2 \, h_n \, \nu_1 + \frac{d\nu_2}{dx}] \, |f_n|^2 \, dx \quad . \qquad (7.36)$$

Since the summation extends over real $h_n$ only, Tai concluded that only the modes with real propagation constants can carry power; the cutoff modes are not capable of carrying power. At first glance Tai's proof is rather convincing; however, it is wrong because equation 7.33 is not true. Equation 7.33 does not follow from equation 7.32 because $h_n = \bar{h}_m$ for some $n \neq m$ .

Recall the eigenvalue equation 5.3 which defines $f_n$ and $h_n$ ,

$$Lf \equiv \frac{d}{dx} \nu_1 \frac{df}{dx} + \omega^2 \varepsilon \mu_o f = (h^2 \nu_1 + h \frac{d\nu_2}{dx}) f. \qquad f(0) = f(1) = 0 \qquad (7.37)$$

Since $\nu_1, \nu_2$ are real and L is a real operator, the complex conjugate of the eigenvalue problem 7.37 is

$$L\bar{f} = (\bar{h}^2 \nu_1 + \bar{h}\frac{d\nu_2}{dx})\bar{f} \qquad \bar{f}(0) = \bar{f}(1) = 0 \ . \qquad (7.38)$$

Since systems 7.37 and 7.38 are identical, it is clear that if $h_m$ is an eigenvalue of 7.37 with eigenfunction $f_m$ , then $\bar{h}_m$ is also an eigenvalue with eigenfunction $\bar{f}_m$ . Thus $h_n = \bar{h}_m$ for some $n \neq m$ . Hence equation 7.33, used by Tai to prove that the cutoff modes cannot carry power does not follow from equation 7.32.

The power calculations may be carried out correctly by splitting the expansion of the electric field into two summations such that $h_n \neq \bar{h}_m$ for $n \neq m$ . We may write

$$E_y(x,z) = \sum_{\mathrm{Im}\, h_n \geq 0} a_n f_n e^{ih_n z} + \sum_{\mathrm{Im}\, h_n > 0} \bar{b}_n \bar{f} e^{i\bar{h}_n z} \qquad (7.39)$$

where under each summation it is understood that $h_n \neq h_m$ for $n \neq m$. Applying equations 7.30 and 5.4, integrating by parts, using the boundary conditions on the eigenfunctions, and formally interchanging the summations and integrations, yields

$$\begin{aligned} P(z) = + \frac{1}{4\omega\mu_o} \Big[ & \sum_{\substack{\mathrm{Im}\, h_n \geq 0 \\ \mathrm{Im}\, h_m \geq 0}} a_n \bar{a}_m e^{i(h_n - \bar{h}_m)z} \int_0^1 [(\bar{h}_m + h_n)\nu_1 + \frac{d\nu_2}{dx}] f_n \bar{f}_m dx \\ & + \sum_{\substack{\mathrm{Im}\, h_n \geq 0 \\ \mathrm{Im}\, h_m > 0}} a_n b_m e^{i(h_n - h_m)z} \int_0^1 [(h_m + h_n)\nu_1 + \frac{d\nu_2}{dx}] f_n f_m dx \\ & + \sum_{\substack{\mathrm{Im}\, h_m \geq 0 \\ \mathrm{Im}\, h_n > 0}} \bar{a}_m \bar{b}_n e^{i(\bar{h}_n - \bar{h}_m)z} \int_0^1 [(\bar{h}_m + \bar{h}_n)\nu_1 + \frac{d\nu_2}{dx}] \bar{f}_n \bar{f}_m dx \\ & + \sum_{\substack{\mathrm{Im}\, h_n > 0 \\ \mathrm{Im}\, h_m > 0}} b_m \bar{b}_n e^{i(\bar{h}_n - h_m)z} \int_0^1 [(h_m + \bar{h}_n)\nu_1 + \frac{d\nu_2}{dx}] \bar{f}_n f_m dx \Big] \ . \end{aligned} \qquad (7.40)$$

Applying BOTH relationships 7.32 and 4.79 to equation 7.40 then yields

$$P(z) = \frac{1}{4\omega\mu_o}\Big[ \sum_{\substack{n \\ \text{real } h_n}} |a_n|^2 \int_o^1 (2\, h_n\nu_1 + \frac{d\nu_2}{dx}) |f_n|^2 \, dx$$

$$+ 2 \sum_{\substack{n \\ \text{Im } h_n > o}} \text{Re } a_n b_n \int_o^1 (2\, h_n\nu_1 + \frac{d\nu_2}{dx}) f_n^2 \, dx \Big] \qquad (7.41)$$

or

$$P(z) = \Sigma \text{ real } h_n + \Sigma \text{ complex } h_n \quad .$$

It is clear from equation 7.41 that Tai's error led to an erroneous conclusion. The cutoff modes actually can carry power in a ferrite filled guide.

Let us now apply equation 7.41 to the problem of the paradox as presented in figure 5.3. The proposed electric field in the ferrite section of the guide is given by

$$E_y(x,z) = a_o f_o e^{ih_o z} + \sum_{\text{Im } h_n > o} a_n f_n e^{ih_n z} + \sum_{\text{Im } h_n > o} \bar{b}_n \bar{f}_n e^{i\bar{h}_n z} \qquad (7.42)$$

where $a_o f_o e^{ih_o z}$ is the unidirectional mode and the expansion coefficients, $a_n$ , are chosen such that $E_y(x,\ell) = 0$. Now notice from equation 7.41 that $P(z)$ is not a function of $z$, as certainly must be the case for a lossless guide. Since $E_y(x,\ell) = 0$, it follows that $P(\ell) = 0$ or $P(z) \equiv 0$. Thus according to equation 7.41

$$0 = |a_o|^2 \int_0^1 (2 h_o \nu_1 + \frac{d\nu_2}{dx}) |f_o|^2 dx + 2 \sum_{Im h_n > o} Re\, a_n b_n \int_0^1 (2 h_n \nu_1 + \frac{d\nu_2}{dx}) f_n^2 dx \tag{7.43}$$

and the power carried by the unidirectional mode $a_o f_o e^{ih_o z}$ clearly returns via the cutoff modes.

In this section we have conclusively shown that all previous proofs which attempt to show that the power cannot return via the cutoff modes are incorrect. We have calculated the actual power flow and shown that it is quite reasonable to assume that the power does indeed return via the cutoff modes as suggested by Kales. We have thus shown that there is absolutely no reason to assume the problem of the "paradox" constitutes any violation of basic thermodynamic laws.

It appears that there are two reasons why the problem of the paradox has remained a mystery for so many years. First, the early works by Seidel and Fletcher, Tai, and Bresler which "proved" the existence of the paradox remained unquestioned; second, no one was able to obtain a mathematical solution of the boundary value problem on which the paradox has been based (figure 5.3). If a rigorous mathematical solution to this boundary value problem could have been easily obtained, there would have been no paradox. In the remaining two chapters we will study this boundary value problem.

## 8.0 Formal Derivation of Solutions to the Boundary Value Problem of the Paradox

In this chapter we will study the boundary value problem on which the so-called thermodynamic paradox has been based (figure 5.3). As previously stated, no one to date has actually solved this boundary value problem and found any violation of the basic laws of thermodynamics. The paradox has been based only on conjecture about the form of the possible solutions. In the previous chapter it was proven that all these conjectures are wrong; hence, there is absolutely no justification in assuming that the problem constitutes any violation of the basic laws of thermodynamics.

Although the work in the preceding chapter conclusively resolved the paradox, it did not present the solution to the original boundary value problem. In this chapter and the next we will derive the form of, and prove the existence of, solutions to this boundary value problem. Our objective is to develop an understanding of the form of these solutions such that it will be even more apparent that, although the solutions are somewhat complicated, there is nothing paradoxical about them.

In order to simplify our calculations, we will translate the guide in figure 5.3 a distance $\ell$ along the positive z axis. After this translation the problem appears as in figure 8.1 For $z < -\ell$ the guide is empty; for $-\ell < z < 0$, it contains a ferrite slab,

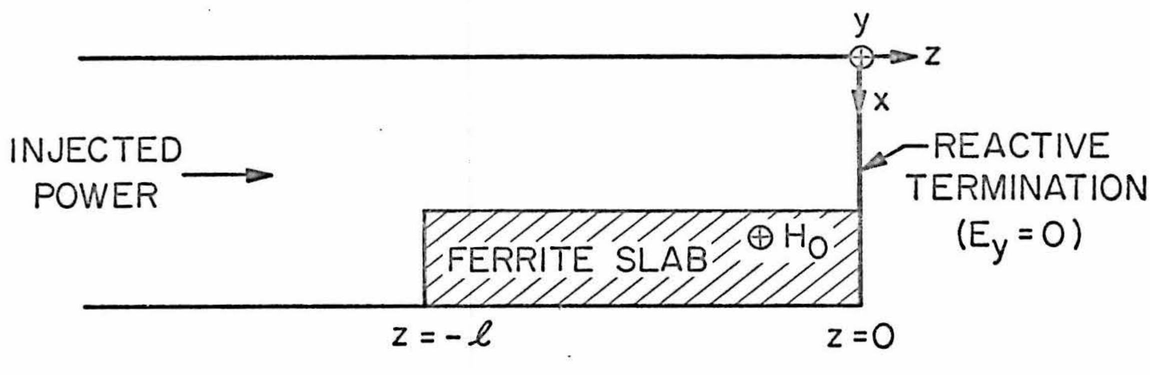


Figure 8.1

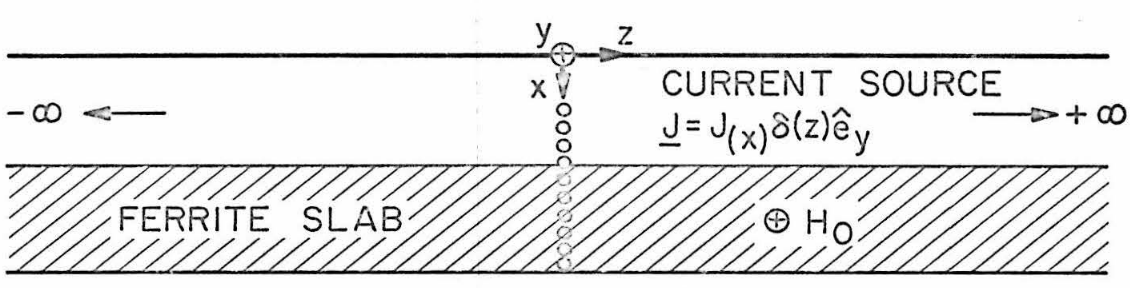


Figure 8.2

The shorting plane is now located at $z = 0$.

In the ferrite loaded section of the guide ($-\ell < z < 0$) the modal expansion of the electric field is

$$E_y(x,z) = \Sigma\, c_n f_n e^{ih_n z}$$

where the $\{ f_n(x)\}$ are the modes of the infinite ferrite filled guide and are defined by the non-self-adjoint eigenvalue equation 5.3. Now consider the problem of matching the boundary condition at $z = 0$ ($E_y(x,o) = 0$). Assuming the necessary uniform convergence, the problem reduces to finding a set of coefficients $\{c_n\}$ such that $\Sigma\, c_n f_n = 0$. We know from the work in part I that the set $\{c_n\}$ cannot be unique and that, if the ferrite fills the guide in an inhomogeneous and continuous manner, the set $\{c_n\}$ is given by equation 4.69 with $E_y(x,o) = 0$:

$$c_n = \frac{-\int_o^1 i\, \nu_1 \frac{\partial E_y(x,o)}{\partial z} f_n(x)\, dx}{\int_o^1 (2\, h_n \nu_1 + \frac{d\nu_2}{dx})\, f_n^2\, dx}, \qquad (8.1a)$$

or

$$c_n = \frac{\int_o^1 J(x)\, f_n(x)\, dx}{\int_o^1 (2\, h_n\, \nu_1 + \frac{d\nu_2}{dx})\, f_n^2\, dx} \qquad (8.1b)$$

where J(x) is an arbitrary function.

