ENERGY INEQUALITIES AND ERROR ESTIMATES FOR AXISYMMETRIC TORSION OF THIN ELASTIC SHELLS OF REVOLUTION

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ABSTRACT

The problem motivating this investigation is that of pure axisymmetric torsion of an elastic shell of revolution. The analysis is carried out within the framework of the three-dimensional linear theory of elastic equilibrium for homogeneous, isotropic solids. The objective is the rigorous estimation of errors involved in the use of approximations based on thin shell theory.

The underlying boundary value problem is one of Neumann type for a second order elliptic operator. A systematic procedure for constructing pointwise estimates for the solution and its first derivatives is given for a general class of second-order elliptic boundary-value problems which includes the torsion problem as a special case.

The method used here rests on the construction of "energy inequalities" and on the subsequent deduction of pointwise estimates from the energy inequalities. This method removes certain drawbacks characteristic of pointwise estimates derived in some investigations of related areas.

Special interest is directed towards thin shells of constant thickness. The method enables us to estimate the error involved in a stress analysis in which the exact solution is replaced by an approximate one, and thus provides us with a means of assessing the quality of approximate solutions for axisymmetric torsion of thin shells.

Finally, the results of the present study are applied to the stress analysis of a circular cylindrical shell, and the quality of stress estimates derived here and those from a previous related publication are discussed.

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I. STATEMENT OF THE PROBLEM

1. Introduction

The present thesis is concerned with the problem of axisymmetric torsion by terminal loads of a class of elastic solids which we shall call shells of revolution. Such solids occupy a region of three-dimensional space, which consists of all points whose distances from a given surface of revolution -- called the midsurface -- do not exceed h/2 ; h is the shell thickness.[1] Our interest is directed especially to the case in which the shell is thin in a sense to be made precise later. The analysis is based on the classical linear theory of elastic equilibrium for homogeneous and isotropic materials, and it may be regarded as an extension of that reported in [1]. The present study represents a continuation to the development of methods for the assessment of the quality of approximate solutions of thin shell problems. The problem of axisymmetric torsion is perhaps the simplest one suitable for this purpose, and simple approximate solutions, constructed from two-dimensional shell theories or otherwise are known.[2] Our results provide estimates, based on three-dimensional elasticity theory, for the error involved in a stress analysis when the exact solution is replaced by an approximate one.

From another point of view, the present work described here may be regarded as an extension of that in [3], where a systematic procedure was given for constructing a pointwise estimate of the solution of a problem of Neumann type for a class of second order

[1] Precise geometrical details are given in the next section.

[2] See, for example, Love [2].

elliptic boundary value problems in two independent variables. In order to assess the stresses in the problem of axisymmetric torsion of solids of revolution, a pointwise estimate is required for the first derivatives of the solution of such a boundary value problem. While such stress estimates were given in [1] for a certain class of axisymmetric torsion problems, they were based on mean value theorems, as in previous recent work concerning Saint-Venant's principle [4, 5]. Pointwise estimates constructed on this basis suffer significant drawbacks, as pointed out elsewhere [4, 5, 6]; the estimates for the solution itself constructed in [3] avoid these drawbacks. The major extensions necessary to obtain estimates for the first derivatives which are comparable in character to those derived in [3] are presented here.

Because we are concerned directly with the calculation of bounds on errors due to approximate solutions of the differential equations involved, we shall encounter circumstances which did not arise in [1] or [3].

The method which we use rests on the derivation of "energy inequalities" and on the deduction of pointwise estimates from the results pertaining to energies. We shall obtain most of our results in the general context of second-order elliptic operators of divergence type on rectangular two-dimensional domains as in [3]. These general results are presented in Chapters II - IV, following the formulation in Chapter I of the axisymmetric torsion problem and the questions we wish to ask about it. In Chapter V we return to the shell problem for the detailed application of the general estimates derived in Chapters II - IV.

In §2 of the present chapter, we introduce the natural coordinate system for shells of revolution and state the necessary geometrical preliminaries. The differential equations of elasticity in shell coordinates and the boundary value problems to be considered for these equations are formulated in §3. Although Michell's theory of axisymmetric torsion [7] has been considered in detail for cylindrical coordinates in [1] and elsewhere,[3] we rederive it here in §4 in the form appropriate for shell coordinates. In order to exhibit the small parameter which measures the thinness of the shell, we introduce dimensionless variables and reduce the boundary value problem to its final form in §5. In §6 we construct an approximate solution to the axisymmetric torsion problem for a thin shell, and we frame the "residual problem" appropriate to the difference between the exact and the approximate solution.

Finally, we summarize our results in §7, and in §8 we relate them to other work in this general area.

2. Geometric Preliminaries. Shell Coordinates.

To describe the shell of revolution, we shall first describe its meridional cross-section. Let r, θ, z be cylindrical coordinates, and let θ be fixed. Let C be a smooth curve in this half-plane of constant θ with parametric equations

$$C: \qquad r = r_o(\xi), \quad z = z_o(\xi), \quad 0 \le \xi \le \ell, \tag{2.1}$$

where ξ is arc length on C, ℓ is the length of C, and r_o and z_o

[3] See, for example, Mindlin and Salvadori[8].

are twice continuously differentiable functions on $[0, \ell]$ with $z_o(0) = 0$. (See Figure 1.) If h is a given constant, the meridional cross-section of the shell of revolution consists of all points (in the half-plane of fixed θ) which lie on some line segment of length h whose midpoint lies on C and which is perpendicular to C at their common point. The set of all points in this plane which are interior points of the meridional cross-section is denoted by M ; the closed cross-section is denoted by $\overline{M}$.

The shell of revolution is now generated by rotating $\overline{M}$ about the z-axis (see Figure 2). It should be noted that the "ends" of the shell are not plane; they are conical surfaces which we shall call the terminal surfaces. The terminal surfaces corresponding to the ends $\xi = 0$ and $\xi = \ell$ of C are denoted by π_1 and π_2 respectively. The remaining portions of the surface forming the boundary of the shell are surfaces of revolution parallel to and equidistant from the mid-surface S obtained by rotating C about the z-axis. Let S_1 denote the outer surface; S_2 the inner surface. The region consisting of interior points of the shell is denoted by $\mathcal{R}$; its closure is $\overline{\mathcal{R}}$. The boundary $\partial\mathcal{R}$ of $\mathcal{R}$ is $\partial\mathcal{R} = \pi_1 + \pi_2 + S_1 + S_2$.

In shell theory, it is customary[4] to use an orthogonal curvilinear coordinate system which we shall call shell coordinates; the coordinates of a typical point P are denoted by (ξ, θ, ζ). Here, ξ and θ are, respectively, the arc length along C and the cylindrical polar angle already introduced, while the coördinate ζ is the perpendicular distance from P to the midsurface S (see Figure 1). Points on one

[4] See, for example, [9, 10].

side of S are associated with positive values of ζ, while those on the other side of S are associated with negative values. The shell coordinates of P are related to its cylindrical coordinates r, θ, z by

$$\left.\begin{aligned} r &= r(\xi, \zeta) = r_o(\xi) - \zeta \sin\beta(\xi) \\ z &= z(\xi, \zeta) = z_o(\xi) + \zeta \cos\beta(\xi) \end{aligned}\right\}, \qquad \begin{aligned} &0 \le \xi \le \ell \ , \\ &-h/2 \le \zeta \le h/2 \ , \end{aligned} \tag{2.2}$$

where

$$\beta(\xi) = \tan^{-1}\left[\frac{z_o'(\xi)}{r_o'(\xi)}\right] \tag{2.3}$$

is the angle measured from the "r-axis" in the plane of (r, z) to the tangent to C in the direction of increasing ξ (see Figure 1). The metric of the orthogonal curvilinear coordinate system[5] (ξ, θ, ζ) is given by the differential form

$$ds^2 = \left(1 + \frac{\zeta}{R_\xi}\right)^2 d\xi^2 + [r_o(\xi)]^2\left(1 + \frac{\zeta}{R_\theta}\right)^2 d\theta^2 + d\zeta^2 \ , \tag{2.4}$$

where ds refers to the local Euclidian distance and

$$\frac{1}{R_\xi(\xi)} = -\beta'(\xi) \ , \quad \frac{1}{R_\theta(\xi)} = -\frac{\sin\beta(\xi)}{r_o(\xi)} \ , \tag{2.5}$$

are the principal curvatures of the midsurface; the superscripted comma indicates differentiation with respect to the argument.

Some further notation is convenient. Denote by Γ_1 and Γ_2 the respective intersections of the surfaces S_1 and S_2 with the meridional half-plane of fixed θ, and let L_1 and L_2 be the corresponding intersections of π_1 and π_2 with this half-plane (see Figure 1).

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[5] The fact that $r_o(\xi)$ and $z_o(\xi)$ are twice continuously differentiable on $[0, \ell]$ assures that (ξ, θ, ζ) do form a coordinate system for sufficiently small h. The orthogonality is easily demonstrated. See [9, 10].

3. The Elastostatic Boundary Value Problem.

In terms of the shell coordinates (ξ, θ, ζ) the basic field equations of linear elastistatics take the following form in the case of rotational symmetry.[6]

(i) Equation of equilibrium:

$$\frac{\partial}{\partial\xi}\left[r_o\left(1+\frac{\zeta}{R_\theta}\right)\tau_{\xi\xi}\right]+r_o\frac{\partial}{\partial\zeta}\left[\left(1+\frac{\zeta}{R_\xi}\right)\left(1+\frac{\zeta}{R_\theta}\right)\tau_{\xi\zeta}\right]+$$

$$+\tau_{\xi\zeta}\frac{r_o}{R_\xi}\left(1+\frac{\zeta}{R_\theta}\right)-\tau_{\theta\theta}\frac{\partial}{\partial\xi}\left[r_o\left(1+\frac{\zeta}{R_\theta}\right)\right]=0, \tag{3.1}$$

$$\frac{\partial}{\partial\xi}\left[r_o\left(1+\frac{\zeta}{R_\theta}\right)\tau_{\xi\theta}\right]+r_o\frac{\partial}{\partial\zeta}\left[\left(1+\frac{\zeta}{R_\xi}\right)\left(1+\frac{\zeta}{R_\theta}\right)\tau_{\theta\zeta}\right]+$$

$$+\tau_{\xi\theta}\frac{\partial}{\partial\xi}\left[r_o\left(1+\frac{\zeta}{R_\theta}\right)\right]+\tau_{\theta\zeta}\frac{r_o}{R_\theta}\left(1+\frac{\zeta}{R_\xi}\right)=0, \tag{3.2}$$

$$\frac{\partial}{\partial\xi}\left[r_o\left(1+\frac{\zeta}{R_\theta}\right)\tau_{\xi\zeta}\right]+r_o\frac{\partial}{\partial\zeta}\left[\left(1+\frac{\zeta}{R_\xi}\right)\left(1+\frac{\zeta}{R_\theta}\right)\tau_{\zeta\zeta}\right]+$$

$$-\frac{r_o}{R_\xi}\left(1+\frac{\zeta}{R_\theta}\right)\tau_{\xi\xi}-\frac{r_o}{R_\theta}\left(1+\frac{\zeta}{R_\xi}\right)\tau_{\theta\theta}=0. \tag{3.3}$$

In these equations, $\tau_{\xi\xi}$, etc., denote the components in the coordinate system (ξ, θ, ζ) of the stress tensor $\underline{\tau}$. We assume that $\underline{\tau}$ is independent of θ.

(ii) Stress-strain relations.

$$\tau_{\xi\xi}=(\lambda+2\mu)e_{\xi\xi}+\lambda(e_{\theta\theta}+e_{\zeta\zeta}), \tag{3.4}$$

$$\tau_{\theta\theta}=(\lambda+2\mu)e_{\theta\theta}+\lambda(e_{\zeta\zeta}+e_{\xi\xi}), \tag{3.5}$$

$$\tau_{\zeta\zeta}=(\lambda+2\mu)e_{\zeta\zeta}+\lambda(e_{\xi\xi}+e_{\theta\theta}), \tag{3.6}$$

$$\tau_{\xi\theta}=2\mu e_{\xi\theta}, \tag{3.7}$$

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6 See [10].

$$\tau_{\theta\zeta} = 2\mu e_{\theta\zeta} \, , \tag{3.8}$$

$$\tau_{\zeta\xi} = 2\mu e_{\zeta\xi} \, , \tag{3.9}$$

where $e_{\xi\xi}$, etc., denote the components in shell coordinates of the strain tensor $\underline{e}$, and λ and μ are the Lamé constants characteristic of a homogeneous, isotropic elastic material.

(iii) Strain-displacement relations.

$$e_{\xi\xi} = \frac{1}{(1+\frac{\zeta}{R_\xi})} \left(\frac{\partial u_\xi}{\partial \xi} + \frac{u_\zeta}{R_\xi} \right) , \tag{3.10}$$

$$e_{\theta\theta} = \frac{1}{r_o(1+\frac{\zeta}{R_\theta})} \left[u_\xi (1+\frac{\zeta}{R_\xi}) \frac{\partial}{\partial \xi} \{ r_o (1+\frac{\zeta}{R_\theta}) \} + r_o \frac{u_\zeta}{R_\theta} \right] , \tag{3.11}$$

$$e_{\zeta\zeta} = \frac{\partial u_\zeta}{\partial \zeta} \, , \tag{3.12}$$

$$e_{\xi\theta} = \frac{1}{2} \, \frac{1+\frac{\zeta}{R_\theta}}{1+\frac{\zeta}{R_\xi}} \, \frac{\partial}{\partial \xi} \left[\frac{u_\theta}{r_o(1+\frac{\zeta}{R_\theta})} \right] \, , \tag{3.13}$$

$$e_{\theta\zeta} = \frac{1}{2} (1+\frac{\zeta}{R_\theta}) \frac{\partial}{\partial \zeta} \left(\frac{u_\theta}{1+\frac{\zeta}{R_\theta}} \right) \, , \tag{3.14}$$

$$e_{\zeta\xi} = \frac{1}{2} \, \frac{1}{1+\frac{\zeta}{R_\xi}} \, \frac{\partial u_\zeta}{\partial \xi} + (1+\frac{\zeta}{R_\xi}) \frac{\partial}{\partial \zeta} \left(\frac{u_\xi}{1+\frac{\zeta}{R_\xi}} \right) \, , \tag{3.15}$$

where u_ξ, u_θ, and u_ζ are the components in shell coordinates of the displacement vector $\underline{u}$.

In equations (3.1) through (3.15), it has been assumed that u_ξ, u_θ, and u_ζ, and therefore the components of strain and stress, do not depend on θ.

Our objective is to determine a displacement field $\underline{u}$, a strain field $\underline{e}$, and a stress field $\underline{\tau}$, each defined and continuously differentiable on $\overline{\mathcal{R}}$, and satisfying (3.1) - (3.15). In addition, we require that the outer and inner surfaces be traction free, so that

$$\tau_{\zeta\xi} = \tau_{\zeta\zeta} = \tau_{\zeta\theta} = 0 \quad \text{on } S_1 \text{ and } S_2, \tag{3.16}$$

while over the terminal surfaces π_1 and π_2 we impose the boundary conditions as follows:

$$\tau_{\xi\zeta} = \tau_{\xi\xi} = 0, \quad \tau_{\xi\theta} = f_i \quad \text{on } \pi_i, \quad i = 1, 2. \tag{3.17}$$

Here, f_i is a given function defined and continuously differentiable on $[-h/2, h/2]$.

A necessary condition for the above boundary value problem to have a solution is that torques produced on the terminal surfaces by the applied shear tractions f_i be self-equilibrating. Thus, it is necessary that

$$\int_{L_1} r^2 f_1 d\zeta = \int_{L_2} r^2 f_2 d\zeta = \frac{T}{2\pi}. \tag{3.18}$$

Here, T stands for the scalar torque due to the applied tractions.

If a solution $\{\underline{u}, \underline{e}, \underline{\tau}\}$ of the foregoing boundary value problem exists, it is unique, apart from arbitrary additive rigid-body displacements, provided the shear modulus μ and Poisson's ratio ν satisfy

$$\mu > 0, \quad -1 < \nu < \tfrac{1}{2}. \tag{3.19}$$

4. Michell's Theory in Shell Coordinates.

To reduce the problem formulated in the preceding section, we restrict $\underline{u}$ by requiring that

$$u_\xi = u_\zeta = 0 \ , \quad u_\theta(\xi,\zeta) = r(\xi,\zeta)\varphi(\xi,\zeta) \ , \tag{4.1}$$

where $r(\xi,\zeta)$ is given by the first of (2.2). We retain the terminology of [1] and refer to φ as the twist function. From (4.1) and (3.4) - (3.9) we find that

$$\tau_{\xi\xi} = \tau_{\theta\theta} = \tau_{\zeta\zeta} = \tau_{\zeta\xi} = 0 \ , \tag{4.2}$$

and

$$\tau_{\zeta\theta} = \mu r \frac{\partial\varphi}{\partial\zeta} \ , \qquad \tau_{\xi\theta} = \frac{\mu r \frac{\partial\varphi}{\partial\xi}}{1+\frac{\zeta}{R_\xi}} \ . \tag{4.3}$$

For stress fields $\underline{\tau}$ of the form (4.2), (4.3), two of the equilibrium equations (3.1) and (3.3) are identically satisfied, while the remaining one (3.2) can be reduced to

$$\frac{\partial}{\partial\xi}(r^2\tau_{\xi\theta}) + \frac{\partial}{\partial\zeta}[r^2(1+\frac{\zeta}{R_\xi})\tau_{\zeta\theta}] = 0 \ . \tag{4.4}$$

Substituting from (4.3) into (4.4) provides the differential equation satisfied by φ:

$$\frac{\partial}{\partial\xi}[r^3(1+\frac{\zeta}{R_\xi})^{-1}\frac{\partial\varphi}{\partial\xi}] + \frac{\partial}{\partial\zeta}[r^3(1+\frac{\zeta}{R_\xi})\frac{\partial\varphi}{\partial\zeta}] = 0 \ . \tag{4.5}$$

The boundary conditions associated with φ are obtained from (3.16), (3.17), (4.2), and (4.3). On the lateral surfaces

$$\zeta = \pm\frac{h}{2} : \qquad \frac{\partial\varphi}{\partial\zeta} = 0 \ . \tag{4.6}$$

On the terminal surfaces

$$\xi = 0 : \qquad \frac{r^3}{1+\frac{\zeta}{R_\xi}}\frac{\partial\varphi}{\partial\xi} = \frac{r^2}{\mu}f_1(\zeta) \ , \tag{4.7}$$

$$\xi = \ell : \qquad \frac{r^3}{1+\frac{\zeta}{R_\xi}}\frac{\partial\varphi}{\partial\xi} = \frac{r^2}{\mu}f_2(\zeta) \ . \tag{4.8}$$

We require a function φ, continuously differentiable once on $\overline{M}$ and twice on M, which satisfies (4.5) on M and the boundary conditions (4.6) through (4.8). To fix the arbitrary constant which may be added to any solution of the foregoing boundary value problem, we must add a normalization condition. Many different conditions, such as the requirement that φ have a zero average on M, will serve this purpose. Since the type of normalization best suited to our purposes is not clear until some of the subsequent analysis has been carried out, we defer until later an explicit statement of this condition.

We now deduce from (4.5), (4.6), and (3.18) a useful formula describing the conservation of torque on sections ξ = constant. Integrating (4.5) with respect to ζ from $\zeta = -h/2$ to $\zeta = +h/2$ and using (4.6), we find that

$$\int_{-h/2}^{h/2} \frac{r^3}{1+\frac{\zeta}{R_\xi}} \frac{\partial\varphi}{\partial\xi} d\zeta = \text{constant}, \quad 0 \le \xi \le \ell .$$

Using (3.18) and (4.7), we conclude that

$$\int_{-h/2}^{h/2} \frac{r^3}{1+\frac{\zeta}{R_\xi}} \frac{\partial\varphi}{\partial\xi} d\zeta = \frac{T}{2\pi\mu}, \quad 0 \le \xi \le \ell . \tag{4.9}$$

The above reduction of the torsion problem thus leads to a second-order boundary value problem of Neumann type for the twist function φ. An alternative reduction to a second order problem of Dirichlet type begins with the following representation, guaranteed by (4.4) and the simple connectivity of M, of the stresses in terms of a stress function ψ.

$$\tau_{\xi\theta} = \frac{\mu}{r^2}\frac{\partial\psi}{\partial\zeta}, \quad \tau_{\zeta\theta} = -\frac{\mu\frac{\partial\psi}{\partial\xi}}{r^2\left(1+\frac{\zeta}{R_\xi}\right)} . \tag{4.10}$$

A comparison of (4.10) with (4.3), followed by the elimination of φ, furnishes the differential equation for ψ in the form

$$\frac{\partial}{\partial\xi}\left[r^{-3}\left(1+\frac{\zeta}{R_\xi}\right)^{-1}\frac{\partial\psi}{\partial\xi}\right]+\frac{\partial}{\partial\zeta}\left[r^{-3}\left(1+\frac{\zeta}{R_\xi}\right)\frac{\partial\psi}{\partial\zeta}\right] = 0 \quad \text{on } M . \tag{4.11}$$

The boundary conditions associated with (4.11) follow from (3.16), (3.17), (4.2), and (4.10).

$$\zeta = h/2 : \quad \psi = C_1 , \tag{4.12}$$

$$\zeta = -h/2 : \quad \psi = C_2 , \tag{4.13}$$

$$\xi = 0: \quad \psi = \frac{1}{\mu}\int_{-h/2}^{\zeta} r^2 f_1(\zeta')d\zeta' + C_3 , \tag{4.14}$$

$$\xi = \ell: \quad \psi = \frac{1}{\mu}\int_{-h/2}^{\zeta} r^2 f_2(\zeta')d\zeta' + C_4 , \tag{4.15}$$

where C_1, C_2, C_3, and C_4 are integration constants. As a consequence of the assumed continuity of ψ on $\overline{M}$ and the overall equilibrium condition (3.18), we must have

$$C_2 = C_3 = C_4 = C_1 - \frac{T}{2\pi\mu} . \tag{4.16}$$

Since ψ in (4.10) is defined up to an arbitrary additive constant, one of the four constants in (4.16) is at our disposal. Choosing $C_1 = T/2\pi\mu$, we obtain the boundary conditions for (4.11) in the simpler form

$$\zeta = h/2 : \quad \psi = \frac{T}{2\pi\mu} , \tag{4.17}$$

$$\zeta = -h/2 : \quad \psi = 0 , \tag{4.18}$$

$$\xi = 0: \qquad \psi = \frac{1}{\mu}\int_{-h/2}^{\zeta} r^2 f_1(\zeta')d\zeta' \,, \tag{4.19}$$

$$\xi = \ell: \qquad \psi = \frac{1}{\mu}\int_{-h/2}^{\zeta} r^2 f_2(\zeta')d\zeta' \,. \tag{4.20}$$

Equations (4.11) and (4.17) through (4.20) describe a problem of Dirichlet type for ψ.

If the shell is a hollow circular cylinder, the curvature $1/R_\xi$ vanishes, and the differential equations (4.5) and (4.11) are identical with the equations of Michell's theory in cylindrical coordinates [1].

5. Dimensionless Variables. The Boundary Value Problem's Final Form.

Since our ultimate interest is in the thin shell, we shall introduce new independent variables which make clear the sense in which the notion of thinness is intended. Let

$$R = \min\left\{ \min_{[0,\ell]} |R_\xi(\xi)|, \min_{[0,\ell]} |R_\theta(\xi)| \right\} \tag{5.1}$$

be the minimum principal radius of curvature of the midsurface of the shell, and let

$$L = \min(R, \ell) \,, \tag{5.2}$$

where ℓ is the length of the meridian curves of the midsurface S. Define

$$\varepsilon = h/L \,. \tag{5.3}$$

We speak of the shell as thin if $\varepsilon << 1$. To put in evidence the role played by ε, we use the following dimensionless quantities:

$$x = \xi/L \,, \quad y = \zeta/h \,, \quad \bar{\ell} = \ell/L \,. \tag{5.4}$$

The open meridional domain M now corresponds to the rectangular domain $\mathcal{R}$ given by

$$\mathcal{R}: \qquad 0 < x < \overline{\ell} \ , \qquad -\tfrac{1}{2} < y < \tfrac{1}{2} \ . \tag{5.5}$$

It is also convenient to set

$$p(x, y; \epsilon) = \epsilon^2 r^3(Lx, hy)\left[1 + \frac{\epsilon y}{\rho(x)} \right]^{-1} , \tag{5.6a}$$

$$q(x, y; \epsilon) = r^3(Lx, hy)\left[1 + \frac{\epsilon y}{\rho(x)} \right] , \tag{5.6b}$$

where

$$\rho(x) = R_{\xi}(Lx)/L \ . \tag{5.7}$$

We note that, according to (5.1), the dimensionless radius of curvature $\rho(x)$ satisfies

$$|\rho(x)| \geq 1 \ , \qquad 0 \leq x \leq \overline{\ell} \ . \tag{5.8}$$

In the new notation, the differential equation (4.5) for the twist function φ takes the following form.[7]

$$[p(x, y;\epsilon)\varphi_x(x, y;\epsilon)]_x + [q(x, y;\epsilon)\varphi_y(x, y;\epsilon)]_y = 0 \qquad \text{on } \mathcal{R} \ . \tag{5.9}$$

The boundary conditions (4.6) - (4.8) become

$$y = \pm\tfrac{1}{2}: \quad \varphi_y = 0 \ , \tag{5.10}$$

$$x = 0: \qquad p\varphi_x = \frac{\epsilon^2 L r^2(0, hy)}{\mu} f_1(hy) \ , \tag{5.11}$$

$$x = \overline{\ell}: \qquad p\varphi_x = \frac{\epsilon^2 L r^2(\ell, hy)}{\mu} f_2(hy) \ . \tag{5.12}$$

The torque conservation formula (4.9) now appears to be

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} p(x, y;\epsilon)\varphi_x(x, y;\epsilon)dy = \frac{\epsilon T}{2\pi\mu} \ , \qquad 0 \leq x \leq \overline{\ell} \ . \tag{5.13}$$

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[7] We now regard φ as a function of x, y and ϵ without introducing the new notation which is suggested by the transformation (5.4).

In (5. 9) through (5. 13) and in the sequel, the subscript x or y attached to a quantity indicates differentiation with respect to the corresponding coordinate.

While it is clearly possible to recast the Dirichlet type problem for ψ in the new variables, we shall not do so explicitly because our main interest is in the problem for φ.

6. An Approximate Solution for the Thin Shell.

In the case of the thin shell, ϵ is small compared to unity, and p is therefore small compared to q, according to (5. 6). To construct an approximate solution of the boundary value problem which takes advantage of the thinness of the shell, it is therefore natural to investigate the result of neglecting the first term in the differential equation (5. 9). Using a tilde to connote an approximation to φ, we consider the following mutilated version of (5. 9):

$$(q\tilde{\varphi}_y)_y = 0 \qquad \text{on } \mathfrak{R} . \tag{6. 1}$$

In replacing (5. 9) by (6. 1), we lose the capacity of satisfying all of the original boundary conditions, as is commonly the case with approximating procedures which alter the type or reduce the order of the governing differential equation. To (6. 1) we add the boundary conditions (5. 10),

$$y = \pm \tfrac{1}{2} : \qquad \tilde{\varphi}_y = 0 , \tag{6. 2}$$

but we discard the boundary conditions (5. 11) and (5. 12) at the ends of the shell. We thus expect that the approximation $\tilde{\varphi}$ will be of poor quality near the ends. This corresponds to the anticipated "boundary-layer" character of the exact solution.

Together, (6.1) and (6.2) are equivalent to the statement that $\widetilde{\varphi}$ is independent of y. To complete the determination of $\widetilde{\varphi}$, we enforce the torque conservation formula (5.13), which reads as follows when applied to $\widetilde{\varphi}$:

$$\widetilde{\varphi}_x(x;\varepsilon)\int_{-\frac{1}{2}}^{\frac{1}{2}} p(x,y;\varepsilon)dy = \frac{\varepsilon T}{2\pi\mu} . \tag{6.3}$$

Thus,

$$\widetilde{\varphi} = \frac{\varepsilon T}{2\pi\mu}\int_0^x \left[\int_{-\frac{1}{2}}^{\frac{1}{2}} p(x',y;\varepsilon)dy\right]^{-1} dx' + \text{constant} , \tag{6.4}$$

where the arbitrary additive constant corresponds to an infinitesimal rigid body rotation and can be taken as zero.

