INVESTIGATION OF EFFECT OF ENERGY DISSIPATION BEHIND A DETACHED SHOCK WAVE ON TOTAL HEAD MEASUREMENTS

Thesis
by
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ABSTRACT

The flow field behind a detached shock wave, created by a blunt body in a supersonic air stream, is analyzed with respect to energy dissipation through the action of viscosity in the fluid stress field near the axis of symmetry. This energy dissipation is related to the rise in entropy, and consequently to the additional drop in reservoir pressure beyond that given by the Rayleigh pitot tube formula. The method is applied to the calculations of an apparent defect in reservoir pressure, for more precise calculations of free stream Mach numbers using total head tube measurements.
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SYMBOLS

\(a\) - radius of pitot tube at nose
\(c\) - speed of sound
\(c_p\) - specific heat at constant pressure
\(f()\) - function of
\(h\) - wedge stagger
\(k_1\) - constant defined by Equation (24)
\(k_2\) - constant defined by Equation (25)
\(L\) - parameter, reduced entropy level at \(x = \delta\)
\(\eta\) - parameter defined by Equation (19)
\(M\) - Mach number
\(p\) - pressure
\(Q\) - rate of inflow of heat per unit volume due to thermal conductivity
\(R\) - gas constant
\(Re\) - Reynolds number, defined by Equation (22)
\(\Delta s\) - change in entropy
\(\delta\) - boundary layer thickness at stagnation point
\(T\) - temperature
\(u\) - velocity in \(x\) direction
\(U\) - velocity at infinity, potential solution
\(v\) - velocity in \(y\) direction
\(w\) - velocity in \(z\) direction
\(x'\) - coordinates, orthogonal axes
\(\phi_0\) - angle of shock wave to free stream flow direction
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\( \rho \) - parameter, boundary layer problem

\( \gamma \) - ratio of specific heats \( c_v/c_p = 1.40 \)

\( \delta \) - distance of detachment of shock from nose of pitot tube

\( \varepsilon \) - location of stagnation point for potential solution

\( \eta \) - parameter in boundary layer problem

\( \theta \) - wedge half angle

\( \lambda \) - coefficient of thermal conductivity

\( \mu \) - coefficient of viscosity

\( \nu \) - kinematic viscosity = \( \mu/\rho \)

\( \rho \) - density

\( \sigma \) - Frandtl number = \( \mu c_p /\lambda \)

\( \phi \) - dissipation function, defined on page 11

(\( \gamma \)) - reservoir conditions

(\( \gamma \)) - free stream condition

(\( \gamma \)) - free stream conditions after normal shock

(\( \gamma \)) - reservoir conditions after normal shock

(\( \gamma \)) - reservoir conditions at nose of pitot tube
I. Introduction

A. General

The most direct method for determining the state of a fluid at some arbitrary point requires knowledge of three fundamental flow parameters, which may be, for instance, pressure, temperature, and velocity. Therefore, three items of data must be available. However, it is possible to determine one or two of these parameters at some point in the fluid other than the one for which a complete description of the state of the fluid is desired. This possibility is dependent on the availability of this supplementary information, and on the existence of a relationship, in a convenient form, between the state parameters for these two points.

In a subsonic wind tunnel, the local reservoir pressure and temperature, except in the boundary layer, are assumed to be equal to the pressure and temperature that actually exist in the reservoir. This assumption implies a condition of isentropy for the entire flow. The static pressure may be determined by a local measurement at the wall without disturbing the flow, and assumed constant across the section of the tunnel. Thus, applying the condition of isentropy, the local Mach number, or velocity, may be determined from the following relations:

\[
\frac{p}{p_0} = \left(1 + \frac{\gamma - 1}{2} \frac{M^2}{\gamma - 1}\right)^{-\frac{1}{\gamma - 1}}
\]  

\[(1a)\]
\[ c = \sqrt{\frac{Y R T}{c}} \]  
\[ \frac{T}{T_0} = 1 + \frac{\gamma-1}{2} M_c^2 \]  
\[ u_c = M_c c \]  

If the local stagnation pressure is not known, owing to entropy changes along the channel either by cooling or by dissipation in a boundary layer, an additional measurement must be made. A total head, or pitot tube, serves for this purpose. However, here it is assumed that the flow decelerates isentropically to the stagnation point of the tube. In this case, the local conditions may again be determined from the relations above.

If the flow velocity is greater than a Mach number of 1.0, these same relations are applicable. However, if the local stagnation pressure is not known, the determination of this pressure is not as convenient as in the subsonic case. The deceleration to the nose of the total head tube is not isentropic, for a detached shock wave must form in front of the tube. Immediately in front of the stagnation point of this body the shock wave is normal to the axis of the body; consequently, the fluid can be treated as if it actually traversed a normal shock. Applying the known relations for the change in the total head across a normal shock, and further assuming that the flow behind the shock can again be treated as an isentropic phenomenon, the local
reservoir pressures in the fluid before and after the de-
tached shock are related by the following expression:

$$\frac{p}{p_0} = \left( \frac{2X}{Y+1} \frac{M^2}{1 - \frac{2X}{Y+1}} \right)^{\frac{Y}{Y-1}} \left[ \frac{1 + \frac{2X}{Y+1} M^2}{\frac{2X}{Y+1} M^2} \right]^{\frac{Y}{2}}$$  \hspace{1cm} (2)

From the relation between $p$ and $\rho$ given above, the Rayleigh pitot formula is directly deducible.

$$\frac{p}{p_0} = \left( \frac{2X}{Y+1} \frac{M^2}{1 - \frac{2X}{Y+1}} \right)^{\frac{Y}{Y-1}} \left( \frac{X}{Y+1} \frac{M^2}{M^2} \right)^{\frac{Y}{2}}$$  \hspace{1cm} (3)

If the local static and pitot total head pressures are known, the Mach number can be determined from Equation (3).