However, we cannot blindly apply equation 4.69 to guides containing ferrite slabs because 4.69 was derived under the assumption that the ferrite parameters $\nu_1(x)$, $\nu_2(x)$ were continuous functions. (More precisely, equation 4.69 was derived from equations 4.47-4.48 which were derived from equations 4.39-4.46. The latter were derived by Langer under the assumption that the parameters $\nu_1(x)$,$\nu_2$ (x) were continuous functions. Clearly 4.39 is undefined for $\nu_1 = 1 + (A-1)\ H\ (x-d)$ and $\nu_1 = B\ H(x-d)$.) It is interesting to note that equation 4.69 is well defined when $\nu_1$ and $\nu_2$ are discontinuous even though it was derived from equations which are undefined when $\nu_1$ and $\nu_2$ are discontinuous. Thus one might suspect that perhaps the derivation of 4.69 could be generalized to include cases where $\nu_1$ and $\nu_2$ are discontinuous. Such a generalization would not be easy, it would require investigation of the completeness properties and expansion theorem for a larger class of parameters.

Rather than generalize the derivation of equation 4.69 to include discontinuous parameters, we will present a "trick" based on physical arguments which will yield a set of coefficients $\{c_n\}$ such that $\Sigma\ c_n\ f_n = 0$. Our approach will not require a detailed investigation of the completeness properties of the $f_n$ as would be necessary in finding a set $\{c_n\}$ such that $\Sigma c_n\ f_n = F(x)$ for arbitrary $F\ (x)$. Our approach will lead to a proposed solution in a purely formal manner;

in the next chapter this proposed solution will be justified.

Consider an infinitely long parallel plate guide containing a ferrite slab such that a single unidirectional mode exists. Assume that the parameters of the guide are identical to those of the guide in figure 8.1 for the region $-\ell < z < 0$. Now assume a current source distribution of the form $\underline{J}(x,y,z) = J(x)\,\delta(z)\,\hat{e}_y$ is placed in this guide. Figure 8.2. Clearly only $TE_{no}$ modes will be generated. As in figure 8.1, let us assume without loss of generality that the single unidirectional mode carries power to the right and is denoted by $a_o f_o(x)\, e^{ih_o z}$.

If a modal expansion of the solution exists, it will have the form

$$E_y(x,z) = \begin{cases} a_o f_o\, e^{ih_o z} + \sum\limits_{Im\, h_n > o} a_n f_n\, e^{ih_n z} & z > 0 \\ \sum\limits_{Im\, h_n < o} b_n f_n\, e^{ih_n z} & z < 0 \end{cases} \tag{8.3}$$

where the expansion coefficients $a_n$ , $b_n$ are functionals of the source current $J(x)$. Now suppose the coefficients $a_n$ and $b_n$ are determined. By continuity of $E_y(x,z)$ at $z = 0$,

$$E_y(x,z+0) - E_y(x,z-0) = 0$$

or formally

$$a_o f_o + \sum_{Im\, h_n > o} a_n f_n - \sum_{Im\, h_n > o} b_n f_n = 0 \tag{8.4a}$$

or

$$\sum_{\text{all } h_n} c_n f_n = 0 \quad \text{where } c_n = \begin{cases} a_n & \text{for} \quad \operatorname{Im} h_n \geq 0 \\ -b_n & \text{for} \quad \operatorname{Im} h_n < 0 \end{cases} \tag{8.4b}$$

and thus the coefficients $a_n$ , $b_n$ may be applied to satisfy the boundary condition at $z = 0$ for the paradox problem of figure 8.1. Since the coefficients are functionals of the current $J(x)$, which is actually an arbitrary function, they yield an infinite set of solutions of the problem in figure 8.1 which satisfy the boundary condition $E_y(x,o) = 0$. The final solution of the problem in figure 8.1 would then be obtained by choosing the function $J(x)$ such that the fields match at $z = -\ell$ .

We will now determine the coefficients $a_n$ , $b_n$ such that the modal expansion 8.3 is the solution of the problem in figure 8.2. In chapter 2 it was shown that $TE_{no}$ propagation in a source free region may be described by the scalar partial differential equation 2.9. It may be shown in an analogous manner that Maxwell's equations reduce to the following scalar equation if current sources of the form $\underline{J}(x,y,z) = J(x) \;\; \delta(z) \;\; \hat{e}_y$ are permitted:

$$\frac{J(x) \;\; \delta'(z)}{-\, i\omega\varepsilon} + E_y = \frac{1}{\omega^2 \varepsilon \mu_o}\Big( -\frac{\partial}{\partial z}\nu_1 \frac{\partial E_y}{\partial z} + i\frac{\partial}{\partial z}\nu_2 \frac{\partial E_y}{\partial x} - i\frac{\partial}{\partial x}\nu_2 \frac{\partial E_y}{\partial z} - \frac{\partial}{\partial x}\nu_1 \frac{\partial E_y}{\partial x}\Big) \tag{8.5}$$

Assuming, in the usual manner, the existence of a Fourier transformable Green's function $G(x,\xi,z)$ such that

$$E_y(x,z) = -i\,\omega\mu_o \int_0^1 J(\xi)\; G(x,\xi,z)\, d\xi\,, \tag{8.6}$$

$$G(x,\xi,z) = \int_{-\infty}^{\infty} g(x,\xi,h)\, e^{ihz}\, dh\,, \tag{8.7}$$

equation 8.5 may be formally reduced to

$$\frac{d}{dx}\nu_1\frac{dg}{dx} + \left(\omega^2\varepsilon\mu_o - h^2\,\nu_1 - h\frac{d\nu_2}{dx}\right)\, g(x,\xi,h) = \delta(x-\xi). \tag{8.8}$$

Applying the parameters 5.2 of the ferrite slab

$$\nu_1(x) = 1 + (A - 1)\, H(x - d)$$

$$\nu_2(x) = B\, H(x - d)$$

yields the following equation for $g(x,\xi,h)$

$$\frac{d}{dx}[\,1+(A-1)H(x-d)]\,\frac{dg}{dx} + \{\omega^2\varepsilon\mu_o - h^2[1+(A-1)H(x-d)]$$

$$-hB\delta(x-d)\}\, g=\delta(x-\xi)\quad . \tag{8.9}$$

Thus for the region $x < d$ , $g(x,\xi,h)$ must satisfy

$$\frac{d^2g}{dx^2} + (\omega^2\varepsilon\mu_o - h^2)\, g = 0 \qquad x \neq \xi\;; \tag{8.10a}$$

and for the region $x > d$,

$$\frac{d^2 g}{dx^2} + \left( \frac{\omega^2 \varepsilon \mu_o}{A} - h^2 \right) g = 0 \qquad x \neq \xi \quad . (8.10b)$$

Integrating equation 8.9 through small regions across the discontinuities $x = \xi$ and $x = d$ and assuming $g(x,\xi,h)$ is a continuous function, yields the following matching conditions:

$$g(o,\xi,h) = g(1,\xi,h) = 0 \qquad (8.11a)$$

$$g(d+o,\xi,h) = g(d-o,\xi,h) \qquad \xi \neq d \quad (8.11b)$$

$$g(\xi+o,\xi,h) = g(\xi-o,\xi,h) \qquad \xi \neq d \quad (8.11c)$$

$$A \frac{d}{dx} g(d+o,\xi,h) - \frac{d}{dx} g(d-o,\xi,h) = hBg(d,\xi,h) \qquad \xi \neq d \quad (8.11d)$$

$$\frac{d}{dx} g(\xi+o,\xi,h) - \frac{d}{dx} g(\xi-o,\xi,h) = \begin{cases} 1 & \xi < d \\ \frac{1}{A} & \xi > d \end{cases} \quad (8.11e)$$

The solution of equations 8.10 subject to boundary conditions 8.11 is as follows:

For $x < \xi < d$

$$g = g_1 \equiv \left[ - \coth k^{(1)} (\xi - d) - \frac{k^{(2)}}{k^{(1)}} A \coth k^{(2)} (d-1) + \frac{hB}{k^{(1)}} \right] \frac{\sinh k^{(1)} x \sinh k^{(1)} (\xi - d)}{T \sinh k^{(1)} d} \qquad (8.12a)$$

For $\xi < x < d$

$$g = g_2 \equiv \left[ -\coth k^{(1)}(x-d) - \frac{k^{(2)}}{k^{(1)}} A \coth k^{(2)}(d-1) \right. \tag{8.12b}$$

$$\left. + \frac{hB}{k^{(1)}} \right] \frac{\sinh k^{(1)}\xi \sinh k^{(1)}(x-d)}{T \sinh k^{(1)} d}$$

For $\xi < d < x$

$$g = g_3 \equiv -\frac{\sinh k^{(1)}\xi \sinh k^{(2)}(x-1)}{T \sinh k^{(1)} d \sinh k^{(2)}(d-1)} \tag{8.12c}$$

For $x < d < \xi$

$$g = g_4 \equiv -\frac{\sinh k^{(1)} x \sinh k^{(2)}(\xi-1)}{T \sinh k^{(2)}(d-1) \sinh k^{(1)} d} \tag{8.12d}$$

For $d < x < \xi$

$$g = g_5 \equiv \left[ -\coth k^{(2)}(x-d) - \frac{k^{(1)}}{Ak^{(2)}} \coth k^{(1)} d \right. \tag{8.12e}$$

$$\left. - \frac{hB}{Ak^{(2)}} \right] \frac{\sinh k^{(2)}(\xi-1) \sinh k^{(2)}(x-d)}{T \sinh k^{(2)}(d-1)}$$

For $d < \xi < x$

$$g = g_6 \equiv \left[ - \coth k^{(2)} (\xi - d) - \frac{k^{(1)}}{Ak^{(2)}} \coth k^{(1)} d - \frac{hB}{Ak^{(2)}} \right] \frac{\sinh k^{(2)} (x-1) \sinh k^{(2)} (\xi - d)}{T \sinh k^{(2)} (d-1)} \tag{8.12f}$$

where $k^{(1)} = \sqrt{h^2 - \omega^2 \varepsilon \mu_o}$ and $k^{(2)} = \sqrt{h^2 - \dfrac{\omega^2 \varepsilon \mu_o}{A}}$

and

$$T(h) = k^{(1)} \coth k^{(1)} d - k^{(2)} A \coth k^{(2)} (d-1) + hB \quad . \tag{8.13}$$

These expressions for $g(x,\xi,h)$ may now be used to determine

$$E_y(x,z) = - i \omega \mu_o \int_0^1 \int_{-\infty}^{\infty} J(\xi)\, g(x,\xi,h)\, e^{ihz}\, dh\, d\xi \quad . \tag{8.14}$$

Formal interchange of the order of integrations yields

$$E_y(x,z) = - i \omega \mu_o \int_{-\infty}^{\infty} e^{ihz}\, I(x,h)\, dh \tag{8.15}$$

where

$$I(x,h) = \int_0^1 J(\xi)\, g(x,\xi,h)\, d\xi \quad . \tag{8.16}$$

Using the derived expressions for $g(x,\xi,h)$, equation 8.16 may be written as

$$I(x,h) = \begin{cases} \int_0^x J(\xi)\, g_2 d\xi + \int_x^d J(\xi)\, g_1 d\xi + \int_d^1 J(\xi)\, g_4 d\xi & \text{for } x < d \\ \int_0^d J(\xi)\, g_3 d\xi + \int_d^x J(\xi)\, g_6 d\xi + \int_x^1 J(\xi)\, g_5 d\xi & \text{for } x > d\ . \end{cases} \qquad (8.17)$$

Now let us confine our consideration to the region $x < d$. Applying the calculated values for $g_1$, $g_2$, $g_4$, and performing some rather long and tedious elementary manipulations yields the following results for $I(x,h)$ in the region $x < d$ :

$$I(x,h) = -\frac{\sinh k^{(1)}x}{T(h)} \left[ \int_0^d J(\xi) \frac{\sinh k^{(1)}\xi}{\sinh^2 k^{(1)}d} d\xi + \int_d^1 J(\xi) \frac{\sinh k^{(2)}(\xi-1)\ d\xi}{\sinh k^{(1)}d \sinh k^{(2)}(d-1)} \right]$$

$$+ \frac{1}{k^{(1)}} \left[ \frac{\sinh k^{(1)}(x-d)}{\sinh k^{(1)}d} \int_0^x J(\xi) \sinh k^{(1)}\xi d\xi - \frac{\sinh k^{(1)}x}{\sinh k^{(1)}d} \int_d^x J(\xi) \sinh k^{(1)}(\xi-d)\ d\xi \right] \qquad (8.18)$$

It may be shown that the function $I(x,h)$ remains bounded for all $x$ $(0 \leq x < d)$ and all $h$ bounded away from $T(h) = 0$ (the zeros of $T(h)$ are the poles of $I(x,h)$).* Thus the integral in equation 8.15 may be evaluated by completing the contour in the upper half plane for $z > 0$ and in the lower half plane for $z < 0$, providing the contours may be completed without passing through the poles of $I(x,h)$.** It should be noted that the integral 8.15 is undefined if $T(h) = 0$ for some real $h$. As was discussed in a previous section, such integrals are defined by inserting loss into the system, noting the manner in which the poles shift, and then defining the integral accordingly. For the problem of unidirectional propagation which we are considering, $T(h) = 0$ for one real value of $h$; thus, there is one pole on the real $h$ axis. The assumption that the power associated with the unidirectional mode carries energy in the positive $z$ direction is equivalent to assuming this pole shifts upward when loss is inserted. Hence, the contour in the upper half plane will include this unidirectional mode.