To convert the formula (6.4) to the notation pertaining to the original geometry, we use (5.6a), (5.7), and (5.4) to write

$$\widetilde{\varphi} = \frac{T}{2\pi\mu}\int_0^{\xi}\left[\int_{-h/2}^{h/2} \frac{r^3(\xi',\zeta)d\zeta}{1+\zeta/R_\xi(\xi')}\right]^{-1} d\xi' . \tag{6.5}$$

In [2], Love derives a general, two-dimensional approximate theory of shells and applies it, in particular, to deformations of shells of revolution. For the case of axisymmetric torsion,[8] his theory leads to a formula for the value at the midsurface of the circumferential displacement. If in (6.5) we neglect terms of the form ζ/R in comparison with unity, so that $r^3(\xi,\zeta)$ is replaced by $r_o^3(\xi)$ and $1+\frac{\zeta}{R_\xi(\xi)}$ is replaced by unity, we are led immediately to Love's formula.

The approximate stress associated with the approximation $\widetilde{\varphi}$ of (6.4) [or (6.5)] would be computed from (4.3) after replacing φ by

[8] See Love [2], page 567.

$\widetilde{\varphi}$. Thus,

$$\widetilde{\tau}_{\zeta\theta} = 0 \,, \quad \widetilde{\tau}_{\xi\theta} = \frac{T}{2\pi} \frac{r(\xi,\zeta)}{1+\zeta/R_\xi(\xi)} \left[\int_{-h/2}^{h/2} \frac{r^3(\xi,\zeta)d\zeta}{1+\zeta/R_\xi(\xi)} \right]^{-1} \,. \tag{6.6}$$

The methods to be developed in the sequel permit us to estimate rigorously the errors $\varphi - \widetilde{\varphi}$, $\tau_{\zeta\theta} - \widetilde{\tau}_{\zeta\theta}$, and $\tau_{\xi\theta} - \widetilde{\tau}_{\xi\theta}$ at each point in the domain. We will find that the anticipated boundary-layer character of the solution is confirmed, and at points away from the terminal surfaces, the twist- and stress-errors are small as $\epsilon \to 0$. Moreover, our upper bounds on the errors will exhibit the order of this smallness in ϵ explicitly. Thus, $\widetilde{\varphi}$ is an "interior approximation" to the exact solution φ. Set

$$\hat{\varphi} = \varphi - \widetilde{\varphi} \qquad \text{on } \overline{\mathcal{R}} \;; \tag{6.7}$$

the analysis to follow is applied to the boundary-value problem satisfied by the error $\hat{\varphi}$. From (5.9) and (6.1), we find that $\hat{\varphi}$ satisfies the differential equation

$$(p\hat{\varphi}_x)_x + (q\hat{\varphi}_y)_y = F \qquad \text{on } \mathcal{R} \,, \tag{6.8}$$

where the known nonhomogeneous F is given by

$$F(x,y;\epsilon) = -[p(x,y;\epsilon)\widetilde{\varphi}_x(x;\epsilon)]_x \,. \tag{6.9}$$

The boundary conditions satisfied by $\hat{\varphi}$ are obtained from (5.10) - (5.12) and (6.2). They are

$$y = \pm \tfrac{1}{2} : \qquad \hat{\varphi}_y = 0 \,, \tag{6.10}$$

$$x = 0 : \qquad p\hat{\varphi}_x = \frac{\epsilon^2 L r^2}{\mu} f_1 - p\widetilde{\varphi}_x \equiv g_1(y;\epsilon) \tag{6.11}$$

$$x = \overline{\ell} : \qquad p\hat{\varphi}_x = \frac{\epsilon^2 L r^2}{\mu} f_2 - p\widetilde{\varphi}_x \equiv g_2(y;\epsilon) \,. \tag{6.12}$$

Since the approximate twist function $\widetilde{\varphi}$ was forced to satisfy the torque conservation formula (5.13), and since f_1 and f_2 satisfy (3.18), it follows that the boundary values g_1 and g_2 appearing in (6. 11) and (6. 12) are "self-equilibrated":

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} g_1(y;\varepsilon)dy = \int_{-\frac{1}{2}}^{\frac{1}{2}} g_2(y;\varepsilon)dy = 0 . \qquad (6.13)$$

This is an important feature of the boundary value problem (6. 8) - (6. 12). In fact, this is the principal reason that we employ $\widetilde{\varphi}$ as given in (6. 4) as our approximate solution, rather than the function obtained from φ by retaining only the leading terms in ε. If this latter function were to be used in place of $\widetilde{\varphi}$, the torque conservation formula would be violated, the functions g_1 and g_2 appearing in the foregoing "residual boundary value problem" would not be self-equilibrated, and the subsequent analysis would be more difficult.

In view of the self-equilibration (6. 13) of the "end-loads" in the "residual boundary-value problem," it is to be expected that a necessary condition for the existence of a solution to this problem is that F satisfies

$$\int_{0}^{\overline{\ell}} \int_{-\frac{1}{2}}^{\frac{1}{2}} F(x,y;\varepsilon)dydx = 0 . \qquad (6.14)$$

The necessity of this condition is readily confirmed. Moreover, it is easily verified with the aid of (6. 3) that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} F(x,y;\varepsilon)dy = 0 , \qquad 0 \le x \le \overline{\ell} , \qquad (6.15)$$

from which (6. 14) follows.

7. Summary of Results.

In the sequel, we shall aim at quantitative estimates, valid on the closure $\overline{M}$ of M, for the error stresses $\hat{\tau}_{\xi\theta}$ and $\hat{\tau}_{\zeta\theta}$, which are proportional to the derivatives of the error $\hat{\varphi}$ according to (4.3).[9] The main results of this investigation state[10] that on $\overline{M}$

$$|\hat{\varphi}(x,y)| \le K_1+K_2 e^{-\frac{k_o}{4\varepsilon}x}+K_3 e^{-\frac{k_o}{\varepsilon}(\overline{\ell}-x)}, \qquad (7.1)$$

$$|\hat{\varphi}_x(x,y)| \le K_4+K_5 e^{-\frac{k_o x}{\varepsilon 8}}+K_6 e^{-\frac{k_o}{\varepsilon}(\overline{\ell}-x)}, \qquad (7.2)$$

$$|\hat{\varphi}_y(x,y)| \le K_7+K_8 e^{-\frac{k_o x}{\varepsilon 8}}+K_9 e^{-\frac{k_o}{\varepsilon}(\overline{\ell}-x)}, \qquad (7.3)$$

where

$$k_o = \pi \sqrt{\frac{r_2^3\left(1-\frac{\varepsilon}{2}\frac{1}{\hat{\rho}}\right)}{r_1^3\left(1+\frac{\varepsilon}{2}\frac{1}{\hat{\rho}}\right)}}, \qquad (7.4)$$

and the constants ε and $\overline{\ell}$ are given by (5.3) and (5.4), respectively.

In (7.4), r_1, r_2, and $\hat{\rho}$ are constants such that

$$r_2 \le r(x,y) \le r_1 \quad , \quad \hat{\rho} \le |\rho(x)| \quad \text{on } \overline{M}, \qquad (7.5)$$

with $\rho(x)$ defined by (5.7) and in (7.1) - (7.3), the constants K_1 through K_9 are fully determined. In fact, we can show from (29.9) - (29.17) that for fixed (x, y) on $0 \le x \le \overline{\ell}$ and $-\frac{1}{2} \le y \le \frac{1}{2}$, K_1 through K_9 satisfy

$$K_1 = O(\varepsilon^2)+O(\varepsilon^{1/2})\exp[-O(\varepsilon^{-1})\overline{\ell}] \qquad \text{as } \varepsilon \to 0, \qquad (7.6)$$

---

[9] See also (32.1).

[10] See (23.1) - (23.4) and (30.12) - (30.14).

$$K_2 = O(\varepsilon^{3/4}) , \; K_3 = O(\varepsilon^{3/4}) , \; K_4 = O(\varepsilon^{3/2}) \quad \text{as } \varepsilon \to 0 , \tag{7.7}$$

$$K_5 = O(1) , \; K_6 = O(1) \qquad \text{as } \varepsilon \to 0 , \tag{7.8}$$

$$K_7 = O(\varepsilon^{5/2}) + O(\varepsilon) \exp[-O(\varepsilon^{-1})\bar{\ell}] \qquad \text{as } \varepsilon \to 0 , \tag{7.9}$$

and

$$K_8 = O(\varepsilon^{3/2}) , \; K_9 = O(\varepsilon^{3/2}) \qquad \text{as } \varepsilon \to 0 . \tag{7.10}$$

Inequalities (7.1) - (7.3) are derived from energy-like inequalities which are given in detail in Chapters II and III. When (7.2) and (7.3) are coupled with the basic relations between stresses and the twist function, they yield, in terms of the variables (x, y),

$$|\hat{\tau}_{x\theta}(x, y)| \le \frac{\mu r}{L}\left[1 + \frac{\varepsilon y}{\rho(x)}\right]^{-1}\left[K_4 + K_5 e^{-\frac{k_o x}{\varepsilon 8}} + K_6 e^{-\frac{k_o}{\varepsilon}(\bar{\ell}-x)}\right], \tag{7.11}$$

$$|\hat{\tau}_{y\theta}(x, y)| \le \frac{\mu r}{\varepsilon L}\left[K_7 + K_8 e^{-\frac{k_o x}{\varepsilon 8}} + K_9 e^{-\frac{k_o}{\varepsilon}(\bar{\ell}-x)}\right], \tag{7.12}$$

where μ is shear modulus, and L is defined in (5.2).

In §32, we shall further show that (7.11) and (7.12) remain bounded as the point (x, y) approaches the boundary of $\bar{M}$. This important result implies that the estimates for the error stresses are uniformly valid on the closure $\bar{M}$ of M.

8. Relations to Previous Work.

Energy inequalities of the type (10.2) were first derived in connection with problems of elasticity theory independently by Knowles [4] and Toupin [5]. The main purpose of the analyses in [4] and [5] was to precisely formulate certain quantitative versions of Saint Venant's principle applicable to a class of elastic solids.

This type of Saint Venant's principle was also investigated in [1] for the problem of axisymmetric torsion for elastic solids of revolution with plane terminal sections. Two particular cases of bodies of revolution were considered; a hollow body and a solid body. Based on the methods developed in [4], stress inequalities of the following type were constructed in [1]:

$$|\tau(r,\theta,z)| \le 30\sqrt{\frac{\mu}{\pi}\frac{U(0)}{c(\nu)}}\;\frac{1}{\delta^{3/2}}\;\exp\left[-\int_0^{z-\delta}\sqrt{k(\zeta)}\,d\zeta\right], \tag{8.1}$$

where τ stands for any one of the stress components in cylindrical coordinates (r, θ, z); μ and ν are shear modulus and Poisson's ratio, respectively; $c(\nu)$ is a constant defined by

$$c(\nu) = \min_{-\frac{1}{2}<\nu<1}\left[1, \frac{1-2\nu}{1+\nu}\right]. \tag{8.2}$$

The constant $U(0)$ represents the total strain energy contained in the body and can be bounded in terms of the load data and the geometry of the body; the function $k(\zeta)$ is determined by the geometry and is positive for $\zeta \in [0, \ell]$ where ℓ is the length of the cylinder; and finally, the constant δ represents the radius of a sphere which has the interior point (r, θ, z) as its center and which lies within the elastic solid.

We first note that the stress inequality given by (8.1) is characterized by a pure exponential decaying term. In contrast, the stress inequalities given by (7.11) and (7.12) contain non-decaying constant terms as well as pure exponential decaying terms. This difference arises from the nature of the basic differential equations dealt with. The differential equation, (6.8), governing the error $\hat{\varphi}$

in the approximation, is not homogeneous and the nonhomogeneous term is responsible for the non-decaying parts of the stress inequalities. On the other hand, the basic differential equations dealt with in [1] are homogeneous, and this fact is responsible for the absence of non-decaying terms in the results of that reference.

The stress inequality (8. 1) was obtained by combining an energy inequality of exponential decaying type with a suitable mean value theorem of elasticity theory. As was mentioned in §1, stress estimates constructed on this basis suffer significant drawbacks. We can now clarify this point with the help of (8. 1), where the distance δ from the boundary to the point at which the estimate is to be made always occurs in the denominator when mean value theorems are used. As the point at which stress estimates are required approaches the boundary of the elastic body, δ tends to zero, and the estimates fail. To amend such defects for the problem treated by Toupin in [5], a method based on the assessment of $\mathcal{L}^2$ norms of the derivatives of the unknown functions, and also on Sobolev's lemma [12], was used by Roseman in [6]. To avoid such drawbacks for the type of problem treated in [1], we develop a method which is an extension of that employed in [3]. This method leads to the establishment of useful pointwise estimates for the stresses; see (7. 11) and (7. 12). They are fully determined and uniformly valid up to the boundary of the cylinder.

II. ENERGY INEQUALITIES FOR A CLASS OF SECOND-ORDER BOUNDARY-VALUE PROBLEMS

9. Preliminaries.

Let $\mathfrak{R}$ denote the open rectangle $0 < x < \ell$, $-\frac{1}{2} < y < \frac{1}{2}$ in the x, y plane, and let p and q be positive, continuously differentiable functions on the closure $\overline{\mathfrak{R}}$ of $\mathfrak{R}$. Define the operator L by writing

$$Lu = (pu_x)_x + (qu_y)_y \qquad \text{on } \mathfrak{R}, \tag{9.1}$$

for any function u which is twice continuously differentiable on $\mathfrak{R}$. In this chapter and the following one, we shall be concerned with the boundary value problem for L which we now state.

$$Lu = F \qquad \text{on } \mathfrak{R}, \tag{9.2}$$

$$pu_x = g_1 \qquad \text{at } x = 0, \tag{9.3}$$

$$pu_x = g_2 \qquad \text{at } x = \ell, \tag{9.4}$$

$$u_y = 0 \qquad \text{at } y = \pm\tfrac{1}{2}, \tag{9.5}$$

where F is a given continuous function on $\overline{\mathfrak{R}}$, and g_1 and g_2 are given and continuous on $[-\frac{1}{2}, \frac{1}{2}]$. A necessary condition which must be satisfied by g_1, g_2 and F for the existence of a solution of the foregoing boundary-value problem is easily obtained by integrating (9. 2) over $\overline{\mathfrak{R}}$ and using the boundary conditions. There follows

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} (g_2 - g_1)\,dy = \int_{\mathfrak{R}} F\,dA. \tag{9.6}$$

We assume the existence of a solution u of the boundary value problem which is continuously differentiable once on $\overline{\mathfrak{R}}$ and twice on $\mathfrak{R}$. For some of the results to be derived in the sequel, we shall require more restrictive smoothness assumptions than those stated above.

These will be stated explicitly as the need arises.

The "residual boundary-value problem" (6.8) - (6.12) governing the difference $\hat{\varphi}$ between exact and approximate solutions of the shell problem is a special case of the foregoing general problem.[1] The necessary condition (9.6) is fulfilled in the shell problem; in fact, reference to (6.13) and (6.15) shows that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} g_1 dy = \int_{-\frac{1}{2}}^{\frac{1}{2}} g_2 dy = 0 , \qquad (9.7)$$

and

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} F(x, y) dy = 0 , \qquad 0 \le x \le \ell , \qquad (9.8)$$

so that the three integrals in (9.6) vanish separately. Therefore, in our subsequent discussion of the general problem (9.2) - (9.5), we shall expressly assume that the given functions F, g_1, and g_2 satisfy (9.7) and (9.8). This important assumption permits an essential decomposition of the basic boundary-value problem into "subproblems," as we now indicate.

We let

$$u = v + w , \qquad (9.9)$$

where v and w are solutions of the following two boundary-value problems.

<u>Problem I.</u>

$$Lv = 0 \qquad \text{on } \mathcal{R} , \qquad (9.10)$$

---

[1] In the shell problem of §6, the functions p, q, F, g_1 and g_2 depend on the parameter ϵ. Since this parameter is of no direct importance in the arguments used in the present chapter, we suppress it in the notation.

$$pv_x = 0 \qquad \text{at } x = 0 , \tag{9.11}$$

$$pv_x = g_2 \qquad \text{at } x = \ell , \tag{9.12}$$

$$v_y = 0 \qquad \text{at } y = \pm \tfrac{1}{2} . \tag{9.13}$$

Problem II.

$$Lw = F \qquad \text{on } \mathfrak{R} , \tag{9.14}$$

$$pw_x = g_1 \qquad \text{at } x = 0 , \tag{9.15}$$

$$pw_x = 0 \qquad \text{at } x = \ell , \tag{9.16}$$

$$w_y = 0 \qquad \text{at } y = \pm \tfrac{1}{2} . \tag{9.17}$$

The remainder of this chapter is devoted to the derivation of energy inequalities for Problems I and II which are roughly analogous to those of [3]. The present analysis differs from that of [3] in several important respects. Apart from a change of coordinates, Problem I is precisely the problem examined in [3]. The objective in [3], however, was to obtain a satisfactory pointwise estimate for v, and this required the consideration of first and second order energies (see below). We are now concerned with pointwise estimates for the first derivatives of v_x and v_y, and this will require the use of an associated third order energy.

Problem II involves the nonhomogeneous term F in the differential equation, and its treatment accordingly requires significant modifications of the analysis as given in [3].

10. First Order Energy for Problem I.

We define the first order energy V_1 associated with Problem I by writing

$$V_1(z) = \int_0^z \int_{-\frac{1}{2}}^{\frac{1}{2}} (pv_x^2 + qv_y^2) dydx \,. \tag{10.1}$$

Problem I is identical with the problem considered in [3], except that the nonhomogeneous boundary condition occurs at $x = 0$ in [3], rather than at $x = \ell$. We can, thus, appropriate directly the results of [3] after changing the variables in (9. 10) - (9. 13) from x, y to x', y', where $x' = \ell - x$, $y' = y$. After such a transformation,[2] we conclude from §3 of [3] that

$$V_1(z) \leq V_1(\ell) e^{-2k(\ell - z)}, \tag{10.2}$$

where

$$k = \pi \sqrt{q_o / p_1} \,, \tag{10.3}$$

and q_o, p_1 are positive constants such that

$$p(x, y) \leq p_1 \,, \quad q(x, y) \geq q_o \quad \text{on } \overline{\mathcal{R}} \,. \tag{10.4}$$

The constant $V_1(\ell)$ represents the total (first order) energy associated with the boundary value problem. An upper bound for $V_1(\ell)$ is required before the inequality (10. 2) becomes fully determined. We shall repeatedly encounter the question of finding an upper bound for total energies. We defer the calculation of such bounds until a later chapter.

Expression (10. 2) provides mean square estimates of the first derivatives of v.

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[2] We omit the details of the transformation and we do not repeat in detail the arguments in [3].

11. Second Order Energy for Problem I.

In analogy with the procedure in [3], the second order energy $V_2(z)$ associated with the boundary value problem is defined by

$$V_2(z) = \int_0^z \int_{-\frac{1}{2}}^{\frac{1}{2}} (pv_{yx}^2 + qv_{yy}^2)dydx \, . \tag{11.1}$$

In order to obtain an exponential decay inequality for $V_2(z)$, we need more stringent assumptions concerning the smoothness of the functions p, q, and g_2 appearing in the statement of Problem I, as well as the solution v. We assume, in fact, that p and q are twice continuously differentiable on $\overline{\mathcal{R}}$, that g_2 is continuously differentiable on $[-\frac{1}{2}, \frac{1}{2}]$, and that v is continuously differentiable twice on $\overline{\mathcal{R}}$ and three times on $\mathcal{R}$. The assumed continuity of v_{xy} on $\overline{\mathcal{R}}$, in particular, implies that the condition

$$p(\ell, \pm\tfrac{1}{2})g_2'(\pm\tfrac{1}{2}) - p_y(\ell, \pm\tfrac{1}{2})g_2(\pm\tfrac{1}{2}) = 0 \, , \tag{11.2}$$

must be satisfied by p and g_2.

According to §4 of [3], $V_2(z)$ satisfies the inequality

$$V_2(z) \le [V_2(\ell) + 2k(\alpha_1+\alpha_2)(\ell-z)V_1(\ell)]e^{-2k(\ell-z)} \, , \tag{11.3}$$

where the decay constant k is again given by (10.3) and $V_1(\ell)$ and $V_2(\ell)$ represent the total first- and second-order energies respectively. The constants α_1 and α_2 are such that on $\overline{\mathcal{R}}$

$$\frac{1}{\sqrt{pq}} \left| \frac{p_x p_y}{p} - p_{xy} \right| \le \alpha_1 \qquad \text{on } \overline{\mathcal{R}} \, , \tag{11.4}$$

and

$$\frac{1}{2} \left| \frac{p_y q_y}{pq} - \frac{q_{yy}}{q} - \frac{p_{yy}}{p} + \frac{p_y^2}{p^2} \right| \le \alpha_2 \quad \text{on } \overline{\mathcal{R}} \, . \tag{11.5}$$

When the total energies $V_1(\ell)$ and $V_2(\ell)$ are estimated, (11.3) provide mean square estimates of v_{xy} and v_{yy}. A mean square estimate of v_{xx} can be computed by using (11. 3), (10. 2), and the differential equation (9. 10).

12. Third Order Energy for Problem I.

The inequalities (10. 2) and (11. 3) provide mean square estimates of first and second derivatives of v respectively. These are sufficient to establish pointwise decay estimates for v itself, as shown in [3]. If pointwise information concerning the first derivatives of v is required, it is necessary to analyze the _third order energy_ defined by

$$V_3(z) = \int_0^z \int_{-\frac{1}{2}}^{\frac{1}{2}} (pv_{xyx}^2 + qv_{xyy}^2)dydx \quad . \tag{12. 1}$$

To establish a decay inequality for $V_3(z)$, we begin by setting

$$\overline{v} = v_{xy} \quad . \tag{12. 2}$$

We assume that the given functions p, q are three times continuously differentiable on $\overline{\mathcal{R}}$, and that the solution v of Problem I is continuously differentiable three times on $\overline{\mathcal{R}}$ and four times on $\mathcal{R}$.

By differentiating the basic differential equation (9. 10) once with respect to x and once with respect to y, we find that $\overline{v}$ satisfies an equation of the form

$$L\overline{v} = H \ , \qquad \text{on } \mathcal{R} \ , \tag{12. 3}$$

where H is given by

$$H = -p_{xxy}v_x - q_{xyy}v_y - (p_{xx}+q_{yy})v_{xy} - 2q_{xy}v_{yy} - 2p_{xy}v_{xx} - p_y v_{xxx} - q_y v_{xyy} -$$

$$-p_x v_{xxy} - q_y v_{yyy} \ . \qquad (12.4)$$

From (12.2) and the boundary conditions (9.11) - (9.13) satisfied by v , we derive the boundary conditions satisfied by $\bar{v}$:

$$\bar{v} = 0 \qquad \text{at } x = 0 \ , \qquad (12.5)$$

$$p^2(\ell, y)\bar{v}(\ell, y) = p(\ell, y)g_2'(y) - p_y(\ell, y)g_2(y) \qquad \text{at } x = \ell \ , \qquad (12.6)$$

$$\bar{v} = 0 \qquad \text{at } y = \pm\tfrac{1}{2} \ . \qquad (12.7)$$

Thus, $\bar{v}$ is the solution of a problem of Dirichlet type described by (12.3), (12.5) - (12.7). Continuity of the boundary value of $\bar{v}$ is assumed by the smoothness assumptions already made concerning p and g_2, and by the assumption (11.2).

The definition (12.1) can be written

$$V_3(z) = \int_0^z \int_{-\frac{1}{2}}^{\frac{1}{2}} (p\bar{v}_x^2 + q\bar{v}_y^2)dydx \ . \qquad (12.8)$$

Now

$$V_3(z) = \int_0^z \int_{-\frac{1}{2}}^{\frac{1}{2}} [(p\bar{v}_x\bar{v})_x + (q\bar{v}_y\bar{v})_y - \bar{v}L\bar{v}]dydx \ ,$$

so that integration by parts and reference to (12.3), (12.5), and (12.7) yields

$$V_3(z) = \int_{-\frac{1}{2}}^{\frac{1}{2}} (p\bar{v}\bar{v}_x)_{\text{at } x=z} dy - \int_0^z \int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{v}Hdydx \ . \qquad (12.9)$$

Furthermore, differentiation of (12.8) furnishes

$$V_3'(z) = \int_{-\frac{1}{2}}^{\frac{1}{2}} (p\overline{v}_x^2 + q\overline{v}_y^2)_{\text{at } x=z} \, dy \ . \tag{12.10}$$

For any positive constant k_3, we find from (12.9), (12.10)

$$-V_3'(z) + 2k_3V_3(z) = -\int_{-\frac{1}{2}}^{\frac{1}{2}} (p\overline{v}_x^2 + q\overline{v}_y^2 - 2k_3 p\overline{v}\overline{v}_x)_{\text{at } x=z} \, dy - 2k_3 \int_0^z \int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{v} H dy dx$$

$$= -\int_{-\frac{1}{2}}^{\frac{1}{2}} [p(\overline{v}_x - k_3\overline{v})^2 + (q\overline{v}_y^2 - pk_3^2\overline{v}^2)] dy - 2k_3 \int_0^z \int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{v} H dy dx \ . \tag{12.11}$$

Bearing in mind (10.4), we have

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} (q\overline{v}_y^2 - pk_3^2\overline{v}^2) dy \geq q_o \int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{v}_y^2 dy - p_1 k_3^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{v}^2 dy \ . \tag{12.12}$$

Recalling that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{v}_y^2 dy \geq \pi^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{v}^2 dy \ , \tag{12.13}$$

for any function $\overline{v}$ which is continuously differentiable on $[-\frac{1}{2}, \frac{1}{2}]$ and vanishes at the end points,[3] we choose for k_3 in (12.12) the value

$$k_3 = k = \pi\sqrt{q_o/p_1} \tag{12.14}$$

as in (10.3). Combining (12.12), (12.13), and (12.11) then provides the inequality

$$-V_3'(z) + 2kV_3(z) \leq 2k \left| \int_0^z \int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{v} H dy dx \right| \ . \tag{12.15}$$

In Appendix A, it is shown that

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[3] See, for example, [11] or [13].

$$\left| \int_0^z \int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{v} H dy dx \right| \leq \beta_1 [V_1(z) V_2(z)]^{\frac{1}{2}} + \beta_2 V_2(z) + \beta_3 V_2'(z) \quad , \qquad (12.16)$$

where the positive constants β_1 , β_2 , and β_3 are such that on $\overline{\mathfrak{R}}$

$$\left.\begin{aligned}
\beta_1 \geq \max_{\overline{\mathfrak{R}}} \Bigg[& \frac{|p_{xxy}|}{p} + \frac{2|p_{xy}p_x|}{p^2} + \frac{2|p_{xy}q_y|}{p^{3/2}q^{1/2}} + \frac{|p_y|}{p^2} \left| \frac{2p_x^2}{p} - p_{xx} \right| \\
& + \frac{|p_y|}{p^{3/2}q^{1/2}} \left| \frac{2p_x q_y}{p} - q_{xy} \right| + \frac{|q_{xyy}|}{\sqrt{pq}} + \frac{|q_x|}{pq} \left| \frac{p_x p_y}{p} - p_{xy} \right| + \\
& + \frac{|q_x|}{p^{1/2}q^{3/2}} \left| \frac{p_y q_y}{p} - q_{yy} \right| \Bigg] \quad , \\
\beta_2 \geq \max_{\overline{\mathfrak{R}}} \Bigg[& \frac{|p_{xx}+q_{yy}|}{p} + \frac{2|p_{xy}|\sqrt{q}}{p^{3/2}} + \frac{|p_y q_y|}{p^2} + \frac{|p_y|}{p^{3/2}q^{1/2}} \left| \frac{2p_x q}{p} - q_x \right| \\
& + \frac{1}{2} \left| \left(\frac{p_y q}{p} \right)_y \right| \frac{1}{p} + \frac{2|q_{xy}|}{\sqrt{pq}} + \frac{1}{2} \frac{|q_{yy}|}{p} + \frac{1}{2} \frac{|p_{xx}|}{p} + \\
& + \frac{|q_x|}{p^{1/2}q^{3/2}} \left| \frac{q p_y}{p} - q_y \right| + \frac{|p_x q_x|}{pq} + \frac{|q_x q_y|}{p^{1/2}q^{3/2}} + \frac{1}{2}\frac{1}{p} \left| \left(\frac{p q_x}{q} \right)_x \right| \Bigg] \quad , \\
\beta_3 \geq \max_{\overline{\mathfrak{R}}} \Bigg[& \frac{1}{2} \frac{|p_x|}{p} + \frac{1}{2} \frac{|q_x|}{q} \Bigg] \quad .
\end{aligned}\right\} \qquad (12.17)$$

Substitution of (12.16) into (12.15) provides the differential inequality

$$-V_3'(z) + 2kV_3(z) \leq 2k[\beta_1 \{V_1(z) V_2(z)\}^{\frac{1}{2}} + \beta_2 V_2(z) + \beta_3 V_2'(z)] \quad . \qquad (12.18)$$

From this it follows immediately by integration that

$$V_3(z) \leq V_3(\ell) e^{-2k(\ell - z)} + 2k e^{2kz} \int_z^{\ell} \beta_1 e^{-2k\zeta} [V_1(\zeta) V_2(\zeta)]^{\frac{1}{2}} d\zeta +$$

$$+ 2k e^{2kz} \int_z^{\ell} [\beta_2 e^{-2k\zeta} V_2(\zeta) + \beta_3 e^{-2k\zeta} V_2'(\zeta)] d\zeta \ . \qquad (12.19)$$

If the last term in the integral on the right side of (12.19) is integrated by parts, we find

$$V_3(z) \leq [V_3(\ell)+2k\beta_3V_2(\ell)]e^{-2k(\ell-z)}-2k\beta_3V_2(z)$$

$$+2ke^{2kz}\int_z^{\ell} e^{-2k\zeta}\{\beta_1[V_1(\zeta)V_2(\zeta)]^{\frac{1}{2}}+(\beta_2+2k\beta_3)V_2(\zeta)\}d\zeta . \quad (12.20)$$

Replacing $V_1(\zeta)$ and $V_2(\zeta)$ in the integral in (12. 20) by their upper bounds according to (10. 2) and (11. 3), and discarding the term $-2k\beta_3V_2(z)$ in (12. 20), we obtain

$$V_3(z) \leq [V_3(\ell) + 2k\beta_3V_2(\ell)]e^{-2k(\ell-z)} +$$

$$+2ke^{-2k(\ell-z)}\int_z^{\ell}\{\beta_1V_1^{\frac{1}{2}}(\ell)[V_2(\ell)+2k(\alpha_1+\alpha_2)(\ell-\zeta)V_1(\ell)]^{\frac{1}{2}}$$

$$+(\beta_2+2k\beta_3)[V_2(\ell)+2k(\alpha_1+\alpha_2)(\ell-\zeta)V_1(\ell)]\}d\zeta .$$

In the remaining ζ-integral, we replace z by zero throughout to establish the final <u>third order</u> decay inequality

$$V_3(z) \leq C e^{-2k(\ell-z)} , \quad (12.21)$$

where the constant C is given by

$$C = V_3(\ell)+2k[\ell\beta_2+(1+k\ell)\beta_3]V_2(\ell)+4k^2(\beta_2+2k\beta_3)(\alpha_1+\alpha_2)\ell^2V_1(\ell)+$$

$$+2k\ell\beta_1[V_1(\ell)]^{\frac{1}{2}}[V_2(\ell)+2k(\alpha_1+\alpha_2)\ell V_1(\ell)]^{\frac{1}{2}} . \quad (12.22)$$

In this formula, $V_1(\ell)$, $V_2(\ell)$, and $V_3(\ell)$ represent the <u>total energies</u> of various orders; k is given by (12. 14), β_1, β_2, and β_3 by (12. 17), and α_1 and α_2 are defined through (11. 4) and (11. 5).

