PARTICLE FLUID MECHANICS IN SHEAR FLOWS, ACOUSTIC WAVES, AND SHOCK WAVES

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ABSTRACT

Three different categories of flow problems of a fluid containing small particles are being considered here. They are: (i) a fluid containing small, non-reacting particles (Parts I and I); (ii) a fluid containing reacting particles (Parts III and IV); and (iii) a fluid containing particles of two distinct sizes with collisions between two groups of particles (Part V).

Part I

A numerical solution is obtained for a fluid containing small particles flowing over an infinite disc rotating at a constant angular velocity. It is a boundary layer type flow,and the boundary layer thickness for the mixture is estimated. For large Reynolds number, the solution suggests the boundary layer approximation of a fluid-particle mixture by assuming $W = W_p$. The error introduced is consistent with the Prandtl's boundary layer approximation. Outside the boundary layer, the flow field has to satisfy the "inviscid equation" in which the viscous stress terms are absent while the drag force between the particle cloud and the fluid is still important. Increase of particle concentration reduces the boundary layer thickness and the amount of mixture being transported outwardly is reduced. A new parameter, $\beta = 1/\Omega\tau_v$, is introduced which is also proportional to μ. The secondary flow of the particle cloud depends very much on β. For small values of β, the particle cloud velocity attains its maximum value on the surface of the disc,and for infinitely large values of β, both the radial and axial particle velocity components vanish on the surface of the disc.

Part II.

The "inviscid" equation for a gas-particle mixture is linearized to describe the flow over a wavy wall. Corresponding to the Prandtl-Glauert equation for pure gas, a fourth order partial differential equation in terms of the velocity potential φ is obtained for the mixture. The solution is obtained for the flow over a periodic wavy wall. For equilibrium flows where λ_v and λ_T approach zero and frozen flows in which λ_v and λ_T become infinitely large, the flow problem is basically similar to that obtained by Ackeret for a pure gas. For finite values of λ_v and λ_T, all quantities except v are not in phase with the wavy wall. Thus, the drag coefficient C_D is present even in the subsonic case, and similarly, all quantities decay exponentially for supersonic flows. The phase shift and the attenuation factor increase for increasing particle concentration.

Part III.

Using the boundary layer approximation, the initial development of the combustion zone between the laminar mixing of two parallel streams of oxidizing agent and small, solid, combustible particles suspended in an inert gas is investigated. For the special case when the two streams are moving at the same speed, a Green's function exists for the differential equations describing first order gas temperature and oxidizer concentration. Solutions in terms of error functions and exponential integrals are obtained. Reactions occur within a relatively thin region of the order of λ_D. Thus, it seems advantageous in the general study of two-dimensional laminar flame problems to intro-

duce a chemical boundary layer of thickness λ_D within which reactions take place. Outside this chemical boundary layer, the flow field corresponds to the ordinary fluid dynamics without chemical reaction.

Part IV.

The shock wave structure in a condensing medium of small liquid droplets suspended in a homogeneous gas-vapor mixture consists of the conventional compressive wave followed by a relaxation region in which the particle cloud and gas mixture attain momentum and thermal equilibrium. Immediately following the compressive wave, the partial pressure corresponding to the vapor concentration in the gas mixture is higher than the vapor pressure of the liquid droplets and condensation sets in. Farther downstream of the shock, evaporation appears when the particle temperature is raised by the hot surrounding gas mixture. The thickness of the condensation region depends very much on the latent heat. For relatively high latent heat, the condensation zone is small compared with Λ_D.

For solid particles suspended initially in an inert gas, the relaxation zone immediately following the compression wave consists of a region where the particle temperature is first being raised to its melting point. When the particles are totally melted as the particle temperature is further increased, evaporation of the particles also plays a role.

The equilibrium condition downstream of the shock can be calculated and is independent of the model of the particle - gas mixture interaction.

Part V.

For a gas containing particles of two distinct sizes and satisfying certain conditions, momentum transfer due to collisions between the two groups of particles can be taken into consideration using the classical elastic spherical ball model. Both in the relatively simple problem of normal shock wave and the perturbation solutions for the nozzle flow, the transfer of momentum due to collisions which decreases the velocity difference between the two groups of particles is clearly demonstrated. The difference in temperature as compared with the collisionless case is quite negligible.

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LIST OF SYMBOLS

a	speed of sound
a_e	equilibrium speed of sound
a_c	speed of sound in the rocket chamber
a_0	speed of sound at some reference state
c	specific heat of the particle cloud
c_1	specific heat of the particle cloud with radius
c_2	specific heat of the particle cloud with radius
c_l	specific heat of liquid particles
c_s	specific heat of solid particles
c_p	gas specific heat
$\mathcal{C}_i$	chemical symbol of the inert gas
$\mathcal{C}_o$	chemical symbol of the oxidizing agent
$\mathcal{C}_p$	chemical symbol of the particles
D	diffusion coefficient
F	forces exerted on a unit volume of fluid by particles
F_x, F_y, F_z	components of F along x, y, z axis
F_r, F_θ	radial and angular components of
$\mathcal{F}$	forces due to particle-particle collisions
$f_{\alpha i}$	components of "supply of momentum" for species
h	enthalpy of the gas or gas mixture
h_i, h_i^0	enthalpy and reference enthalpy of the inert gas
h_o, h_o^0	enthalpy and reference enthalpy of the oxidizing agent
h_p, h_p^0	enthalpy and reference enthalpy of the particle cloud
h_v, h_v^0	enthalpy and reference enthalpy of the vapor

h_p^s, h_{ps}^{s0}	enthalpy and reference enthalpy of the solid particles
h_p^l, h_p^{l0}	enthalpy and reference enthalpy of the liquid particles
h_α, h_α^0	enthalpy and reference enthalpy of species
$L^e(T_p)$	latent heat of evaporation at temperature
$L^s(T_p)$	latent heat of sublimation at temperature
Le	Lewis number
m_i	velocity components associated with the reacting gas
m_p	velocity components associated with the particle cloud
$\dot{m}$	mass flux
$\mathcal{M}_i$	atomic weight of the inert gas
$\mathcal{M}_o$	atomic weight of the oxidizing agent
$\mathcal{M}_p$	atomic weight of the particles
$m.$	mass of the particle
m_o	mass of the particles with radius σ_o
M	Mach number
M_e	equilibrium Mach number
M_f	frozen Mach number
M_o	reference Mach number
n_p	particle number density
P	gas pressure
P_R	pressure at which reaction takes place
Pr	Prandtl number
Q	rate of heat transfer between gas and particle cloud
Q_1	rate of heat transfer between gas and particle cloud of radius σ_1
Q_2	rate of heat transfer between gas and particle cloud of radius σ_2

R	universal gas constant
$\bar{R}$	gas constant of the mixture
Re	Reynolds number
Sc	Schmidt number
T	gas temperature
T_p	particle temperature
T_{p1}	temperature of particles of radius
T_{p2}	temperature of particles of radius
T_c	rocket chamber temperature
T_∞	equilibrium temperature of gas and particle cloud
T_R	temperature at which reaction occurs
T_s	$T - T_p$
T_{s1}	$T - T_{p1}$
T_{s2}	$T - T_{p2}$
$u_i : (u, v, w)$	gas velocity components
$u_{pi} : (u_p, v_p, w_p)$	particle velocity components
u_0	reference velocity
$u_{p\infty}$	equilibrium velocity
u_{p1}	x -component of particle velocity of radius σ_1
u_{p2}	x -component of particle velocity of radius σ_2
u_s	$u - u_p$
u_{s1}	$u - u_{p1}$
u_{s2}	$u - u_{p2}$
v_s	$v - v_p$
w_s	$w - w_p$
u_{s1}	$u - u_{p1}$
u_{s2}	$u - u_{p2}$

$\alpha = 1, 2, \cdots N$	gas species
β	(angular velocity of disc Ω × characteristic time $)^{-1}$
β_e	$(M_e^2 - 1)^{1/2}$
β_f	$(M_f^2 - 1)^{1/2}$
γ	specific heat ratio of gas
$\bar{\gamma}$	specific heat ratio of gas and particle in equilibrium
γ_p	$\nu_p m_p / \nu_0 m_0$
γ_i	$\nu_i m_i / \nu_0 m_0$
δ	boundary layer thickness
ϵ_α	"supply of energy" of species
κ, κ_1	vapor or oxidizer concentration
κ_1^*	vapor or oxidizer concentration in the concentration boundary layer
κ_e^l	equilibrium vapor concentration of liquid droplets
κ_e^s	equilibrium vapor concentration of solid particles
λ	characteristic length
λ_v	$\frac{ma}{6\pi\mu\sigma}$ or $\frac{mU_0}{6\pi\mu\sigma}$
λ_T	$\frac{3}{2} P_r \lambda_v$
λ_D	$\frac{3}{2} S_c \lambda_v$
λ_{v_0}	$\frac{m_0 a}{6\pi\mu\sigma_0}$ or $\frac{mU_0}{6\pi\mu\sigma_0}$
λ_{T_0}	$\frac{3}{2} \lambda_{v_0}$
λ_{D_0}	$\frac{3}{2} \lambda_{v_0}$
Λ_v	λ/λ_v
Λ_T	λ/λ_T

Λ_D	λ/λ_D
μ	coefficient of viscosity
μ_0	coefficient of viscosity at some reference state
ν	kinematic viscosity
$\bar{\nu}$	kinematic viscosity of mixture at equilibrium
ν_0, ν_0', ν_0''	stochiometric coefficient of oxidizer
ν_i, ν_i', ν_i''	stochiometric coefficient of inert gas
ν_p, ν_p', ν_p''	stochiometric coefficient of particle
ρ	gas density
ρ_p	particle density
ρ_{p1}	particle density of radius σ_1
ρ_{p2}	particle density of radius σ_2
$\bar{\rho}$	$\rho + \rho_p$
ρ_0	gas density at some reference state
ρ_∞	gas density at the equilibrium state
ρ_s	$1 - \rho_p/\chi\rho$
ρ_{s1}	$1 - \rho_{p1}/\chi_1\rho$
ρ_{s2}	$1 - \rho_{p2}/\chi_2\rho$
σ	particle radius
σ_1, σ_2	radii of particles of two distinct sizes
σ_0	particle radius at some reference state
σ_∞	particle radius when gas and particle cloud attain equilibrium
τ_v	$m/6\pi\mu\sigma$
τ_{ij}	stress tensor
τ_w	shear stress on the disc surface

φ	gas velocity potential
φ_p	particle velocity potential
χ	ρ_p/ρ at some reference state
χ_1	ρ_{p1}/ρ at some reference state
χ_2	ρ_{p2}/ρ at some reference state
χ_∞	ρ_p/ρ for equilibrium state
ω	rate of mass production for gas
ω_α	rate of mass production for species
ω_p	rate of mass production for particles
Ω	angular velocity of the disc

PART I.

LAMINAR FLOW OVER A ROTATING DISC

I. INTRODUCTION

Much has been accomplished in the study of the dynamics of fluids containing small solid particles. Because of the complexity of the problem, most solutions involve some perturbation scheme(1, 2) or approximation methods(3, 4). For a laminar flow of a fluid containing small solid particles over a disc rotating at constant velocity, a similarity solution can be found and the solution can be obtained by numerical integration.

Von Kármán(5) first investigated the flow of an incompressible, viscous fluid over an infinite disc rotating at a constant velocity. He conceived of a true similarity solution and was able to reduce the governing equations to a set of ordinary differential equations. Using the integral method, he was able to obtain an approximate solution. More accurate values were obtained by Cochran(6) who used numerical integration methods to find the solution close to the disc and matched to an asymptotic series solution valid far away from the disc. Cochran's solution, which is quoted in some textbooks(7), has a sign error in the pressure distribution.

The flow field obtained by von Kármán and Cochran is of a boundary layer type. In fact, von Kármán used this solution to check the accuracy of Prandtl's boundary layer equations, and confirmed the main assumptions of the boundary layer theory. Perhaps this rotating disc is also the first study involving secondary flows. Due to the boundary condition of the fluid on the disc, the layer of fluid close to the disc is constantly being transported in spiral paths along the surface from the axis of rotation to the outer edges. This is compensated by the fluid flowing along the axial direction towards the disc, and is again in turn being transported towards the outer edge. The disc thus acts very much like a centrifugal pump, and the solution illustrates very well the characteristics of a secondary flow in three-dimensional boundary layers.

A similarity solution still exists in the case when the fluid contains a cloud of small particles. The problem is certainly useful in designing centrifugal pumps and other areas of chemical engineering. The solution illustrates particle-fluid boundary layer flows as well as the secondary flows involving a cloud of small particles.

II. BASIC EQUATIONS

The governing equations describing the flow of a fluid containing small particles have been discussed by many authors. Here, we will mainly follow those given by Marble$^{(1)}$. The following problem is being investigated. A half infinite space $z > 0$ is filled with an incompressible fluid containing small particles. The boundary plane $z = 0$ is rotating with a constant angular velocity Ω . From symmetry arguments, we can easily see that all the quantities are independent of the angle θ .

Denote (u, v, w) as the radial, angular, and axial velocities of the fluid and (u_p, v_p, w_p) those of the particle cloud, and the mass density of the fluid and the particle cloud by ρ and ρ_p . The continuity equations for the two phases are

$$\frac{\partial}{\partial r}(r u) + \frac{\partial}{\partial z}(r w) = 0 \tag{1. 1}$$

$$\frac{\partial}{\partial r}(r \rho_p u_p) + \frac{\partial}{\partial z}(r \rho_p w_p) = 0 \tag{1. 2}$$

The corresponding momentum equations are

$$u\frac{\partial u}{\partial r} - \frac{v^2}{r} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r}\frac{u}{r} + \frac{\partial^2 u}{\partial z^2}\right) + F_r \tag{1. 3}$$

$$u\frac{\partial v}{\partial r} + \frac{uv}{r} + w\frac{\partial v}{\partial z} = \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{\partial}{\partial r}\frac{v}{r} + \frac{\partial^2 v}{\partial z^2}\right) + F_\theta \tag{1. 4}$$

$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right) + F_z \tag{1. 5}$$

$$u_p \frac{\partial u_p}{\partial r} - \frac{v_p^2}{r} + w_p \frac{\partial u_p}{\partial z} = -\frac{1}{\rho_p} F_r \qquad (1.6)$$

$$u_p \frac{\partial v_p}{\partial r} + \frac{u_p v_p}{r} + w_p \frac{\partial v_p}{\partial z} = -\frac{1}{\rho_p} F_\theta \qquad (1.7)$$

$$u_p \frac{\partial w_p}{\partial r} + w_p \frac{\partial w_p}{\partial z} = -\frac{1}{\rho_p} F_z \qquad (1.8)$$

where p is the local gas pressure and F_r , F_θ , F_z are the forces in the radial, angular, and axial directions exerted on a unit volume of fluid by the particles. Using Stokes drag law as a first order approximation to describe the particle-fluid interaction,

$$F_r = n_p 6 \pi \mu \sigma (u_p - u) = \frac{1}{\tau_v} \rho_p (u_p - u) \qquad (1.9)$$

where n_p is the particle number density of the particle radius σ and mass m and $n_p m = \rho_p$. τ_v is defined as

$$\tau_v = m / 6 \pi \mu \sigma \qquad (1.10)$$

Physically, τ_v is a characteristic time during which the relative velocity between the fluid and the particles is reduced to e^{-1} of its initial value when Stokes drag is the only force acting on the particle cloud.

Similarly, F_θ and F_z are

$$F_\theta = \frac{1}{\tau_v} \rho_p (v_p - v) \qquad (1.11)$$

$$F_z = \frac{1}{\tau_v} \rho_p (w_p - w) \qquad (1.12)$$

The boundary conditions are:

$$
\begin{aligned}
&u(r,0)=0 && u(r,\infty)=0 \\
&v(r,0)=\Omega r && v(r,\infty)=0 \\
&w(r,0)=0 && w(r,\infty)=\text{CONSTANT} \\
& && p(r,\infty)=p_\infty \\
&u_p(r,\infty)=0 && \\
&v_p(r,\infty)=0 && \\
&w_p(r,\infty)=w(r,\infty) && \\
&\rho_p(r,\infty)=\rho\chi &&
\end{aligned}
\tag{1.13}
$$

$w(r,\infty)$ has a finite limiting value because the mass flux $\rho(r,\infty)\,w(r,\infty)$ balances the radial outflow near the disc as discussed in the previous section. χ is the density ratio at $z=\infty$, $\chi=\rho_p(\infty)/\rho$.

III. SOLUTION FOR THE EQUILIBRIUM FLOW

From the definition of τ_v, (eq. 1.10), the limiting case when $\tau_v \to 0$ corresponds to either very small particles or the viscosity of the fluid μ is very large. Under such circumstances, the velocity difference between the particle cloud and the fluid is expected to be very small. This is the case when we examine equations 1.6, 1.7, and 1.8. Define

$$\begin{aligned} U_s &= U_p - U \\ V_s &= V_p - V \\ W_s &= W_p - W \end{aligned} \qquad (1.14)$$

The momentum equations for the particle cloud can be expressed as

$$U \frac{\partial U}{\partial r} - \frac{V^2}{r} + W \frac{\partial U}{\partial z} = -\frac{U_s}{\tau_v} + O(U_s) \qquad (1.15)$$

$$U \frac{\partial V}{\partial r} + \frac{UV}{r} + W \frac{\partial V}{\partial z} = -\frac{V_s}{\tau_v} + O(V_s) \qquad (1.16)$$

$$U \frac{\partial W}{\partial r} + W \frac{\partial W}{\partial z} = -\frac{W_s}{\tau_v} + O(W_s) \qquad (1.17)$$

The continuity equation is

$$\frac{\partial}{\partial r}(r U \rho_p) + \frac{\partial}{\partial z}(r W \rho_p) = O(U_s, W_s) \qquad (1.18)$$

Since both the velocities and their spatial derivatives are finite, the limiting case when $\tau_v \to 0$ requires $U_s \to 0$, $V_s \to 0$, and $W_s \to 0$. Thus, equations 1.15 - 1.17 are reduced to

$$U \frac{\partial U}{\partial r} - \frac{V^2}{r} + W \frac{\partial U}{\partial z} = -\frac{U_s}{\tau_v} = -\frac{U_p - U}{\tau_v} \qquad (1.19)$$

$$u\frac{\partial V}{\partial r} + \frac{uV}{r} + W\frac{\partial V}{\partial r} = -\frac{V_s}{\tau_v} = -\frac{V_p - V}{\tau_v} \qquad (1.20)$$

$$u\frac{\partial W}{\partial r} \qquad + W\frac{\partial W}{\partial z} = -\frac{W_s}{\tau_v} = -\frac{W_p - W}{\tau_v} \qquad (1.21)$$

From the continuity equation we observe that $\rho_p = constant = \rho \chi$.

The momentum equations for the whole system, in the limiting case when $\tau_v \rightarrow 0$, can be expressed as

$$u\frac{\partial u}{\partial r} - \frac{V^2}{r} + W\frac{\partial u}{\partial z} = -\frac{1}{\rho^*}\frac{\partial P}{\partial r} + \nu^*\left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r}\frac{u}{r} + \frac{\partial^2 u}{\partial z^2}\right) \qquad (1.22)$$

$$u\frac{\partial V}{\partial r} + \frac{uV}{r^2} + W\frac{\partial V}{\partial z} = \qquad \nu^*\left(\frac{\partial^2 V}{\partial z^2} + \frac{\partial}{\partial r}\frac{V}{r} + \frac{\partial^2 V}{\partial z^2}\right) \qquad (1.23)$$

$$u\frac{\partial W}{\partial r} \qquad + W\frac{\partial W}{\partial z} = -\frac{1}{\rho^*}\frac{\partial P}{\partial z} + \nu^*\left(\frac{\partial^2 W}{\partial z^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{\partial^2 W}{\partial z^2}\right) \qquad (1.24)$$

where

$$\rho^* = \rho + \rho_p(\infty) = \rho(1 + \chi)$$

$$\nu^* = \mu/(\rho + \rho_p(\infty)) = \mu/(1+\chi)\rho \qquad (1.25)$$

The solution for $\tau_v \rightarrow 0$ is therefore identical with the flow without particles with an effective density $\rho^* = \rho(1 + \chi)$. In the literature, this limiting case is usually referred to as the "equilibrium flow."

The other limiting case when $\tau_v \rightarrow \infty$, the presence of the particle cloud does not change the fluid quantities. The particles are moving with constant velocities $U_p = 0$, $V_p = 0$, $W_p = W_p(r, \infty)$ and $\rho_p = \rho\chi$. This case is usually referred to as the "frozen flow."

IV. METHOD OF SOLUTION

To investigate the problem for finite values of T_v, it is convenient to introduce a dimensionless distance variable η. η is defined as

$$\eta = \left(\frac{\Omega}{\nu}\right)^{1/2} z \tag{1.26}$$

Furthermore, we assume that the velocity components as well as the particle density ρ_p and the local fluid pressure p can be expressed in the following form:

$$\begin{aligned}
u(r, z) &= \Omega\, r\, F(\eta) \\
v(r, z) &= \Omega\, r\, G(\eta) \\
w(r, z) &= (\Omega \nu)^{1/2} H(\eta) \\
p(r, z) - p(r, \infty) &= \rho \nu \Omega\, P(\eta) \\
\rho_p(r, z)/\rho(r, z) &= \rho_r(r, z) = Q(\eta) \\
\rho_r(r, z)\, u_p(r, z) &= \Omega\, r\, F_p(\eta) \\
\rho_r(r, z)\, v_p(r, z) &= \Omega\, r\, G_p(\eta) \\
\rho_r(r, z)\, w_p(r, z) &= (\Omega \nu)^{1/2} H_p(\eta)
\end{aligned} \tag{1.27}$$

Using 1.26 and 1.27, eqs. 1.1 to 1.8 can be reduced to a set of ordinary differential equations

$$\frac{dH}{d\eta} + 2F = 0 \tag{1.28}$$

$$\frac{d^2F}{d\eta^2} - F^2 - H\frac{dF}{d\eta} + G^2 + \beta(F_p - QF) = 0 \tag{1.29}$$

$$\frac{d^2G}{d\eta^2} - 2FG - H\frac{dG}{d\eta} + \beta(G_p - QG) = 0 \tag{1.30}$$

$$\frac{dP}{d\eta} - 2HF + 2\frac{dE}{d\eta} + \beta(-H_p + QH) = 0 \tag{1.31}$$

$$\frac{dH_p}{d\eta} + 2F_p = 0$$

$$-H_p Q\frac{dF_p}{d\eta} + H_p\frac{dQ}{d\eta}F_p + G_p^2 Q - QF_p^2 - \beta(Q^2F_p - Q^3F) = 0 \tag{1.32}$$

$$-H_p Q\frac{dG_p}{d\eta} - 2QG_pF_p + H_pG_p\frac{dQ}{d\eta} - \beta(Q^2G_p - Q^3G) = 0 \tag{1.33}$$

$$H_p^2\frac{dQ}{d\eta} + 2H_pQF_p - \beta(Q^2H_p - Q^3H) = 0 \tag{1.34}$$

and

$$\beta = (\Omega\tau_v)^{-1} \tag{1.35}$$

The boundary conditions according to equations 1.13 read

$$\begin{aligned}
F(0) &= 0 & F(\infty) &= 0 \\
G(0) &= 1 & G(\infty) &= 0 \\
H(0) &= 0 \\
P(\infty) &= 0 \\
Q(\infty) &= \chi \\
F_p(\infty) &= 0 \\
G_p(\infty) &= 0 \\
H_p(\infty) &= \chi H(\infty)
\end{aligned} \tag{1.36}$$

Since the problem involves a set of non-linear differential equations, numerical integration is the only means to find an exact solution. The problem is further complicated in that it is a two point boundary value problem. From eq. 1.36 we observe that three of the ten boundary conditions are prescribed at $\eta = 0$ while the remaining seven are at $\eta = \infty$. Thus, we propose to integrate the equations numerically from large values of η by first finding an asymptotic solution which is valid for large η . The asymptotic solution involves three parameters, namely A , B , and C . By varying A , B , and C , we can satisfy the boundary conditions $F(0) = 0$, $G(0) = 1$, and $H(0) = 0$.

To find the asymptotic solution for large values of η , we assume

$$
\begin{aligned}
F(\eta) &= F^{(0)}(\eta) + \epsilon F^{(1)}(\eta) + \epsilon^2 F^{(2)}(\eta) \\
G(\eta) &= G^{(0)}(\eta) + \epsilon G^{(1)}(\eta) + \epsilon^2 G^{(2)}(\eta) \\
H(\eta) &= H^{(0)}(\eta) + \epsilon H^{(1)}(\eta) + \epsilon^2 H^2(\eta) \\
F_p(\eta) &= F_p^{(0)}(\eta) + \epsilon F_p^{(1)}(\eta) + \epsilon^2 F_p^{(2)}(\eta) \\
G_p(\eta) &= G_p^{(0)}(\eta) + \epsilon G_p^{(1)}(\eta) + \epsilon^2 G_p^{(2)}(\eta) \\
H_p(\eta) &= H_p^{(0)}(\eta) + \epsilon H_p^{(1)}(\eta) + \epsilon^2 H_p^{(2)}(\eta) \\
Q(\eta) &= Q^{(0)}(\eta) + \epsilon Q^{(1)}(\eta) + \epsilon^2 Q^{(2)}(\eta) \\
P(\eta) &= P^{(0)}(\eta) + \epsilon P^{(1)}(\eta) + \epsilon^2 P^{(2)}(\eta)
\end{aligned}
\qquad (1.37)
$$

From the boundary condition 1.36, we obtain

$$\begin{aligned} F^{(0)}(\eta) &= 0 \\ G^{(0)}(\eta) &= 0 \\ H^{(0)}(\eta) &= H(\infty) = -C \\ F_p^{(0)}(\eta) &= 0 \\ G_p^{(0)}(\eta) &= 0 \\ H_p^{(0)}(\eta) &= -\chi C \\ Q^{(0)}(\eta) &= \chi \\ P^{(0)}(\eta) &= 0 \end{aligned} \tag{1.38}$$

C is related to the fluid velocity along the axial direction for $\eta = \infty$, and is a function of χ.

Substitute eqs. 1.37 into eqs. 1.28 to 1.34; using 1.38, we obtain the first order equations.

$$\frac{dH^{(1)}}{d\eta} + 2F^{(1)} = 0 \tag{1.39}$$

$$\frac{d^2F^{(1)}}{d\eta^2} + C\frac{dF^{(1)}}{d\eta} + \beta(F_p^{(1)} - \chi F^{(1)}) = 0 \tag{1.40}$$

$$\frac{d^2G^{(1)}}{d\eta^2} + C\frac{dG^{(1)}}{d\eta} + \beta(G_p^{(1)} - \chi G^{(1)}) = 0 \tag{1.41}$$

$$\frac{d^2P^{(1)}}{d\eta} + 2CF^{(1)} + 2\frac{dF^{(1)}}{d\eta} + \beta(-H_p^{(1)} - C\,Q^{(1)}) = 0 \tag{1.42}$$

$$-C\frac{dF_p^{(1)}}{d\eta} + \beta(F_p^{(1)} - \chi F^{(1)}) = 0 \tag{1.43}$$

$$-c\frac{dG_p^{(1)}}{d\eta} + \beta(G_p^{(1)} - \chi G^{(1)}) = 0 \tag{1.44}$$

$$c^2\frac{dQ^{(1)}}{d\eta} - 2cF_p^{(1)} - \beta(H_p^{(1)} - 2cQ^{(1)} - \chi H^{(1)} + 3cQ^{(1)}) = 0 \tag{1.45}$$

$$\frac{dH_p^{(1)}}{d\eta} + 2F_p^{(1)} = 0 \tag{1.46}$$

The boundary conditions are simply

$$\begin{aligned}
&H^{(1)}(\infty) = 0\\
&F^{(1)}(\infty) = 0\\
&G^{(1)}(\infty) = 0\\
&P^{(1)}(\infty) = 0\\
&F_p^{(1)}(\infty) = 0\\
&G_p^{(1)}(\infty) = 0\\
&H_p^{(1)}(\infty) = 0\\
&Q^{(1)}(\infty) = 0\\
&\frac{dF^{(1)}}{d\eta}(\infty) = 0\\
&\frac{dG^{(1)}}{d\eta}(\infty) = 0
\end{aligned} \tag{1.47}$$

Solutions to eqs. 1.39 to 1.46 can very easily be found. They are

$$F^{(1)}(\eta) = Ae^{s\eta}$$

$$G^{(1)}(\eta) = Be^{s\eta}$$

$$\frac{dF^{(1)}}{d\eta}(\eta) = sAe^{s\eta}$$

$$\frac{dG^{(1)}(\eta)}{d\eta} = sBe^{s\eta}$$

$$H^{(1)}(\eta) = -\frac{2}{s}Ae^{s\eta}$$

$$F_p^{(1)}(\eta) = -(1+\frac{s}{c})Ae^{s\eta} \tag{1.48}$$

$$G_p^{(1)}(\eta) = -(1+\frac{s}{c})Be^{s\eta}$$

$$H_p^{(1)}(\eta) = \frac{2}{s}(1+\frac{s}{c})Ae^{s\eta}$$

$$Q^{(1)}(\eta) = 0$$

$$P^{(1)}(\eta) = \frac{1}{s}\left[\frac{2AB\chi}{s} - 2AC - 2AS - A\beta(1+\frac{s}{c})\right]e^{s\eta}$$

with

$$s = -\frac{1}{2}\left[(c-\frac{\beta}{c}) + \left\{((c-\frac{\beta}{c})^2 + 4(\beta+\chi\beta)\right\}^{1/2}\right] \tag{1.49}$$

Using the zeroth order and first order solutions, we can easily find second order quantities $F^{(2)}(\eta)$, etc. It is apparent that the second order functions are of order $e^{2s\eta}$, and third order functions are of order $e^{3s\eta}$. Thus, for large values of η, we have

$$
\begin{aligned}
F(\eta) &= Ae^{s\eta} + O(e^{2s\eta}) \\
G(\eta) &= Be^{s\eta} + O(e^{2s\eta}) \\
\frac{dF(\eta)}{d\eta} &= sAe^{s\eta} + O(e^{2s\eta}) \\
\frac{dG(\eta)}{d\eta} &= sBe^{s\eta} + O(e^{2s\eta}) \\
H(\eta) &= -C - \frac{2}{s}Ae^{s\eta} + O(e^{2s\eta}) \\
F_p(\eta) &= -(1+s/c)Ae^{s\eta} + O(e^{2s\eta}) \qquad (1.50) \\
G_p(\eta) &= -(1+s/c)Be^{s\eta} + O(e^{2s\eta}) \\
H_p(\eta) &= -\chi C + \frac{2}{s}(1+s/c)Ae^{s\eta} + O(e^{2s\eta}) \\
Q(\eta) &= \chi + O(e^{2s\eta}) \\
P(\eta) &= \frac{1}{s}\left[\frac{2A\chi B}{s} - 2AC - 2As - A\beta(1+s/c)\right]e^{s\eta} + O(e^{2s\eta})
\end{aligned}
$$

Numerical integration is carried out from the point $\eta = \eta_0$. η_0 is chosen such that $e^{2s\eta_0}$ is negligible. The expressions in 1.50 contain three parameters A, B, and C. The boundary conditions at $\eta = 0$ are satisfied by varying A, B, and C. An iteration procedure is used by considering $F(\eta=0)$, $G(\eta=0)$, and $H(\eta=0)$ as functions of A, B, and C.

$$\begin{aligned} F(\eta=0) &= F(A,B,C) \\ G(\eta=0) &= G(A,B,C) \\ H(\eta=0) &= H(A,B,C) \end{aligned} \tag{1.51}$$

With initial trial values of A_0, B_0, and C_0, the boundary values at η_0 are obtained using 1.50. Equations 1.28 - 1.34 are integrated numerically from $\eta = \eta_0$. The values for $F(\eta)$, $G(\eta)$, and $H(\eta)$ at $\eta = 0$ are $F(A_0, B_0, C_0)$, $G(A_0, B_0, C_0)$, $H(A_0, B_0, C_0)$. Corrections to A_0, B_0, and C_0 are to be found such that $F(\eta)$, $G(\eta)$, and $H(\eta)$ satisfy the boundary conditions at $\eta = 0$. Let h_1, h_2, and h_3 be the corrections to A_0, B_0, and C_0. Thus

$$\begin{aligned} F(A_0+h_1, B_0+h_2, C_0+h_3) &\simeq F(A_0,B_0,C_0) + \frac{\partial F}{\partial A} h_1 + \frac{\partial F}{\partial B} h_2 + \frac{\partial F}{\partial C} h_3 = 0 \\ G(A_0+h_1, B_0+h_2, C_0+h_3) &\simeq G(A_0,B_0,C_0) + \frac{\partial G}{\partial A} h_1 + \frac{\partial G}{\partial B} h_2 + \frac{\partial G}{\partial C} h_3 = 1 \\ H(A_0+h_1 + B_0+h_2 + C_0+h_3) &\simeq H(A_0,B_0,C_0) + \frac{\partial H}{\partial A} h_1 + \frac{\partial H}{\partial B} h_2 + \frac{\partial H}{\partial C} h_3 = 0 \end{aligned} \tag{1.52}$$

Thus

$$\frac{\partial F}{\partial A} h_1 + \frac{\partial F}{\partial B} h_2 + \frac{\partial F}{\partial C} h_3 = -F(A_0, B_0, C_0) \tag{1.53}$$

$$\frac{\partial G}{\partial A} h_1 + \frac{\partial G}{\partial B} h_2 + \frac{\partial G}{\partial C} h_3 = 1 - G(A_0, B_0, C_0) \tag{1.54}$$

$$\frac{\partial H}{\partial A}h_1 + \frac{\partial H}{\partial B}h_2 + \frac{\partial H}{\partial C}h_3 = -H(A_0, B_0, C_0) \tag{1.55}$$

The derivatives can be found numerically and h_1 , h_2 , and h_3 can be obtained by solving eqns. 1.53 - 1.55. The procedure can be repeated until the boundary conditions at $\eta = 0$ are satisfied. This method converges rather fast and is usually within one per cent error after two or three iterations.

Numerical integration is performed by using California Institute of Technology's IBM 7094 using DEQ subroutine. Two sets of solutions were obtained by using various values of χ and β , where χ and β were defined previously as

$$\chi = \rho_p(\infty)/\rho(\infty)$$
$$\beta = (\Omega \tau_v)^{-1}$$

The first set is for $\beta = 0.5$ and values of χ equal to 0.0, 0.5, 1.0, 1.5, 2.0, and 2.5 were used. The second set is for $\chi = 0.5$ and values of $\beta = 0.0$, 0.2, 0.5, 1.0, 2.0, and ∞ were used. The numerical solutions were plotted on Figures I-1 to I-12. If we define Δ as

$$\Delta = [F^2(0) + (G(0)-1)^2 + H^2(0)]^{1/2} \tag{1.56}$$

the solution is accurate to $\Delta \leq 0.01$.