B. Mach Number Determination in a Supersonic Wind Tunnel

There exist four direct and relatively accurate methods by which it is possible to make a fairly accurate determina-
tion of the Mach number in the test section of a supersonic wind tunnel.

The first method is the simplest, requiring no more
than a measurement of the reservoir pressure and a measure-
ment of the static pressure with an orifice located at a
convenient point in the wall of the test section. The Mach number can be determined from Equation (1a). For high pre-
cision, or for complete axial surveys of the variation of
Mach number along the center line of the test section, this simple procedure is not very useful.
The second method is actually a refinement of the first. The reservoir pressure is measured as before; however, the static pressure is now measured at orifices located on both sides of a sharp wedge or small total included angle. Adjustment of the wedge in pitch to zero the difference between the

![Figure 1](image)

*Figure 1*

*Single Wedge Method for Determination of Free Stream Mach Number*

two static pressures \( p_1 \) and \( p_2 \) guarantees a flow direction parallel to the axis of wedge, and assures a change in flow direction for either side to equal the half angle of the wedge. Assuming isentropic flow everywhere, except across the oblique shock, the true Mach number ahead of the shock wave can be determined from the following oblique shock relations:

\[
\frac{p_{s_2}}{p} = \frac{\gamma}{\gamma+1} M_i^\gamma \sin^2 \beta_o - \frac{\gamma-1}{\gamma+1} \quad (4)
\]

\[
\frac{1}{M_i^2} = \sin^2 \beta_o - \frac{\gamma+1}{\gamma} \frac{\sin \beta_o \cos \Theta}{\cos (\beta - \Theta)} \quad (5)
\]

and the isentropic channel pressure relation expressed in Equation (1a).
For specific values of $\rho$, $\Theta$, and $\beta$, it is possible to determine $\rho$, $M$, and $\beta$. This method has the additional advantage of making possible a complete survey of pressure and Mach number continuously along any line parallel to the center line of the tunnel, throughout the entire test section. The method is still applicable if the wedge is not adjustable in pitch; here, however, the interpretation of data is a little more lengthy.

The third method is more elaborate. In addition to the wedge of the previous method, an additional fixed wedge (II) is mounted in the tunnel just above and downstream of the first wedge (I), and arranged such that the stagger ($h$) and angle of pitch of wedge (I) are both independently adjustable. An initial determination of $\beta_s$ is made when the entire disturbance caused by wedge (I) lies completely downstream from the $\beta_s$ orifice. Wedge (I), with the upper surface approximately parallel to the flow, is then moved axially upstream until the orifice $\beta_s$ lies on the character-
istic of the orifice $\beta_z$. This position is determined by observing the point where the orifice $\beta_z$, connected to a high vacuum, has the maximum effect on the value of $\beta_z$. The value of $\beta_z$, after wege (I) is pitched until $\beta_z$, agrees with the initial determination of $\beta_z$, is then the free stream static pressure. The measurement is actually made behind a double family of Mach waves and expansion waves created by the boundary layer along the upper surface of the wedge. The cumulative effect of these waves is sensibly zero, as indicated by the return of $\beta_z$ to its original value. Again, the Mach number may be quickly determined by Equation (1a).

The fourth method is simply a determination of the total head behind a detached shock wave. By directly applying the Rayleigh Pitot tube formula, the Mach number can be determined.

C. Experimental Evidence

In the calibration of the 12-inch supersonic wind tunnel at the Jet Propulsion Laboratory, the single static wedge method was used for Mach number surveys. However, both the double wedge method and the total head tube method were employed as checks on the primary method. Very good agreement between the two static wedge methods was observed at $M = 2.174$, and a minor departure from
this value ($M = 2.176$), not creditable to lack of instru-
ment precision, was observed for the total head tube method.
The observed data indicated an apparent defect in $\rho'$, as
measured by the total head tube.

At higher Mach numbers, there is evidence that the
disagreement is more pronounced.

The fundamental objective of this thesis is to investi-
gate the effect of energy dissipation behind the detached
shock wave as the cause for this discrepancy, and to de-
termine the order of magnitude of this effect in supersonic
total head measurements.
II. INVESTIGATION OF ENTROPY RISE BEHIND A DETACHED SHOCK WAVE

A. Technique and Assumptions

For pitot tubes, or other blunt bodies of revolution in a supersonic flow, the distances of detachment of the shock waves are usually very small. This is especially true for very high Mach numbers. Figure 5, extracted from Reference (1), illustrates the variation of this distance with Mach number, as measured in experimental programs. Within the detachment span, the flow must adjust from the conditions which prevail immediately downstream from the detached shock to the conditions which prevail on the surface of the body. This action indicates that for sufficiently high supersonic velocities the flow field in this region is characterized by large velocity stresses. If the viscosity of the fluid is not zero, energy dissipation may account for an apparent defect in $\rho$.