13. First Order Energy for Problem II.

In order to establish suitable inequalities for the energy distributions of various orders of Problem II, it is convenient to convert the problem from one with boundary conditions of Neumann type to one of Dirichlet type. Referring to (1.14) - (9.17), we accordingly introduce a new unknown function t through the equations

$$\left.\begin{aligned} t_x &= -qw_y \\ t_y &= pw_x + G \end{aligned}\right\} , \qquad (13.1)$$

where

$$G(x, y) = \int_x^{\ell} F(\xi, y)d\xi \qquad \text{on } \overline{\mathcal{R}} . \qquad (13.2)$$

The existence of a solution t of (13.1) is assumed by the fact that w satisfies (9.14). Elimination of w from (13.1) yields the differential equation for t :

$$L^{*}t \equiv \left(\frac{1}{q}\, t_x\right)_x + \left(\frac{1}{p}\, t_y\right)_y = \left(\frac{1}{p}\, G\right)_y \qquad \text{on } \mathcal{R} . \qquad (13.3)$$

Expressing the derivatives of w in (9.15) - (9.17) in terms of G and the derivatives of t with the aid of (13.1) provides boundary conditions for t in a form which can be easily integrated. Performing this integration and adjusting the constants of integration to assure the continuity of boundary values of t, we find[4]

$$t = \int_{-\frac{1}{2}}^{y} [g_1(\eta) + G(0, \eta)]d\eta \qquad \text{at } x = 0 , \qquad (13.4)$$

$$t = 0 \qquad \text{at } x = \ell , \qquad (13.5)$$

$$t = 0 \qquad \text{at } y = \pm \tfrac{1}{2} . \qquad (13.6)$$

---

[4] The arbitrary additive constant which may be added to any solution t of (13.1) has been chosen so that the constant value of t along $x = \ell$ is zero.

We now direct our attention to the boundary value problem (13.3) - (13.6). It is convenient to decompose the problem by writing

$$t = n + s \ , \tag{13.7}$$

where n satisfies

$$L^{*}n = 0 \qquad \text{on } \mathcal{R} \ , \tag{13.8}$$

$$n = \int_{-\frac{1}{2}}^{y} [g_1(\eta) + G(0,\eta)]d\eta \qquad \text{at } x = 0 \ , \tag{13.9}$$

$$n = 0 \qquad \text{at } x = \ell \ , \tag{13.10}$$

$$n = 0 \qquad \text{at } y = \pm \tfrac{1}{2} \ , \tag{13.11}$$

while s satisfies

$$L^{*}s = (G/p)_y \qquad \text{on } \mathcal{R} \ , \tag{13.12}$$

$$s = 0 \qquad \text{at } x = 0 \ , \ x = \ell \ , \tag{13.13}$$

$$s = 0 \qquad \text{at } y = \pm \tfrac{1}{2} \ . \tag{13.14}$$

The legitimacy of this decomposition of the boundary value problem for t is a consequence of (9.7), (9.8), and (13.2).

We now introduce the <u>first order energy</u> distributions associated with the boundary value problems formulated in this section. Define <u>first order energies</u> $T_1(z)$, $N_1(z)$, and $S_1(z)$ by setting

$$T_1(z) = \int_z^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} [\frac{1}{q} t_x^2 + \frac{1}{p}(t_y - G)^2]dydx \ , \tag{13.15}$$

$$N_1(z) = \int_z^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} (\frac{1}{q} n_x^2 + \frac{1}{p} n_y^2)dydx \ , \tag{13.16}$$

$$S_1(z) = \int_z^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} [\frac{1}{q} s_x^2 + \frac{1}{p}(s_y - G)^2]dydx \ . \tag{13.17}$$

An easy application of the Schwartz inequality shows that

$$T_1(z) \le 2N_1(z) + 2S_1(z) \quad . \tag{13.18}$$

As an upper bound for $S_1(z)$, we employ the simple estimate

$$S_1(z) \le S_1(0) , \tag{13.19}$$

based on the monotone decreasing character of $S_1(z)$. Since the differential equation satisfied by s is nonhomogeneous, we would not expect $S_1(z)$ to decrease exponentially.

To analyze $N_1(z)$, we proceed as in [3]. An integration by parts in (13.16), together with the boundary conditions (13.10) and (13.11), shows that

$$N_1(z) = - \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{q} n n_x\right)_{\text{at } x=z} dy \quad . \tag{13.20}$$

Moreover, we also have from (13.15) the formula

$$N_1'(z) = - \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{q} n_x^2 + \frac{1}{p} n_y^2\right) dy \quad , \qquad 0 \le z \le \ell \; , \tag{13.21}$$

for the derivative $N_1'(z)$. Thus, for any non-negative constant k_1,

$$N_1'(z) + 2k_1 N_1(z) = - \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{q} n_x^2 + \frac{1}{p} n_y^2 + \frac{2k_1}{q} n n_x\right)_{\text{at } x=z} dy$$

$$= - \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\frac{1}{q}(n_x + k_1 n)^2 + \frac{1}{p} n_y^2 - \frac{k_1^2}{q} n^2\right] dy$$

$$\le - \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{p} n_y^2 - \frac{k_1^2}{q} n^2\right) dy \quad . \tag{13.22}$$

Recalling the definitions (10.4) of the constants p_1 and q_o, we infer from (13. 22) the inequality

$$N_1'(z) + 2k_1N_1(z) \le -\left\{ \frac{1}{p_1}\int_{-\frac{1}{2}}^{\frac{1}{2}} n_y^2 dy - \frac{k_1^2}{q_o}\int_{-\frac{1}{2}}^{\frac{1}{2}} n^2 dy \right\} . \tag{13.23}$$

Since

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} n_y^2 dy \ \ge \ \pi^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} n^2 dy \ , \tag{13.24}$$

for any continuously differentiable function of y on $-\frac{1}{2} \le y \le \frac{1}{2}$ which vanishes at the endpoints,[5] we may again choose

$$k_1 \ = \ k \ = \ \pi\sqrt{q_o/p_1} \tag{13.25}$$

and conclude that

$$N_1'(z) + 2kN_1(z) \le 0 \ . \tag{13.26}$$

Thus,

$$N_1(z) \le N_1(0)\, e^{-2kz} \ . \tag{13.27}$$

Combining (13. 19) and (13. 27) in (13. 18), we have

$$T_1(z) \le 2[S_1(0) + N_1(0)\, e^{-2kz}] , \qquad 0 \le z \le \ell . \tag{13.28}$$

In the analysis of the present section, we have assumed that p, q, and G are continuously differentiable on $\overline{\mathcal{R}}$, and that each of the boundary value problems for n and s possesses a solution which is continuously differentiable once on $\overline{\mathcal{R}}$ and twice on $\mathcal{R}$.

14. Second Order Energy for Problem II.

We define $\overline{w}$ on $\overline{\mathcal{R}}$ by

$$\overline{w}(x, y) \ = \ w_y(x, y) \ , \tag{14.1}$$

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[5] See [11] or [13].

and we obtain the differential equation satisfied by $\bar{w}$ by differentiating (9.14) with respect to y. Thus,

$$L\bar{w} = \bar{H} \qquad \text{on } \bar{\mathcal{R}} , \tag{14.2}$$

where

$$\bar{H} = (\frac{p_x p_y}{p} - p_{xy})w_x + (\frac{p_y q_y}{p} - q_{yy})w_y + (\frac{qp_y}{p} - q_y)w_{yy} + F_y - \frac{Fp_y}{p} . \tag{14.3}$$

In addition, the boundary conditions satisfied by $\bar{w}$ are found from (9.15) - (9.17) and (14.1) to be

$$p(0,y)\bar{w}_x(0,y) = g_1'(y) - \frac{g_1(y)}{p(0,y)} p_y(0,y) \qquad \text{at } x = 0 , \tag{14.4}$$

$$\bar{w}_x = 0 \qquad \text{at } x = \ell , \tag{14.5}$$

$$\bar{w} = 0 \qquad \text{at } y = \pm\tfrac{1}{2} . \tag{14.6}$$

We now define a <u>second order energy</u> $W_2(z)$ by

$$W_2(z) = \int_z^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} (p\bar{w}_x^2 + q\bar{w}_y^2)dydx . \tag{14.7}$$

Again,

$$W_2'(z) = -\int_{-\frac{1}{2}}^{\frac{1}{2}} (p\bar{w}_x^2 + q\bar{w}_y^2)_{\text{at } x=z} dy , \tag{14.8}$$

while an integration by parts and (14.2), (14.5), and (14.6) give

$$W_2(z) = -\int_{-\frac{1}{2}}^{\frac{1}{2}} (p\bar{w}\bar{w}_x)_{\text{at } x=z} dy - \int_z^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{w}\bar{H}dydx . \tag{14.9}$$

For any constant k_2, we therefore have

$$W_2'(z)+2k_2W_2(z) = -\int_{-\frac{1}{2}}^{\frac{1}{2}} (p\bar{w}_x^2+q\bar{w}_y^2+2k_2p\bar{w}\bar{w}_x)_{\text{at } x=z}dy - 2k_2\int_z^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{w}\bar{H}dydx . \tag{14.10}$$

Arguing as in the preceding section, we choose $k_2 = k$ as in (13.25) and find that

$$W_2'(z) + 2kW_2(z) \le 2k\left| \int_z^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{w}\overline{H}dydx \right| . \tag{14.11}$$

We now require a useful estimate of the integral appearing in (14.11). In Appendix B, it is shown that

$$\left| \int_z^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{w}\overline{H}dydx \right| \le \gamma_1 T_1(z) + \gamma_2[T_1(z)]^{\frac{1}{2}} , \tag{14.12}$$

where $T_1(z)$ is the <u>first order energy</u> defined by (13.15) and γ_1 and γ_2 are such that

$$\left.\begin{aligned} \alpha_1 + \alpha_2 &\le \gamma_1 && \text{on } \overline{\mathfrak{R}} , \\ \left[\int_{\mathfrak{R}} \left(F_y - \frac{Fp_y}{p}\right)^2 \frac{1}{q}\, dA\right]^{\frac{1}{2}} &\le \gamma_2 && \text{on } \overline{\mathfrak{R}} . \end{aligned}\right\} \tag{14.13}$$

In (14.13), α_1, α_2 are given by (11.4) and (11.5), while F is given in (9.14).

From (14.11), (14.12), and (13.27) we find

$$W_2'(z)+2kW_2(z) \le 2k\gamma_1[S_1(0)+N_1(0)e^{-2kz}]+2k\gamma_2[S_1(0)+N_1(0)e^{-2kz}]^{\frac{1}{2}}$$

$$\le 2k\{\gamma_1[S_1(0)+N_1(0)e^{-2kz}]+\gamma_2[S_1(0)]^{\frac{1}{2}}+\gamma_2[N_1(0)]^{\frac{1}{2}}e^{-kz}\} . \tag{14.14}$$

Integrating (14.14) provides

$$W_2(z) \le \gamma_1 S_1(0)+\gamma_2[S_1(0)]^{\frac{1}{2}}+2\gamma_2[N_1(0)]^{\frac{1}{2}}e^{-kz}+\{W_2(0)+2k\ell\gamma_1 N_1(0)-$$

$$-2\gamma_2[N_1(0)]^{\frac{1}{2}}-\gamma_1 S_1(0)-\gamma_2[S_1(0)]^{\frac{1}{2}}\}e^{-2kz} . \tag{14.15}$$

Here, k is given by (13.25). Expression (14.15) represents our main result concerning the second order energy for Problem II. In order to obtain it, we have had to assume that p, q and G are twice continuously differentiable on $\overline{\mathcal{R}}$, and that w is continuously differentiable twice on $\overline{\mathcal{R}}$ and three times on $\mathcal{R}$.

15. Third Order Energy for Problem II.

The final inequality pertaining to the distribution of energy which we require is obtained by considering the function $\bar{\bar{w}}$ defined on $\overline{\mathcal{R}}$ by

$$\bar{\bar{w}} = \bar{w}_x = w_{xy} \quad . \tag{15.1}$$

We assume that the given functions p, q, and G are three times continuously differentiable on $\overline{\mathcal{R}}$, and that the solution w of Problem II is continously differentiable three times on $\overline{\mathcal{R}}$ and four times on $\mathcal{R}$.

Differentiating the differential equation (14.2) with respect to x, we find that $\bar{\bar{w}}$ satisfies

$$L\bar{\bar{w}} = \bar{\bar{H}} \qquad \text{on } \mathcal{R} , \tag{15.2}$$

where

$$\bar{\bar{H}} = F_{xy} - p_{xxy}w_x - q_{xyy}w_y - (p_{xx}+q_{yy})w_{xy} - 2q_{xy}w_{yy} - 2p_{xy}w_{xx} - p_y w_{xxx} - \\ - q_y w_{xyy} - p_x w_{xxy} - q_y w_{yyy} \quad . \tag{15.3}$$

The boundary conditions satisfied by $\bar{\bar{w}}$ are easily determined from (14.4) - (14.6) to be

$$\bar{\bar{w}}(0,y) = \frac{g_1'(y)}{p(0,y)} - \frac{g_1(y)}{p^2(0,y)}\, p_y(0,y) \qquad \text{at } x = 0 , \tag{15.4}$$

$$\bar{\bar{w}} = 0 \qquad \text{at } x = \ell , \tag{15.5}$$

$$\bar{\bar{w}} = 0 \qquad \text{at } y = \pm\tfrac{1}{2} . \tag{15.6}$$

In analogy with the discussion of the preceding section, we find that the <u>third order energy</u> defined by

$$W_3(z) = \int_z^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} (p\bar{\bar{w}}_x^2 + q\bar{\bar{w}}_y^2)\,dy\,dx \tag{15.7}$$

satisfies the differential inequality

$$W_3'(z) + 2kW_3(z) \le 2k\left|\int_z^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{\bar{w}}\bar{\bar{H}}\,dy\,dx\right| \tag{15.8}$$

where

$$k = \pi\sqrt{q_o/p_1}\ , \tag{15.9}$$

as in (13. 25). In Appendix C, it is shown that

$$\left|\int_z^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{\bar{w}}\bar{\bar{H}}\,dy\,dx\right| \le \beta_1[T_1(z)W_2(z)]^{\frac{1}{2}} + \beta_2 W_2(z) - \beta_3 W_2'(z) + \beta_4[W_2(z)]^{\frac{1}{2}}\ , \tag{15.10}$$

where β_1, β_2, β_3 are given by (12. 17) and

$$\beta_4 \ge \max_{\bar{\mathcal{R}}}\left[\frac{|q_x|\cdot|F_y|}{p^{1/2}q} + \frac{|F_{xy}|}{\sqrt{p}} + \frac{|p_y|}{p^{3/2}}\cdot\left|\frac{2Fp_x}{p} - F_x\right| + \frac{2|p_{xy}F|}{p^{3/2}}\right]. \tag{15.11}$$

(15. 10) in turn converts (15. 8) into the form

$$W_3'(z)+2kW_3(z) \le 2k\{\beta_1[T_1(z)W_2(z)]^{\frac{1}{2}}+\beta_2 W_2(z)-\beta_3 W_2'(z)+\beta_4[W_2(z)]^{\frac{1}{2}}\}\ . \tag{15.12}$$

Integrating (15. 12) yields

$$W_3(z) \le [W_3(0)+2k\beta_3 W_2(0)]e^{-2kz}+2ke^{-2kz}\Big\{\beta_1\int_0^z e^{2k\zeta}[T_1(\zeta)W_2(\zeta)]^{\frac{1}{2}}d\zeta +$$

$$+(\beta_2+2k\beta_3)\int_0^z e^{2k\zeta}W_2(\zeta)d\zeta+\beta_4\int_0^z e^{2k\zeta}[W_2(\zeta)]^{\frac{1}{2}}d\zeta\Big\}\ . \tag{15.13}$$

Substituting (13.27) and (14.15) into (15.13) and simplifying the result, we find

$$W_3(z) \le \varkappa_1+\varkappa_2 e^{-kz/2}+\varkappa_3 e^{-kz}+\varkappa_4 e^{-2kz} , \qquad (15.14)$$

where the constants $\varkappa_1$, $\varkappa_2$, $\varkappa_3$, and $\varkappa_4$ are such that

$$\varkappa_1 = (\beta_2+2k\beta_3)\{\gamma_1 S_1(0)+\gamma_2[S_1(0)]^{\frac{1}{2}}\}+\beta_4\sqrt{\gamma_1 S_1(0)+\gamma_2[S_1(0)]^{\frac{1}{2}}} + $$
$$+ \sqrt{2}\,\beta_1\sqrt{\gamma_1[S_1(0)]^2+\gamma_2[S_1(0)]^{3/2}} , \qquad (15.15)$$

$$\varkappa_2 = \frac{4}{3}\Big\{\beta_4\sqrt{2\gamma_2[N_1(0)]^{\frac{1}{2}}}+\sqrt{2}\,\beta_1 S_1(0)[2\gamma_2\sqrt{N_1(0)}+W_2(0)]+$$
$$+\sqrt{2}\,(1+2k\ell)\beta_1\gamma_1 S_1(0)N_1(0)+\sqrt{2}\,\beta_1\gamma_2 N_1(0)[\sqrt{S_1(0)}+\sqrt{N_1(0)}]+$$
$$+\sqrt{2}\,\beta_1 N_1(0)[W_2(0)+2k\ell\gamma_1 N_1(0)]\Big\} , \qquad (15.16)$$

$$\varkappa_3 = 4\gamma_2(\beta_2+2k\beta_3)[N_1(0)]^{\frac{1}{2}}+2\beta_4[W_2(0)+2k\ell\gamma_1 N_1(0)]^{\frac{1}{2}} , \qquad (15.17)$$

and

$$\varkappa_4 = 2k\ell(\beta_2+2k\beta_3)[W_2(0)+2k\ell\gamma_1 N_1(0)]+[W_3(0)+2k\beta_3 W_2(0)] . \qquad (15.18)$$

We now turn to the question of estimating the total energies of various orders appearing in the energy inequalities derived in this chapter.

III. BOUNDS FOR TOTAL ENERGIES

16. Fundamental Minimum Principles

The inequalities pertaining to the distribution of energies of various orders which we have derived in the preceding chapter involve the total energies as undetermined quantities. While it is not expected that these can be computed exactly, it is possible to obtain upper bounds for them by applying suitable minimum principles. This was done in [1], [3], and [4] for the problems considered in these references, and our procedure here is similar to that of [3]. In the present section, we collect the minimum principles necessary for our purposes. Throughout this discussion, L represents an operator of the form (9. 1) where p and q are positive continuously differentiable functions on the closure $\overline{\mathfrak{R}}$ of $\mathfrak{R}$.

Theorem 16. 1.

Let φ be continuously differentiable once on $\overline{\mathfrak{R}}$ and twice on $\mathfrak{R}$, and suppose

$$L\varphi = \Phi \qquad \text{on } \mathfrak{R}, \tag{16. 1}$$

where Φ is continuous on $\overline{\mathfrak{R}}$. Then for any $\hat{\varphi}$ which is continuously differentiable on $\overline{\mathfrak{R}}$ and satisfies $\varphi = \hat{\varphi}$ on the boundary $\partial\mathfrak{R}$ of $\mathfrak{R}$ we have

$$\int_{\mathfrak{R}} (p\hat{\varphi}_x^{\,2} + q\hat{\varphi}_y^{\,2} + 2\hat{\varphi}\Phi)\,dA \geq \int_{\mathfrak{R}} (p\varphi_x^{\,2} + q\varphi_y^{\,2} + 2\varphi\Phi)\,dA\,, \tag{16. 2}$$

with equality holding if and only if $\hat{\varphi} \equiv \varphi$.

We shall apply this theorem to the problems of Dirichlet type formulated in Sections 12, 13, and 15.

Theorem 16.2.

Let ψ be continuously differentiable once on $\overline{\mathcal{R}}$ and twice on $\mathcal{R}$, and suppose that

$$L\psi = \Psi \qquad \text{on } \mathcal{R}, \tag{16.3}$$

where Ψ is continuous on $\overline{\mathcal{R}}$, and that

$$\psi = 0 \qquad \text{at } y = \pm\tfrac{1}{2}. \tag{16.4}$$

Then for any $\hat{\psi}$ which is continuously differentiable on $\overline{\mathcal{R}}$ and for which

$$\hat{\psi}_y = -p\psi_x \qquad \text{at } x = 0, \ell \tag{16.5}$$

we have

$$\int_{\mathcal{R}} \left(\frac{1}{q}\hat{\psi}_x^{\,2} + \frac{1}{p}\hat{\psi}_y^{\,2}\right)dA \geq \int_{\mathcal{R}} (p\psi_x^{\,2} + q\psi_y^{\,2} + 2\psi\Psi)dA\ , \tag{16.6}$$

with equality holding if and only if

$$\hat{\psi}_y = -p\psi_x \quad \text{and} \quad \hat{\psi}_x = q\psi_y \quad \text{on } \mathcal{R}.$$

We shall apply this inequality to the boundary value problems formulated in Section 14.

The proofs of the two theorems stated above follow along standard lines. See, for example, [11] and [14]; see also Section 6 of [3].

17. Total First Order Energy for Problem I.

From (10.1), the total first order energy $V_1(\ell)$ for Problem I is given by

$$V_1(\ell) = \int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} (pv_x^{\,2} + qv_y^{\,2})dydx\ . \tag{17.1}$$

By the application of an appropriate minimum principle, it has been shown in [3] that

$$V_1(\ell) \leq \left(\frac{\ell}{p_o} + \frac{1}{\ell q_o}\right) m_2^2 , \tag{17.2}$$

where m_2 is such that

$$m_2 \geq |g_2(y)| , \qquad -\tfrac{1}{2} \leq y \leq \tfrac{1}{2} , \tag{17.3}$$

and

$$p(x, y) \geq p_o > 0 , \qquad q(x, y) \geq q_o > 0 \qquad \text{on } \bar{\mathcal{R}} . \tag{17.4}$$

18. Total Second Order Energy for Problem I.

The total second order energy $V_2(\ell)$ for Problem I is found from (11.1) to be

$$V_2(\ell) = \int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} (p v_{xy}^2 + q v_{yy}^2) dy dx . \tag{18.1}$$

It was shown in [3] by an application of Theorem 16.2 that

$$V_2(\ell) \leq \left(\frac{\ell}{p_o} + \frac{1}{\ell q_o}\right)(m_3^2 + 2\gamma_1 m_2^2) , \tag{18.2}$$

where m_3 is such that

$$m_3 \geq \left| g_2'(y) - \frac{g_2(y)}{p(\ell, y)} p_y(\ell, y)\right| \quad \text{on } [-\tfrac{1}{2}, \tfrac{1}{2}] , \tag{18.3}$$

and γ_1 and m_2 are given by (14.13) and (17.3), respectively.

19. Total Third Order Energy for Problem I.

Referring to (12.1) and (12.8), we have the expressions

$$V_3(\ell) = \int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} (p v_{xyx}^2 + q v_{xyy}^2) dy dx = \int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} (p \bar{v}_x^2 + q \bar{v}_y^2) dy dx , \tag{19.1}$$

for the total third order energy. In order to establish $V_3(\ell)$ with the aid of Theorem 16.1, we identify φ and Φ of that theorem with $\bar{v}$ and H of (12.3), respectively. Inequality (16.2) then shows that

$$V_3(\ell) \le \int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} (p\hat{\varphi}_x^2 + q\hat{\varphi}_y^2)dydx + 2\left|\int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{v}Hdydx\right| + 2\left|\int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{\varphi}Hdydx\right| , \qquad (19.2)$$

for any $\hat{\varphi}$ admissible under the hypothesis of Theorem 16.1.

In Appendix D, it is shown that

$$\left|\int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{\varphi}Hdydx\right| = \left|\int_{\Re(0)} \hat{\varphi}Hda\right| \le \nu_3 + \nu_1[V_1(\ell)]^{\frac{1}{2}} + \nu_2[V_2(\ell)]^{\frac{1}{2}}, \qquad (19.3)$$

where

$$\nu_3 \ge \left|\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{p_x(\ell,y)\hat{\varphi}(\ell,y)}{p(\ell,y)}\left[g_2'(y) - \frac{p_y(\ell,y)g_2(y)}{p(\ell,y)}\right]dy\right| +$$

$$+\left|\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{q_y(\ell,y)}{q(\ell,y)}\hat{\varphi}(\ell,y)\left[g_2'(y) - \frac{p_y(\ell,y)g_2(y)}{p(\ell,y)}\right]dy\right| , \qquad (19.4)$$

and ν_1 and ν_2 are given by (22.4) and (22.5), respectively.