V. DISCUSSION

From the curves describing the velocities and densities, both for the particle cloud and the fluid, this problem of a rotating disc is very much a boundary layer type flow. Furthermore, the boundary layer thickness for the fluid and the particle cloud is approximately of the same order. This thickness, δ , corresponding to the layer of mixture carried around by the disc, can be estimated by equating the centrifugal force and the shearing stresses acting on the mixture(8). The centrifugal force acting on an element $(\rho_p+\rho)\, dr\, r\, d\theta\, \delta$ at a distance r from the axis is $(\rho_p+\rho)\, r\, \Omega^2\, dr\, r d\theta\, \delta$. The same element is also acted upon by a shearing stress τ_w . If ϕ is the angle between τ_w and the circumferential velocity, the net force on the element due to the shearing stress is $\tau_w \sin\phi\, dr\, r d\theta$. τ_w is proportional to the velocity gradient,

$$\tau_w \cos\phi \sim \mu\, r\, \Omega / \delta$$

Equating the two forces

$$(\rho_p+\rho)\, r \left(\frac{V_0}{R_0}\right)^2 dr\, r d\theta\, \delta \sim \mu\, r\, \Omega \tan\phi\, r d\theta\, dr \frac{1}{\delta}$$

or

$$\delta^2 = \nu^* \tan\phi \frac{1}{\Omega}$$

If it is assumed that the direction of slip close to the wall is approximately a constant, the thickness is

$$\delta \sim \left(\frac{\nu^*}{\Omega}\right)^{1/2} = \frac{r}{Re^{1/2}} \tag{1.57}$$

where Re is the Reynolds number defined as $\frac{\nu^* r^2}{\Omega}$. Thus, in the case of large Reynolds number, the effect of the wall on the mixture is confined within a thin layer of thickness δ . The axial components of the fluid velocity and the particle cloud velocity as compared with the radial or angular components are of the order $Re^{1/2}$ within this boundary layer δ . Thus, Prandtl's boundary layer theory can very easily be extended to the case of a fluid containing small particles by suppressing the axial velocities of both phases. Corresponding to neglecting $\frac{\partial P}{\partial y}$ in the boundary layer theory, an additional relationship

$$w(r,z) - w_p(r,z) = 0 \tag{1.58}$$

is introduced. The error in assuming eq. 1.58 is of the order $Re^{1/2}$, which is consistent with the boundary layer theory. Outside this boundary layer δ , the flow field has to satisfy the "inviscid" equations in which the viscous stress terms are absent. These "inviscid" equations correspond to the Euler's equations in Prandtl's boundary layer theory.

The boundary layer thickness δ decreases with increasing particle concentration. This implies that the amount of particle-fluid mixture being transported outwardly is reduced, which in turn reduces the velocity component in the axial direction flowing towards the disc as shown in Figure I. 1 and Figure I. 3.

The secondary flows for the particle cloud and the fluid are quite different for small values of β . There are two forces acting on the particles -- the particle-fluid drag force and the centrifugal force. Close to the disc, the centrifugal force dominates and particles are constantly being thrown outward. The radial velocity of the particle cloud acquires a maximum on the disc (Figures I. 4 and I. 10), while the fluid radial velocity is zero because of the boundary condition at the wall. For larger values of β , the particle- fluid force plays a more important role. For $\beta = 2.0$, the radial velocity of the particle cloud acquires a maximum away from the disc, as shown in Figure I. 10.

When the radius R_0 of the disc is large compared with the boundary layer thickness, as is almost always the case in practice, the boundary conditions along the edge of the disc can be neglected. The solution for the infinite disc can be used in this case, and the turning moment M for the disc can be evaluated.

$$M = -\int_0^{R_0} 2\pi r^2 \tau_{\theta z}\, dr$$

The shearing stress $\tau_{\theta z}$ is $\mu \frac{\partial V}{\partial z}$. Using eq. 1. 27, the shearing stress can be expressed as

$$\tau_{\theta z} = \left(\frac{V_0}{R_0}\right)^{3/2}\left(\frac{1}{\nu}\right)^{1/2} r \left.\frac{dG}{d\eta}\right|_{\eta=0} = \Omega^{3/2}\left(\frac{1}{\nu}\right)^{1/2} r \left.\frac{dG}{d\eta}\right|_{\eta=0}$$

Introducing a dimensionless coefficient of turning moment

$$C_M = \frac{M}{\frac{1}{2}\rho\,\Omega^2 R_0^5}$$

this gives

$$C_M = \frac{-\pi \left.\frac{dG}{d\eta}\right|_0}{(R_0^2\Omega/\nu)^{1/2}}$$

$$= \frac{-\pi \left.\frac{dG}{d\eta}\right|_0}{R_e^{\circ}} \qquad (1.59)$$

The coefficient of turning moment is plotted on Figure I. 12. If the disc is immersed in a fluid containing small particles, the net moment is twice as large.

The problem is solved under the assumption that the amount of particles reflected from the disc is negligible. From Figures I. 3 and

I. 9, we see that W_p on the disc is indeed small except for small values of β. Furthermore, since $W_p \sim Re^{-1/2}$, this assumption is certainly justified for large values of Reynolds number. In fact, in the boundary layer approximation to the particle-fluid dynamics, to be consistent with the boundary layer assumptions, the particles reflected from the wall are neglected. To study the problem with particles colliding with the disc, additional differential equations describing the reflected particles are necessary with proper particle-disc collision model. Numerical integration in this case will be very much involved.

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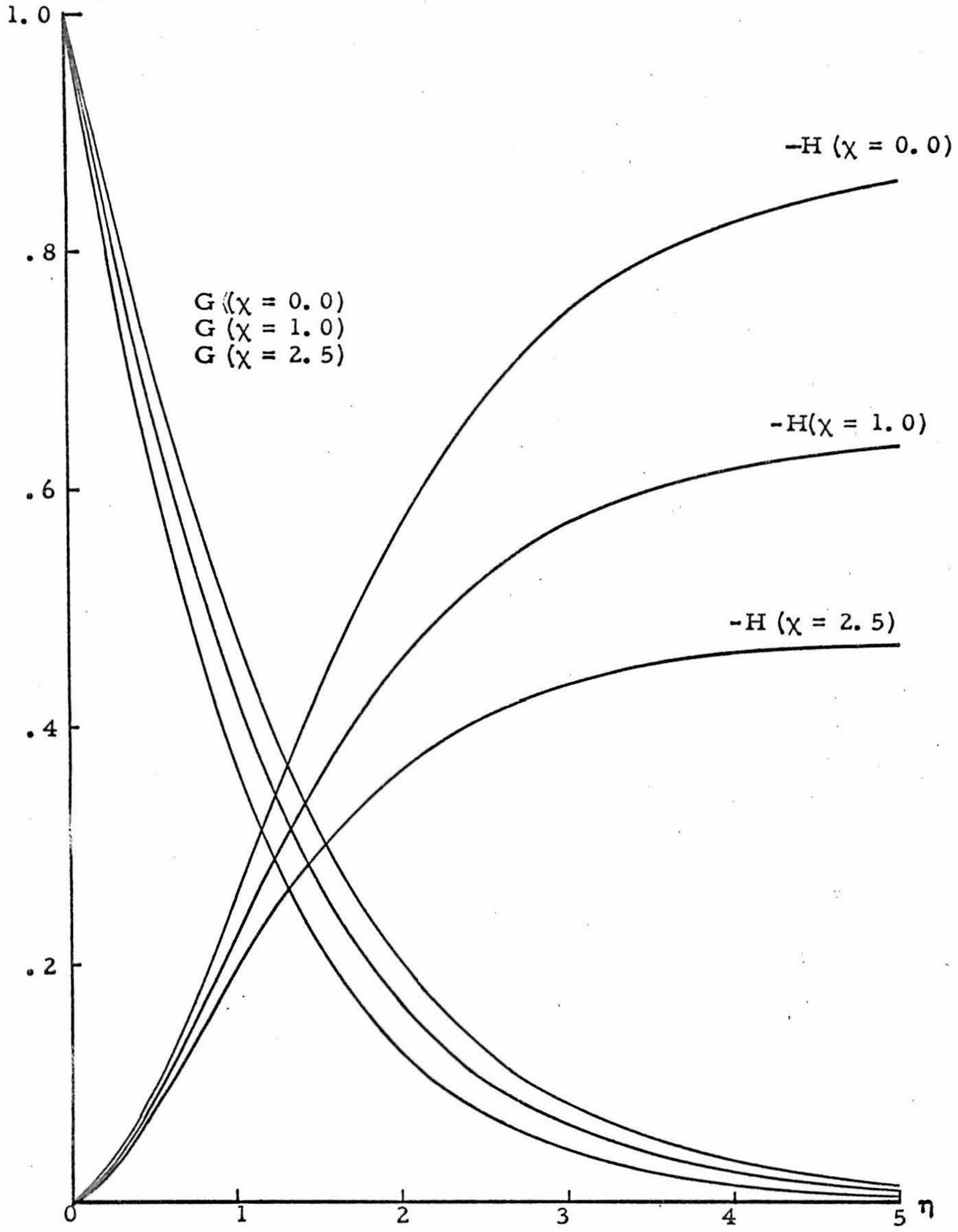

Fig. II.1. Normalized Fluid Velocities in Tangential and Axial Directions ($\beta = 0.5$).

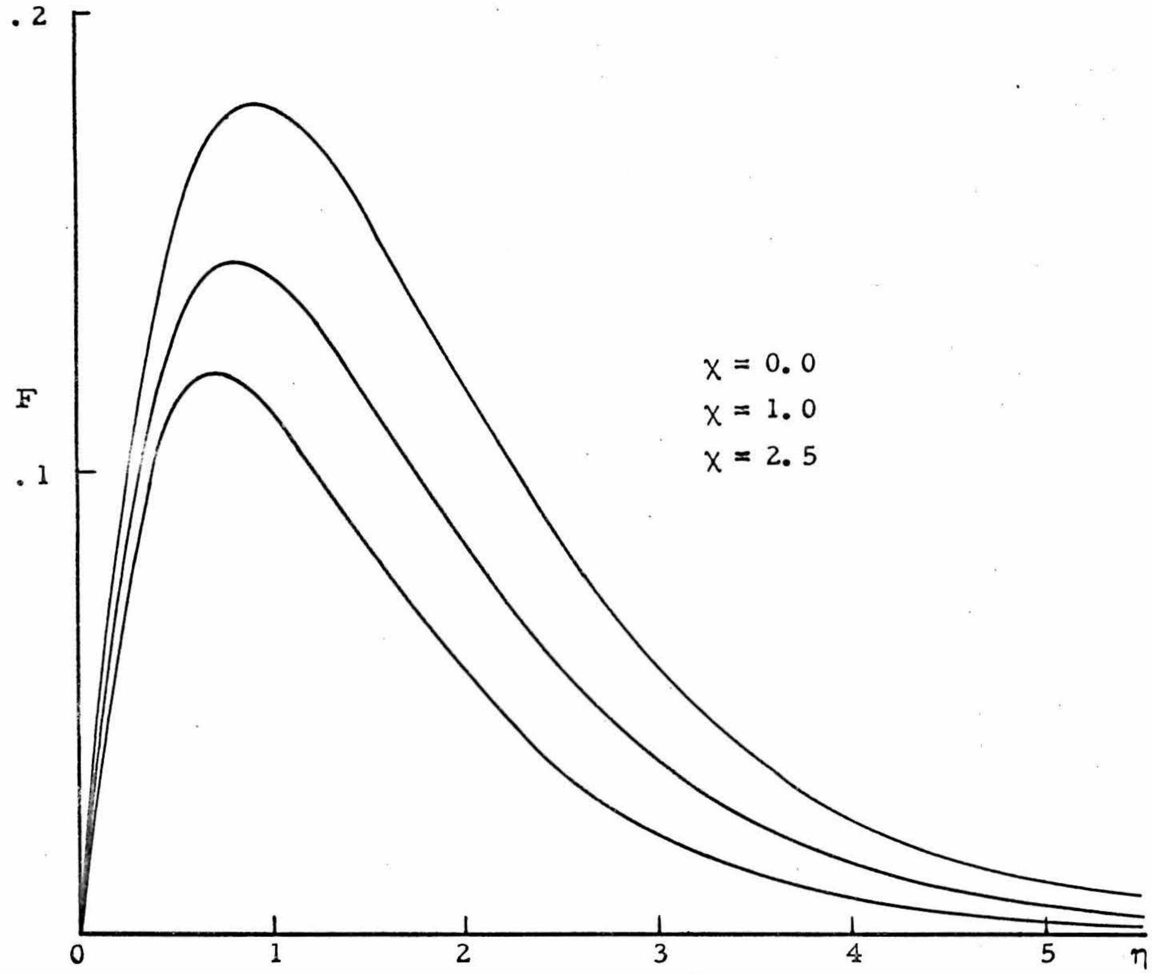

Fig. 1.2. Normalized Fluid Velocity in Radial Direction ($\beta = 0.5$).

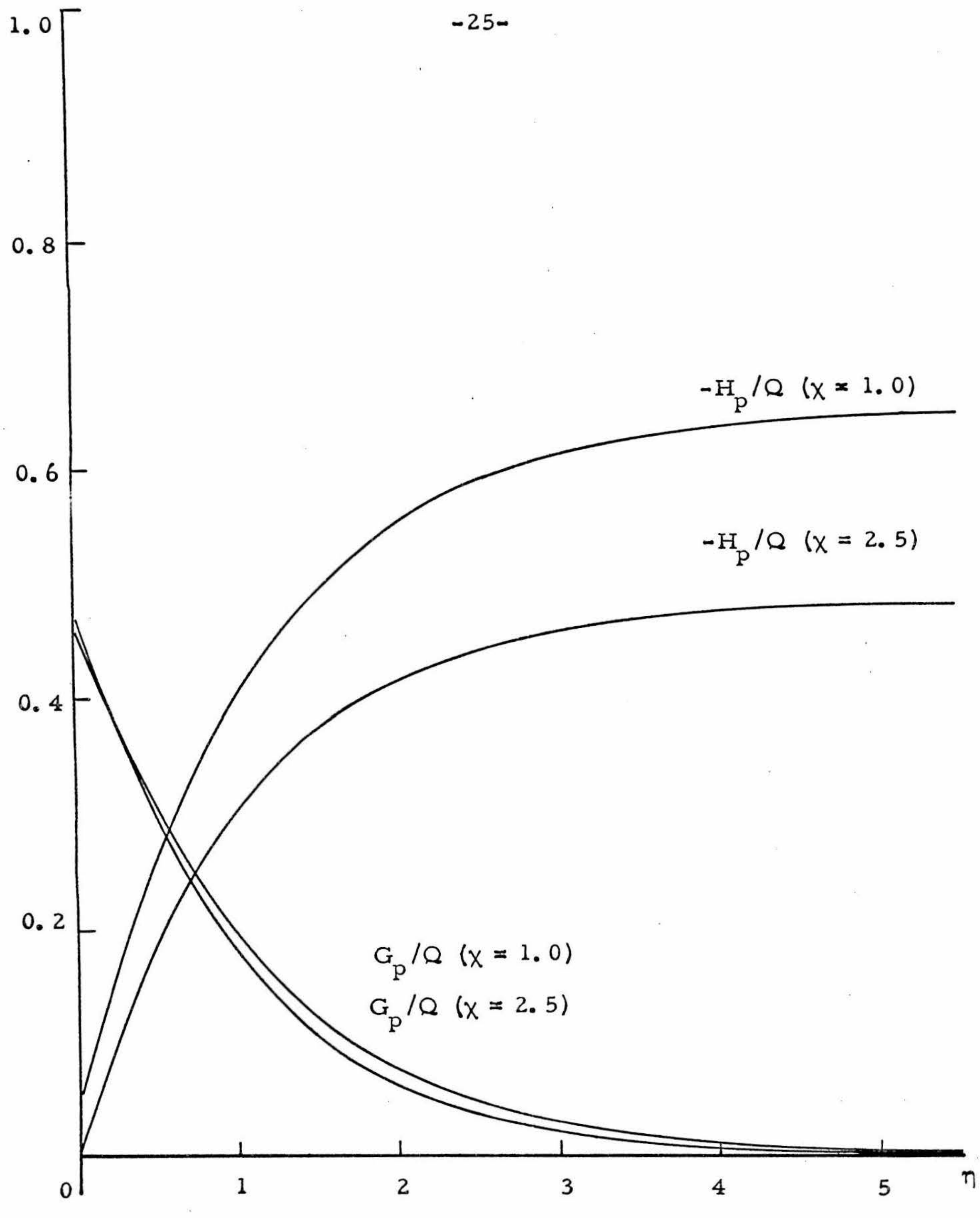

Fig. 1.3. Normalized Particle Velocities in Tangential and Axial Directions (β = 0.5).

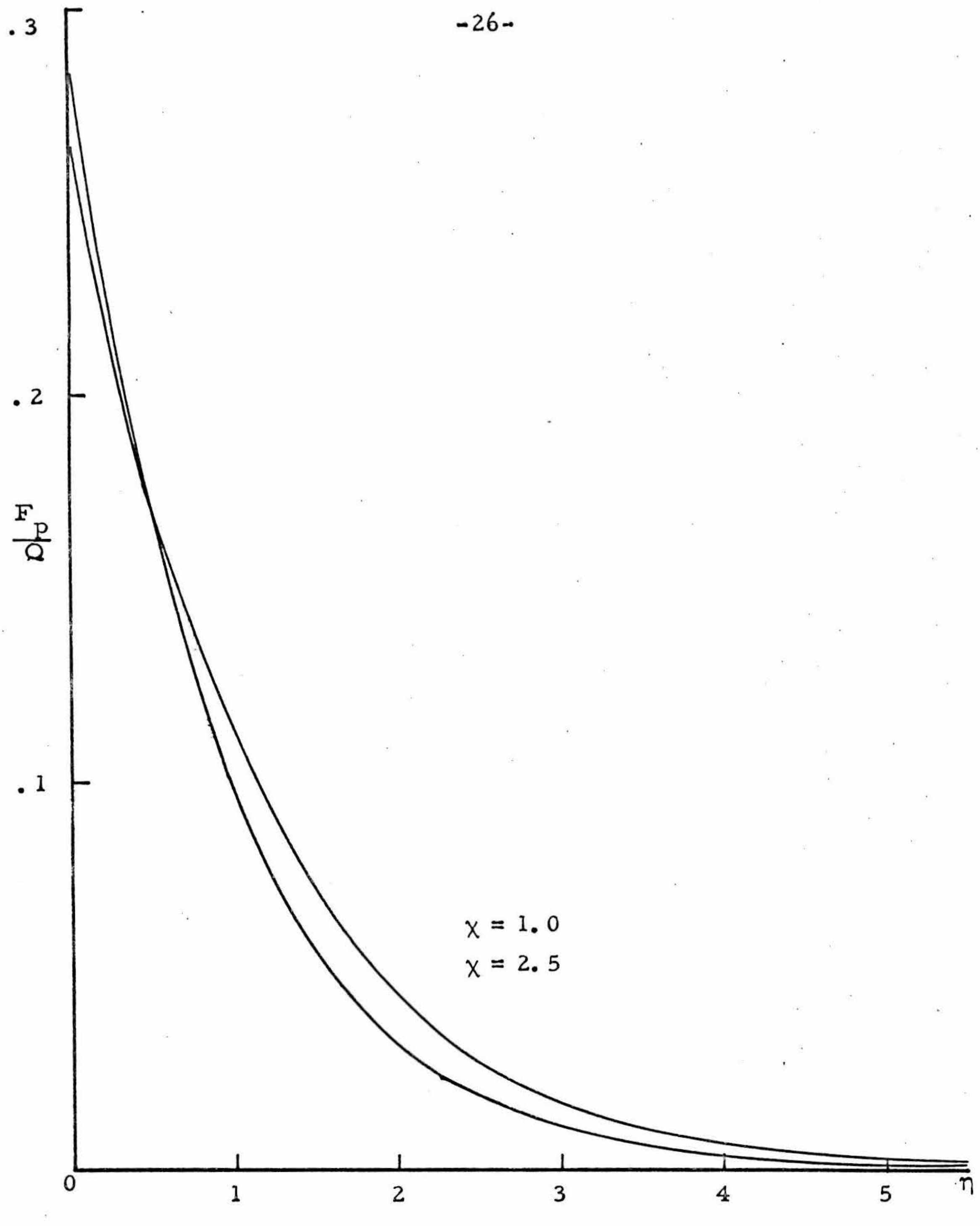

Fig. 1.4. Normalized Particle Velocity in Radial Direction ($\beta = 0.5$).

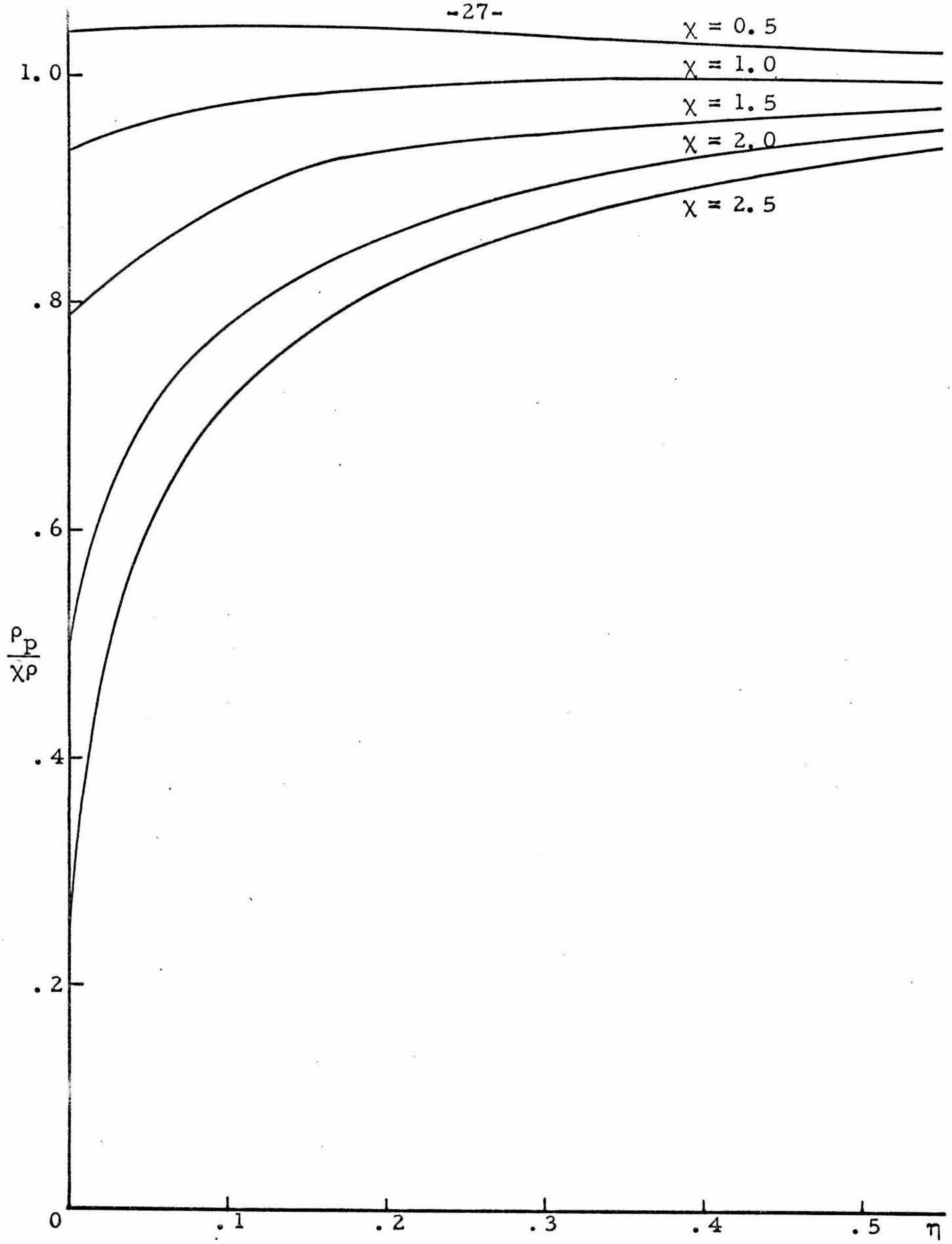

Fig. 1.5. Particle Density (β = 0.5).

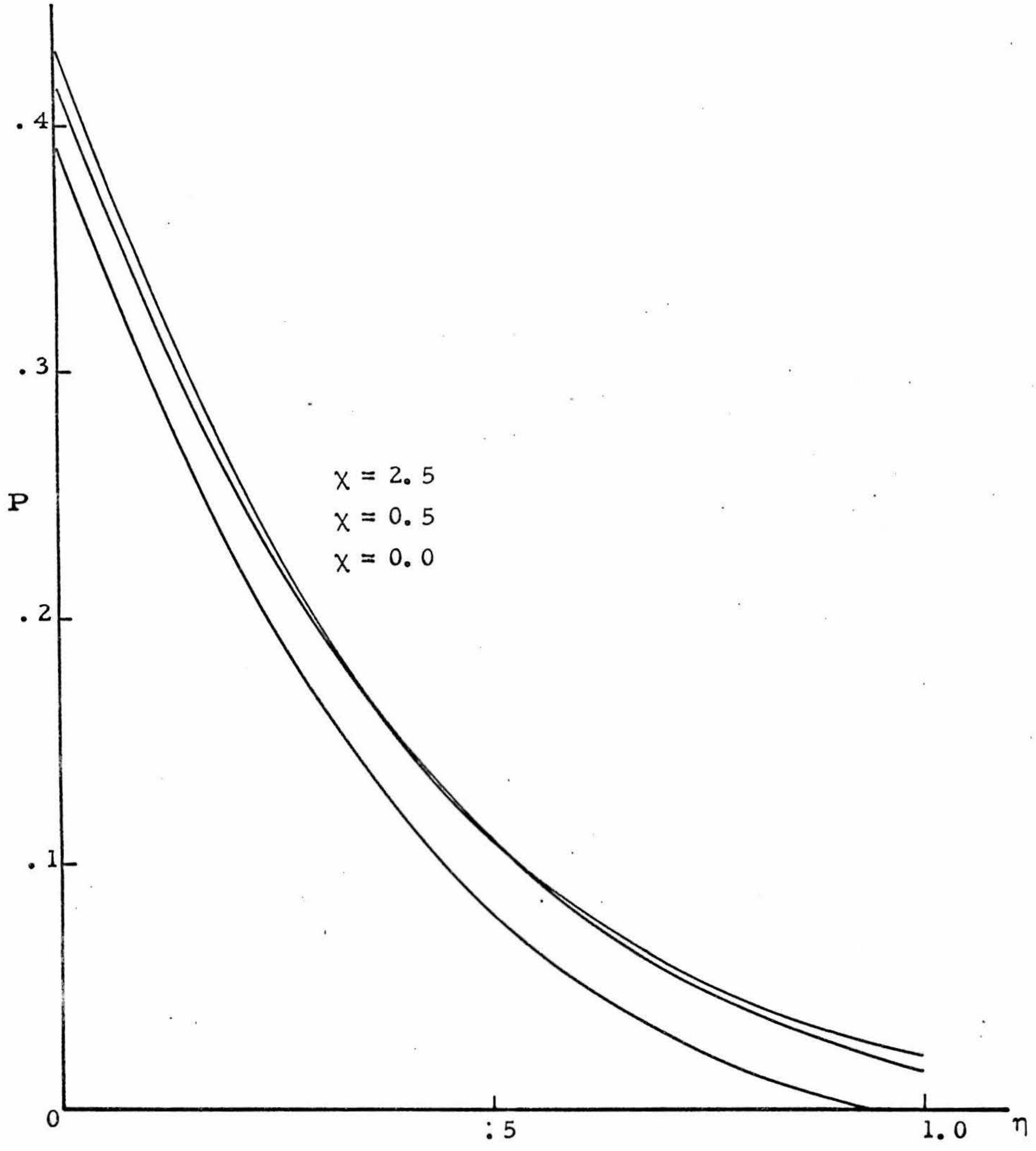

Fig. 1.6. Normalized Pressure ($\beta = 0.5$).

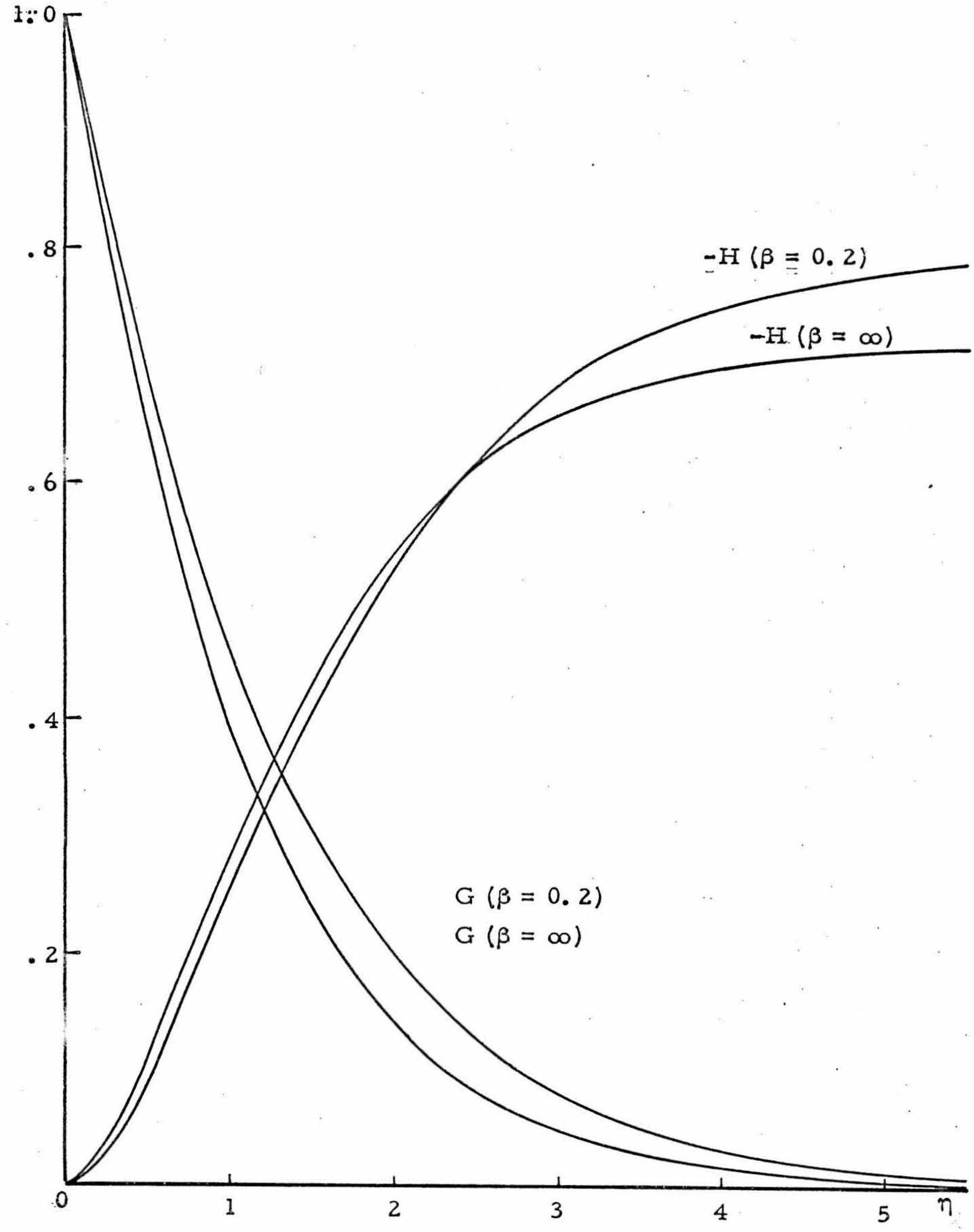

Fig. 1.7. Normalized Fluid Velocities in Tangential and Radial Directions ($\chi = 0.5$).

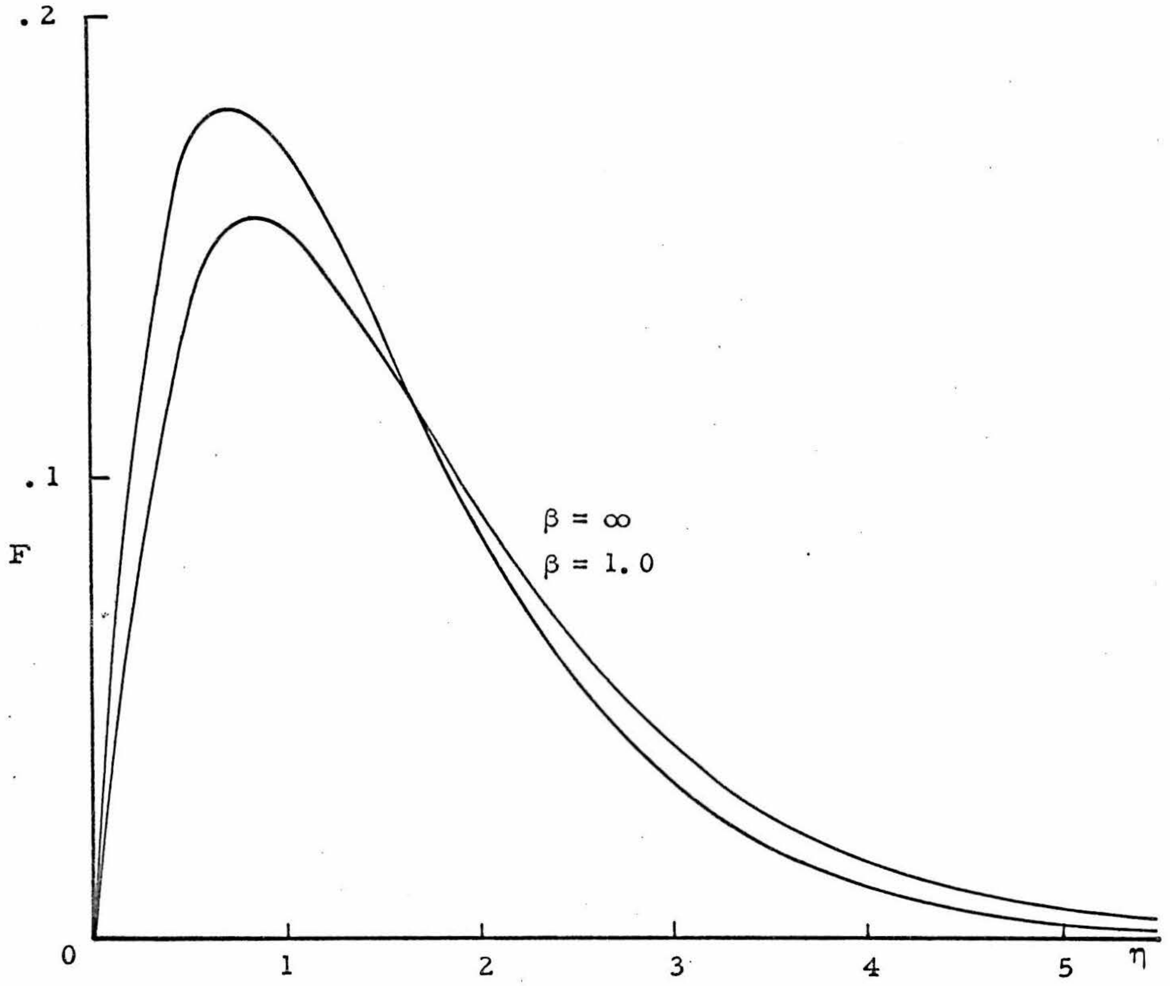

Fig. 1.8. Normalized Fluid Velocity in Radial Direction ($\chi = 0.5$).