In the vicinity of the axis of symmetry, the region within the detachment span is separated into two zones, depending on the role of viscosity in determining the character of the flow. The upstream zone is considered to possess the potential type flow pattern of a fluid without viscosity. In this zone the flow velocity along the axis of symmetry is constructed to match the velocity
of the fluid immediately downstream from a normal shock wave, at the detachment distance \( \delta \) from the nose of the body. In this zone, the subsequent introduction of viscosity is considered to produce a negligible effect on the velocity pattern given by the potential solution, such that, to a first approximation, the dissipation of energy throughout this zone can be conveniently obtained. This is effectively the first step of an iteration procedure to obtain the exact nature of the flow in a region where the viscosity is sufficiently low to account for only a small perturbation on the flow field as determined for a perfect fluid.

The downstream zone is the boundary layer on the nose of the body. The boundary layer thickness is defined to be that distance from the nose of the body where the effect of viscosity in shaping the form of the flow becomes negligibly small. For this application, it is considered to be that distance beyond which the velocity gradient along the axis of symmetry, as given by the boundary layer solution, approaches a constant within 0.1%. At this distance, the parameter included in the boundary layer problem can be adjusted to provide the matching of velocities of the potential solution to that of the boundary layer solution. Since the potential solution of the upstream part is constructed under the assumption of zero viscosity, while the
boundary layer solution considers a finite viscosity for all of the fluid, it is impossible to provide a matching of both the velocity and the velocity gradient at a common point. However, a matching of the velocity gradients together with the velocities can occur at two different points, one pertaining to the potential solution and the other to the boundary layer solution. This can be interpreted as a virtual increase in the radius of the body, amounting to the distance between these points, in the construction of the potential solution.

Figure 3

Axial Velocity Profile in Region of Nose of Pitot Tube for both Potential and Boundary Layer Solutions

The change in entropy for an adiabatic process is given by

\[ T \frac{\partial s}{\partial t} = \frac{1}{\rho} \frac{\partial P}{\partial t} + \frac{1}{\rho} \frac{\partial Q}{\partial t} \]  \hspace{1cm} (6)

where

\[ \frac{\partial s}{\partial t} = \frac{\partial t}{\partial t} + u \frac{\partial s}{\partial x} + v \frac{\partial s}{\partial y} + w \frac{\partial s}{\partial z} \]
and where \( \phi \), the dissipation function, is given by

\[
\phi = -\frac{2}{3} \mu \left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right]^2 + 2 \mu \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 \right] \\
+ \mu \left[ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right]^2 + \mu \left[ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right]^2 + \mu \left[ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right]^2
\]

See Reference (2), article 358. \( \dot{Q} \) represents the rate of inflow of heat per unit volume due to thermal conductivity.

In the vicinity of the axis of symmetry, all velocities and velocity gradients, except \( u \) and \( \frac{du}{dx} \), are taken to be zero. Also, all temperature gradients, except \( \frac{dT}{dx} \), are assumed zero. This is inferred from the symmetrical character of the flow in this region.

Equation (6) then reduces, for steady state flow, to

\[
u \tau \frac{dT}{dx} = \nu \frac{4}{3} \left( \frac{du}{dx} \right)^2 + \frac{1}{\rho} \frac{dt}{dx} \lambda \left( \frac{dT}{dx} \right)
\]

(6a)

where \( \lambda \) is the coefficient of thermal conductivity.

Assuming \( \lambda \) constant and also assuming that

\[
\sigma = \frac{\mu \sqrt{\text{Gr}}}{A} = 1
\]

(Frãndtl number)

Equation (6a) can be written in either of the two following forms:

\[
u \tau \frac{dT}{dx} = \nu \left( \frac{4}{3} \left( \frac{du}{dx} \right)^2 + \frac{1}{\rho} \frac{dt}{dx} \lambda \left( \frac{dT}{dx} \right) \right) - \nu \left[ \mu \frac{d^2 u}{dx^2} + \left( \frac{du}{dx} \right)^2 \right]
\]

(7)
The second form of Equation (7) follows for no heat transfer through the body.

From this, it appears that the effect of conductivity may be of the same order of magnitude as the effect of viscous dissipation. Since no simple integral of the boundary layer equations (corresponding to Appendix A) is known which considers the effect of heat transfer, it is not possible to carry out the complete solution; consequently the effects of heat transfer will be neglected in these calculations.

Neglecting variation in $T$ and $\nu$ and neglecting the heat transfer terms, Equation (7) becomes

$$
\Delta S = \frac{4L}{3T} \int \frac{L}{\alpha} \left( \frac{du}{dx} \right)^2 dx
$$

Using the velocity and velocity gradients as determined by the method described above, the entropy change through the detachment span can be evaluated. From this change the drop in reservoir pressure can be determined, using

$$
\Delta S = R \log \frac{\rho'}{\rho_0'}
$$

For $\Delta S \ll \mathcal{E}$, this equation reduces to

$$
\frac{\rho'}{\rho_0} = 1 + \frac{\Delta S}{\mathcal{E}}
$$

The change in entropy through the detachment span will obviously be a function of the free stream Mach number $M_1$. Consequently, in line with the assumption that the change in entropy is small, the approximation that $\rho' = \rho_0'$ and the
application of the Rayleigh pitot formula (Eq. 3) allow
the determination of an uncorrected $M_1$ for use in evalu-
ating $\Delta S$, and subsequently lead to the evaluation of a
corrected $M_1$ through Equation (9a).

