HARMONIC OSCILLATIONS OF A NARROW DELTA WING IN SUPERSONIC FLOW

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ABSTRACT

A theory is presented for the calculation of the velocity potential of a harmonically oscillating delta wing having subsonic leading edges in a supersonic flow. The velocity potential is expanded in a power series in powers of the reduced frequency. Two modes of oscillation, plunging and pitching, are considered. For both modes the analysis is carried through the term linear in reduced frequency, this being generally sufficient for dynamic stability analyses. The results thus obtained for the pitching mode verify those of Miles (Ref. 9) obtained by an integral transformation of the steady-state solution. In addition, the term that is quadratic in the reduced frequency is presented for the plunging mode to illustrate the general procedure.

Lift and pitching moment coefficients are calculated from the velocity potential and numerical results valid for low frequency oscillations are presented.

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I. INTRODUCTION

As a consequence of the current rate of increase of the high speed performance of aircraft, the problems of dynamic stability and flutter of finite wings moving at supersonic speeds are becoming increasingly important. These problems have yielded to analysis for certain planforms (Ref. 1, Ref. 2, Ref. 3). However, although procedures for their solution in the case of the delta wing with both leading edges inside the Mach cone from the wing vertex have been outlined (Ref. 4 and Ref. 5) and have, by one author, been partially carried out (Ref. 6), no practical engineering solution has been obtained. It is the purpose of this thesis to present such procedures for a narrow delta wing performing (i) harmonic plunging oscillations and (ii) harmonic pitching oscillations in a supersonic stream.

II. ASSUMPTIONS

The development presented in this thesis is based upon the following assumptions:

i) The wing under consideration is flat.

ii) The wing has zero thickness.

iii) Small perturbations are assumed:

a) The fluid at each point on the surface of the wing moves with a velocity whose vector representation makes a small angle with the plane $z = 0$.

b) The motion of the wing is such that at all times any point on the wing lies at a distance from the plane $z = 0$ which is small compared with c, so that all boundary conditions in the plane of the wing may be applied in the plane $z = 0$.

iv) The flow is isentropic and non-viscous.

Assumptions iii) and iv) allow the existence of a velocity potential, ϕ, and the application of tangency of flow to the wing surface as a boundary condition.

Because of assumption iii) the equation that must be solved is linear.

The boundary condition off of the wing but in the plane of the wing follows from the fact that, because of assumption ii), ϕ_z is even in z and ϕ_x and ϕ_y are odd in z.

III. GENERAL THEORY

The coordinate system and wing that will be considered are shown in Figure 1.

Making use of the assumptions stated in Part I, the equations of motion, continuity and energy may be combined into

$$\frac{1}{a^2}\phi_{\bar{t}\bar{t}} + \frac{2U}{a^2}\phi_{x\bar{t}} + \beta^2\phi_{xx} - \phi_{yy} - \phi_{zz} = 0 \tag{3-1}$$

Harmonic oscillations of the wing are to be considered, therefore a potential of the form $\phi\,(x,y,z,\bar{t}) = \varphi\,(x,y,z)\,e^{i\omega\bar{t}}$ is assumed, and if the space coordinates are normalized by the introduction of $\xi = \frac{x}{c}$, $\eta = \frac{y}{c}$, $\zeta = \frac{z}{c}$ equation (3-1) becomes

$$\beta^2\varphi_{\xi\xi} - \varphi_{\eta\eta} - \varphi_{\zeta\zeta} = \frac{\omega^2c^2}{a^2}\varphi - i\,\frac{2U\omega c}{a^2}\varphi_{\xi} \tag{3-2}$$

Introducing the notation $\nu = \frac{\omega c}{a}$, $M = \frac{U}{a}$ the equation is, finally

$$\beta^2\varphi_{\xi\xi} - \varphi_{\eta\eta} - \varphi_{\zeta\zeta} = \nu^2\varphi - 2iM\nu\varphi_{\xi} \tag{3-3}$$

Initially, an expansion of $\varphi(\xi,\eta,\zeta)$ in the powers of the reduced frequency, ν , will be assumed to exist,

$$\varphi(\xi,\eta,\zeta) = \sum_{n=0}^{\infty}\nu^n\varphi^{(n)}(\xi,\eta,\zeta) \tag{3-4}$$

Substitution of this expansion into equation (3-3) and the equating of like powers of ν on both sides of the equation yields the following infinite set of equations

$$\begin{aligned}
&(a) \quad \beta^2 \varphi^{(0)}_{\xi\xi} - \varphi^{(0)}_{\eta\eta} - \varphi^{(0)}_{\zeta\zeta} = 0 \\
&(b) \quad \beta^2 \varphi^{(1)}_{\xi\xi} - \varphi^{(1)}_{\eta\eta} - \varphi^{(1)}_{\zeta\zeta} = -2iM\varphi^{(0)}_{\xi} \qquad (3\text{-}5) \\
&(c) \quad \beta^2 \varphi^{(n)}_{\xi\xi} - \varphi^{(n)}_{\eta\eta} - \varphi^{(n)}_{\zeta\zeta} = \varphi^{(n-2)} - 2iM\varphi^{(n-1)}_{\xi}\,, \quad n \geq 2
\end{aligned}$$

IV. PLUNGING OSCILLATION

Consider a narrow delta wing performing pure harmonic plunging oscillations. The downwash on the wing surface is given by $w = w_0 e^{i\omega\bar{t}}$ and therefore, in addition to the specification that the potential and all perturbation velocities vanish on the Mach cone, boundary conditions can be specified for the following regions:

(i) S ⌙ the wing surface

(ii) R ⌙ the region in the plane of the wing that lies between the wing leading edge and the Mach cone

A. Boundary Conditions

The boundary conditions for equation (3-1) are

On S: $\phi_z = -w(x,y)e^{i\omega\bar{t}} = \varphi_z e^{i\omega\bar{t}}$ where $w = w_0$ = constant

On R: Following the consequences of assumption ii), the pressure must vanish since no discontinuities are tenable in the fluid off of the wing.

The boundary conditions for equations (3-5) follow from those given above and the transformations employed in deriving equations (3-5).

On S: $\varphi_\zeta^{(o)} = -w_0 c$ = constant; $\varphi_\zeta^{(n)} = 0$ for $n \geq 1$

On R: $\frac{p_0 - p}{\rho_0} = \phi_{\bar{t}} + U\phi_x = e^{i\omega\bar{t}}\frac{a}{c}\left[i\nu\varphi + M\varphi_\xi\right] = 0$

Employing equation (3-4) and equating like powers of ν :

$$i\varphi^{(o)} + M\varphi_\xi^{(1)} = 0$$

$\varphi^{(o)}$ is the potential for the narrow delta wing in stationary flow and is equal to zero on R.

Therefore,

$$\varphi_\xi^{(1)} = 0 \qquad \underline{\text{on R}}$$

Repetition of this procedure of equating like powers of ν yields

$$\varphi_\xi^{(n)} = \varphi^{(n)} = 0 \qquad \underline{\text{on R}}$$

To summarize the boundary conditions for equations (3-5)

(i) <u>On S:</u> $\varphi_\xi^{(o)} = -W_o C$ = constant

for n ≥ 1

(4-1)

(ii) <u>On R:</u> $\varphi^{(n)} = \varphi_\xi^{(n)} = 0$ for n ≥ 0

(iii) <u>On Mach cone :</u> $\varphi^{(n)} = \varphi_\xi^{(n)} = \varphi_\eta^{(n)} = \varphi_\zeta^{(n)} = 0$ for n ≥ 0

B. Solutions of Equations (3-5)

$\varphi^{(o)}$, satisfying equation (3-5a) and its boundary conditions, is the potential for a narrow delta wing in stationary supersonic flow (Ref. 7).

Since equations (3-5b) and (3-5c) are linear and non-homogeneous, the solution of each may be considered to be the sum of a particular integral, $\varphi_p^{(n)}$, and a complementary solution, $\varphi_c^{(n)}$. The boundary conditions to be satisfied by $\varphi_c^{(n)}$ are dictated by equations (4-1) and the values assumed by $\varphi_p^{(n)}$ and $\varphi_{p_\zeta}^{(n)}$ on the appropriate boundaries. In deriving the particular integrals extensive use is made of the fact that, from equations (3-5), equations (4-1) and the knowledge that $\varphi^{(o)}$ is homogeneous of degree 1, $\varphi^{(n)}$ is homogeneous of degree (n+1) in the space variables. The detailed derivations of pertinent particular integrals is presented in the appendix; only final results are presented below.

$\varphi^{(1)}$: The particular integral for equation (3-5b) is

$$\varphi_p^{(1)} = -\frac{iM}{\beta^2}\zeta\varphi^{(o)} \tag{4-2}$$

Now, $\varphi_p^{(1)}$ satisfies boundary conditions (4-1ii) and (4-1iii), but on S

$$\varphi_{p_\zeta}^{(1)} = \frac{iMw_oc}{\beta^2}\zeta \tag{4-3}$$

Therefore, a solution, $\varphi_c^{(1)}$, of the homogeneous wave equation must be found such that

$$\begin{array}{ll} \text{(i) } \underline{\text{On S:}} & \varphi_{c_\zeta}^{(1)} = -\frac{iM}{\beta^2}w_oc\zeta \\ \text{(ii) } \underline{\text{On R:}} & \varphi_c^{(1)} = 0 \\ \text{(iii) } \underline{\text{On Mach cone:}} & \varphi_c^{(1)} = 0 \end{array} \tag{4-4}$$

Such a solution is that presented in reference 8 for a pitching narrow delta wing where the "pitching" velocity, q, of that paper is to be replaced by $-\frac{iM}{\beta^2}w_oc$. From reference 8, then

$$\varphi_c^{(1)} = \frac{iMw_oc}{\beta^2 b}C_1\zeta\sqrt{\xi^2-b^2\eta^2} \qquad \underline{\text{on S}} \tag{4-5}$$

where $C_1 = \left[\frac{1-2k^2}{1-k^2}E(k')+\frac{k^2}{1-k^2}K(k')\right]^{-1}$ (see Fig.3). Therefore,

$$\varphi^{(1)} = -\frac{iM}{\beta^2}\zeta\varphi^{(o)} + \frac{iMw_oc}{\beta^2 b}C_1\zeta\sqrt{\xi^2-b^2\eta^2} \qquad \underline{\text{on S}} \tag{4-6}$$

and, defining $u^{(n)}=\varphi_x^{(n)}=\frac{1}{c}\varphi_\xi^{(n)}$,

$$u^{(1)}=-\frac{iM}{\beta^2 c}\left[\varphi^{(0)}+\xi\varphi_\xi^{(0)}\right]+\frac{iMw_0}{\beta^2 b}C_1\left[\sqrt{\xi^2-b^2\eta^2}+\frac{\xi^2}{\sqrt{\xi^2-b^2\eta^2}}\right] \qquad \underline{\text{on S}} \qquad (4\text{-}7)$$

$\varphi^{(n)}$ $(n \geq 2)$: Use will be made of their homogeneity properties in determining these components.