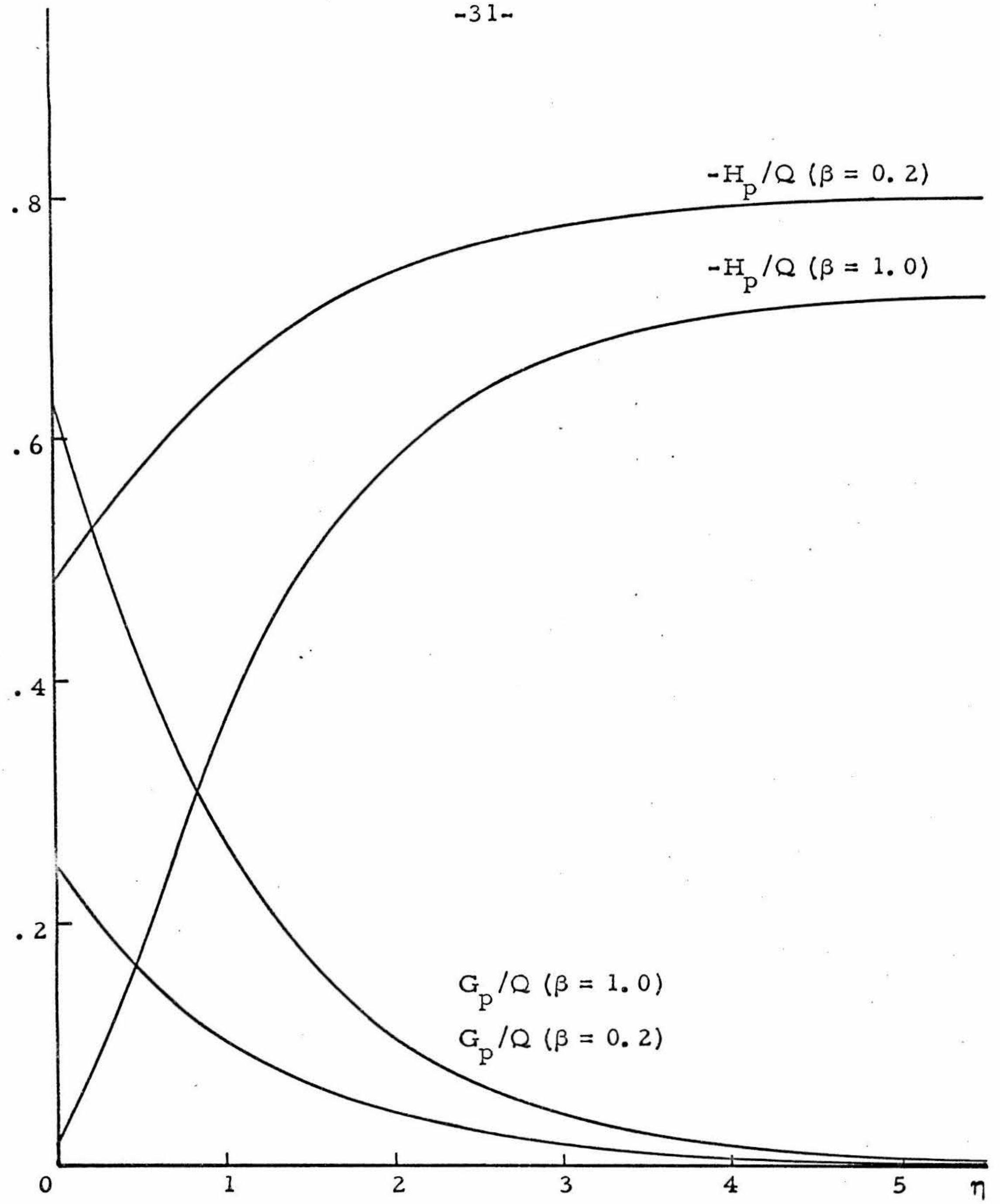

Fig. 1.9. Normalized Particle Velocities in Tangential and Radial Directions ($\chi = 0.5$).

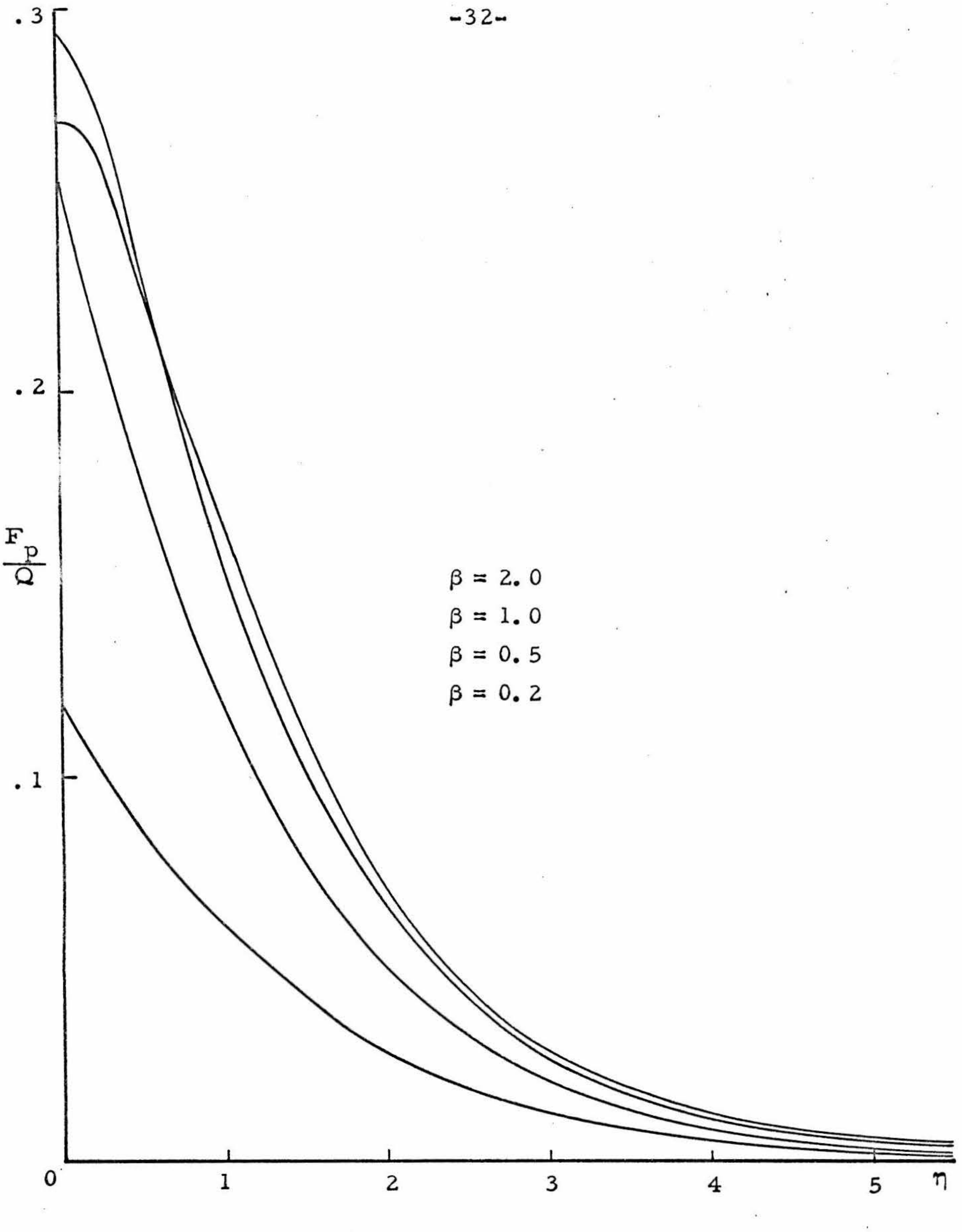

Fig. 1.10. Normalized Particle Velocity in Radial Direction ($\chi = 0.5$).

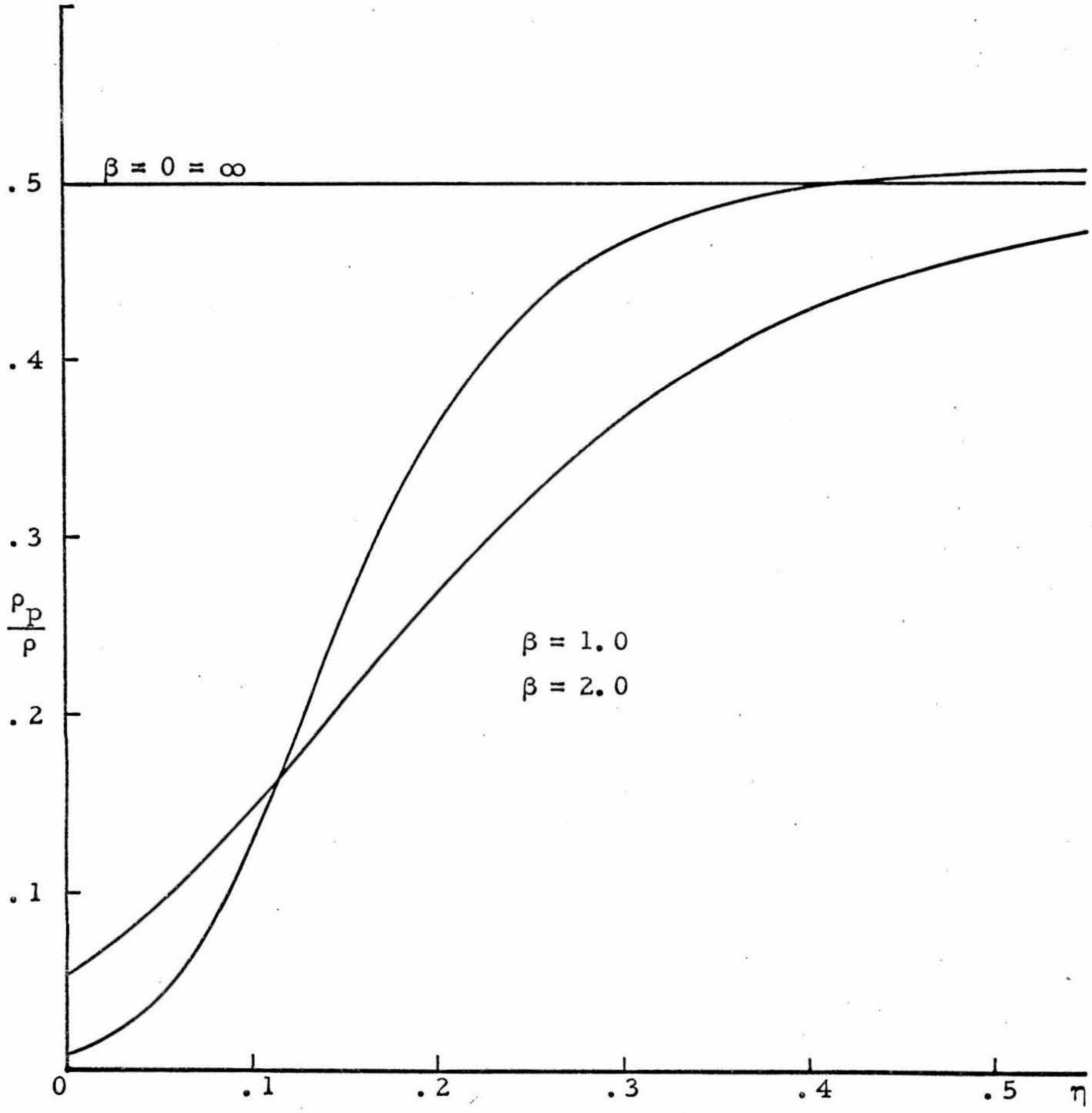

Fig. 1.11. Particle Density (χ = 0. 5).

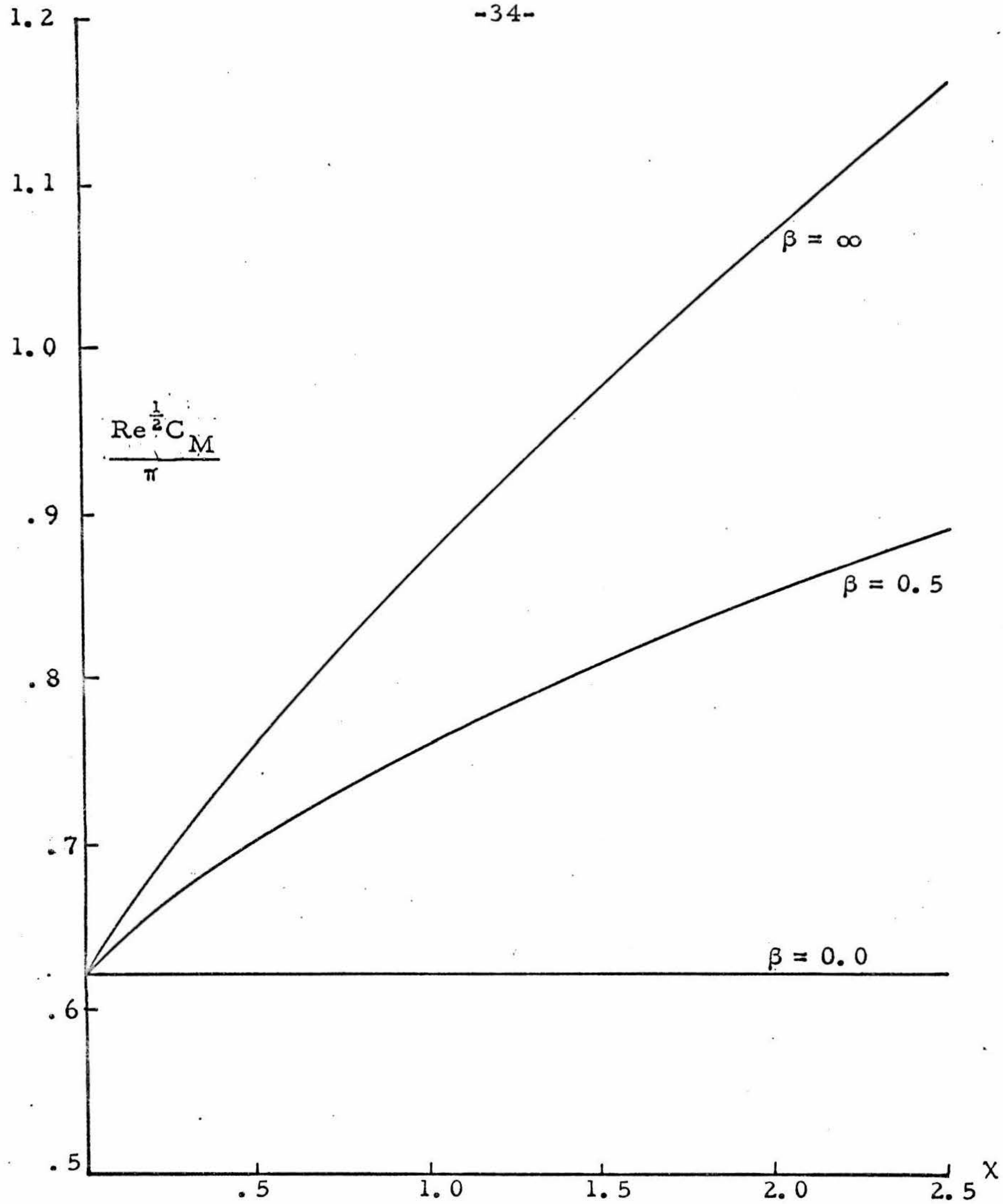

Fig. 1.12. Normalized Shearing Stress on the Disc.

PART II.

TWO PHASE FLOW OVER A WAVY WALL

I. INTRODUCTION

The perturbation technique applied to problems of aerodynamics was formally introduced by Prandtl(1) and independently by Glauert(2), resulting in the well known Prandtl-Glauert equation. Ackeret(3) was the first one to apply the theory to both subsonic and supersonic cases.

The same problem extended to the case containing small particles was first investigated by Marble(4). Marble introduced velocity potentials, which seems to be the natural variable in the spirit of small perturbation theory in aerodynamics, and obtained the corresponding Prandtl-Glauert equation for two-phase flow. Chu and Parlange(5) applied the same basic principle to the propagation of sound in a two-phase medium. Attenuation and dispersion of sound based on Chu and Parlange's theory was obtained by Temkin and Dobbins(6). The case investigated by Chu and Parlange is, in the true sense, an acoustic problem, rather than a problem in aerodynamics. Lick(7) investigated a rather similar problem of propagation of small disturbances in a radiating gas. An asymptotic solution using Laplace transform was obtained.

A small disturbance solution of a steady, two-dimensional, two-phase flow over a wavy wall using the equations derived by Marble is obtained here. Resler and McCune(8) treated the magneto-aerodynamic case, and Vincenti(9) solved the non-equilibrium flow over a wavy wall, but the governing equations for all these cases are quite different.

II. BASIC EQUATIONS

We begin by stating the equations of a steady, two-dimensional, 'inviscid'flow of a fluid containing small particles. The problem is not inviscid in the true sense. Although the viscous stress terms and the heat flux are neglected, the viscous interaction and heat conduction between the particle cloud and the fluid are still retained in the formulation. Following the notations given by Marble(9), the equations of state and the equations of conservation of mass, momentum, and energy for the particle cloud and a perfect gas, both with constant specific heats, are

$$p = \rho R T \tag{2.1}$$

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \tag{2.2}$$

$$\frac{\partial \rho_p u_p}{\partial x} + \frac{\partial \rho_p v_p}{\partial y} = 0 \tag{2.3}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_x \tag{2.4}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_y \tag{2.5}$$

$$\rho_p u_p \frac{\partial u_p}{\partial x} + \rho_p v_p \frac{\partial u_p}{\partial y} = -F_x \tag{2.6}$$

$$\rho_p u_p \frac{\partial v_p}{\partial x} + \rho_p v_p \frac{\partial v_p}{\partial y} = -F_y \tag{2.7}$$

$$\begin{aligned} \rho c_p u \frac{\partial T}{\partial x} + \rho c_p v \frac{\partial T}{\partial y} &= u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + Q \\ &\quad + (u_p - u)F_x + (v_p - v)F_y \end{aligned} \tag{2.8}$$

$$\rho_p C u_p \frac{\partial T_p}{\partial x} + \rho_p C v_p \frac{\partial T_p}{\partial y} = - Q \qquad (2.9)$$

where ρ, T, u, v are respectively the gas density, temperature, and velocity components, while ρ_p, T_p, u_p, and v_p are those pertaining to the particle cloud. P is the local gas pressure. F_x and F_y are the forces exerted on a unit volume of gas by the particle cloud and Q is the heat exchange rate per unit volume between the gas and the particles. If the drag force between the particles and the gas follows the Stokes law, F_x can be written as

$$F_x = n_p 6\pi\mu\sigma (u_p - u) = \frac{1}{\lambda_v} \rho_p (u_p - u) u_0 \qquad (2.10)$$

where n_p is the particle number density, and

$$\lambda_v = \frac{m u_0}{6\pi\mu\sigma} \qquad (2.11)$$

u_0 is some characteristic velocity. λ_v is the characteristic length in which the relative velocity between the fluid and the particle cloud is reduced to e^{-1} of its initial value. Similarly,

$$F_y = \frac{1}{\lambda_v} \rho_p u_0 (v - v_p) \qquad (2.12)$$

Similar consideration holds for the heat transfer between the particle cloud and the fluid when the temperature of the two phases is not at equilibrium:

$$Q = n_p 4\pi\sigma^2 \frac{k}{\sigma} (T_p - T) = \frac{1}{\lambda_T} \rho_p C_p u_0 (T_p - T) \qquad (2.13)$$

When

$$\lambda_T = \frac{3}{2} P_r \frac{m u_0}{6\pi\mu\sigma} = \frac{3}{2} P_r \lambda_v \qquad (2.14)$$

λ_T has very much the same physical meaning as λ_v. λ_T is the distance in which the temperature difference between the two phases is reduced to e^{-1} of its initial value. The terms $(u_p-u)F_x$ and $(v_p-v)F_y$ are the energy dissipation terms due to the momentum transfer between the particle cloud and the fluid. Here, we assume that this dissipation term is totally absorbed by gas.

Since we are concerned with the flow of small disturbances due to a wavy wall, eqs. 2. 1 to 2. 9 can be linearized in the usual manner. We assume as usual that the flow can be described by a small perturbation on a uniform flow with velocity u_0 parallel to the x-axis, and with the undisturbed temperature T_0, density ρ_0, and pressure p_0. The particle cloud and the gas are assumed to be at equilibrium in the undisturbed flow. Thus, the unperturbed particle velocity, temperature, and density have values u_0, T_0, and $\chi\rho_0$ where χ is a constant. Denote perturbed quantities by primes; the velocity components are given by $u=u_0+u'$, $u_p=u_0+u_p'$, $v=v'$, $v_p=v_p'$. Similarly, the temperatures, densities, and the local pressure of the gas can be written as $T=T_0+T'$, $T_p=T_0+T_p'$, $\rho=\rho_0+\rho'$, $\rho_p=\chi\rho_0+\rho_p'$ and $p=p_0+p'$. Neglecting higher order terms, the linearized form of equations 2. 1 - 2. 9 takes the following form:

$$\frac{p'}{p_0}=\frac{\rho'}{\rho_0}+\frac{T'}{T_0} \tag{2. 15}$$

$$u_0\frac{\partial\rho'}{\partial x}+\rho_0\frac{\partial u'}{\partial x}+\rho_0\frac{\partial v'}{\partial y}=0 \tag{2. 16}$$

$$u_0\frac{\partial\rho_p'}{\partial x}+\chi\rho_0\left[\frac{\partial u_p'}{\partial x}+\frac{\partial v_p'}{\partial y}\right]=0 \tag{2. 17}$$

$$u_0 \frac{du'}{dx} = -\frac{1}{\rho_0}\frac{dP'}{dx} + \chi \frac{(u_p' - u')u_0}{\lambda_v} \qquad (2.18)$$

$$u_0 \frac{dv'}{dx} = -\frac{1}{\rho_0}\frac{dP'}{\partial x} + \chi \frac{(v_p' - v')u_0}{\lambda_v} \qquad (2.19)$$

$$u_0 \frac{du_p'}{dx} = -\frac{1}{\lambda_v} u_0 (u_p' - u') \qquad (2.20)$$

$$u_0 \frac{\delta v_p'}{dx} = -\frac{1}{\lambda_v} u_0 (v_p' - v') \qquad (2.21)$$

$$\rho_0 C_p u_0 \frac{dT'}{dx} = u_0 \frac{dP'}{dx} + \chi \rho_0 C_p u_0 \frac{1}{\tau_T}(T_p' - T') \qquad (2.22)$$

$$C u_0 \frac{dT_p'}{dx} = -C_p \frac{1}{\lambda_T} u_0 (u_p' - u') \qquad (2.23)$$

Velocity potentials for the perturbed velocities of the particle cloud and the gas can be introduced. Rewriting the momentum equations in vectorial form

$$u_0 \frac{d\bar{V}'}{dx} = -\frac{1}{\rho_0}\nabla p' + \chi u_0 \frac{1}{\lambda_v}(\bar{V}_p' - \bar{V}') \qquad (2.24)$$

$$u_0 \frac{d\bar{V}_p'}{dx} = -\frac{1}{\lambda_v} u_0 (\bar{V}_p' - \bar{V}) \qquad (2.25)$$

Taking the curl of the sum of equations 2.24 and 2.25, we obtain

$$u_0 \frac{d}{dx}\left[\nabla \times \bar{V}' + \chi \nabla \times \bar{V}_p'\right] = 0 \qquad (2.25)$$

Since it is initially a uniform flow in which the vorticity is zero, it follows from equation 2.26 that curl $\bar{V}'$ and curl $\bar{V}_p'$ remain zero. The velocities of the gas and the particle cloud can thus be represented as the gradient of their respective potentials:

$$\bar{V} = \nabla\varphi \tag{2.27}$$

$$\bar{V_p} = \nabla\varphi_p \tag{2.28}$$

The linearized equations (2.16 - 2.23) in terms of the velocity potentials are

$$u_0 \frac{\partial\rho'}{\partial x} + \rho_0\left[\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right] = 0 \tag{2.29}$$

$$u_0 \frac{\partial\rho_p'}{\partial x} + \chi\rho_0\left[\frac{\partial^2\varphi_p}{\partial x^2} + \frac{\partial^2\varphi_p}{\partial y^2}\right] = 0 \tag{2.30}$$

$$u_0 \frac{\partial^2\varphi}{\partial x^2} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} + \frac{\chi u_0}{\lambda_v}\left[\frac{\partial\varphi_p}{\partial x} - \frac{\partial\varphi}{\partial x}\right] \tag{2.31}$$

$$u_0 \frac{\partial^2\varphi}{\partial x\partial y} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} + \frac{\chi u_0}{\lambda_v}\left(\frac{\partial\varphi_p}{\partial y} - \frac{\partial\varphi}{\partial y}\right) \tag{2.32}$$

$$u_0 \frac{\partial^2\varphi_p}{\partial x^2} = -\frac{u_0}{\lambda_v}\left(\frac{\partial\varphi_p}{\partial x} - \frac{\partial\varphi}{\partial x}\right) \tag{2.33}$$

$$u_0 \frac{\partial^2\varphi_p}{\partial x\partial y} = -\frac{u_0}{\lambda_v}\left(\frac{\partial\varphi_p}{\partial y} - \frac{\partial\varphi}{\partial y}\right) \tag{2.34}$$

$$\rho_0 C_p u_0 \frac{\partial T'}{\partial x} = u_0\frac{\partial p'}{\partial x} + \chi\rho_0 C_p \frac{u_0}{\lambda_T}(T_p' - T) \tag{2.35}$$

$$c\, u_0 \frac{\partial T_p'}{\partial x} = -C_p u_0 \frac{1}{\lambda_T}(T_p' - T') \tag{2.36}$$

Note that the momentum equations 2.31 and 2.32 as well as 2.33 and 2.34 are not longer independent of each other. Using equations 2.29, 2.30, 2.31, 2.33, 2.35, 2.36, and the equation of state 2.15, a partial differential equation in the gas velocity potential φ can be obtained.

$$\lambda_v \lambda_T^* \frac{\partial^2}{\partial x^2}\left[\beta_f^2 \frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^2\varphi}{\partial y^2}\right] + (\lambda_v + \lambda_T^*)\frac{\partial}{\partial x}\left[\beta_1 \frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^2\varphi}{\partial y^2}\right]$$

$$+ \left[\beta_e^2 \frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^2\varphi}{\partial y^2}\right] = 0 \qquad (2.37)$$

with

$$\lambda_T^* = \frac{1}{\varphi/c + \chi}\lambda_v \qquad (2.38)$$

$$\beta_f^2 = M_f^2 - 1 = \left(\frac{u_0}{a_0}\right)^2 - 1 \qquad (2.39)$$

$$\beta_1^2 = M_1^2 - 1 = \frac{u_0^2}{(\lambda_T^* + \lambda_v) P_0/\rho_0}\left[\frac{(1+\chi)\lambda_T^*}{\gamma} + \frac{c_v + \chi c}{c_p + \chi c}\lambda_v\right]^{-1} - 1 \qquad (2.40)$$

$$\beta_e^2 = M_e^2 - 1 = \frac{u_0^2}{\frac{c_p + \chi c}{c_v + \chi c}\frac{P_0}{(1+\chi)\rho_0}} - 1 \qquad (2.41)$$

Equation 2.37 corresponds to the familiar Prandtl-Glauert equation in gas dynamics. The relationship is more apparent if we set λ_v and λ_T^* both equal to zero. The resulting equation describing the equilibrium flow is almost identical to the Prandtl-Glauert equation. The

Mach number corresponding to the equilibrium flow is

$$a_e = \left(\bar{\gamma} \frac{p_0}{\bar{\rho}_0}\right)^{1/2} \tag{2.42}$$

where $\bar{\gamma} = \frac{C_p + \chi c}{C_v + \chi c}$ and $\bar{\rho}_0 = (1+\chi)\rho_0$. a_e is usually referred to as the equilibrium Mach number. In the other limiting case where both λ_v and λ_T^* approach infinity, the resulting equation is essentially the classical Prandtl-Glauert equation and describes the so-called frozen flow. The corresponding Mach number, which is usually referred to as the frozen Mach number in particle-fluid dynamics, is identical with the Mach number in the flow of a pure gas. In the case when either λ_v or λ_T^* is equal to zero, which corresponds to the momentum equilibrium flow or the temperature equilibrium flow, the corresponding Mach numbers are respectively $(\gamma \frac{p_0}{\bar{\rho}_0})^{1/2}$ and $(\bar{\gamma} \frac{p_0}{\rho_0})^{1/2}$.

The differential equation 2.37 is still hyperbolic for $M_f > 1$ and elliptic for $M_f < 1$. However, there is a certain range of u_0 in which $M_e > 1$ and $M_f < 1$. In this range, the characteristics of the equation will change discontinuously from hyperbolic to elliptic when λ_v and λ_T^* approach zero. This singular behavior appears in many other problems in fluid mechanics in which there are two different characteristic lengths. In the problem we are going to study, they are the relaxation lengths λ_v or λ_T^* and the wave length of the wavy wall.

III. SOLUTION FOR A WAVY WALL

The particular problem considered is the steady, two-dimensional flow in the half plane above an infinite sinusoidal wall of contour $\epsilon \sin \frac{2\pi}{\lambda} x$ where ϵ is the amplitude and λ is the wave length. An exact solution is particularly easy to obtain because of the periodic boundary condition in x. The boundary condition on the wall is

$$v' = \frac{\partial \varphi}{\partial y} = 2\pi \frac{\epsilon}{\lambda} U_0 \cos \frac{2\pi x}{\lambda} = 2\pi \frac{\epsilon}{\lambda} U_0 \, R_e \left(e^{i \frac{2\pi x}{\lambda}} \right) \qquad (2.43)$$

where R_e denotes the real part of the expression in the bracket. The other boundary conditions are u' and v', etc. being bounded at $y = \infty$. The solution for φ can be written in the following form

$$\varphi = - R_e \left[\frac{\epsilon U_0}{k} e^{-\frac{2\pi}{\lambda} k y} e^{i \frac{2\pi}{\lambda} x} \right] \qquad (2.44)$$

k can be found by substituting expression 2.44 into the differential equation 2.37:

$$k^2 = (k_1 + i k_2)^2 = - \frac{\Lambda_V \Lambda_T \beta_f^2 + i(\Lambda_V + \Lambda_T)\beta_1^2 - \beta_e^2}{\Lambda_V \Lambda_T + i(\Lambda_V + \Lambda_T) - 1} \qquad (2.45)$$

with

$$\Lambda_V = \frac{\lambda_V}{\lambda} 2\pi \qquad (2.46)$$

$$\Lambda_T = \frac{\lambda_T^*}{\lambda} 2\pi \qquad (2.47)$$

k_1 and k_2 can be solved by equating the real and imaginary parts of eq. 2.35. φ can therefore be expressed as

$$\varphi = - R_e \left[\frac{\epsilon U_0}{k_1 + i k_2} e^{-\frac{2\pi}{\lambda} k_1 y} e^{i \frac{2\pi}{\lambda}(x - k_2 y)} \right] \qquad (2.48)$$

φ_p can easily be obtained by integrating eq. 2. 33

$$\varphi_p = -\,Re\left[\frac{\epsilon U_0}{k_1 + i k_2}\,\frac{1}{1 + i\Lambda_v}\,e^{-\frac{2\pi}{\lambda}k_1 y}\,e^{i\frac{2\pi}{\lambda}(x - k_2 y)}\right] \qquad (2.\ 49)$$

The velocities as defined by eqs. 2. 27 and 2. 28 are

$$u = -\,2\pi\frac{\epsilon}{\lambda}\,\frac{U_0}{(k_1^2 + k_2^2)^{1/2}}\,e^{-\frac{2\pi}{\lambda}k_1 y}\cos\left[\frac{2\pi}{\lambda}(x - k_2 y) + \delta_1\right] \qquad (2.\ 50)$$

$$\tan\delta_1 = \frac{k_1}{k_2} \qquad (2.\ 51)$$

$$V = -\,2\pi\frac{\epsilon}{\lambda}U_0\,e^{-\frac{2\pi}{\lambda}k_1 y}\cos\frac{2\pi}{\lambda}(x - k_2 y) \qquad (2.\ 52)$$

$$u_p = \frac{-\,2\pi\frac{\epsilon}{\lambda}U_0}{[(k_1^2 + k_2^2)(1 + \Lambda_v^2)]^{1/2}}\,e^{-\frac{2\pi}{\lambda}k_1 y}\cos\left[\frac{2\pi}{\lambda}(x - k_2 y) + \delta_2\right] \qquad (2.\ 53)$$

$$\tan\delta_2 = \frac{k_1 - \Lambda_v k_2}{k_1 + \Lambda_v k_2} \qquad (2.\ 54)$$

$$V_p = \frac{-\,2\pi\frac{\epsilon}{\lambda}U_0}{(1 + \Lambda_v^2)^{1/2}}\,e^{-\frac{2\pi}{\lambda}k_1 y}\cos\left[\frac{2\pi}{\lambda}(x - k_2 y) + \delta_3\right] \qquad (2.\ 55)$$

$$\tan\delta_3 = -\,\Lambda_v \qquad (2.\ 56)$$

Similarly, all the other quantities can be found. They are all of the form

$$A_i\,e^{-\frac{2\pi}{\lambda}k_1 y}\cos\left[\frac{2\pi}{\lambda}(x - k_2 y) + \delta_i\right] \qquad (2.\ 57)$$

IV. DISCUSSION OF THE SOLUTION

All the perturbation quantities decay along the straight line with slope $\frac{dy}{dx} = \frac{1}{k_2}$. The rate of decay is exponentially proportional to k_1 . Along the straight line $x = k_2 y$, v is in phase with the wall, and all the other quantities have a phase change with respect to the wall given by δ_i .

Before solving k_1 and k_2 , we first examine the relations between the two relaxation lengths λ_v and λ_T . The ratio of these two lengths is

$$\frac{\lambda_v}{\lambda_T} = \frac{3}{2}\left(\frac{C_p \mu}{k}\right)\frac{C_s}{C_p}$$

$$= \frac{3}{2} P_r \frac{C_s}{C_p} \qquad (2.58)$$

P_r is the fluid Prandtl number. For most gases and metal particles, this ratio is very close to unity. Consequently, the momentum relaxation length is approximately equal to the temperature relaxation length. Furthermore, for some alkali metals, C_p is approximately equal to C . With $\lambda_v = \lambda_T$ and $C_p = C$, k_1 and k_2 , from eqn. 2.45, are

$$\left.\begin{matrix} k_1 \\ k_2 \end{matrix}\right\} = \frac{1}{\sqrt{2}}\left[\mp\left(\beta_f^2 + \frac{\gamma K M_f^2}{\Lambda_v^2+1}\right) + \sqrt{\frac{(\beta_f^2+\gamma K M_f^2)^2}{\Lambda_v^2+1} + \frac{\Lambda_r^2 \beta_f^4}{\Lambda_v^2+1}}\right]^{1/2} \qquad (2.59)$$

In the limiting case of frozen flow where $\Lambda_v \to \infty$, eq. 2.59 reduces to

$$\left.\begin{matrix} k_1 \\ k_2 \end{matrix}\right\} = \frac{1}{\sqrt{2}}\left(\mp \beta_f^2 + |\beta_f^2|\right)^{1/2} = \frac{1}{\sqrt{2}}\left(\mp(M_f^2-1) + |M_f^2 - 1\right) \qquad (2.60)$$

In the equilibrium flow where $\Lambda_v \rightarrow 0$, we have

$$\begin{matrix} k_1 \\ k_2 \end{matrix} = \frac{1}{\sqrt{2}}\left[\mp(\beta_f^2 + \gamma K M_f^2) + |\beta_f^2 + \gamma M_f^2 K|\right] = \frac{1}{\sqrt{2}}\left[\mp(M_e^2-1) + |M_e^2 - 1|\right] \qquad (2.61)$$

Equations 2.60 and 2.61 can be expressed as in the following table.