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* The proof that $I(x,h)$ remains bounded for all $h(T(h) \neq 0)$ is rather straight forward and we do not feel justified in presenting it here.

** In the next chapter the asymptotic location of the roots $T(h) = 0$ for $|h| \to \infty$ are determined and it is found that they do not approach a continuum as $|h| \to \infty$; hence a contour may always be chosen to pass around the poles of $I(x,h)$.

Completing the contour and evaluating the integral in equation 8.15 by applying the residue theory of complex variables, we obtain

$$E_y(x,z) = \begin{cases} 2\pi\omega\mu_o \sum_{U.H.P.} \lim_{h\to h_n} \frac{h-h_n}{T(h)} e^{ihz} \; T(h)\, I(x,h) , & z>o \\ -2\pi\omega\mu_o \sum_{L.H.P.} \lim_{h\to h_n} \frac{h-h_n}{T(h)} e^{ihz} \; T(h)\, I(x,h) , & z<o \end{cases} \tag{8.19}$$

where $\sum_{U.H.P.}$ denotes the sum over $h_n$ such that $T(h_n) = 0$ and $\mathrm{Im}\, h_n \geqslant 0$; $\sum_{L.H.P.}$, the sum over $h_n$ such that $T(h_n) = 0$ and $\mathrm{Im}\, h_n < 0$. Expanding $T(h)$ about $h_n$ shows that $\lim_{h\to h_n} \frac{h-h_n}{T(h)} = \frac{1}{T'(h_n)}$, hence equation 8.19 reduces to

$$E_y(x,z) = \begin{cases} -2\pi\omega\mu_o \sum_{U.H.P.} \Psi_n(x,z) & z>0 \\ 2\pi\omega\mu_o \sum_{L.H.P.} \Psi_n(x,z) & z<0 \end{cases} \tag{8.20}$$

where

$$\Psi_n(x,z) = \frac{e^{ih_n z} \sinh k_n^{(1)} x}{T'(h_n) \sinh k_n^{(1)} d} \left[ \int_o^d J(\xi) \frac{\sinh k_n^{(1)} \xi}{\sinh k_n^{(1)} d} d\xi + \int_d^1 J(\xi) \frac{\sinh k_n^{(2)} (\xi-1)}{\sinh k_n^{(2)} (d-1)} d\xi \right]$$

and where $k_n^{(1)} = \sqrt{h_n^2 - \omega^2 \varepsilon \mu_o}$ and $k_n^{(2)} = \sqrt{h_n^2 - \omega^2 \varepsilon \mu_o / A}$

In a like manner the calculations may be carried out for $x > d$. The final results are

$$E_y(x,z) = \begin{cases} \sum\limits_{U.H.P.} \dfrac{-2\pi\omega\mu_o}{T'(h_n)} e^{ih_n z} f_n(x) \int_o^1 J(\xi)\, f_n(\xi)\, d\xi & , z > 0 \\ \sum\limits_{L.H.P.} \dfrac{2\pi\omega\mu_o}{T'(h_n)} e^{ih_n z} f_n(x) \int_o^1 J(\xi)\, f_n(\xi)\, d\xi & , z < 0 \end{cases} \tag{8.21}$$

where $f_n$, $h_n$ are the eigenfunctions and eigenvalues respectively of the original non-self-adjoint eigenvalue equation 5.3,

$$f_n(x) = \begin{cases} \dfrac{\sinh k_n^{(1)} x}{\sinh k_n^{(1)} d} & x < d \\ \dfrac{\sinh k_n^{(2)} (x-1)}{\sinh k_n^{(2)} (d-1)} & x > d \end{cases} .$$

Equation 8.21 may be written as

$$E_y(x,z) = \begin{cases} a_o f_o\, e^{ih_o z} + \sum\limits_{Im\, h_n > o} a_n f_n\, e^{ih_n z} & z > 0 \\ \sum\limits_{Im\, h_n < o} b_n f_n\, e^{ih_n z} & z < 0 \end{cases} \tag{8.22}$$

where $a_o f_o \, e^{ih_o z}$ is the single unidirectional mode and

$$a_n = - \frac{2\pi\omega\mu_o}{T'(h_n)} \int_o^1 J(\xi) \, f_n(\xi) \, d\xi$$

$$b_n = \frac{2\pi\omega\mu_o}{T'(h_n)} \int_o^1 J(\xi) \, f_n(\xi) \, d\xi \quad . \tag{8.23}$$

We have thus formally solved the problem described in figure 8.2. Expressions 8.22 and 8.23 yield the fields generated by a current source in an infinitely long guide partially filled with ferrite. It is interesting to observe the effect of the non-orthogonality of the modes. A current source distribution J(x) having a distribution identical to the $j^{th}$ mode ($J(x) = \text{const.} \; f_j(x)$) generates not only the $j^{th}$ mode but many other modes, because, in general, $a_n = \frac{-2\pi\omega\mu_o}{T'(h_n)} \int_o^1 f_j(\xi) \, f_n(\xi) \, d\xi \neq 0$ . A current source may not be chosen to generate a single mode.

We will now use the above solution of the problem in figure 8.2 to obtain the general form of solutions to the problem in figure 8.1 Applying the continuity of $E_y(x,z)$ at z = 0 to equation 8.22 yields

$$\sum_{\text{Im } h_n < o} b_n f_n - (a_o f_o + \sum_{\text{Im } h_n > o} a_n f_n) = 0 \tag{8.24}$$

or

$$\sum_{\text{all } h_n} c_n f_n = 0 \tag{8.25}$$

where

$$c_n = \left\{ \begin{array}{ll} -a_n, & \text{Re } h_n \geqslant 0 \\ b_n, & \text{Re } h_n < 0 \end{array} \right\} = \frac{2\pi\omega\mu_o}{T'(h_n)} \int_o^1 J(x)\, f_n(x)\, dx \ . \tag{8.26}$$

Then, formally

$$E_y(x,z) = \sum_{\text{all } h_n} c_n f_n\, e^{ih_n z} \tag{8.27}$$

converges to 0 for $z \to 0$ and hence satisfies the boundary condition at $z = 0$ in figure 8.1.

Since equation 8.25 is valid for arbitrary $J(x)$, we can simplify expression 8.26 by defining a new current $J'(x) = 2\pi\omega\mu_o\ J(x)$ to obtain

$$c_n = \frac{1}{T'(h_n)} \int_o^1 J(x)\, f_n(x)\, dx \tag{8.28}$$

where the prime has been dropped.

In this chapter we have obtained the proposed solution 8.27 with expansion coefficients $\{c_n\}$ defined by 8.28 in terms of an arbitrary function $J(x)$. Formally, this proposed solution represents an infinite number of solutions which match the boundary condition at $z = 0$. We would expect that within this infinite set of solutions lie all possible solutions matching the boundary conditions at $z = -\ell$; that is, by choosing $J(x)$ properly, it is possible to match arbitrary boundary conditions at $z = -\ell$. In the next chapter we will rigorously justify our formal solution, this will require rigorously showing $\Sigma\, c_n f_n = 0$, uniform convergence of the modal expansion, etc.

Before proceeding, let us point out one encouraging observation concerning the proposed expansion coefficients $\{c_n\}$. The identity

$$T'(h_n) = \int_0^1 \left(2\, h_n \nu_1 + \frac{d\nu_2}{dx}\right) f_n^{\,2} dx \tag{8.29}$$

may be easily verified by substituting the known values of $f_n(x)$, $\nu_1(x)$, $\nu_2(x)$, and differentiating $T(h_n)$. Thus, the proposed expansion coefficients 8.28 could just as well be written in the form

$$c_n = \frac{\int_0^1 J(x)\, f_n(x)\, dx}{\int_0^1 \left(2\, h_n \nu_1 + \frac{d\nu_2}{dx}\right) f_n^{\,2} dx} \; . \tag{8.30}$$

This is exactly the same expression as we obtained by applying the rigorous work from part I for guides filled with ferrite in a continuous manner. (Compare with equation 8.1b).

## 9.0 Rigorous Justification of the Formal Solution

In the preceding chapter we have considered the problem in figure 8.1. An infinite set of solutions to Maxwell's equations which satisfy the boundary condition at $z = 0$ was formally derived. No attempt was made to satisfy the boundary condition at $z = -\ell$; it was simply assumed that each solution of this infinite set of solutions corresponds to a particular matching condition at $z = -\ell$. In this chapter we will rigorously justify our formally obtained solution,

$$E_y(x,z) = \Sigma\, c_n f_n\, e^{ih_n z} \tag{9.1a}$$

where

$$c_n = \frac{1}{T'(h_n)} \int_0^1 J(\xi)\, f_n(\xi)\, d\xi \tag{9.1b}$$

and $J(\xi)$ is an arbitrary function.

In order to justify this proposed solution it is necessary to prove that 9.1 satisfies Maxwell's equations in the region $0 < x < 1$, $-\ell < z < 0$, and that $E_y(x,o) = E_y(o,z) = E_y(1,z) = 0$. It is sufficient to prove that

1. The series $\Sigma\, c_n f_n\, e^{ih_n z}$, $\Sigma\, c_n h_n f_n\, e^{ih_n z}$, $\Sigma\, c_n f_n'\, e^{ih_n z}$, are uniformly convergent for $0 < x < 1$, $-\ell < z < 0$. If these series are uniformly convergent, the proposed solution clearly satisfies Maxwell's equations since the first and second partial derivatives

of $E_y(x,z)$ with respect to x and z can be performed on the individual modes before summing.

2. The series $\Sigma\, c_n\, f_n\, e^{ih_n z}$ is uniformly convergent in x and z for $0 \leqslant x \leqslant 1$ and $-\ell < z \leqslant 0$. This will permit the interchange of the limits $z \to 0$ and $x \to 0$ and $x \to 1$ with the summation such that the boundary conditions are satisfied,

$$\lim_{z\to 0} \Sigma\, c_n\, f_n\, e^{ih_n z} = \Sigma c_n\, f_n$$

$$\lim_{x\to 0} \Sigma\, c_n\, f_n\, e^{ih_n z} = \Sigma c_n\, f_n(o)\, e^{ih_n z} = 0$$

$$\lim_{x\to 1} \Sigma\, c_n\, f_n\, e^{ih_n z} = \Sigma c_n\, f_n(1)\, e^{ih_n z} = 0$$

3. $\Sigma c_n\, f_n = 0$. In conjunction with property 2 this will prove that the proposed solution converges to satisfy the boundary condition at $z = 0$.