B. Matching Potential and Boundary Layer Solutions

\[ \begin{align*}
&\text{Shock Wave} \\
&\text{Boundary Layer}
\end{align*} \]

Figure 4
Sketch of Flow Geometry in Region of
Nose of Pitot Tube

In accordance with the definition given in Section A
for the boundary layer thickness, from Appendix A

\[ \eta = 2.00 \]

\[ t = 2 \left( \frac{\mu}{\rho} \right)^{\frac{1}{2}} \]  \hspace{1cm} (10) \]
Hence, at \( x = t \)

\[
\begin{align*}
  u_2 &= -2.90 \ (\beta \nu)^{\frac{1}{2}} \\
  \frac{du_2}{dx} &= -2 \beta
\end{align*}
\]

From the potential solution for the region outside the boundary layer, the velocity and the velocity gradient along the axis of symmetry are

\[
\begin{align*}
  u_2 &= -U + U \frac{a^3}{(x+a)^3} \\
  \frac{du_2}{dx} &= -3 U \frac{a^3}{(x+a)^4}
\end{align*}
\]

where \(-U\) is the velocity at infinity for the potential flow. At \( x = \epsilon \)

\[
\begin{align*}
  u_2 &= -U + U \frac{a^3}{(\epsilon+a)^3} \\
  \frac{du_2}{dx} &= -3 U \frac{a^3}{(\epsilon+a)^4}
\end{align*}
\]

Equating velocities and velocity gradients at these points,

\[
\begin{align*}
  -2.90 \ (\beta \nu)^{\frac{1}{2}} &= -U + U \frac{a^3}{(\epsilon+a)^3} \\
  -2 \beta &= 3 U \frac{a^3}{(\epsilon+a)^4}
\end{align*}
\]

from which \( \beta \) and \( \epsilon \) can be determined. Assuming \( \epsilon/a < 1 \),

\[
\beta = \frac{2}{a} \frac{U}{\epsilon} \left( 1 + 2 \frac{\epsilon}{a} + O\left(\frac{\epsilon}{a}\right)^2 \right) \tag{11}
\]
\[ \varepsilon = 0.73 \, t = 1.45 \left( \frac{U}{a} \right) \]

(12)

At \( x = \delta - \varepsilon \), the velocity along the axis of symmetry in the potential solution is matched to the velocity of the fluid immediately downstream from the detached shock wave. Thus \( U \) can be expressed in terms of the detachment distance \( \delta \), the radius of the tube, and the free stream velocity.

\[
-U_z = -U + U \left[ 1 - 5 \left( \frac{\varepsilon - \delta}{a} \right) + O \left( \left( \frac{\varepsilon - \delta}{a} \right)^2 \right) \right]
\]

\[
U \left( \frac{\varepsilon - \delta}{a} \right)^2 - \left( 2u_z \frac{\varepsilon - \delta}{a} + 12 \frac{u_z}{a} \right) U + U_z = 0
\]

Since for most practical cases

\[ u_z \frac{\delta}{a} > \frac{\delta}{a} \]

the second factor of \( U \) can be neglected, leaving

\[ U = u_z \frac{a}{\delta - \delta} \]

(13)

From the basic equation for the velocity change across a normal shock wave

\[ u_z = c^* \sqrt{\gamma - 1} \]

\[ u_z = \frac{c_0}{M_1} \gamma \left( 1 + \frac{\gamma - 1}{2} M_1^2 \right)^{\frac{1}{2}} \]
Thus

\[ U = \frac{\rho_0}{J(z+1)} \frac{\rho}{M} \left( 1 + \frac{1}{2} M^2 \right)^{\frac{1}{2}} \]  

(14)

For \( M_1 > 3 \), \( J/\alpha \) approaches 0.14 with negligible variation in this value for higher Mach numbers. See Figure 5.

With this restriction, Equation (13) can be written

\[ U = \frac{2 \rho_0}{M} \left( 1 + \frac{1}{2} M^2 \right)^{\frac{1}{2}} \]  

(15)

C. Solution for Entropy Rise, Using Potential Form for Velocity Distribution

The entropy change through the detachment span, except for the boundary layer, can be determined from Equation (8)

\[ \Delta S = \frac{\gamma}{\gamma - 1} \int_{\alpha}^{\varepsilon} \left( \frac{dU}{dx} \right)^2 dx \]

where, from potential form of velocity pattern,

\[ U_2 = \frac{U_0 \rho_0}{(x+a)^{\gamma}} - U \]

\[ \frac{dU}{dx} = -U \frac{\gamma \rho_0}{(x+a)^{\gamma}} \]

The integral can then be simply evaluated.

\[ \Delta S = \frac{12 \sqrt{\gamma - 1}}{\gamma - 1} \frac{U_0}{a} \left[ -\frac{1}{4(1 + \frac{1}{2})^{\gamma}} - \frac{1}{1 + \frac{1}{2}} + \frac{1}{6} \log \left( 1 + \frac{2}{\gamma} \right)^{\gamma} \right] \right]_{\alpha}^{\varepsilon}