Supersonic flow	$(M_e > 1,\ M_f > 1)$	$\Lambda_v = 0$	$\Lambda_v = \infty$
	$k_1 \quad =$	0	0
	$k_2 \quad =$	$(M_e^2 - 1)^{1/2}$	$(M_f^2 - 1)^{1/2}$
Subsonic flow	$(M_0 < 1,\ M_f < 1)$		
	$k_1 \quad =$	$(1 - M_e^2)^{1/2}$	$(1 - M_f^2)^{1/2}$
	$k_2 \quad =$	0	0

This is exactly the form as would be obtained based on the Prandtl-Glauert equation for a pure fluid. In the case where $\Lambda_v = 0$, equilibrium Mach number M_e is used instead. The third case is when $M_e > 1$ and $M_f < 1$,

$(M_e > 1,\ M_f < 1)$	$\Lambda_v = 0$	$\Lambda_v = \infty$
$k_1 \quad =$	0	$(1 - M_f^2)^{1/2}$
$k_2 \quad =$	$(M_e^2 - 1)^{1/2}$	0

The variation of k_1 and k_2, for various values of Λ_v, are shown in Figures 2.1 and 2.2. $M_f^2 = 2.0$ corresponds to the supersonic case where both $M_f > 1$ and $M_e > 1$. For $M_f^2 = 0.5$

and 0.8, $M_e > 1$. This corresponds to the third case we have discussed. k_1 and k_2, as a function of the Mach number M_f, are shown in Figures 2.3 and 2.4. Here, both k_1 and k_2 are non-zero throughout the whole range of the Mach number. Exponential decay and the phase shift both exist for the perturbation quantities for subsonic and supersonic flows. For limiting values of Λ_v, both k_1 and k_2 approach the result obtained by Ackeret(3) with respect to their various Mach numbers. In the range of Λ_v of order unity, both k_1 and k_2 vary significantly for different values of χ as shown in Figures 2.5 and 2.8. Both the attenuation factor and the phase shift increase for increasing particle concentration, as would be expected. Values of k_1 and k_2, for $\lambda_v \neq \lambda_T$, and $C_p \neq C$ are shown in Figures 2.6 to 2.10.

The pressure coefficient $C_p = \frac{P - P_\infty}{\frac{1}{2}\rho_\infty U_0^2}$ can be easily calculated. For two-dimensional flows, consistent with the perturbation theory, the coefficient of pressure is simply

$$C_p = -\frac{2u'}{U_0} \tag{2.62}$$

Using eq. 2.50,

$$C_p = 4\pi\frac{\epsilon}{\lambda}\frac{1}{(k_1^2+k_2^2)^{1/2}}\,e^{-\frac{2\pi}{\lambda}k_1 y}\cos\left[\frac{2\pi}{\lambda}(x-k_2 y)+\delta_1\right] \tag{2.63}$$

On the wall,

$$C_p\Big|_{WALL} = \frac{4\pi\frac{\epsilon}{\lambda}}{(k_1^2+k_2^2)^{1/2}}\cos\left(\frac{2\pi}{\lambda}x+\delta_1\right) \tag{2.64}$$

where

$$\delta_1 = \tan^{-1} \frac{k_1}{k_2} \tag{2.65}$$

Except for the limiting cases of frozen flow and equilibrium flow, δ_1 has a non-zero value for any Mach number. Thus, C_p is never in phase with the wavy wall and a drag force is always present even in the subsonic flow, in contrast with zero drag in Acketer's calculation.

The drag coefficient per unit wave length along the wall is

$$\begin{aligned}
C_d &= \int_0^{\lambda} \epsilon \frac{2\pi}{\lambda} C_p \cos \frac{2\pi}{\lambda} x \, dx \\
&= 8 \frac{2}{\pi} \left(\frac{\epsilon}{\lambda}\right)^2 \frac{1}{k_1^2 + k_2^2} \int_0^1 \cos\left(2\pi \frac{x}{\lambda} + \delta_1\right) \cos \frac{2\pi}{\lambda} x \, d\left(\frac{x}{\lambda}\right) \\
&= 4\pi^2 \left(\frac{\epsilon}{\lambda}\right)^2 \frac{1}{k_1^2 + k_2^2} \cos \delta_1 \\
&= 4\pi^2 \left(\frac{\epsilon}{\lambda}\right)^2 \frac{k_2}{k_1^2 + k_2^2}
\end{aligned} \tag{2.66}$$

In the limiting case of equilibrium and frozen flows, eq. 2.66 reduces to the result obtained by Ackeret. In the subsonic flow, $k_2 = 0$ and $C_d = 0$. In the supersonic flow, for $\Lambda_V = \infty$

$$C_d = 4\pi^2 \frac{\epsilon^2}{\lambda^2} \left(M_f^2 - 1\right)^{-1/2} \tag{2.67}$$

and $\Lambda_V = 0$,

$$C_d = 4\pi^2 \frac{\epsilon^2}{\lambda^2} \left(M_e^2 - 1\right)^{-1/2} \tag{2.68}$$

In the case where $M_e > 1$ and $M_f < 1$, C_d is zero for $\Lambda_v = \infty$ since the flow is basically subsonic and $C_d = 4\pi^2 \frac{\epsilon^2}{\lambda^2}(M_f^2 - 1)^{-1/2}$ for $\Lambda_v = 0$. Using eq. 2. 59,

$$C_d = 4\pi^2 \frac{\epsilon^2}{\lambda^2} \frac{\left[\beta_f^2 + \frac{\gamma K M_f^2}{\Lambda_v^2 + 1} + \sqrt{\frac{(\beta_f^2 + \gamma K M_f^2)^2}{\Lambda_v^2 + 1} + \frac{\Lambda_v^2 \beta_f^4}{\Lambda_v^2 + 1}}\right]}{\sqrt{2}\left[\frac{(\beta_f^2 + \gamma K M_f^2)^2}{\Lambda_v^2 + 1} + \frac{\Lambda_v^2 \beta_f^4}{\Lambda_v^2 + 1}\right]^{1/2}} \tag{2. 69}$$

The variation of C_d with respect to M_f^2 is shown in Figure 2. 13. The characteristic singularity of Acketer's solution close to the sonic region is being removed, although the present equation is still not valid for transonic flows. In the case of equilibrium or frozen flows, C_d is zero for subsonic flows and is given by eqs. 2. 67 and 2. 68 for supersonic flows, which are similar to the expressions obtained by Ackeret. The same expression, eq. 2. 69, is plotted against Λ_T in Figure 2. 14. In supersonic flow, C_d increases monotonically from $4\pi^2(\frac{\epsilon}{\lambda})^2(M_f^2 - 1)^{-1/2}$ to $4\pi^2(\frac{\epsilon}{\lambda})^2(M_e^2 - 1)^{-1/2}$ as shown by the curve $M_f^2 = 2.0$.

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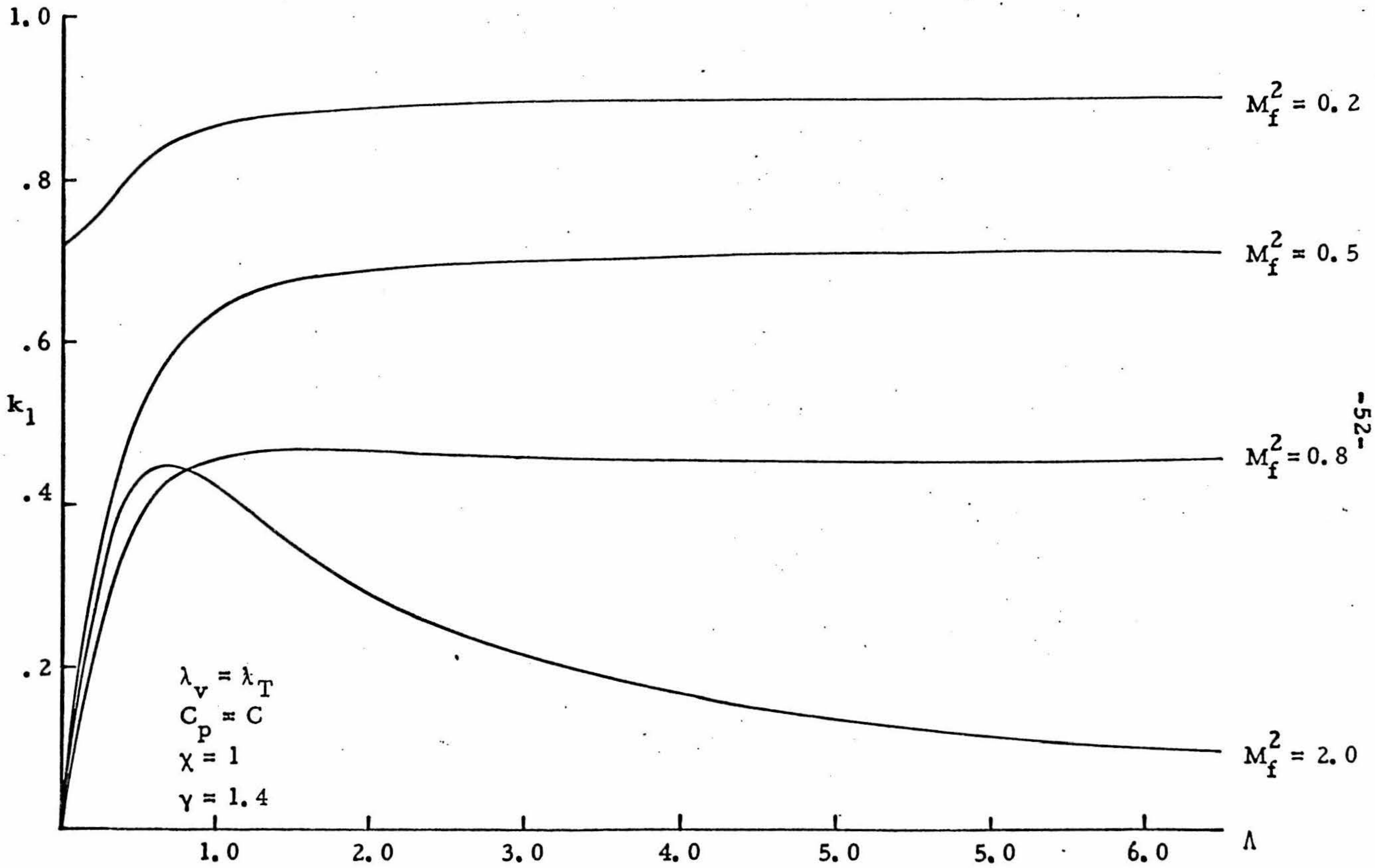

Fig. 2.1. k_1 as a Function of Λ for Various Frozen Mach Numbers.

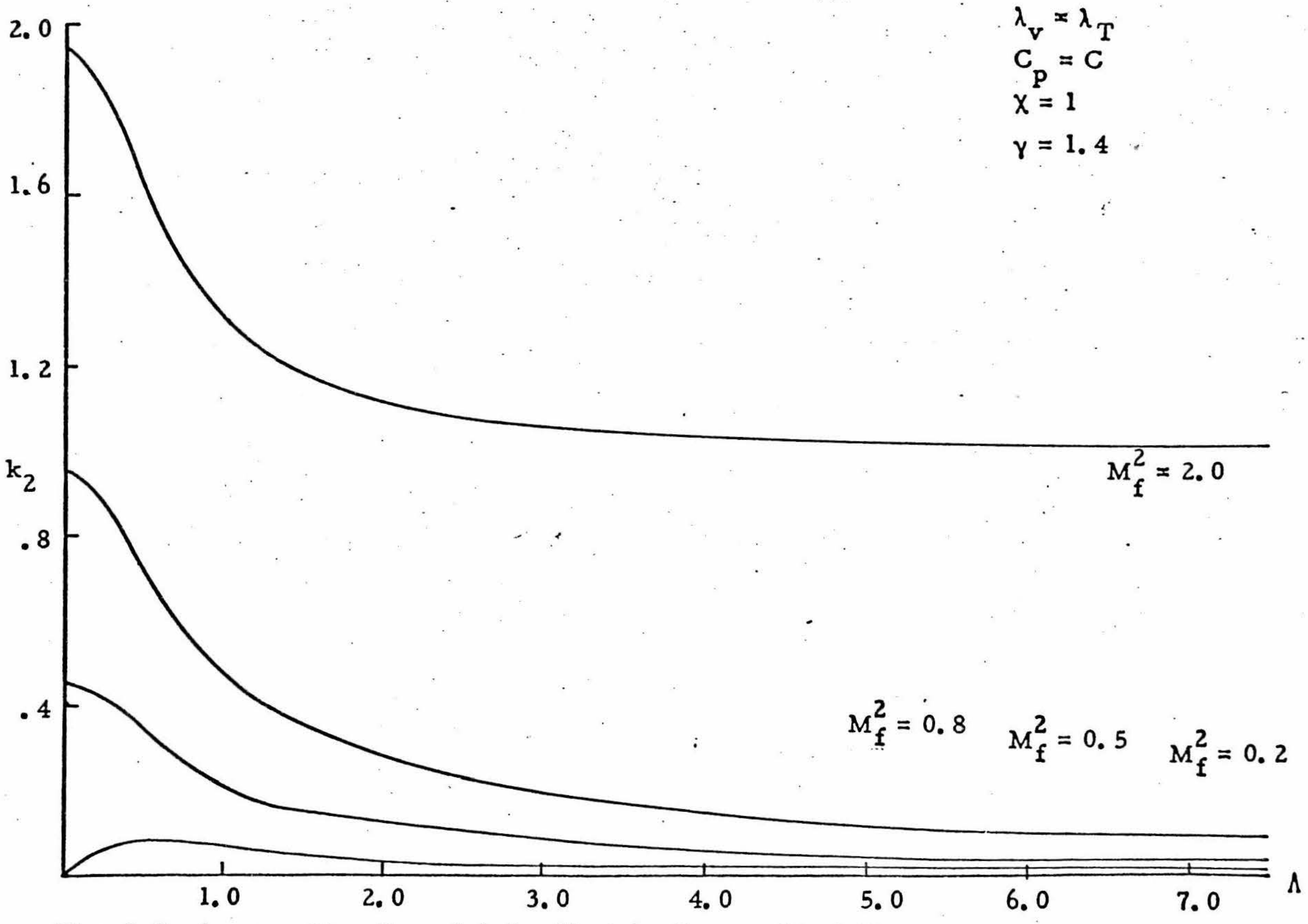

Fig. 2.2. k_2 as a Function of Λ for Various Frozen Mach Numbers.

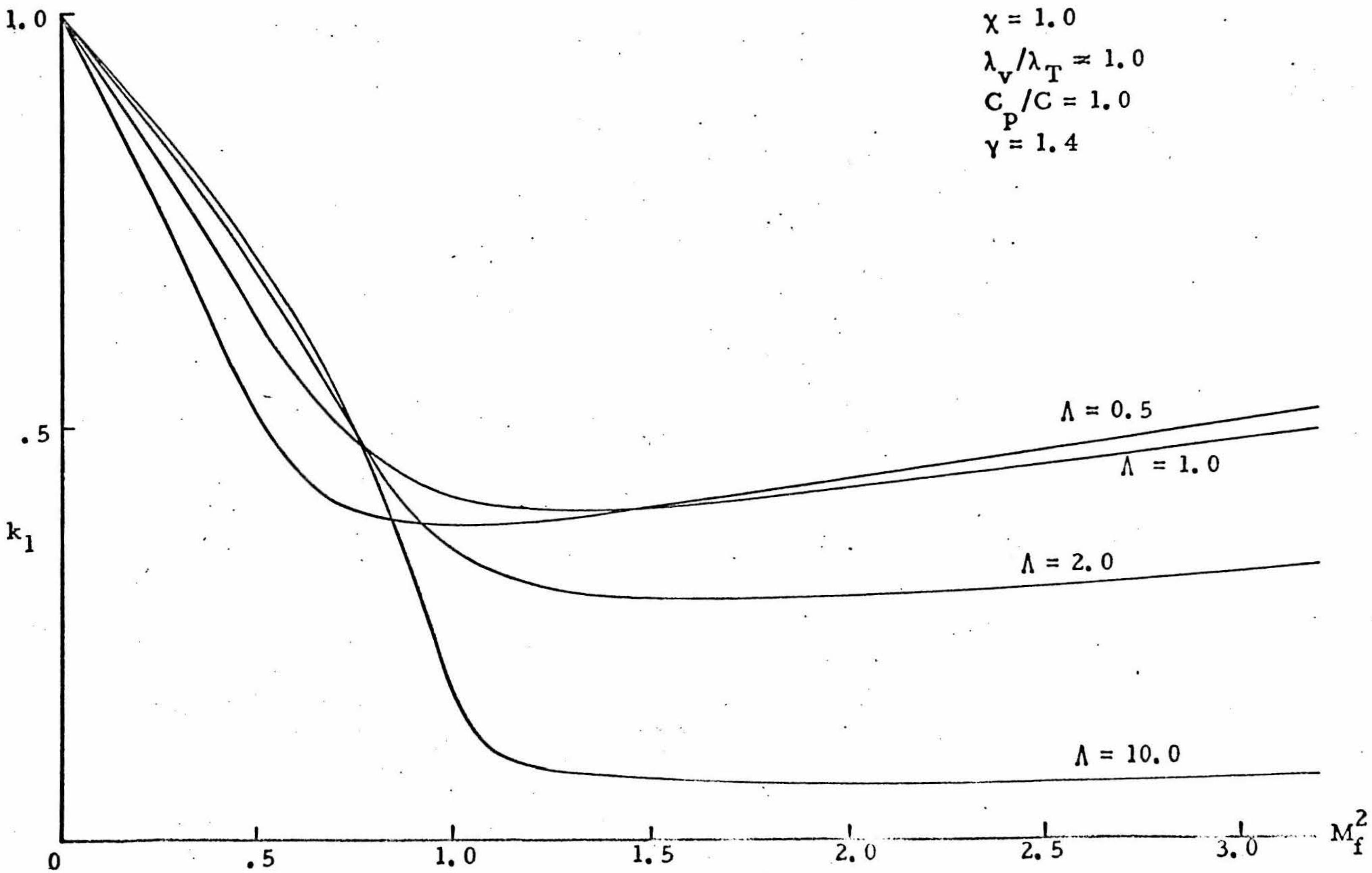

Fig. 2.3. k_1 as a Function of Frozen Mach Number for Various Λ.

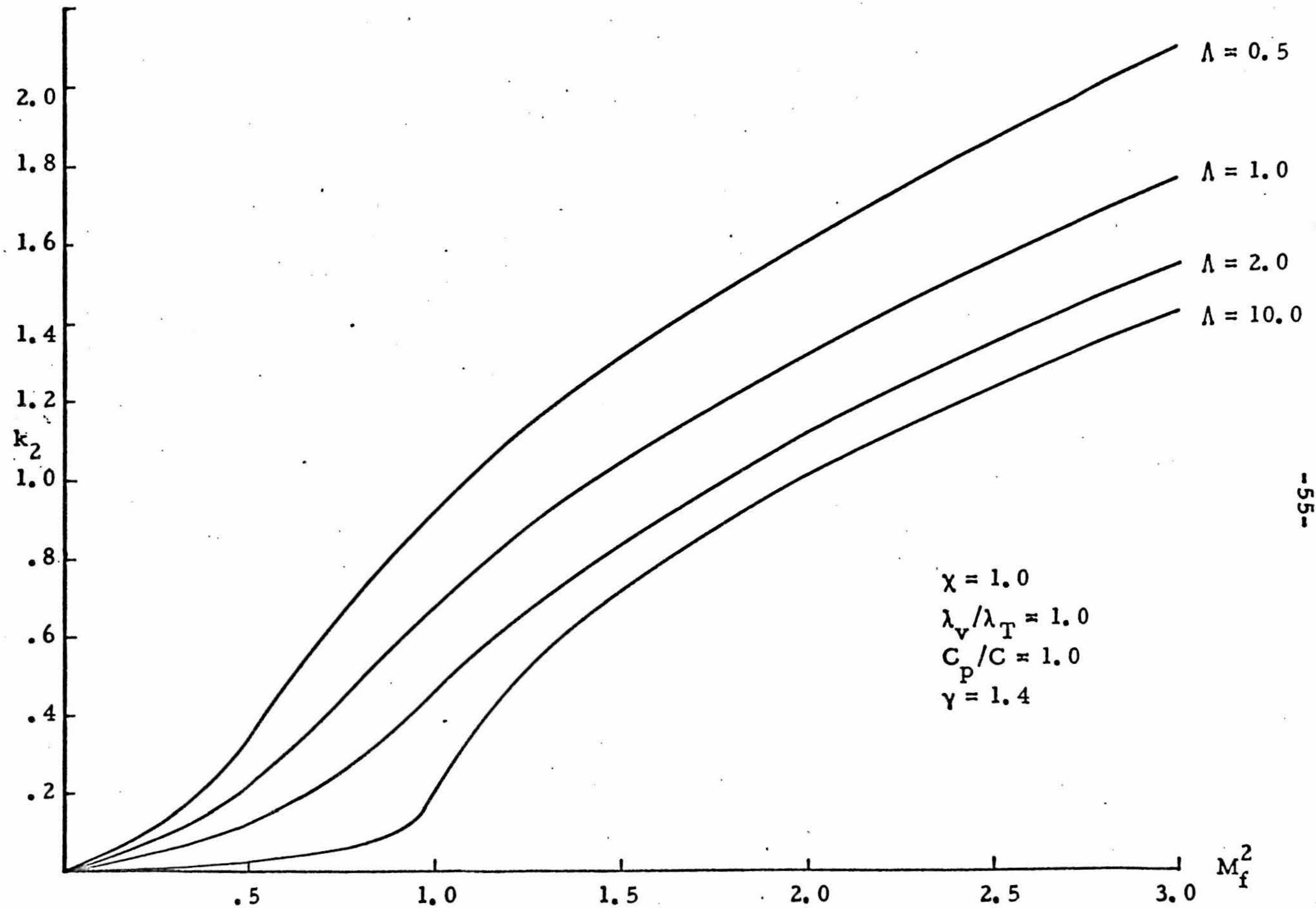

Fig. 2.4. k_2 as a Function of Frozen Mach Number for Various Λ.

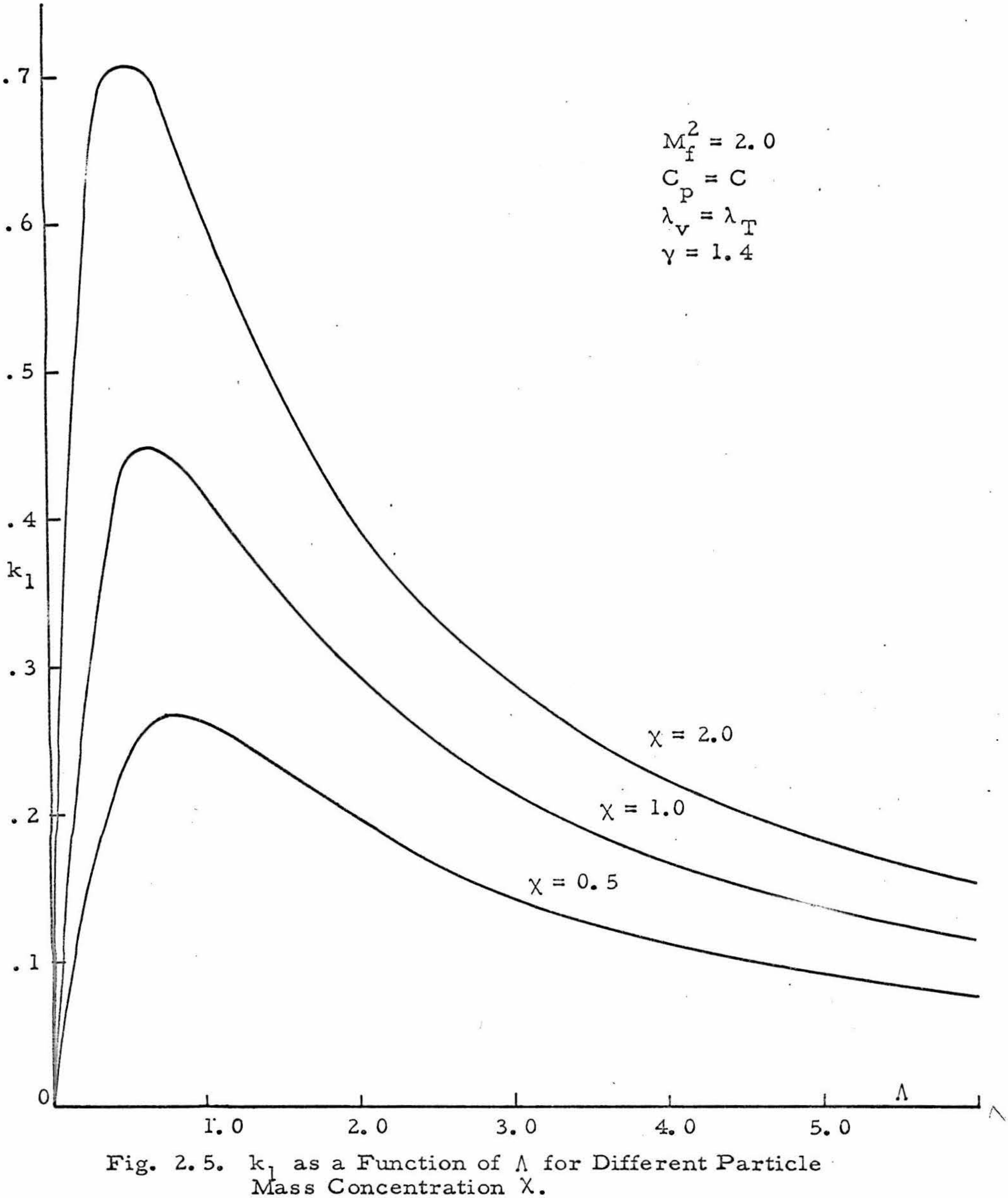

Fig. 2.5. k_1 as a Function of Λ for Different Particle Mass Concentration χ.

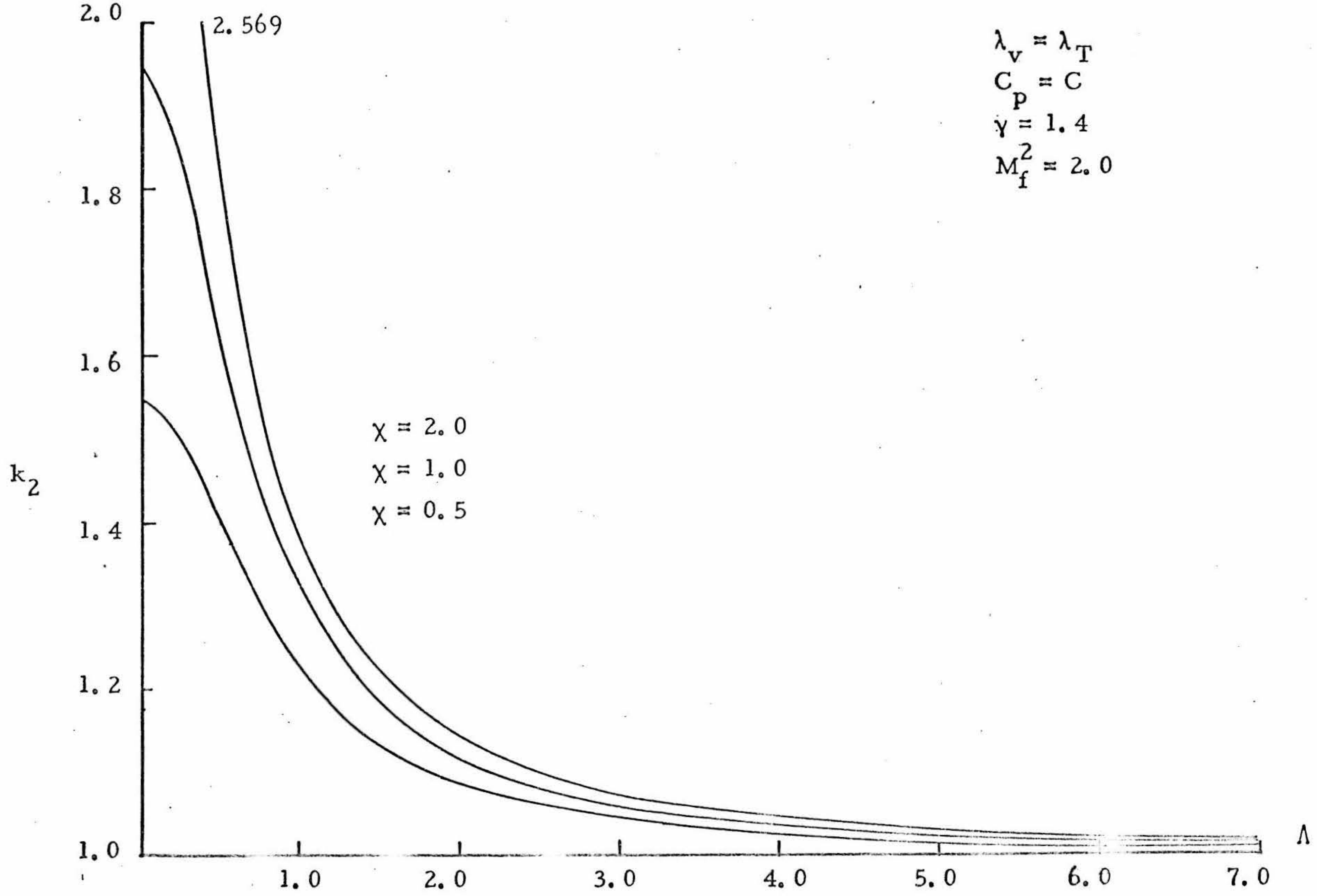

Fig. 2.6. k_2 as a Function of Λ for Various Particle Mass Concentration χ.

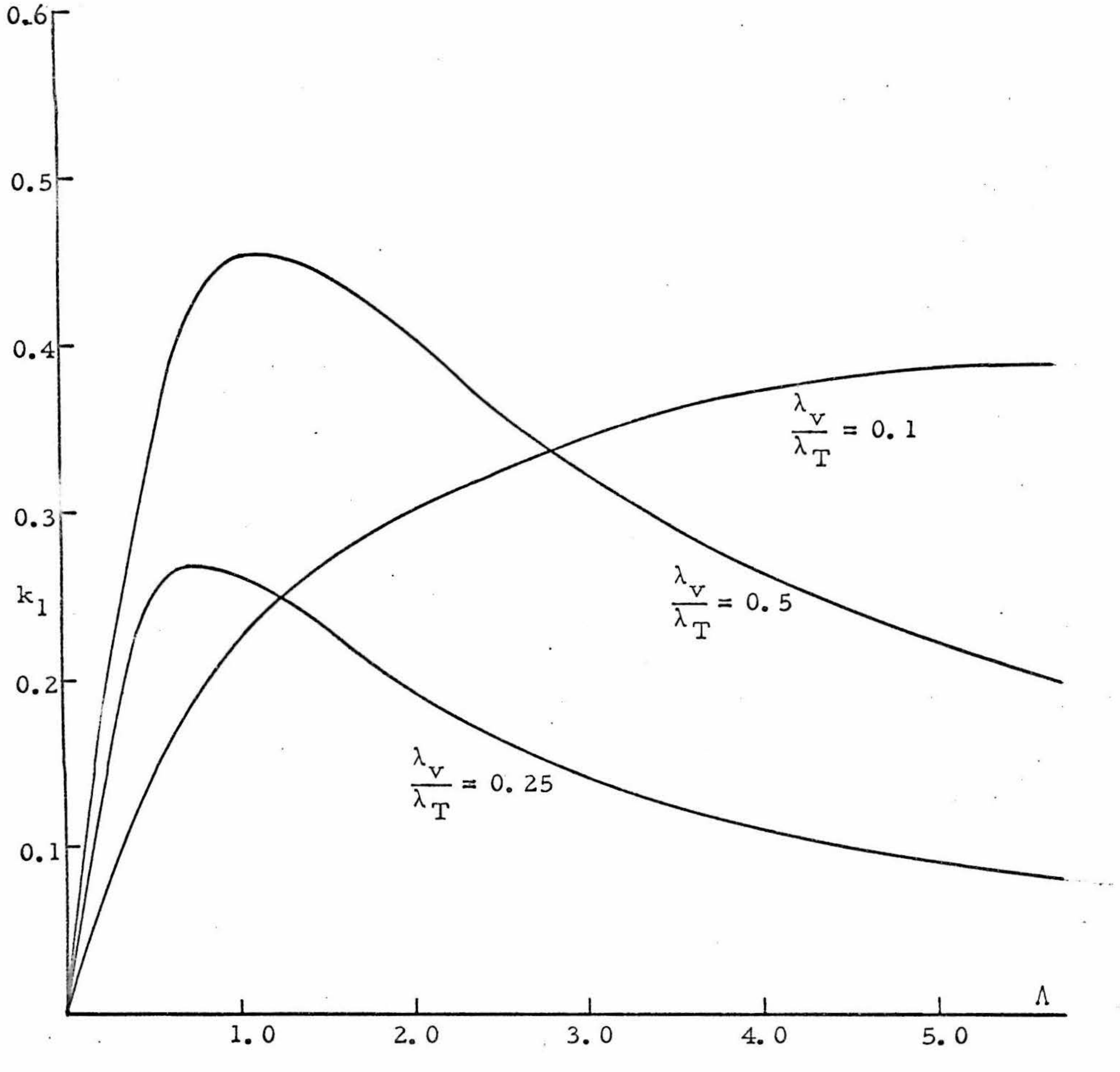

Fig. 2.7. k_1 as a Function of Λ for Different Ratios of λ_v/λ_T.

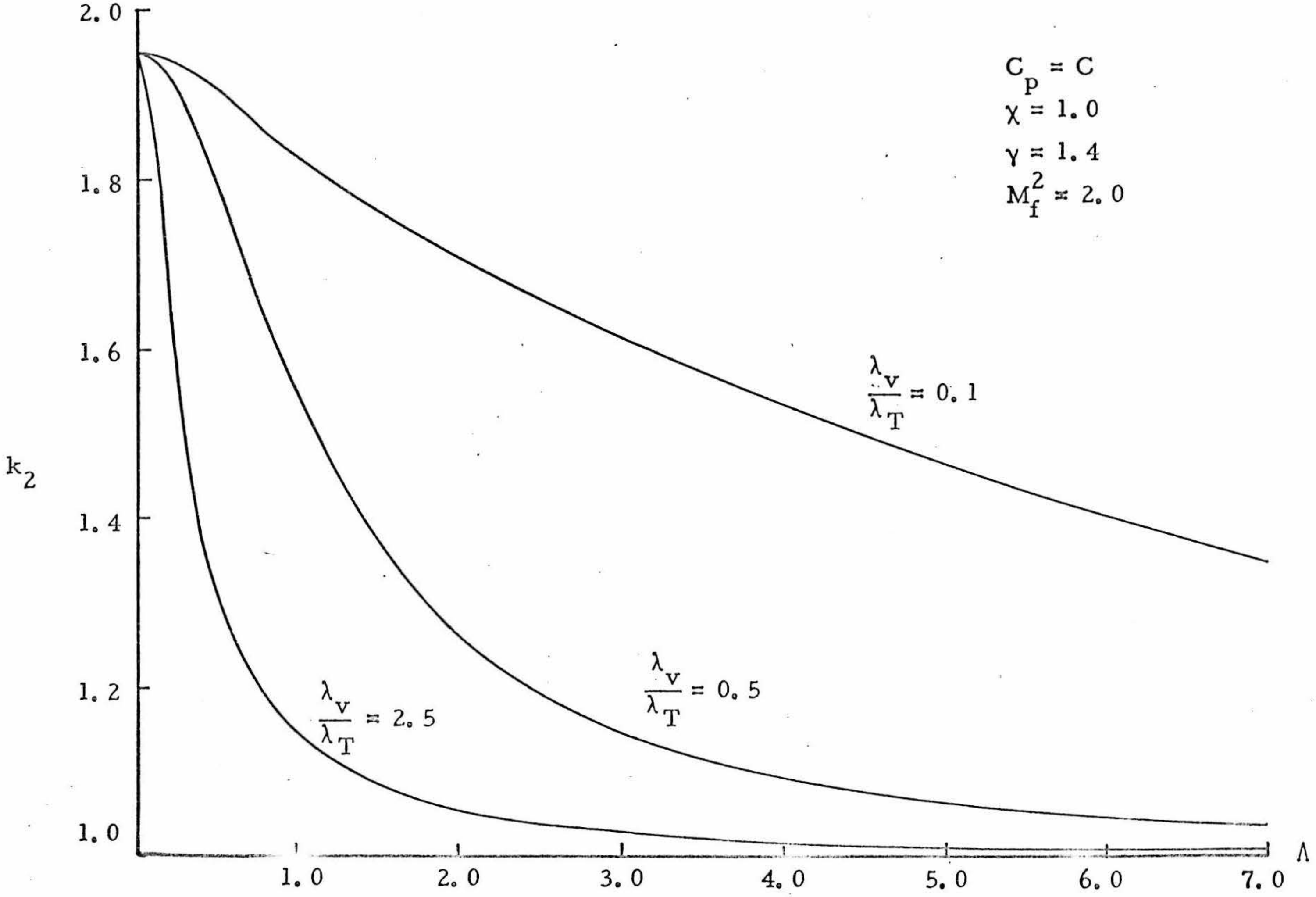

Fig. 2.8. k_2 as a Function of Λ for Various Ratios of λ_v/λ_T.