In order to prove the above properties it is first necessary to determine the asymptotic location of the propagation constants, or eigenvalues, $h_n$. The set $\{ h_n\}$ is defined by

$$T(h) \equiv k^{(1)} \coth k^{(1)} d - k^{(2)} A \coth k^{(2)} (d-1) + hB = 0 \qquad (9.2)$$

We will first make a crude estimate of $h_n$ as $|h_n| \to \infty$. As $|h| \to \infty$

$$k^{(1)} = \sqrt{h^2 - \omega^2 \mu_o \varepsilon} \quad \to \quad h$$
$$k^{(2)} = \sqrt{h^2 - \frac{\omega^2 \mu_o \varepsilon}{A}} \quad \to \quad h$$

and thus

$$T(h) \to \coth hd - A \coth h\,(d-1) + B = 0 \quad .$$

Writing $h = \beta + i\alpha$ ($\alpha,\beta$ both real) and equating the real and imaginary parts of equation 9.2 yields the following relationships for $\alpha$ and $\beta$ :

$$\frac{\sinh 2\beta d}{\cosh 2\beta d - \cos 2\alpha d} - A\,\frac{\sinh 2\beta(d-1)}{\cosh 2\beta(d-1) - \cos 2\alpha(d-1)} + B = 0 \qquad (9.3a)$$

$$\frac{\sin 2\alpha d}{\cosh 2\beta d - \cos 2\alpha d} - A\,\frac{\sin 2\alpha(d-1)}{\cosh 2\beta(d-1) - \cos 2\alpha(d-1)} = 0 \qquad (9.3b)$$

We are interested in the asymptotic roots $|h| = \sqrt{\alpha^2 + \beta^2} \to \infty$. Either $|\alpha|$ or $|\beta|$ or both $|\alpha|$ and $|\beta|$ go to infinity as $|h| \to \infty$. Suppose $|\beta| \to \infty$ ; equation 9.3a approaches

$$\tanh 2\beta d - A \tanh 2\beta(d-1) + B = 0 \quad . \qquad (9.4)$$

A simple sketch of the graph of 9.4 shows that equation 9.4 has a unique and bounded solution for $\beta$ if $|1 + A| > |B|$ . (This condition is satisfied for the paradox problem.) Thus, by contradiction, $|\beta|$ does not approach infinity. The roots must therefore be confined to the region of the complex plane $-N \leq \mathrm{Re}\, h \leq N$ for some $|N| < \infty$ .

Now that we have confined the region of the complex plane in which the h must lie, we will attempt to find the asymptotic locations as $|h| \to \infty$ or $|\alpha| \to \infty$ . Determining the asymptotic $\alpha,\beta$ is very difficult in general. However, for the particular case $d = \frac{1}{2}$ (the ferrite slab half fills the guide), equations 9.3 reduce to

$$(1 + A)\frac{\sinh \beta}{\cosh \beta - \cos \alpha} + B = 0 \tag{9.5a}$$

$$(1 + A)\frac{\sin \alpha}{\cosh \beta - \cos \alpha} = 0 \tag{9.5b}$$

which may be easily solved. Assuming $A \neq -1$, the solutions are

$$h_n = \beta + i\, n\, \pi \tag{9.6}$$

where n is a large odd integer if $\frac{A+1}{B} > 1$; a large even integer if $\frac{A+1}{B} < 1$ and $\beta \neq 0$ is given by

$$\frac{A+1}{B}\sinh \beta = \cos n\pi - \cosh \beta \,. \tag{9.7}$$

The asymptotic location of the roots for arbitrary d will have a considerably more complicated form than $\beta + i\, n\, \pi$. We feel that, at the present time, the asymptotic locations for arbitrary d are not sufficiently important to warrant our consideration. The boundary value problem we are considering is of interest because of of the presence of a unidirectional mode. This mode exists for the half filled guide; hence, we are not excluding the interesting problem of the "paradox" by requiring $d = \frac{1}{2}$ .

In order to verify that the roots of T(h) are indeed asymptotically near $\beta + in\pi$ we will make a Taylor series expansion of T(h) about the points $\beta + in\pi$ and then revert the resulting series. Let us denote the true roots of T(h) by $h_n^{(T)}$ $(T(h_n^{(T)}) = 0)$ and the proposed asymptotic roots $\beta + in\pi$ by $h_n^{(P)}$ . Making a Taylor series expansion of T(h) about $h_n^{(P)}$ yields

$$T(h_n^{(T)}) = T(h_n^{(P)}) + T'(h_n^{(P)}) \, (h_n^{(T)} - h_n^{(P)}) + \frac{T''(h_n^{(P)})}{2!} (h_n^{(T)} - h_n^{(P)})^2 + \cdots \quad (9.8)$$

Using the definition $T(h_n^{(T)}) = 0$ and subtracting $T(h_n^{(P)})$ from both sides of equation 9.8 yields

$$-T(h_n^{(P)}) = T'(h_n^{(P)}) \, (h_n^{(T)} - h_n^{(P)}) + \frac{T''(h_n^{(P)})}{2!} (h_n^{(T)} - h_n^{(P)})^2 + \cdots \quad (9.9)$$

Expansion 9.9 is in a form suitable for reversion. Reversion of the series yields

$$h_n^{(P)} - h_n^{(T)} = \frac{1}{T'(h_n^{(P)})} T(h_n^{(P)}) + \frac{T''(h_n^{(P)})}{2!\ (T'(h_n^{(P)}))^3} T^2(h_n^{(P)}) + \dots \quad (9.10)$$

In appendix D it is proven that as $n \to \infty$

1. $T(h_n^{(P)}) = O(1)$ (9.11)

2. $T'(h_n^{(P)}) = h_n^{(P)}\ B \operatorname{csch}\beta \cos n\pi + o\ (h_n^{(P)})$ (9.12)

3. $T^{(\ell)}_{(h_n^{(P)})} = O(h_n^{(P)}) \qquad \ell = 1,2, \dots$ (9.13)

Thus the reverted series 9.10 becomes

$$h_n^{(T)} = h_n^{(P)} + \frac{\text{const}}{h_n^{(P)}} + o\ \left(\frac{1}{h_n^{(P)}}\right) \text{ as } n \to \infty \quad (9.14)$$

and hence

$$h_n^{(T)} \sim h_n^{(P)} = \beta + in\pi \quad . \quad (9.15)$$

In the future we will omit the superscript and simply write

$$h_n \sim \beta + in\pi \quad . \quad (9.16)$$

The asymptotic form of the eigenvalues $h_n$ may now be used to obtain the asymptotic form of the expansion coefficients $c_n$ . Assuming $J(\xi)$ is a continuous function so that $J'(\xi) = \frac{dJ}{d\xi}$ exists, the integral in equation 9.1b may be integrated by parts to yield

$$c_n = c_n^{(1)} - c_n^{(2)} \tag{9.17}$$

where

$$c_n^{(1)} = \frac{J(1/2)\cosh k_n^{(1)}\frac{1}{2} - J(o)}{T'(h_n)\, k_n^{(1)}\sinh k_n^{(1)}\frac{1}{2}} + \frac{J(\frac{1}{2})\cosh k_n^{(2)}\frac{1}{2} - J(1)}{T'(h_n)\, k_n^{(2)}\sinh k_n^{(2)}\frac{1}{2}} \tag{9.18}$$

and

$$c_n^{(2)} = \frac{1}{T'(h_n)k_n^{(1)}} \int_o^{1/2} J'(\xi)\frac{\cosh k_n^{(1)}\xi}{\sinh k_n^{(1)}\frac{1}{2}}\, d\xi$$

$$- \frac{1}{T'(h_n)\, k_n^{(2)}} \int_{1/2}^{1} J'(\xi)\frac{\cosh k_n^{(2)}(\xi-1)}{\sinh k_n^{(2)}\frac{1}{2}}\, d\xi \quad . \tag{9.19}$$

Since $h_n \sim \beta + in\pi$ , it is clear that as $n \to \infty$

$$\cosh k_n^{(i)}\frac{1}{2} = \text{const} + o(1) \tag{9.20a}$$

$$\sinh k_n^{(i)}\frac{1}{2} = \text{const} + o(1) \qquad i = 1,2 \tag{9.20b}$$

$$k_n^{(i)} = h_n + o\left(\frac{1}{h_n}\right) \quad . \tag{9.20c}$$

Using equations 9.20 and 9.12, it follows that

$$c_n^{(1)} = 0\left(\frac{1}{h_n^2}\right) \qquad \text{as } n \to \infty \qquad . \tag{9.21}$$

Now consider $c_n^{(2)}$. Since neither $\frac{i}{2} k_n^{(1)} = n\pi$ nor $\frac{i}{2} k_n^{(2)} = n\pi$ solve $T(h_n) = 0$ and $h_n \sim \beta + in\pi$ $(\beta \neq 0)$, there exists some constant $N > 0$ independent of n such that $|\sinh k_n^{(1)} \frac{1}{2}| \geqslant N$ and $|\sinh k_n^{(2)} \frac{1}{2}| \geqslant N$. Thus the maximum values of the integrals in equation 9.19 may be used to obtain

$$|c_n^{(2)}| \leqslant \frac{J'_{max}}{|T'(h_n)|N}\left(\frac{\cosh(\frac{1}{2}\,\mathrm{Re}\,k_n^{(1)})}{|k_n^{(1)}|} + \frac{\cosh(\frac{1}{2}\,\mathrm{Re}\,k_n^{(2)})}{|k_n^{(2)}|}\right) \tag{9.22}$$

where $J'_{max} = \max\limits_{1>\xi>0} |J'(\xi)|$. Applying the asymptotic values of $k_n^{(1)}$, $k_n^{(2)}$, and $T'(h_n)$, it is clear that

$$|c_n^{(2)}| \leqslant \frac{\text{const.}}{|h_n|^2} + o\left(\frac{1}{h_n^2}\right) \qquad \text{as } n \to \infty \tag{9.23}$$

or

$$c_n^{(2)} = 0\left(\frac{1}{h_n^2}\right) \qquad \text{as } n \to \infty \qquad . \tag{9.24}$$

Thus

$$c_n = 0\left(\frac{1}{h_n^2}\right) \qquad \text{as } n \to \infty \qquad . \tag{9.25}$$

Now that we have obtained the asymptotic form of $c_n$ and $h_n$ we are prepared to rigorously justify the proposed solution $E_y(x,z) = \Sigma\, c_n f_n e^{ih_n z}$. We must prove that the proposed solution satisfies the three requirements presented at the beginning of this chapter. First, we will consider the third requirement, $\Sigma\, c_n f_n = 0$.

Let us define

$$F(x,z) = \frac{-i}{2\pi} \int_{-\infty}^{\infty} e^{ihz}\, I(x,h)\, dh \tag{9.26}$$

where I(x,h) is defined by equations 8.16-8.18. In appendix E we prove that F(x,z) is a continuous function of z for all z. We may therefore write

$$\lim_{z\to o^+} F(x,z) = \lim_{z\to o^-} F(x,z) \quad . \tag{9.27}$$

As already discussed in the preceding chapter, I(x,h) remains bounded for all x ($o \leq x \leq 1$) and all h, except at simple poles defined by $T(h) = 0$. We have already determined that the zeros of $T(h)$ for $|h| \to \infty$ are given by $\beta + in\pi$, hence the integral 9.26 may be closed in the upper half plane for $z > o$, and in the lower half plane for $z < o$.

$$\lim_{z\to o^+} \left( \frac{-i}{2\pi} \int_{\frown} e^{ihz} I(x,h)\, dh \right) = \lim_{z\to o^-} \left( \frac{-i}{2\pi} \int_{\smile} e^{ihz} I(x,h)dh \right) \quad (9.28)$$

Evaluation of the integrals by the residue theory of complex variables yields*

$$\lim_{z\to o^+} \sum_{U.H.P.} \lim_{h\to h_n} \frac{h-h_n}{T(h)} e^{ihz}\, T(h) I(x,z) \quad (9.29)$$

$$= - \lim_{z\to o^-} \sum_{L.H.P.} \lim_{h\to h_n} \frac{h-h_n}{T(h)} e^{ihz}\, T(h)\, I(x,h)$$

where $\sum_{U.H.P.}$ denotes the sum over $h_n$ such that $T(h_n) = 0$ and $\mathrm{Im}\, h_n \geq 0$; $\sum_{L.H.P.}$, the sum over $h_n$ such that $T(h_n) = 0$ and $\mathrm{Im}\, h_n < 0$. Using definition 8.18 of $I(x,h)$ and the fact that $\lim_{h\to h_n} \frac{h-h_n}{T(h)} = \frac{1}{T'(h_n)}$

$$\lim_{z\to o^+} \sum_{U.H.P.} c_n f_n\, e^{ih_n z} = \lim_{z\to o^-} \sum_{L.H.P.} - c_n f_n\, e^{ih_n z} \quad (9.30)$$

where

$$c_n = \frac{1}{T'(h_n)} \int_0^1 J(\xi)\, f_n(\xi)\, d\xi \quad (9.31)$$

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*Poles on the real h axis are interpreted by inserting loss as discussed in chapters 7 and 8; the contour in the upper half plane will include the unidirectional mode.

as given by the proposed solution $E_y(x,z) = \Sigma\, c_n f_n\, e^{ih_n z}$.