+ \sqrt{\frac{\gamma}{3}} \tan^{-1} \left( -2 \sqrt{3} \frac{a}{a} - \sqrt{3} \right) \right]_{\alpha}^{\varepsilon} \]
Figure 6 is a plot of the value of this expression as a function of \( \delta/\alpha \). For

\[
\epsilon = \sqrt{\frac{5}{\beta}} \left( \frac{\nu'}{\beta} \right)^{\frac{1}{4}} \tag{12}
\]

\[
\beta = \frac{2}{\epsilon} \frac{U}{\alpha} \tag{11}
\]

the change in entropy across this portion of the detachment span can be expressed in the form

\[
\Delta S = \frac{12.5 \nu' U}{T_0 \alpha} \int \frac{1}{\epsilon} \log \left( \frac{2.14 \alpha U}{\nu'} \right) - 1.855 - \frac{L}{\epsilon} \int \tag{16}
\]

where

\[
\frac{12.5 \nu' U}{T_0 \alpha} \int \tag{16}
\]

represents the value of the integral at \( x = \delta \).

D. Solution for Entropy Rise Through Boundary Layer

The entropy change through the boundary layer can be evaluated in two ways. The simplest method, and perhaps in practice the most accurate method, is to integrate stepwise, using the tabulated data from the boundary layer solution. The other method is to choose enough terms of the series solution for the velocity in the boundary layer to provide an accurate representation of this velocity, and then integrate as before, using Equation (5). If two terms are selected, then
\[ u_e = -1.3172 \left( \frac{f}{f'} \right)^2 \beta x^2 + 0.3333 \frac{f}{f'} \beta x^2 \]

\[ \frac{du_e}{dx} = -2.6341 \left( \frac{f}{f'} \right)^2 \beta x + \frac{f}{f'} \beta x^2 \]

as given in Appendix A. Substitution into Equation (8) with proper limits

\[ \Delta S = \frac{1}{f_0} \int \left( \frac{du_e}{dx} \right)^2 dx \]

where

\[ t = 2 \left( \frac{f'}{f} \right)^2 \]

\[ \beta = \frac{3}{2} \frac{U}{a} \]

The result is, after the integration is carried out,

\[ \Delta S = 11.15 \frac{f_0' U}{f_0'a} \]

To integrate stepwise, Equation (8) written in terms of the boundary layer solution nomenclature becomes

\[ \Delta S = \frac{\varepsilon f_0' \beta}{f_0'} \int_{\eta}^{2} \left[ \frac{f(\eta)}{f_0(\eta)} \right]^2 d\eta \]

Evaluating from data in Appendix A, the result is

\[ \Delta S = 12 \frac{f_0' U}{f_0'a} \quad (17) \]

The evaluation of the change in entropy using the two term approximation for the boundary layer velocity agrees fairly well with this value.
E. Total Entropy Rise Equation

The total change in entropy through the entire detachment span is then the sum of the entropy changes given by Equations (16) and (17).

\[
\Delta S = \frac{12 \alpha' \ell}{\tau_o} \left[ \frac{L}{6} \log 2 \pi \frac{\alpha'}{\nu} - 0.855 - L \right]
\]

(18)

To reduce Equation (18) to a simple usable relation, Equation (14) is rewritten as

\[
U = c_0 \eta
\]

where

\[
\eta = \frac{1}{3/(Y+1) \delta_a} \frac{L}{M_o} \left( 1 + \frac{Y-1}{2} M_o^2 \right)^{\frac{1}{2}}
\]

(19)

In addition, since \( T_o' = T_o \) within the approximation of constant \( T_o' \) throughout the entire detachment span, the kinematic viscosity within the detachment span can be approximated by

\[
\nu_o' = \nu_o \frac{p_o}{p_o}
\]

(20)

where \( \nu_o \) is the kinematic viscosity based upon reservoir conditions. This assumption is valid for the entire detachment span provided that

\[
\frac{p_o'}{p_o} = 1
\]

(21)
which is in line with the assumption that the entropy change through the detachment span is small, and also provided that

\[ \nu' = \nu'' \]

Letting

\[ R_e = \frac{a}{c_0} \frac{V}{u} \]  \hspace{1cm} (22)

represent a Reynolds number based upon the radius of the pitot tube head, the reservoir speed of sound, and the reservoir kinematic viscosity, Equation (18) then reduces to

\[ \Delta s = \frac{4822}{R_e} \left( \frac{\eta}{\nu} \right) \left[ \log R_e + \log \frac{\eta}{\nu} \frac{\beta'}{\rho_0} - 4.37 - 6.1 \right] \]  \hspace{1cm} (23)

Defining

\[ K_1 = \frac{\eta}{\nu} \frac{\beta'}{\rho_0} \] \hspace{1cm} (24)

\[ K_2 = \log \frac{\eta}{\nu} \frac{\beta'}{\rho_0} - 4.37 - 6.1 \] \hspace{1cm} (25)

both of which are functions of the Mach number, Equation (23) can be written

\[ \Delta s = \frac{4822}{R_e} K_1 \left[ \log R_e + K_2 \right] \] \hspace{1cm} (26)
From Equation (9a), continuing the assumption of Equation (21)

$$\frac{p_e'}{p_o''} = 1 + \frac{C_0 K_1}{K_0} \left[ \frac{1}{\log Re + K_2} \right]$$