Substituting the inequalities (19.3) and (A.24)of Appendix A into (19.2), we find that

$$V_3(\ell) \le \int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} (p\hat{\varphi}_x^2 + q\hat{\varphi}_y^2)dydx + 2\{\beta_o + \beta_1[V_1(\ell)V_2(\ell)]^{\frac{1}{2}} + \beta_2 V_2(\ell)\} +$$

$$+ 2\{\nu_3 + \nu_1[V_1(\ell)]^{\frac{1}{2}} + \nu_2[V_2(\ell)]^{\frac{1}{2}}\} . \qquad (19.5)$$

One admissible $\hat{\varphi}$ is given by

$$\hat{\varphi}(x,y) = \frac{x}{\ell}\left[\frac{g_2'(y)}{p(\ell,y)} - \frac{p_y(\ell,y)g_2(y)}{p^2(\ell,y)}\right] . \qquad (19.6)$$

Compute $\hat{\varphi}_x$ and $\hat{\varphi}_y$ from (19.6) and upon substitution of these quantities and (19.6) into (19.5), we find that

$$V_3(\ell) \le \left(\frac{p_1}{p_o^2}\frac{m_3^2}{\ell} + q_1\ell m_4^2\right) + 2\{\beta_o + \beta_1[V_1(\ell)V_2(\ell)]^{\frac{1}{2}} + \beta_2 V_2(\ell)\} +$$

$$+ 2\{\bar{\nu}_3 + \bar{\nu}_1[V_1(\ell)]^{\frac{1}{2}} + \bar{\nu}_2[V_2(\ell)]^{\frac{1}{2}}\} , \qquad (19.7)$$

where

$$m_4 \geq \left| \left[\frac{g_2'(y)}{p(\ell, y)} - \frac{g_2(y) p_y(\ell, y)}{p^2(\ell, y)} \right]_y \right| \qquad \text{on} \quad [-\tfrac{1}{2}, \tfrac{1}{2}] \; , \tag{19.8}$$

$$p_1 \geq p(x, y) \; , \quad q_1 \geq q(x, y) \qquad \text{on } \overline{\mathcal{R}} \; , \tag{19.9}$$

p_o and m_3 are given by (17.4) and (18.3), respectively, and $\overline{\nu}_3$ is such that

$$\overline{\nu}_3 \geq \frac{m_3^2}{p_o^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} |p_x(\ell, y)| dy + \frac{m_3^2}{p_o q_o} \int_{-\frac{1}{2}}^{\frac{1}{2}} |q_y(\ell, y)| dy \; , \tag{19.10}$$

while $\overline{\nu}_1$ and $\overline{\nu}_2$ are obtained from ν_1' and ν_2' respectively after replacing m_5 by m_3 and m_6 by m_4 in (22.10) and (22.11).

20. Total First Order Energy for Problem II.

To complete the energy inequality (13.28) for the first order energy $T_1(z)$ associated with Problem II, it is sufficient to provide upper bounds for the total energies $N_1(0)$ and $S_1(0)$ associated with the boundary value problems (13.8) - (13.11) and (13.12) - (13.14), respectively.

The boundary value problem (13.8) - (13.11) for n is one of Dirichlet type. From (13.16), the associated total energy is given by

$$N_1(0) = \int_0^{\ell} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{q} n_x^2 + \frac{1}{p} n_y^2 \right) dy dx \; . \tag{20.1}$$

In order to find an upper bound for $N_1(0)$, we shall apply Theorem 16.1 with $\Phi \equiv 0$, $\varphi = n$ and with the operator L in that theorem replaced by L* as defined in (13.3). Thus, if $\hat{\varphi}$ is a continuously differentiable function which coincides on the boundary of $\mathcal{R}$ with the solution n of (13. 8)-(13. 11), inequality (16. 2) asserts that

$$\int_{\mathcal{R}}\left(\frac{1}{q}\hat{\varphi}_x^2+\frac{1}{p}\hat{\varphi}_y^2\right)dA \geq \int_{\mathcal{R}}\left(\frac{1}{q}n_x^2+\frac{1}{p}n_y^2\right)dA = N_1(0)\,. \tag{20. 2}$$

To obtain an explicit estimate of $N_1(0)$, we choose

$$\hat{\varphi}(x,y) = \left(\frac{\ell-x}{\ell}\right)\int_{-\frac{1}{2}}^{y}[g_1(\eta)+G(0,\eta)]d\eta\ . \tag{20. 3}$$

The fact that $\hat{\varphi}$ is admissible follows from the smoothness properties of g_1 and G and from (9. 7), (9. 8), and (13. 2). Substitution of (20. 3) into (20. 2) leads in a straightforward way to

$$N_1(0) \leq 2(m_1^2+M^2)\left(\frac{\ell}{p_o}+\frac{1}{\ell q_o}\right), \tag{20. 4}$$

where m_1 and M are such that

$$\begin{aligned} |g_1(y)| &\leq m_1 \quad \text{on } [-\tfrac{1}{2},\tfrac{1}{2}]\,, \\ |G(x,y)| &\leq M \quad \text{on } \overline{\mathcal{R}}\,, \end{aligned} \tag{20. 5}$$

and p_o, q_o are as in (12. 4). Expression (20. 4) provides a bound for $N_1(0)$ in terms of quantities pertaining to the given data.

To complete the estimate (13. 28) we must also consider the total energy $S_1(0)$ associated with the boundary value problem (13.12)-(13. 14) for s. From (13. 17)

$$\begin{aligned} S_1(0) &= \int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left[\frac{1}{q}s_x^2+\frac{1}{p}(s_y-G)^2\right]dydx = \int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{q}s_x^2+\frac{1}{p}s_y^2-\frac{2G}{p}s_y+\frac{G^2}{p}\right)dydx \\ &= \int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left[\frac{1}{q}s_x^2+\frac{1}{p}s_y^2+2s\left(\frac{G}{p}\right)_y\right]dydx+\int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{G^2}{p}\,dydx\,. \end{aligned} \tag{20. 6}$$

In this computation, we have used the boundary conditions (13. 14) satisfied by s at $y = \pm \frac{1}{2}$ in the integration by parts.

To estimate $S_1(0)$, we again use Theorem 16.1 with L replaced by L^*, with $\varphi = s$ and with

$$\Phi = (G/p)_y . \tag{20.7}$$

Thus, from (20.6) and (16.2), we have

$$S_1(0) \le \int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} [\frac{1}{q}\hat{\varphi}_x^{\,2} + \frac{1}{p}\hat{\varphi}_y^{\,2} + 2\hat{\varphi}(\frac{G}{p})_y]dydx + \int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{G^2}{p}\,dydx , \tag{20.8}$$

for any sufficiently smooth $\hat{\varphi}$ which coincides with s on the boundary of $\mathfrak{R}$. An integration by parts in (20.8), together with the vanishing of $\hat{\varphi}$ at $y = \pm\frac{1}{2}$ converts (20.8) to

$$S_1(0) \le \int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} [\frac{1}{q}\hat{\varphi}_x^{\,2} + \frac{1}{p}(\hat{\varphi}_y - G)^2]dydx . \tag{20.9}$$

To make (20.9) explicit, we choose

$$\hat{\varphi}(x,y) = \frac{x}{x+\epsilon}\int_{-\frac{1}{2}}^{y} G(x,\eta)\,d\eta . \tag{20.10}$$

Clearly $\hat{\varphi} = 0$ at $y = -\frac{1}{2}$; from (9.8) and (13.2), it follows that $\hat{\varphi}(x, \frac{1}{2}) = 0$, $0 \le x \le \ell$. Finally, the definition (13.2) implies that $\hat{\varphi} = 0$ when $x = \ell$. Thus, $\hat{\varphi}$ is admissible. Substituting (20.10) into (20.9) leads, after some manipulation, to

$$S_1(0) \le \frac{2\ell}{q_o}(M^2 + \ell^2 M'^2) + \frac{2\ell}{p_o}(1+\ell^2)M^2 , \tag{20.11}$$

where M is given by the second of (20.5), and M' is such that

$$M' \ge |G_x| = |F| \qquad \text{on } \bar{\mathfrak{R}} . \tag{20.12}$$

When (20.11) and (20.4) are combined in (13.28), there results a decay inequality for the first order energy $T_1(z)$ associated with Problem II. Furthermore, setting $z = 0$, we find the total first order energy to be

$$T_1(0) \leq 4[\frac{\ell}{q_o}(M^2+\ell^2 M'^2) + \frac{\ell}{p_o}(1+\ell^2)M^2 + (m_1^2+M^2)(\frac{\ell}{p_o} + \frac{1}{\ell q_o})] . \tag{20.13}$$

21. Total Second Order Energy for Problem II.

The total second order energy $W_2(0)$ for Problem II is found from (14.7) to be

$$W_2(0) = \int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}}(p\bar{w}_x^2 + q\bar{w}_y^2)dydx \quad . \tag{21.1}$$

Along lines similar to those used in establishing the estimate for $V_2(\ell)$ in §18, we find by an application of Theorem 16.2 that

$$W_2(0) \leq \int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}}(\frac{1}{q}\hat{\psi}_x^2 + \frac{1}{p}\hat{\psi}_y^2)dydx + 2\{\gamma_1 T_1(0) + \gamma_2[T_1(0)]^{\frac{1}{2}}\} . \tag{21.2}$$

An admissible $\hat{\psi}$ is given by

$$\hat{\psi}(x,y) = (\frac{x-\ell}{\ell})\int_{-\frac{1}{2}}^{y}[g_1'(\eta) - \frac{g_1(\eta)}{p(0,\eta)}p_y(0,\eta)]d\eta . \tag{21.3}$$

Substituting (21.3) into (21.2), we find

$$W_2(0) \leq (\frac{\ell}{p_o} + \frac{1}{\ell q_o})m_5^2 + 2\{\gamma_1 T_1(0) + \gamma_2[T_1(0)]^{\frac{1}{2}}\} , \tag{21.4}$$

where p_o, q_o are given by (17.4), γ_1 and γ_2 by (14.13), and $T_1(0)$ by (20.13), while m_5 is such that

$$m_5 \geq \left|g_1'(y) - \frac{g_1(y)}{p(0,y)}p_y(0,y)\right| \quad \text{on } [-\tfrac{1}{2}, \tfrac{1}{2}] . \tag{21.5}$$

22. Total Third Order Energy for Problem II.

Along lines analogous to those used in establishing the estimate for $V_3(\ell)$ in §19, we find, with the aid of Theorem 16.1, that the total third order energy $W_3(0)$ for Problem II satisfies

$$W_3(0) \le \int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} (p\hat{\varphi}_x^2 + q\hat{\varphi}_y^2)dydx + 2\left|\int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{\bar{w}}\bar{\bar{H}}dydx\right| + 2\left|\int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{\varphi}\bar{\bar{H}}dydx\right|, \qquad (22.1)$$

for any $\hat{\varphi}$ admissible under the hypothesis of Theorem 16.1.

In Appendix D, it is shown that

$$\left|\int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{\varphi}\bar{\bar{H}}dydx\right| = \left|\int_{\Re(0)} \hat{\varphi}\bar{\bar{H}}da\right| \le \nu_o + \nu_1[T_1(0)]^{\frac{1}{2}} + \nu_2[W_2(0)]^{\frac{1}{2}}, \qquad (22.2)$$

where

$$\nu_o \ge \left|\int_{\Re(0)} F_{xy}\hat{\varphi}da\right| + 2\left|\int_{\Re(0)} \frac{p_{xy}}{p}F\hat{\varphi}da\right| + \left|\int_{\Re(0)} p_y\left(\frac{F}{p}\right)_x\hat{\varphi}da\right| + \left|\int_{\Re(0)} q_y\hat{\varphi}\left(F_y - \frac{Fp_y}{p}\right)da\right|$$

$$+\left|\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{p_x(0,y)\hat{\varphi}(0,y)}{p(0,y)}[g_1'(y) - \frac{p_y(0,y)g_1(y)}{p(0,y)}]dy\right| +$$

$$+\left|\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{q_y(0,y)}{q(0,y)}\hat{\varphi}(0,y)[g_1'(y) - \frac{p_y(0,y)g_1(y)}{p(0,y)}]dy\right|, \qquad (22.3)$$

$$\nu_1 \ge \left[\int_{\Re(0)} (p_{xxy})^2\frac{\hat{\varphi}^2}{p}da\right]^{\frac{1}{2}} + \left[\int_{\Re(0)} (q_{xyy})^2\frac{\hat{\varphi}^2}{q}da\right]^{\frac{1}{2}} + 2\left[\int_{\Re(0)} (p_{xy}p_x)^2\frac{\hat{\varphi}^2}{p^3}da\right]^{\frac{1}{2}}$$

$$+2\left[\int_{\Re(0)} (p_{xy}q_y)^2\frac{\hat{\varphi}^2}{p^2q}da\right]^{\frac{1}{2}} + \left[\int_{\Re(0)} \frac{(p_y)^2}{p^3}\left(\frac{2p_x^2}{p} - p_{xx}\right)^2\hat{\varphi}^2da\right]^{\frac{1}{2}}$$

$$+\left[\int_{\Re(0)} \frac{(p_y)^2}{p^2q}\left(\frac{2p_xq_y}{p} - q_{xy}\right)^2\hat{\varphi}^2da\right]^{\frac{1}{2}} + \left[\int_{\Re(0)} \left(\frac{q_y}{q}\right)^2\left(\frac{p_xp_y}{p} - p_{xy}\right)^2\frac{\hat{\varphi}^2}{p}da\right]^{\frac{1}{2}}$$

$$+\left[\int_{\Re(0)} (q_y)^2\left(\frac{p_yq_y}{p} - q_{yy}\right)^2\frac{\hat{\varphi}^2}{q^3}da\right]^{\frac{1}{2}}, \qquad (22.4)$$

and

$$\nu_2 \geq \left[\iint_{\Re(0)} (p_{xx}+q_{yy})^2 \frac{\hat{\varphi}^2}{p}\, da\right]^{\frac{1}{2}} + 2\left[\iint_{\Re(0)} (q_{xy})^2 \frac{\hat{\varphi}^2}{q}\, da\right]^{\frac{1}{2}} + 2\left[\iint_{\Re(0)} (p_{xy})^2 \frac{q\hat{\varphi}^2}{p^2}\, da\right]^{\frac{1}{2}} +$$

$$+\left[\iint_{\Re(0)} \frac{(p_y q_y)^2}{p^3}\hat{\varphi}^2 da\right]^{\frac{1}{2}} + \left[\iint_{\Re(0)} \frac{(p_y)^2}{p^2 q}\left(\frac{2qp_x}{p} - q_x\right)^2 \hat{\varphi}^2 da\right]^{\frac{1}{2}} + \left\{\iint_{\Re(0)} \left[\left(\frac{p_y q\hat{\varphi}}{p}\right)_y\right]^2 \frac{da}{q}\right\}^{\frac{1}{2}}$$

$$+\left\{\iint_{\Re(0)} [(q_y\hat{\varphi})_y]^2 \frac{da}{q}\right\}^{\frac{1}{2}} + \left\{\iint_{\Re(0)} [(p_x\hat{\varphi})_x]^2 \frac{da}{p}\right\}^{\frac{1}{2}} + \left[\iint_{\Re(0)} (q_y)^2\left(\frac{qp_y}{p} - 2q_y\right)^2 \frac{\hat{\varphi}^2}{q^3} da\right]^{\frac{1}{2}} +$$

$$+\left[\iint_{\Re(0)} (q_y p_x)^2 \frac{\hat{\varphi}^2}{pq^2}\, da\right]^{\frac{1}{2}} + \left\{\iint_{\Re(0)} \left[\left(q_y \frac{p}{q}\hat{\varphi}\right)_x\right]^2 \frac{da}{p}\right\}^{\frac{1}{2}} . \tag{22.5}$$

Substituting (22.2) and (C.11) of Appendix C into (22.1), we find

$$W_3(0) \leq \int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} (p\hat{\varphi}_x^2 + q\hat{\varphi}_y^2)dydx + 2\{\beta_1[T_1(0)W_2(0)]^{\frac{1}{2}} + \beta_2 W_2(0) + \beta_4[W_2(0)]^{\frac{1}{2}} + \beta_5\} +$$

$$+2\{\nu_o + \nu_1[T_1(0)]^{\frac{1}{2}} + \nu_2[W_2(0)]^{\frac{1}{2}}\} . \tag{22.6}$$

An admissible $\hat{\varphi}$ is given by

$$\hat{\varphi}(x,y) = \left(\frac{\ell - x}{\ell}\right)\left[\frac{g_1'(y)}{p(0,y)} - \frac{p_y(0,y)g_1(y)}{p^2(0,y)}\right] . \tag{22.7}$$

Compute $\hat{\varphi}_x$ and $\hat{\varphi}_y$ from (22.7), and upon substitution of these quantities and (22.7) into (22.6) and after elementary reductions, we find

$$W_3(0) \leq \left(\frac{p_1}{p_o^2}\frac{m_5^2}{\ell} + q_1\ell m_6^2\right) + 2\{\beta_1[T_1(0)W_2(0)]^{\frac{1}{2}} + \beta_2 W_2(0) + \beta_4[W_2(0)]^{\frac{1}{2}} + \beta_5\} +$$

$$+2\{\nu_o' + \nu_1'[T_1(0)]^{\frac{1}{2}} + \nu_2'[W_2(0)]^{\frac{1}{2}}\} , \tag{22.8}$$

where p_1 and q_1 are given by (19.7); β_1 and β_2, β_4 and β_5 are given respectively by (12.17), (15.11), and (C.12) of Appendix C; ν_o', ν_1', and ν_2' are such that

$$\nu_o' \geq \frac{m_5}{p_o}\Big[\int_{\Re(0)} |F_{xy}|da + 2\int_{\Re(0)} \frac{|p_{xy}F|}{p}da + \int_{\Re(0)} |p_y(\frac{F}{p})_x|da + \int_{\Re(0)} |q_y(F_y - \frac{Fp_y}{p})|da$$

$$+ \frac{m_5}{p_o}\int_{-\frac{1}{2}}^{\frac{1}{2}} |p_x(0,y)|da + \frac{m_5}{q_o}\int_{-\frac{1}{2}}^{\frac{1}{2}} |q_y(0,y)|dy\Big] , \qquad (22.9)$$

$$\nu_1' \geq \frac{m_5}{p_o}\Big\{\Big[\int_{\Re(0)}(p_{xxy})^2\frac{da}{p}\Big]^{\frac{1}{2}} + \Big[\int_{\Re(0)}(q_{xyy})^2\frac{da}{q}\Big]^{\frac{1}{2}} + 2\Big[\int_{\Re(0)}(p_{xy}p_x)^2\frac{da}{p^3}\Big]^{\frac{1}{2}} +$$

$$+ 2\Big[\int_{\Re(0)}(p_{xy}q_y)^2\frac{da}{p^2q}\Big]^{\frac{1}{2}} + \Big[\int_{\Re(0)}(p_y)^2\Big(\frac{2p_x^2}{p} - p_{xx}\Big)^2\frac{da}{p^3}\Big]^{\frac{1}{2}} +$$

$$+ \Big[\int_{\Re(0)}(p_y)^2\Big(\frac{2p_xq_y}{p} - q_{xy}\Big)^2\frac{da}{p^2q}\Big]^{\frac{1}{2}} + \Big[\int_{\Re(0)}\Big(\frac{q_y}{q}\Big)^2\Big(\frac{p_xp_y}{p} - p_{xy}\Big)^2\frac{da}{p}\Big]^{\frac{1}{2}} +$$

$$+ \Big[\int_{\Re(0)}(q_y)^2\Big(\frac{p_yq_y}{p} - q_{yy}\Big)^2\frac{da}{q^3}\Big]^{\frac{1}{2}}\Big\} , \qquad (22.10)$$

$$\nu_2' \geq \frac{m_5}{p_o}\Big\{\Big[\int_{\Re(0)}(p_{xx}+q_{yy})^2\frac{da}{p}\Big]^{\frac{1}{2}} + 2\Big[\int_{\Re(0)}(q_{xy})^2\frac{da}{q}\Big]^{\frac{1}{2}} + 2\Big[\int_{\Re(0)}(p_{xy})^2 q\frac{da}{p^2}\Big]^{\frac{1}{2}} +$$

$$+ \Big[\int_{\Re(0)}(p_yq_y)^2\frac{da}{p^3}\Big]^{\frac{1}{2}} + \Big[\int_{\Re(0)}(p_y)^2\Big(\frac{2qp_x}{p} - q_x\Big)^2\frac{da}{p^2q}\Big]^{\frac{1}{2}} + \sqrt{\frac{2}{q_o}}\int_{\Re(0)}\Big|\Big(\frac{p_yq}{p}\Big)_y\Big|da +$$

$$+ \sqrt{\frac{2}{q_o}}\int_{\Re(0)}|q_{yy}|da + \sqrt{\frac{2}{p_o}}\int_{\Re(0)}|p_{xx}|da + \Big[\int_{\Re(0)}(q_y)^2\Big(\frac{qp_y}{p} - 2q_y\Big)^2\frac{da}{q^3}\Big]^{\frac{1}{2}} +$$

$$+ \Big[\int_{\Re(0)}(q_yp_x)^2\frac{da}{pq^2}\Big]^{\frac{1}{2}} + \sqrt{\frac{2}{p_o}}\int_{\Re(0)}\Big|\Big(\frac{p}{q}q_y\Big)_x\Big|da\Big\} +$$

$$+\frac{m_5}{p_o \ell}\left[\sqrt{\frac{2}{p_o}}\int_{\Re(0)}|p_x|da+\sqrt{\frac{2}{p_o}}\int_{\Re(0)}\frac{p}{q}|q_y|da\right]+$$

$$+\, m_6\left[\sqrt{\frac{2}{q_o}}\int_{\Re(0)}\frac{q}{p}|p_y|da+\sqrt{\frac{2}{q_o}}\int_{\Re(0)}|q_y|da\right] \;; \tag{22.11}$$

p_o and q_o are given by (17.4); m_5 is given by (21.5), while m_6 is such that

$$m_6 \geq \left|\left[\frac{g_1'(y)}{p(0,y)} - \frac{g_1(y)p_y(0,y)}{p^2(0,y)}\right]_y\right| \quad \text{on } [-\tfrac{1}{2}, \tfrac{1}{2}] \;. \tag{22.12}$$

When the upper bounds of total energies of various orders derived in this chapter are combined with the inequalities pertaining to the distribution of energies of corresponding orders established in the preceding chapter, the inequalities become pointwise decay estimates in terms of parameters pertaining to the given data of the basic boundary value problem.

IV. POINTWISE ESTIMATES

23. Preliminaries.

In this chapter, we are concerned with pointwise estimates of the values of the solution u and its derivatives u_x and u_y at any point (x, y) in the closure $\bar{\mathcal{R}}$ of $\mathcal{R}$. The results to be obtained state that

$$|u(x, y)| \leq K_1 + K_2 e^{-\frac{kx}{4}} + K_3 e^{-k(\ell - x)}, \tag{23.1}$$

$$|u_x(x, y)| \leq K_4 + K_5 e^{-\frac{kx}{8}} + K_6 e^{-k(\ell - x)}, \tag{23.2}$$

$$|u_y(x, y)| \leq K_7 + K_8 e^{-\frac{kx}{8}} + K_9 e^{-k(\ell - x)}, \tag{23.3}$$

where

$$k = \pi \sqrt{q_o / p_1} \tag{23.4}$$

as in (10.3), and the constants K_i (i = 1, 2, ..., 9) are given explicitly[1] in terms of known data. The pointwise estimates of the solution u and its derivatives u_x and u_y given by (23.1) - (23.3) are thus fully determined, and they do not deteriorate near the boundary of $\mathcal{R}$. In fact, they hold everywhere in the closure $\bar{\mathcal{R}}$ of $\mathcal{R}$.

We first present the method used in establishing (23.1) - (23.3), the details of which will be given in the subsequent sections.

Recall from (9.9) that

$$u = v + w. \tag{23.5}$$

We now differentiate (23.5) with respect to x to find

$$u_x = v_x + w_x, \tag{23.6}$$

and then differentiate (23.5) with respect to y to find

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[1] See (29.9) - (29.17).

$$u_y = v_y + w_y \; . \tag{23.7}$$

In view of the triangular inequality, the following are true:

$$|u| \le |v| + |w| \; , \tag{23.8}$$

$$|u_x| \le |v_x| + |w_x| \; , \tag{23.9}$$

and

$$|u_y| \le |v_y| + |w_y| \; . \tag{23.10}$$

Obviously, in order to estimate u and its derivatives, it is sufficient to estimate v, w, and their respective derivatives, and then take the corresponding sums in accordance with (23.8) - (23.10).

In estimating v and w, the general procedure coincides with that of [3] in a broad sense. We first write

$$v(x, y) = \overline{v}(x) + \hat{v}(x, y) \; , \tag{23.11}$$

and

$$w(x, y) = \overline{w}(x) + \hat{w}(x, y) \; , \tag{23.12}$$

where $\overline{v}$ and $\overline{w}$ are defined by

$$\overline{v}(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} v(x, y) dy \; , \tag{23.13}$$

and

$$\overline{w}(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} w(x, y) dy \; . \tag{23.14}$$

It follows from(23.11) and (23.13) and from (23.12) and (23.14) that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{v}(x, y) dy = 0 \; , \tag{23.15}$$

and

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{w}(x, y)dy = 0 . \tag{23.16}$$

Next, the averages $\overline{v}$ and $\overline{w}$ will be estimated in terms of the first order energies $V_1(x)$ and $T_1(x)$, respectively; the deviations $\hat{v}$ and $\hat{w}$ will be estimated in terms of the first- and second-order energies $V_1(x)$ and $V_2(x)$ and $T_1(x)$ and $W_2(x)$, respectively. We shall then extend the procedure with minor modifications to estimate the derivatives v_x, v_y, w_x, and w_y. It is for these estimates that appropriate normalization conditions on v and w will be introduced to assure their uniqueness.

We shall consider v first, because the associated differential equation is homogeneous and admits a simpler treatment.

24. Pointwise Estimate of v(x, y).

v is the solution of Problem I given by (9.10) - (9.13). Although we can appropriate directly the results of [3] for the pointwise estimate of v as pointed out in §10, we prefer to present the analysis in detail, because it differs significantly from that of [3] and because we shall often refer to it in later sections.

We first introduce the average of p by

$$\overline{p}(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} p(x, y)dy , \tag{24.1}$$

where p is given by (5.6a).