Consider the left side of equation 9.30. In the upper half h plane, Im $h_n > 0$; thus for $z \geq 0$

$$| c_n f_n e^{ih_n z} | \leq | c_n f_n | . \tag{9.32}$$

$f_n(x)$ is bounded for $0 \leq x \leq 1$, thus $|f_n| \leq$ const, or

$$| c_n f_n e^{ih_n z} | \leq \text{const.} | c_n | . \tag{9.33}$$

From our previous work, $c_n = 0 \left(\frac{1}{h_n^2}\right)$ and $h_n \sim \beta + in\pi$, thus

$$\sum_1^\infty | c_n | < \infty , \tag{9.34}$$

and by the Weierstrass M-test the series on the left side of equation 9.30 is uniformly convergent for $0 \leq x \leq 1$, $z \geq 0$. In a like manner it may be shown that the series on the right side of equation 9.30 is uniformly convergent for $z \leq 0$; hence, the limits and summation may be interchanged to yield

$$\sum_{U.H.P.} c_n f_n = - \sum_{L.H.P.} c_n f_n$$

or

$$\sum_{\text{all } h_n} c_n f_n = 0 \quad .$$

It remains to be proven that the proposed solution satisfies the first two requirements presented at the beginning of this chapter. For this purpose it is convenient to write the proposed solution in the "split" form

$$E_y(x,z) = \sum_{\text{Im } h_n \geq o} c_n f_n e^{ih_n z} + \sum_{\text{Im } h_n < o} c_n f_n e^{ih_n z} \quad . \tag{9.35}$$

In order to prove the second requirement, $\sum_{\text{all } h_n} c_n f_n e^{ih_n z}$ is uniformly convergent in x and z for $-\ell < z \leq o$ and $o \leq x \leq 1$, it is sufficient to show the uniform convergence of $\sum_{\text{Im } h_n \geq o} c_n f_n e^{ih_n z}$ and $\sum_{\text{Im } h_n < o} c_n f_n e^{ih_n z}$. Consider the second summation. The eigenfunctions $f_n(x)$ are bounded for $o \leq x \leq 1$ ; and $| e^{ih_n z} | \leq 1$ for $\text{Im } h_n < o$ , $z \leq o$ . Thus

$$| c_n f_n e^{ih_n z} | \leq \text{const.} \; | c_n | \; .$$

However, we know from the previous work that $c_n = 0 \left( \frac{1}{h_n^2} \right)$ where $h_n \sim \beta + in\pi$ ; thus clearly $\sum_{\text{Im } h_n < o} | c_n | < \infty$ and by the Weierstrass M-test $\sum_{\text{Im } h_n < o} c_n f_n e^{ih_n z}$ is uniformly convergent in x and z for $o \leq x \leq 1$ and $-\ell < z \leq o$ .

Now consider $\sum_{\text{Im } h_n \geq o} c_n f_n e^{ih_n z}$. The eigenfunctions $f_n$

are bounded for $o \leqslant x \leqslant 1$ ; and $| e^{ih_n z} | \leqslant e^{\mathrm{Im}\, h_n \ell}$ for $\mathrm{Im}\, h_n \geqslant o$ and $-\ell < z \leqslant o$ . Thus

$$| c_n f_n e^{ih_n z} | \leqslant \text{const.} \, | c_n | \, e^{\mathrm{Im}\, h_n \ell} \; .$$

However, we know from the previous work that $h_n \sim \beta + in\pi$ ; thus, by the Weierstrass M-test, $\sum_{\mathrm{Im}\, h_n \geqslant o} c_n \; f_n \; e^{ih_n z}$ is uniformly convergent in x and z for $o \leqslant x \leqslant 1$ and $-\ell < z \leqslant o$ if

$$c_n = o \; (e^{-\pi \ell n}) \qquad \text{for } \mathrm{Im}\, h_n > o \; .$$

Thus, we have shown that $\sum_{\text{all } h_n} c_n f_n e^{ih_n z}$ is uniformly convergent in x and z for $o \leqslant x \leqslant 1$ and $-\ell < z \leqslant o$ if the arbitrary function J is chosen such that

$c_n = o \; (e^{-n\pi\ell})$ for $\mathrm{Im}\, h_n > o$ .

It now remains to show that the proposed solution satisfies the first requirement; that is $\sum_{\text{all } h_n} c_n h_n f_n e^{ih_n z}$ and $\sum_{\text{all } h_n} c_n f_n' e^{ih_n z}$ are uniformly convergent for $o \leqslant x \leqslant 1$ and $-\ell < x \leqslant \varepsilon < o$ .

Consider $\sum_{\mathrm{Im}\, h_n < o} c_n h_n f_n e^{ih_n z}$ . The eigenfunctions $f_n$ are bounded for $o \leqslant x \leqslant 1$; and $| e^{ih_n z} | \leqslant e^{-| \varepsilon \, \mathrm{Im}\, h_n |}$ for $o \leqslant x \leqslant 1$, $-\ell < z \leqslant \varepsilon < 0$.

Thus

$$| c_n h_n f_n e^{ih_n z} | \leqslant \text{const.} \, | c_n | \, | h_n | \, e^{-| \varepsilon \, \mathrm{Im}\, h_n |} \; .$$

Since $c_n = O \left( \frac{1}{h_n^{\,2}} \right)$ and $h_n \sim \beta + in\pi$ , it is clear that

$$\sum_{\text{Im } h_n < o} | c_n | \, | h_n | \, e^{-|\varepsilon \text{ Im } h_n|} < \infty .$$

Thus $\sum_{\text{Im } h_n < o} c_n h_n f_n e^{i h_n z}$ is uniformly convergent for all $o \leq x \leq 1$ and $- \ell < z \leq \varepsilon < o$.

Now consider $\sum_{\text{Im } h_n \geq o} c_n h_n f_n e^{i h_n z}$. The eigenfunctions $f_n$ are bounded for $o \leq x \leq 1$; and $|e^{i h_n z}| \leq e^{|\text{Im } h_n| \ell}$ for Im $h_n \geq o$, $- \ell < z < o$. Thus

$$| h_n c_n f_n e^{i h_n z} | \leq \text{const.} \, | h_n | \, | c_n | \, e^{|\text{Im } h_n| \ell} .$$

However, $h_n \sim \beta + in\pi$, thus by the Weierstrass M-test, $\sum_{\text{Im } h_n \geq o} c_n h_n f_n e^{i h_n z}$ is uniformly convergent in x and z for $o \leq x \leq 1$ and $- \ell < z < o$ if

$$c_n = o \, (e^{- n\pi\ell}) \qquad \text{for Im } h_n > o .$$

Thus $\sum_{\text{all } h_n} c_n h_n f_n e^{i h_n z}$ is uniformly convergent in x and z for $o \leq x \leq 1$ and $- \ell < z < o$ if $c_n = o \, (e^{- n\pi\ell})$ for Im $h_n > o$.

In a like manner it may be shown that $\sum_{\text{all } h_n} c_n f_n' e^{i h_n z}$ is uniformly convergent under the same conditions.

We have thus conclusively demonstrated that the proposed solution 9.1 yields solutions to the problem in figure 8.1 provided the arbitrary function $J(\xi)$ is restricted to a set such that $c_n = o \, (e^{-n\pi\ell})$ for Im $h_n > o$.

Now consider the problem of the paradox in which there is a single unidirectional mode $f_o \, e^{ih_o z}$ .

$$E_y(x,z) = c_o \, f_o \, e^{ih_o z} + \sum_{\mathrm{Im}\, h_n \neq o} c_n \, f_n \, e^{ih_n z} \quad .$$

In appendix F it is proven that there exists a non-empty set of J such that $c_o \neq o$ and $c_n = o\,(e^{-n\pi\ell})$ for $\mathrm{Im}\, h_n > o$ . That is, there exists a class of functions J such that the proposed solution 9.1 satisfies the required boundary conditions with a non-zero component of the unidirectional mode. Thus it is clear that, although the solutions of the boundary value problem in figure 8.1 are complicated, there is nothing paradoxical about them. Solutions do exist without violating any of the basic laws of thermodynamics.

# SUMMARY

The general problem of electromagnetic wave propagation through rectangular guides filled inhomogeneously in cross-section with transversely magnetized ferrite was studied in part I. The problem was split split into TE and TM parts and scalarized. The TM part did not lead to any new or interesting results. The TE part, however, led to a non-self-adjoint eigenvalue equation for which the classical waveguide techniques do not apply. A rigorous analysis of this non-self-adjoint equation yielded an unusual expansion theorem. According to this expansion theorem, the eigenfunction expansion of an arbitrary function is not unique; the eigenfunctions (TE modes) are not orthogonal and form a "more than complete set." This expansion theorem was used to rigorously solve the most fundamental waveguide problem; that is, the problem of determining the fields at all points within a guide from a knowledge of the fields at one cross-section. It was also shown that the rigorous results could be obtained formally by applying a well-known orthogonality relationship.

The problem of the thermodynamic paradox was considered in part II. The past significant research was reviewed and corrected. It was pointed out that the existence of the paradox is based on the assumption that the power in question cannot return via the cutoff modes. A detailed study was made of the three commonly accepted "proofs" by Seidel and Fletcher, Tai, and Bresler which independently purport to prove the

validity of this crucial assumption. The study revealed that each of the three "proofs" contained serious errors such that their results cannot be accepted. The premise upon which the paradox has been based for so many years was thus destroyed. It clearly followed that there is absolutely no justification in assuming that the problem constitutes any violation of the basic laws of thermodynamics.

The understanding of the nature wave propagation in ferrite loaded guides which was developed in the general theoretical work of part I was then applied to the problem of the paradox. The general form of the solution to the boundary value problem of the paradox was obtained by considering the problem of a current source in an infinite guide. This general form was then rigorously studied and it was shown that, although the solution to the boundary value problem of the paradox is complicated, there is nothing paradoxical about it.

# APPENDIX A

In this section it is shown that equation 2.3 with $H_z \equiv 0$ has a non-zero solution only if $H_x \equiv 0$.