Since $\varphi^{(n)}$ is homogeneous of degree (n+1),

$$(n+1)\varphi^{(n)}=x\varphi_x^{(n)}+y\varphi_y^{(n)}+z\varphi_z^{(n)}=\xi\varphi_\xi^{(n)}+\eta\varphi_\eta^{(n)}+\zeta\varphi_\zeta^{(n)} \qquad (4\text{-}8)$$

On S, where $\zeta=0$

$$\varphi^{(n)}=\frac{1}{n+1}\left[\xi\varphi_\xi^{(n)}+\eta\varphi_\eta^{(n)}\right] \qquad (4\text{-}9)$$

Now, differentiating equation (3-5c) with respect to x and y and defining $u^{(n)}=\varphi_x^{(n)}$ and $v^{(n)}=\varphi_y^{(n)}$, the following equations are obtained:

$$\text{(i)} \quad \beta^2 u_{\xi\xi}^{(n)}-u_{\eta\eta}^{(n)}-u_{\zeta\zeta}^{(n)}=\frac{1}{c}\varphi_\xi^{(n-2)}-2i\frac{M}{c}\varphi_{\xi\xi}^{(n-1)}$$

$$(4\text{-}10)$$

$$\text{(ii)} \quad \beta^2 v_{\xi\xi}^{(n)}-v_{\eta\eta}^{(n)}-v_{\zeta\zeta}^{(n)}=\frac{1}{c}\varphi_\eta^{(n-2)}-2i\frac{M}{c}\varphi_{\xi\eta}^{(n-1)}$$

The boundary conditions for these equations are easily derived from equations (4-1). On R and on the Mach cone $u^{(n)}=v^{(n)}=0$ and on S

$$u_\zeta^{(n)}=v_\zeta^{(n)}=0 \qquad n \geq 0 \qquad (4\text{-}11)$$

In accordance with previous discussion, the solutions of equations (4-10i) and (4-10ii) will be

$$u^{(n)}=u_p^{(n)}+u_c^{(n)}$$

$$(4\text{-}12)$$

$$v^{(n)}=v_p^{(n)}+v_c^{(n)}$$

For n = 2,

$$u_p^{(2)}=-\frac{1}{2\beta^4 c}\xi\varphi^{(0)}+\frac{M^2}{2\beta^4 c}\xi^2\varphi_\xi^{(0)}-\frac{iM}{\beta^2 c}\xi\varphi_\xi^{(1)} \qquad (4\text{-}13)$$

It can be seen that $u_p^{(2)}=0$ on R and on the Mach cone, but on S

$$u_{p_\zeta}^{(2)}=\frac{w_0}{2\beta^4}\xi \qquad (4\text{-}14)$$

Therefore, a solution of the homogeneous wave equation, $U_c^{(2)}$, must be found such that

$$\begin{aligned} &\underline{\text{On S:}} && U_{c_\xi}^{(2)} = -\frac{W_0}{2\beta^4}\xi \\ &\underline{\text{On R:}} && U_c^{(2)} = 0 \\ &\underline{\text{On Mach cone:}} && U_c^{(2)} = 0 \,. \end{aligned} \tag{4-15}$$

The desired solution will again be that presented in reference 8 for the pitching delta wing, but with q replaced by $-\frac{W_0}{2\beta^4}$:

$$U_c^{(2)} = \frac{W_0}{2\beta^4 b} C_1 \xi \sqrt{\xi^2 - b^2\eta^2} \qquad \underline{\text{on S}} \tag{4-16}$$

so that

$$U^{(2)} = -\frac{1}{2\beta^4 c}\xi\varphi^{(0)} + \frac{M^2}{2\beta^4 c}\xi^2\varphi_\xi^{(0)} - \frac{iM}{\beta^2 c}\xi\varphi_\xi^{(1)} + \frac{W_0}{2\beta^4 b} C_1 \xi\sqrt{\xi^2 - b^2\eta^2} \qquad \underline{\text{on S}}\,. \tag{4=17}$$

$V_p^{(2)}$ is given by

$$V_p^{(2)} = \frac{1}{2\beta^2 c}\eta\varphi^{(0)} + \frac{M^2}{2\beta^4 c}\xi^2\varphi_\eta^{(0)} - \frac{iM}{\beta^2 c}\xi\varphi_\eta^{(1)} \,. \tag{4-18}$$

This particular integral equals zero on R and on the Mach cone, but on S

$$V_{p_\xi}^{(2)} = -\frac{W_0}{2\beta^2}\eta \,. \tag{4-19}$$

Therefore, $V_c^{(2)}$ must satisfy the homogeneous wave equation and the following boundary conditions:

$$\begin{aligned} &\underline{\text{On S:}} && V_{c_\xi}^{(2)} = \frac{W_0}{2\beta^2}\eta \\ &\underline{\text{On R:}} && V_c^{(2)} = 0 \\ &\underline{\text{On Mach Cone:}} && V_c^{(2)} = 0 \end{aligned} \tag{4-20}$$

In this case, the desired complementary solution is that presented in reference 8 for the rolling narrow delta wing with the "rolling" velocity, p, of that paper replaced by $\frac{W_0}{2\beta^2}$. Then,

$$V_c^{(2)} = -\frac{W_0}{2\beta^2 b} C_2 \eta \sqrt{\xi^2 - b^2\eta^2} \qquad \underline{\text{on S}} \tag{4-21}$$

where $C_2 = \left[\frac{2-k^2}{1-k^2}E(k') - \frac{k^2}{1-k^2}K(k')\right]^{-1}$ (cf. Fig. 3), and

$$V^{(2)} = \frac{1}{2\beta^2 c}\eta\varphi^{(0)} + \frac{M^2}{2\beta^4 c}\xi^2\varphi_\eta^{(0)} - \frac{iM}{\beta^2 c}\xi\varphi_\eta^{(1)} - \frac{W_0}{2\beta^2 b}C_2\eta\sqrt{\xi^2 - b^2\eta^2} \qquad \underline{\text{on S}} \tag{4-22}$$

Therefore, from equation (4-9),

$$\varphi^{(2)} = \frac{c}{3}\left[-\frac{1}{2\beta^2 c}\xi\varphi^{(0)} + \frac{M^2}{2\beta^4 c}\xi^3\varphi_\xi^{(0)} - \frac{iM}{\beta^2 c}\xi^2\varphi_\xi^{(1)} + \frac{W_0}{2\beta^4 b}C_1\xi^2\sqrt{\xi^2 - b^2\eta^2} + \frac{1}{2\beta^2 c}\eta^2\varphi^{(0)}\right.$$
$$\left. + \frac{M^2}{2\beta^4 c}\xi^2\eta\varphi_\eta^{(0)} - \frac{iM}{\beta^2 c}\xi\eta\varphi_\eta^{(1)} - \frac{W_0}{2\beta^2 b}C_2\eta^2\sqrt{\xi^2 - b^2\eta^2}\right] \qquad \underline{\text{on S}} \tag{4-23}$$

The determination of $\varphi^{(n)}$ for n > 2 is based upon equation (4-9). $U_p^{(n)}$ and $V_p^{(n)}$ are derived in a manner completely analogous to that presented in the appendix for $U_p^{(2)}$ and $V_p^{(2)}$. As stated above, the boundary conditions that must be satisfied by $U_c^{(n)}$ and $V_c^{(n)}$ are partially specified by $U_{p_\xi}^{(n)}$ and $V_{p_\xi}^{(n)}$. The expressions for the latter two functions will be polynomials in ξ and η. The general method presented in reference 8 is then utilized to determine $U_c^{(n)}$ and $V_c^{(n)}$.

To summarize the pertinent values on the wing:

$$\varphi^{(0)} = \frac{W_0 c}{bE(k')}\sqrt{\xi^2 - b^2\eta^2} \tag{4-24}$$

$$\varphi^{(1)} = \frac{iMW_0 c}{\beta^2 b}\left[C_1 - \frac{1}{E(k')}\right]\xi\sqrt{\xi^2 - b^2\eta^2} \tag{4-25}$$

$$\varphi^{(2)} = \frac{c}{3}\left\{\frac{W_0}{\beta^4 b}\left[M^2\left(2C_1 - \frac{3}{2E(k')}\right) + \frac{1}{2}\left(C_1 + \frac{1}{E(k')}\right)\right]\xi^2\sqrt{\xi^2 - b^2\eta^2}\right.$$
$$\left. - \frac{W_0}{2\beta^2 b}\left(C_2 - \frac{1}{E(k')}\right)\eta^2\sqrt{\xi^2 - b^2\eta^2}\right\} \tag{4-26}$$

$$U^{(0)} = \frac{W_0}{bE(k')}\frac{\xi}{\sqrt{\xi^2 - b^2\eta^2}} \tag{4-27}$$

$$U^{(1)} = \frac{iMW_0}{\beta^2 b}\left[C_1 - \frac{1}{E(k')}\right]\left[\sqrt{\xi^2 - b^2\eta^2} + \frac{\xi^2}{\sqrt{\xi^2 - b^2\eta^2}}\right] \tag{4-28}$$

$$U^{(2)} = \frac{W_0}{2\beta^4 b}(1+2M^2)\left[C_1 - \frac{1}{E(k')}\right]\xi\sqrt{\xi^2 - b^2\eta^2} + \frac{M^2 W_0}{\beta^4 b}\left[C_1 - \frac{1}{E(k')}\right]\frac{\xi^3}{\sqrt{\xi^2 - b^2\eta^2}} \tag{4-29}$$

V. PITCHING OSCILLATION

Consider a narrow delta wing pitching harmonically about an axis located a distance mc downstream of the wing vertex (Fig. 2).

A. Boundary Conditions

The instantaneous angle of attack is given by

$$\alpha = \alpha_o e^{i\omega \bar{t}} \tag{5-1}$$

where α_o = constant. The z coordinate of any point on the wing is

$$z_w = -\alpha(x - mc) \tag{5-2}$$

and the downwash necessary to preserve flow tangent to the wing surface is

$$\phi_z = \frac{\partial z_w}{\partial \bar{t}} + U\frac{\partial z_w}{\partial x} = -\alpha_o e^{i\omega \bar{t}}\left[i\omega(x - mc) + U\right] \tag{5-3}$$

or

$$\varphi_z = \frac{iU\alpha_o}{M}\left(m - \frac{x}{c}\right)\nu - U\alpha_o \qquad \underline{\text{on S}} \tag{5-4}$$

Therefore,

$$\varphi_\zeta = \sum_{n=0}^{\infty} \nu^n \varphi_\zeta^{(n)} = \frac{iU\alpha_o c}{M}(m - \zeta)\nu - U\alpha_o \qquad \underline{\text{on S}} \tag{5-5}$$

$\varphi^{(n)} = 0$ on R and on the Mach cone, and the boundary conditions on S for equations (3-5) are obtained by equating like powers of ν in equation (5-5).

$$\underline{\text{on S:}} \qquad \varphi_\xi^{(0)} = -U\alpha_0 c \qquad = \text{constant}$$

$$\varphi_\xi^{(1)} = \frac{iU\alpha_0 c}{M}(m-\xi) \tag{5-6}$$

$$\varphi_\xi^{(n)} = 0 \qquad \text{for } n \geqslant 2$$

B. Solutions of Equations (3-5)

Equation (3-5a) and its boundary conditions developed above determine $\varphi^{(0)}$ as the potential for a narrow delta wing in stationary flow.

The solution of equation (3-5b) for the pitching wing will again be the sum of a particular integral, $\varphi_p^{(1)}$, and a complementary solution, $\varphi_c^{(1)}$. $\varphi_p^{(1)}$ will, of course, be the same as for Part IV:

$$\varphi_p^{(1)} = -\frac{iM}{\beta^2}\xi\varphi^{(0)} \tag{5-7}$$

This, as stated previously, satisfies the boundary conditions on R and on the Mach cone. On S

$$\varphi_{p_\xi}^{(1)} = \frac{iMU\alpha_0 c}{\beta^2}\xi \tag{5-8}$$

Therefore, $\varphi_c^{(1)}$ must be a solution of the homogeneous wave equation that vanishes on R and on the Mach cone such that on S: $$\varphi_{c_\xi}^{(1)} = -i\,\frac{2M^2-1}{M\beta^2}\,U\alpha_0 c\,\xi + \frac{iU\alpha_0 c}{M}\,m \tag{5-9}$$

This complementary solution may be considered to be the sum of two such solutions,

$$\varphi_c^{(1)} = \varphi_c^{(1)\prime} + \varphi_c^{(1)\prime\prime} \tag{5-10}$$

where $\varphi_c^{(1)\prime}$ and $\varphi_c^{(1)\prime\prime}$ vanish on R and on the Mach cone and

on S: $$\begin{aligned} \varphi_{c_\xi}^{(1)\prime} &= \frac{iU\alpha_0 c}{M}\,m \\ \varphi_{c_\xi}^{(1)\prime\prime} &= -i\,\frac{2M^2-1}{M\beta^2}\,U\alpha_0 c\,\xi \end{aligned} \tag{5-11}$$

The first, $\varphi_c^{(1)\prime}$, is evidently a multiple of $\varphi^{(o)}$:

$$\varphi_c^{(1)\prime} = -\frac{im}{M}\,\varphi^{(o)} \tag{5-12}$$

The second is of the same form as $\varphi_c^{(1)}$ for the plunging wing but with a different "pitching" velocity.