Upon differentiating (23.13) with respect to x, we find

$$\overline{v}_x(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} v_x(x, y)dy \quad . \tag{24.2}$$

The product of (24. 1) and (24. 2) can be written as

$$\overline{v}_x\overline{p} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{p}v_x dy \quad . \tag{24.3}$$

The differential equation (9. 10) and the boundary conditions (9. 13) imply that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} p\, v_x\, dy = 0 \quad . \tag{24.4}$$

The difference of (24. 3) and (24. 4) yields

$$\overline{v}_x = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\overline{p}-p}{\overline{p}}\right) v_x dy \quad . \tag{24.5}$$

Upon integrating (24. 5) from $x = 0$ to $x = z$, we find

$$\overline{v}(z) - \overline{v}(0) = \int_0^z \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\overline{p}-p}{\overline{p}}\right) v_x dydx \quad . \tag{24.6}$$

In (24. 6), if $\overline{v}(0)$ vanishes, then $\overline{v}(z)$ can be estimated in terms of the <u>first</u> <u>order</u> <u>energy</u> $V_1(z)$ given by (10. 1) after an application of Schwarz's inequality. At this point it is clear that the natural normalization condition on $v(x, y)$ is

$$\overline{v}(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} v(0, y)dy = 0 \quad . \tag{24.7}$$

(24. 6), (24. 7), and Schwarz's inequality imply

$$\begin{aligned} |\overline{v}(z)| &\leq C_1[V_1(z)]^{\frac{1}{2}} \\ &\leq C_1[V_1(\ell)]^{\frac{1}{2}} e^{-k(\ell-z)} , \end{aligned} \tag{24.8}$$

where the constant C_1 satisfies

$$C_1 \geq \left[\int_0^{\ell} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{p} \left(\frac{\bar{p}-p}{\bar{p}} \right)^2 dy dx \right]^{\frac{1}{2}} . \tag{24.9}$$

Next, we estimate the deviation $\hat{v}$. Consider a point (x, y) fixed in $\overline{\mathcal{R}}$. For any constant δ, such that

$$\delta \neq y \quad \text{and} \quad -\tfrac{1}{2} < \delta < \tfrac{1}{2} , \tag{24.10}$$

the following identity holds true:

$$\hat{v}(x, y) \equiv - \int_y^{\delta} \frac{\partial}{\partial \zeta} \left[\left(1 - \frac{y-\zeta}{y-\delta}\right) \hat{v}(x, \zeta) \right] d\zeta . \tag{24.11}$$

When the indicated differentiation is carried out, Schwarz's inequality and the triangular inequality imply

$$|\hat{v}(x, y)| \leq \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{v}^2 dy \right)^{\frac{1}{2}} |\delta - y|^{-\frac{1}{2}} + \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{v}_y^2 dy \right)^{\frac{1}{2}} |\delta - y|^{\frac{1}{2}} . \tag{24.12}$$

Upon differentiating (23.11) with respect to y, we find

$$v_y(x, y) = \hat{v}_y(x, y) . \tag{24.13}$$

In view of (9.13), we also find

$$\hat{v}_y(x, \pm\tfrac{1}{2}) = 0 . \tag{24.14}$$

It follows from (23.15), (24.14), and the continuous differentiability of $\hat{v}$ on $[-\frac{1}{2}, \frac{1}{2}]$ that[2]

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{v}_y^2 dy \geq \pi^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{v}^2 dy . \tag{24.15}$$

(24.12), (24.13), and (24.15) imply

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[2] See, for example, [11] or [13].

$$|\hat{v}(x,y)| \le \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{v}_y^{\,2} dy\right)^{\frac{1}{2}} \left(\frac{|\delta-y|^{-\frac{1}{2}}}{\pi} + |\delta-y|^{\frac{1}{2}}\right). \tag{24.16}$$

Since the left hand side of (24. 16) is independent of δ, we can minimize the inequality with respect to $|\delta-y|$ over all admissible δ of (24. 10). A straightforward computation shows that this minimum value occurs at

$$|\delta-y|_{\min} = \frac{1}{\pi}\ , \tag{24.17}$$

whence

$$|\hat{v}(x,y)| \le \frac{2}{\sqrt{\pi}}[\overline{V}(x)]^{\frac{1}{2}}\ , \tag{24.18}$$

where

$$\overline{V}(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} v_y^{\,2} dy\ . \tag{24.19}$$

If we differentiate (24. 19) with respect to x and then integrate the result from $x = 0$ to $x = z$, we find

$$\overline{V}(z) = V(0) + 2\int_0^z\int_{-\frac{1}{2}}^{\frac{1}{2}} v_y v_{yx} dydx\ . \tag{24.20}$$

Schwarz's inequality and the triangular inequality, together with (10. 1) and (11. 1), imply

$$\overline{V}(z) \le \overline{V}(0) + 2\left(\int_0^z\int_{-\frac{1}{2}}^{\frac{1}{2}} v_y^{\,2} dydx\right)^{\frac{1}{2}}\left(\int_0^z\int_{-\frac{1}{2}}^{\frac{1}{2}} v_{yx}^{\,2} dydx\right)^{\frac{1}{2}}$$

$$\le \overline{V}(0) + \frac{2}{\sqrt{p_o q_o}}[V_1(z)V_2(z)]^{\frac{1}{2}}\ , \tag{24.21}$$

where the positive constants p_o and q_o are such that

$$p_o \le p \quad \text{and} \quad q_o \le q \quad \text{on } \overline{\mathcal{R}}\ . \tag{24.22}$$

Our task is then reduced to finding a useful upper bound for the constant $\overline{V}(0)$, which can be expressed as

$$\overline{V}(0) \equiv -\int_0^\delta \frac{\partial}{\partial x}[\psi(\frac{x}{\delta})\overline{V}(x)]dx , \qquad (24.23)$$

where $\psi(x)$ is any function which is continuously differentiable on $0 \leq x \leq 1$ and satisfies

$$\psi(0) = 1 \quad , \quad \psi(1) = 0 , \qquad (24.24)$$

while δ is a constant such that

$$0 < \delta < \ell . \qquad (24.25)$$

Now we carry out the indicated differentiation in (24.33), and then computations similar to those leading to (24.21) yield

$$\overline{V}(0) = -2\int_0^\delta\int_{-\frac{1}{2}}^{\frac{1}{2}} \psi(\frac{x}{\delta})v_y v_{yx} dydx - \frac{1}{\delta}\int_0^\delta\int_{-\frac{1}{2}}^{\frac{1}{2}} \psi'(\frac{x}{\delta})v_y^2 dydx$$

$$\leq \frac{2\psi_o}{\sqrt{p_o q_o}}[V_1(\delta)V_2(\delta)]^{\frac{1}{2}} + \frac{\psi_1}{\delta q_o} V_1(\delta) , \qquad (24.26)$$

where the positive constants ψ_o and ψ_1 are such that

$$\psi_o = \max_{[0,1]} |\psi(x)| \qquad \text{and} \qquad \psi_1 = \max_{[0,1]} |\psi'(x)| . \qquad (24.27)$$

Substituting the energy inequalities (10.2) and (11.3) into (24.26) and simplifying the result, we find

$$\overline{V}(0) \leq U_1 Q(\delta) e^{-2k\ell} . \qquad (24.28)$$

In (24.28),

$$Q(\delta) = (1 + \frac{U_2}{\delta})e^{2k\delta} , \qquad (24.29)$$

$$U_1 = \frac{2\psi_o}{\sqrt{p_o q_o}}\{V_1(\ell)[V_2(\ell)+2k\ell(\alpha_1+\alpha_2)V_1(\ell)]\}^{\frac{1}{2}} , \qquad (24.30)$$

and

$$U_2 = \tfrac{1}{2}\sqrt{\frac{p_o}{q_o}}\,\frac{\psi_1}{\psi_o}\left[\frac{V_1(\ell)}{V_2(\ell)+2k\ell(\alpha_1+\alpha_2)V_1(\ell)}\right]^{\frac{1}{2}}. \tag{24.31}$$

The independence of $\overline{V}(0)$ on δ enables us to minimize $Q(\delta)$ over all admissible δ of (24.25). Computations then lead to

$$\delta_{min} = \frac{U_2}{2}\left(\sqrt{1+\frac{2}{kU_2}} - 1\right), \tag{24.32}$$

and

$$Q_{min} = \left[1+kU_2\left(\sqrt{1+\frac{2}{kU_2}}+1\right)\right]\exp\left[kU_2\left(\sqrt{1+\frac{2}{kU_2}}-1\right)\right]. \tag{24.33}$$

If (24.25) and (24.32) are to hold simultaneously, then

$$k = \pi\sqrt{q_o/p_1} > 1/2\ell . \tag{24.34}$$

We shall assume, hereafter, that the given functions p and q satisfy (24.34).

Noting that for any $x > 0$,

$$\sqrt{1+\frac{2}{x}} < 1+\frac{1}{x} , \tag{24.35}$$

and

$$x\left(\sqrt{1+\frac{2}{x}}-1\right) \equiv \frac{2}{\sqrt{1+\frac{2}{x}}+1} , \tag{24.36}$$

we can bound Q_{min} by

$$Q_{min} \le \left[1+kU_2\left(2+\frac{1}{kU_2}\right)\right]\exp\left[\frac{2}{1+\sqrt{1+\frac{2}{kU_2}}}\right]$$

$$\le 6(1+kU_2) . \tag{24.37}$$

Combining (24.28) and (24.37), we find

$$\overline{V}(0) \le 6U_1(1+kU_2)e^{-2k\ell} . \tag{24.38}$$

It follows from (24. 18), (24. 21), and (24. 38) that

$$|\hat{v}(x,y)| \le \frac{2}{\sqrt{\pi}}\left\{\sqrt{6U_1(1+kU_2)}\,e^{-k\ell} + \frac{\sqrt{2}}{(p_o q_o)^{1/4}}\,[V_1(x)V_2(x)]^{1/4}\right\}. \tag{24.39}$$

The expressions (23. 11), (24. 8), (24. 39), (10. 2), and (11. 3) lead to, after simplification,

$$|v(x,y)| \le |\bar{v}(x,y)| + |\hat{v}(x,y)| \le D_1 e^{-k(\ell-x)} + D_2 e^{-k\ell}, \tag{24.40}$$

where

$$D_1 = C_1[V_1(\ell)]^{\frac{1}{2}} + 2\sqrt{\frac{2}{\pi}}(p_o q_o)^{-\frac{1}{4}}\{V_1(\ell)[V_2(\ell)+2k\ell(\alpha_1+\alpha_2)V_1(\ell)]\}^{\frac{1}{4}} \tag{24.41}$$

and

$$D_2 = 4\sqrt{\frac{3}{\pi}}(p_o q_o)^{-\frac{1}{4}}\left\{[V_1(\ell)]^{\frac{1}{4}}[V_2(\ell)+2k\ell(\alpha_1+\alpha_2)V_1(\ell)]^{\frac{1}{4}} + \sqrt{\frac{\pi}{2}}[V_1(\ell)]^{\frac{1}{2}}\right\}. \tag{24.42}$$

25. Pointwise Estimate of $v_x(x, y)$.

For convenience, we let

$$\beta(x,y) = v_x(x,y). \tag{25.1}$$

The arguments used in the pointwise estimation of $\beta(x, y)$ are parallel to those used in estimating $v(x, y)$. We therefore set

$$\beta = \bar{\beta} + \hat{\beta}, \tag{25.2}$$

where the average $\bar{\beta}$ is given by

$$\bar{\beta}(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \beta(x,y)\,dy, \tag{25.3}$$

and the deviation $\hat{\beta}$ satisfies

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{\beta}(x,y)\,dy = 0. \tag{25.4}$$

In view of (9. 11), (25. 1), and (25. 3) we find

$$\bar{\beta}(0) = 0. \tag{25.5}$$

It follows from (24. 4) and (25. 1) that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} p(x,y)\beta(x,y)dy = 0 \ . \tag{25.6}$$

Differentiating (25. 3) and (25. 6) with respect to x , we find

$$\overline{\beta}_x = \int_{-\frac{1}{2}}^{\frac{1}{2}} \beta_x dy \ , \tag{25.7}$$

and

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} (p\beta_x + p_x\beta)dy = 0 \ , \tag{25.8}$$

respectively.

Recalling (24. 1), we can write the product of (24. 1) and (25. 7) as

$$\overline{p}(x)\overline{\beta}_x(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{p}(x)\beta_x(x,y)dy \ . \tag{25.9}$$

(25. 8) and (25. 9) imply

$$\overline{\beta}_x = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\overline{p}-p}{\overline{p}}\right)\beta_x dy - \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{p_x}{\overline{p}}\beta dy \ . \tag{25.10}$$

Integrating (25. 10) from x = 0 to x = z and using (25. 5), we find

$$\overline{\beta}(z) = \int_0^z\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\overline{p}-p}{\overline{p}}\right)\beta_x dydx - \int_0^z\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{p_x}{\overline{p}}\beta dydx \ . \tag{25.11}$$

In view of (9. 10),

$$\beta_x = -\frac{1}{p}(p_x v_x + q v_{yy} + q_y v_y) \ . \tag{25.12}$$

Upon substitution of (25. 12) into the first integral of (25. 11) and an easy application of Schwarz's inequality and the triangular inequality, it follows from (10. 1) that

$$|\overline{\beta}(z)| \le C_2[V_2(z)]^{\frac{1}{2}}+(C_3+C_4)[V_1(z)]^{\frac{1}{2}}$$

$$\le \{C_2[V_2(\ell)+2k\ell(\alpha_1+\alpha_2)V_1(\ell)]^{\frac{1}{2}}+(C_3+C_4)[V_1(\ell)]^{\frac{1}{2}}\}e^{-k(\ell-z)}, \tag{25.13}$$

where the constants C_2, C_3, and C_4 are such that

$$C_2 \ge \left[\int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{\overline{p}-p}{p\overline{p}}\right)^2 q\,dydx\right]^{\frac{1}{2}}, \tag{25.14}$$

$$C_3 \ge \left\{\int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left[\left(\frac{\overline{p}-p}{p\overline{p}}\right)q_y\right]^2 \frac{dydx}{q}\right\}^{\frac{1}{2}}, \tag{25.15}$$

and

$$C_4 \ge \left[\int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{p_x^2}{p^3}\,dydx\right]^{\frac{1}{2}}. \tag{25.16}$$

(9.13) and (25.2) imply

$$\hat{\beta}_y(x, \pm\tfrac{1}{2}) = v_{yx}(x, \pm\tfrac{1}{2}). \tag{25.17}$$

The conditions (24.4) and (25.17) and the continuous differentiability of $\hat{\beta}$ imply (see footnote 2)

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\hat{\beta}_y^2\,dy \ge \pi^2\int_{-\frac{1}{2}}^{\frac{1}{2}}\hat{\beta}^2\,dy, \tag{25.18a}$$

or

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}v_{yx}^2\,dy \ge \pi^2\int_{-\frac{1}{2}}^{\frac{1}{2}}\hat{\beta}^2\,dy. \tag{25.18b}$$

Then, calculations analogous to those leading to (24.18) and (24.21) yield

$$|\hat{\beta}(x,y)| \le \frac{2}{\sqrt{\pi}}[B(x)]^{\frac{1}{2}}, \tag{25.19}$$

where

$$B(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} v_{xy}^2 dy \qquad \text{and} \qquad B(0) = 0 , \tag{25.20}$$

and

$$B(z) \le \frac{2}{p_o} [V_2(z)V_3(z)]^{\frac{1}{2}} . \tag{25.21}$$

(25.9), (25.21), (11.3), and (12.21) imply

$$|\hat{\beta}(x, y)| \le C_5 e^{-k(\ell-x)} , \tag{25.22}$$

where the constant C_5 is given by

$$C_5 \ge 2(\frac{2}{\pi p_o})^{1/2} [V_2(\ell)+2k\ell(\alpha_1+\alpha_2)V_1(\ell)]^{1/4} C^{1/4} . \tag{25.23}$$

Combining (25.2), (25.13), and (25.22), we find

$$|v_x(x, y)| \le D_3 e^{-k(\ell-x)} ; \tag{25.24}$$

D_3 is a constant such that

$$D_3 = C_2[V_2(\ell)+2k\ell(\alpha_1+\alpha_2)V_1(\ell)]^{\frac{1}{2}}+(C_3+C_4)[V_1(\ell)]^{\frac{1}{2}}+C_5 . \tag{25.25}$$

26. Pointwise Estimate of $v_y(x, y)$.

The boundary conditions (9.13) and the twice continuous differentiability of the solution v on $\overline{\mathcal{R}}$ imply[3]

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} v_{yy}^2 dy \ge \pi^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} v_y^2 dy . \tag{26.1}$$

Then, along the same steps leading to (24.18) and (24.21) we reach

$$|v_y(x, y)| \le \frac{2}{\sqrt{\pi}} [X(x)]^{\frac{1}{2}} , \tag{26.2}$$

where

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[3] See, for example, [11] or [13].

$$X(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} v_{yy}^2 dy , \tag{26.3}$$

and

$$X(z) \le X(0) + \frac{2}{q_o}[V_2(z)V_3(z)]^{\frac{1}{2}} , \tag{26.4}$$

and

$$X(0) \le 6U_3(1+kU_4)e^{-2k\ell} , \tag{26.5}$$

where

$$U_3 = \frac{2\psi_o}{q_o}[V_2(\ell)+2k\ell(\alpha_1+\alpha_2)V_1(\ell)]^{\frac{1}{2}}C^{\frac{1}{2}} , \tag{26.6}$$

$$U_4 = \frac{1}{2}\frac{\psi_1}{\psi_o}\left[\frac{V_2(\ell)+2k\ell(\alpha_1+\alpha_2)V_1(\ell)}{C}\right]^{\frac{1}{2}} , \tag{26.7}$$

and C is given by (12.22).

Substituting (26.4) and (26.5) into (26.2), we find

$$|v_y(x; y)| \le D_4 e^{-k(\ell-x)} + D_5 e^{-k\ell} \tag{26.8}$$

where

$$D_4 = \frac{2}{\sqrt{\pi}}\sqrt{\frac{2}{q_o}}\, C^{\frac{1}{4}}[V_2(\ell)+2k\ell(\alpha_1+\alpha_2)V_1(\ell)]^{\frac{1}{4}} , \tag{26.9}$$

and

$$D_5 = \frac{2}{\sqrt{\pi}}[6U_3(1+kU_4)]^{\frac{1}{2}} . \tag{26.10}$$

The inequalities (24.30), (24.31), (26.6), and (26.7) involve the constants ψ_o and ψ_1, which in turn depend on the function $\psi(x)$ defined in (24.24). A function ψ suitable for the present need is given by

$$\psi(x) = (1-x)\exp\left[-\left(\sqrt{\frac{p_1}{p_o}} - 1\right)x\right] , \tag{26.11}$$

from which and (24.27) we find

$$\psi_o = \max_{[0,1]} |\psi(x)| = 1 \ , \qquad \psi_1 = \max_{[0,1]} |\psi'(x)| = \sqrt{\frac{p_1}{p_o}} \ . \qquad (26.12)$$

27. Pointwise Estimate of w(x, y).

The procedure used in estimating $w(x, y)$ is akin to that used in §24, except for some modifications which arise due to the non-homogeneous term, F, in (9.14).

Upon differentiating (23.14) with respect to x, and taking the product of this derivative and (24.1), we find

$$\overline{w}_x \overline{p} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{p} w_x dy \ . \qquad (27.1)$$

Next, the differential equation (9.14), the boundary conditions (9.17), and the assumption (9.8) imply

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} p(x, y) w_x(x, y) dy = 0 \ . \qquad (27.2)$$

It follows from (27.1) and (27.2) that

$$\overline{w}_x = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\overline{p}-p}{\overline{p}}\right) w_x dy \ . \qquad (27.3)$$

Integrating (27.3) from $x = z$ to $x = \ell$, we find

$$\overline{w}(\ell) - \overline{w}(z) = \int_z^{\ell} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\overline{p}-p}{\overline{p}}\right) w_x dy dx \ . \qquad (27.4)$$

We observe that a natural normalization condition for the solution w of Problem II is given by

$$\overline{w}(\ell) = \int_{-\frac{1}{2}}^{\frac{1}{2}} w(\ell, y) dy = 0 \ . \qquad (27.5)$$

Recalling (B. 3) and (B. 4) of Appendix B, we apply Schwarz's inequality to (27. 4) and keep (27. 5) in mind to find

$$|\bar{w}(z)| \le C_1[T_1(z)]^{\frac{1}{2}} \le C_1\{[2S_1(0)]^{\frac{1}{2}}+[2N_1(0)]^{\frac{1}{2}}e^{-kz}\} , \quad (27.6)$$

where the constant C_1 is given by (24. 9).

To estimate the deviation $\hat{w}$, a procedure analogous to that leading to (24. 18) and (24. 19) yields

$$|\hat{w}(x, y)| \le \frac{2}{\sqrt{\pi}}[\overline{W}(x)]^{\frac{1}{2}} , \quad (27.7)$$

where

$$\overline{W}(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} w_y^2 dy \; . \quad (27.8)$$

Differentiating (27. 8) with respect to x and then integrating the result from $x = z$ to $x = \ell$, we find

$$\overline{W}(z) = \overline{W}(\ell) - 2\int_z^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} w_y w_{yx} dy dx \; . \quad (27.9)$$

A result similar to that of (24. 21) states

$$\overline{W}(z) \le \overline{W}(\ell) + \frac{2}{\sqrt{p_o q_o}}[T_1(z)W_2(z)]^{\frac{1}{2}} . \quad (27.10)$$

To find a useful upper bound for the constant $\overline{W}(\ell)$, we introduce an identity as follows.

$$\overline{W}(\ell) \equiv \int_{\ell-\delta}^{\ell} \frac{\partial}{\partial x}[\psi(\frac{\ell-x}{\delta/\ell})\overline{W}(x)]dx \; ; \quad (27.11)$$

where $\psi(x)$ is continuously differentiable on $0 \le x \le \ell$ and satisfies

$$\psi(0) = 1 \quad , \quad \psi(\ell) = 0 , \quad (27.12)$$

and δ is any constant such that

$$0 < \delta < \ell . \quad (27.13)$$

Then, a result similar to that of (24.26) states

$$\overline{W}(\ell) \le \frac{2\psi_o}{\sqrt{p_o q_o}} [T_1(\ell-\delta)W_2(\ell-\delta)]^{\frac{1}{2}} + \frac{\ell}{\delta}\frac{\psi_1}{q_o} T_1(\ell-\delta) , \qquad (27.14)$$

where the constants ψ_o and ψ_1 are given by

$$\psi_o = \max_{[0,\ell]} |\psi(x)| , \qquad \psi_1 = \max_{[0,\ell]} |\psi'(x)| . \qquad (27.15)$$

After substitution of (13.27) and (14.15) into (27.14), we are tempted to minimize the result over all admissible δ as we did in (24.29). If we try this in the present case, we have to determine δ_{min} from a transcendental equation as follows.

$$\exp\{-\frac{k}{2}\delta_{min}\} = a_1 + a_2\delta_{min} + a_3\delta_{min}^2 , \qquad (27.16)$$

where the constants a_1, a_2, and a_3 are known. A closed form of δ_{min} such as that obtained in (24.32) is beyond our means in the present case, but a suitable choice of an admissible δ is given by

$$\delta = \ell/2 . \qquad (27.17)$$

With this choice of δ, (27.14) becomes

$$\overline{W}(\ell) \le \frac{2\psi_o}{\sqrt{p_o q_o}} [T_1(\ell/2)W_2(\ell/2)]^{\frac{1}{2}} + \frac{2\psi_1}{q_o} T_1(\ell/2) . \qquad (27.18)$$

A suitable choice of the function $\psi(x)$ satisfying (27.12) and the smoothness assumption is given by

$$\psi(x) = 1 - \frac{x}{\ell} . \qquad (27.19)$$

Then

$$\left.\begin{aligned} \psi_o &= \max_{[0,\ell]} |\psi(x)| = 1 , \\ \psi_1 &= \max_{[0,\ell]} |\psi'(x)| = 1/\ell . \end{aligned}\right\} \qquad (27.20)$$

In view of (27.7), (27.10), (27.18), and (27.20), we find

$$|\hat{w}(x,y)| \le \frac{2\sqrt{2}}{\sqrt{\pi}}\left\{\frac{1}{(p_o q_o)^{\frac{1}{4}}}\left[T_1(\tfrac{\ell}{2})W_2(\tfrac{\ell}{2})\right]^{\frac{1}{4}} + \frac{1}{(\ell q_o)^{\frac{1}{2}}}\left[T_1(\tfrac{\ell}{2})\right]^{\frac{1}{2}} + \right.$$

$$\left. + \frac{1}{(p_o q_o)^{\frac{1}{4}}}\left[T_1(x)W_2(x)\right]^{\frac{1}{4}}\right\} . \tag{27.21}$$

Combining (23.12), (27.6), (27.21), (13.27), and (14.15), we find, after simplification,

$$|w(x,y)| \le D_6 + D_7 e^{-\frac{k\ell}{8}} + D_8 e^{-\frac{kx}{4}} , \tag{27.22}$$

where the constants D_6, D_7, and D_8 are such that

$$D_6 = C_1[2S_1(0)]^{\frac{1}{2}} + 4\sqrt{\frac{2}{\pi}}\left[\frac{2S_1(0)}{p_o q_o}\right]^{\frac{1}{4}}\left\{\gamma_1 S_1(0) + \gamma_2[S_1(0)]^{\frac{1}{2}}\right\}^{\frac{1}{4}} +$$

$$+ 2\sqrt{\frac{2}{\pi}}\left[\frac{2S_1(0)}{\ell q_o}\right]^{\frac{1}{2}} , \tag{27.23}$$

$$D_7 = 2\sqrt{\frac{2}{\pi}}\left\{\left(\frac{2}{p_o q_o}\right)^{\frac{1}{4}}[S_1(0)+N_1(0)]^{\frac{1}{4}}[2\gamma_2 N_1^{\frac{1}{2}}(0) + W_2(0) + 2k\ell\gamma_1 N_1(0) -\right.$$

$$-2\gamma_2 N_1^{\frac{1}{2}}(0) - \gamma_1 S_1(0) - \gamma_2 S_1^{\frac{1}{2}}(0)]^{\frac{1}{4}} + \left(\frac{2}{p_o q_o}\right)^{\frac{1}{4}}[N_1(0)]^{\frac{1}{4}}[\gamma_1 S_1(0) + \gamma_2 S_1^{\frac{1}{2}}(0)]^{\frac{1}{4}} +$$

$$\left. + \left[\frac{2N_1(0)}{\ell q_o}\right]^{\frac{1}{2}}\right\} , \tag{27.24}$$

and

$$D_8 = 2\sqrt{\frac{2}{\pi}}\left\{\left(\frac{2}{p_o q_o}\right)^{\frac{1}{4}}[S_1(0)+N_1(0)]^{\frac{1}{4}}[2\gamma_2 N_1^{\frac{1}{2}}(0) + W_2(0) + 2k\ell\gamma_1 N_1(0) - 2\gamma_2 N_1^{\frac{1}{2}}(0)\right.$$

$$-\gamma_1 S_1(0) - \gamma_2 S_1^{\frac{1}{2}}(0)]^{\frac{1}{4}} + \left(\frac{2}{p_o q_o}\right)^{\frac{1}{4}}[N_1(0)]^{\frac{1}{4}}[\gamma_1 S_1(0) + \gamma_2 S_1^{\frac{1}{2}}(0)]^{\frac{1}{4}} +$$

$$\left. + C_1[2N_1(0)]^{\frac{1}{2}}\right\} . \tag{27.25}$$

28. Pointwise Estimate of $w_x(x, y)$.

Guided by the procedure used in estimating v_x in §25, we let

$$d(x, y) = w_x(x, y) , \tag{28.1}$$

and define

$$d = \bar{d} + \hat{d} , \tag{28.2}$$

where the average $\bar{d}$ is given by

$$\bar{d}(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} d(x, y)dy , \tag{28.3}$$

and the deviation $\hat{d}$ satisfies

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{d}(x, y)dy = 0 . \tag{28.4}$$

It follows from (9.16), (28.3), and (28.1) that

$$\bar{d}(\ell) = 0 . \tag{28.5}$$

Then, computations analogous to those leading to (25.13) yield

$$|\bar{d}(x)| \leq C_6 + C_2[W_2(z)]^{\frac{1}{2}} + (C_3+C_4)[T_1(z)]^{\frac{1}{2}} , \tag{28.6}$$

where the constants C_2, C_3, and C_4 are given by (25.14) - (25.16) and C_6 is such that

$$C_6 \geq \left| \int_x^{\ell} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\bar{p}-p}{\bar{p}p}\right) F dy dx \right| . \tag{28.7}$$

Similarly, we find

$$|\hat{d}(x, y)| \leq \frac{2}{\sqrt{\pi}} [D(x)]^{\frac{1}{2}} , \tag{28.8}$$

where

$$D(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} w_{xy}^2 dy \quad \text{and} \quad D(\ell) = 0 , \tag{28.9}$$

and also

$$D(x) = -2\int_x^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} w_{xy}w_{xxy}\,dydx \le \frac{2}{p_o}[W_2(x)W_3(x)]^{\frac{1}{2}}. \qquad (28.10)$$

It follows from (28. 8) and (28. 10) that

$$|\hat{d}(x,y)| \le 2\sqrt{\frac{2}{\pi p_o}}[W_2(x)W_3(x)]^{1/4}. \qquad (28.11)$$

Combining (28. 1), (28. 2), (28. 6), (28. 11), (14. 15), and (15. 14), we find

$$|w_x(x,y)| \le D_9 + D_{10}e^{-\frac{kx}{8}}, \qquad (28.12)$$

where the constants D_9 and D_{10} are given by

$$D_9 = C_2\sqrt{\gamma_1 S_1(0)+\gamma_2[S_1(0)]^{\frac{1}{2}}}+\sqrt{2}(C_3+C_4)[S_1(0)]^{\frac{1}{2}}+C_6+$$

$$+2\sqrt{\frac{2}{\pi p_o}}\{\kappa_1\gamma_1 S_1(0)+\kappa_1\gamma_2[S_1(0)]^{\frac{1}{2}}\}^{1/4}, \qquad (28.13)$$

and

$$D_{10} = C_2\sqrt{2\gamma_2[N_1(0)]^{\frac{1}{2}}+W_2(0)+2k\ell\gamma_1 N_1(0)} + (C_3+C_4)\sqrt{2N_1(0)}$$

$$+2\sqrt{\frac{2}{\pi p_o}}\{(\kappa_2+\kappa_3+\kappa_4)(\gamma_1 S_1(0)+\gamma_2[S_1(0)]^{\frac{1}{2}})+$$

$$+\kappa_1\left(2\gamma_2[N_1(0)]^{\frac{1}{2}}+W_2(0)+2k\ell\gamma_1 N_1(0)\right)\}^{1/4}. \qquad (28.14)$$

29. Pointwise Estimate of $w_y(x,y)$.

Finally, we estimate w_y. The main ideas in this case are similar to those of §26 and §27. Guided by the procedures of those two sections, we find

$$|w_y(x,y)| \le \frac{2}{\sqrt{\pi}}[Y(x)]^{\frac{1}{2}}, \qquad (29.1)$$

where

$$Y(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} w_{yy}^2 dy \quad , \tag{29.2}$$

and

$$Y(z) \le Y(\ell) + \frac{2}{q_o} [W_2(z)W_3(z)]^{\frac{1}{2}} . \tag{29.3}$$

The constant $Y(\ell)$ is estimated by

$$Y(\ell) \le \frac{2}{q_o} [W_2(\frac{\ell}{2})W_3(\frac{\ell}{2})]^{\frac{1}{2}} + \frac{2}{\ell q_o} W_2(\frac{\ell}{2}) . \tag{29.4}$$

Then, combining (29.1), (29.3), and (29.4), we find

$$|w_y(x,y)| \le D_{11} + D_{12} e^{-\frac{k\ell}{16}} + D_{13} e^{-\frac{kx}{8}} , \tag{29.5}$$

where the constants D_{11}, D_{12}, and D_{13} are given by

$$D_{11} = \frac{4}{\sqrt{\pi q_o}} (\varkappa_1)^{\frac{1}{4}} \left\{ \gamma_1 S_1(0) + \gamma_2 [S_1(0)]^{\frac{1}{2}} \right\}^{\frac{1}{4}} + 2\sqrt{\frac{2}{\pi \ell q_o}} \sqrt{\gamma_1 S_1(0) + \gamma_2 [S_1(0)]^{\frac{1}{2}}} , \tag{29.6}$$

$$\begin{aligned} D_{12} = 2\sqrt{\frac{2}{\pi q_o}} \{ & (\varkappa_2 + \varkappa_3 + \varkappa_4)[\gamma_1 S_1(0) + \gamma_2 \sqrt{S_1(0)}] \\ & + \varkappa_1 [2\gamma_1 \sqrt{N_1(0)} + W_2(0) + 2k\ell\gamma_1 N_1(0)]\}^{\frac{1}{4}} \\ & + 2\sqrt{\frac{2}{\pi \ell q_o}} [2\gamma_2 \sqrt{N_1(0)} + W_2(0) + 2k\ell\gamma_1 N_1(0)]^{\frac{1}{2}} , \end{aligned} \tag{29.7}$$

and

$$\begin{aligned} D_{13} = 2\sqrt{\frac{2}{\pi q_o}} \{ & (\varkappa_2 + \varkappa_3 + \varkappa_4)[\gamma_1 S_1(0) + \gamma_2 \sqrt{S_1(0)}] \\ & + \varkappa_1 [2\gamma_1 \sqrt{N_1(0)} + W_2(0) + 2k\ell\gamma_1 N_1(0)]\}^{\frac{1}{4}} . \end{aligned} \tag{29.8}$$

If we now apply the triangular inequality to (23.5) - (23.7) and employ the various estimates of the preceding sections suitably, we find (23.1) - (23.3). The constants K_i $(i = 1, 2, \ldots, 9)$ can now be explicitly written down as

$$K_1 = D_2 e^{-k\ell} + D_6 + D_7 e^{-\frac{k\ell}{8}}, \qquad (29.9)$$

$$K_2 = D_8 , \qquad (29.10)$$

$$K_3 = D_1 , \qquad (29.11)$$

$$K_4 = D_9 , \qquad (29.12)$$

$$K_5 = D_{10} , \qquad (29.13)$$

$$K_6 = D_3 , \qquad (29.14)$$

$$K_7 = D_5 e^{-k\ell} + D_{11} + D_{12} e^{-\frac{k\ell}{16}} , \qquad (29.15)$$

$$K_8 = D_{13} , \qquad (29.16)$$

and

$$K_9 = D_{14} . \qquad (29.17)$$

V. THE SHELL PROBLEM

30. Decomposition of the Error $\varphi(x,y)$.

In this chapter we return to the shell problem formulated in Chapter I. We shall apply the results of Chapters II and III to derive pointwise estimates of the first partial derivatives, $\hat{\varphi}_x$ and $\hat{\varphi}_y$, of the error $\hat{\varphi}$. These partial derivatives in turn are proportional to the errors in the shear stresses associated with the approximate solution.