For $H_z \equiv 0$ equation 2.3 becomes

$$H_x = \frac{1}{\omega^2\varepsilon}\left(-\nu_1 \frac{\partial^2 H_x}{\partial z^2} + i\,\nu_2 \frac{\partial^2 H_x}{\partial x \partial z}\right) \tag{A.1a}$$

$$0 = \frac{1}{\omega^2\varepsilon}\left(i\nu_2 \frac{\partial^2 H_x}{\partial z^2} + \nu_1 \frac{\partial^2 H_x}{\partial x \partial z}\right) \tag{A.1b}$$

Integration of equation A.1b over z yields

$$i\,\nu_2 \frac{\partial H_x}{\partial z} + \nu_1 \frac{\partial H_x}{\partial x} = P(x) \tag{A.2}$$

where P(x) is a constant of integration. Equation A.2 may be solved using the standard directional derivative approach. Assuming $\nu_1(x) \neq 0 \quad (o \leq x \leq 1)$, the general solution is

$$H_x(x,z) = \int_o^x \frac{P(\xi)}{\nu_1(\xi)}\, d\xi + H_o\left(z - \int_o^x i\,\frac{\nu_2(\xi)}{\nu_1(\xi)}\, d\xi\right) \tag{A.3}$$

where $H_o$ is an arbitrary function. Substituting equation A.3 into equation A.1a yields

$$\int_0^x \frac{P(\xi)}{\nu_1(\xi)}\, d\xi + H_o\left(z - \int_0^x i\, \frac{\nu_2(\xi)}{\nu_1(\xi)}\, d\xi\right)$$

(A.4)

$$= \frac{\nu_2^{\,2} - \nu_1^{\,2}}{\omega^2 \varepsilon \nu_2} H_o''\left(z - \int_0^x i\, \frac{\nu_2}{\nu_1}\, d\xi\right)$$

where $H_o''(\xi)$ denotes $\frac{d^2}{d\xi^2} H_o(\xi)$ . If we define

$$y = z - \int_0^x i\, \frac{\nu_2(\xi)}{\nu_1(\xi)}\, d\xi \quad ,$$

equation A.4 may be written in the form

$$\frac{\nu_2^2(x) - \nu_1^2(x)}{\omega^2 \varepsilon\, \nu_2(x)} H_o''(y) - H_o(y) = \int_0^x \frac{P(\xi)}{\nu_1(\xi)}\, d\xi \, . \qquad \text{(A.5)}$$

Clearly this is possible only for $\int_0^x \frac{P(\xi)}{\nu_1(\xi)}\, d\xi + H_o(y) \equiv 0$ ;
or using equation A.3,

$$H_x(x,z) \equiv 0 \quad .$$

## APPENDIX B

This section contains the proof that Cohen's expansion theorem can be reduced to the following simple statement:

> Any arbitrary bounded, continuous function F(x) may be expanded in the uniformly convergent series

$$F(x) = \sum_n \frac{\int_0^1 (\nu_1(\xi)\, h_n F(\xi) + K(\xi))\, f_n(\xi)\, d\xi}{\int_0^1 (2h_n \nu_1(\xi) + \frac{d\nu_2}{d\xi})\, f_n^2(\xi)\, d\xi}\, f_n(x) \qquad (B.1)$$

where $f_n(x)$ are eigenfunctions of the system

$$\frac{d}{dx}\nu_1(x)\frac{df}{dx} + (\omega^2\varepsilon - h^2\,\nu_1(x) - h\frac{d\nu_2}{dx})\, f(x) = 0 \qquad (B.2)$$

$$f(o) = f(1) = 0$$

and $K(\xi)$ is an arbitrary bounded, continuous function.

According to Cohen, any arbitrary bounded, continuous function F(x) may be expanded in the uniformly convergent series

$$F(x) = \sum_n \frac{\int_0^1 Z_n R \begin{bmatrix} F(\xi) \\ k(\xi) \end{bmatrix} d\xi}{\int_0^1 Z_n R\, Y_n\, d\xi}\, \nu_1(x)^{1/2}\, e^{\int_0^x \frac{1}{2\nu_1(\xi)} \frac{d\nu_2(\xi)}{d\xi} d\xi}\, f_n(x) \qquad (B.3)$$

where $f_n(x)$ are the eigenfunctions of system B.2, $k(x)$ is an arbitrary continuous function, and $Y_n = \begin{bmatrix} y_n^{(1)} \\ y_n^{(2)} \end{bmatrix}$ is defined by

$$Y'(x) - [-hR + B(x)]\, Y(x) = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} Y(o) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} Y(1) = 0 \tag{B.4}$$

and $Z_n = [z_n^{(1)}, z_n^{(2)}]$ is defined by the adjoint system

$$Z'(x) + Z(x)\, [-hR + B(x)] = 0$$

$$-Z(o) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + Z(1) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0 \tag{B.5}$$

and

$$R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{B.6}$$

and

$$B(x) = \begin{bmatrix} 0 & e^{\int_o^x \frac{1}{\nu_1(\xi)} \frac{d\nu_2}{d\xi} d\xi} \\ q(x)\, e^{-\int_o^x \frac{1}{\nu_1(\xi)} \frac{d\nu_2}{d\xi} d\xi} & 0 \end{bmatrix} \tag{B.7}$$

and

$$q(x) = \frac{1}{2\nu_1}\left(\frac{d^2\nu_1}{dx^2} + \frac{d^2\nu_2}{dx^2}\right) - \frac{1}{4\nu_1^2}\left(\frac{d\nu_1}{dx} + \frac{d\nu_2}{dx}\right)^2 - \frac{\omega^2\varepsilon}{\nu_1} \qquad (B.8)$$

and the functions $f_n$ , $y_n$ are related by

$$f_n(x) = \nu_1^{-1/2}(x)\; e^{-\int_0^x \frac{1}{2\nu_1(\xi)} \frac{d\nu_2}{d\xi} d\xi}\; y_n^{(1)}(x) \;. \qquad (B.9)$$

According to Cohen's results, the function $F(x)\nu_1^{1/2} e^{\int_0^x \frac{-}{\nu_1} \frac{d\nu_2}{d\xi} d\xi}$ may be expanded as

$$\sum_n \frac{\int_0^1 Z_n(\xi)\; R \left[\frac{F(\xi)\;\nu_1^{1/2}(\xi)\; W^{-1}(\xi)}{k(\xi)}\right] d\xi}{\int_0^1 Z_n(\xi)\; R\; Y_n(\xi)\; d\xi}\; \nu_1^{1/2}(x)\; W^{-1}(x)\; f_n(x)$$

(B.10)

where we have defined

$$W(x) = e^{-\int_0^x \frac{1}{2\nu_1(\xi)} \frac{d\nu_2}{d\xi} d\xi} \;.$$

Dividing equation B.10 by $\nu_1^{1/2}(x)\; W^{-1}(x)$ yields

$$F(x) = \sum_n \frac{\int_0^1 Z_n(\xi)\; R \left[\frac{F(\xi)\;\nu_1^{1/2}(\xi)\; W^{-1}(\xi)}{k(\xi)}\right] d\xi}{\int_0^1 Z_n(\xi)\; R\; Y_n(\xi)\; d\xi}\; f_n(x) \;.$$

(B.11)

We will now prove that the proposed expansion B.1 is identical to the rigorously derived expansion B.11; that is, we will prove

$$C_n \equiv \frac{-\int_0^1 Z_n(x)\; R \begin{bmatrix} F(x)\; \nu_1^{1/2}\; W^{-1}(x) \\ k(x) \end{bmatrix} dx}{-\int_0^1 Z_n(x)\; R\; Y_n(x)\; dx}$$

$$\equiv \frac{\int_0^1 (\; \nu_1(x)\; h_n\; F(x) + K(x)\;)\; f_n(x)\; dx}{\int_0^1 (\; 2\; h_n\; \nu_1(x) \pm \frac{d\nu_2}{dx}\;)\; f_n^2\; dx} \tag{B.12}$$

In order to prove identity B.12 it is first necessary to notice that there is a very simple relationship between the vector components of system B.4 and those of system B.5. The transpose of system B.5 is

$$Z^{T'}(x) + [\; -h\; R + B(x)\; ]^T\; Z^T(x) = 0 \tag{B.13a}$$

$$- \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} Z^T(0) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} Z^T(1) = 0 \tag{B.13b}$$

where T denotes the matrix transpose. Premultiplying the differential equation B.13 by the matrix $A \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and the boundary conditions B.13b by $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ yields

$$A\, Z^{T'}(x) + A\, [-h\,R + B(x)]^T Z^T(x) = 0$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} Z^T(0) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} Z^T(1) = 0 \quad . \qquad (B.14)$$

The particular form of the matrices A, B, R, permit us to write

$$A\,[-h\,R + B]^T = -[-h\,R + B]\,A$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} A \qquad (B.15)$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} A$$

and thus system B.14 may be written as

$$A\, Z^{T'}(x) - [-h\,R + B]\, A\, Z^T(x) = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} A\; Z^T(0) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} A\; Z^T(1) = 0 \quad . \qquad (B.16)$$

Systems B.4 and B.16 are identical, hence

$$Y_n(x) \equiv A\, Z_n^T(x) \qquad (B.17)$$

or, in component form,

$$y_n^{(1)} = z_n^{(2)}$$

$$y_n^{(2)} = -z_n^{(1)} \quad . \qquad (B.18)$$

We will now use relationships B.18 to prove that the denominators of the rigorous and proposed expansion coefficients $C_n$ given by expression B.12 are identical.

Writing $Z_n$ and $Y_n$ in component form, the denominators of the rigorous expansion coefficients are given by

$$-\int_0^1 Z_n R Y_n \, dx = \int_0^1 ( z_n^{(2)} y_n^{(2)} - z_n^{(1)} y_n^{(1)} ) \, dx \, . \tag{B.19}$$

Applying relationships B.18 to eliminate $z_n^{(1)}$ and $z_n^{(2)}$ yields

$$-\int_0^1 Z_n R Y_n \, dx = 2\int_0^1 y_n^{(1)} y_n^{(2)} \, dx \quad . \tag{B.20}$$

Using system B.4 to write $y_n^{(2)}$ in terms of $y_n^{(1)}$ we obtain

$$y_n^{(2)} = ( y_n^{(1)'} + h_n y_n^{(1)} ) \overset{2}{W}(x) \quad . \tag{B.21}$$

Substituting equation B.21 into equation B.20

$$-\int_0^1 Z_n R Y_n \, dx = 2 \int_0^1 h_n (y_n^{(1)})^2 \overset{2}{W}(x) \, dx + 2 \int_0^1 y_n^{(1)} y_n^{(1)'} \overset{2}{W}(x) \, dx \quad . \tag{B.22}$$

The second integral on the right side of equation B.22 can be integrated by parts

$$\int_0^1 y_n^{(1)} y_n^{(1)'} \overset{2}{W}(x) \, dx = (y_n^{(1)})^2 \overset{2}{W}(x) \Big|_{x=0}^{x=1} - \int_0^1 \left[ y_n^{(1)} y_n^{(1)'} + \frac{1}{\nu_1} \frac{d\nu_2}{dx} y_n^{(1)2} \right] \overset{2}{W}(x) \, dx. \tag{B.23}$$

Using the boundary conditions on $y_n^{(1)}$ as given by system B.4 to eliminate the first term on the right and rearranging

$$\int_0^1 y_n^{(1)} y_n^{(1)'} W^2(x)\, dx = \int_0^1 \frac{1}{2\nu_1} \frac{d\nu_2}{dx} \left( y_n^{(1)} W(x) \right)^2 dx \ . \qquad \text{(B.24)}$$

Substituting this expression into equation B.22

$$-\int_0^1 Z_n R Y_n\, dx = \int_0^1 \left(2 h_n + \frac{1}{\nu_1}\frac{d\nu_2}{dx}\right) \left( y_n^{(1)} W(x) \right)^2 dx \ . \qquad \text{(B.25)}$$

Using equation B.9 to relate $y_n^{(1)}$ and $f_n$

$$-\int_0^1 Z_n R Y_n\, dx = \int_0^1 \left(2 h_n \nu_1 + \frac{d\nu_2}{dx}\right) f_n^2\, dx \ . \qquad \text{(B.26)}$$

Thus we have shown that the denominators of the rigorous and proposed expansion coefficients as given by expressions B.12 are identical. Consider the numerators of the rigorous expansion coefficients,

$$N \equiv -\int_0^1 Z_n(x)\, R \begin{bmatrix} F(x)\, \nu_1^{1/2}\, W^{-1}(x) \\ \\ k(x) \end{bmatrix} dx \qquad \text{(B.27)}$$

or in component form,

$$N \equiv -\int_0^1 \left( z_n^{(1)} F(x)\, \nu_1^{1/2}\, W^{-1}(x) - z_n^{(2)} k(x) \right) dx \ . \qquad \text{(B.28)}$$