On S: $$\varphi_c^{(1)\prime\prime} = i\,\frac{2M^2-1}{M\beta^2}\,\frac{U\alpha_0 c}{b}\,K\,\xi\sqrt{\xi^2-b^2\eta^2} \tag{5-13}$$

Therefore, on S

$$\varphi^{(1)} = -\frac{iM}{\beta^2}\,\xi\varphi^{(o)} - \frac{im}{M}\,\varphi^{(o)} + i\,\frac{2M^2-1}{M\beta^2}\,\frac{U\alpha_0 c}{b}\,K\,\xi\sqrt{\xi^2-b^2\eta^2} \tag{5-14}$$

Then,

$$u^{(1)} = -\frac{iM}{\beta^2 c}\left[\varphi^{(o)} + \xi\varphi_\xi^{(o)}\right] - \frac{im}{Mc}\,\varphi_\xi^{(o)} + i\,\frac{2M^2-1}{M\beta^2}\,\frac{U\alpha_0}{b}\,K\left[\sqrt{\xi^2-b^2\eta^2} + \frac{\xi^2}{\sqrt{\xi^2-b^2\eta^2}}\right] \tag{5-15}$$

Only the first two terms of the series will be presented for the pitching case since the velocity potential may be represented as $\varphi^{(0)}+\nu\varphi^{(1)}$ with sufficient accuracy for use in stability analyses.

The values of the pertinent functions on the wing surface are

$$\varphi^{(0)}=\frac{U\alpha_0 c}{bE(k')}\sqrt{\xi^2-b^2\eta^2} \tag{5-16}$$

$$\varphi^{(1)}=-\left[\frac{iM}{\beta^2}\xi+\frac{im}{M}\right]\frac{U\alpha_0 c}{bE(k')}\sqrt{\xi^2-b^2\eta^2}+i\,\frac{2M^2-1}{M\beta^2}\,\frac{U\alpha_0 c}{b}K\xi\sqrt{\xi^2-b^2\eta^2} \tag{5-17}$$

$$u^{(0)}=\frac{U\alpha_0}{bE(k')}\,\frac{\xi}{\sqrt{\xi^2-b^2\eta^2}} \tag{5-18}$$

$$u^{(1)}=-\left[\frac{iM}{\beta^2}\xi+\frac{im}{M}\right]\frac{U\alpha_0}{bE(k')}\,\frac{\xi}{\sqrt{\xi^2-b^2\eta^2}}-\frac{iM}{\beta^2}\,\frac{U\alpha_0}{bE(k')}\sqrt{\xi^2-b^2\eta^2}$$
$$+\,i\,\frac{2M^2-1}{M\beta^2}\,\frac{U\alpha_0}{b}K\left[\sqrt{\xi^2-b^2\eta^2}+\frac{\xi^2}{\sqrt{\xi^2-b^2\eta^2}}\right] \tag{5-19}$$

VI. AERODYNAMIC COEFFICIENTS

The determination of the lift and pitching moment coefficients may be outlined as follows:

(A) Calculation of overpressure on the wing

(B) Integration of the overpressure over the wing with suitable weighting factors to obtain lift and moment

(C) Conversion of the lift and moment to coefficient form

(D) Division of the lift and pitching moment coefficients by the quasi-steady coefficients

In detail:

A) The overpressure is given by

$$\frac{p_0-p}{\rho_0} = \phi_t + U\phi_x = (i\omega\varphi + U\varphi_x)e^{i\omega t} \tag{6-1}$$

Introducing equation (3-4)

$$\frac{p_0-p}{\rho_0} = \left[i\omega\sum_{n=0}^{\infty}\nu^n\varphi^{(n)} + U\sum_{n=0}^{\infty}\nu^n u^{(n)}\right]e^{i\omega t} \tag{6-2}$$

This may be expressed as

$$\frac{p_0-p}{\rho_0} = \sum_{n=0}^{\infty}\left(\frac{p_0-p}{\rho_0}\right)_n \tag{6-3}$$

where

$$\left(\frac{p_0-p}{\rho_0}\right)_n = \left[i\omega\nu^n\varphi^{(n)} + U\nu^n u^{(n)}\right]e^{i\omega t} \tag{6-4}$$

B)

$$\frac{L}{p_0} = \iint\limits_{Wing} \frac{p_0 - p}{p_0}\, dx\, dy \tag{6-5}$$

The integration is simplified by the introduction of new coordinates

$$\begin{aligned} \bar{\xi} &= \xi \\ t &= \frac{\eta}{\xi} \end{aligned} \tag{6-6}$$

Then

$$\frac{L}{p_0} = 4c^2 \int_0^1 \int_0^{\frac{1}{b}} \bar{\xi}\, \frac{p_0 - p}{p_0}\, dt\, d\bar{\xi} \tag{6-7}$$

The diving moment about the wing vertex is given by

$$\frac{\mathcal{M}}{p_0} = \iint\limits_{Wing} x\, \frac{p_0 - p}{p_0}\, dx\, dy \tag{6-8}$$

Upon transforming to $\bar{\xi}$, t coordinates, this becomes

$$\frac{\mathcal{M}}{p_0} = 4c^3 \int_0^1 \int_0^{\frac{1}{b}} \bar{\xi}^{\,2}\, \frac{p_0 - p}{p_0}\, dt\, d\bar{\xi} \tag{6-9}$$

These functions may be expressed as

$$\frac{L}{p_0} = \sum_{n=0}^{\infty} \left(\frac{L}{p_0}\right)_n \quad \text{and} \quad \frac{\mathcal{M}}{p_0} = \sum_{n=0}^{\infty} \left(\frac{\mathcal{M}}{p_0}\right)_n \tag{6-10}$$

where

$$\left(\frac{L}{p_0}\right)_n = 4c^2 \int_0^1 \int_0^{\frac{1}{b}} \bar{\xi} \left(\frac{p_0 - p}{p_0}\right)_n dt\, d\bar{\xi}$$

and

$$\left(\frac{\mathcal{M}}{\rho_0}\right)_n = 4c^3 \int_0^1 \int_0^{\frac{1}{b}} \bar{\xi}^2 \left(\frac{p_0 - p}{\rho_0}\right)_n dt\, d\bar{\xi} \tag{6-11}$$

C) The coefficients are defined in the conventional manner, being based upon wing area and maximum chord:

$$C_L = \frac{2b}{U^2 c^2} \frac{L}{\rho_0}$$

$$C_m = \frac{2b}{U^2 c^3} \frac{\mathcal{M}}{\rho_0} \tag{6-12}$$

D) The quasi-steady lift coefficient, C_{L_s}, is obtained by multiplying the stationary value of C_{L_α} as found in Ref. 7 by the instantaneous angle of attack, $\alpha_0 e^{i\omega \bar{t}}$

$$C_{L_s} = \frac{2\pi}{b E(k')} \alpha_0 e^{i\omega \bar{t}} \tag{6-13}$$

The stationary C_{m_α} was derived from the results of Ref. 7.

$$C_{m_s} = \frac{4\pi}{3b E(k')} \alpha_0 e^{i\omega \bar{t}} \tag{6-14}$$

VII. CALCULATIONS AND RESULTS

The equations for the first three potential components and the corresponding free stream perturbation velocity components have been determined. In addition, expressions for C_L/C_{L_s} and C_m/C_{m_s} were obtained using (i) $\varphi^{(0)}$, (ii) $\varphi^{(0)}+\nu\varphi^{(1)}$, and (iii) $\varphi^{(0)}+\nu\varphi^{(1)}+\nu^2\varphi^{(2)}$. All of these expressions are tabulated below.

a. Plunging Oscillation

$$\begin{aligned}
\varphi^{(0)} &= \frac{w_0 c}{bE(k')}\sqrt{\xi^2-b^2\eta^2} \\
\varphi^{(1)} &= \frac{iMw_0 c}{\beta^2 b}\left[C_1-\frac{1}{E(k')}\right]\xi\sqrt{\xi^2-b^2\eta^2} \\
\varphi^{(2)} &= \frac{w_0}{\beta^4 b}\left[M^2\left(2C_1-\frac{3}{2E(k')}\right)+\frac{1}{2}\left(C_1+\frac{1}{E(k')}\right)\right]\xi^2\sqrt{\xi^2-b^2\eta^2} \\
&\quad -\frac{w_0}{2\beta^2 b}\left(C_2-\frac{1}{E(k')}\right)\eta^2\sqrt{\xi^2-b^2\eta^2} \\
C_1 &= \left[\frac{1-2k^2}{1-k^2}E(k')+\frac{k^2}{1-k^2}K(k')\right]^{-1} \quad ; \quad C_2=\left[\frac{2-k^2}{1-k^2}E(k')-\frac{k^2}{1-k^2}K(k')\right]^{-1}
\end{aligned} \tag{7-1}$$

$$\begin{aligned}
u^{(0)} &= \frac{w_0}{bE(k')}\frac{\xi}{\sqrt{\xi^2-b^2\eta^2}} \\
u^{(1)} &= \frac{iMw_0}{\beta^2 b}\left[C_1-\frac{1}{E(k')}\right]\left[\sqrt{\xi^2-b^2\eta^2}+\frac{\xi^2}{\sqrt{\xi^2-b^2\eta^2}}\right] \\
u^{(2)} &= \frac{w_0}{2\beta^4 b}\left[(1+2M^2)\left(C_1-\frac{1}{E(k')}\right)\right]\xi\sqrt{\xi^2-b^2\eta^2}+\frac{M^2 w_0}{\beta^4 b}\left(C_1-\frac{1}{2E(k')}\right)\frac{\xi^3}{\sqrt{\xi^2-b^2\eta^2}}
\end{aligned} \tag{7-2}$$

$$\begin{aligned}
(C_L/C_{L_s})_0 &= 1+\frac{i\nu}{3M} \\
(C_L/C_{L_s})_0+(C_L/C_{L_s})_1 &= 1+\frac{i\nu}{M}\left\{\frac{M^2}{\beta^2}\left[C_1E(k')-1\right]+\frac{1}{3}\right\}-\frac{\nu^2}{4\beta^2}\left[C_1E(k')-1\right] \\
(C_L/C_{L_s})_0+(C_L/C_{L_s})_1+(C_L/C_{L_s})_2 &= 1+\frac{i\nu}{M}\left\{\frac{M^2}{\beta^2}\left[C_1E(k')-1\right]+\frac{1}{3}\right\} \\
&\quad +\frac{\nu^2}{2\beta^2}\left\{\frac{1}{2}(C_1E(k')-1)+\frac{1+2M^2}{4\beta^2}(C_1E(k')-1)+\frac{M^2}{\beta^2}\left(C_1E(k')-\frac{1}{2}\right)\right\} \\
&\quad +\frac{i\nu^3}{15M\beta^4}\left\{M^2\left[2C_1E(k')-\frac{3}{2}\right]+\frac{1}{2}\left[C_1E(k')+1\right]-\frac{k^2}{8}\left[C_2E(k')-1\right]\right\}
\end{aligned} \tag{7-3}$$