(27)

The parameters $K_1$ and $K_2$ are plotted against $M_1$ in Figures 7 and 8. The values of $C_0$ used in the evaluation of these parameters as a function of Mach number are selected from the curve drawn in Figure 5.
III. RESULTS

To provide some measure for evaluating the significance of Equation (27) in predicting the drop in reservoir pressure ratio across the detachment span, several computations were made based upon assumed initial conditions. The results are as follows:

Case I

\[ T_0 = 540 \degree \text{ Rankine} \]
\[ \rho_0 = 93.3 \text{ lb/sq. in. absol.} \]
\[ \rho'_0 = 0.0145 \text{ slugs/cubic ft.} \]
\[ a = 0.0117 \text{ feet} \]
\[ M_i = 1.70 \]

whereupon

\[ Re = 5.0 \times 10^5 \]
\[ \Delta S = 0.10 \text{ ft}^2/\text{sec}^2 \text{ Rankine} \]
\[ \frac{h_1}{h_0} = 1.000063 \]
\[ M_i = 1.70 \]

The values selected here for initial conditions agree with those specified in Appendix B, or Reference (5). The step integration technique employed in Appendix B yielded the values

\[ \Delta S = 0.056 \text{ ft}^2/\text{sec}^2 \text{ Rankine} \]
\[ \frac{h'_1}{h_0} = 1.000050 \]
\[ M_i = 1.70 \]
These two results are in fair agreement. However, it is to be noted that if corrections were made to the values of \( k_1 \) and \( k_2 \), which are strictly necessary owing to the lack of correspondence in the value of \( \frac{\partial}{\partial x} \) used in their evaluation to that of Appendix B, the results by the theoretical approach of Equations (26) and (27) become

\[
\Delta S = 0.086 \text{ ft}^2/\text{sec}^2 \text{ Rankine} \\
\rho_1'/\rho_0' = 1.000050 \\
M_i = 1.70
\]

The agreement is then very good.

Case II

Using the same initial conditions of Case I except that now \( M_i = 9.00 \),

\[
Re = 5.0 \times 10^5 \\
\Delta S = 14.4 \text{ ft}^2/\text{sec}^2 \text{ Rankine} \\
\rho_1'/\rho_0' = 1.0084 \\
M_i = 9.017
\]

Case III

Using the same initial conditions of Case II, except that now the radius of the tube end is reduced by a factor of 5,

\[
Re = 1.0 \times 10^5 \\
\Delta S = 57.6 \text{ ft}^2/\text{sec}^2 \text{ Rankine} \\
\rho_1'/\rho_0' = 1.0336 \\
M_i = 9.067
\]
IV. CONCLUSIONS

The results of this investigation indicate that the energy dissipation in the flow field behind a detached shock wave, created by a blunt body in a supersonic airstream, is quite small and produces only a minor change of the reservoir pressure at the stagnation point from the reservoir pressure existing immediately downstream from the detached shock wave. This condition is especially true for the majority of cases where the Mach number is only moderately high (around $M_1 = 5$ or 6, or less), and the pitot tubes are of average dimensions ($a = 0.125$ inch or greater). Here the errors obtained in the computation of the true Mach number $M_1$ are well beyond experimental observation in ordinary engineering applications. This investigation consequently does not explain the apparent defect in $M_1$ experienced in the calibration of the 12-inch supersonic wind tunnel at the Jet Propulsion Laboratory, as mentioned in Section I-C.

For very high Mach numbers, and for small pitot tubes, this investigation indicates that the dissipation of energy within the detachment span may account for a small but measurable error in the determination of the true free stream Mach number. For experiments demanding a high degree of precision, the effect should be taken into account.
REFERENCES


(3) S. Goldstein, "Modern Developments in Fluid Dynamics", Oxford University Press, 1938.


(5) Ladenberg, Winckler and Van Voorhis, "Interferometer Studies of Faster than Sound phenomena", The Physical Review, Vol. 73, No. 11, June 1, (1948).

**Figure 5**

Detached Shock Wave Distances for Spheres

+ Supersonic Wind Tunnel Data
- Ballistic Range Data

Mach No. $M_1$
$L = \frac{S}{\nu g U} \frac{1}{t' a}$

**Parametric Value of Entropy Rise Through Potential Type Flow Region**

**Figure 6**
Plot of Parameter $K_1$

Equation (24)

Figure 7

Mach No. $M_1$
Plot of Parameter $K_2$

Equation (25)

Figure 6
APPENDIX A

BOUNDARY LAYER SOLUTION NEAR A FORWARD STAGNATION POINT
BOUNDARY LAYER SOLUTION NEAR A FORWARD STAGNATION POINT

Figure I A

Description of Coordinates for Boundary Layer Problem

The equation of continuity, in curvilinear orthogonal coordinates possessing one axis of symmetry can be written

$$\frac{1}{\rho} \left[ \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) \right] = 0$$

where \( x, y \) are coordinates as indicated above. Since \( \rho \neq \rho_0 \neq \rho \) in the vicinity of the stagnation point, the equation of continuity becomes

$$\frac{\partial}{\partial x} (x u) + \frac{\partial}{\partial y} (x v) = 0$$

This equation implies a stream function \( \psi \) such that

$$u = \frac{1}{x} \frac{\partial \psi}{\partial y}$$

$$v = -\frac{1}{x} \frac{\partial \psi}{\partial x}$$
which identically satisfies the continuity equation.