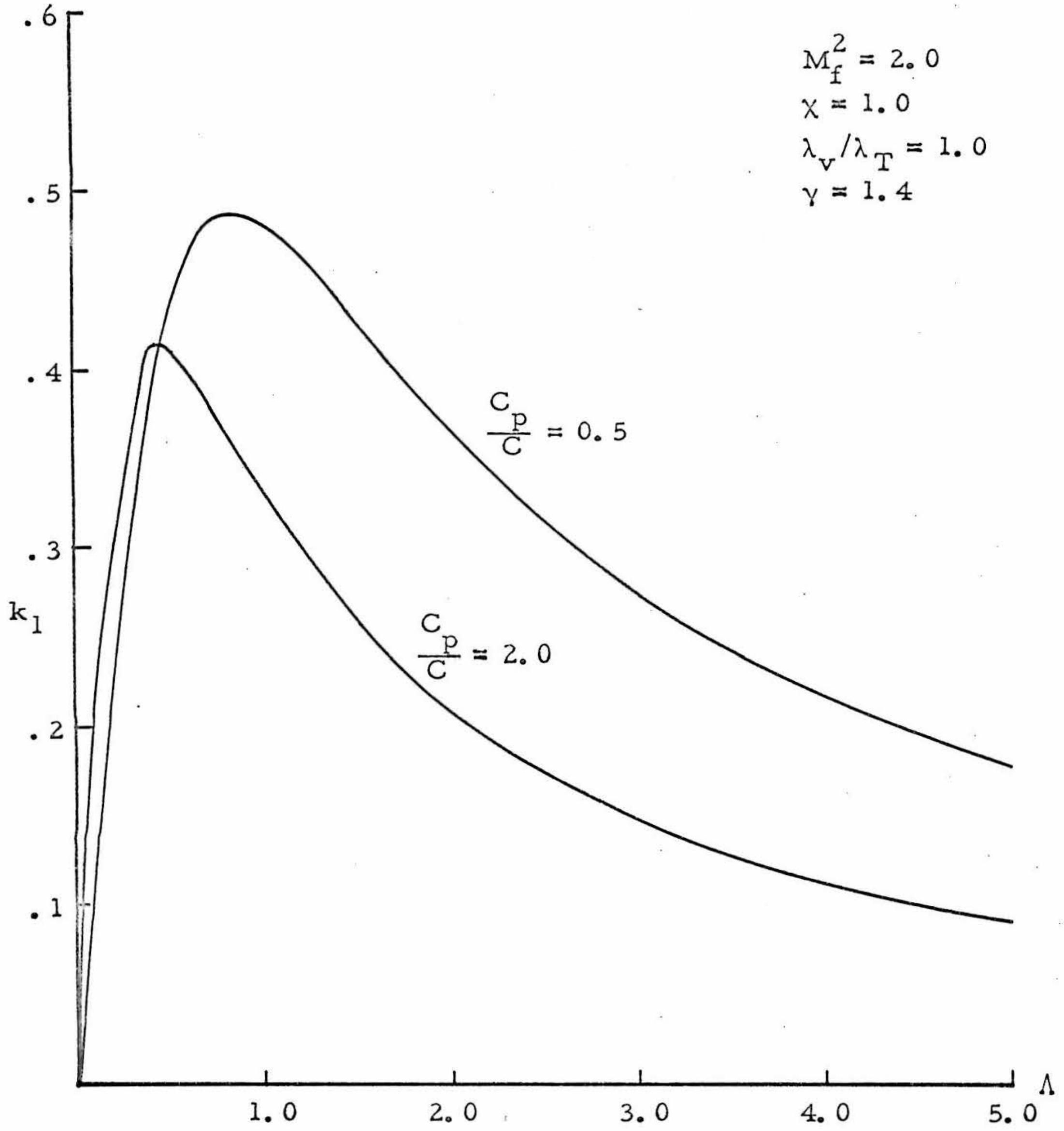

Fig. 2.9. k_1 as a Function of Λ for Different Values of C_p/C.

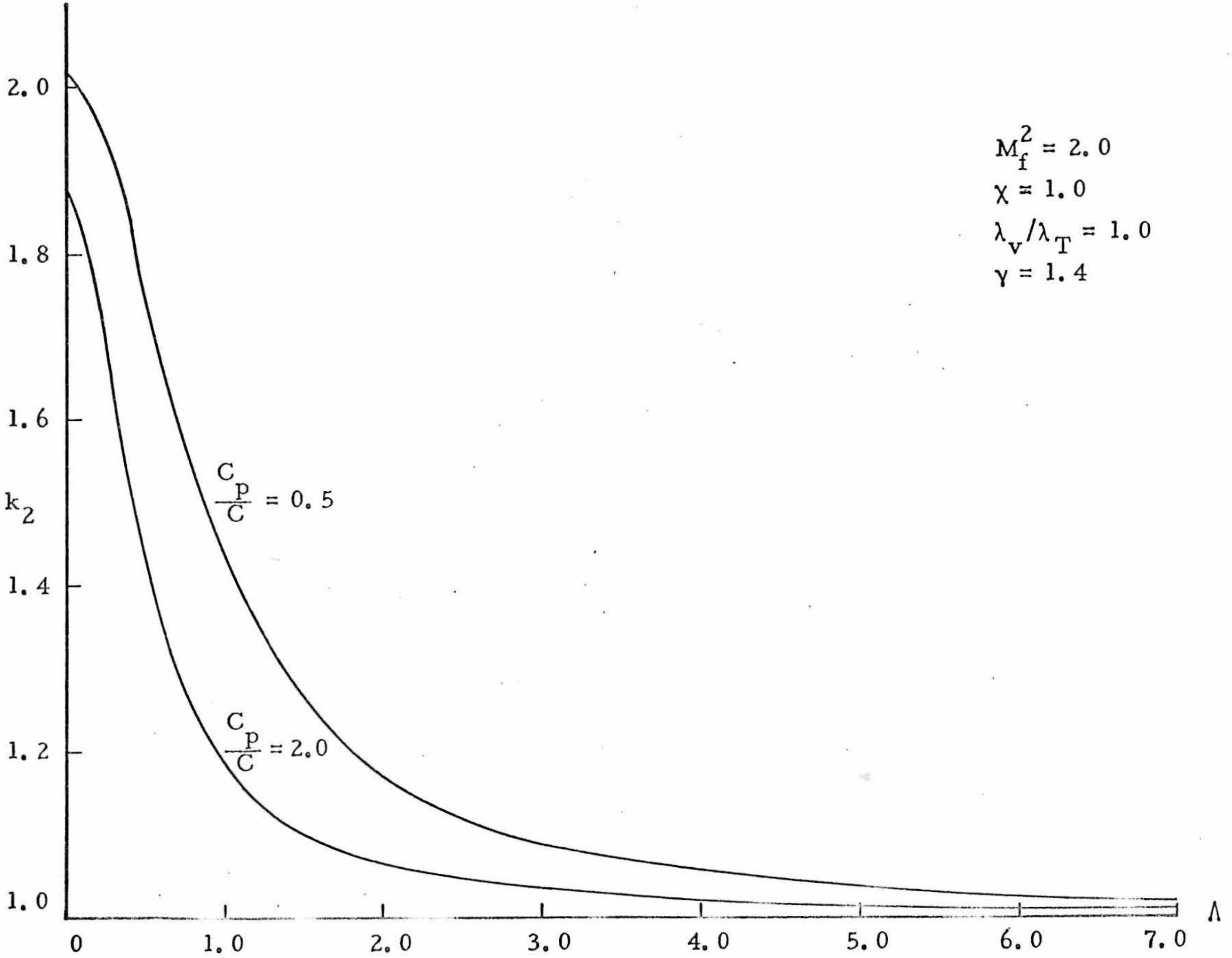

Fig. 2.10. k_2 as a Function of Λ for Different Values of C_p/C.

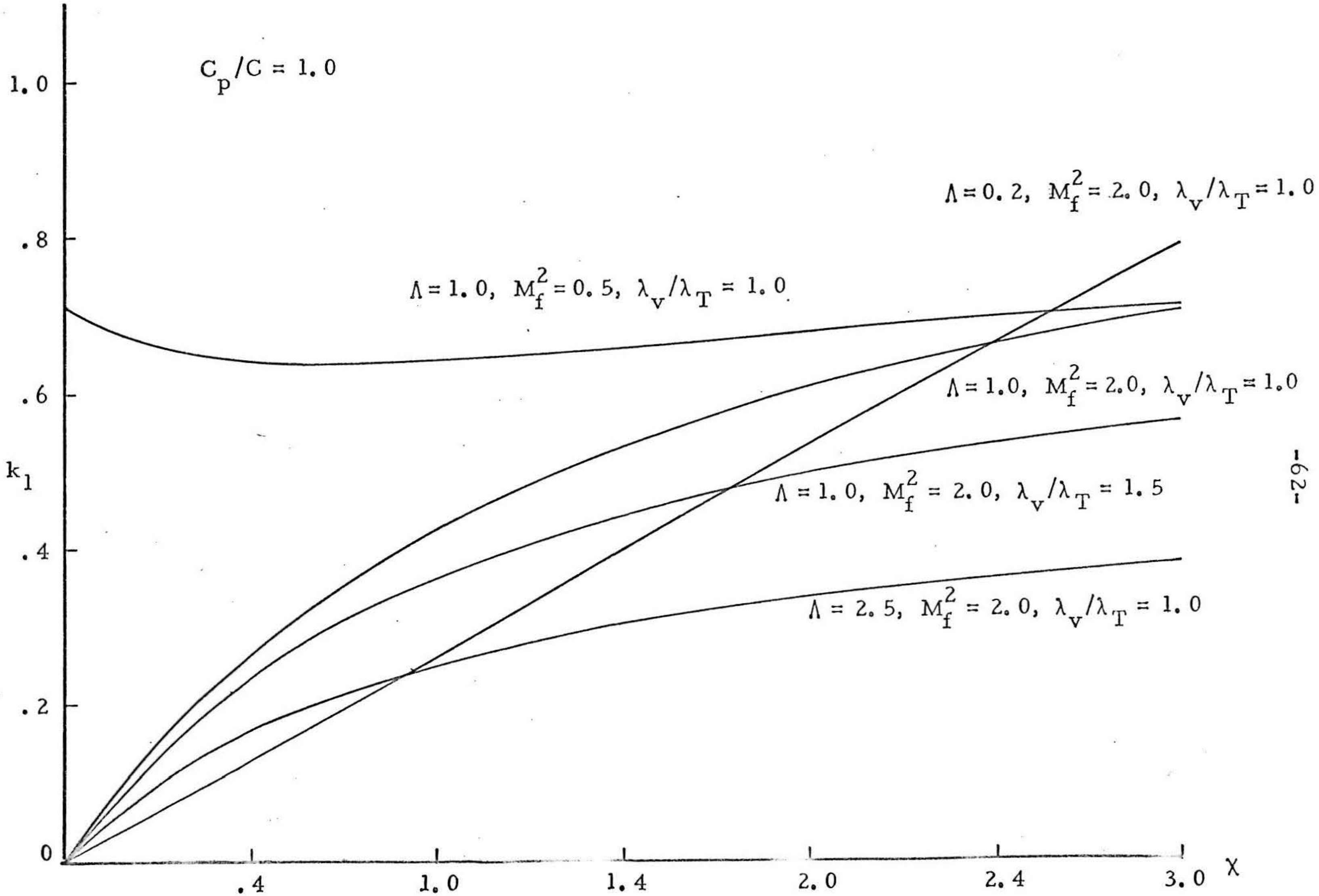

Fig. 2.11. Values of k_1 as a Function of χ.

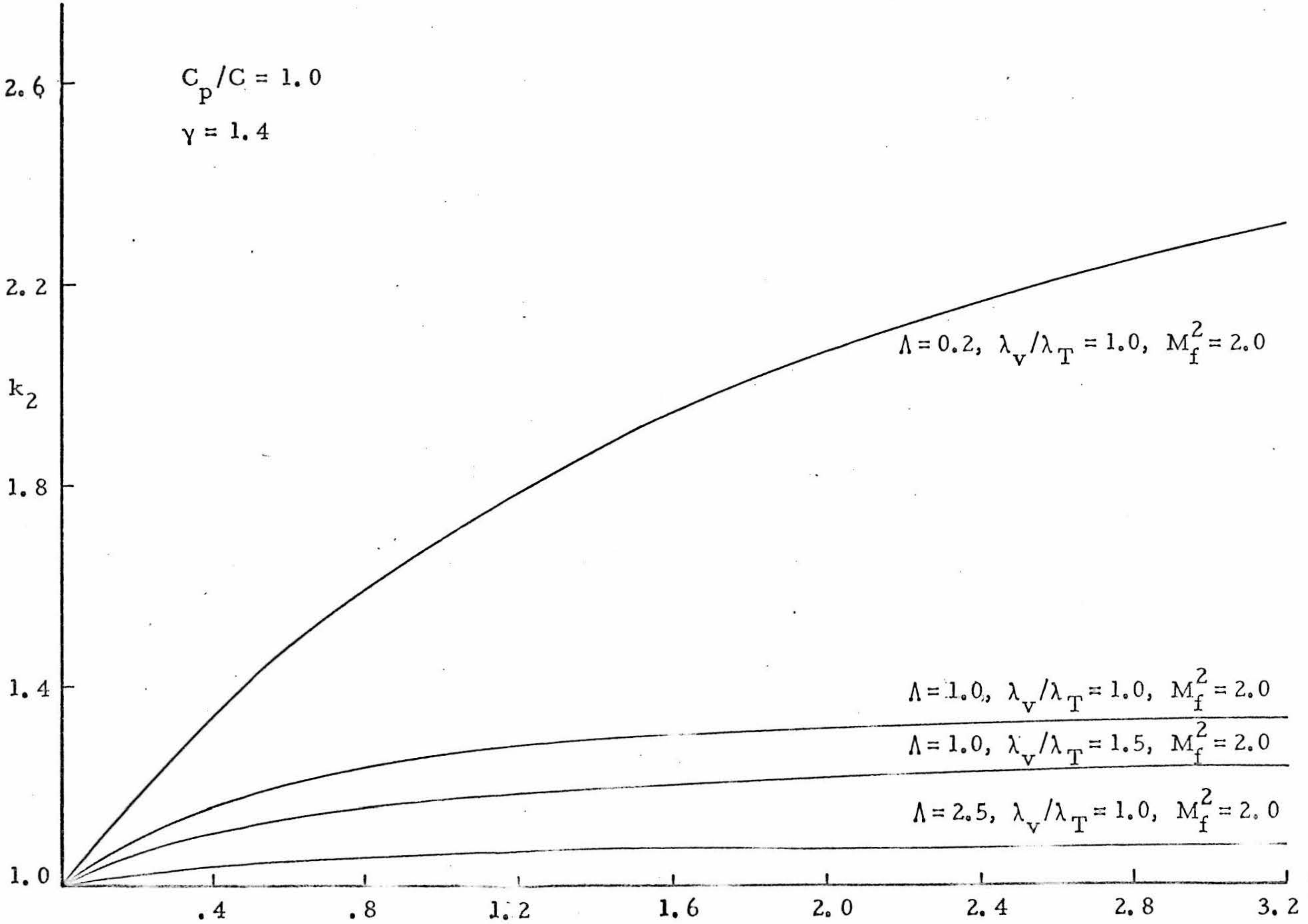

Fig. 2.12. Values of k_2 as a Function of χ.

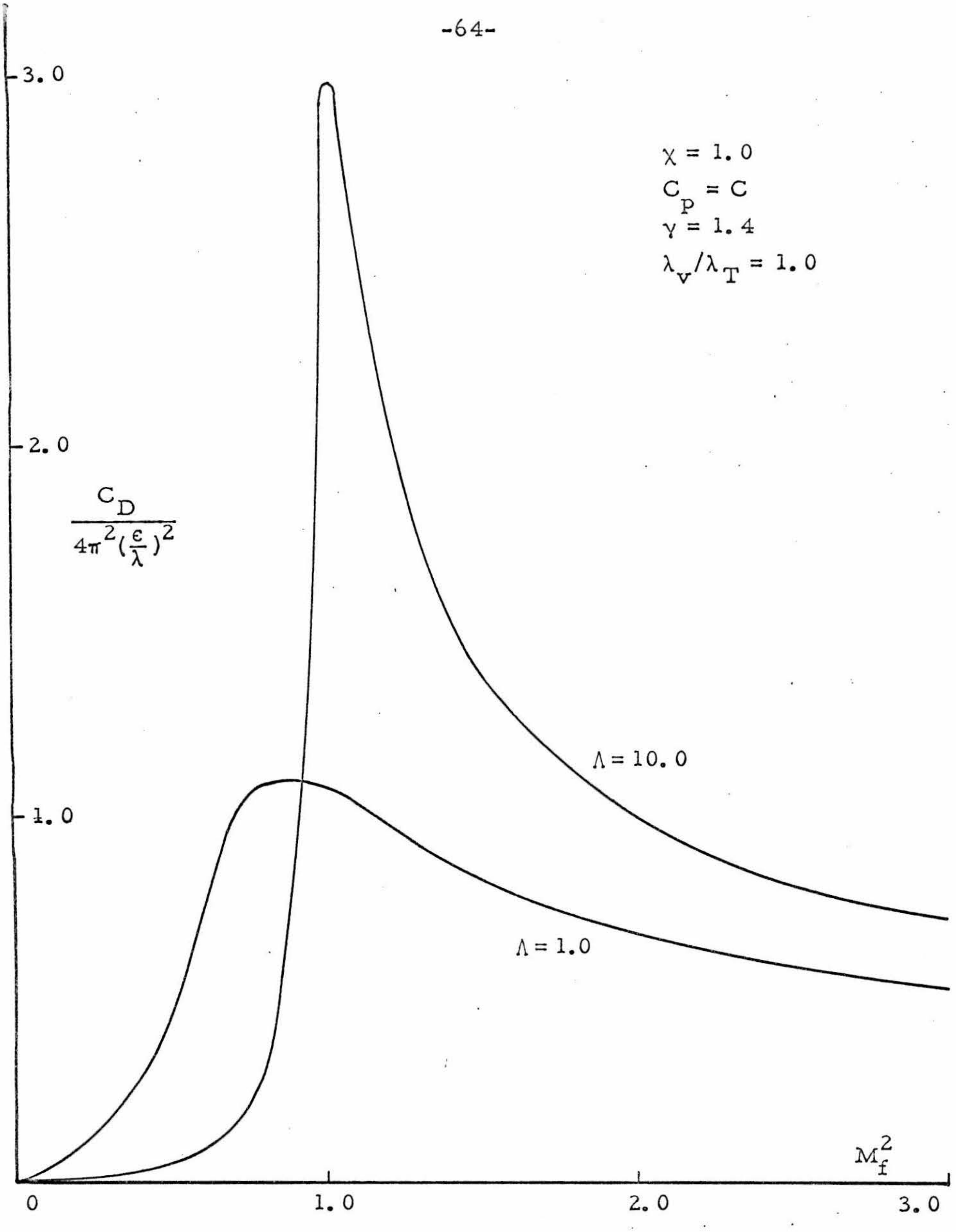

Fig. 2.13. Drag Coefficient as a Function of Frozen Mach Number.

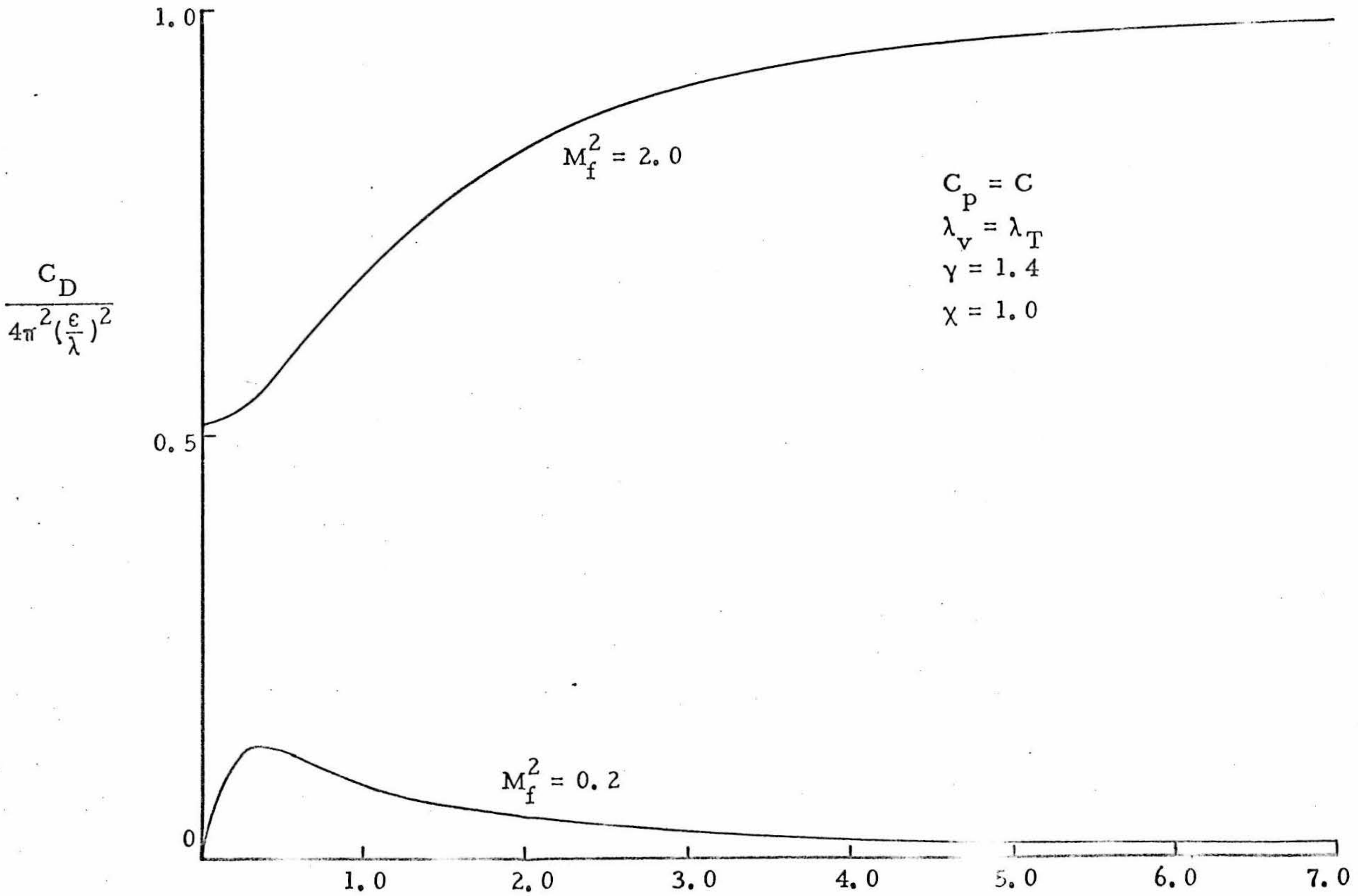

Fig. 2.14. Drag Coefficient as a Function of Λ.

PART III.

COMBUSTION OF SOLID PARTICLES IN A LAMINAR MIXING ZONE

I. GENERAL EQUATIONS

Marble and Adamson$^{(1)}$ used the boundary layer approximation to treat the problem of ignition and development of a flame front in the laminar mixing region between parallel streams of combustible gas and hot products of combustion. This is probably the first analytical investigation of a two-dimensional, laminar combustion process, and introduces the investigation of a new class of combustion processes through the concept of boundary layer theory. A rather similar problem is being investigated here by considering the burning of solid fuel particles.

In many problems of thermal jet systems, small solid fuel particles suspended in some inert gas are ignited by mixing with another gas containing some oxidizing agent. This process occurs when solid unburnt fuel particles are being injected into the atmosphere. The particles react with the oxygen in the atmosphere.

Denoting the gas density and the mean velocity components of the gas mixture by ρ' , u' , and v' , the time-independent two-dimensional continuity equation reads as:

$$\frac{\partial \rho' u'}{\partial x'} + \frac{\partial \rho' v'}{\partial y'} = \omega' \tag{3.1}$$

ω' is the rate of mass production for the gas mixture. If ω_o' and ω_i' are the rates of mass production for the oxidizer and the inert gas, then

$$\omega' = \omega_o' + \omega_i' \tag{3.2}$$

The continuity equation for the particle cloud is

$$\frac{\partial}{\partial x'}(\rho_p' u_p') + \frac{\partial}{\partial y'}(\rho_p' v_p') = \omega_p' \tag{3.3}$$

where the subscript p denotes the quantities pertaining to the particle cloud. Since mass is conserved,

$$\omega_p' + \omega' = 0 \tag{3.4}$$

Using Fick's law, the continuity equation for the oxidizer can be written as

$$\rho' u' \frac{\partial K}{\partial x'} + \rho' v' \frac{\partial K}{\partial y'} = \frac{\partial}{\partial x'}\left(\rho' D \frac{\partial K}{\partial x'}\right) + \frac{\partial}{\partial y'}\left(\rho' D \frac{\partial K}{\partial y'}\right) - K\omega' + \omega_0' \tag{3.5}$$

where K is the oxidizer concentration and D is the binary diffusion coefficient between the oxidizer and the inert gas.

The momentum equation can be written as

$$\rho' u' \frac{\partial u'}{\partial x'} + \rho' v' \frac{\partial u'}{\partial y'} = F_x' - \frac{\partial p'}{\partial x'} + \frac{\partial \tau_{x'x'}'}{\partial x'} + \frac{\partial \tau_{x'y'}'}{\partial y'} + \omega'(m_x' - u') \tag{3.6}$$

$$\rho' u' \frac{\partial v'}{\partial x'} + \rho' v' \frac{\partial v'}{\partial y'} = F_y' - \frac{\partial p'}{\partial y'} + \frac{\partial \tau_{y'x'}'}{\partial x'} + \frac{\partial \tau_{y'y'}'}{\partial y'} + \omega'(m_y' - v') \tag{3.7}$$

p' is the local pressure for the gas mixture and $\tau_{x'y'}'$ is defined by the usual rate of stress-strain relationship, but the viscosity μ is modified to include the apparent stresses caused by diffusion. $m_{x'}'$ is the velocity associated with ω' and F_x' and F_y' are the forces exerted on a unit volume of gas mixture by the particles. If it is assumed that the Stokes drag law gives a reasonable approximation to the particle-gas interaction,

$$F_x' = n_p 6\pi\mu'\sigma(u_p' - u') = \frac{1}{\lambda_{r_0}} \rho_p'(u_p' - u') u_1 \left(\frac{\sigma}{\sigma_0}\right)^2 \frac{\mu}{\mu_1} \tag{3.8}$$

u_1 and μ_1 are some reference velocity and the viscosity at some reference state. σ_0 is the initial radius of the particles. Similarly,

$$F_y' = \frac{1}{\lambda_{r_0}} \rho_p' (v_p' - v') u_1 \left(\frac{\sigma_0}{\sigma}\right)^2 \frac{\mu}{\mu_1} \tag{3.9}$$

The momentum equations for the particle cloud are

$$\rho_p' u_p' \frac{\partial u_p'}{\partial x'} + \rho_p' v_p' \frac{\partial u_p'}{\partial y'} = -F_x' + \omega_p' (m_{px'}' - u_p') \tag{3.10}$$

$$\rho_p' u_p' \frac{\partial v_p'}{\partial x'} + \rho_p' v_p' \frac{\partial v_p'}{\partial y'} = -F_y' + \omega_p' (m_{py'}' - v_p') \tag{3.11}$$

$m_{px'}'$ and $m_{py'}'$ are the velocities associated with ω_p'. Since momentum is conserved for the whole system, we have

$$\omega' m_{x'}' + \omega_p' m_{px'} = 0 \tag{3.12}$$

$$\omega' m_{y'}' + \omega_p' m_{y'}' = 0 \tag{3.13}$$

In the energy equations we neglect the kinetic energies and the viscous dissipation terms and assume that they are small compared with thermal energies and the heat of reaction. For calorically perfect gas, the enthalpy of the mixture is

$$h = K (C_{p_0} T + h_0^\circ) + (1 - K)(C_{p_i} T + h_i^\circ) \tag{3.14}$$

where C_{p_0} and C_{p_i} are the specific heats of the oxidizer and inert gas, h_0° and h_i° are their respective reference enthalpies. Similarly, the enthalpy of the particle cloud can be written as

$$h_p = C T_p + h_p^\circ \tag{3.15}$$

where C and h_p° are the specific heat and the reference enthalpy of the particle cloud. In the present problem, we will assume that

$$C_{p_0} = C_{p_i} = C \equiv C_p \tag{3.16}$$

This assumption greatly simplifies the mathematical analysis while it still retains the essences of the problem. The energy equation for the gas mixture can thus be written as

$$\rho' u' c_p \frac{\partial T'}{\partial x'} + \rho' v' c_p \frac{\partial T'}{\partial y'} = \frac{\partial}{\partial x'}\left(k \frac{\partial T'}{\partial x'}\right) + \frac{\partial}{\partial y'}\left(k \frac{\partial T'}{\partial y'}\right)$$

$$+ u' \frac{\partial p'}{\partial x'} + v' \frac{\partial p'}{\partial y'} + Q' - \omega_p' \left[c_p T_p' - c_p T' \right]$$

$$- \left[\omega_p' h_p^{o} + \omega_o' h_o^{o} + \omega_i h_i^{o} \right] \qquad (3.17)$$

Q' is the rate of heat transfer per unit volume between the particle cloud and the gas mixture. To the same approximation as the Stokes drag law, this rate of heat transfer can be expressed as

$$Q = n_p \frac{k}{\sigma} 4\pi\sigma^2 (T_p' - T') = \rho_p' c_p u_1 \left[(T_p' - T') \frac{1}{\lambda_{T_0}} \left(\frac{\bar{\sigma}_0}{\sigma_0}\right)^2 \frac{\mu}{\mu_1} \right] \qquad (3.18)$$

where

$$\lambda_{T_0} = \frac{3}{2} P_r \lambda_{v_0} \qquad (3.19)$$

The energy equation for the particle cloud reads as

$$\rho_p' u_p' c_p \frac{\partial T_p'}{\partial x'} + \rho_p' v_p' c_p \frac{\partial T_p'}{\partial y'} = - Q' \qquad (3.20)$$

The equation of state for the gas mixture is

$$p' = \rho' R_M T' \qquad (3.21)$$

where

$$R_M = \left(\frac{K}{\mathcal{M}_o} + (1-K) \frac{1}{\mathcal{M}_i} \right) \bar{R} \qquad (3.22)$$

$\bar{R}$ is the universal gas constant and $\mathcal{M}_o$ and $\mathcal{M}_i$ are the molecular weights of oxidizer and the inert gas. When $\mathcal{M}_o$ and $\mathcal{M}_i$ are approximately equal, eqn. 3.21 can be written as

$$p' = \rho' R T \qquad (3.23)$$

where $R = \bar{R}/\mathcal{M}_0$

Due to the chemical reaction between the particles and the oxidizer, the particle size is decreasing. The equation governing the particle radius is

$$u_p' \frac{\partial}{\partial x'}\left(\frac{\sigma}{\sigma_0}\right)^3 + v_p' \frac{\partial}{\partial y'}\left(\frac{\sigma}{\sigma_0}\right)^3 = \frac{1}{n_p m_0}\,\omega_p' \qquad (3.24)$$

The reaction mechanism is basically governed by the rate of oxidizer diffusion towards the particles and the kinetic rate of chemical reaction. We will assume that the chemical rate of reaction is much faster than the rate of diffusion. Following the first order reaction

$$\nu_0 C_0 + \nu_p C_p = \nu_i C_i \qquad (3.25)$$

where ν_0 , ν_p , and ν_i are stochiometric coefficients for the oxidizer, the particles and the inert gas and C_0 , C_p , and C_i are their respective chemical symbols, the rate of particle cloud being consumed is

$$\omega_p' = -\frac{\nu_p \mathcal{M}_p}{\nu_0 \mathcal{M}_0}\,\frac{\rho_p' \kappa u_0}{\lambda_{D_0}}\left(\frac{\sigma_0}{\sigma}\right)^2 \frac{\mu}{\mu_1}$$

$$= -\gamma_p \frac{\rho_p' \kappa u_0}{\lambda_{D_0}}\left(\frac{\sigma_0}{\sigma}\right)^2 \frac{\mu}{\mu_1} \qquad (3.26)$$

and

$$\lambda_{D_0} = 3/2\; S_c \lambda_{v_0} \qquad (3.27)$$

Because of the very high molecular collision frequency, the gas mixture within a thin layer around the particle is actually in equilibrium with the particle. Thus, the velocity associated with the rate of mass being produced has the local average particle velocity:

$$m'_{x\cdot} = m'_{px\cdot} = u'_p \tag{3.28}$$

$$m'_{y\cdot} = m'_{py\cdot} = v'_p \tag{3.29}$$

II. PROBLEM OF A LAMINAR MIXING ZONE

The problem being investigated here consists of two half infinite streams. The upper stream is a gas mixture of oxidizer and an inert gas. The lower stream consists of a cloud of solid fuel particles suspended in an inert gas. The two streams initially have constant temperatures T_1 and T_2 moving with constant velocities u_1 and u_2 parallel to the x' axis. Let $x'=0$ be the point where two streams start to mix. For $x'>0$, oxidizer from the upper stream diffuses into the lower stream and combustion takes place where there are solid particles.