$$(C_m/C_{m_s})_0 = 1 + \frac{3i\nu}{8M}$$

$$(C_m/C_{m_s})_0 + (C_m/C_{m_s})_1 = 1 + \frac{3i\nu}{8M}\left\{\frac{3M^2}{\beta^2}\left[C_1E(k')-1\right]+1\right\} - \frac{3\nu^2}{10\beta^2}\left[C_1E(k')-1\right]$$

$$\begin{aligned}(C_m/C_{m_s})_0 + (C_m/C_{m_s})_1 + (C_m/C_{m_s})_2 = {} & 1 + \frac{3i\nu}{8M}\left\{\frac{3M^2}{\beta^2}\left[C_1E(k')-1\right]+1\right\} \\ & + \frac{3\nu^2}{5\beta^2}\left\{\frac{1}{2}\left[C_1E(k')-1\right] + \frac{1+2M^2}{4\beta^2}\left[C_1E(k')-1\right] + \frac{M^2}{\beta^2}\left[C_1E(k') - \frac{1}{2}\right]\right\} \\ & + \frac{i\nu^3}{12M\beta^4}\left\{M^2\left[2C_1E(k') - \frac{3}{2}\right] + \frac{1}{2}\left[C_1E(k')+1\right] - \frac{k^2}{8}\left[C_2E(k')-1\right]\right\}\end{aligned} \tag{7-4}$$

b. Pitching Oscillation

$$\begin{aligned}\varphi^{(0)} &= \frac{U\alpha_0 c}{bE(k')}\sqrt{\xi^2 - b^2\eta^2} \\ \varphi^{(1)} &= \frac{iU\alpha_0 c}{M\beta^2 b}\left[(2M^2-1)C_1 - \frac{M^2}{E(k')}\right]\xi\sqrt{\xi^2-b^2\eta^2} - \frac{iU\alpha_0 c}{bE(k')}\frac{m}{M}\sqrt{\xi^2-b^2\eta^2}\end{aligned} \tag{7-5}$$

$$\begin{aligned}u^{(0)} &= \frac{U\alpha_0}{bE(k')}\frac{\xi}{\sqrt{\xi^2-b^2\eta^2}} \\ u^{(1)} &= \frac{iU\alpha_0}{M\beta^2 b}\left[(2M^2-1)C_1 - \frac{M^2}{E(k')}\right]\left[\sqrt{\xi^2-b^2\eta^2} + \frac{\xi^2}{\sqrt{\xi^2-b^2\eta^2}}\right] - \frac{iU\alpha_0}{bE(k')}\frac{m}{M}\frac{\xi}{\sqrt{\xi^2-b^2\eta^2}}\end{aligned} \tag{7-6}$$

$$\begin{aligned}(C_L/C_{L_s})_0 &= 1 + \frac{i\nu}{3M} \\ (C_L/C_{L_s})_0 + (C_L/C_{L_s})_1 &= 1 + \frac{i\nu}{M}\left[\frac{(2M^2-1)C_1E(k')-M^2}{\beta^2} + \frac{1}{3} - m\right] + \frac{\nu^2}{M^2}\left[\frac{m}{3} - \frac{(2M^2-1)C_1E(k')-M^2}{4\beta^2}\right]\end{aligned} \tag{7-7}$$

$$\begin{aligned}(C_m/C_{m_s})_0 &= 1 + \frac{3i\nu}{8M} \\ (C_m/C_{m_s})_0 + (C_m/C_{m_s})_1 &= 1 + \frac{i\nu}{M}\left[\frac{9}{8}\,\frac{(2M^2-1)C_1E(k')-M^2}{\beta^2} + \frac{3}{8} - m\right] + \frac{3\nu^2}{2M^2}\left[\frac{m}{4} - \frac{(2M^2-1)C_1E(k')-M^2}{5\beta^2}\right]\end{aligned} \tag{7-8}$$

Termination of the potential series with the $\varphi^{(1)}$ term should introduce only a slight inaccuracy into the aerodynamic coefficients for $\nu \leq 0.50$. Since, for dynamic stability analyses, the reduced frequency is considerably lower than 0.50 $(\varphi^{(0)} + \nu\varphi^{(1)})e^{i\omega\bar{t}}$ should represent the velocity potential with sufficient accuracy for this purpose. In practical applications of this theory to engineering problems the most useful form for the force and moment coefficient ratios is that in which the partially represented highest order frequency terms are omitted. Thus

a. Plunging Oscillation

$$
\begin{aligned}
&(C_L/C_{L_s})_0 = 1 \\
&(C_L/C_{L_s})_0 + (C_L/C_{L_s})_1 = 1 + \frac{i\nu}{M}\left\{\frac{M^2}{\beta^2}\left[C_1E(k')-1\right] + \frac{1}{3}\right\} \\
&(C_L/C_{L_s})_0 + (C_L/C_{L_s})_1 + (C_L/C_{L_s})_2 = 1 + \frac{i\nu}{M}\left\{\frac{M^2}{\beta^2}\left[C_1E(k')-1\right] + \frac{1}{3}\right\} \\
&\qquad + \frac{\nu^2}{2\beta^2}\left\{\frac{1}{2}\left[C_1E(k')-1\right] + \frac{1+2M^2}{4\beta^2}\left[C_1E(k')-1\right] + \frac{M^2}{\beta^2}\left[C_1E(k')-\frac{1}{2}\right]\right\}
\end{aligned}
\qquad (7\text{-}9)
$$

$$
\begin{aligned}
&(C_m/C_{m_s})_0 = 1 \\
&(C_m/C_{m_s})_0 + (C_m/C_{m_s})_1 = 1 + \frac{3i\nu}{8M}\left\{\frac{3M^2}{\beta^2}\left[C_1E(k')-1\right] + 1\right\} \\
&(C_m/C_{m_s})_0 + (C_m/C_{m_s})_1 + (C_m/C_{m_s})_2 = 1 + \frac{3i\nu}{8M}\left\{\frac{3M^2}{\beta^2}\left[C_1E(k')-1\right] + 1\right\} \\
&\qquad + \frac{3\nu^2}{5\beta^2}\left\{\frac{1}{2}\left[C_1E(k')-1\right] + \frac{1+2M^2}{4\beta^2}\left[C_1E(k')-1\right] + \frac{M^2}{\beta^2}\left[C_1E(k')-\frac{1}{2}\right]\right\}
\end{aligned}
\qquad (7\text{-}10)
$$

b. Pitching Oscillation

$$(C_L/C_{L_s})_o = 1$$
$$(C_L/C_{L_s})_o + (C_L/C_{L_s})_1 = 1 + \frac{i\nu}{M}\left[\frac{(2M^2-1)C_1E(k')-M^2}{\beta^2} + \frac{1}{3} - m\right] \tag{7-11}$$

$$(C_m/C_{m_s})_o = 1$$
$$(C_m/C_{m_s})_o + (C_m/C_{m_s})_1 = 1 + \frac{i\nu}{M}\left[\frac{9}{8}\,\frac{(2M^2-1)C_1E(k')-M^2}{\beta^2} + \frac{3}{8} - m\right] \tag{7-12}$$

are the values of $(C_L/C_{L_s})_o + (C_L/C_{L_s})_1$, and $(C_m/C_{m_s})_o + (C_m/C_{m_s})_1$, to be used for stability analyses.

In the case of the pitching wing, the diving moment coefficient ratios based upon the moments about the axis of pitch are of great interest. They are useful in determining whether the damping is stable.

$$(C_m/C_{m_s})_p = (C_m/C_{m_s}) - m\,(C_L/C_{m_s}) = (C_m/C_{m_s}) - m\,(C_{L_s}/C_{m_s})(C_L/C_{L_s}) \tag{7-13}$$

Since this is useful only for dynamic stability studies, this ratio will be presented here using $(C_L/C_{L_s}) = (C_L/C_{L_s})_o + (C_L/C_{L_s})_1$, and $(C_m/C_{m_s}) = (C_m/C_{m_s})_o + (C_m/C_{m_s})_1$, as found in equations (7-11) and (7-12).

$$(C_m/C_{m_s})_p = 1 - \frac{3m}{2} + \frac{3i\nu}{2M}\left\{m^2 - m\left[1 + \frac{(2M^2-1)C_1E(k')-M^2}{\beta^2}\right] + \frac{1}{4} + \frac{3}{4}\,\frac{(2M^2-1)C_1E(k')-M^2}{\beta^2}\right\} \tag{7-14}$$

The damping is stable $(C_{m_{\dot{\theta}}} < 0)$ if $Im\,(C_m/C_{m_s})_b > 0$.

It can be seen from equation (7-14) and the stability criterion above that below a certain critical Mach number, the damping will be negative (unstable) for pitch axes located between two points lying fore and aft of mid-chord (m = 0.5). At the Mach number specified by

$$M = \sqrt{\frac{C_1 E(k')}{2C_1 E(k') - 1}} \tag{7-15}$$

these two points coincide at m = 0.5. For greater Mach numbers $Im\,(C_m/C_{m_s})_b > 0$ for all values of m. These results, presented in figures 10 and 11, agree with those of reference 9 and fair smoothly into the results of reference 2 as k approaches 1.

VIII. CONCLUSIONS

By expanding the velocity potential in a series in powers of the reduced frequency one can, within the limitations of linearized theory, develop a practical engineering solution for the harmonically oscillating narrow delta wing in supersonic flow. The method of solution is, in fact, applicable to any wing planform for which the stationary supersonic flow velocity potential is known. In general, for stability analyses the series may be terminated with the term linear in the reduced frequency without great loss of accuracy.

The results for the harmonic pitching oscillation indicates that for the wing moving at supersonic speed there is a critical Mach number above which there is no pitching axis about which the damping in pitch is unstable for sufficiently small frequencies.

It is to be noted that Brown's (Ref. 12) quasi-stationary results were obtained by retaining only part of the terms that are linear in frequency. In the present paper, all first order frequency terms are retained.

NOMENCLATURE

A	Wing Area
a	Acoustic Velocity
b	cot γ
c	Maximum Wing Chord
$E(k')$	Complete Elliptic Integral of the Second kind of Modulus k'
$K(k')$	Complete Elliptic Integral of the First Kind of Modulus k'
i	$\sqrt{-1}$
K	Constant, defined on page
k	β/b
k'	$\sqrt{1-k^2}$
L	Lift
m	Defined on page
M	Free Stream Mach Number
$\mathcal{M}$	Pitching Moment
p	Static pressure
$q = \frac{1}{2}\rho_o U^2$	Dynamic pressure
R	Defined on Page
S	Defined on Page
$t = y/x = \eta/\xi$	Conical Coordinate
$\bar{t}$	Time
u, v, w	Perturbation Velocities in the x, y, directions respectively

U	Free stream velocity
x, y, z	Wing coordinates
α	Angle of attack
β	$\sqrt{M^2-1}$
ω	Vibrational frequency
ρ	Density
ν	Reduced frequency
ϕ	Complete Perturbation Velocity Potential
φ	Time Free Perturbation Velocity Potential
γ	Wing Semi-vertex angle
$\xi = x/c$ $\eta = y/c$ $\zeta = z/c$	Non-dimensional coordinates
$\bar{\xi}$	Defined on Page

Superscripts:

$(\ )^{(n)}$	Component in series expansion of φ or u having ν^n as a coefficient

Subscripts:

$(\ )_o$	Free Stream
$(\ )_s$	Quasi-Stationary Value
$(\ )_p$	Particular Integral

$(\ )_c$	Complementary Solution
$(\ )_p$	About the axis of pitch

Coefficients:

C_L	Lift Coefficient
C_m	Pitching Moment Coefficient

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APPENDIX

Derivations of $\varphi_p^{(1)}$, $u_p^{(2)}$, and $v_p^{(2)}$

The differential operator, L, will be defined as follows

$$L[F] = \beta^2 F_{\xi\xi} - F_{\eta\eta} - F_{\zeta\zeta} \tag{A-1}$$

It will be noted that

$$\begin{aligned}
&\text{(i)} \quad L[\xi F] = 2\beta^2 F_\xi + \xi L[F] \\
&\text{(ii)} \quad L[\xi^2 F] = 2\beta^2 F + 4\beta^2 \xi F_\xi + \xi^2 L[F] \\
&\text{(iii)} \quad L[\eta F] = -2F_\eta + \eta L[F] \\
&\text{(iv)} \quad L[\xi\eta F] = 2\beta^2 \eta F_\xi - 2\xi F_\eta + \xi\eta L[F] \\
&\text{(v)} \quad L[\varphi^{(0)}] = 0 \\
&\text{(vi)} \quad L[\varphi^{(1)}] = -2iM\varphi_\xi^{(0)}
\end{aligned} \tag{A-2}$$

Since $\varphi_p^{(1)}$ is homogeneous of degree 2, it may be of the form $\int_{-\infty}^{\xi} \varphi^{(0)} dx_1$, $\xi\varphi^{(0)}$, $\xi^2\varphi_\xi^{(0)}$, $\xi^3\varphi_{\xi\xi}^{(0)}$, and so on. $\xi\varphi^{(0)}$ will be investigated first. From equations (A-2i) and (A-2v),

$$L[\xi\varphi^{(0)}] = 2\beta^2\varphi_\xi^{(0)} \tag{A-3}$$

Therefore, since it is desired that

$$\varphi_p^{(1)} = -\frac{iM}{\beta^2}\xi\varphi^{(0)} \tag{A-4}$$

Because $\varphi_p^{(2)}$ is homogeneous of degree 3, $u_p^{(2)}$ and $v_p^{(2)}$ are homogeneous of degree 2. It is desired to find a function, $u_p^{(2)}$, such that

$$L[u_p^{(2)}] = \frac{1}{c}\varphi_\xi^{(0)} - 2i\frac{M}{c}\varphi_{\xi\xi}^{(1)} \tag{A-5}$$

Assuming $\varphi^{(1)}$ to be known before $u^{(2)}$ and $v^{(2)}$ are to be determined, a wider selection of functions that are homogeneous of degree 2 is available for use in this task; $\xi\varphi^{(0)}, \xi^2\varphi_\xi^{(0)}, \xi\varphi_\xi^{(1)}, \eta\varphi^{(0)}, \xi^2\varphi_\eta^{(0)}, \xi\eta\varphi_\xi^{(0)}$, and so on. The function $\xi\varphi_\xi^{(1)}$ will be examined first. From equations (A-2i) and (A-2iv),

$$L[\xi\varphi_\xi^{(1)}] = 2\beta^2\varphi_{\xi\xi}^{(1)} - 2iM\xi\varphi_{\xi\xi}^{(0)} \tag{A-6}$$

Therefore, $L\left[-\frac{iM}{\beta^2 c}\xi\varphi_\xi^{(1)}\right]$ yields one of the required terms. Equation (A-2ii) indicates that a function of the form $\frac{M^2}{2\beta^4 c}\xi^2\varphi_\xi^{(0)}$ will allow cancellation of the undesirable term in equation (A-6):

$$L\left[\frac{M^2}{2\beta^4 c}\xi^2\varphi_\xi^{(0)}\right] = \frac{M^2}{\beta^2 c}\varphi_\xi^{(0)} + \frac{2M^2}{\beta^2 c}\xi\varphi_{\xi\xi}^{(0)} \tag{A-7}$$

From equations (A-2i) and (A-2v),

$$L[\xi\varphi^{(0)}] = 2\beta^2\varphi_\xi^{(0)} \tag{A-8}$$

so that $L\left[-\frac{M^2}{2\beta^4 c}\xi\varphi^{(0)}\right]$ will cancel the undesired $\frac{M^2}{\beta^2 c}\varphi_\xi^{(0)}$ term of equation (A-7) and $L\left[\frac{1}{2\beta^2 c}\xi\varphi^{(0)}\right]$ will complete the requirements of equation (A-5). Therefore,

$$u_p^{(2)} = -\frac{1}{2\beta^4 c}\xi\varphi^{(0)} + \frac{M^2}{2\beta^4 c}\xi^2\varphi_\xi^{(0)} - \frac{iM}{\beta^2 c}\xi\varphi_\xi^{(1)} \tag{A-9}$$

$V_p^{(2)}$ must satisfy the following relation.

$$L[V_p^{(2)}] = \frac{1}{c}\varphi_\eta^{(0)} - 2i\frac{M}{c}\varphi_{\xi\eta}^{(1)} \tag{A-10}$$

From equations (A-2i) and (A-2vi),

$$L[-\frac{iM}{\beta^2 c}\xi\varphi_\eta^{(1)}] = -2i\frac{M}{c}\varphi_{\xi\eta}^{(1)} - 2\frac{M^2}{\beta^2 c}\xi\varphi_{\xi\eta}^{(0)} \tag{A-11}$$

To cancel the undesired term in equation (A-11), consider equations (A-2ii) and (A-2v),

$$L[\frac{M^2}{2\beta^4 c}\xi^2\varphi_\eta^{(0)}] = \frac{M^2}{\beta^2 c}\varphi_\eta^{(0)} + \frac{2M^2}{\beta^2 c}\xi\varphi_{\xi\eta}^{(0)} \tag{A-12}$$

There remains the task of eliminating a term of the form $\frac{M^2}{\beta^2 c}\varphi_\eta^{(0)}$ and adding a term of the form $\frac{1}{c}\varphi_\eta^{(0)}$.

$$L[\eta\varphi^{(0)}] = -2\varphi_\eta^{(0)} \tag{A-13}$$

Therefore,

$$L[(-\frac{1}{2c} + \frac{M^2}{2\beta^2 c})\eta\varphi^{(0)}] = \frac{1}{c}\varphi_\eta^{(0)} - \frac{M^2}{\beta^2 c}\varphi_\eta^{(0)}. \tag{A-14}$$

$$V_p^{(2)} = \frac{1}{2\beta^2 c}\eta\varphi^{(0)} + \frac{M^2}{2\beta^4 c}\xi^2\varphi_\eta^{(0)} - \frac{iM}{\beta^2 c}\xi\varphi_\eta^{(1)} \tag{A-15}$$

The particular solutions for all higher order components are obtained in a completely analogous manner.

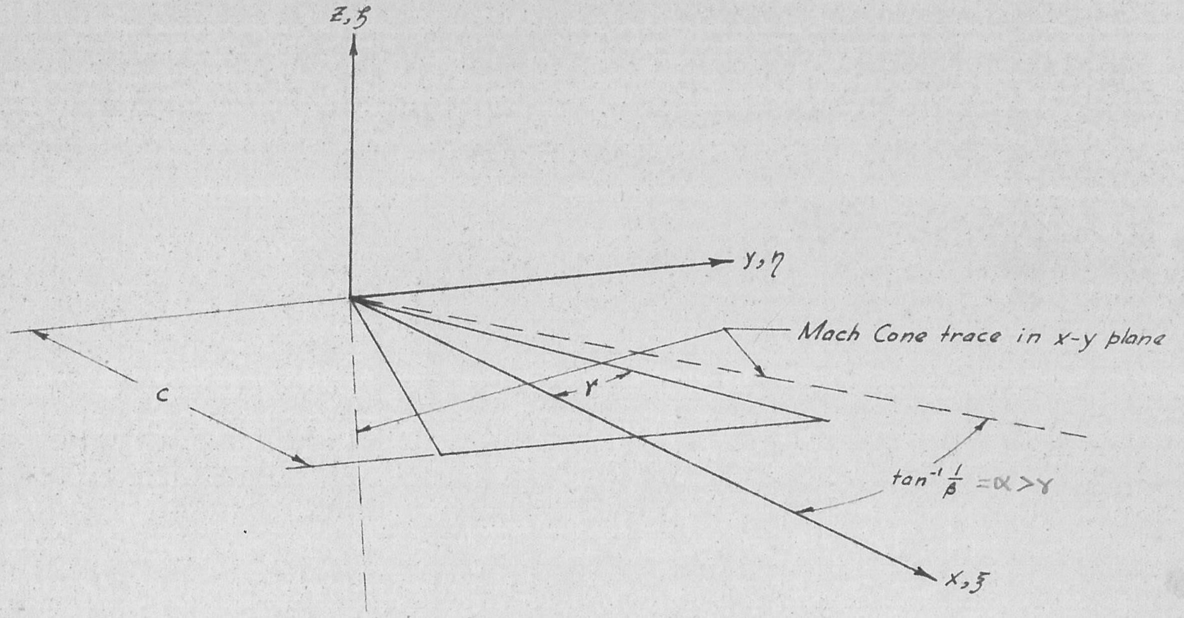


Figure 1

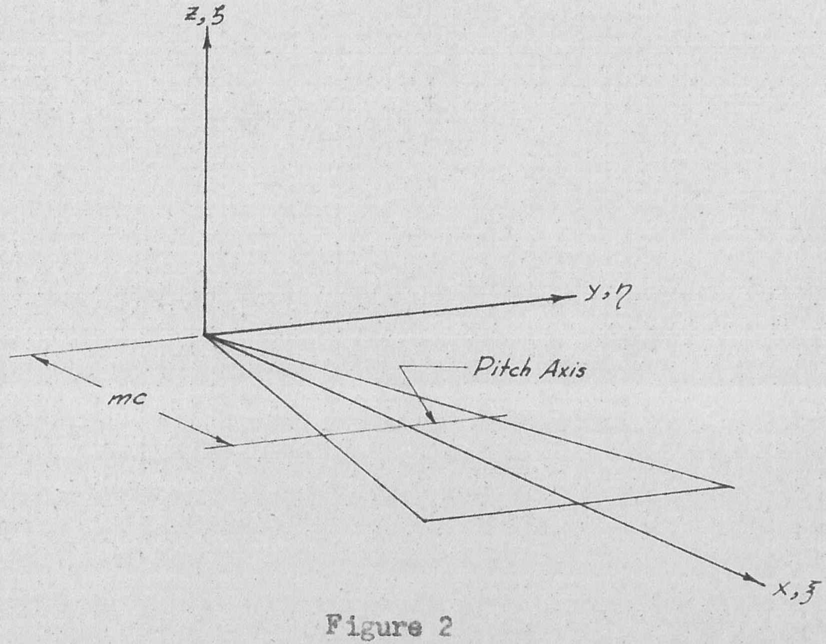
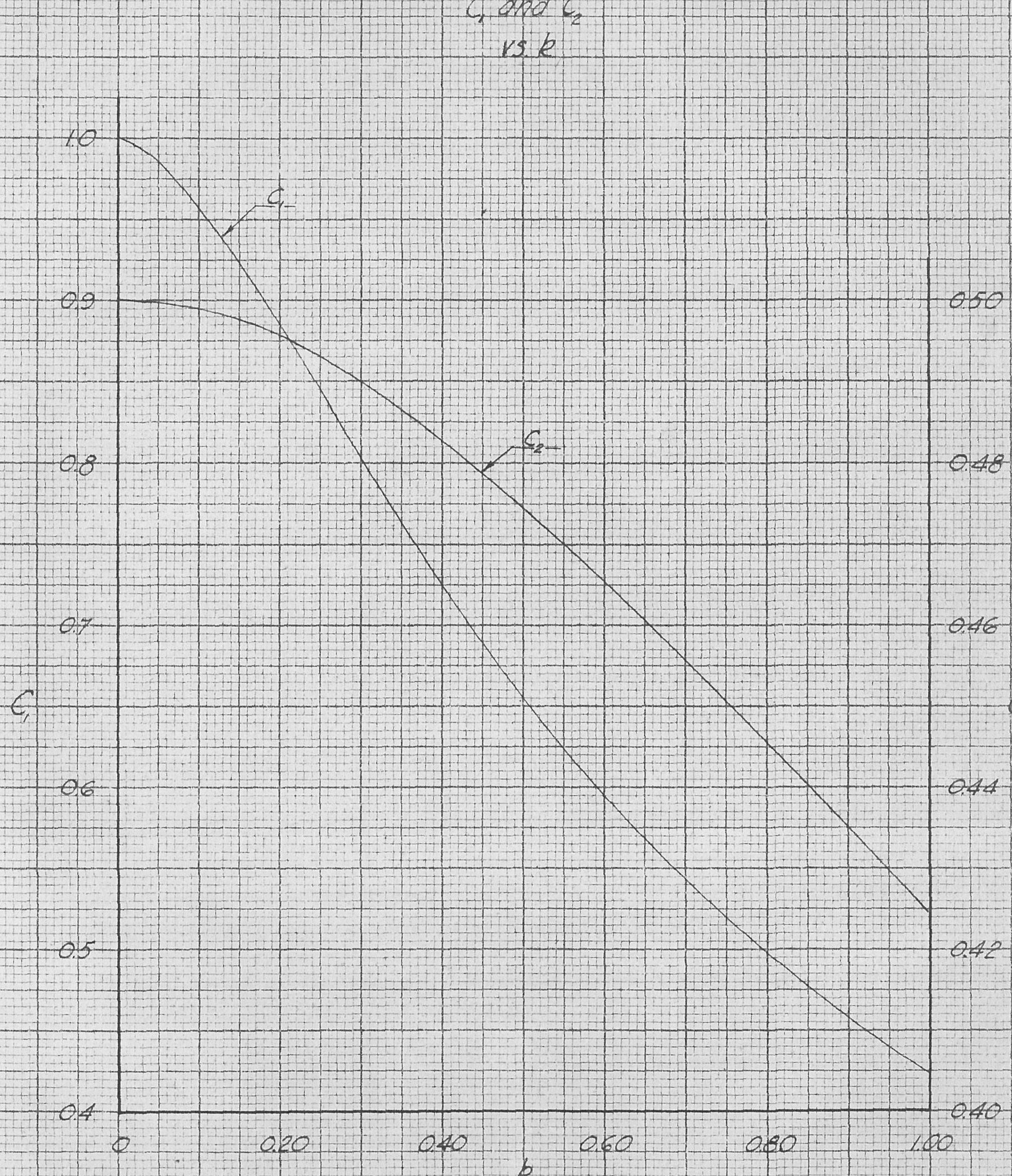