The equations of motion in two dimensions are

\[ u \frac{du}{dx} + v \frac{dv}{dy} = -\frac{1}{\rho} \frac{dp}{dy} + \nu \frac{d^2u}{dx^2} + \nu \frac{d^2v}{dy^2} \]

\[ u \frac{dv}{dx} + v \frac{du}{dy} = -\frac{1}{\rho} \frac{dp}{dx} + \nu \frac{d^2v}{dx^2} + \nu \frac{d^2u}{dy^2} \]

In the boundary layer, these equations reduce to

\[ -\frac{1}{\rho} \frac{dp}{dy} = O(\delta) = 0 \]

\[ u \frac{du}{dx} + v \frac{dv}{dy} = \frac{1}{\rho} \frac{dp}{dx} + \nu \frac{d^2u}{dy^2} \]

by the method of establishing the equations, by Prandtl.

Since \( \rho \) is independent of \( y \),

\[ -\frac{1}{\rho} \frac{dp}{dx} = u_0 \frac{du}{dx} \]

where \( u_0 \) is the velocity outside the boundary layer.

The momentum equation can then be written

\[ u \frac{du}{dx} + v \frac{dv}{dy} = u_0 \frac{du}{dx} + \nu \frac{d^2u}{dy^2} \]

Near the forward stagnation point the velocity \( u_0 \),
just outside the boundary layer, may be put equal to \( \rho x \),
where \( x \) is the distance along meridian curve from the
stagnation point.
Hence with

\[ u = \frac{1}{x} \frac{\partial u}{\partial y} \]

\[ v = -\frac{1}{x} \frac{\partial v}{\partial x} \]

\[ u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} = \beta^2 x + \nu \frac{\partial^2 u}{\partial y^2} \]

a solution is obtained by putting

\[ \eta = (\beta x^2) \frac{1}{\nu} y \]

\[ \psi = (\beta x^2) \frac{1}{\nu} x^2 f(\eta) \]

Thus

\[ u = \beta x f(\eta) \]

\[ v = -2 (\beta x^2) \frac{1}{\nu} f(\eta) \]

\[ \beta^2 x f(\eta) - 2 (\beta x^2) \frac{1}{\nu} f(\eta) = \beta^2 x + \beta^2 x f(\eta) \]

Calling \( f(\eta) = f \), this equation reduces to

\[ f'' + 2 f' f = 1 + f'' \]

Since

\[ u = v = 0 \quad \text{at} \quad y = 0 \]

and

\[ u = u_0 = \beta x \quad \text{at} \quad y = \infty \]
the boundary conditions for $f$ are

$$f(0) = 0$$
$$f'(0) = 0$$
$$f'(\infty) = 1$$

Choosing a solution for $f$ of the form

$$f = \sum_{n=0}^{\infty} a_n \eta^n$$

then

$$f' = \sum_{n=0}^{\infty} n a_n \eta^{n-1}$$
$$f'' = \sum_{n=0}^{\infty} n(n-1) a_n \eta^{n-2}$$
$$f''' = \sum_{n=0}^{\infty} n(n-1)(n-2) a_n \eta^{n-3}$$

Substituting in the differential equation, and collecting coefficients of like powers of $\eta$, the values of $a_n$ may be evaluated. The values for $a$, through $a_8$ as determined by Somann (Ref. 3) are

$$a_0 = 0$$
$$a_1 = 0$$
$$a_2 = \text{arbitrary}$$
$$a_3 = -0.16667$$
$$a_4 = 0$$
$$a_5 = 0$$
$$a_6 = 0.005556 a_2$$
$$a_7 = -0.0003968$$
$$a_8 = 0$$
Thus

\[ f = a_2 \eta^2 - 0.1667 \eta^3 + 0.005556 a_2 \eta^4 - 0.0067 \eta^5 + \ldots \]

To determine \( a_2 \), from boundary condition

\[ \eta = \infty \quad f' = 1 \]

let

\[ f = f_0 + f_1 \]

\[ f' = f_0' + f_1' \]

\[ f_0' = 1 \quad , \quad f_1' = 0 \quad a + \eta = \infty \]

Thus

\[ f = \eta + f_1 \]

Substituting in original differential equation,

\[ (1 + f_1')^2 - 2 (\eta + f_1) f_1'' = 1 + f_1''' \]

\[ 2 f_1' + f_1'^2 - 2 \eta f_1'' - 2 f_1 f_1'' = f_1''' \]

neglecting \( f_1 \) and \( f_1'^2 \) in comparison with the other terms

\[ f_1''' + 2 \eta f_1'' - 2 f_1' = 0 \]

This equation has the solution

\[ f_1' = A_1 \eta \int_{-\infty}^{\eta} \frac{1}{e^{x^2}} e^{-x^2} \, dx \]
\[ f_i = A_2 + \int_0^\eta A_3 \eta \int_\tau^\eta e^{-\tau^2} d\tau d\eta \]
\[ f_r = A_2 - A_3 \int_0^\eta e^{-\tau^2} d\tau - 2A_1 \int_\tau^\eta d\tau \int_0^\eta e^{-u^2} du \]