Introducing non-dimensional quantities with respect to the values at $y'=+\infty$, we have $u=u'/u_1$, $v^*=v'/u_1$, $u_p=u_p'/u_1$, $v_p^*=v_p'/u_1$, $T=T'/T_1$, $\rho=\rho'/\rho_1$, $T_p=T_p'/T_1$, $\rho_p=\rho_p'/\rho_1$, $P=P'/\rho_1 u_1^2$, $x^*=x'/\lambda$, $y^*=y'/\lambda$, where λ is some characteristic length of the problem as one of the relaxation lengths, λ_{D_0}. Defining $\bar{H}=\frac{1}{C_p T_1}\left[\gamma_p h_p^\circ + h_o^\circ - \gamma_i h_i^\circ\right]$ where $\gamma_p=\frac{\nu_p m_p}{\nu_0 m_0}$ and $\nu_i=\frac{\nu_i m_i}{\nu_0 m_0}$ and assuming both Schmidt number and Prandtl number as constants, the governing equations read as follows:

$$\frac{\partial}{\partial x^*}(\rho u)+\frac{\partial}{\partial y^*}(\rho v)=\gamma_p \Lambda_D \rho_p K\left(\frac{\sigma_0}{\sigma}\right)^2\frac{\mu}{\mu_1} \tag{3.30}$$

$$\frac{\partial}{\partial x^*}(\rho_p u_p)+\frac{\partial}{\partial y^*}(\rho_p v_p)=-\gamma_p \Lambda_D \rho_p K\left(\frac{\sigma_0}{\sigma}\right)^2\frac{\mu}{\mu_1} \tag{3.31}$$

$$\rho u \frac{\partial u}{\partial x^*} + \rho v^* \frac{\partial u}{\partial y^*} = \Lambda_v \rho_p (u_p - u) \left(\frac{\sigma_0}{\sigma}\right)^2 \frac{\mu}{\mu_1} - \frac{\partial P}{\partial x^*}$$

$$- \Lambda_D \rho_p K \left(\frac{\sigma_0}{\sigma}\right)^2 \frac{\mu}{\mu_1} (u - u_p') + \frac{1}{Re} \left[\frac{\partial}{\partial x^*} \left(\frac{\mu}{\mu_1} \frac{\partial u}{\partial x^*} \right) + \frac{\partial}{\partial y^*} \left(\frac{\mu}{\mu_1} \frac{\partial u}{\partial y^*} \right) \right] \qquad (3.32)$$

$$\rho_p u_p \frac{\partial u_p}{\partial x^*} + \rho_p v_p^* \frac{\partial u_p}{\partial y^*} = - \gamma_p \Lambda_v \rho_p \left(\frac{\sigma_0}{\sigma}\right)^2 \frac{\mu}{\mu_1} (u_p - u) \qquad (3.33)$$

$$\rho u \frac{\partial v^*}{\partial x^*} + \rho v^* \frac{\partial v^*}{\partial y^*} = \Lambda_v \rho_p (v_p^* - v^*) \left(\frac{\sigma_0}{\sigma}\right)^2 \frac{\mu}{\mu_1} - \frac{\partial P}{\partial y^*}$$

$$+ \frac{1}{Re} \left[\frac{\partial}{\partial x^*} \left(\frac{\mu}{\mu_1} \frac{\partial v^*}{\partial x^*} \right) + \frac{\partial}{\partial y^*} \left(\frac{\mu}{\mu_1} \frac{\partial v^*}{\partial y^*} \right) \right] + \gamma_p \Lambda_D \rho_p K \left(\frac{\sigma_0}{\sigma}\right)^2 \frac{\mu}{\mu_1} (v_p^* - v^*) \qquad (3.34)$$

$$\rho_p u_p \frac{\partial v_p^*}{\partial x^*} + \rho_p v_p^* \frac{\partial v_p^*}{\partial y^*} = - \Lambda_v \rho_p \left(\frac{\sigma_0}{\sigma}\right)^2 \left(\frac{\mu}{\mu_1}\right) (v_p^* - v^*) \qquad (3.35)$$

$$\rho u \frac{\partial T}{\partial x^*} + \rho v^* \frac{\partial T}{\partial y^*} = (\gamma - 1) M_1^2 \left(u \frac{\partial P}{\partial x^*} + v \frac{\partial P}{\partial y^*} \right)$$

$$+ \Lambda_T \rho_p \left(\frac{\sigma_0}{\sigma}\right)^2 \left(\frac{\mu}{\mu_1}\right) (T_p - T) + \frac{1}{Re Pr} \Big[\frac{\partial}{\partial x^*} \left(\frac{\mu}{\mu_1} \frac{\partial T}{\partial x^*} \right) + \frac{\partial}{\partial y^*} \left(\frac{\mu}{\mu_1} \frac{\partial T}{\partial y^*} \right)$$

$$+ \gamma_p \Lambda_D \rho_p K \left(\frac{\sigma_0}{\sigma}\right)^2 \frac{\mu}{\mu_1} \left[T_p - T + \bar{H} \right] \qquad (3.36)$$

$$\rho_p u_p \frac{\partial T_p}{\partial x^*} + \rho_p v_p^* \frac{\partial T_p}{\partial y^*} = - \Lambda_T \rho_p \left(\frac{\sigma_0}{\sigma}\right)^2 \frac{\mu}{\mu_1} (T_p - T) \qquad (3.37)$$

$$\rho u \frac{\partial K}{\partial x^*} + \rho v^* \frac{\partial K}{\partial y^*} = \frac{1}{Sc\,Re}\left[\frac{\partial}{\partial x^*}\left(\frac{\mu}{\mu_1}\frac{\partial K}{\partial x^*}\right) + \frac{\partial}{\partial y^*}\left(\frac{\mu}{\mu_1}\frac{\partial K}{\partial y^*}\right)\right]$$

$$- (1 + \nu_p K)\Lambda_D \rho_p K \left(\frac{\sigma_0}{\sigma}\right)^2 \frac{\mu}{\mu_1} \tag{3.38}$$

$$u_p \frac{\partial}{\partial x^*}\left(\frac{\sigma}{\sigma_0}\right)^3 + v_p^* \frac{\partial}{\partial y^*}\left(\frac{\sigma}{\sigma_0}\right)^3 = -\frac{2}{3}\nu_p \Lambda_D K \frac{\mu}{\mu_1} \tag{3.39}$$

$$p = \frac{1}{\gamma M_1^2}\rho T \tag{3.40}$$

The momentum equations, as well as the energy equations and the continuity equation governing the oxidizer concentration K, can be simplified by using boundary layer approximation. As was shown in the problem for the flow over an infinite rotating disc, the momentum boundary layer thickness for the particle cloud and the fluid is of the same order as expected. Also, both v^* and v_p^* are of the order of $(Re)^{-1/2}$. Thus, to be consistent, we have to suppress the momentum equation along the y-axis for the particle cloud as well as for the gas. In addition, in neglecting $\partial p/\partial y$ in the Prandtl's boundary layer theory, an additional condition that

$$v^* = v_p^* \tag{3.41}$$

is introduced. The error introduced in assuming eqn. 3.41 is consistent with the boundary layer approximation. The equations governing the gas phase are thus:

$$\frac{\partial}{\partial x^*}(\rho u) + \frac{\partial}{\partial y^*}(\rho v) = \nu_p \Lambda_D \rho_p K \left(\frac{\sigma_0}{\sigma}\right)^2 \frac{\mu}{\mu_1} \tag{3.42}$$

$$\rho u \frac{\partial u}{\partial x^*} + \rho v \frac{\partial u}{\partial y^*} = \frac{1}{Re} \frac{\partial}{\partial y^*}\left(\frac{\mu}{\mu_1} \frac{\partial u}{\partial y^*}\right) + \Lambda_v \rho_p (u_p - u)\left(\frac{\sigma_0}{\sigma}\right)^2 \frac{\mu}{\mu_1}$$

$$+ r_p \Lambda_D \rho_p K \left(\frac{\sigma_0}{\sigma}\right)^2 \frac{\mu}{\mu_1} (u_p - u) \tag{3. 43}$$

$$\rho u \frac{\partial T}{\partial x^*} + \rho v \frac{\partial T}{\partial y^*} = \frac{1}{Pr\, Re} \frac{\partial}{\partial y^*}\left(\frac{\mu}{\mu_1} \frac{\partial T}{\partial y^*}\right) + \Lambda_T \rho_p \left(\frac{\sigma_0}{\sigma}\right)^2 \frac{\mu}{\mu_1} (T_p - T)$$

$$+ r_p \Lambda_D \rho_p K \left(\frac{\sigma_0}{\sigma}\right)^2 \frac{\mu}{\mu_1} \left(T_p - T + \bar{H}\right) \tag{3. 44}$$

$$\rho u \frac{\partial K}{\partial x^*} + \rho v^* \frac{\partial K}{\partial y^*} = \frac{1}{Sc\, Re} \frac{\partial}{\partial y^*}\left(\frac{\mu}{\mu_1} \frac{\partial K}{\partial y^*}\right)$$

$$- \left(1 + r_p K\right) \Lambda_D \rho_p K \left(\frac{\sigma_0}{\sigma}\right)^2 \frac{\mu}{\mu_1} \tag{3. 45}$$

Similarly, the governing equations for the upper stream are:

$$\frac{\partial}{\partial x^*}(\rho u) + \frac{\partial}{\partial y^*}(\rho v^*) = 0 \tag{3. 46}$$

$$\rho u \frac{\partial u}{\partial x^*} + \rho v \frac{\partial u}{\partial y^*} = \frac{1}{Re} \frac{\partial}{\partial y^*}\left(\frac{\mu}{\mu_1} \frac{\partial u}{\partial y^*}\right) \tag{3. 47}$$

$$\rho u \frac{\partial T}{\partial x^*} + \rho v^* \frac{\partial T}{\partial y^*} = \frac{1}{Re\, Pr} \frac{\partial}{\partial y^*}\left(\frac{\mu}{\mu_1} \frac{\partial T}{\partial y^*}\right) \tag{3. 48}$$

$$\rho u \frac{\partial K}{\partial x^*} + \rho v^* \frac{\partial K}{\partial y^*} = \frac{1}{Re\, Sc} \frac{\partial}{\partial y^*}\left(\frac{\mu}{\mu_1} \frac{\partial K}{\partial y^*}\right) \tag{3. 49}$$

The problem has to satisfy the free stream conditions at $y^* = +\infty$ and $y^* = -\infty$, and the gas mixture density ρ, velocity u, temperature T, and oxidizer K and their appropriate derivatives are continuous across the interface.

III. SOLUTION FOR THE INITIAL DEVELOPMENT OF THE COMBUSTION ZONE

For Maxwellian molecules, the viscosity μ for the gas - mixture can be assumed to be simply proportional to the gas - mixture temperature. Thus,

$$\frac{\mu}{\mu_1} = \frac{T'}{T_1} = T \tag{3.50}$$

We further assume that the molecular weights of the gas species are approximately equal. Since the pressure is a constant according to the boundary layer approximation, the gas - mixture density is inversely proportional to the temperature according to the equation of state

$$\rho = \frac{1}{T} \tag{3.51}$$

Howarth's$^{(4)}$ transformation is particularly convenient to treat compressible boundary layer flow when eqns. 3.50 and 3.51 are true. Introducing a new set of coordinate axes such that

$$\begin{aligned} x &= x^* \\ y &= \int^{y^*} \rho \, dy^* \end{aligned} \tag{3.52}$$

and two new variables V and V_p with

$$V = \rho v^* + u \frac{\partial y}{\partial x^*} \tag{3.53}$$

$$V_p = \rho_p v_p^* + u_p \frac{\partial y}{\partial x^*} \tag{3.54}$$

With eqns. 3.52, 3.53, and 3.54, we observe that

$$u\frac{\partial}{\partial x^*} + v^*\frac{\partial}{\partial y^*} = u\left[\frac{\partial}{\partial x} + \frac{\partial y}{\partial x^*}\frac{\partial}{\partial y}\right] + \frac{1}{\rho}\left[v - u\frac{\partial y}{\partial x^*}\right]\rho\frac{\partial}{\partial y}$$

$$= u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \qquad (3.55)$$

$$u_p\frac{\partial}{\partial x^*} + v_p^*\frac{\partial}{\partial y^*} = u_p\left[\frac{\partial}{\partial x} + \frac{\partial y}{\partial x^*}\frac{\partial}{\partial y}\right] + \frac{1}{\rho}\left[v_p - u_p\frac{\partial y}{\partial x^*}\right]\rho\frac{\partial}{\partial y}$$

$$= u_p\frac{\partial}{\partial x} + v_p\frac{\partial}{\partial y} \qquad (3.56)$$

$$\frac{\partial}{\partial y^*}\left(\frac{\mu}{\mu_1}\frac{\partial}{\partial y^*}\right) = \rho\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) = \rho\frac{\partial^2}{\partial y^2} \qquad (3, 57)$$

After some simple calculation, we also obtain the following relations:

$$\frac{\partial}{\partial x^*}(\rho u) + \frac{\partial}{\partial y^*}(\rho v^*) = \rho\frac{\partial u}{\partial x} + \rho\frac{\partial v}{\partial y} \qquad (3.58)$$

$$\frac{1}{\rho}\left[\frac{\partial}{\partial x^*}(\rho_p u_p) + \frac{\partial}{\partial y^*}(\rho_p v_p)\right] = \frac{\partial}{\partial x}\left(\frac{\rho_p}{\rho}u_p\right) + \frac{\partial}{\partial y}\left(\frac{\rho_p}{\rho}v_p\right) \qquad (3.59)$$

Defining $\rho_r = \frac{\rho_p}{\rho}$, the governing equations after the Howarth transformation read:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \gamma_p \Lambda_D \rho_r K \left(\frac{\sigma_0}{\sigma}\right)^2 T \qquad (3.60)$$

$$\frac{\partial}{\partial x}\rho_r u_p + \frac{\partial}{\partial y}\rho_r v_p = -\gamma_p \Lambda_D \rho_r K \left(\frac{\sigma_0}{\sigma}\right)^2 T \qquad (3.61)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{Re}\frac{\partial^2 u}{\partial y^2} + \Lambda_v P_r (u_p - u)\, T \left(\frac{\sigma_0}{\sigma}\right)^2$$
$$+ \gamma_p \Lambda_0 P_r K\, T\, (u_p - u)\left(\frac{\sigma_0}{\sigma}\right)^2 \qquad (3.62)$$

$$u_p\frac{\partial u_p}{\partial x} + v_p\frac{\partial u_p}{\partial x} = -\Lambda_v (u_p - u)\, T\left(\frac{\sigma_0}{\sigma}\right)^2 \qquad (3.63)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{Re Pr}\frac{\partial^2 T}{\partial y^2} + \Lambda_T P_r (T_p - T)\, T\left(\frac{\sigma_0}{\sigma}\right)^2$$

$$+ r_p \Lambda_0 P_p K\, T\left(\frac{\sigma_0}{\sigma}\right)^2 (T_p - T + \bar{H}) \qquad (3.64)$$

$$u_p\frac{\partial T_p}{\partial x} + v_p\frac{\partial T_p}{\partial y} = -\Lambda_T (T_p - T)\, T\left(\frac{\sigma_0}{\sigma}\right)^2 \qquad (3.65)$$

$$u\frac{\partial K}{\partial x} + v\frac{\partial K}{\partial y} = \frac{1}{Ro\, Sc}\frac{\partial^2 K}{\partial y^2} - (1 + \gamma_p K)\Lambda_D P_r K\, T\left(\frac{\sigma_0}{\sigma}\right)^2 \qquad (3.66)$$

$$u_p\frac{\partial}{\partial x}\left(\frac{\sigma}{\sigma_0}\right)^2 + v_p\frac{\partial}{\partial y}\left(\frac{\sigma}{\sigma_0}\right)^2 = -\,2/3\, \gamma_p \Lambda_0 K\, T \qquad (3.67)$$

The governing equations for the upper stream are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (3.68)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{Re}\frac{\partial^2 u}{\partial y^2} \qquad (3.69)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{Re\, Pr}\frac{\partial^2 T}{\partial y^2} \qquad (3.70)$$

$$u\frac{\partial K}{\partial x} + v\frac{\partial K}{\partial y} = \frac{1}{Re\,Sc}\frac{\partial^2 K}{\partial y^2} \qquad (3.71)$$

The simplification given by using the Howarth transformation coupled with the condition $\rho\mu = \mu_1$ is now apparent. For the flow field where there is no particle cloud, Howarth's transformation reduces the governing equations describing the compressible flow into a set of equations describing incompressible flow. Here, the momentum equation is decoupled from the energy equation and the effect of temperature variation is introduced only through transforming the solution back to the "compressible form" by using eqns. 3. 52, 3. 53, and 3. 54. The transformation only changes the vertical scale and the horizontal, or flow scale, remains unchanged, i. e., $x = x^*$. For the flow stream consisting of a particle cloud, the momentum equation and the temperature equation are still coupled, but the equations are of a simpler form. Furthermore, instead of considering ρ_p and ρ separately, we only need to consider one function $\rho_r = \rho_p/\rho$.

From eqns. 3. 60 and 3. 61, we can introduce two stream functions φ and φ_p , φ for the gas velocities and φ_p for the particle cloud velocities.

$$u = \frac{\partial \varphi}{\partial y} \qquad (3.72)$$

$$v = -\frac{\partial \varphi}{\partial x} + W \qquad (3.73)$$

$$u_p = \frac{1}{\rho_r}\frac{\partial \varphi_p}{\partial y} \qquad (3.74)$$

$$v_p = \frac{-1}{\rho_r}\left(\frac{\partial \varphi_p}{\partial x} + W\right) \qquad (3.75)$$

and

$$W = \int_{-\infty}^{y} \gamma_p \Lambda_0 K \rho_r T \left(\frac{\sigma_0}{\sigma}\right)^2 dy \tag{3.76}$$

Introduce two new independent variables ξ and η :

$$\xi = x \tag{3.77}$$

$$\eta = \left(\frac{Re}{x}\right)^{1/2} y = \left(\frac{Re}{\xi}\right)^{1/2} y$$

and let

$$\varphi = \left(\frac{\xi}{Re}\right)^{1/2} f(\xi, \eta) \tag{3.78}$$

$$\varphi_p = \left(\frac{\xi}{Re}\right)^{1/2} f_p(\xi, \eta) \tag{3.79}$$

the velocities for the gas-mixture are

$$u = \frac{\partial \varphi}{\partial y} = \frac{\partial f}{\partial \eta} \tag{3.80}$$

$$v = \left(\frac{\xi}{Re}\right)^{1/2}\left(-\frac{\partial f}{\partial \xi} + \frac{1}{2}\frac{\eta}{\xi}\frac{\partial f}{\partial \eta} - \frac{1}{2\xi} f\right) + W \tag{3.81}$$

$$u_p = \frac{1}{\rho_r}\frac{\partial f_p}{\partial \eta} \tag{3.82}$$

$$v_p = \frac{1}{\rho_r}\left(\frac{\xi}{Re}\right)^{1/2}\left(-\frac{\partial f_p}{\partial \xi} + \frac{1}{2}\frac{\eta}{\xi}\frac{\partial f_p}{\partial \eta} - \frac{1}{2\xi} f_p\right) - \frac{1}{\rho_r} W \tag{3.83}$$

where W can be expressed as

$$W = \left(\frac{\xi}{Re}\right)^{1/2} \int_{-\infty}^{\eta} \gamma_p \Lambda_0 \rho_r K T \left(\frac{\sigma_0}{\sigma}\right)^2 d\eta$$

Expressing eqns. 3.62 - 3.67 in ξ and η and using eqns. 3.80 - 3.82, we obtain:

$$\frac{\partial^3 f}{\partial \eta^3} + \frac{f}{2}\frac{\partial^2 f}{\partial \eta^2} = \xi\left(\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \eta \partial \xi} - \frac{\partial f}{\partial \xi}\frac{\partial^2 f}{\partial \eta^2}\right) + \xi \Lambda_D \frac{\partial^2 f}{\partial \eta^2}\int_{-\infty}^{\eta} \gamma_p P_r K T \left(\frac{\sigma_0}{\sigma}\right)^2 d\eta$$

$$- \xi \Lambda_v P_r T \left(\frac{\sigma_0}{\sigma}\right)^2 \left(1 + \gamma_p \frac{\Lambda_D}{\Lambda_v} K\right)\left(\frac{1}{P_r}\frac{\partial f_p}{\partial \eta} - \frac{\partial f}{\partial \eta}\right) \tag{3.84}$$

$$\frac{f_p}{P_r 2}\frac{\partial}{\partial \eta}\left(\frac{1}{P_r}\frac{\partial f_p}{\partial \eta}\right) = \xi\left[\frac{1}{P_r}\frac{\partial f_p}{\partial \eta}\frac{\partial}{\partial \xi}\left(\frac{1}{P_r}\frac{\partial f_p}{\partial \eta}\right) - \frac{1}{P_r}\frac{\partial f_p}{\partial \xi}\frac{\partial}{\partial \eta}\left(\frac{1}{P_r}\frac{\partial f_p}{\partial \eta}\right)\right]$$

$$- \xi \Lambda_D \frac{1}{P_r}\frac{\partial}{\partial \eta}\left(\frac{1}{P_r}\frac{\partial f_p}{\partial \eta}\right)\int_{-\infty}^{\eta} \gamma_p \Lambda_D P_r K T \left(\frac{\sigma_0}{\sigma}\right)^2 d\eta + \Lambda_v T \left(\frac{\sigma_0}{\sigma}\right)^2 \left(\frac{1}{P_r}\frac{\partial f_p}{\partial \eta} - \frac{\partial f}{\partial \eta}\right) \tag{3.85}$$

$$\frac{1}{P_r}\frac{\partial^2 T}{\partial \eta^2} + \frac{f}{2}\frac{\partial T}{\partial \eta} = \xi\left(\frac{\partial f}{\partial \eta}\frac{\partial T}{\partial \xi} - \frac{\partial f}{\partial \xi}\frac{\partial T}{\partial \eta}\right) + \Lambda_D \frac{\partial T}{\partial \eta}\int_{-\infty}^{\eta} \gamma_p P_r K T \left(\frac{\sigma_0}{\sigma}\right)^2 d\eta$$

$$- \xi \Lambda_T P_r T (T_p - T)\left(\frac{\sigma_0}{\sigma}\right)^2 - \xi \Lambda_D \gamma_p P_r K T \left(\frac{\sigma_0}{\sigma}\right)^2 (T_p - T + \bar{H}) \tag{3.86}$$

$$\frac{1}{2 P_r} f_p \frac{\partial T_p}{\partial \eta} = \xi\left(\frac{1}{P_r}\frac{\partial f_p}{\partial \eta}\frac{\partial T}{\partial \xi} - \frac{1}{P_r}\frac{\partial f_p}{\partial \xi}\frac{\partial T}{\partial \eta}\right) + \xi \Lambda_T \left(\frac{\sigma_0}{\sigma}\right)^2 (T_p - T) T$$

$$- \xi \Lambda_D \frac{1}{P_r}\frac{\partial T}{\partial \eta}\int_{-\infty}^{\eta} \gamma_p P_r K T \left(\frac{\sigma_0}{\sigma}\right)^2 d\eta \tag{3.87}$$

$$\frac{1}{S_c}\frac{\partial^2 K}{\partial \eta^2} + \frac{f}{2}\frac{\partial K}{\partial \eta} = \left(\frac{\partial f}{\partial \eta}\frac{\partial K}{\partial \xi} - \frac{\partial f}{\partial \xi}\frac{\partial K}{\partial \eta}\right) + \xi \Lambda_D \frac{\partial K}{\partial \eta}\int_{-\infty}^{\eta} \gamma_p P_r K T \left(\frac{\sigma_0}{\sigma}\right)^2 d\eta$$

$$+ \xi \Lambda_D (1 + \gamma_p K) P_r K T \left(\frac{\sigma_0}{\sigma}\right)^2 \tag{3.88}$$

$$\frac{1}{2\rho_r} f_p \frac{\partial}{\partial\eta}\left(\frac{\sigma}{\sigma_0}\right)^2 = \xi\left[\frac{1}{\rho_r}\frac{\partial f_p}{\partial\eta}\frac{\partial}{\partial\xi}\left(\frac{\sigma}{\sigma_0}\right)^2 - \frac{1}{\rho_r}\frac{\partial f_p}{\partial\xi}\frac{\partial}{\partial\eta}\left(\frac{\sigma}{\sigma_0}\right)^2\right]$$

$$-\xi\Lambda_0\frac{1}{\rho_r}\frac{\partial}{\partial\eta}\left(\frac{\sigma}{\sigma_0}\right)^2\int_{-\infty}^{\eta} r_p K T \rho_r \left(\frac{\sigma_0}{\sigma}\right)^2 d\eta + \frac{2}{3}\xi\Lambda_0 K T r_p \qquad (3.89)$$

The relation $v^* = v_p^*$ takes the following form:

$$\frac{1}{\rho_r}\left[-\frac{\partial f_p}{\partial\xi} + \frac{1}{2}\frac{\eta}{\xi}\frac{\partial f_p}{\partial\eta} - \frac{1}{2} f_p\right] - \left[-\frac{\partial f}{\partial\xi} + \frac{1}{2}\frac{\eta}{\xi}\frac{\partial f}{\partial\eta} - \frac{f}{2\xi}\right]$$

$$= \left(1 - \frac{1}{\rho_r}\right)\int_{-\infty}^{\eta} r_p \Lambda_0 K T \left(\frac{\sigma_0}{\sigma}\right)^2 d\eta - \left(\frac{Re}{\xi}\right)^{1/2}\frac{\partial y}{\partial x^*}\left(\frac{\partial f}{\partial\eta} - \frac{1}{\rho_r}\frac{\partial f_p}{\partial\eta}\right) \qquad (3.90)$$

The region where $\rho_r = 0$, the equations are

$$\frac{\partial^3 f}{\partial\eta^3} + \frac{f}{2}\frac{\partial^2 f}{\partial\eta^2} = \left[\frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial\eta\partial\xi} - \frac{\partial f}{\partial\xi}\frac{\partial^2 f}{\partial\eta^2}\right] \qquad (3.91)$$

$$\frac{1}{P_r}\frac{\partial^2 T}{\partial\eta^2} + \frac{f}{2}\frac{\partial T}{\partial\eta} = \left[\frac{\partial f}{\partial\eta}\frac{\partial T}{\partial\xi} - \frac{\partial f}{\partial\xi}\frac{\partial T}{\partial\eta}\right] \qquad (3.92)$$

$$\frac{1}{S_c}\frac{\partial^2 K}{\partial\eta^2} + \frac{f}{2}\frac{\partial K}{\partial\eta} = \xi\left[\frac{\partial f}{\partial\eta}\frac{\partial K}{\partial\xi} - \frac{\partial f}{\partial\xi}\frac{\partial K}{\partial\eta}\right] \qquad (3.93)$$

To demonstrate the presence of the particles and the chemical reaction between the particles and the oxidizing agent in the gas-mixture, it is sufficient to solve the special case where $U_1 = U_2$. From eqns. 3.84 and 3.85, we obtain that

$$f = \eta \qquad (3.94)$$

and

$$\frac{1}{\rho_r}\frac{\partial f_p}{\partial\eta} = 1 \qquad (3.95)$$

and streamlines are lines y = constant. With eqns. 3.94 and 3.95, the governing equations are simplified to the following form:

$$\frac{1}{P_r}\frac{\partial^2 T}{\partial \eta^2}+\frac{\eta}{2}\frac{\partial T}{\partial \eta} = \xi\frac{\partial T}{\partial \xi}+(\xi\Lambda_D)\frac{\partial T}{\partial \eta}\int_{-\infty}^{\eta} r_p \rho_r K T \left(\frac{\sigma_0}{\sigma}\right)^2 d\eta - \xi\Lambda_T \rho_r T(T_p-T)\left(\frac{\sigma_0}{\sigma}\right)^2$$
$$- \xi\Lambda_D r_p \rho_r K T \left(\frac{\sigma_0}{\sigma}\right)^2 (T_p - T + \bar{H}) \qquad (3.96)$$

$$\frac{1}{2\rho_r} f_p \frac{\partial T_p}{\partial \eta} = \xi\left[\frac{\partial T_p}{\partial \xi} - \frac{1}{\rho_r}\frac{\partial f_p}{\partial \xi}\frac{\partial T_p}{\partial \eta}\right] - \xi\Lambda_D\frac{\partial T_p}{\partial \eta}\int_{-\infty}^{\eta} r_p K T\left(\frac{\sigma_0}{\sigma}\right)^2 d\eta$$
$$+ \xi\Lambda_T \rho_r T(T_p - T)\left(\frac{\sigma_0}{\sigma}\right)^2 \qquad (3.97)$$

$$\frac{1}{S_c}\frac{\partial^2 K}{\partial \eta^2}+\frac{f}{2}\frac{\partial K}{\partial \eta} = \xi\frac{\partial K}{\partial \xi} + \xi\Lambda_D\frac{\partial K}{\partial \eta}\int_{-\infty}^{\eta} r_p \rho_r K T\left(\frac{\sigma_0}{\sigma}\right)^2 d\eta$$
$$+ \xi\Lambda_D(1+r_p K)\rho_r K T\left(\frac{\sigma_0}{\sigma}\right)^2 \qquad (3.98)$$

$$\frac{1}{2\rho_r} f_p \frac{\partial}{\partial \eta}\left(\frac{\sigma}{\sigma_0}\right)^2 = \xi\left[\frac{\partial}{\partial \xi}\left(\frac{\sigma}{\sigma_0}\right)^2 - \frac{1}{\rho_r}\frac{\partial f_p}{\partial \xi}\frac{\partial}{\partial \eta}\left(\frac{\sigma}{\sigma_0}\right)^2\right] + \frac{2}{3}\xi\Lambda_D K T r_p$$
$$- \xi\Lambda_D\frac{1}{\rho_r}\frac{\partial}{\partial \eta}\left(\frac{\sigma}{\sigma_0}\right)^2\int_{-\infty}^{\eta} r_p K T \rho_r \left(\frac{\sigma}{\sigma_0}\right)^2 d\eta \qquad (3.99)$$

$$\frac{-1}{2\rho_r} f_p + \frac{\eta}{2} = \xi\frac{1}{\rho_r}\frac{\partial f_p}{\partial \xi} + \Lambda_D\left(1+\frac{1}{\rho_r}\right)\int_{-\infty}^{\eta}\Lambda_D K T\left(\frac{\sigma_0}{\sigma}\right)^2 d\eta \qquad (3.100)$$

and for $\rho_r = 0$:

$$\frac{1}{P_r}\frac{\partial^2 T}{\partial \eta^2} + \frac{\eta}{2}\frac{\partial T}{\partial \eta} = \frac{\partial T}{\partial \xi} \qquad (3.101)$$

$$\frac{1}{S_c}\frac{\partial^2 K}{\partial \eta^2} + \frac{\eta}{2}\frac{\partial K}{\partial \eta} = \frac{\partial K}{\partial \eta} \qquad (3.102)$$

In the initial portion of the mixing zone, the heat of combustion added to the gas mixture is still small. Also, the particle cloud temperature differs very little from the temperature of the gas mixture. The flow field is more like the mixing of two gas streams containing no particles and without chemical reaction. Thus, the presence of the particle cloud and its chemical reaction with the oxidizing agent in the gas mixture can be treated as a perturbation.

In the present treatment, it might be more convenient to use λ_{D_0} as the characteristic length λ. Thus we have

$$\begin{aligned} \Lambda_D &= 1 \\ \Lambda_T &= P_r / S_c = L_e \\ \Lambda_V &= 3/2 \; S_c \end{aligned} \qquad (3.103)$$

and for small values of ξ, or for $x << \lambda_{D_0}$, the perturbation scheme is to express every term in its Taylor series of ξ

$$\begin{aligned} f_p(\xi,\eta) &= f_p^{(0)}(\eta) + \xi f_p^{(1)}(\eta) + \cdots\cdots \\ T(\xi,\eta) &= T^{(0)}(\eta) + \xi T^{(1)}(\eta) + \cdots\cdots \\ T_p(\xi,\eta) &= T_p^{(0)}(\eta) + \xi T_p^{(1)}(\eta) + \cdots\cdots \\ K(\xi,\eta) &= K^{(0)}(\eta) + \xi K^{(1)}(\eta) + \cdots\cdots \\ \rho_r(\xi,\eta) &= \rho_r^{(0)}(\eta) + \xi \rho_r^{(1)}(\eta) + \cdots\cdots \\ \frac{\sigma^2}{\sigma_0}(\xi,\eta) &= g^{(0)}(\eta) + \xi g^{(1)}(\eta) + \cdots\cdots \end{aligned} \qquad (3.104)$$

Using 3.104, the zeroth and the first order equations can be easily obtained.

$$\frac{1}{P_r}\frac{d^2T^{(0)}}{d\eta^2} + \frac{\eta}{2}\frac{dT^{(0)}}{d\eta} = 0 \qquad (3.105)$$

$$\frac{1}{2\rho_r^{(0)}} f_p^{(0)} \frac{dT_p^{(0)}}{d\eta} = 0 \qquad (3.106)$$

$$\frac{1}{S_c}\frac{d^2K^{(0)}}{d\eta^2} + \frac{\eta}{2}\frac{dK^{(0)}}{d\eta} = 0 \qquad (3.107)$$

$$\frac{1}{2\rho_r^{(0)}} f_p^{(0)} \frac{dg^{(0)}}{d\eta} = 0 \qquad (3.108)$$

$$\frac{df_p^{(0)}}{d\eta} = \rho_r^{(0)} \qquad (3.109)$$

$$f_p^{(0)} = \rho_r^{(0)}\eta \qquad (3.110)$$

For the region where $\rho_r^{(0)} = 0$, the equations for temperature and concentration K are identical with eqns. 3.105 and 3.107. The first order equations are

$$\frac{1}{P_r}\frac{d^2T^{(1)}}{d\eta^2} + \frac{\eta}{2}\frac{dT^{(1)}}{d\eta} - T^{(1)} = \frac{dT^{(0)}}{d\eta}\int_{-\infty}^{\eta} \gamma_p \rho_r^{(0)} K^{(0)} T^{(0)} g^{(0)}\, d\eta \qquad (3.111)$$

$$- Le\, \rho_r^{(0)} T^{(0)} (T_p^{(0)} - T^{(0)})\, g^{(0)} - \gamma_p \rho_r^{(0)} K^{(0)} T^{(0)} g^{(0)} (T_p^{(0)} - T^{(0)} + \bar{H})$$

$$\frac{1}{Sc}\frac{d^2K^{(1)}}{d\eta^2} + \frac{\eta}{2}\frac{dK^{(1)}}{d\eta} - K^{(1)} = \frac{dK^{(0)}}{d\eta}\int_{-\infty}^{\eta} r_p K^{(0)} T^{(0)} g^{(0)}\, d\eta$$

$$+ (1 + r_p K^{(0)})\rho_r^{(0)} K^{(0)} T^{(0)} g^{(0)} \tag{3.112}$$

$$\frac{f_p^{(0)}}{2\rho_r^{(0)}}\frac{dT_p^{(1)}}{d\eta} - T_p^{(1)} + \frac{1}{\rho_r^{(0)}}\frac{dT_p^{(0)}}{d\eta} f_p^{(1)} = -\frac{dT^{(0)}}{d\eta}\int_{-\infty}^{\eta} r_p K^{(0)} T^{(0)} g^{(0)}\, d\eta$$

$$+ Le\, \rho_r^{(0)} T^{(0)} (T_p^{(0)} - T^{(0)}) g^{(0)} \tag{3.113}$$

$$\frac{f_p^{(0)}}{2\rho_r^{(0)}}\frac{dg^{(1)}}{d\eta} - g^{(1)} + \frac{1}{\rho_r^{(0)}}\frac{dg^{(0)}}{d\eta} f_p^{(1)} = -\frac{1}{\rho_r^{(0)}}\frac{dg^{(0)}}{d\eta}\int_{-\infty}^{\eta} r_p K^{(0)} T^{(0)} \rho_r^{(0)} g^{(0)}\, d\eta$$

$$+ \frac{2}{3} K^{(0)} T^{(0)} r_p \tag{3.114}$$

$$\frac{df_p^{(1)}}{d\eta} = \rho_r^{(1)} \tag{3.115}$$

$$\frac{1}{2}\frac{\rho_r^{(1)}}{\rho_r^{(0)}}\frac{f_p^{(0)}}{\rho_r^{(0)}} - \frac{1}{2\rho_r^{(0)}} f_p^{(1)} - \frac{1}{\rho_r^{(0)}} f_p^{(1)} = (1 + \frac{1}{\rho_r^{(0)}}))\int_{-\infty}^{\eta} K^{(0)} T^{(0)} g^{(0)}\, d\eta \tag{3.116}$$

For the region where $\rho_r = 0$, the first order equations are simply

$$\frac{1}{Pr}\frac{d^2T^{(1)}}{d\eta^2} + \frac{\eta}{2}\frac{dT^{(1)}}{d\eta} - T^{(1)} = 0 \tag{3.117}$$

$$\frac{1}{Sc}\frac{d^2K^{(1)}}{d\eta^2} + \frac{\eta}{2}\frac{dK^{(1)}}{d\eta} - K^{(1)} = 0 \tag{3.118}$$