Guided by the results of Chapter II, we decompose the error $\hat{\varphi}$ into:

$$\hat{\varphi}(x,y) = \hat{\varphi}_1(x,y) + \hat{\varphi}_2(x,y)\,, \tag{30.1}$$

where $\hat{\varphi}_1$ and $\hat{\varphi}_2$ satisfy

$$(p\hat{\varphi}_{1x})_x + (q\hat{\varphi}_{1y})_y = 0 \qquad \text{on } \mathcal{R}\,, \tag{30.2}$$

$$p\hat{\varphi}_{1x} = 0 \qquad \text{at } x = 0\,, \tag{30.3}$$

$$p\hat{\varphi}_{1x} = g_2 \qquad \text{at } x = \overline{\ell}\,, \tag{30.4}$$

$$\hat{\varphi}_{1y} = 0 \qquad \text{at } y = \pm\tfrac{1}{2}\,, \tag{30.5}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} p(x,y)\hat{\varphi}_{1x}(x,y)\,dy = 0\,, \qquad 0 \le x \le \overline{\ell}\,; \tag{30.6}$$

and

$$(p\hat{\varphi}_{2x})_x + (q\hat{\varphi}_{2y})_y = F \qquad \text{on } \mathcal{R}\,, \tag{30.7}$$

$$p\hat{\varphi}_{2x} = g_1 \qquad \text{at } x = 0\,, \tag{30.8}$$

$$p\hat{\varphi}_{2x} = 0 \qquad \text{at } x = \overline{\ell}\,, \tag{30.9}$$

$$\hat{\varphi}_{2y} = 0 \qquad \text{at } y = \pm\tfrac{1}{2}\,, \tag{30.10}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} p(x,y)\hat{\varphi}_{2x}(x,y)dy = 0 \ , \qquad 0 \le x \le \bar{\ell} \ , \tag{30.11}$$

respectively. Here, p, q, F, g_1, and g_2 are given by (5.6a, b), (6.9), (6.11), and (6.12). We note that the "residual boundary value problem" (6.8) - (6.12) for $\hat{\varphi}$ has, in fact, been decomposed into two subsidiary problems for $\hat{\varphi}_1$ and $\hat{\varphi}_2$ along lines similar to those used in connection with the problem for u in Chapter II.

In §9, we pointed out that the "residual boundary value problem" (6.8) - (6.12) was a special case of the general problem (9.2) - (9.6) and became identical when (9.7) and (9.8) held. The fact that g_1, g_2, and F in (30.8), (30.4), and (30.7) satisfy (6.13) and (6.15) and that the necessary conditions (30.6) and (30.11) must hold enables us to consider the problems governing $\hat{\varphi}_1$ and $\hat{\varphi}_2$ as parallel cases to Problems I and II of §9.

In our subsequent discussion of the solutions $\hat{\varphi}_1$ and $\hat{\varphi}_2$, we shall expressly set

$$\hat{\varphi}_1 = v \ , \tag{30.12}$$

and

$$\hat{\varphi}_2 = w \ ,[1] \tag{30.13}$$

so that it is appropriate to write

$$\hat{\varphi} \equiv u \ . \tag{30.14}$$

[1] In adjusting the constants of integration associated with the solution t of (13.1), we have chosen

$$G(x,y;\varepsilon) = p(x,y;\varepsilon)\,\tilde{\varphi}_x(x;\varepsilon) - p(\bar{\ell},y;\varepsilon)\tilde{\varphi}_x(\bar{\ell};\varepsilon) \ ; \tag{30.15}$$

see (6.9) and (13.2). (30.15) implies $w(\bar{\ell}, y) = 0$. See footnote 4 of §13.

We can now apply the results in Chapters II-IV for the solution u after changing the constant ℓ to $\overline{\ell}$. Such an application immediately gives us pointwise estimates for $\hat{\varphi}_x$ and $\hat{\varphi}_y$.

31. Uniqueness of the Twist Function φ.

In §4, we pointed out that the twist function φ satisfying the boundary value problem (4.5) - (4.8) is unique up to an arbitrary constant whose determination has been postponed. We can determine this constant as follows.

The twist function φ, the approximate solution $\tilde{\varphi}$, and the error $\hat{\varphi}$ are related by

$$\varphi = \tilde{\varphi} + \hat{\varphi} \ . \tag{31.1}$$

From (30.1) we also have

$$\varphi = \tilde{\varphi} + \hat{\varphi}_1 + \hat{\varphi}_2 \ , \tag{31.2}$$

where, according to (30.12) - (30.13), (24.7), and (27.5), $\hat{\varphi}_i$ $(i = 1, 2)$ are made unique by normalization conditions

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{\varphi}_1(0, y)dy = 0 \ , \tag{31.3}$$

and

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{\varphi}_2(\overline{\ell}, y)dy = 0. \tag{31.4}$$

Conditions (31.3) and (31.4), when converted to the notation pertaining to the original geometry, become

$$\int_{-h/2}^{h/2} \hat{\varphi}_1(0, \zeta)d\zeta = 0 \ , \tag{31.5}$$

and

$$\int_{-h/2}^{h/2} \hat{\varphi}_2(\ell, \zeta)d\zeta = 0 . \tag{31.6}$$

Equations (6.5), (31.5), and (31.6) define $\tilde{\varphi}$, $\hat{\varphi}_1$, and $\hat{\varphi}_2$, respectively, and hence also φ through the relation (31.2). Thus, φ is uniquely determined.

32. Limiting Estimates for the Error Stresses as $\epsilon \to 0$.

The error stresses $\hat{\tau}_{\xi\theta}$ and $\hat{\tau}_{\zeta\theta}$ are computed in terms of the error $\hat{\varphi}$ from (4.3) by

$$\hat{\tau}_{\xi\theta} = \frac{\mu r \frac{\partial\hat{\varphi}}{\partial\xi}}{1 + \frac{\zeta}{R_\xi}} , \qquad \hat{\tau}_{\zeta\theta} = \mu r \frac{\partial\hat{\varphi}}{\partial\zeta} . \tag{32.1}$$

In this section, we shall compute the limiting estimates of the error stresses for points (x, y) with $0 \le x \le \bar{\ell}$ and $-\frac{1}{2} \le y \le \frac{1}{2}$ in terms of the thinness parameter ϵ as ϵ approaches zero. During this limiting process, the positions (x, y) relative to each other, of the points in the shell, remain fixed. The algebraic calculations for such estimates are lengthy but straightforward. We shall not exhibit these calculations in detail, but shall point out the main steps leading to the final results.

In our procedure, we shall first determine the limiting estimates for the energies $V_i(x)$ and $W_i(x)$ $(i = 1, 2, 3)$ from the results of Chapters II and III after replacing ℓ by $\bar{\ell}$. We then apply the limiting energy estimates to obtain bounds for the error stresses in accordance with the formulas established in Chapter IV and the relations (32.1).

To estimate $V_1(x)$, we first read off from (5.6a, b) and (10.3) that

$$p = O(\varepsilon^2) \ , \quad q = O(1) \qquad \text{as } \varepsilon \to 0 \ , \tag{32.2}$$

and

$$k = O(\varepsilon^{-1}) \qquad \text{as } \varepsilon \to 0 \ . \tag{32.3}$$

Next, we assume that the load-functions f_i $(i = 1, 2)$ which are introduced in (3.17) satisfy

$$f_i = O(1) \quad (i = 1, 2) \qquad \text{as } \varepsilon \to 0 \ . \tag{32.4}$$

From (32.2) - (32.4), we find

$$T = O(\varepsilon) \qquad \text{as } \varepsilon \to 0 \ , \tag{32.6}$$

$$G = O(\varepsilon^3) \qquad \text{as } \varepsilon \to 0 \ , \tag{32.7}$$

and

$$g_i = O(\varepsilon^2) \quad (i = 1, 2) \qquad \text{as } \varepsilon \to 0 \ , \tag{32.8}$$

where the scalar torque T is given by (3.18); G and g_i are given by (30.15), (6.11), and (6.12), respectively. (32.2), (32.8), and (17.2), after replacing ℓ by $\overline{\ell}$, imply

$$V_1(\overline{\ell}) = O(\varepsilon^2) \qquad \text{as } \varepsilon \to 0 \ . \tag{32.9}$$

We can now observe from (10.2), (32.3), and (32.9) that the <u>first order energy</u> associated with $\hat{\varphi}_1$ satisfies

$$V_1(x) = O(\varepsilon^2)\exp[-O(\varepsilon^{-1})(\overline{\ell}-x)] \qquad \text{as } \varepsilon \to 0 \ . \tag{32.10}$$

Similarly, we find that γ_1 of (14.13) and $V_2(\overline{\ell})$ of (18.2) satisfy

$$\gamma_1 = O(\varepsilon^2) \qquad \text{as } \varepsilon \to 0 \ , \tag{32.11}$$

and

$$V_2(\overline{\ell}) = O(\varepsilon^3) \qquad \text{as } \varepsilon \to 0 \ . \tag{32.12}$$

Then (11.3) asserts that the <u>second order energy</u> of $\hat{\varphi}_1$ satisfies

$$V_2(x) = O(\varepsilon^3)\exp[-O(\varepsilon^{-1})(\bar{\ell}-x)] \quad \text{as } \varepsilon \to 0 . \tag{32.13}$$

In estimating the third order energy $V_3(x)$ of $\hat{\varphi}_1$, it is convenient for us first to summarize the following results. From (11.4), (11.5), (A.25) of Appendix A, (12.17), (17.4), (19.8), and (19.7), we find

$$\alpha_1 + \alpha_2 = O(\varepsilon^2) \quad \text{as } \varepsilon \to 0 , \tag{32.14}$$

$$\beta_o = O(\varepsilon^4) , \quad \beta_1 = O(\varepsilon) , \quad \beta_2 = O(1) , \quad \beta_3 = O(1) \quad \text{as } \varepsilon \to 0 , \tag{32.15}$$

$$m_3 = O(\varepsilon^3) , \quad m_4 = O(\varepsilon^2) \quad \text{as } \varepsilon \to 0 , \tag{32.16}$$

and

$$V_3(\bar{\ell}) = O(\varepsilon^3) \quad \text{as } \varepsilon \to 0 . \tag{32.17}$$

Combining (12.21) and (12.22) with (32.14) - (32.17) and the limiting estimates for $V_1(\bar{\ell})$, $V_2(\bar{\ell})$, and k, we find

$$V_3(x) = O(\varepsilon)\exp[-O(\varepsilon^{-1})(\bar{\ell}-x)] \quad \text{as } \varepsilon \to 0 . \tag{32.18}$$

Likewise, limiting estimates from (14.13), (15.11), (C.13) of Appendix C, (20.5), (20.12), (21.5), and (22.12) give

$$\gamma_2 = O(\varepsilon^3) \quad \text{as } \varepsilon \to 0 , \tag{32.19}$$

$$\beta_4 = O(\varepsilon^2) , \quad \beta_5 = O(\varepsilon^4) \quad \text{as } \varepsilon \to 0 , \tag{32.20}$$

$$m_1 = O(\varepsilon^2) , \quad M = O(\varepsilon^3) \quad \text{as } \varepsilon \to 0 , \tag{32.21}$$

$$M' = O(\varepsilon^3) \quad \text{as } \varepsilon \to 0 , \tag{32.22}$$

and

$$m_5 = O(\varepsilon^3) , \quad m_6 = O(\varepsilon^2) \quad \text{as } \varepsilon \to 0 . \tag{32.23}$$

Computations analogous to those leading to (32.10), (32.13), and (32.18) then yield

$$W_1(x) = O(\varepsilon^5) + O(\varepsilon^2)\exp[-O(\varepsilon^{-1})x] \quad \text{as } \varepsilon \to 0 , \qquad (32.24)$$ [2]

$$W_2(x) = O(\varepsilon^{11/2}) + O(\varepsilon^3)\exp[-O(\varepsilon^{-1})x] \quad \text{as } \varepsilon \to 0 , \qquad (32.25)$$

and

$$W_3(x) = O(\varepsilon^{9/2}) + O(\varepsilon)\exp[-O(\varepsilon^{-1})x] \quad \text{as } \varepsilon \to 0 , \qquad (32.26)$$

where $W_i(x)$ $(i = 1, 2, 3)$ are the three orders of energies associated with $\hat{\varphi}_2$.

With the energy estimates given by (32.10), (32.13), (32.18), and (32.24) - (32.26) at our disposal, we find, from (25.24), (28.12), (26.8), (29.5), (30.12), and (30.13), that

$$\hat{\varphi}_x = O(\varepsilon^{3/2}) + O(1)\exp[-O(\varepsilon^{-1})x] + O(1)\exp[-O(\varepsilon^{-1})(\bar{\ell}-x)] \quad \text{as } \varepsilon \to 0 , \qquad (32.27)$$

and

$$\hat{\varphi}_y = O(\varepsilon^{5/2}) + O(\varepsilon)\exp[-O(\varepsilon^{-1})x] + O(\varepsilon)\exp[-O(\varepsilon^{-1})(\bar{\ell}-x)] + O(\varepsilon)\exp[-O(\varepsilon^{-1})\bar{\ell}] \quad \text{as } \varepsilon \to 0 . \qquad (32.28)$$

Finally, the two non-vanishing error stresses $\hat{\tau}_{\xi\theta}$ and $\hat{\tau}_{\zeta\theta}$, in terms of the variables x and y, become, after combining (32.1), (32.27), and (32.28),

$$\hat{\tau}_{x\theta} = O(\varepsilon^{3/2}) + O(1)\exp[-O(\varepsilon^{-1})x] + O(1)\exp[-O(\varepsilon^{-1})(\bar{\ell}-x)] \quad \text{as } \varepsilon \to 0 , \qquad (32.29)$$

$$\hat{\tau}_{y\theta} = O(\varepsilon^{3/2}) + O(1)\exp[-O(\varepsilon^{-1})x] + O(1)\exp[-O(\varepsilon^{-1})(\bar{\ell}-x)] + O(1)\exp[-O(\varepsilon^{-1})\bar{\ell}] \quad \text{as } \varepsilon \to 0 . \qquad (32.30)$$

In (6.6), the approximate stress $\tilde{\tau}_{\zeta\theta} = 0$; hence, (32.30) actually gives the estimate for the exact stress component $\tau_{y\theta}$.

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[2] Note that $W_1(x) = T_1(x)$. See (B.4) of Appendix B.

33. Application to a Cylindrical Shell.

In the last section, we obtained limiting estimates for the stresses; now we shall exhibit stronger estimates by finding upper bounds for the stresses in terms of the thinness parameter ϵ.

To minimize the algebraic complexity, we shall deal with a circular cylindrical shell whose meridional cross section is shown in Figure 3. In Figure 3, ρ, r_1 and r_2 are constants. They stand for the distances respectively from the mid-surface and the lateral surfaces of the shell to the axis of symmetry. The angle $\beta(\xi)$ (Figure 1) becomes $\pi/2$ in this case.

From (2.5), we have $|R_\xi| = \infty$ and $|R_\theta| = \rho$. Then, according to (5.2), we find

$$L = \min(\rho, \ell) . \tag{33.1}$$

In order to simplify the algebra which follows, and for the convenience of comparison of the results to be derived here with those from previous publications, we assume hereafter that $\ell < \rho$ and, thus, take

$$L = \ell . \tag{33.2}$$

Expressions (5.4) and (33.2) imply

$$\bar{\ell} = 1 . \tag{33.3}$$

It follows from (5.6 a, b) that

$$p(x, y; \epsilon) = \epsilon^2(\rho - \ell y \epsilon)^3 , \quad q(x, y; \epsilon) = (\rho - \ell y \epsilon)^3 . \tag{33.4}$$

From Figure 3, we can write down immediately that

$$r_1 = \rho + \frac{\ell\epsilon}{2} , \quad r_2 = \rho - \frac{\ell\epsilon}{2} . \tag{33.5}$$

Next, from (6.4), after setting the constant of integration equal to zero, we find

$$\widetilde{\varphi}(x) = \frac{T'}{2\pi\mu}\left(\rho^3 + \frac{3}{4}\rho L^2\varepsilon^2\right)^{-1} x \ , \tag{33.6}$$

where

$$T' = T/\varepsilon \ . \tag{33.7}$$

(32.6) and (33.7) imply that

$$T' = O(1) \qquad \text{as } \varepsilon \to 0 \ . \tag{33.8}$$

Since p and q in (33.4) are independent of x, and $\widetilde{\varphi}$ in (33.6) is linear in x, we conclude at once from (6.9) and (13.2) that

$$F(x, y;\varepsilon) = G(x, y;\varepsilon) = 0 \ , \tag{33.9}$$

which implies that both of the solutions $\hat{\varphi}_1$ and $\hat{\varphi}_2$ in (30.1) satisfy homogeneous differential equations (30.2) and (30.7) after setting F equal to zero in (30.7). The boundary value problem (30.2) - (30.6) of $\hat{\varphi}_1$ and that of (30.7) - (30.11) of $\hat{\varphi}_2$ are almost identical except for the different boundary conditions at x = 0, 1.[3] We shall first find upper bounds in terms of the thinness parameter ε for quantities associated with the solution $\hat{\varphi}_2$ and shall then directly appropriate these results to find the upper bounds for the quantities associated with the solution $\hat{\varphi}_1$.

From (33.4) and (33.5), we take

$$p_1 = \varepsilon^2 r_1^3 \ , \qquad p_o = \varepsilon^2 r_2^3 \ , \tag{33.10}$$

and

$$q_1 = r_1^3 \ , \qquad q_o = r_2^3 \ , \tag{33.11}$$

where p_1, p_o, q_1, and q_o are such that

$$p_o \le p \le p_1 \ , \qquad q_o \le q \le q_1 \ , \tag{33.12}$$

as defined in (10.4), (17.4), and (19.9). (10.3) and (33.10) - (33.11)

[3] $\bar{\ell} = 1$ according to (33.3).

imply

$$k = \frac{\pi}{\varepsilon}\left(\frac{r_2}{r_1}\right)^{3/2} . \tag{33.13}$$

We now proceed to estimate the various orders of energies associated with the solution $\hat{\varphi}_2$. Since $\hat{\varphi}_2$ for the circular cylinder satisfies a homogeneous differential equation, we can omit considering $S_1(z)$ of (13.17) in estimating the first order energy $T_1(z)$ and can simply take

$$T_1(z) = N_1(z) , \tag{33.14}$$

where $T_1(z)$ and $N_1(z)$ are given by (13.15) - (13.16) with $G = 0$ in (13.15) according to (33.9).

After lengthy but straightforward computations, we find from (20.4), (21.4), and (22.8) that

$$N_1(0) \le \frac{4r_1^4}{\mu^2 r_2^3}\left(\ell^2 h_1^2 + \frac{T'^2}{4\pi^2}\frac{r_1^2}{\rho^6}\right)\varepsilon^2 , \tag{33.15}$$

$$W_2(0) \le \frac{4\ell^2}{\mu^2}\frac{r_1^4}{r_2^5}\left(14\ell^2 h_1^2 + \frac{T'^2}{4\pi^2}\frac{r_1^2}{\rho^6}\right)\varepsilon^4 , \tag{33.16}$$

and

$$W_3(0) \le \frac{\ell^4}{\mu^2 r_2}\left(843\frac{\ell^2 r_1^4}{r_2^6} h_1^2 + 4h_1^2 + \frac{60T'\ell}{\pi\rho^3}\frac{r_1^3}{r_2^4} h_1 + \right.$$

$$\left. + \frac{6T'^2 r_1^6}{\pi^2\rho^6 r_2^6}\right)\varepsilon^4 , \tag{33.17}$$

where the constant h_1 is such that

$$h_1 = \max_{[-\frac{1}{2},\frac{1}{2}]}\left(|f_1|, r_1|f_1'|, r_1^2|f_1''|\right) . \tag{33.18}$$

We now appropriate the results of (33.15) - (33.17) to conclude that the various orders of total energies $V_i(\ell)$ $(i = 1, 2, 3)$ associated with the solution ϕ_1 satisfy

$$V_1(1) \le \frac{4r_1^4}{\mu^2 r_2^3} \left(\ell^2 h_2^2 + \frac{T'^2}{4\pi^2}\frac{r_1^2}{\rho^6}\right)\varepsilon^2 , \tag{33.19}$$

$$V_2(1) \le \frac{4\ell^2}{\mu^2}\frac{r_1^4}{r_2^5}\left(14\ell^2 h_2^2 + \frac{T'^2}{4\pi^2}\frac{r_1^2}{\rho^6}\right)\varepsilon^4 , \tag{33.20}$$

and

$$V_3(1) \le \frac{\ell^4}{\mu^2 r_2}\left(843\ell^2 \frac{r_1^4}{r_2^6} h_2^2 + 4h_2^2 + \frac{60T'\ell}{\pi\rho^3}\frac{r_1^3}{r_2^4} h_2 + \frac{6T'^2 r_1^6}{\pi^2\rho^6 r_2^6}\right)\varepsilon^4 , \tag{33.21}$$

where h_2 is given by

$$h_2 = \max_{[-\frac{1}{2},\frac{1}{2}]} \left(|f_2| , r_1|f_2'| , r_1^2|f_2''|\right) . \tag{33.22}$$

For the sake of completeness, we record below the estimates for the constants β_i $(i = 1, 2, 3, 4)$ and γ_j $(j = 1, 2)$ which are defined in (12.17), (15.11), and (14.13):

$$\beta_1 = \beta_3 = \beta_4 = 0 , \qquad \beta_2 \le \frac{21\ell^2}{r_2^2} , \tag{33.23}$$

$$\gamma_1 = \alpha_1 + \alpha_2 \le \frac{3\ell^2}{r_2^2}\varepsilon^2 , \quad \gamma_2 = 0 . \tag{33.24}$$

Now, to simplify the algebra in the sequel, we define a new constant f which is such that

$$f = \max(h_1, h_2) . \tag{33.25}$$

Combining (33.14) and (13.27) with (33.15), (14.15) with (33.15) and (33.16), and (15.14) - (15.15) with (33.15) - (33.17), and keeping in mind (33.23) - (33.25), we find

$$T_1(x) \le \left[\frac{4r_1^4}{\mu^2 r_2^3}\,(\ell^2 f^2 + \frac{T'^2}{4\pi^2}\frac{r_1^2}{\rho^6})\,\epsilon^2\right] e^{-2kx}\,, \qquad (33.26)$$

$$W_2(x) \le \frac{\ell^2}{\mu^2}\frac{r_1^4}{r_2^5}\left[8\ell^2 f^2(3\epsilon^3+7\epsilon^4)+\frac{T'^2}{\pi^2}\frac{r_1^2}{\rho^6}(6\epsilon^3+\epsilon^4)\right] e^{-2kx}\,, \qquad (33.27)$$

and

$$W_3(x) \le \frac{\ell^4}{\mu^2}\frac{r_1^3}{r_2^5}\left[336\ell^2\frac{r_1}{r_2^2} f^2(3\epsilon^2+7\epsilon^3+\frac{5}{2}\epsilon^4)+\frac{42T'^2}{\pi^2\rho^6}\frac{r_1^3}{r_2^2}(6\epsilon^2+\epsilon^3+\frac{\epsilon^4}{6}) + r_1 f^2\epsilon^4+\frac{60T'\ell}{\pi^2\rho^3} f\epsilon^4\right] e^{-2kx}\,. \qquad (33.28)$$

Similarly, we find

$$V_1(x) \le \left[\frac{4r_1^4}{\mu^2 r_2^3}\,(\ell^2 f^2 + \frac{T'^2}{4\pi^2}\frac{r_1^2}{\rho^6})\epsilon^2\right] e^{-2k(1-x)}, \qquad (33.29)$$

$$V_2(x) \le \frac{\ell^2}{\mu^2}\frac{r_1^4}{r_2^5}\left[8\ell^2 f^2(3\epsilon^3+7\epsilon^4)+\frac{T'^2}{\pi^2}\frac{r_1^2}{\rho^6}(6\epsilon^3+\epsilon^4)\right] e^{-2k(1-x)}\,, \qquad (33.30)$$

$$V_3(x) \le \frac{\ell^4}{\mu^2}\frac{r_1^3}{r_2^5}\left[336\ell^2\frac{r_1}{r_2^2} f^2(3\epsilon^2+7\epsilon^3+\frac{5}{2}\epsilon^4)+\frac{42T'^2}{\pi^2\rho^6}\frac{r_1^3}{r_2^2}(6\epsilon^2+\epsilon^3+\frac{\epsilon^4}{6}) + r_1 f^2\epsilon^4+\frac{60T'\ell}{\pi^2\rho^3} f\epsilon^4\right] e^{-2k(1-x)}\,. \qquad (33.31)$$

To further simplify the algebra, we assume

$$\epsilon \le \min\left[\frac{1}{64}\frac{r_2}{r_1}\,,\ \frac{1}{32}\sqrt{\frac{r_2 T'}{\pi\ell f\rho^3}}\,,\ \frac{\ell}{4}\sqrt{\frac{r_1}{r_2}}\,,\ \frac{1}{116}\sqrt{\frac{r_2^3}{r_1^3}}\,,\ \left(\frac{r_2}{208 r_1}\right)^2\right]. \qquad (33.32)$$

Then, from (25.24) and (26.8), we find

$$|\hat{\varphi}_x(x,y;\epsilon)| \le \frac{9\ell^2 r_1^4}{\mu\rho r_2^6}(\ell f+\frac{T' r_1}{2\pi\rho^3})(2+\epsilon)[e^{-kx}+e^{-k(1-x)}]\,, \qquad (33.33)$$

and

$$|\hat{\phi}_y(x, y;\varepsilon)| \le 24\sqrt{\frac{2}{\pi}}\frac{\ell r_1}{\mu r_2^3}(\ell f + \frac{T' r_1}{2\pi\rho^3})\left\{\sqrt{\frac{\ell}{r_2}}\,\varepsilon^{5/4}[e^{-kx}+e^{-k(1-x)}] + \right.$$
$$\left. + \sqrt{3}(1+3\sqrt{\frac{\ell}{r_2}})\varepsilon e^{-k}\right\} . \qquad (33.34)$$

Finally, from (32.1), (33.33), and (33.34), we find, in terms of the variables x and y, the pointwise estimates of the error stresses to be

$$|\hat{\tau}_{x\theta}(x, y;\varepsilon)| \le \frac{9\ell}{\rho}\frac{r_1^4}{r_2^6}(\ell f + \frac{T' r_1}{2\pi\rho^3})(2+\varepsilon)(\rho - \ell y\varepsilon)[e^{-kx}+e^{-k(1-x)}] , \qquad (33.35)$$

and

$$|\hat{\tau}_{y\theta}(x, y;\varepsilon)| \le 24\sqrt{\frac{2}{\pi}}\frac{r_1}{r_2^3}(\ell f + \frac{T'}{2\pi}\frac{r_1}{\rho^3})(\rho - \ell y\varepsilon)\left\{\sqrt{\frac{\ell}{r_2}}\,\varepsilon^{1/4}[e^{-kx}+e^{-k(1-x)}] + \right.$$
$$\left. + \sqrt{3}(1+3\sqrt{\frac{\ell}{r_2}})e^{-k}\right\} . \qquad (33.36)$$

Since $\tilde{\tau}_{\zeta\theta} = 0$ in (6.6), it follows that (33.36) actually gives the upper bound for the exact stress component $\tau_{y\theta}$ in terms of the variables x and y.