From Homann, \( A_1 = 2.16492 \), \( A_2 = -0.557611 \) and \( a_3 = 0.659619 \), determined by equating
\[ f = \sum a_n \eta^n = \eta + f_i \]

Thus
\[
\begin{align*}
a_0 &= 0 \\
a_1 &= 0 \\
a_2 &= +0.658619 \\
a_3 &= -0.16647 \\
a_4 &= 0 \\
a_5 &= 0 \\
a_6 &= 0.00366 \\
a_7 &= -0.0003968 \\
a_8 &= 0 \\
\end{align*}
\]

The final expression for \( f \) is then
\[ f = 0.6586 \eta^2 - 0.16647 \eta^3 + 0.00366 \eta^6 - 0.0003968 \eta^7 + \ldots \]

and is graphically represented, together with \( f' \) and \( \frac{f''}{f} \), in Figure 2A.
From the definitions for \( u \) and \( v \) given above, these velocities may be written

\[
\begin{align*}
u &= 1.3172 \beta \left( \frac{x}{\nu} \right)^{3/2} x y - 5.000 \frac{\beta^2}{\nu} x y^2 \\
v &= -1.3172 \beta \left( \frac{x}{\nu} \right)^{3/2} y^2 + 0.3333 \frac{\beta^2}{\nu} y^3
\end{align*}
\]

where only two terms of the series expansion for \( f \) are employed.

The notation used in this appendix is standard boundary layer nomenclature, and differs from that used in the main body of this thesis. For use there, the two equations above for \( u \) and \( v \) are then written as

\[
\begin{align*}
u &= -1.3172 \beta \left( \frac{x}{\nu} \right)^{3/2} x^2 + 0.3333 \frac{\beta^2}{\nu} x^3 \\
v &= 1.3172 \beta \left( \frac{x}{\nu} \right)^{3/2} x y - 0.5000 \frac{\beta^2}{\nu} y x^2
\end{align*}
\]
Graphical Representation

Solution of Boundary Layer Problem

\[ \left( \frac{f'(\eta)}{f(\eta)} \right)^2 \]

\[ f(\eta), f'(\eta) \]

Figure 2A
APPENDIX B

GRAPHICAL SOLUTION FROM DATA OBTAINED FROM AN INTERFEROMETER STUDY OF DETACHED SHOCK WAVES
GRAPHICAL SOLUTION FROM DATA OBTAINED FROM AN INTERFEROMETER STUDY OF DETACHED SHOCK WAVES

Figure 24 of Reference 5, reproduced here in part as Figure 13, is an isoplethal chart for air flow about a sphere in a supersonic jet of Mach number 1.70. In this chart the contours are labeled in milligrams per cubic centimeters. These data were computed from an interferogram of air flow about a 9/32 inch diameter sphere in a homogeneous jet.

Figure 25 is a plot of the density against distance along the axis of symmetry, taken from Figure 13.

The equations used for the determination of the pressure $p_2$, the temperature $T_2$, and the velocity $u_2$, as given in Reference (5), are

$$p_2 = 0.414 \rho_2^0 \text{ atmos}$$

$$T_2 = 144 \rho_2^{0.1} \text{ °Kelvin}$$

$$u_2 = 533 \left( 0.063 - \rho_2^{0.1} \right)^{1/3} \text{ meters/sec}$$

Using the plotted values of the density, the temperatures and the velocities may be determined conveniently from these relations. The variation of the temperature and the velocity along the axis of symmetry is graphically represented in figures 35 and 45. Figure 35 and Figure 45
are actually the same plot; however, in Figure 46 the state parameters which are necessary for a stepwise integration have been more smoothly fairied, and have been expressed in more convenient units.

From equation (8)

$$A s = \frac{d v}{3} \int_{-1}^{0} \frac{1}{u_0} \left( \frac{d u_2}{d x} \right)^2 dx$$

and using the values of density, temperature, and velocity plotted in Figure 4b, a stepwise integration was performed, and the entropy rise along the axis of symmetry was determined. For this computation, the value of $\mu$ was fixed at $37 \times 10^{-9}$ slugs/ft.$^2$.sec. This procedure then gives

$$A s = 0.086 \text{ ft}^2/\text{sec}^2 \circ \text{Rankine}$$

$$\frac{\rho_0'}{\rho_0} = 1.000050$$
ISOPYCNAL CHART
FOR AIR FLOW AROUND A SPHERE:
FROM INTERFEROMETER STUDY

\[ M = 1.70 \]

FIGURE 1B
VARIATION OF DENSITY ALONG AXIS OF SYMMETRY

DISTANCE IN MILLIMETERS ALONG AXIS OF SYMMETRY

FIGURE 2B
VARIATION OF TEMPERATURE AND VELOCITY ALONG AXIS OF SYMMETRY

$T_2$

$U_z$

Distance in Millimeters along Axis of Symmetry

Figure 3B
IDEALIZED VARIATION
OF STATE PARAMETERS
ALONG AXIS OF SYMMETRY

DISTANCE IN FEET
ALONG AXIS OF SYMMETRY

Figure 48