The boundary conditions for the zeroth order functions are

$$
\begin{aligned}
&K^{(0)}(+\infty) = \bar{K} \qquad && K^{(0)}(-\infty) = 0 \\
&T^{(0)}(+\infty) = 1 && T^{(0)}(-\infty) = T_2/T_1 \\
& && T_p^{(0)}(-\infty) = T_2/T_1 \\
& && \rho_r^{(0)}(-\infty) = \chi \\
& && g^{(0)}(-\infty) = 1
\end{aligned}
\qquad (3.119)
$$

while all the perturbation quantities have to vanish at $\eta = \pm\infty$

The zeroth order quantities can be easily integrated. Defining

$$erf = \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-s^2}\,ds$$

we have

$$T^{(0)} = \frac{1}{2}\left(1 - \frac{T_2}{T_1}\right)\left[erf\left(\frac{P_r^{1/2}\eta}{2}\right) + 1\right] + \frac{T_2}{T_1} \qquad -\infty < \eta < \infty \qquad (3.120)$$

$$K^{(0)} = \frac{\bar{K}}{2}\left[erf\left(\frac{P_r^{1/2}\eta}{2}\right) + 1\right] \qquad -\infty < \eta < \infty \qquad (3.121)$$

$$T_p^{(0)} = \frac{T_2}{T_1} \qquad \eta < 0 \qquad (3.122)$$

$$g^{(0)} = 1 \qquad \eta < 0 \qquad (3.123)$$

$$\rho_r^{(0)} = \chi \qquad \eta < 0 \qquad (3.124)$$

$$f_p^{(0)} = \eta \qquad \eta < 0 \qquad (3.125)$$

To find the first order functions, we assume that $P_r = S_c$ or $L_e = 1$ and $C_p = C$. This assumption is mainly a mathematical convenience, for if the above assumption is removed, the solution will involve more complex integrations. Define a new variable

$$J = \frac{P_r^{1/2}\eta}{2} = \frac{S_c^{1/2}\eta}{2} \qquad (3.126)$$

$$\frac{1}{4}\frac{d^2T^{(1)}}{d\zeta^2}+\frac{\zeta}{2}\frac{dT^{(1)}}{d\zeta}-T^{(1)}=R_1(\zeta) \tag{3.127}$$

$$\frac{1}{4}\frac{dK^{(1)}}{d\zeta}+\frac{\zeta}{2}\frac{dK^{(1)}}{d\zeta}-K^{(1)}=R_2(\zeta) \tag{3.128}$$

$$\begin{aligned}
R^{(1)}(\zeta) = {} & \frac{dT^{(0)}}{d\zeta}\int_{-\infty}^{\zeta} r_p\bar{K}\chi\Big[\frac{1}{4}\big(1-\frac{T_2}{T_1}\big)(\operatorname{erf}\zeta+1)^2+\frac{1}{2}\frac{T_2}{T_1}(\operatorname{erf}\zeta+1)\,d\zeta \\
& +\frac{1}{2}\chi\big(1-\frac{T_2}{T_1}\big)\Big[\frac{1}{2}\big(1-\frac{T_2}{T_1}\big)(\operatorname{erf}\zeta+1)^2+\frac{T_2}{T_1}(\operatorname{erf}\zeta+1)\Big] \\
& -\frac{1}{2}r_p\chi\bar{K}\bar{H}\Big[\frac{1}{2}\big(1-\frac{T_2}{T_1}\big)(\operatorname{erf}\zeta+1)^2+\frac{T_2}{T_1}(\operatorname{erf}\zeta+1)\Big] \\
& +\frac{1}{4}r_p\chi\bar{K}\big(1-\frac{T_2}{T_1}\big)\Big[\frac{1}{2}\big(1-\frac{T_2}{T_1}\big)(\operatorname{erf}\zeta+1)^3+\frac{T_2}{T_1}(\operatorname{erf}\zeta+1)^2\Big] \\
= {} & \frac{1}{2}\chi\bar{K}r_p\big(1-\frac{T_2}{T_1}\big)e^{-\zeta^2}\Big[\frac{1-\frac{T_2}{T_1}}{2\sqrt{\pi}}\zeta(\operatorname{erf}\zeta+1)^2+\frac{1}{\pi}\big(1-\frac{T_2}{T_1}\big) \\
& e^{-\zeta^2}(\operatorname{erf}\zeta+1)-\frac{1}{2\sqrt{\pi}}\big(1-\frac{T_2}{T_1}\big)(\operatorname{erf}\sqrt{2}\,\zeta+1)+\frac{T_2}{T_1}\frac{1}{\pi}e^{-\zeta^2} \\
& +\frac{1}{\sqrt{\pi}}\frac{T_2}{T_1}\zeta(\operatorname{erf}\zeta+1)\Big]+\frac{\chi}{2}\frac{T_2}{T_1}\Big[\big(1-\frac{T_2}{T_1}\big)-r_p\bar{K}\bar{H}\Big](\operatorname{erf}\zeta+1) \\
& +\frac{1}{4}\chi\big(1-\frac{T_2}{T_1}\big)\Big\{\Big[\big(1-\frac{T_2}{T_1}\big)-r_p\bar{K}\bar{H}+r_p\bar{K}\frac{T_2}{T_1}\Big](\operatorname{erf}\zeta+1)^2\Big\} \\
& +\frac{\chi}{8}r_p\bar{K}\big(1-\frac{T_2}{T_1}\big)^2(\operatorname{erf}\zeta+1)^3 \qquad \text{for } \zeta<0 \\
= {} & 0 \qquad \text{for } \zeta>0
\end{aligned} \tag{3.129}$$

$$\begin{aligned} R_2(\zeta) = {} & \frac{\chi}{2}\bar{K}^2 r_p e^{-\zeta^2}\Big[\frac{1-\frac{T_2}{T_1}}{2\sqrt{\pi}}\zeta(\operatorname{erf}\zeta+1)^2+\frac{1}{\pi}\Big(1-\frac{T_2}{T_1}\Big)e^{-\zeta^2}(\operatorname{erf}\zeta+1) \\ & -\frac{1-\frac{T_2}{T_1}}{2\sqrt{\pi}}(\operatorname{erf}\sqrt{2}\,\zeta+1)+\frac{1}{\sqrt{\pi}}\frac{T_2}{T_1}\zeta(\operatorname{erf}\zeta+1)+\frac{1}{\pi}\frac{T_2}{T_1}e^{-\zeta^2}\Big] \\ & +\chi\frac{\bar{K}}{2}\frac{T_2}{T_1}(\operatorname{erf}\zeta+1)+\frac{\bar{K}}{4}\Big[r_p\bar{K}\beta\frac{T_2}{T_1}+\chi\Big(1-\frac{T_2}{T_1}\Big)\Big](\operatorname{erf}\zeta+1)^2 \\ & +\frac{\bar{K}^2}{2}r_p\chi\Big(1-\frac{T_2}{T_1}\Big)(\operatorname{erf}\zeta+1)^3 \qquad \text{for } \zeta<0 \\ = {} & \quad 0 \qquad \text{for } \zeta>0 \end{aligned} \tag{3.130}$$

The remaining first order equations are

$$\frac{\zeta}{2}\frac{dT_p^{(1)}}{d\zeta}-T_p^{(1)}=\frac{-1}{2}\chi\Big(1-\frac{T_2}{T_1}\Big)\Big[\frac{1}{2}\Big(1-\frac{T_2}{T_1}\Big)(\operatorname{erf}\zeta+1)^2+\frac{T_2}{T_1}(\operatorname{erf}\zeta+1)\Big] \tag{3.131}$$

$$\frac{\zeta}{2}\frac{dg^{(1)}}{d\zeta}-g^{(1)}=\frac{\bar{K}}{3}r_p\Big[\frac{1}{2}\Big(1-\frac{T_2}{T_1}\Big)(\operatorname{erf}\zeta+1)^2+\frac{T_2}{T_1}(\operatorname{erf}\zeta+1)\Big] \tag{3.132}$$

From eqn. 3.115 and eqn. 3.116,

$$\frac{\zeta}{2}\frac{d\rho_r^{(1)}}{d\zeta}-\rho_r^{(1)}=\frac{\bar{K}}{2}\Big(1+\frac{1}{\chi}\Big)\Big[\frac{1}{2}\Big(1-\frac{T_2}{T_1}\Big)(\operatorname{erf}\zeta+1)^2+\frac{T_2}{T_1}(\operatorname{erf}\zeta+1)\Big] \tag{3.133}$$

The homogeneous equation of 1.128 or 1.129 has two linearly independent solutions$^{(3)}$

$$H_1(\zeta)=2\zeta e^{-\zeta^2}+2\sqrt{\pi}\Big(\frac{1}{2}+\zeta^2\Big)(\operatorname{erf}\zeta-1) \tag{3.134}$$

$$H_2(\zeta)=2\zeta e^{-\zeta^2}+2\sqrt{\pi}\Big(\frac{1}{2}+\zeta^2\Big)(\operatorname{erf}+1) \tag{3.135}$$

A Green's function $G(\zeta,\bar{\zeta})$ can be easily found using H_1 and H_2.

If $\qquad G(\zeta,\bar{\zeta}) = G_1(\zeta,\bar{\zeta}) \qquad \text{for } \zeta>\bar{\zeta}$

$$= G_2(\zeta,\bar{\zeta}) \qquad \text{for } \zeta<\bar{\zeta}$$

the jump condition is

$$\frac{1}{4}\frac{\partial G_1}{\partial \zeta}\bigg|_{\bar{\zeta}} - \frac{1}{4}\frac{\partial G_2}{\partial \zeta}\bigg|_{\bar{\zeta}} = 1 \tag{3.136}$$

with

$$G_1(\zeta,\bar{\zeta}) = G_2(\zeta,\bar{\zeta}) \tag{3.137}$$

The Green's function is

$$G(\zeta,\bar{\zeta}) = G_1(\zeta,\bar{\zeta}) = \frac{1}{2\sqrt{\pi}} H_1(\zeta) H_2(\bar{\zeta}) e^{\bar{\zeta}^2} \qquad \text{for } \zeta > \bar{\zeta} \tag{3.138}$$

$$G(\zeta,\bar{\zeta}) = G_2(\zeta,\bar{\zeta}) = \frac{1}{2\sqrt{\pi}} H_1(\bar{\zeta}) H_2(\zeta) e^{\bar{\zeta}^2} \qquad \text{for } \zeta < \bar{\zeta} \tag{3.139}$$

Using the Green's function, $T^{(1)}$ can be written as

$$T^{(1)}(\zeta) = \int_{-\infty}^{\infty} G(\zeta,\bar{\zeta}) R_1(\bar{\zeta})\, d\bar{\zeta}$$

$$= \int_{-\infty}^{\zeta} G_1(\zeta,\bar{\zeta}) R_1(\bar{\zeta})\, d\bar{\zeta} + \int_{\zeta}^{0} G_2(\zeta,\bar{\zeta}) R_1(\bar{\zeta})\, d\bar{\zeta} \qquad \text{for } \zeta < 0$$

$$= \int_{-\infty}^{0} G_1(\zeta,\bar{\zeta}) R_1(\bar{\zeta})\, d\bar{\zeta} \qquad \text{for } \zeta > 0$$

Similarly,

$$K^{(1)}(\zeta) = \int_{-\infty}^{\zeta} G_1(\zeta,\bar{\zeta}) R_2(\bar{\zeta})\, d\bar{\zeta} + \int_{\zeta}^{0} G_2(\zeta,\bar{\zeta}) R_2(\bar{\zeta})\, d\bar{\zeta} \qquad \text{for } \zeta < 0$$

$$= \int_{-\infty}^{0} G_1(\zeta,\bar{\zeta}) R_2(\bar{\zeta})\, d\bar{\zeta}$$

Carrying out the integration, the solutions for $T^{(1)}(\zeta)$ and $K^{(1)}(\zeta)$ are

$$\begin{aligned} T^{(1)}(\zeta) &= \frac{\chi \bar{R} \gamma_p}{4\pi}\left(1-\frac{T_2}{T_1}\right)^2 H_3(\zeta) + \frac{1}{8\sqrt{\pi}} \chi \bar{R} \gamma_p \left(1-\frac{T_2}{T_1}\right)\left(\frac{T_2}{T_1}\right) H_4(\zeta) \\ &+ \frac{\chi}{2}\frac{T_2}{T_1}\left[\left(1-\frac{T_2}{T_1}\right) - \gamma_p \bar{R} \bar{A}\right] H_5(\zeta) + \frac{\chi}{4}\left(1-\frac{T_2}{T_1}\right)\left[\left(1-\frac{T_2}{T_1}\right) - \gamma_p \bar{R} \bar{A} \right. \\ &\left. + \gamma_p \bar{R} \frac{T_2}{T_1}\right] H_6(\zeta) + \frac{\chi}{8} \gamma_p \bar{R} \left(1-\frac{T_2}{T_1}\right)^2 H_7(\zeta) \end{aligned} \tag{3.140}$$

$$K^{(1)}(\zeta)=\frac{\chi\bar{K}^2\gamma_p}{4\pi}(1-\frac{T_2}{T_1})H_3(\zeta)+\frac{\chi\bar{K}^2\gamma_p}{8\sqrt{\pi}}\frac{T_2}{T_1}H_4(\zeta)+\frac{\chi\bar{K}}{2}\frac{T_2}{T_1}H_5(\zeta)$$

$$+\frac{\chi\bar{K}}{4}\Big[\gamma_p\bar{K}\frac{T_2}{T_1}+(1-\frac{T_2}{T_1})\Big]H_6(\zeta)+\frac{\chi\gamma_p\bar{K}^2}{8}(1-\frac{T_2}{T_1})H_7(\zeta) \qquad (3.141)$$

The remaining first order quantities are:

$$T_p^{(1)}=-\frac{1}{2}\chi(1-\frac{T_2}{T_1})H_8(\zeta)+\frac{\chi}{2\sqrt{\pi}}(1-\frac{T_2}{T_1})\frac{T_2}{T_1}H_2(\zeta) \qquad (3.142)$$

$$g^{(1)}=\frac{\bar{K}}{3}\gamma_p(1-\frac{T_2}{T_1})H_8(\zeta)-\frac{T_2}{T_1}\frac{\bar{K}}{3\sqrt{\pi}}\gamma_p H_2(\zeta) \qquad (3.143)$$

$$\rho_r^{(1)}=\frac{\bar{K}}{2}(1+\frac{1}{\beta})(1-\frac{T_2}{T_1})H_8(\zeta)-\frac{\bar{K}}{2\sqrt{\pi}}(1+\frac{1}{\beta})\frac{T_2}{T_1}H_2(\zeta) \qquad (3.144)$$

with

$$H_1(\zeta)=2\zeta e^{-\zeta^2}+2\sqrt{\pi}(\frac{1}{2}+\zeta^2)(\operatorname{erf}\zeta+1)$$

$$H_2(\zeta)=2\zeta e^{-\zeta^2}+2\sqrt{\pi}(\frac{1}{2}+\zeta^2)(\operatorname{erf}\zeta-1)$$

$$H_3(\zeta)=\frac{1}{4\sqrt{3\pi}}H_2(\sqrt{3}\zeta)+\frac{1}{48\sqrt{\pi}}(20-16\sqrt{2}+18\sqrt{3}-13\pi)H_2(\zeta)$$

$$-(\frac{5}{6}+2\zeta^2)\sqrt{3}(\operatorname{erf}\sqrt{3}\zeta+1)+\frac{\pi}{3}(\frac{1}{2}+\zeta^2)(\operatorname{erf}\zeta+1)^3$$

$$+\frac{\sqrt{\pi}}{2}\zeta e^{-\zeta^2}(\operatorname{erf}\zeta+1)^2+\frac{2\sqrt{2}}{3}e^{-\zeta^2}(\operatorname{erf}\sqrt{2}\zeta+1)-\frac{2}{\sqrt{\pi}}\zeta e^{-3\zeta^2}$$

for $\zeta<0$

$$H_3(\zeta)=\frac{1}{48\sqrt{\pi}}(5\pi+18\sqrt{3}-20-16\sqrt{2})H_1(\zeta) \qquad \text{for } \zeta>0$$

$$H_4(J) = (\operatorname{erf} J - 1/4)\, H_2(J) \qquad \text{for } J < 0$$

$$H_4(J) = \frac{1}{4} H_1(J) \qquad \text{for } J > 0$$

$$H_5(J) = \frac{1}{4\sqrt{\pi}}\left[3 H_2(J) - \frac{d^2}{dJ^2} H_2(J)\right] \qquad \text{for } J < 0$$

$$H_5(J) = \frac{1}{4\sqrt{\pi}} H_1(J) \qquad \text{for } J > 0$$

$$H_6(J) = \frac{1}{8\pi}\left(\frac{d}{dJ} H_2(J)\right)^2 - \frac{1}{2\sqrt{\pi}}\left(\frac{2}{\pi} + 1\right) H_2(J) \qquad \text{for } J < 0$$

$$H_6(J) = \frac{1}{2\sqrt{\pi}}\left(1 - \frac{2}{\pi}\right) H_1(J) \qquad \text{for } J > 0$$

$$H_7(J) = \frac{3}{2\pi} e^{-J^2} (\operatorname{erf} J + 1) \frac{d}{dJ} H_2(J) + 2J^2 (\operatorname{erf} J + 1)^3$$

$$- \frac{2\sqrt{\pi 3}}{\pi^{3/2}} (\operatorname{erf} \sqrt{3}\, J + 1) - \frac{\sqrt{3}}{\pi\sqrt{\pi}} H_2(\sqrt{3}\, J)$$

$$+ \left(\frac{9}{2\sqrt{3}\,\pi^{3/2}} - \frac{3}{4\sqrt{\pi}} - \frac{3}{2\pi\sqrt{\pi}}\right) H_2(J) \qquad \text{for } J < 0$$

$$H_7(J) = \left[\frac{3}{4\sqrt{\pi}} - \frac{9}{2\pi^{3/2}}\left(1 - \frac{1}{\sqrt{3}}\right)\right] H_1(J) \qquad \text{for } J > 0$$

$$H_8(J) = -(\operatorname{erf} J + 1)\left[\left(\frac{1}{2} + J^2\right)(\operatorname{erf} J + 1) + \frac{2J}{\sqrt{\pi}} e^{-J^2}\right] - \frac{2J^2}{\pi} E_i(2J^2)$$

$$\text{for } J < 0$$

$$H_8(J) = 0 \qquad \text{for } J > 0$$

IV. CONCLUDING REMARKS

For the special case when the two streams are moving at the same velocity, perturbation solutions for the zeroth order and the first order can be obtained relatively easily. Since $\bar{H}$ is usually much greater than unity, heat of reaction is the dominant factor in increasing the gas temperature. From the graphs on the perturbed temperature $T^{(1)}$ in Figure 3. 2, we observe that most of the reaction occurs within a rather narrow zone of the order of a few diffusion thicknesses λ_{D_0}, and maximum $T^{(1)}$ occurs at about $y = -\frac{1}{2} P_r \lambda_{D_0}$. Thus, it might seem advantageous in the general studies of the two-dimensional laminar flame problems to introduce a chemical reaction boundary layer where combustion takes place, while outside this layer, the flow field corresponds to the ordinary fluid dynamics without chemical reaction.

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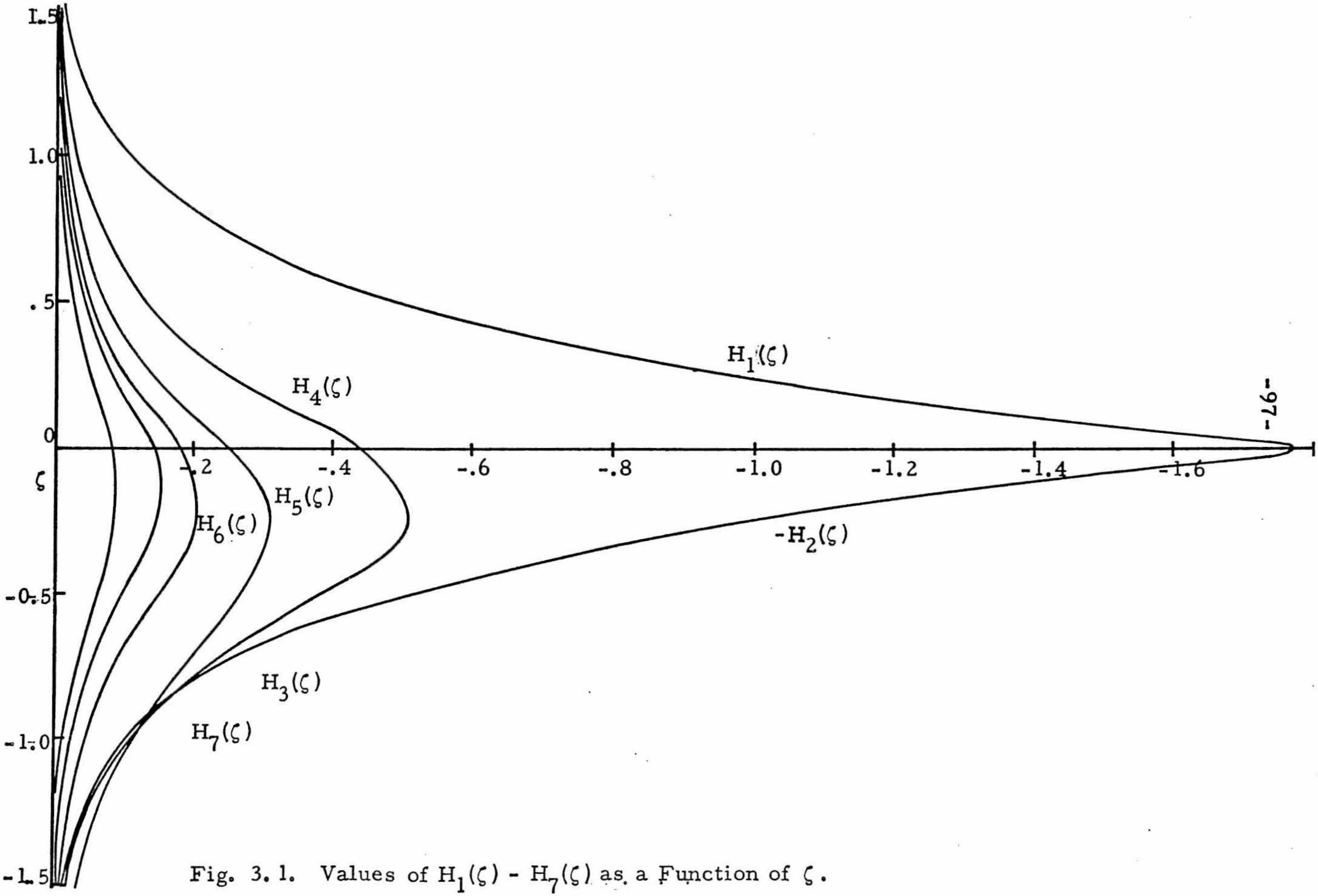

Fig. 3.1. Values of $H_1(\zeta)$ - $H_7(\zeta)$ as a Function of ζ.

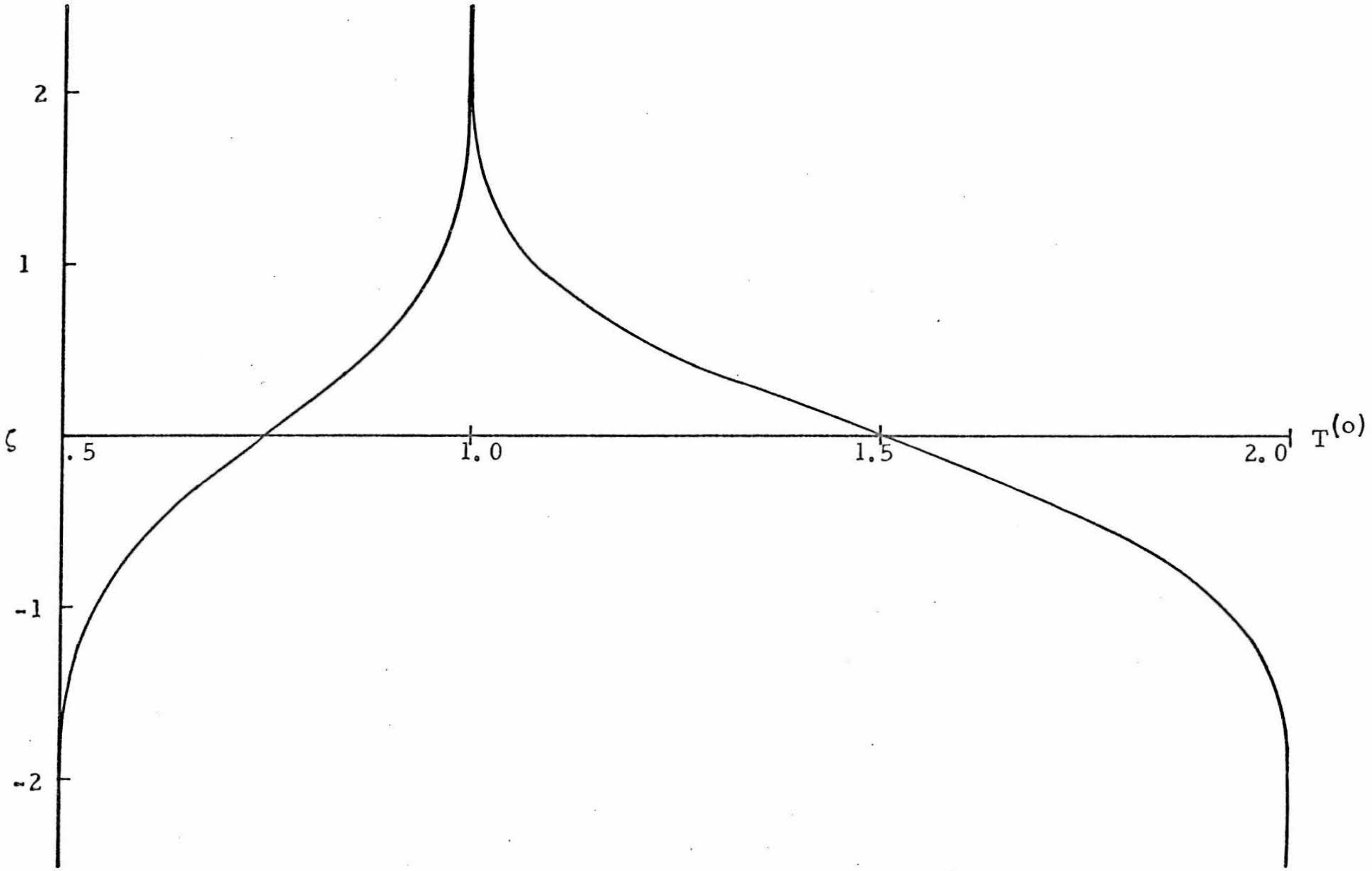

Fig. 3.2. Zeroth Order Temperature Profile.

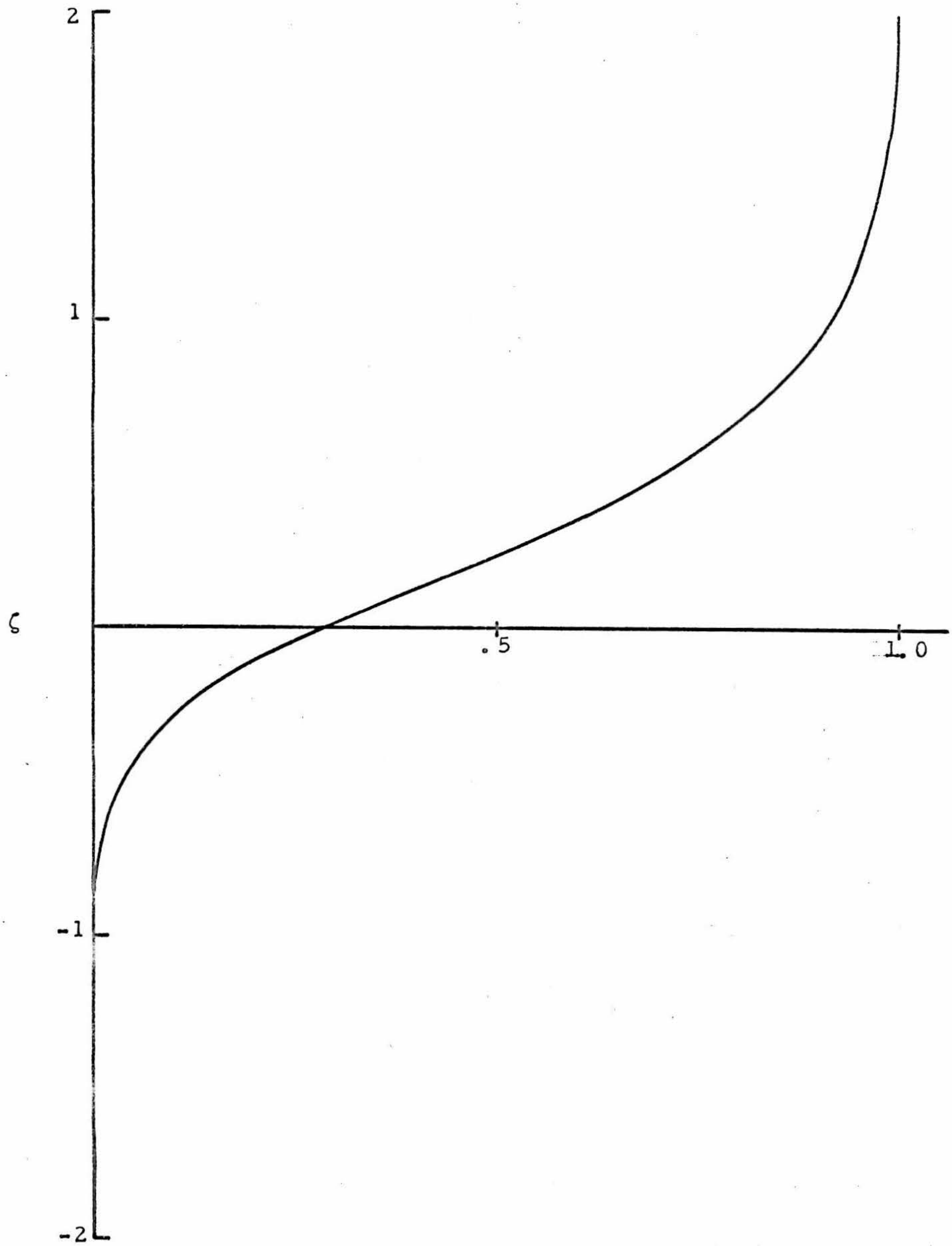

Fig. 3.3. Zeroth Order Concentration Profile.

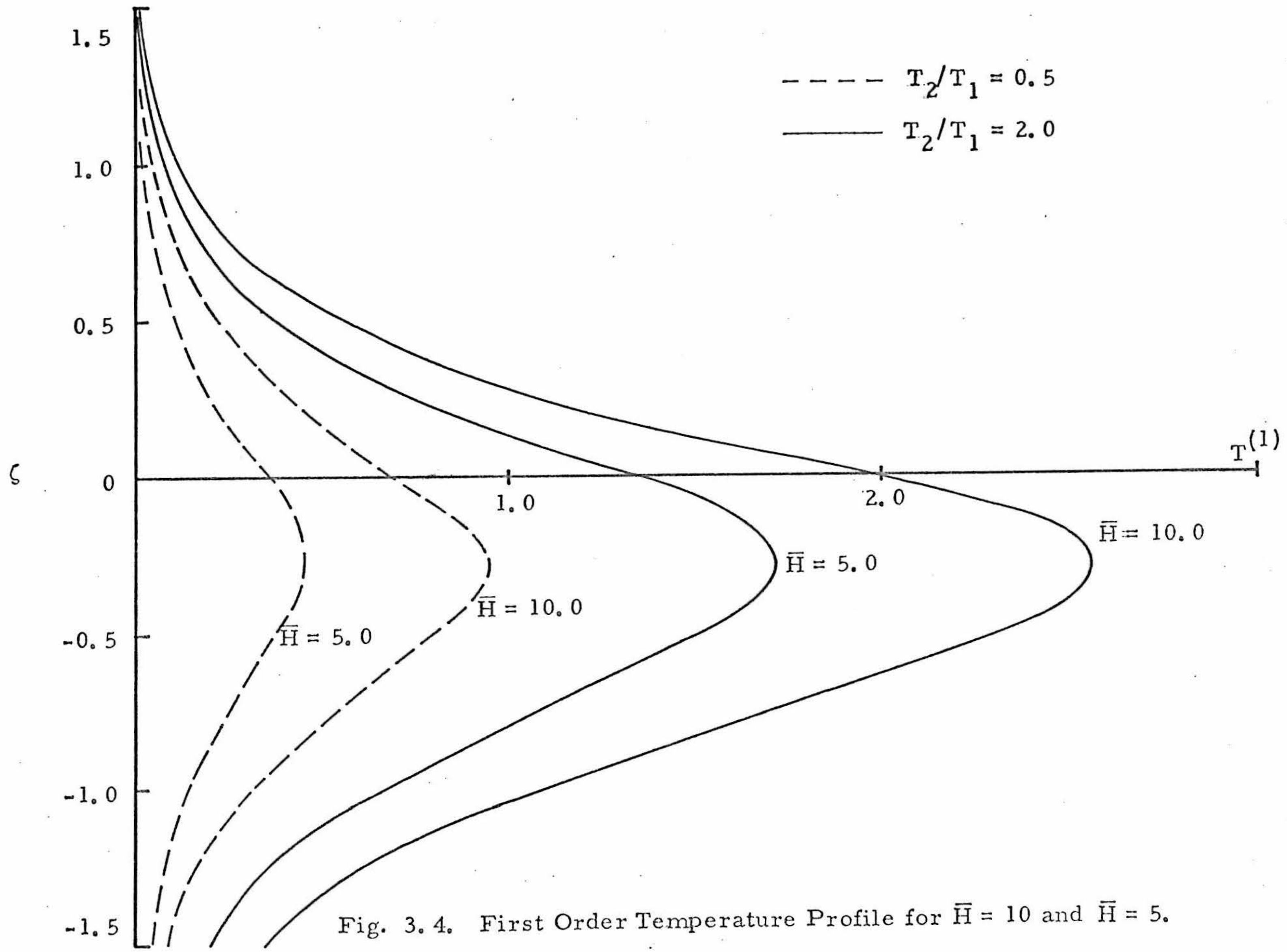

Fig. 3.4. First Order Temperature Profile for $\overline{H} = 10$ and $\overline{H} = 5$.