Figure 2



C_1 and C_2
vs. k
1.0
0.9
0.8
0.7
0.6
0.5
0.4
C_1
C_2
0.50
0.48
0.46
0.44
0.42
0.40
0
0.20
0.40
0.60
0.80
1.00
k

Figure 3

Plunging Oscillation

$b = 3$

$M = \sqrt{2}$ ————

$M = \sqrt{10}$ ————

$\varphi = \varphi^{(0)} + \nu\,\varphi^{(1)}$

Neglecting quadratic and higher powers of ν

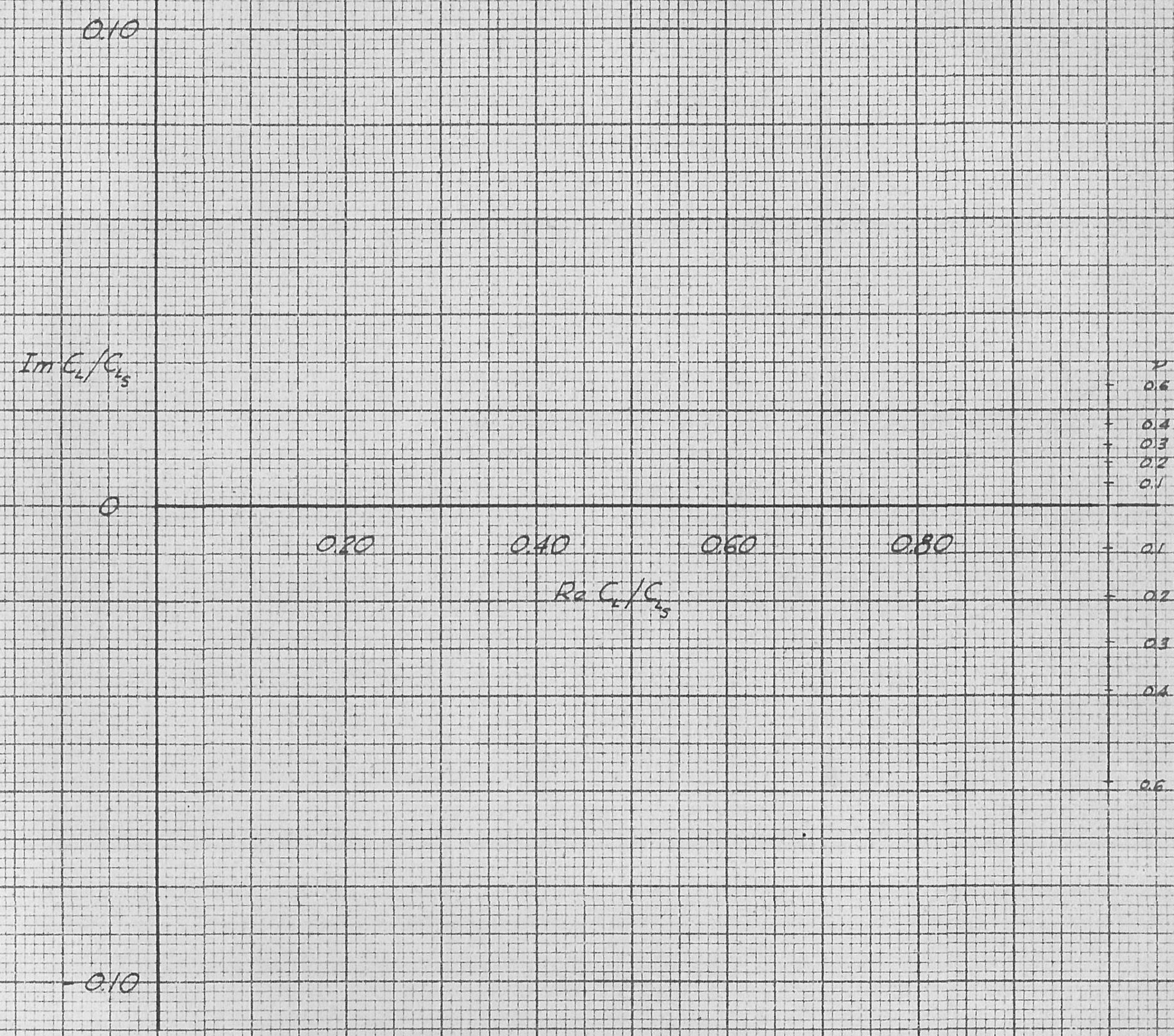


Figure 4

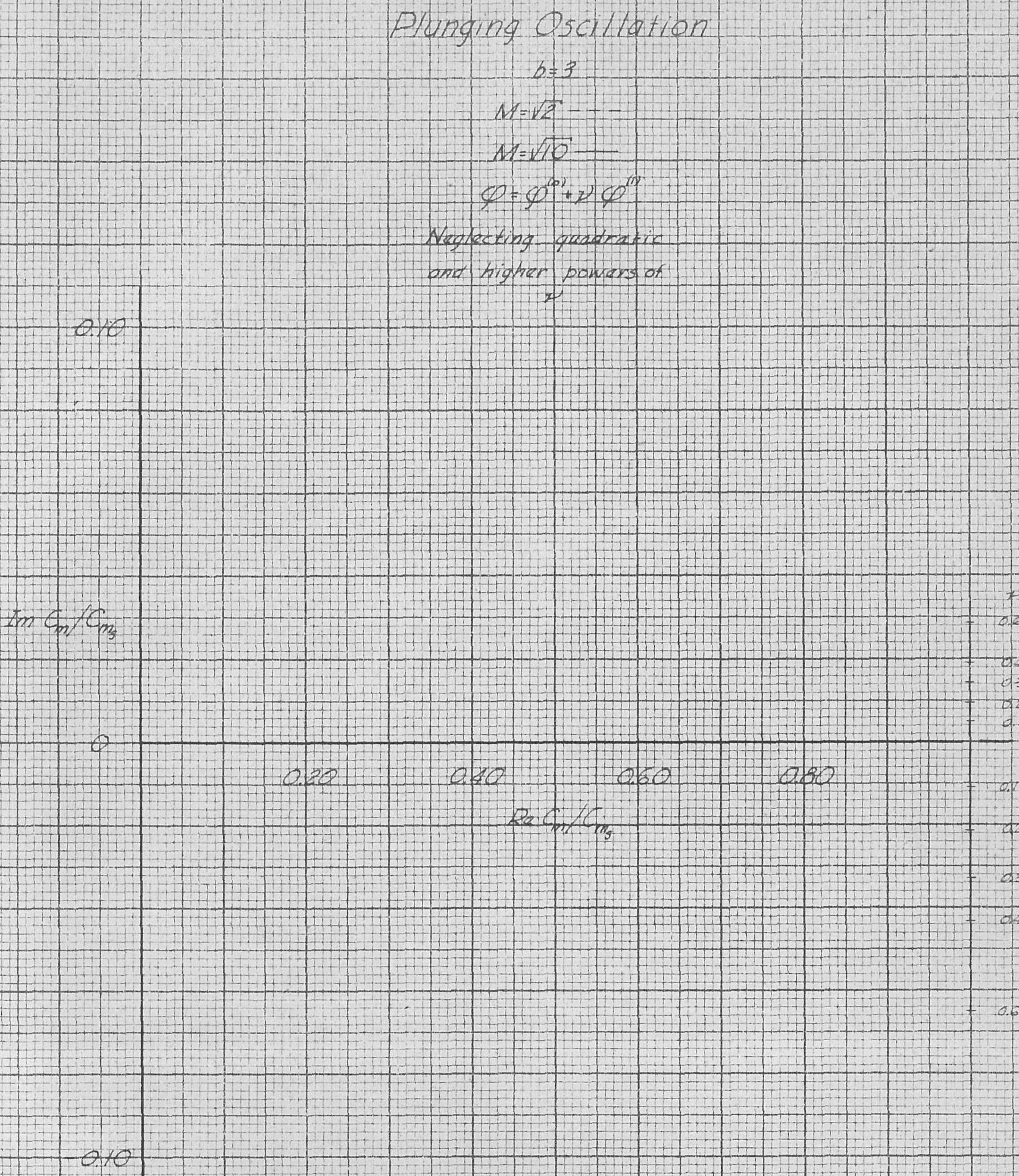

Plunging Oscillation
b=3
M=√2 - - -
M=√10 ——
φ=φ(0)+ν φ(1)
Neglecting quadratic
and higher powers of
ν
0.10
Im Cm/Cm3
0
-0.10
0.20
0.40
0.60
0.80
Re Cm/Cm3
ν
0.6
0.4
0.3
0.2
0.1
0.1
0.2
0.3
0.4
0.6