34. Comparison with Previous Work.

In [1], the upper bounds for stresses in axisymmetric torsion are found by an energy method coupled with a mean value theorem of linear elasticity. Such estimates are of the form

$$|\tau(x, y)| \le C\sqrt{\frac{h}{\delta^3}}\exp\left[-\int_0^{x-\delta}\sqrt{\lambda_o(\zeta)}\,d\zeta\right] , \qquad (34.1)$$

where

(1) C is a known constant which depends on the load data and the geometry of the problem;

(2) h is a constant to be defined as follows. The hollow body of revolution considered in [1] is of varying[4] thickness along the axis of symmetry which coincides with the z-axis. Let $r_1(z)$ and $r_2(z)$ measure the distances from the z-axis to the outer and inner lateral surfaces of the body. Then

$$h = \max_{0 \le z \le \ell} |r_1(z) - r_2(z)| . \qquad (34.2)$$

(3) δ is the radius of a solid sphere centered at (x, θ, z) and lying wholly in the body; and

(4) λ_o is the decay function replacing the constant k of the present work.

In (34. 1), $\tau(x, y)$ can stand for either $\tau_{x\theta}$ or $\tau_{y\theta}$.

Let us apply (34. 1) to a circular elastic shell of constant thickness h (see Figure 3). We shall examine two limiting processes.

Case (1): Let (x, y) be the dimensionless coordinates and (ξ, ζ) be the coordinates pertaining to the original geometry of the elastic body. The two systems of coordinates are related as given in (5. 4). Then (5. 3) and (5. 4) imply that (x, y) depend on the thinness parameter ϵ. Now, we consider the limiting case of stresses at a fixed interior point (ξ, ζ) of the body as $\epsilon \to 0$. We find, from (34. 1), in terms of (x, y),

$$\delta = O(\epsilon) , \quad \lambda_o = O(\epsilon^{-1}) \qquad \text{as } \epsilon \to 0 , \qquad (34.3)$$

and

$$\tau(x, y) = O(\epsilon^{-1})\exp[-O(\epsilon^{-1})x + O(1)] \quad \text{as } \epsilon \to 0 . \quad (34.4)$$

---

4 See Figure 1, Case 1 of [1].

Case (2): Let the thickness h be fixed, and let the point (x, y) approach the boundary, namely, the lateral surfaces of the shell. This implies that δ approaches zero. Then, (34.1) deteriorates and breaks down completely on the boundary. This drawback is common to all stress estimates obtained with the aid of a mean value theorem.

The problem just considered is a special case of that treated in [1]; it is clearly also a special case of the shell problem of the present thesis upon setting $f_2 = 0$ in (3.17), $\widetilde{\varphi} = 0$ in (6.4), and the shell to be circular cylindrical.

Applying the techniques developed in this thesis, we find, from (33.35) and (33.36), that

$$\tau(x, y) = O(1)\exp[-O(\epsilon^{-1})x] \qquad \text{as } \epsilon \to 0 , \tag{34.5}$$

whenever Case (1) is concerned; and the upper bounds of $\tau_{x\theta}$ and $\tau_{y\theta}$ are given by (33.35) and (33.36) with the term $e^{-k(1-x)}$ deleted, since $f_2 = 0$ whenever Case (2) is concerned. We see that these estimates remain valid up to and including the boundary points. Hence, we have repaired the deficiency of the pointwise estimates obtained in [1]. Furthermore, by direct comparison, (34.5) is a better estimate than (34.4).

APPENDIX A. Estimate of $\left|\int_0^z \int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{v} H dy dx\right|$.

We derive here expression (12. 16), after introducing further notation. Let

$$\int_{\mathfrak{R}(z)} dA = \int_0^z \int_{-\frac{1}{2}}^{\frac{1}{2}} dy dx \ . \qquad (A.1)$$

Then from (12. 4) we have

$$\left|\int_{\mathfrak{R}(z)} \bar{v} H dA\right| \leq \left|\int_{\mathfrak{R}(z)} p_{xxy} v_x \bar{v} dA\right| + \left|\int_{\mathfrak{R}(z)} q_{xyy} v_y \bar{v} dA\right| + \left|\int_{\mathfrak{R}(z)} (p_{xx}+q_{yy}) v_{xy} \bar{v} dA\right| +$$

$$+ \left|\int_{\mathfrak{R}(z)} 2q_{xy} v_{yy} \bar{v} dA\right| + \left|\int_{\mathfrak{R}(z)} 2p_{xy} v_{xx} \bar{v} dA\right| + \left|\int_{\mathfrak{R}(z)} p_y v_{xxx} \bar{v} dA\right| +$$

$$+ \left|\int_{\mathfrak{R}(z)} q_y v_{xyy} \bar{v} dA\right| + \left|\int_{\mathfrak{R}(z)} p_x v_{xxy} \bar{v} dA\right| + \left|\int_{\mathfrak{R}(z)} q_y v_{yyy} \bar{v} dA\right| \ . \qquad (A.2)$$

Upper bounds for each of the nine integrals on the right of (A. 2) are to be derived as follows:

From Schwarz's inequality, (10. 1), (11. 1), (12. 2), and (A. 1), we have

$$\left|\int_{\mathfrak{R}(z)} p_{xxy} v_x \bar{v} dA\right| \leq \int_{\mathfrak{R}(z)} \left|\frac{p_{xxy}}{p}\right| \cdot \left|p^{\frac{1}{2}} v_x\right| \cdot \left|p^{\frac{1}{2}} v_{xy}\right| dA \leq$$

$$\leq \left(\max_{\bar{\mathfrak{R}}} \left|\frac{p_{xxy}}{p}\right|\right) \left(\int_{\mathfrak{R}(z)} p v_x^2 dA\right)^{\frac{1}{2}} \left(\int_{\mathfrak{R}(z)} p v_{xy}^2 dA\right)^{\frac{1}{2}} \leq \left(\max_{\bar{\mathfrak{R}}} \left|\frac{p_{xxy}}{p}\right|\right) V_1^{\frac{1}{2}}(z) V_2^{\frac{1}{2}}(z). \qquad (A.3)$$

Similarly,

$$\left|\int_{\mathcal{R}(z)} q_{xyy} v_y \bar{v}\, dA\right| \le \left(\max_{\overline{\mathcal{R}}} \left|\frac{q_{xyy}}{\sqrt{pq}}\right|\right) V_1^{\frac{1}{2}}(z) V_2^{\frac{1}{2}}(z)\,, \tag{A.4}$$

$$\left|\int_{\mathcal{R}(z)} (p_{xx}+q_{yy}) v_{xy} \bar{v}\, dA\right| \le \left(\max_{\overline{\mathcal{R}}} \left|\frac{p_{xx}+q_{yy}}{p}\right|\right) V_2(z)\,, \tag{A.5}$$

$$\left|\int_{\mathcal{R}(z)} 2q_{xy} v_{yy} \bar{v}\, dA\right| \le \left(\max_{\overline{\mathcal{R}}} \left|\frac{2q_{xy}}{\sqrt{pq}}\right|\right) V_2(z)\,. \tag{A.6}$$

In estimating $\left|\int_{\mathcal{R}(z)} 2p_{xy} v_{xx} \bar{v}\, dA\right|$, we first find v_{xx} from (9.10) and obtain

$$v_{xx} = -\frac{1}{p}\left(p_x v_x + q_y v_y + q v_{yy}\right)\,. \tag{A.7}$$

Then,

$$\left|\int_{\mathcal{R}(z)} 2p_{xy} v_{xx} \bar{v}\, dA\right| \le 2\int_{\mathcal{R}(z)} \left|\frac{p_{xy}p_x}{p}\right| \cdot |v_x| \cdot |\bar{v}|\, dA + \int_{\mathcal{R}(z)} \left|\frac{q_y p_{xy}}{p}\right| \cdot |v_y| \cdot |\bar{v}|\, dA +$$

$$+ \int_{\mathcal{R}(z)} \left|\frac{p_{xy} q}{p}\right| \cdot |v_{yy}| \cdot |\bar{v}|\, dA$$

$$\le 2\left[\left(\max_{\overline{\mathcal{R}}} \left|\frac{p_{xy}p_x}{p^2}\right|\right) V_1^{\frac{1}{2}}(z) V_2^{\frac{1}{2}}(z) + \left(\max_{\overline{\mathcal{R}}} \left|\frac{p_{xy}q_y}{p^{\frac{3}{2}} q^{\frac{1}{2}}}\right|\right) V_1^{\frac{1}{2}}(z) V_2^{\frac{1}{2}}(z) + \right.$$

$$\left. + \left(\max_{\overline{\mathcal{R}}} \left|\frac{p_{xy} q^{\frac{1}{2}}}{p^{\frac{3}{2}}}\right|\right) V_2(z)\right] \tag{A.8}$$

Differentiating (A.7) with respect to x and rearranging, we have

$$v_{xxx} = \left(\frac{2p_x^2}{p^2} - \frac{p_{xx}}{p}\right) v_x + \left(\frac{2p_x q_y}{p^2} - \frac{q_{xy}}{p}\right) v_y - \frac{q_y}{p} v_{xy} +$$

$$+ \left(\frac{2qp_x}{p^2} - \frac{q_x}{p}\right) v_{yy} - \frac{q}{p} v_{xyy}\,. \tag{A.9}$$

Then

$$\left|\int_{\mathcal{R}(z)} p_y v_{xxx}\bar{v}\,dA\right| \le \int_{\mathcal{R}(z)} \left|\frac{p_y}{p}\right|\cdot\left|\frac{2p_x^2}{p} - p_{xx}\right|\cdot|v_x|\cdot|\bar{v}|\,dA + \int_{\mathcal{R}(z)} \left|\frac{p_y q_y}{p}\right| v_{xy}^2\,dA +$$

$$+\int_{\mathcal{R}(z)} \left|\frac{p_y}{p}\right|\cdot\left|\frac{2p_x q_y}{p} - q_{xy}\right|\cdot|v_y|\cdot|\bar{v}|\,dA + \int_{\mathcal{R}(z)} \left|\frac{p_y}{p}\right|\cdot\left|\frac{2qp_x}{p} - q_x\right|\cdot|v_{yy}|\cdot|\bar{v}|\,dA +$$

$$+\left|\int_{\mathcal{R}(z)} \frac{qp_y}{p} v_{xyy}\bar{v}\,dA\right| \le \left(\max_{\overline{\mathcal{R}}}\left|\frac{p_y}{p^2}\right|\cdot\left|\frac{2p_x^2}{p} - p_{xx}\right|\right) V_1^{\frac{1}{2}}(z) V_2^{\frac{1}{2}}(z) +$$

$$+\left(\max_{\overline{\mathcal{R}}}\left|\frac{p_y q_y}{p^2}\right|\right) V_2(z) + \left(\max_{\overline{\mathcal{R}}}\left|\frac{p_y}{p}\right|\cdot\left|\frac{2p_x q_y}{p} - q_{xy}\right|\frac{1}{\sqrt{pq}}\right) V_1^{\frac{1}{2}}(z) V_2^{\frac{1}{2}}(z) +$$

$$+\left(\max_{\overline{\mathcal{R}}}\left|\frac{p_y}{p}\right|\cdot\left|\frac{2qp_x}{p} - q_x\right|\frac{1}{\sqrt{pq}}\right) V_2(z) + \left(\max_{\overline{\mathcal{R}}}\left|\left(\frac{qp_y}{p}\right)_y \frac{1}{2p}\right|\right) V_2(z) . \qquad (A.10)$$

The last term on the right of (A.10) is obtained by integration by parts as follows:

$$\int_{\mathcal{R}(z)} \frac{qp_y}{p} v_{xyy}\bar{v}\,dA = \int_0^z \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{qp_y}{p} v_{xyy} v_{xy}\,dy\,dx =$$

$$= \int_0^z \left[\frac{qp_y}{p} v_{xy}^2 \Big|_{y=-\frac{1}{2}}^{y=\frac{1}{2}} - \int_{-\frac{1}{2}}^{\frac{1}{2}} v_{xy}\left\{\left(\frac{qp_y}{p}\right)_y v_{xy} + \frac{qp_y}{p} v_{xyy}\right\} dy\right]$$

$$= -\tfrac{1}{2}\int_{\mathcal{R}(z)} \left(\frac{qp_y}{p}\right)_y v_{xy}^2\,dA \quad . \qquad (A.11)$$

Hence,

$$\left|\int_{\mathcal{R}(z)} \frac{qp_y}{p} v_{xyy}\bar{v}\,dA\right| \le \tfrac{1}{2}\int_{\mathcal{R}(z)} \left(\frac{qp_y}{p}\right)_y \frac{1}{p}(pv_{xy}^2)\,dA \le \left(\max_{\overline{\mathcal{R}}}\left|\left(\frac{qp_y}{p}\right)_y \frac{1}{2p}\right|\right) V_2(z) . \qquad (A.12)$$

Similarly,

$$\left|\int_{\mathcal{R}(z)} q_y v_{xyy}\bar{v}\,dA\right| \le \left(\max_{\overline{\mathcal{R}}}\left|\frac{q_{yy}}{2p}\right|\right) V_2(z) \quad . \qquad (A.13)$$

To estimate $\left|\int_{\Re(z)} p_x v_{xxy}\bar{v}dA\right|$, let us consider

$$\int_{\Re(z)} p_x v_{xxy}\bar{v}dA = \int_0^z\int_{-\frac{1}{2}}^{\frac{1}{2}} p_x v_{xxy} v_{xy} dydx \,. \tag{A. 14}$$

Integrating (A. 14) by parts with respect to x and using (12. 5), we have

$$\int_{\Re(z)} p_x v_{xxy}\bar{v}dA = -\tfrac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} p_x v_{xy}^2\Big|_{x=z} dy - \tfrac{1}{2}\int_{\Re(z)} p_{xx} v_{xy}^2 dA \,. \tag{A. 15}$$

Then,

$$\left|\int_{\Re(z)} p_x v_{xxy}\bar{v}dA\right| \le \tfrac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} |p_x|\cdot v_{xy}^2\Big|_{x=z} dy + \tfrac{1}{2}\int_{\Re(z)} |p_{xx}| v_{xy}^2 dA$$

$$\le \tfrac{1}{2}\left(\max_{\overline{\Re}}\left|\frac{p_x}{p}\right|\right)\int_{-\frac{1}{2}}^{\frac{1}{2}} p v_{xy}^2\Big|_{x=z} dy + \tfrac{1}{2}\left(\max_{\overline{\Re}}\left|\frac{p_{xx}}{p}\right|\right)\int_{\Re(z)} p v_{xy}^2 dA$$

$$\le \tfrac{1}{2}\left(\max_{\overline{\Re}}\left|\frac{p_x}{p}\right|\right)V_2'(z) + \tfrac{1}{2}\left(\max_{\overline{\Re}}\left|\frac{p_{xx}}{p}\right|\right)V_2(z) \,. \tag{A. 16}$$

Now, differentiating (9. 10) with respect to y, we obtain v_{yyy} given by

$$v_{yyy} = \frac{1}{q}(av_x + bv_y + 2cv_{yy} - p_x v_{xy} - pv_{xxy} - q_y v_{yy}) \,, \tag{A. 17}$$

where

$$\left.\begin{aligned} a &= \frac{p_x p_y}{p} - p_{xy} \,, \qquad b = \frac{p_y q_y}{p} - q_{yy} \,, \\ c &= \tfrac{1}{2}\left(\frac{qp_y}{p} - q_y\right) . \end{aligned}\right\} \tag{A. 18}$$

From (A. 17) and (A. 16), we have

$$\left|\int_{\mathcal{R}(z)} q_x v_{yyy}\bar{v}dA\right| \leq \int_{\mathcal{R}(z)}\left|\frac{q_x}{q}a\right|\cdot|v_x|\cdot|\bar{v}|dA + \int_{\mathcal{R}(z)}\left|\frac{q_x}{q}b\right|\cdot|v_y|\cdot|\bar{v}|dA$$

$$+2\int_{\mathcal{R}(z)}\left|\frac{q_x}{q}c\right|\cdot|v_{yy}|\cdot|\bar{v}|dA + \int_{\mathcal{R}(z)}\left|\frac{q_x p_x}{q}\right|v_{xy}^2 dA + \left|\int_{\mathcal{R}(z)}\left(\frac{q_x}{q}p\right)v_{xxy}\bar{v}dA\right| +$$

$$+\int_{\mathcal{R}(z)}\left|\frac{q_x q_y}{q}\right|\cdot|v_{yy}|\cdot|\bar{v}|dA \leq \left[\max_{\bar{\mathcal{R}}}\left(\left|\frac{q_x a}{pq}\right| + \left|\frac{q_x b}{\sqrt{pq^3}}\right|\right)\right]V_1^{\frac{1}{2}}(z)V_2^{\frac{1}{2}}(z)+$$

$$+\left(\max_{\bar{\mathcal{R}}}\left|\frac{q_x}{2q}\right|\right)V_2'(z) + \left[\max_{\bar{\mathcal{R}}}\left(2\left|\frac{q_x c}{\sqrt{pq^3}}\right| + \left|\frac{p_x q_y}{pq}\right| + \frac{1}{2p}\left|\left(\frac{q_x p}{q}\right)_x\right| +\right.\right.$$

$$\left.\left.+\left|\frac{q_x q_y}{\sqrt{pq^3}}\right|\right)\right]V_2(z) \quad . \tag{A.19}$$

Substituting (A. i) , (i = 3, 4, 5, 6, 8, 10, 11, 13, 16, and 19), into (A. 2) yields (12. 16):

$$\left|\int_0^z\int_{-\frac{1}{2}}^{\frac{1}{2}}\bar{v}Hdydx\right| \leq \beta_1[V_1(z)V_2(z)]^{\frac{1}{2}} + \beta_2V_2(z) + \beta_3V_2'(z) \; , \tag{12. 16}$$

where the constants β_1, β_2, and β_3 are given by (12. 17).

When $z = \ell$ in (12. 16), we can improve the estimate of $\left|\int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}}\bar{v}Hdydx\right|$ as follows. Replace z by ℓ in (A. j), (j = 3, 4, 5, 6, 8, 10, and 13), but we rewrite (A. 14) as

$$\int_{\mathcal{R}(\ell)} p_x v_{xxy}\bar{v}dA = \int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} p_x v_{xxy} v_{xy} dydx \quad . \tag{A.20}$$

Integrating (A. 20) by parts with respect to x and using (12. 5) and (12. 6), we have

$$\int_{\mathcal{R}(\ell)} p_x v_{xxy}\bar{v}\,dA = -\tfrac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{p_x(\ell,y)}{p^4(\ell,y)}\,[p(\ell,y)g_2'(y)-p_y(\ell,y)g_2(y)]^2 dy$$

$$-\tfrac{1}{2}\int_{\mathcal{R}(\ell)} p_{xx} v_{xy}^2\, dA \quad . \tag{A. 21}$$

Then,

$$\left|\int_{\mathcal{R}(\ell)} p_x v_{xxy}\bar{v}\,dA\right| \le \tfrac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{p_x(\ell,y)}{p^4(\ell,y)}\right| [p(\ell,y)g_2'(y)-p_y(\ell,y)g_2(y)]^2 dy +$$

$$+\tfrac{1}{2}(\max_{\overline{\mathcal{R}}}\left|\frac{p_{xx}}{p}\right|)V_2(\ell) \; . \tag{A. 22}$$

Similarly, we find

$$\left|\int_{\mathcal{R}(\ell)} q_x v_{yyy}\bar{v}\,dA\right| \le \left[\max_{\overline{\mathcal{R}}}(\left|\frac{q_x a}{pq}\right| + \left|\frac{q_x b}{\sqrt{pq^3}}\right|)\right] V_1^{\frac{1}{2}}(\ell)V_2^{\frac{1}{2}}(\ell) +$$

$$+\tfrac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{p(\ell,y)\,|q_x(\ell,y)|}{q(\ell,y)p^4(\ell,y)}\,[p(\ell,y)g_2'(y)-p_y(\ell,y)g_2(y)]^2 dy +$$

$$+\left[\max_{\overline{\mathcal{R}}}(2\left|\frac{q_x c}{\sqrt{pq^3}}\right| + \left|\frac{p_x q_y}{pq}\right| + \frac{1}{2p}\left|\left(\frac{q_x p}{q}\right)_x\right| + \left|\frac{q_x q_y}{\sqrt{pq^3}}\right|)\right] V_2(\ell) \; . \tag{A. 23}$$

It easily follows that

$$\left|\int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{v}H\,dy\,dx\right| \le \beta_o + \beta_1[V_1(\ell)V_2(\ell)]^{\frac{1}{2}} + \beta_2 V_2(\ell) \, , \tag{A. 24}$$

where

$$\beta_o \ge \tfrac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{|p_x(\ell,y)|}{p^4(\ell,y)} + \frac{p(\ell,y)\,|q_x(\ell,y)|}{q(\ell,y)p^4(\ell,y)}\right)[p(\ell,y)g_2'(y)-p_y(\ell,y)g_2(y)]^2 dy \, , \tag{A. 25}$$

while β_1 and β_2 are given by (12. 17).

APPENDIX B. Estimate of $\left|\int_z^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{w}\bar{H}dydx\right|$.

Expression (14. 12) will be derived here after introducing further notation. Let

$$\int_{\mathcal{R}(z)} da = \int_z^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} dydx \quad . \tag{B. 1}$$

From (14. 3) we have

$$\left|\int_{\mathcal{R}(z)} \bar{w}\bar{H}da\right| \le \left|\int_{\mathcal{R}(z)} \left(\frac{p_x p_y}{p} - p_{xy}\right) w_x \bar{w} da\right| + \left|\int_{\mathcal{R}(z)} \left(F_y - \frac{Fp_y}{p}\right)\bar{w}da\right| +$$

$$+ \left|\int_{\mathcal{R}(z)} \left[\left(\frac{p_y q_y}{p} - q_{yy}\right)w_y + \left(\frac{qp_y}{p} - q_y\right)w_{yy}\right]\bar{w}da\right| \; . \tag{B. 2}$$

Now we digress here to define

$$W_1(z) = \int_z^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} (pw_x^2 + qw_y^2)dydx \quad . \tag{B. 3}$$

It follows immediately from (13. 1) and (13. 15) that

$$W_1(z) = T_1(z) \quad . \tag{B. 4}$$

Arguing as in Appendix A, bounds of the integrals in (B. 2) are found to be:

$$\left|\int_{\mathcal{R}(z)} \left(\frac{p_x p_y}{p} - p_{xy}\right) w_x \bar{w} da\right| \le \int_{\mathcal{R}(z)} \frac{1}{\sqrt{pq}} \left|\frac{p_x p_y}{p} - p_{xy}\right| \cdot |p^{\frac{1}{2}} w_x| \cdot |q^{\frac{1}{2}} w_y| da$$

$$\le \alpha_1 \left(\int_{\mathcal{R}(z)} pw_x^2 da\right)^{\frac{1}{2}} \left(\int_{\mathcal{R}(z)} qw_y^2 da\right)^{\frac{1}{2}}$$

$$\le \alpha_1 T_1(z) \, , \tag{B. 5}$$

$$\left|\int_{\Re(z)} (F_y - \frac{Fp_y}{p})\bar{w}da\right| \le \int_{\Re(z)} (F_y - \frac{Fp_y}{p})\frac{1}{\sqrt{q}}\left|\sqrt{q}\, w_y\right|da$$

$$\le \left[\int_{\Re(z)} \left(F_y - \frac{Fp_y}{p}\right)^2 \frac{1}{q}\, da\right]^{\frac{1}{2}} \left(\int_{\Re(z)} qw_y^2 da\right)^{\frac{1}{2}}$$

$$\le \gamma_2 [T_1(z)]^{\frac{1}{2}} \;, \qquad \text{(B. 6)}$$

and after integration by parts, the last integral in (B. 2) yields

$$\left|\int_{\Re(z)} [(\frac{p_y q_y}{p} - q_{yy})w_y + (\frac{qp_y}{p} - q_y)w_{yy}]\bar{w}da\right| =$$

$$= \left|\int_{\Re(z)} \tfrac{1}{2}(\frac{p_y q_y}{pq} - \frac{q_{yy}}{q} - \frac{p_{yy}}{p} + \frac{p_y^2}{p^2})(qw_y^2)da\right| \le \alpha_2 T_1(z) \,. \qquad \text{(B. 7)}$$

In (B. 5) - (B. 7), α_1, α_2, and γ_2 are given by (11. 4), (11. 5), and (14. 13), respectively.

Substituting (B. 5) - (B. 7) into (B. 2) and setting $\gamma_1 \ge \alpha_1 + \alpha_2$, we obtain expression (14. 12).

APPENDIX C. Estimate of $\left|\int_z^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{\bar{w}}\,\bar{\bar{H}}\,dydx\right|$.

Except for the terms arising from the nonhomogeneous term, F(x, y), in the differential equation (9. 14), we can appropriate the argument leading to (12. 16) for the derivation of (15. 10).