Using equations B.18 to write $z_n^{(1)}$, $z_n^{(2)}$ in terms of $y_n^{(1)}$ , $y_n^{(2)}$

and equation B.21 to write $y_n^{(2)}$ in terms of $y_n^{(1)}$

$$N = \int_0^1 (y_n^{(1)})' F(x)\, \nu_1^{1/2}\, W(x)\, dx + \int_0^1 (h_n y_n^{(1)} F\, \nu_1^{1/2} W + y_n^{(1)} k(x))dx \qquad (B.29)$$

The first integral on the right may be integrated by parts to yield

$$-\int_0^1 \left(\frac{dF}{dx}\nu_1^{1/2} + .5F\nu_1^{-1/2} - .5F\nu_1^{-1/2}\frac{d\nu_2}{dx}\right) W(x)\, y_n^{(1)}\, dx \quad . \qquad (B.30)$$

Substituting this expression into equation B.29 yields

$$N = \int_0^1 \left(h_n F\nu_1 + \frac{dF}{dx}\nu_1 + F/2 - \frac{F}{2}\frac{d\nu_2}{dx}\right) \nu_1^{-1/2}\, W(x)\, y_n^{(1)}\, dx + \int_0^1 k(x)\, y_n^{(1)}\, dx \quad . \qquad (B.31)$$

Using equation B.9 to relate $y_n^{(1)}$ and $f_n$

$$N = \int_0^1 \nu_1\, h_n\, f_n(x)\, F(x)\, dx + \int_0^1 \left[\frac{dF}{dx}\nu_1 + F/2 - \frac{F}{2}\frac{d\nu_2}{dx} + k(x)\nu_1^{1/2} W^{-1}(x)\right] f_n(x)\, dx \quad . \qquad (B.32)$$

Since k(x) is an arbitrary function of x, the quantity under the brackets in the second integral is also an arbitrary function of x. Let us denote this arbitrary quantity by K(x). Equation B.32 becomes

$$N = \int_0^1 (\nu_1\, h_n\, F(x) + K(x))\, f_n(x)\, dx \quad . \qquad (B.33)$$

This is identical to the numerator of the proposed expansion coefficients, hence we have shown that the proposed expansion B.1 is identical to the rigorously derived expansion B.11.

## APPENDIX C

In this section it is shown that the orthogonality relationship $\int_0^1 Z_n R Y_m \, dx = 0$ is equivalent to $\int_0^1 [(h_n+h_m)\nu_1 + \frac{d\nu_2}{dx}] f_n f_m \, dx = 0.$

Let us define $N \equiv -\int_0^1 Z_n R Y_m \, dx$ .

We will prove $N \equiv \int_0^1 [(h_n+h_m)\nu_1 + \frac{d\nu_2}{dx}] f_n f_m \, dx = 0$

By definition

$$N \equiv -\int_0^1 [\, z_n^{(1)} \quad z_n^{(2)} \,] \begin{bmatrix} y_m^{(1)} \\ -y_m^{(2)} \end{bmatrix} dx \quad .$$

Applying relationship B.18 of appendix B to eliminate $z_n^{(1)}$ and $z_n^{(2)}$ yields

$$N \equiv \int_0^1 (y_m^{(1)} y_n^{(2)} + y_n^{(1)} y_m^{(2)}) \, dx \quad .$$

Using relationship B.21 of appendix B to write $y_i^{(2)}$ in terms of $y_i^{(1)}$ yields

$$N \equiv \int_0^1 (h_n+h_m) \, y_n^{(1)} \, y_m^{(1)} \, W^2(x) \, dx$$

$$+ \int_0^1 ( y_m^{(1)} y_n^{(1)'} + y_n^{(1)} y_m^{(1)'} ) \, W^2(X) \, dx \quad .$$

Integrating the second integral by parts and using the fact that $y_i^{(1)}(0) = y_i^{(1)}(1) = 0$, we obtain

$$N \equiv \int_0^1 [(h_n+h_m)\nu_1 + \frac{d\nu_2}{dx}]\nu_1^{-1} \, y_n^{(1)} \, y_m^{(1)} \, W^2(X) \, dx \quad .$$

Using relationship B.9 to write $y_i'$ in terms of $f_i$

$$N \equiv \int_0^1 [(h_n+h_m)\nu_1 + \frac{d\nu_2}{dx}]\, f_n\, f_m\, dx \quad .$$

## APPENDIX D

In this section we will prove that for $n \to \infty$

1. $T(h_n) = O(1)$ (D.1)

2. $T'(h_n) = h_n B \operatorname{csch} \beta \cos n\pi + o(h_n)$ (D.2)

3. $T^{(\ell)}(h_n) = O(h_n)$ , $\ell = 1, 2, \ldots$ (D.3)

where $T(h)$ is defined by

$$T(h) = k^{(1)} \coth k^{(1)} \frac{1}{2} + k^{(2)} A \coth k^{(2)} \frac{1}{2} + h B = 0 \tag{D.4}$$

where

$$k^{(1)} = (h^2 - \omega^2 \mu_o \varepsilon)^{1/2} \text{ and } k^{(2)} = (h^2 - \frac{\omega^2 \mu_o \varepsilon}{A})^{1/2}$$

and

$$h_n = \beta + in\pi \tag{D.5}$$

where $\beta$ is defined by

$$\frac{A + 1}{B} \sinh \beta = \cos n\pi - \cosh \beta \tag{D.6}$$

and $n$ is a large odd integer if $\frac{A+1}{B} > 1$ ; a large even integer if $\frac{A+1}{B} < 1$ .

In order to prove statement D.1 let us define

$$S(h,\gamma) = \sqrt{h^2-\gamma^2} \quad \coth \frac{1}{2} \sqrt{h^2-\gamma^2} \quad . \tag{D.7}$$

Expanding $\coth \frac{1}{2}\sqrt{h^2-\gamma^2}$ for large $|h|$ we obtain

$$\coth \frac{1}{2} \sqrt{h^2-\gamma^2} = \coth \frac{h}{2} + \frac{\gamma^2}{4h}(\coth^2 \frac{h}{2} - 1) \left(1 + \frac{\coth \frac{h}{2} + o(1)}{\frac{4h}{\gamma^2} - \coth \frac{h}{2} + o(1)}\right) \quad \text{as } |h| \to \infty \quad . \tag{D.8}$$

Setting $h = h_n$ in expansion D.8 and noting that equation D.6 implies

$$\coth \frac{h_n}{2} = \coth \frac{1}{2} (\beta + in\pi ) = K \tag{D.9}$$

where $K = \frac{-B}{A+1}$, yields

$$\coth \frac{1}{2} \sqrt{h_n^2-\gamma^2} = K + \frac{\gamma^2}{4h_n} (K^2 - 1)(1 + \frac{K + o(1)}{\frac{4h_n}{\gamma^2} - K + o(1)}) \quad \text{as } n \to \infty \tag{D.10}$$

or

$$\coth \frac{1}{2} \sqrt{h_n^2 - \gamma^2} = K + \frac{\gamma^2}{4h_n} (K^2 - 1) + o \left(\frac{1}{h_n}\right) \quad \text{as } n \to \infty \quad . \tag{D.11}$$

Expanding $\sqrt{h_n^2-\gamma^2}$ for large $|h_n|$

$$\sqrt{h_n^2 - \gamma^2} = h_n - \frac{\gamma^2}{2h_n} + o\left(\frac{1}{h_n}\right) \quad \text{as } |h_n| \to \infty \quad . \tag{D.12}$$

Using expansions D.11 and D.12 in equation D.7 defining $S(h,\gamma)$ leads to

$$S(h_n,\gamma) = h_n K + \frac{\gamma^2}{4}(K^2 - 1) + o(1) \qquad \text{as } n \to \infty \quad . \tag{D.13}$$

By definition D.4

$$T(h_n) = S(h_n, \omega\sqrt{\mu_o\varepsilon}) + A\, S\left(h_n, \omega\frac{\sqrt{\mu_o\varepsilon}}{A}\right) + h_n B \quad . \tag{D.14}$$

Thus, using expansion D.13,

$$T(h_n) = h_n (K + A K + B) + O(1) \qquad \text{as } n \to \infty \, . \tag{D.15}$$

Since $K = \frac{-B}{A+1}$, the quantity (K+AK+B) is identically zero; thus

$$T(h_n) = O(1) \qquad \text{as } n \to \infty \, , \tag{D.16}$$

and the proof of statement D.1 is complete.

Now let us consider the proof of statement D.2. Taking the derivative of $S(h,\gamma)$ with respect to h yields

$$\frac{dS(h,\gamma)}{dh} = \frac{h}{h^2 - \gamma^2} S(h,\gamma) - \frac{h}{2} \operatorname{csch}^2 \frac{1}{2}\sqrt{h^2 - \gamma^2} \quad . \tag{D.17}$$

Setting $h = h_n$, expanding $\operatorname{csch} \frac{1}{2}\sqrt{h_n^2 - \gamma^2}$, and using expansion D.13 for $S(h_n,\gamma)$ yields

$$\frac{dS(h_n,\gamma)}{d\ h} = -\frac{h_n}{2}\,\mathrm{csch}^2\,\frac{h_n}{2} + o\ (h_n) \qquad \text{as } n \to \infty \quad . \quad (D.18)$$

From definition D.4,

$$T'(h_n) = \frac{dS(h_n,\omega\sqrt{\mu_o\varepsilon})}{d\ h} + A\ \frac{dS(h_n,\omega\sqrt{\mu_o\varepsilon})}{d\ h} + B; \qquad (D.19)$$

thus using expansion D.18,

$$T'(h_n) = -\frac{(A+1)}{2}\ h_n\ \mathrm{csch}^2\frac{h_n}{2} + o(h_n) \qquad \text{as } n \to \infty. \quad (D.20)$$

From definition D.6 it may be shown that

$$\mathrm{csch}^2\ \frac{h_n}{2} = -\frac{2B}{A+1}\ \mathrm{csch}\ \beta\ \cos\ n\pi \quad . \qquad (D.21)$$

Thus, expansion D.20 reduces to

$$T'\ (h_n) = h_n\ B\ \mathrm{csch}\ \beta\ \cos n\pi + o\ (h_n) \quad , \qquad (D.22)$$

and the proof of statement D.2 is complete.

Let us now consider the proof of the remaining statement, D.3. From the definition of $S(h,\gamma)$ it follows that

$$\frac{d^\ell S(h,\gamma)}{dh^\ell} = \sum_{i=o}^{\ell} c_i\ \left(\frac{d^i}{dh^i}\sqrt{h^2-\gamma^2}\right)\left(\frac{d^j}{dh^j}\ \coth\frac{1}{2}\sqrt{h^2-\gamma^2}\right) \qquad (D.23)$$

where $j = \ell - i$ and $c_i$ are constants. It may be shown that for $n \to \infty$

$$\frac{d^j}{dh^j} \coth \frac{1}{2} \sqrt{h^2 - \gamma^2}\Bigg|_{h=h_n} = O(1) \quad , \quad j = o,1,\ldots \quad \text{as } n \to \infty \tag{D.24}$$

and

$$\frac{d^i}{dh^i} \sqrt{h^2 - \gamma^2}\Bigg|_{h=h_n} = \begin{cases} O(1) & \text{for} \quad i > o \\ O(h_n) & \text{for} \quad i = o \end{cases} \quad \text{as } n \to \infty \ . \tag{D.25}$$

Thus

$$\frac{d^\ell S(h,\gamma)}{dh^\ell}\Bigg|_{h=h_n} = O(h_n) \quad . \tag{D.26}$$

From the definition of T(h)

$$T^{(\ell)}_{(h_n)} = \frac{d^\ell S(h_n,\omega\sqrt{\mu_o\varepsilon}\,)}{dh^\ell} + A\,\frac{d^\ell S(h_n,\omega\,\frac{\sqrt{\mu_o\varepsilon}}{A})}{dh^\ell} + B \quad , \tag{D.27}$$

therefore

$$T^{(\ell)}_{(h_n)} = O(h_n) \quad . \tag{D.28}$$

# APPENDIX E

In this section it is proven that

$$F(x,z) = \int_{-\infty}^{\infty} e^{ihz} I(x,h)\, dh \tag{E.1}$$

is a continuous function of z for I(x,h) defined by equations 8.16-8.18.