Figure 5

Plunging Oscillation

$b=3$

$M=\sqrt{2}$ - - -

$M=\sqrt{10}$ ———

$\varphi=\varphi^{(0)}+\nu\varphi^{(1)}+\nu^2\varphi^{(2)}$

Neglecting cubic and higher powers of ν

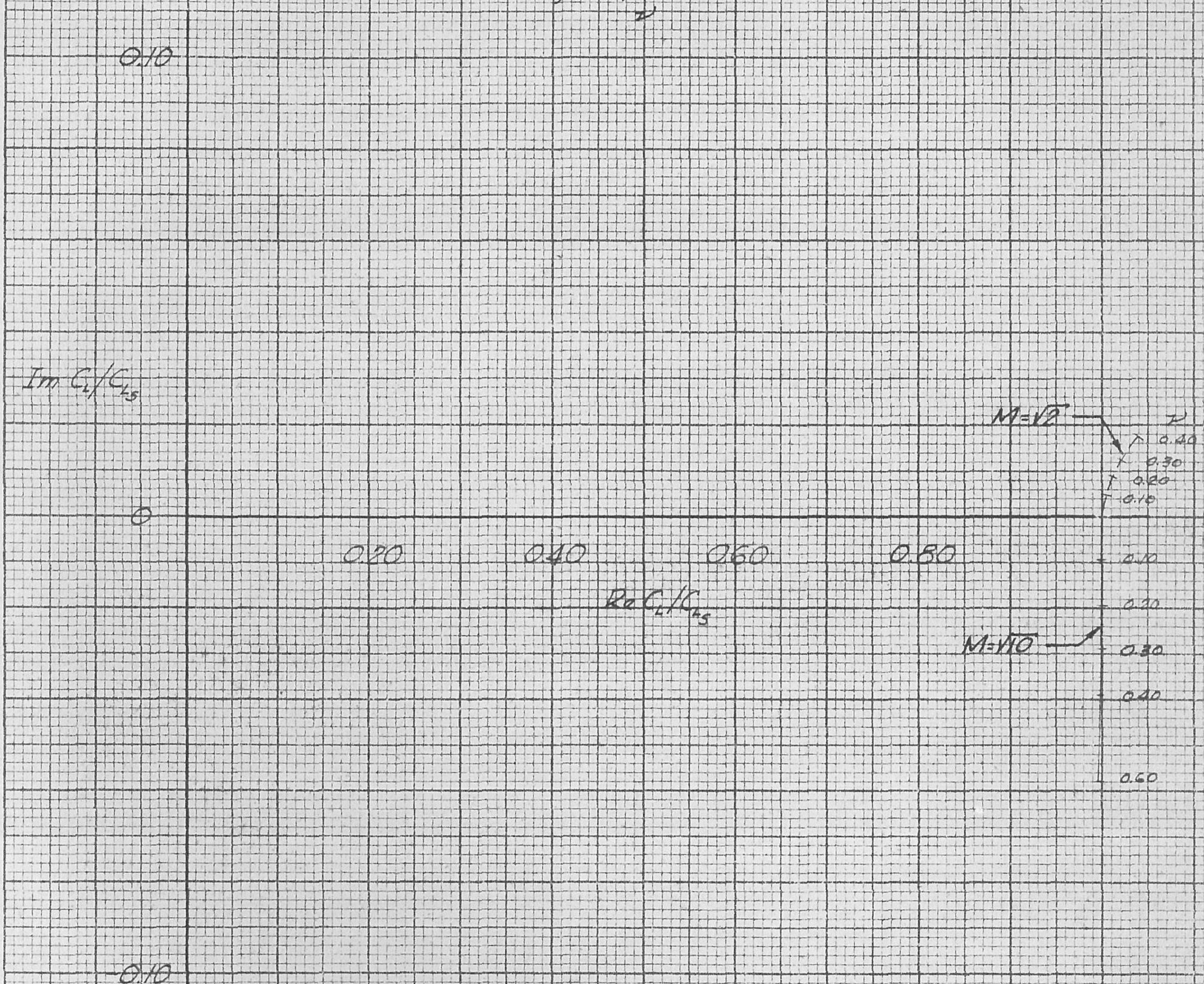


Figure 6

Plunging Oscillation

$b = 3$

$M = \sqrt{2}$ - - -

$M = \sqrt{10}$ ———

$\varphi = \varphi^{(0)} + \nu\varphi^{(1)} + \nu^2\varphi^{(2)}$

Neglecting cubic and higher powers of ν

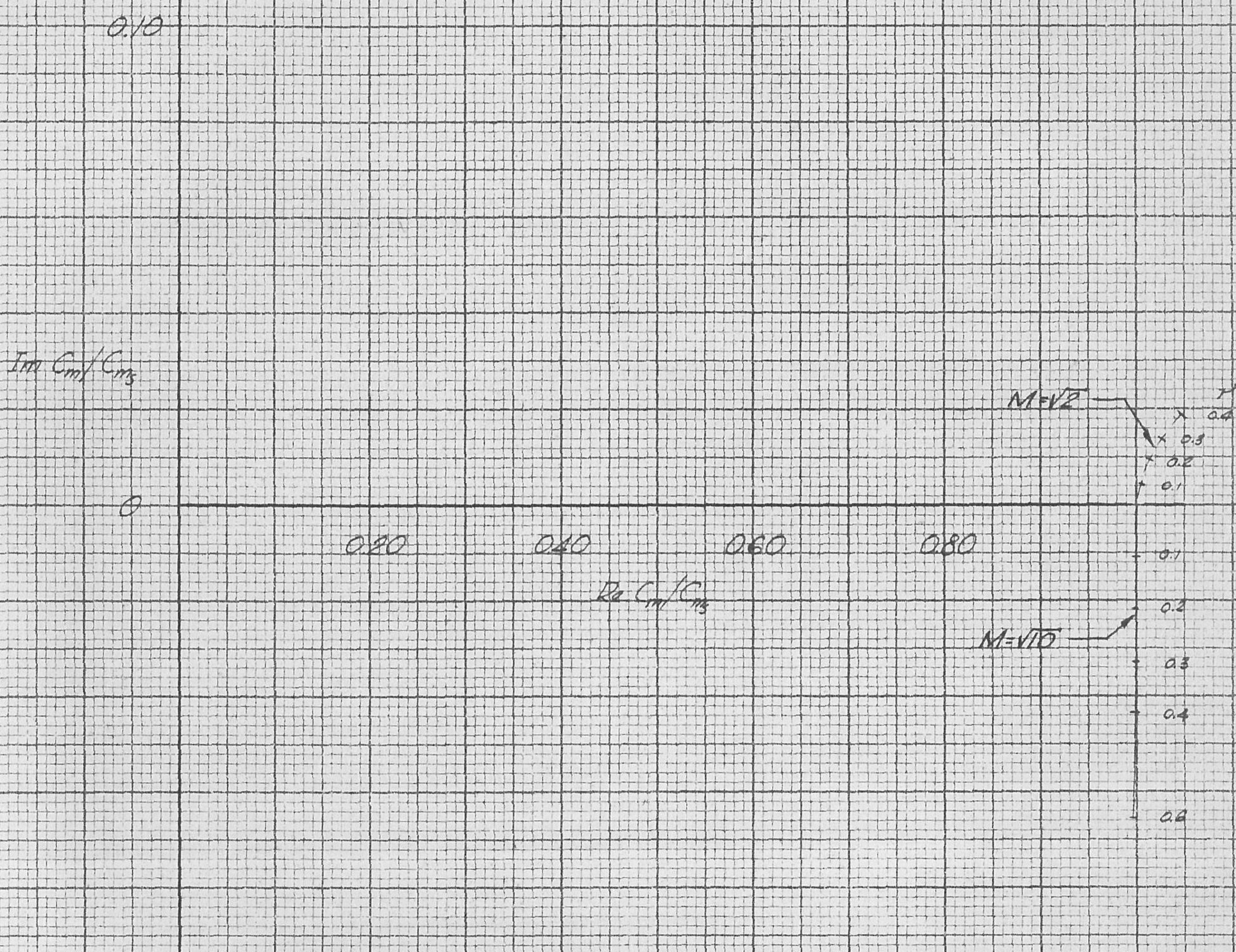


Figure 7

Pitching Oscillation

$b = 3$

$M = \sqrt{2}$

$m = 0.25$

$\Phi = \varphi^{(0)} + \nu\varphi^{(1)}$

Neglecting quadratic and higher powers of ν