From (15. 3) and Appendix A, we can write down immediately

$$\left|\int_{\Re(z)} \bar{\bar{w}}\bar{\bar{H}}da\right| \le \left|\int_{\Re(z)} F_{xy}\bar{\bar{w}}da\right| + \left|\int_{\Re(z)} p_{xxy}w_x\bar{\bar{w}}da\right| + \left|\int_{\Re(z)} q_{xyy}w_y\bar{\bar{w}}da\right| +$$

$$+\left|\int_{\Re(z)} (p_{xx}+q_{yy})w_{xy}\bar{\bar{w}}da\right| + \left|\int_{\Re(z)} 2q_{xy}w_{yy}\bar{\bar{w}}da\right| + \left|\int_{\Re(z)} 2p_{xy}w_{xx}\bar{\bar{w}}da\right| +$$

$$+\left|\int_{\Re(z)} p_y w_{xxx}\bar{\bar{w}}da\right| + \left|\int_{\Re(z)} q_y w_{xyy}\bar{\bar{w}}da\right| + \left|\int_{\Re(z)} p_x w_{xxy}\bar{\bar{w}}da\right| +$$

$$+\left|\int_{\Re(z)} q_y w_{yyy}\bar{\bar{w}}da\right| \quad , \tag{C. 1}$$

whence

$$\left|\int_{\Re(z)} F_{xy}\bar{\bar{w}}da\right| \le \int_{\Re(z)} \frac{|F_{xy}|}{\sqrt{p}}\cdot\left|\sqrt{p}\,\bar{w}_x\right|da \le \left(\int_{\Re(z)} \frac{F_{xy}^2}{p}da\right)^{\frac{1}{2}}[W_2(z)]^{\frac{1}{2}} , \tag{C. 2}$$

$$\left|\int_{\Re(z)} p_{xxy}w_x\bar{\bar{w}}da\right| \le \left(\max\left|\frac{p_{xxy}}{p}\right|\right)[T_1(z)W_2(z)]^{\frac{1}{2}} \quad , \tag{C. 3}$$

$$\left|\int_{\Re(z)} q_{xyy}w_y\bar{\bar{w}}da\right| \le \left(\max\left|\frac{q_{xyy}}{\sqrt{pq}}\right|\right)[T_1(z)W_2(z)]^{\frac{1}{2}} \quad , \tag{C. 4}$$

$$\left|\int_{\Re(z)} (p_{xx}+q_{yy})w_{xy}\bar{\bar{w}}da\right| \le \left(\max\frac{|p_{xx}+q_{yy}|}{p}\right)W_2(z) \quad , \tag{C.5}$$

$$\Big|\int_{\Re(z)} 2p_{xy}w_{xx}\bar{\bar{w}}da\Big| \le 2\Big[\Big(\int_{\Re(z)}\frac{(p_{xy}F)^2}{p^3}da\Big)^{\frac{1}{2}}W_2^{\frac{1}{2}}(z)+(\max_{\overline{\Re}}\Big|\frac{p_{xy}q^{\frac{1}{2}}}{p^{\frac{3}{2}}}\Big|)W_2(z)$$

$$+\Big\{(\max_{\overline{\Re}}\Big|\frac{p_{xy}p_x}{p^2}\Big|)+(\max_{\overline{\Re}}\Big|\frac{p_{xy}q_y}{\sqrt{p^3q}}\Big|)\Big\}\{T_1(z)W_2(z)\}^{\frac{1}{2}}\Big], \tag{C.6}$$

$$\Big|\int_{\Re(z)} p_y w_{xxx}\bar{\bar{w}}da\Big| \le \Big[\int_{\Re(z)}\frac{p_y^2}{p^3}\Big(\frac{2Fp_x}{p}-F_x\Big)^2 da\Big]^{\frac{1}{2}}[W_2(z)]^{\frac{1}{2}}+$$

$$+\Big[(\max_{\overline{\Re}}\Big|\frac{p_y}{p^2}\Big|\cdot\Big|\frac{2p_x^2}{p}-p_{xx}\Big|)+(\max_{\overline{\Re}}\Big|\frac{p_y}{p}\Big|\cdot\Big|\frac{2p_xq_y}{p}-q_{xy}\Big|\frac{1}{\sqrt{pq}})\Big]\cdot$$

$$[T_1(z)W_2(z)]^{\frac{1}{2}}+\Big[(\max_{\overline{\Re}}\Big|\frac{p_yq_y}{p^2}\Big|)+(\max_{\overline{\Re}}\Big|\frac{p_y}{p}\Big|\cdot\Big|\frac{2qp_x}{p}-q_x\Big|\frac{1}{\sqrt{pq}})+$$

$$+(\max_{\overline{\Re}}\Big|\frac{1}{2p}\Big(\frac{qp_y}{p}\Big)_y\Big|)\Big]W_2(z), \tag{C.7}$$

$$\Big|\int_{\Re(z)} q_y w_{xyy}\bar{\bar{w}}da\Big| \le (\max_{\overline{\Re}}\Big|\frac{q_{yy}}{2p}\Big|)W_2(z), \tag{C.8}$$

$$\Big|\int_{\Re(z)} p_x w_{xxy}\bar{\bar{w}}da\Big| \le -\tfrac{1}{2}(\max_{\overline{\Re}}\Big|\frac{p_x}{p}\Big|)W_2'(z)+\tfrac{1}{2}(\max_{\overline{\Re}}\Big|\frac{p_{xx}}{p}\Big|)W_2(z), \tag{C.9}$$

and

$$\Big|\int_{\Re(z)} q_x w_{yyy}\bar{\bar{w}}da\Big| \le \Big(\int_{\Re(z)}\frac{(q_xF_y)^2}{pq^2}da\Big)^{\frac{1}{2}}W_2^{\frac{1}{2}}(z)-(\max_{\overline{\Re}}\Big|\frac{q_x}{2q}\Big|)W_2'(z)$$

$$+\Big[\max_{\overline{\Re}}\Big(\Big|\frac{q_xa}{\sqrt{pq}}\Big|+\Big|\frac{q_xb}{\sqrt{pq^3}}\Big|\Big)\Big][T_1(z)W_2(z)]^{\frac{1}{2}}+$$

$$+\Big[\max_{\overline{\Re}}\Big(2\Big|\frac{q_xc}{\sqrt{pq^3}}\Big|+\Big|\frac{p_xq_y}{pq}\Big|+\frac{1}{2p}\Big|\Big(\frac{q_xp}{q}\Big)_x\Big|+\Big|\frac{q_xq_y}{\sqrt{pq^3}}\Big|\Big)\Big]W_2(z). \tag{C.10}$$

Substituting (C. 2) - (C. 10) into (C. 1), we obtain (15. 10).

When $z = 0$ in (15. 10), we can similarly derive

$$\left| \int_0^{\ell} \int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{\bar{w}}\bar{\bar{H}} dy dx \right| \leq \beta_1 [T_1(0) W_2(0)]^{\frac{1}{2}} + \beta_2 W_2(0) + \beta_4 [W_2(0)]^{\frac{1}{2}} + \beta_5 \,, \qquad (C.\ 11)$$

where β_1 , β_2, and β_4 are given by (12. 17) and (15. 11), while β_5 is such that

$$\beta_5 \geq \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\frac{|p_x(0,y)|}{p^4(0,y)} + \frac{p(0,y)|q_x(0,y)|}{q(0,y)p^4(0,y)} \right] [p(0,y)g_1'(y) - p_y(0,y)g_1(y)]^2 dy \,. \qquad (C.\ 12)$$

APPENDIX D. Estimates of $\left|\int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}}\hat{\varphi}\bar{\bar{H}}da\right|$ and $\left|\int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}}\hat{\varphi}Hda\right|$.

We first derive here the expression (22. 2). Recalling (B. 1) of Appendix B and replacing $\bar{\bar{w}}$ by $\hat{\varphi}$ in (C. 1) of Appendix C, we find that

$$\left|\int_{\Re(0)}\hat{\varphi}\bar{\bar{H}}da\right| \le \left|\int_{\Re(0)}F_{xy}\hat{\varphi}da\right| + \left|\int_{\Re(0)}p_{xxy}w_x\hat{\varphi}da\right| + \left|\int_{\Re(0)}q_{xyy}w_y\hat{\varphi}da\right| +$$

$$+\left|\int_{\Re(0)}(p_{xx}+q_{yy})w_{xy}\hat{\varphi}da\right| + \left|\int_{\Re(0)}2q_{xy}w_{yy}\hat{\varphi}da\right| + \left|\int_{\Re(0)}2p_{xy}w_{xx}\hat{\varphi}da\right| +$$

$$+\left|\int_{\Re(0)}p_yw_{xxx}\hat{\varphi}da\right| + \left|\int_{\Re(0)}q_yw_{xyy}\hat{\varphi}da\right| + \left|\int_{\Re(0)}p_xw_{xxy}\hat{\varphi}da\right| + \left|\int_{\Re(0)}q_yw_{yyy}\hat{\varphi}da\right|. \tag{D. 1}$$

To estimate the integrals on the right of (D. 1), we use Schwarz's inequality, (B. 3), and (B. 4) of Appendix B and (14. 7), with z replaced by zero. Then

$$\left|\int_{\Re(0)}p_{xxy}w_x\hat{\varphi}da\right| \le \left[\int_{\Re(0)}(p_{xxy})^2\frac{\hat{\varphi}^2}{p}da\right]^{\frac{1}{2}}\left(\int_{\Re(0)}pw_x^2da\right)^{\frac{1}{2}}$$

$$\le \left[\int_{\Re(0)}(p_{xxy})^2\frac{\hat{\varphi}^2}{p}da\right]^{\frac{1}{2}}[T_1(0)]^{\frac{1}{2}}. \tag{D. 2}$$

$$\left|\int_{\Re(0)}q_{xyy}w_y\hat{\varphi}da\right| \le \left[\int_{\Re(0)}(q_{xyy})^2\frac{\hat{\varphi}^2}{q}da\right]^{\frac{1}{2}}\left[\int_{\Re(0)}qw_y^2da\right]^{\frac{1}{2}}$$

$$\le \left[\int_{\Re(0)}(q_{xyy})^2\frac{\hat{\varphi}^2}{q}da\right]^{\frac{1}{2}}[T_1(0)]^{\frac{1}{2}}, \tag{D. 3}$$

$$\left|\int_{\mathfrak{R}(0)}(p_{xx}+q_{yy})w_{xy}\hat{\varphi}da\right| \le \left[\int_{\mathfrak{R}(0)}(p_{xx}+q_{yy})^2\frac{\hat{\varphi}^2}{p}da\right]^{\frac{1}{2}}\left[\int_{\mathfrak{R}(0)}p\bar{w}_x^2da\right]^{\frac{1}{2}}$$

$$\le\left[\int_{\mathfrak{R}(0)}(p_{xx}+q_{yy})^2\frac{\hat{\varphi}^2}{p}da\right]^{\frac{1}{2}}[W_2(0)]^{\frac{1}{2}} . \tag{D. 4}$$

$$\left|\int_{\mathfrak{R}(0)}2q_{xy}w_{yy}\hat{\varphi}da\right| \le 2\left[\int_{\mathfrak{R}(0)}(q_{xy})^2\frac{\hat{\varphi}^2}{q}da\right]^{\frac{1}{2}}\left[\int_{\mathfrak{R}(0)}q\bar{w}_y^2da\right]^{\frac{1}{2}}$$

$$\le 2\left[\int_{\mathfrak{R}(0)}(q_{xy})^2\frac{\hat{\varphi}^2}{q}da\right]^{\frac{1}{2}}[W_2(0)]^{\frac{1}{2}} . \tag{D. 5}$$

From (9. 14), we find

$$w_{xx} = \frac{1}{p}(F-p_xw_x-q_yw_y-qw_{yy}) . \tag{D. 6}$$

Then

$$\left|\int_{\mathfrak{R}(0)}2p_{xy}w_{xx}\hat{\varphi}da\right| \le 2\Bigg\{\left|\int_{\mathfrak{R}(0)}\frac{p_{xy}}{p}F\hat{\varphi}da\right|+\left[\int_{\mathfrak{R}(0)}(p_{xy}p_x)^2\frac{\hat{\varphi}^2}{p^3}da\right]^{\frac{1}{2}}\left[\int_{\mathfrak{R}(0)}pw_x^2da\right]^{\frac{1}{2}}$$

$$+\left[\int_{\mathfrak{R}(0)}(p_{xy}q_y)^2\frac{\hat{\varphi}^2}{p^2q}da\right]^{\frac{1}{2}}\left[\int_{\mathfrak{R}(0)}qw_y^2da\right]^{\frac{1}{2}}+$$

$$+\left[\int_{\mathfrak{R}(0)}(p_{xy})^2\frac{q\hat{\varphi}^2}{p^2}da\right]^{\frac{1}{2}}\left[\int_{\mathfrak{R}(0)}q\bar{w}_y^2da\right]^{\frac{1}{2}}\Bigg\}$$

$$\le 2\Bigg\{\left|\int_{\mathfrak{R}(0)}\frac{p_{xy}}{p}F\hat{\varphi}da\right|+\left(\left[\int_{\mathfrak{R}(0)}(p_{xy}p_x)^2\frac{\hat{\varphi}^2}{p^3}da\right]^{\frac{1}{2}}+\left[\int_{\mathfrak{R}(0)}(p_{xy}q_y)^2\frac{\hat{\varphi}^2}{p^2q}da\right]^{\frac{1}{2}}\right)\cdot$$

$$[T_1(0)]^{\frac{1}{2}}+\left[\int_{\mathfrak{R}(0)}(p_{xy})^2\frac{q}{p^2}da\right]^{\frac{1}{2}}[W_2(0)]^{\frac{1}{2}}\Bigg\} . \tag{D. 7}$$

Differentiating (D. 10) with respect to x , we find

$$w_{xxx} = \left(\frac{F}{p}\right)_x + \left(\frac{2p_x^2}{p^2} - \frac{p_{xx}}{p}\right)w_x + \left(\frac{2p_xq_y}{p^2} - \frac{q_{xy}}{p}\right)w_y - \frac{q_y}{p}w_{xy} + \left(\frac{2qp_x}{p^2} - \frac{q_x}{p}\right)w_{yy} - \\ - \frac{q}{p}w_{xyy} . \qquad (D.8)$$

Then

$$\left|\int_{\Re(0)} p_y w_{xxx}\hat{\varphi}da\right| \le \left|\int_{\Re(0)} p_y\left(\frac{F}{p}\right)_x\hat{\varphi}da\right| + \left[\int_{\Re(0)} \frac{(p_y)^2}{p^3}\left(\frac{2p_x^2}{p} - p_{xx}\right)^2\hat{\varphi}^2da\right]^{\frac{1}{2}}[T_1(0)]^{\frac{1}{2}} \\ + \left[\int_{\Re(0)} \frac{(p_y)^2}{p^2q}\left(\frac{2p_xq_y}{p} - q_{xy}\right)^2\hat{\varphi}^2da\right]^{\frac{1}{2}}[T_1(0)]^{\frac{1}{2}} + \left[\int_{\Re(0)} \frac{(p_yq_y)^2}{p^3}\hat{\varphi}^2da\right]^{\frac{1}{2}} \cdot \\ [W_2(0)]^{\frac{1}{2}} + \left[\int_{\Re(0)} \frac{(p_y)^2}{p^2q}\left(\frac{2qp_x}{p} - q_x\right)^2\hat{\varphi}^2da\right]^{\frac{1}{2}}[W_2(0)]^{\frac{1}{2}} + \\ + \left[\int_{\Re(0)} \left[\left(\frac{p_yq\hat{\varphi}}{p}\right)_y\right]^2\frac{1}{q}da\right]^{\frac{1}{2}}[W_2(0)]^{\frac{1}{2}} . \qquad (D.9)$$

In obtaining the last term on the right of (D. 9), we have used integration by parts and the boundary conditions $w_{xy}(x, \pm\frac{1}{2}) = 0$. This method also yields

$$\left|\int_{\Re(0)} q_y w_{xyy}\hat{\varphi}da\right| \le \left\{\int_{\Re(0)} \left[(q_y\hat{\varphi})_y\right]^2\frac{1}{q}da\right\}^{\frac{1}{2}}[W_2(0)]^{\frac{1}{2}} . \qquad (D.10)$$

Now consider integration by parts of

$$\int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}} p_x w_{xxy}\hat{\varphi}dydx = \int_{-\frac{1}{2}}^{\frac{1}{2}}\left[p_x\hat{\varphi}\bar{w}_x\Big|_{x=0}^{x=\ell} - \int_0^{\ell}\bar{w}_x(p_x\hat{\varphi})_x dx\right]dy$$

$$= -\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{p_x(0,y)\hat{\varphi}(0,y)}{p(0,y)}\left[g_1'(y) - \frac{p_y(0,y)g_1(y)}{p(0,y)}\right]dy - \int_0^{\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}}\bar{w}_x(p_x\hat{\varphi})_x dydx . \qquad (D.11)$$

We have used (14. 4) to compute the first term on the right of (D. 11).

Then

$$\left|\int_{\Re(0)} p_x w_{xxy}\hat{\varphi}\,da\right| \le \left|\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{p_x(0,y)\hat{\varphi}(0,y)}{p(0,y)}\left[g_1'(y)-\frac{p_y(0,y)g_1(y)}{p(0,y)}\right]dy\right|$$

$$+\left[\int_{\Re(0)} [(p_x\hat{\varphi})_x]^2\frac{1}{p}da\right]^{\frac{1}{2}}[W_2(0)]^{\frac{1}{2}} . \qquad (D. 12)$$

Differentiating (9. 14) with respect to y , we find

$$w_{yyy} = \frac{1}{q}\left[(\frac{p_x p_y}{p} - p_{xy})w_x + (\frac{p_y q_y}{p} - q_{yy})w_y\right.$$

$$\left.+(\frac{qp_y}{p} - 2q_y)w_{yy} - p_x w_{xy} - p w_{xxy} + F_y - \frac{Fp_y}{p}\right]. \qquad (D. 13)$$

Hence

$$\left|\int_{\Re(0)} q_y w_{yyy}\hat{\varphi}\,da\right| \le \left[\int_{\Re(0)} (\frac{q_y}{q})^2(\frac{p_x p_y}{p} - p_{xy})^2\frac{\hat{\varphi}^2}{p}da\right]^{\frac{1}{2}}[T_1(0)]^{\frac{1}{2}}$$

$$+\left[\int_{\Re(0)} (q_y)^2(\frac{p_y q_y}{p} - q_{yy})^2\frac{\hat{\varphi}^2}{q^3}da\right]^{\frac{1}{2}}[T_1(0)]^{\frac{1}{2}}$$

$$+\left[\int_{\Re(0)} (q_y)^2(\frac{qp_y}{p} - 2q_y)^2\frac{\hat{\varphi}^2}{q^3}da\right]^{\frac{1}{2}}[W_2(0)]^{\frac{1}{2}}$$

$$+\left[\int_{\Re(0)} (q_y p_x)^2\frac{\hat{\varphi}^2}{q^2 p}da\right]^{\frac{1}{2}}[W_2(0)]^{\frac{1}{2}}$$

$$+\left|\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{q_y(0,y)}{q(0,y)}\hat{\varphi}(0,y)\left[g_1'(y) - \frac{p_y(0,y)g_1(y)}{p(0,y)}\right]dy\right|$$

$$+\left(\int_{\Re(0)} \left[(q_y\frac{p}{q}\hat{\varphi})_x\right]^2\frac{1}{p}da\right)^{\frac{1}{2}}[W_2(0)]^{\frac{1}{2}} + \left|\int_{\Re(0)} q_y\hat{\varphi}(F_y - \frac{Fp_y}{p})da\right| .$$

(D. 14)

Combining (D. i), (i = 2, 3, 4, 5, 7, 9, 10, 12, and 14), with (D. 1), we have (22. 2).

By comparing the definition of H in (12.4) with that of $\bar{\bar{H}}$ in (15.3), we see that we can deduce the expression (19.3) from that of (22.2) as follows. First, we set F and its derivatives equal to zero in the surface integrals of (22.3) and then we replace the argument $(0, y)$ by that of (ℓ, y) and $g_1(y)$ and $g_1'(y)$ by $g_2(y)$ and $g_2'(y)$, respectively, in the line integrals of (22.3). This establishes ν_3 given by (19.4). Next, we replace $T_1(0)$ by $V_1(\ell)$ and $W_2(0)$ by $V_2(\ell)$ in (22.2) so as to establish (19.3).

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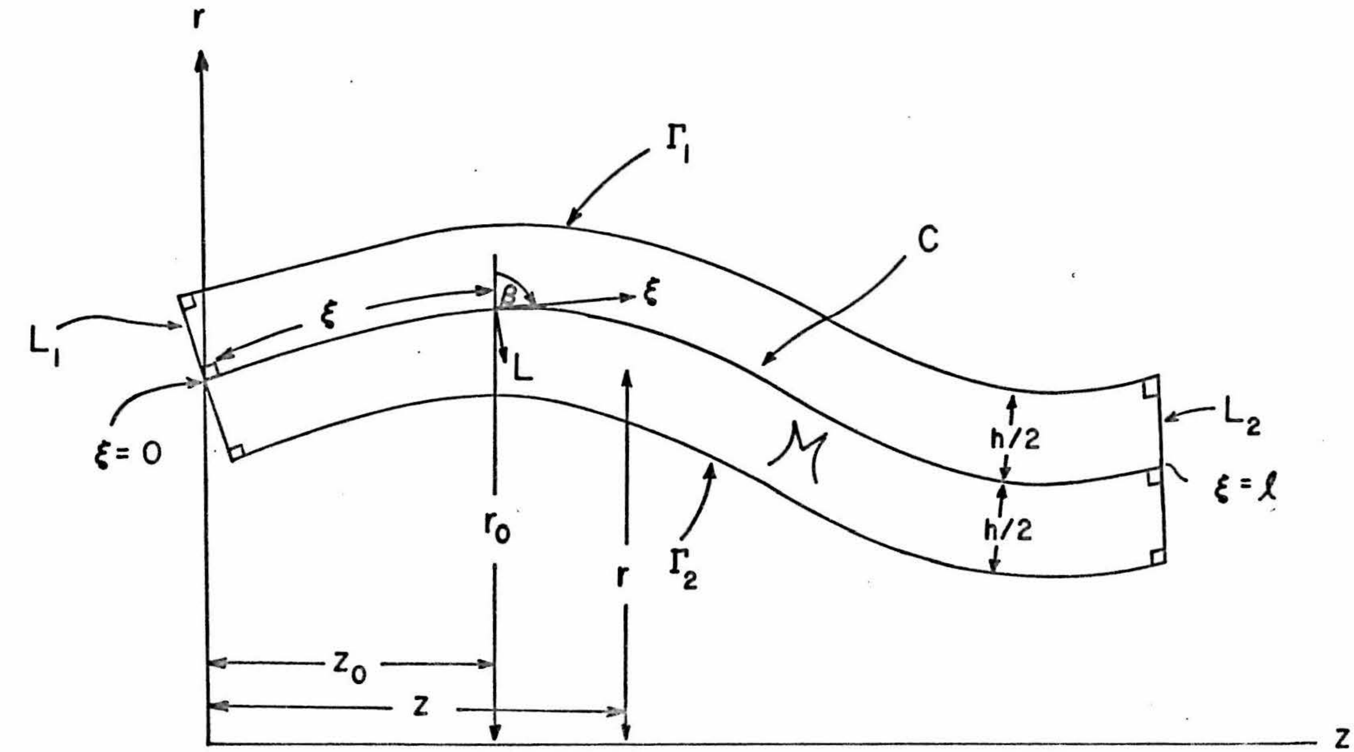

FIGURE I MERIDIONAL HALF PLANE FOR

FIXED Θ

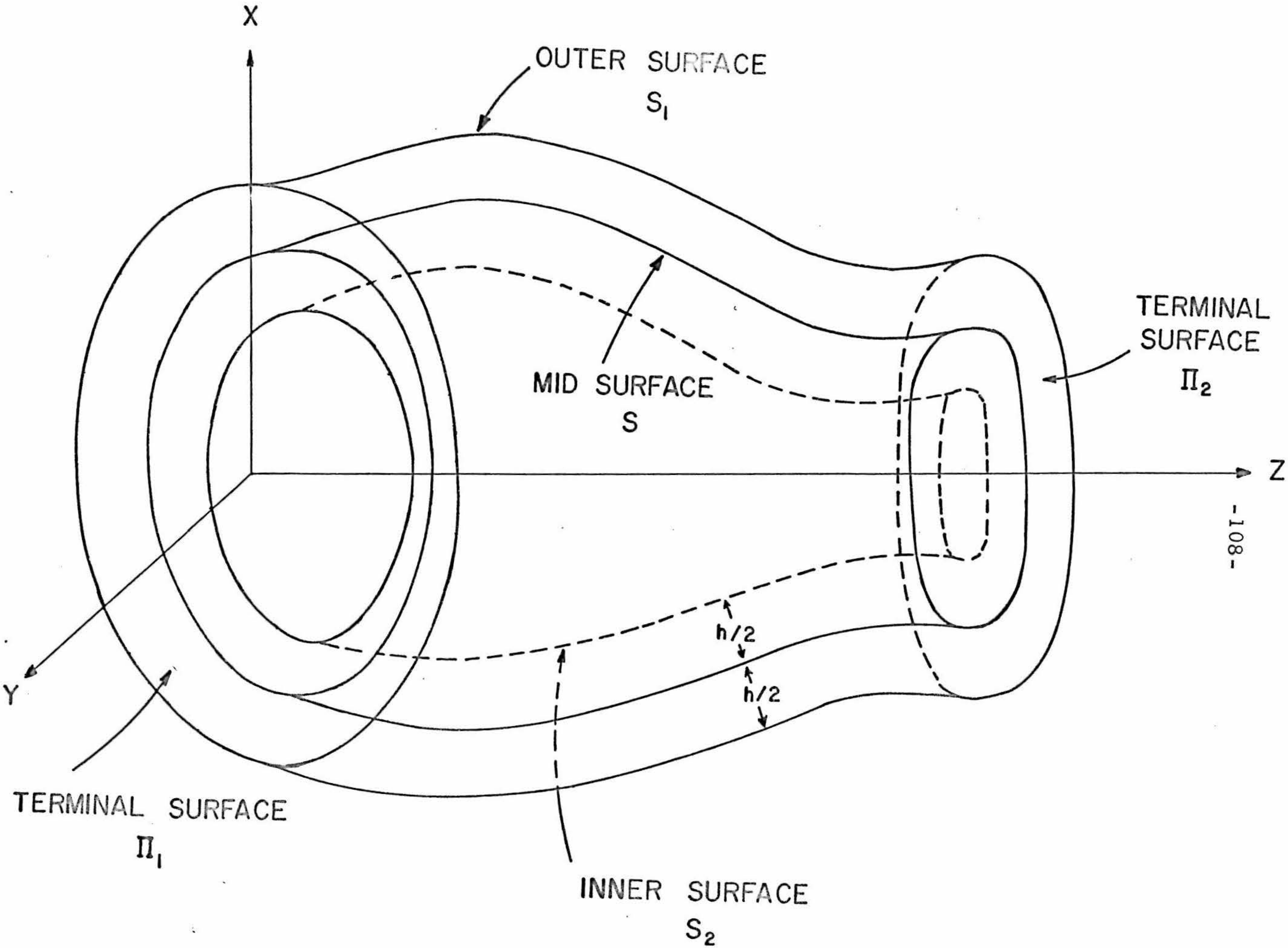

FIGURE 2 ELASTIC BODY OF REVOLUTION

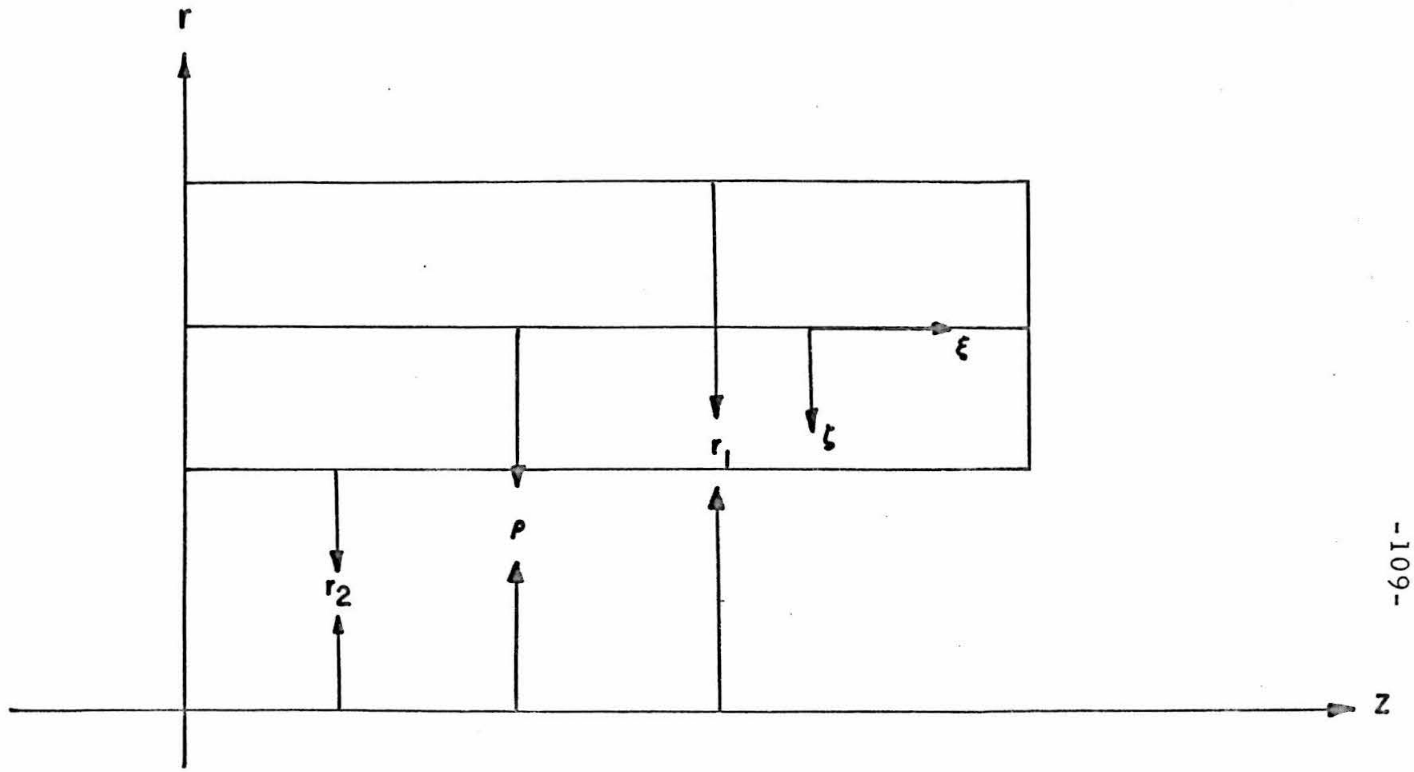

FIGURE 3

MERIDIONAL HALF PLANE OF
CIRCULAR CYLINDRICAL SHELL