Clearly $e^{ihz} I(x,h)$ is continuous in z and $|e^{ihz} I(x,h)| \leq |I(x,h)|$ for all real h and z, thus it is sufficient to show

$$\int_{-\infty}^{\infty} |I(x,h)| \, dh < \infty \quad . \tag{E.2}$$

From the work in chapter 8 we know that I(x,h) is a bounded function for all h such that $T(h) \neq 0$ . (T(h) is defined by equation 8.13.) As already discussed in chapters 7 and 8, if $T(h) = 0$ for some real h, the integral must be interpreted by inserting loss such that $T(h) \neq 0$ for all real h. With this interpretation, I(x,h) is bounded for all real h, $\int_{-L}^{L} |I(x,h)|\, dh < \infty$ for any real $L < \infty$, and thus proving inequality E.2 reduces to proving

$$\int_{-\infty}^{-L} |I(x,h)| \, dh + \int_{+L}^{+\infty} |I(x,h)| \, dh < \infty \tag{E.3}$$

for any finite real $L > 0$.

Applying equation 8.17 for the region $x < d$*

$$|I(x,h)| \leqslant \left| \int_0^x J(\xi) g_2(x,\xi,h)\, d\xi \right| + \left| \int_x^d J(\xi)\, g_1(x,\xi,h) d\xi \right|$$

$$+ \left| \int_d^1 J(\xi)\, g_4(x,\xi,h) d\xi \right|$$

$$\leqslant \quad |I^{(2)}(x,h)| + |I^{(1)}(x,h)| + |I^{(4)}(x,h)| \; . \qquad (E.4)$$

Consider $I^{(2)}(x,h)$ . Clearly there exists a finite real $L > 0$ such that for $|h| \geqslant L$ and $x < d$ both $k^{(1)}$ and $k^{(2)}$ are real and

$$|\coth k^{(1)} (x-d)| < M/3$$

$$\left| \frac{k^{(2)}}{k^{(1)}} A \coth k^{(2)} (d-1) \right| < M/3$$

$$\left| \frac{hB}{k^{(1)}} \right| < M/3$$

$$\left| \frac{1}{\sinh k^{(1)} d} \right| \quad M\, e^{-k^{(1)} d}$$

$$|\sinh k^{(1)} (x-d)| < M\, e^{k^{(1)}(d-x)}$$

$$\left| \frac{1}{T(h)} \right| < \frac{M}{|h|}$$

---

* We will confine our consideration to the region $x < d$; an analogous argument may be applied for $x > d$.

for some finite real number M. Using expression 8.12b for $g_2(x,\xi,h)$ then yields

$$|I^{(2)}(x,h)| < \frac{M^4}{|h|} e^{-k^{(1)}x} \left| \int_o^x J(\xi) \sinh k^{(1)} \xi \, d\xi \right| \qquad (E.5)$$

Assuming $J'(\xi)$ exists for $o \leq \xi \leq x \leq d$, the integral in equation E.5 can be integrated by parts

$$|I^{(2)}(x,h)| < \frac{M^4}{|h|} e^{-k^{(1)}x} \left| \frac{J(x) \cosh k^{(1)}x - J(o)}{k^{(1)}} \right.$$
$$\left. - \frac{1}{k^{(1)}} \int_o^x J'(\xi) \cosh k^{(1)}\xi d\xi \right| . \qquad (E.6)$$

Again applying the maximum values, equation E.6 can be reduced to

$$|I^{(2)}(x,h)| < \frac{\text{const.}}{h^2} \quad . \qquad (E.7)$$

In a like manner it may be shown that there exists a finite real number L such that for $|h| \geq L$

$$|I^{(i)}(x,h)| < \frac{\text{const.}}{h^2} \quad \text{for} \quad i = 1,4 \quad . \qquad (E.8)$$

Thus for $|h| \geq L$

$$|I(x,h)| < \frac{\text{const.}}{h^2} \quad .$$

Clearly

$$\int_{\pm L}^{\pm \infty} \frac{\text{const}}{h^2} \, dh < \infty$$

thus inequality E.3 is true and we have proven that F(x,z) is a continuous function in z.

# APPENDIX F

In this section it is shown that there exists a set of functions J(x) such that

$$c_o \neq 0$$
$$c_n = o(e^{-n\pi\ell}) \qquad \text{for} \quad \text{Im } h_n > 0 \tag{F.1}$$

where the $c_n$ are defined by

$$c_n = \frac{1}{T'(h_n)} \int_o^1 J(x)\ f_n(x)\ dx \tag{F.2}$$

and the $f_n$ $(n \neq o)$ are the eigenfunctions of system 5.3 with complex eigenvalues $h_n$, and $f_o$ is the eigenfunction with real eigenvalue $h_o$ .

We will first prove that it is sufficient to show the $f_n$ (Im $h_n > o$) are independent. If $f_n$ (Im $h_n \geq o$) are independent (this clearly implies the $f_n$ (Im $h_n > o$) are independent), there exists a biothogonal set $S_m$ $(m = o,1,\ldots)$ such that

$$\int_o^1 f_n(x)\ S_m(x)\ dx = \delta_{nm} \quad (\text{Im } h_n \geq 0) \ . \tag{F.3}$$

Let

$$J(x) = \sum_{m=o}^{\infty} d_m\ S_m(x) \ . \tag{F.4}$$

Using the definition F.2 of $c_n$

$$c_n = \frac{1}{T'(h_n)} \int_o^1 \sum_{m=o}^{\infty} d_m f_n(x) S_m(x)\, dx \quad . \tag{F.5}$$

Assuming the $d_m$ are chosen sufficiently small as $m \to \infty$, the summation and integration may be interchanged to yield

$$c_n = d_n/T'(h_n) \tag{F.6}$$

where orthogonality relationship F.3 has been used. Applying the asymptotic form of $T'(h_n)$ as given by equation 9.12, equation F.6 becomes

$$c_n \sim \text{const.}\ d_n/h_n \quad . \tag{F.7}$$

From equation F.7 it is clear that $c_n = o(e^{-n\pi\ell})$ if $d_n = o(h_n e^{-n\pi\ell})$. Thus by choosing $d_o \neq o$ and $d_n = o(h_n\ e^{-n\pi\ell})$ it is clear that the J(x) determined by expansion F.4 yields $c_n$ such that $c_o \neq o$ and $c_n = o(e^{-n\pi\ell})$ for Im $h_n$>o.

Now let us consider another possibility. Suppose the $f_n$ (Im $h_n$>o) are independent but the $f_n$ (Im $h_n \geq o$) are dependent, that is, there exist $p_n$ such that

$$f_o(x) = \sum_{\text{Im}\ h_n > o} p_n\ f_n(x) \quad . \tag{F.8}$$

Since the $f_n$ (Im $h_n$>o) are independent, there exists a set $S_m(m = 1,2,\ldots)$ such that

$$\int_o^1 f_n(x)\ S_m(x)\ dx = \delta_{nm} \qquad (\mathrm{Im}\ h_n>o) \quad . \qquad (F.9)$$

Let

$$J(x) = \sum_{m=1}^{\infty} d_m\ S_m(x) \quad . \qquad (F.10)$$

Then for Im $h_n>o$

$$c_n = d_n/T'(h_n) \quad , \qquad (F.11)$$

and for Im $h_n=o$

$$c_o = \frac{1}{T'(h_n)} \sum_{\mathrm{Im}\ h_n>o} p_n\ d_n \quad . \qquad (F.12)$$

In this case it is also possible to choose a set of $d_n$ defining $J(x) = \sum_{n=1} d_n S_n$ such that $c_o \neq 0$ and $c_n = o(e^{-n\pi\ell})$ for Im $h_n>o$.

We have thus shown that it is sufficient to prove that the $f_n$ (Im $h_n>o$) are independent. In order to show that the $f_n$ (Im $h_n>o$) are independent we will accept without proof the physically reasonable assumption that the fields produced by a current source in an infinitely long ferrite loaded guide are unique. That is, we will assume that the previously discussed problem in figure 8.1 has a unique solution given by

$$E_y(x,z) = \begin{cases} \sum_{\mathrm{Im}\ h_n\geq o} a_n f_n\ e^{ih_n z} & , z>o \\ \sum_{\mathrm{Im}\ h_n<o} b_n f_n\ e^{ih_n z} & , z<o \end{cases} \qquad (F.13)$$

where the $a_n$, $b_n$ are defined by equation 8.23.

Since $E_y(x,z)$ as given in equation F.13 is a solution of the problem in figure 8.1, it must satisfy the following requirements:

(1) $\sum_{\text{Im } h_n \geq o} a_n f_n e^{ih_n z}$ , $\sum_{\text{Im } h_n < o} b_n f_n e^{ih_n z}$ solve Maxwell's equations for z>o and z<o respectively.

(2) $\sum_{\text{Im } h_n \geq o} a_n f_n(x) = \sum_{\text{Im } h_n < o} b_n f_n(x)$ *

(3) $\sum_{\text{Im } h_n \geq o} a_n I_n - \sum_{\text{Im } h_n < o} b_n I_n = \int_o^1 J(x)\, dx$ **

where

$$I_n = \frac{1}{\omega\mu_o} \int_o^1 \left(\nu_1 h_n + \frac{d\nu_2}{dx}\right) f_n(x)\, dx$$

We will now show that if the $f_n$ (Im $h_n$>o) are dependent, the above requirements may be satisfied by solutions other than equation F.13. That is, under this assumption, the solution of the problem in figure 8.1 is not unique.

If the $f_n$ (Im $h_n$>o) are dependent, there exists a set of $e_n$ such that

$$\sum_{\text{Im } h_n \geq o} e_n f_n = 0 \quad \text{where } e_o \equiv o \quad . \tag{F.14}$$

Using this set of $e_n$, consider

$$E_y(x,z) = \begin{cases} \sum_{\text{Im } h_n \geq o} (a_n + \sigma e_n)\, f_n\, e^{ih_n z} & z>o \\ \sum_{\text{Im } h_n < o} (b_n + \gamma \bar{e}_n)\, f_n\, e^{ih_n z} & z<o \quad . \end{cases} \tag{F.15}$$

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*This relationship is obtained by matching tangential E at z=o.

**This relationship is obtained by relating the current and magnetic fields at z=0.

Clearly equation F.15 satisfies requirement (1) providing the $e_n$ are such that the summation F.14 is uniformly convergent. It is also clear that equation F.15 satisfies requirement (2) since by definition $\sum_{Im\, h_n \geq o} e_n f_n = \sum_{Im\, h_n < o} \bar{e}_n f_n = 0$.* It remains to consider requirement (3). If we can find non-zero constants $\sigma$, $\gamma$ such that

$$\sum_{Im\, h_n \geq o} (a_n + \sigma e_n)\, I_n - \sum_{Im\, h_n < o} (b_n + \gamma \bar{e}_n)\, I_n = \int_o^1 J(x)\, dx \qquad (F.16)$$

we must conclude equation F.15 is also a solution of the boundary value problem in figure 8.1. Using condition (3), equation F.16 reduces to

$$\sigma \sum_{Im\, h_n \geq o} e_n I_n - \gamma \sum_{Im\, h_n < o} \bar{e}_n I_n = 0 \quad . \qquad (F.17)$$

Since $\sum_{Im\, h_n \geq o} e_n I_n = \text{const.}$ and $\sum_{Im\, h_n < o} \bar{e}_n I_n = \text{const.}$, it is clear that there exists constants $\sigma$ and $\gamma$ , not both identically zero, which satisfy equation F.17. Thus we must conclude $f_n$ ($Im\, h_n > o$) are independent since if they are not, the assumption of the uniqueness of the fields produced by a current source in an infinitely long ferrite loaded guide is violated.

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*Since $\sum_{Im\, h_n \geq o} e_n f_n = 0$ and the eigenfunctions $f_n$ occur in complex conjugate pairs, it follows that $\sum_{Im\, h_n < 0} \bar{e}_n f_n = o$ .

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