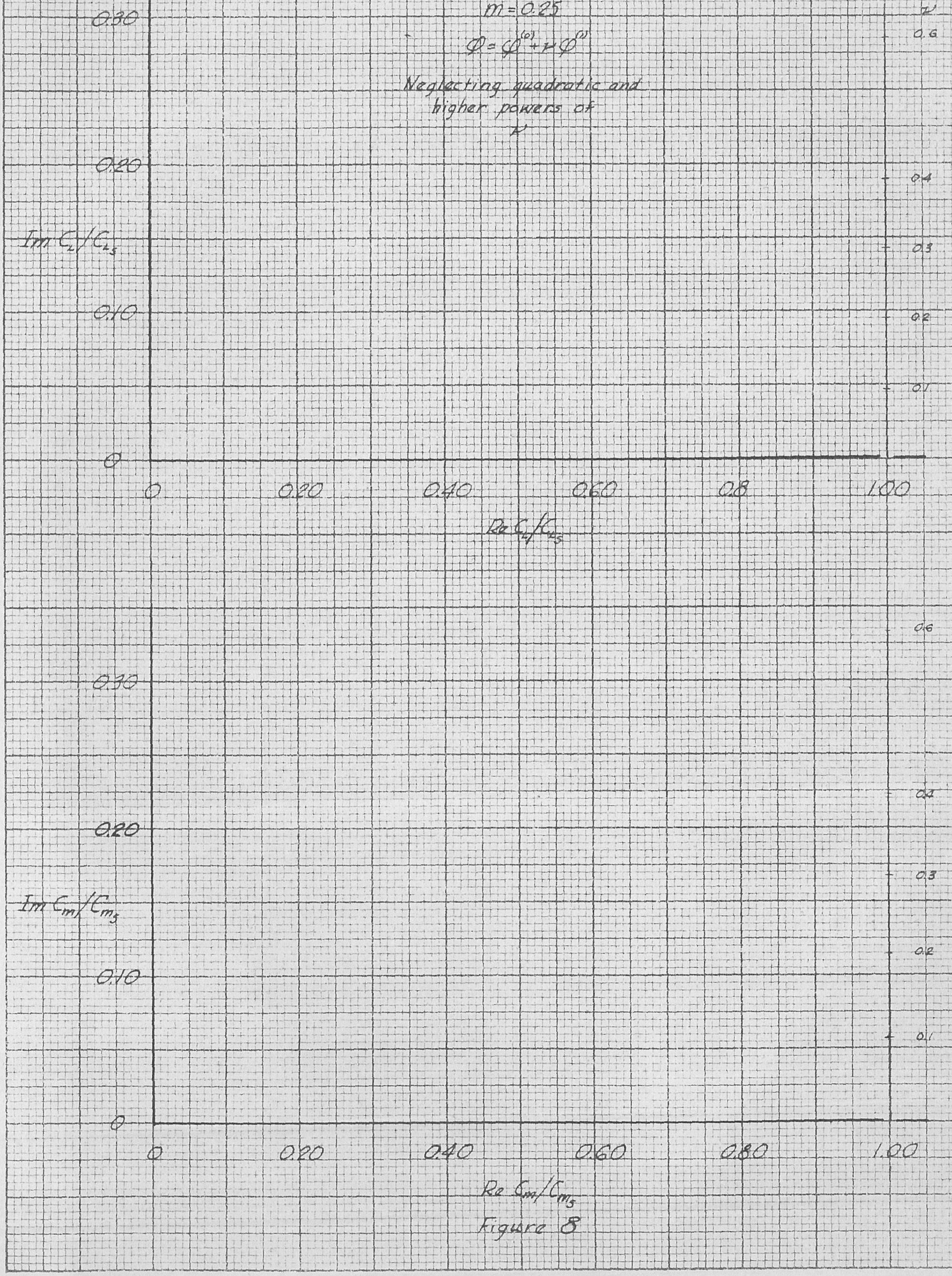


Figure 8

Pitching Oscillation

$b=3$

$M=\sqrt{2}$

$m=0.5$

$\varphi=\varphi^{(0)}+\nu\varphi^{(1)}$

Neglecting quadratic and higher powers of ν

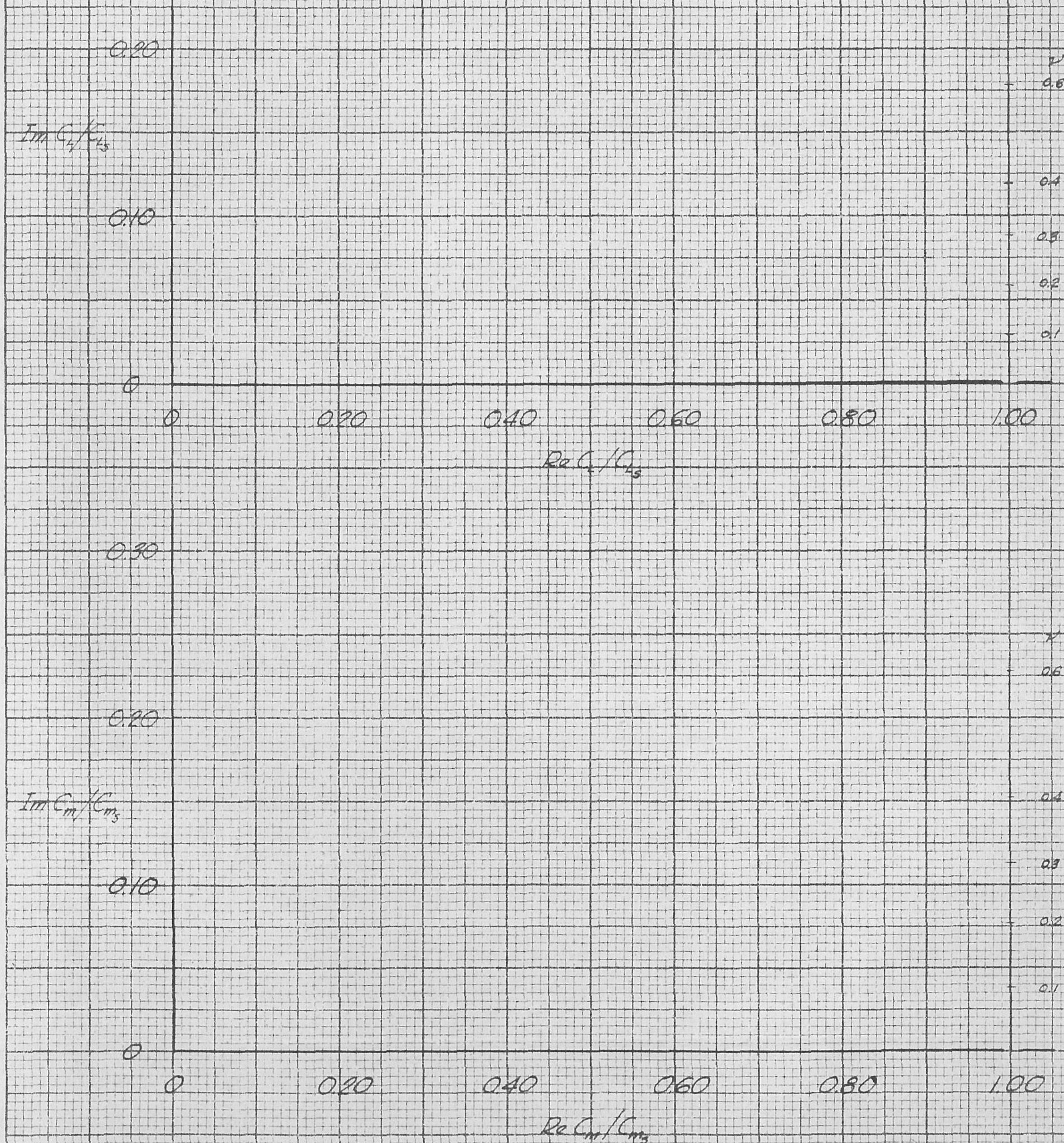


Figure 9

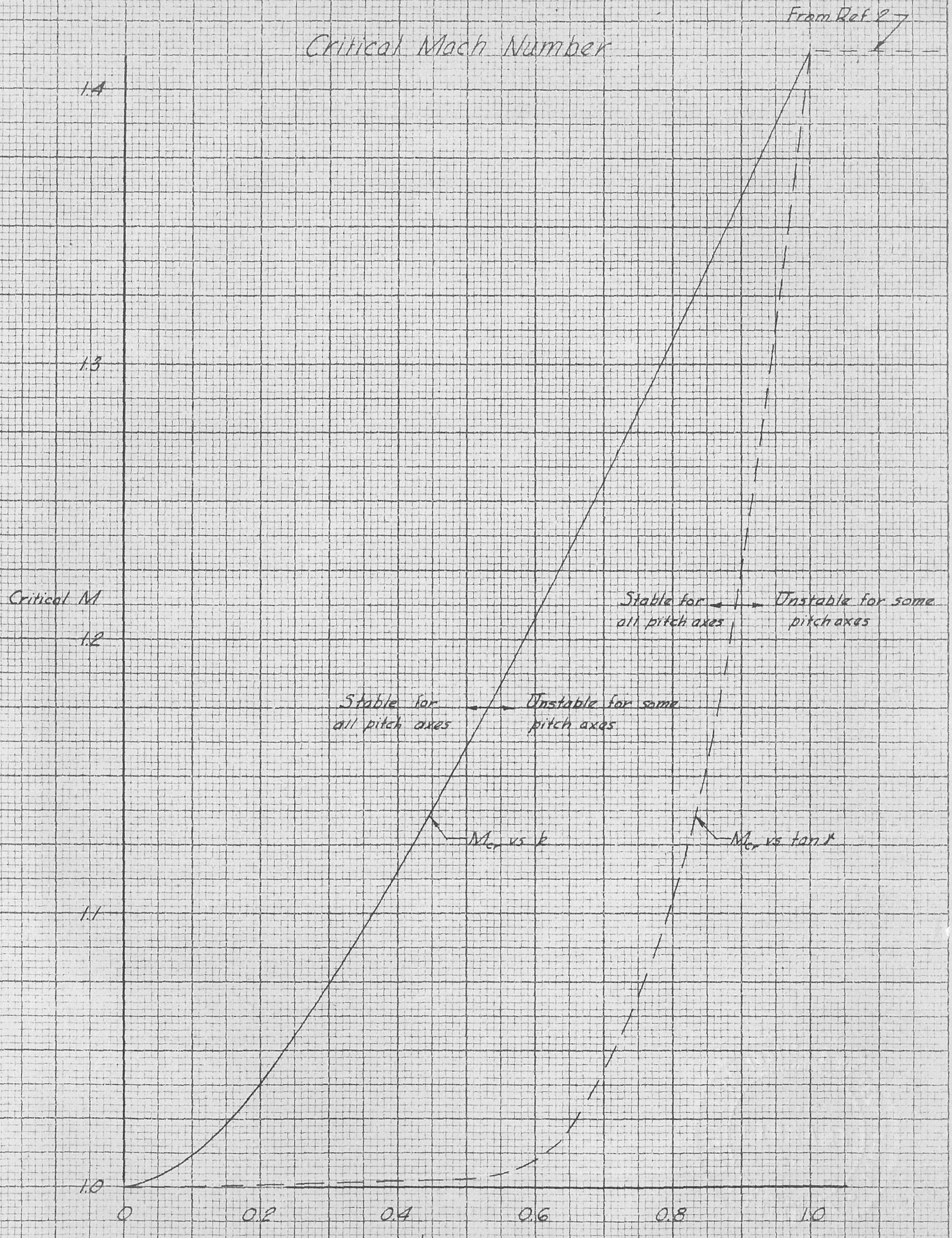

From Ref 2
Critical Mach Number
1.4
1.3
Critical M
1.2
1.1
1.0
Stable for
all pitch axes
Unstable for some
pitch axes
Stable for
all pitch axes
Unstable for some
pitch axes
M_{cr} vs k
M_{cr} vs $\tan\gamma$
0
0.2
0.4
0.6
0.8
1.0
k and $\tan\gamma$

Figure 10

M vs. m
for Zero Damping
in Pitch

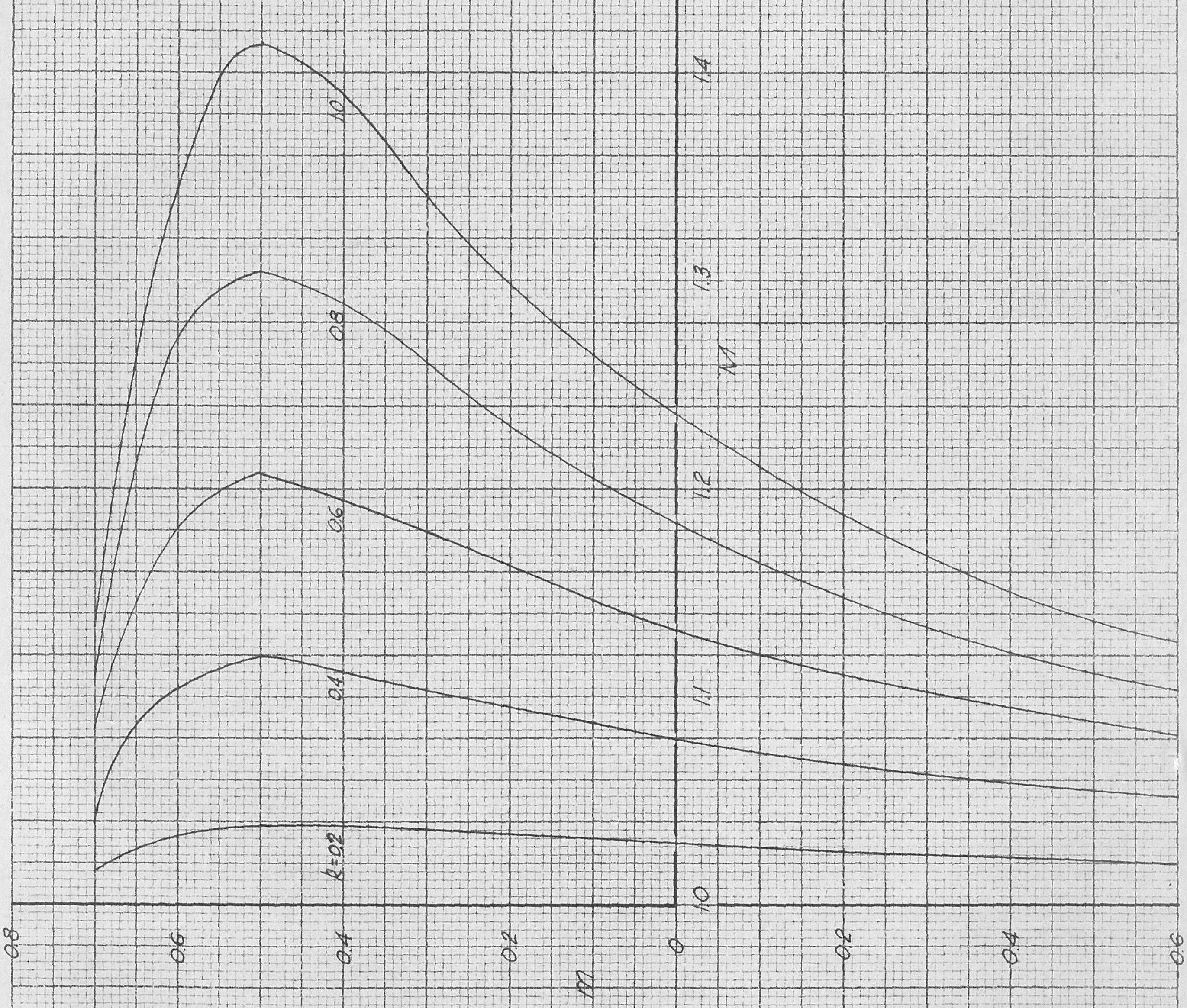


Figure 11