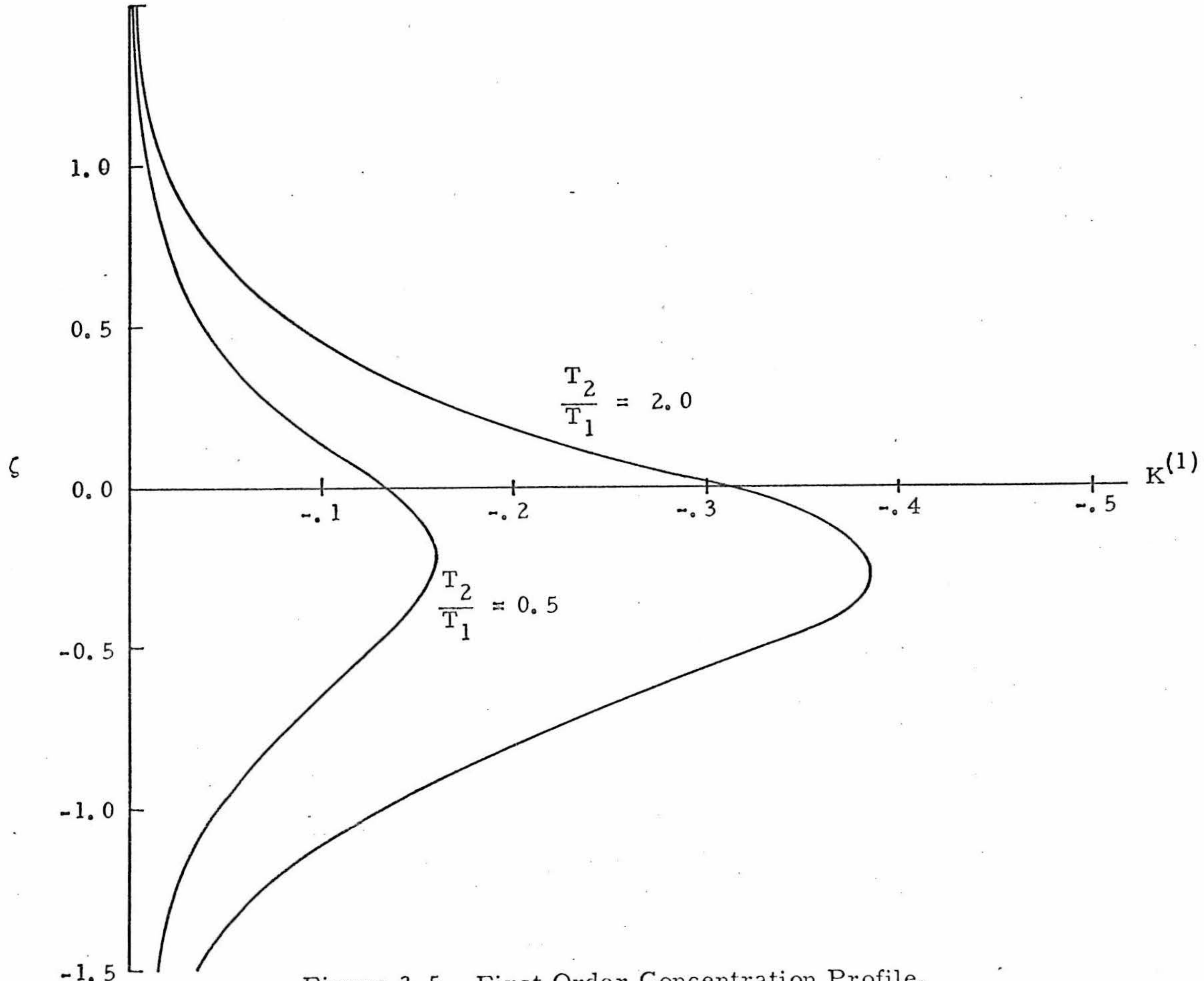

Figure 3.5. First Order Concentration Profile.

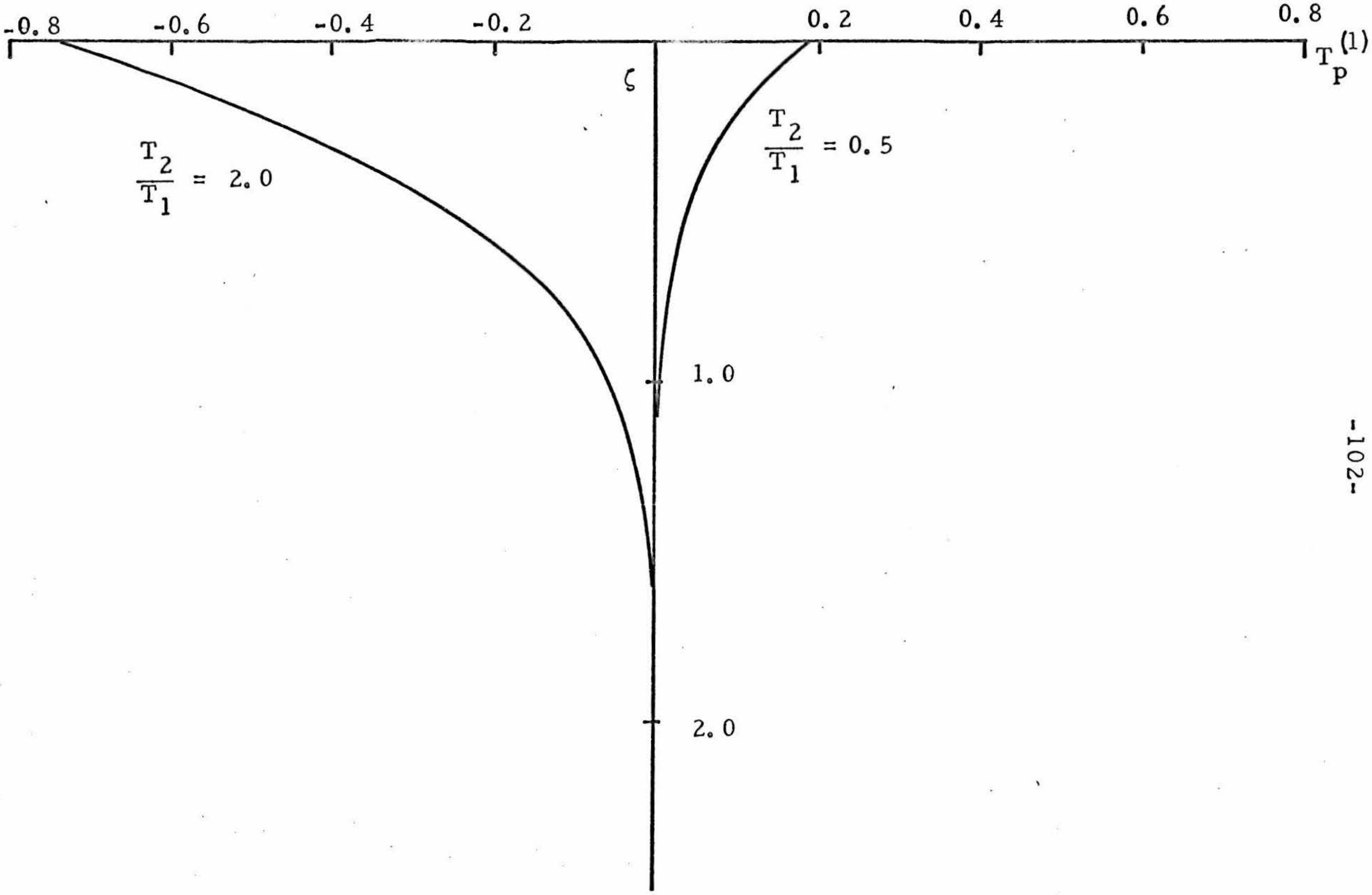

Fig. 3.6. First Order Particle Temperature Profile.

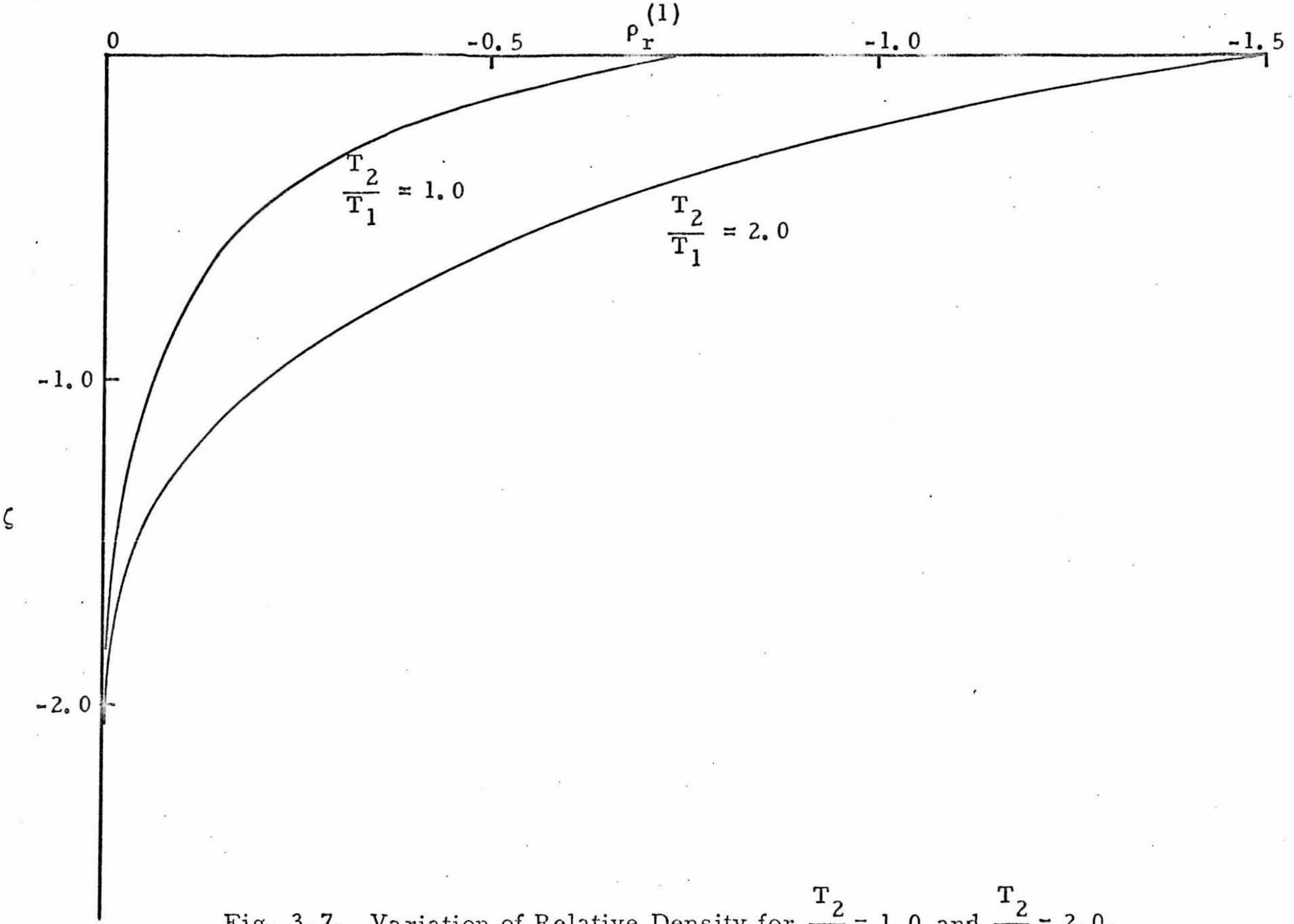

Fig. 3.7. Variation of Relative Density for $\frac{T_2}{T_1} = 1.0$ and $\frac{T_2}{T_1} = 2.0$.

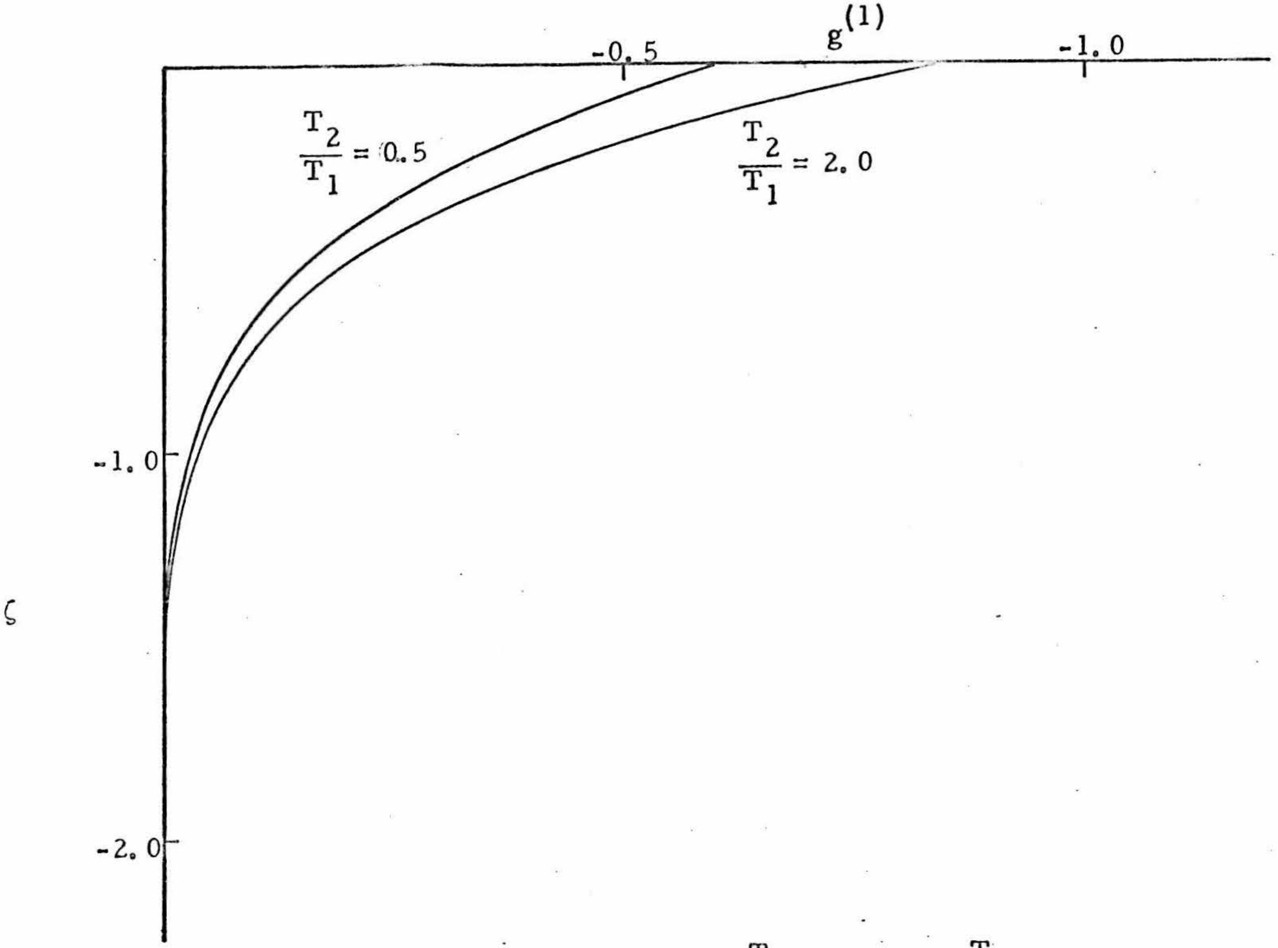

Fig. 3.8. Variation of $g^{(1)}$ for $\frac{T_2}{T_1} = 0.5$ and $\frac{T_2}{T_1} = 2.0$.

PART IV.

SHOCK WAVES IN A CONDENSING MEDIUM

1. INTRODUCTION

The problem of a shock wave propagating through a gas medium containing small particles was first investigated by Carrier(1). The actual shock wave structure, as pointed out by Marble(2), involves a relaxation zone in addition to the compression wave of the usual gas-dynamic shock. In the case of a condensing medium, the shock wave structure is quite different. Two different cases are being investigated here. The first case is when the particles are liquid droplets. The change in pressure and temperature across the shock wave causes the condensation of vapor on the liquid droplets, as well as evaporation of droplets further downstream of the shock. The second case is when the flow field contains small solid particles. The particle temperature within the relaxation zone is first being raised to the melting point followed by the melting of the particles,and eventually evaporation takes place. Since the governing equations involve a set of non-linear differential equations, numerical solution depending on a set of parameters is obtained.

2. NORMAL SHOCK WAVE WITH LIQUID DROPLETS

Consider a medium consisting of (i) a cloud of small liquid droplets suspended in a homogeneous gas mixture composed of (ii) a chemically inert gas with respect to the liquid droplets, and (iii) the vapor phase of the liquid droplets. The problem is to investigate the flow field when a shock wave passes through such a medium.

Since we have assumed that the two gas species are thoroughly mixed, diffusion or peculiar velocity in the macroscopic sense is absent. Upstream of the shock wave the particle cloud and the gas mixture are at equilibrium. Both phases have the same velocity and temperature and the vapor concentration of the gas mixture corresponds to the vapor pressure of the particle cloud. The shock structure first consists of a compression wave, corresponding to the conventional shock in gas dynamics. We assume that the presence of the particles does not change the one-dimensional configuration of this compression wave. The thickness of this wave is of the order of a few mean free paths, which is much smaller than the particle size. In passing through this compression wave, we can anticipate that the velocity and temperature of the particle cloud still basically retain the upstream values. Thus, the change of the properties of the gas mixture across the compression wave corresponds to the usual normal shock relations. Following the compression wave there is a relaxation zone in which the particle cloud and gas mixture eventually attain velocity and temperature equilibrium. Condensation of vapor on the liquid particles appears first because of high gas pressure immediately following the compression wave. This is eventually dominated by evaporation when the particle cloud temperature is being raised. In this problem we assume nucleation as negligible.

Denoting the gas density and the mean velocity of the gas mixture by ρ' and u', and similarly quantities with subscript p pertaining to those of the particle cloud, the continuity equations for the two phases are

$$\frac{d}{dx'}(\rho' u') = \omega' \tag{4.1}$$

$$\frac{d}{dx'}(\rho_p' u_p') = \omega_p' \tag{4.2}$$

ω' and ω_p' are the rate of mass production for the gas mixture and the particle. Since mass is conserved, we have

$$\omega' + \omega_p' = 0 \tag{4.3}$$

Denoting the vapor concentration in the gas mixture by K, the continuity equation for the vapor is

$$\rho' u' \frac{dK}{dx'} = (1 - K)\omega' \tag{4.4}$$

Diffusion velocity is absent in eqn. 4.4 since the gas mixture is thoroughly mixed.

The momentum equations can be written as

$$\rho' u' \frac{du'}{dx'} = F' + \omega'(m' - u') + u'\frac{dp'}{dx'} \tag{4.5}$$

$$\rho_p' u_p' \frac{du_p'}{dx'} = -F_p' + \omega_p'(m_p' - \omega') \tag{4.6}$$

P' is the local pressure for the gas mixture. m' and m_p' are the velocity associated with ω' and ω_p'. F' is the force exerted on a unit volume of gas mixture by the particle cloud. If it is assumed that the force between the gas mixture and the particle cloud follows the Stokes drag law, F' takes the following simple form

$$F' = n_p 6\pi\mu\sigma(u_p' - u') = \frac{1}{\lambda_{v_0}} \rho_p'(u_p' - u')\left(\frac{\sigma}{\sigma_0}\right)^2 a_0 \tag{4.7}$$

$$\lambda_{v_0} = \frac{m_0 a_0}{6\pi\mu\sigma_0} \tag{4.8}$$

where m_0, σ_0 are the mass and radius of the particles upstream of the shock, and a_0 is the upstream Mach number. λ_{v_0} is the characteristic length within which the particle fluid velocity difference is reduced to e^{-1} of its initial value for particles with mass m_0 traveling at sonic velocity a_0.

Assuming the mixture consists of calorically perfect gases, for the case when the specific heat of the vapor and the inert gas are approximately equal, the energy equation for the gas mixture is

$$\begin{aligned}\rho' u' C_p \frac{dT'}{dx'} - u' \frac{dp'}{dx'} = F'(u_p' - u') + Q' + \omega'[C_p(T_R' - T') \\ + \tfrac{1}{2}(m^{2\prime} - u^{2\prime}) + u'(u' - m')]\end{aligned} \tag{4.9}$$

$F'(u_p' - u')$ is the energy dissipation term due to the particle-fluid drag force. Here, it is assumed that this term is totally absorbed by the gas. Q' is the heat transfer per unit volume between the gas mixture and the particle cloud. To the same approximation as the Stokes drag law, Q' can be written as

$$Q' = n_p\left(\frac{k}{\sigma}\right)4\pi\sigma^2(T_p' - T') = \rho_p' C_p a_0 (T_p' - T')\frac{1}{\lambda_{T_0}}\left(\frac{\sigma_0}{\sigma}\right)^2 \tag{4.10}$$

λ_{T_0} has the same physical meaning as λ_{v_0} and they are related as

$$\lambda_{T_0} = 1.5 P_r \lambda_{v_0} \tag{4.11}$$

T_R is the temperature at which condensation or evaporation takes place.

The energy equation for the particle cloud is

$$\rho_p' u_p' C_\ell \frac{dT_p'}{dx'} = -Q' + \omega_p' L^\ell(T_p) \tag{4.12}$$

C_ℓ is the specific heat of particles in the liquid phase. $L^\ell(T_p)$ is the latent heat and is defined as the difference of enthalpies of the vapor phase and the liquid phase.

The equation of state for the gas mixture is

$$P = \rho R_M T \tag{4.13}$$

and

$$R_M = \left(\frac{K}{m_v} + \frac{1-K}{m_i}\right)\bar{R} \tag{4.14}$$

m_v , m_i are the molecular weight of the vapor and the inert gas and $\bar{R}$ is the universal gas constant. When m_v and m_i are approximately equal, R_M is reduced to a constant and is equal to $R_M = \bar{R}/m$.

The equation describing the particle radius or liquid droplet radius is

$$u_p' \frac{d}{dx'}\left(\frac{\sigma}{\sigma_0}\right)^3 = \frac{1}{n_p m_0} w_p' \tag{4.15}$$

The rate of condensation depends on three rate processes, namely, the rate of heat exchange and the rate of vapor diffusion towards the particles and the kinetic rate of condensation. We will assume that the latter is much faster as compared with the first two rates and that infinite kinetic condensation rate is a good approximation. Due to the very large molecular collision frequency, the gas mixture is actually in thermal and momentum equilibrium with the particles. Thus, $m' = n_p' = u_p'$ and $T_R' = T_p'$ and the equilibrium vapor pressure on the liquid droplet surface K_e^{ℓ} (the superscript denotes the equilibrium vapor pressure of liquid) is described by the Clausius-Clapeyron relation

$$K_e^{\ell}(T_p') = \frac{P_e'(T_p')}{P'} = K_e^{\ell}(T_0)\frac{P_0}{P'} \exp -\left[\int_{T_0}^{T_p'} \frac{1}{R} L^{\ell}(T_p')\, d\left(\frac{1}{T_p'}\right)\right] \tag{4.16}$$

where P_0 , T_0 , and $K_e^{\ell}(T_0)$ are the upstream pressure, temperature, and the equilibrium vapor concentration at temperature T_0 . Using this model, the rate of mass production can be easily derived as

$$\omega_p' = n_p 4\pi\sigma\rho' D\left[K(T) - K_e^{\ell}(T_p')\right] = \frac{\rho_p'}{\lambda_{D_0}} a_0 \left(\frac{\sigma_0}{\sigma}\right)^2 \left[K(T') - K_e^{\ell}(T_p')\right] \tag{4.17}$$

and

$$\lambda_{D_0} = \frac{m_0 a_0}{4\pi\sigma_0 \rho' D} = \frac{3}{2} S_c \lambda_{v_0} \tag{4.18}$$

D is the binary diffusion coefficient between the vapor and the inert gas.

Nondimensionalize the quantities with respect to the upstream quantities, $u = u'/a_0$, $u_p = u_p'/a_0$, $T = T'/T_0$, $T_p = T_p'/T_0$ $\rho = \rho'/\rho_0$, $\rho_p = \rho_p'/\rho_0$, $P = P'/\rho_0 a_0^2$, $x = x'/\lambda$ and define $\Lambda_D = \lambda/\lambda_{D_0}$, $\Lambda_V = \lambda_{v_0}/\lambda$, $\Lambda_T = \lambda/\lambda_{T_0}$ where λ is some characteristic length of the problem; the governing equations take the following form.

$$\frac{d}{dx}(\rho u) = -\Lambda_D \rho_p (K - K_e^{\ell})\left(\frac{\sigma_0}{\sigma}\right)^2 \tag{4.19}$$

$$\frac{d}{dx}(\rho_p u_p) = + \Lambda_D \rho_p (K - K_e^{\ell})\left(\frac{\sigma_0}{\sigma}\right)^2 \tag{4.20}$$

$$\rho u \frac{du}{dx} + \frac{dp}{dx} = \Lambda_V \rho_p (u_p - u)\left(\frac{\sigma_0}{\sigma}\right)^2 - \Lambda_D \rho_p [K(T) - K_e^{\ell}(T_p)](u_p - u)\left(\frac{\sigma_0}{\sigma}\right)^2 \tag{4.21}$$

$$\rho_p u_p \frac{du_p}{dx} = - \Lambda_V \rho_p (u_p - u)\left(\frac{\sigma_0}{\sigma}\right)^2 \tag{4.22}$$

$$\frac{1}{\gamma-1} \rho u \frac{dT}{dx} - u \frac{dp}{dx} = \Lambda_V \rho_p (u_p - u)^2 \left(\frac{\sigma_0}{\sigma}\right)^2 + \frac{1}{\gamma-1} \Lambda_T \rho_p (T_p - T)\left(\frac{\sigma_0}{\sigma}\right)^2$$

$$- \Lambda_D \rho_p [K - K_e^{\ell}(T_p)]\left(\frac{\sigma_0}{\sigma}\right)^2 \left[\frac{1}{\gamma-1}(T_p - T) + \frac{1}{2}(u_p^2 - u^2) + u(u - u_p)\right] \tag{4.23}$$

$$\frac{1}{\gamma-1} \frac{C_e}{C_p} \rho_p u_p \frac{d}{dx} T_p = \frac{-1}{\gamma-1} \Lambda_T \rho_p (T_p - T)\left(\frac{\sigma_0}{\sigma}\right)^2 + \Lambda_D \rho_p [K - K_e^{\ell}(T_p)]\left(\frac{\sigma_0}{\sigma}\right)^2$$

$$\left[\frac{1}{\gamma R T_0} L^{\ell}(T_p)\right] \tag{4.24}$$

$$\rho u \frac{dK}{dx} = - (1 - K) \Lambda_D \rho_p [K(T) - K_e^{\ell}(T_p)]\left(\frac{\sigma_0}{\sigma}\right)^2 \tag{4.25}$$

$$u_p \frac{d}{dx}\left(\frac{\sigma}{\sigma_0}\right)^2 = \frac{2}{3} \Lambda_D [K(T) - K_e^{\ell}(T_p)] \tag{4.26}$$

$$K_e^{\ell}(T_p) = K_e^{\ell}(T_0) \frac{1}{\gamma p} \exp\left[- \int_{T_0}^{T_p} \frac{1}{R} L^{\ell}(T_p)\, d\left(\frac{1}{T_p}\right)\right] \tag{4.27}$$

$$p = \frac{1}{\gamma} \rho T \tag{4.28}$$

Denote χ as the density ratio of the particle cloud and gas mixture upstream of the shock wave and M_e as the upstream Mach number. Then the general continuity and momentum integrals can be written as

$$\rho u + \rho_p u_p = M_0 (1 + \chi) \tag{4.29}$$

$$\rho u^2 + \rho_p u_p^2 = P = M_0^2 (1 + \chi) + \frac{1}{\gamma} \tag{4.30}$$

If we define h_g^0, h_v^0, and $h_p^{\ell_0}$ as the reference enthalpies of the gas, the vapor, and the liquid particles, respectively, the energy integral is

$$\rho u \left[\frac{T}{\gamma-1} + \frac{1}{2}u^2 + \frac{1}{\gamma R T_0}\left(\kappa h_v^0 + (1-\kappa) h_g^0\right)\right] + \rho_p u_p \left[\frac{C_\ell}{C_p}\frac{T_p}{\gamma-1} + \frac{1}{2}u_p^2 + \frac{h_p^{\ell_0}}{\gamma R T_0}\right]$$

$$= M_0\left[\frac{1}{\gamma-1} + \frac{1}{2}M_0^2 + \frac{\kappa_e^\ell(T_0)}{\gamma R T_0} h_v^0 + \frac{1-\kappa_e^\ell(T_0)}{\gamma R T_0} h_g^0\right] + \chi M_0 \left(\frac{1}{\gamma-1} + \frac{M_0^2}{2} + \frac{h_p^{\ell_0}}{\gamma R T_0}\right) \tag{4.31}$$

From eqns. 4.22, 4.26, and 4.19, 4.25, the following relationships are obtained

$$\rho_p u_p = \chi M_0 \left(\frac{\sigma}{\sigma_0}\right)^3 \tag{4.32}$$

$$\rho u = M_0 \frac{1 - \kappa_e^\ell(T_0)}{1 - \kappa} \tag{4.33}$$

Far downstream of the shock where the particle cloud and the gas mixture have attained momentum and thermal equilibrium, eqns. 4.29 - 4.33 are reduced to the following form with subscript ∞ denoting quantities far downstream:

$$(1 + \chi_\infty) \rho_\infty u_\infty = M_0 (1 + \chi) \tag{4.34}$$

$$(1 + \chi_\infty) \rho_\infty u_\infty^2 + \frac{1}{\gamma} \rho_\infty T_\infty = M_0^2 (1 + \chi) + \frac{1}{\gamma} \tag{4.35}$$

$$\rho_\infty u_\infty \left[\frac{1}{\gamma-1}\left(1+\frac{c_\ell}{c_p}\chi_\infty\right)T_\infty + \frac{1+\chi_\infty}{2}u_\infty^2 + \frac{1}{\gamma R T_0}\left(K_\infty h_v^\circ + (1-K_\infty)h_g^\circ + \chi_\infty h_p^{\ell_0}\right]$$

$$= M_0\left[\frac{1+\chi}{\gamma-1} + \frac{1+\chi}{2}M_0^2 + \frac{1}{\gamma R T_0}\left\{K_e^\ell(T_0)h_v^\circ + (1-K_e^\ell(T_0))h_g^\circ + \chi\, h_p^{\ell_0}\right\}\right] \tag{4.36}$$

$$\chi_\infty \rho_\infty u_\infty = \chi M_0 \left(\frac{\sigma_\infty}{\sigma_0}\right)^3 \tag{4.37}$$

$$\rho_\infty u_\infty = M_0 \frac{1-K_e^\ell(T_0)}{1-K_\infty} \tag{4.38}$$

The vapor concentration far downstream K_∞ should correspond to the equilibrium vapor pressure of the particle cloud. Thus,

$$K_\infty = K_e^\ell(T_0)\frac{1}{\rho_\infty T_\infty}\exp - \int_{T_0}^{T_\infty}\frac{1}{R}L^\ell(T_p)\,d\left(\frac{1}{T_p}\right)$$

For the case when $L^\ell(T_p)$ can be approximated by

$$L^\ell(T_p) = h_v^\circ - h_p^{\ell_0}$$

K_∞ takes the following form

$$K_\infty = K_e^\ell(T_0)\frac{1}{\rho_\infty T_\infty}\exp - \frac{h_v^\circ - h_p^{\ell_0}}{R T_0}\left[\frac{T_0}{T_\infty} - 1\right] \tag{4.39}$$

Equations 4.34 - 4.39 enable us to find the flow quantities far downstream of the shock.

The region immediately following the conventional gas dynamic shock can be obtained by numerically integrating eqns. 4.20, 4.22, and 4.24 along with eqns. 4.27 - 4.33. As a matter of convenience, it is assumed that $c_\ell = c_p$ and $L^\ell(T_p) = h_v^\circ - h_p^\circ$.

3. NORMAL SHOCK WAVE WITH SOLID PARTICLES

For solid particles with relatively high vapor pressure and easy sublimation, the governing equations are very similar to those for liquid droplets as derived in Section 2. The vapor concentration corresponding to the equilibrium vapor pressure of the solid $K_e^s(T_p)$ rather than the liquid is used. $K_e^s(T_p)$ is also defined by the Clausius-Clapeyron relation

$$K_e^s(T_p) = K_e^s(T_o)\frac{p_o}{p}\exp - \left[\int_{T_o}^{T_p} L^s(T_p)\frac{1}{R}\,d\left(\frac{1}{T_p}\right)\right] \qquad (4.40)$$

where $L^s(T_p)$ is the enthalpy difference between the vapor and solid phases.

$$L^s(T_p) = h_v(T_p) - h_p^s(T_p) \qquad (4.41)$$

For solid particles with low vapor pressure, sublimation is indeed negligible. The governing equations as derived in Section 2 are modified by letting $\omega_p = 0$.

The problem being investigated here is the propagation of a shock wave through a homogeneous mixture of a pure gas and small solid particles. Immediately following the compression wave, the particle temperature is raised until the melting temperature T_m is reached. After the particles are completely melted, and when the particle temperature is further increased, evaporation of the particles becomes important.

The governing equations for the solid particles immediately following the compression wave are

$$\frac{d}{dx}(\rho u) = 0 \qquad (4.42)$$

$$\frac{d}{dx}(\rho_p u_p) = 0 \tag{4.43}$$

$$\rho u \frac{du}{dx} + \frac{dP}{dx} = \Lambda_v \rho_p (u_p - u) \tag{4.44}$$

$$\rho_p u_p \frac{du_p}{dx} = - \Lambda_v \rho_p (u_p - u) \tag{4.45}$$

$$\frac{1}{\gamma-1} \rho u \frac{dT}{dx} - u \frac{dP}{dx} = \Lambda_v \rho_p (u_p - u)^2 + \frac{1}{\gamma-1} \Lambda_T \rho_p (T_p - T) \tag{4.46}$$

$$\frac{1}{\gamma-1} \frac{C_s}{C_p} \rho_p u_p \frac{dT_p}{dx} = \frac{-1}{\gamma-1} \Lambda_T \rho_p (T_p - T) \tag{4.47}$$

$$P = \frac{1}{\gamma} \rho T \tag{4.48}$$

When T_p reaches the melting point T_m at $x = x_m$, eqn. 4.47 is replaced by

$$\frac{dT_p}{dx} = 0 \tag{4.49}$$

The particles will be completely melted at $x = x_1$ where x_1 can be found by evaluating the following integral:

$$\int_{x_m}^{x_1} \frac{-1}{\rho_p u_p} \frac{\Lambda_T}{\gamma-1} \rho_p (T_p - T)\, dx = h_p^{\ell}(T_m) - h_p^{s}(T_m) \tag{4.50}$$

where superscripts ℓ and s denote the enthalpies of the liquid and solid phases of the particles. Evaporation eventually takes place when the particle temperature is further increased. This is especially true for strong shock waves. Thus, for $x > x_1$, the governing equations derived in Section 2 will be used.

To find the equilibrium conditions far downstream of the gas dynamic shock wave, similar equations (4.34 - 4.39) can be obtained by considering solid particles instead of liquid droplets.

4. DISCUSSION OF SOLUTION

For pure liquid droplets, the relaxation region consists of a "condensation zone" and an "evaporation zone." Immediately following the compression wave, the pressure of the gas mixture increases according to the normal shock relations, while the particle cloud temperature still has the upstream values. The partial pressure corresponding to the vapor concentration in the gas mixture is higher than the vapor pressure of the liquid droplets. Condensation sets in, and this is indicated by positive values of ω_p in Figure 4. 1 accompanied by increase of the particle size, as shown in Figure 4. 2. The thickness of this "condensation zone" depends very much on the latent heat of vaporization of the liquid droplets under consideration. As shown in Figure 4. 1, the condensation zone for $(h_v^\circ - h_p^{l\circ})/\gamma R T_0 = 3.0$ actually has a thickness corresponding to the velocity or temperature relaxation length, while for $(h_v^\circ - h_p^{l\circ})/\gamma R T_0 = 10.0$, the thickness of the condensation zone is of an order smaller than the equilibrium length. The particle cloud within this condensation zone is gaining energy through two sources: (i) energy of heat conduction from the gas mixture, and (ii) the latent heat of condensation associated with the condensing vapor molecules. This is shown in Figure 4. 3 as the area between the two curves. Evaporation eventually sets in when the vapor pressure of the particle droplets is higher than the partial pressure of the vapor molecules in the gas mixture. Within this "evaporation zone," while the particle cloud is gaining energy of heat of conduction from the gas mixture, it is at the same time losing energy to the gas in the form of latent heat associated with the evaporating liquid molecules. The difference of these two terms, corresponding to the net energy gained by the particle cloud, approaches zero in the evaporation zone, as shown in Figure 4. 3. Thus, the particle cloud tends to have a uniform temperature within the evaporation zone, as shown in Figure 4. 4. This is especially true for large values of latent heat. The velocity field, as shown in Figure 4. 5, is basically similar to the case when the particles are chemically inert(2).

From the above discussion, we observe that the shock structure in a condensing medium can be considered as composed of three different regions. First, there is the compression wave. The thickness of the compression wave ℓ_1 is only a few mean free paths. Immediately following the compression wave is a "condensation zone." The thickness of this condensation zone ℓ_2 is quite a few orders of magnitude larger than the mean free path. For relatively large values of latent heat, $\ell_2 << \lambda_{v_0}$. The thickness of the evaporation zone ℓ_3 is of the same order as λ_{v_0}. Thus, we have

$$\ell_1 << \ell_2 << \ell_3 \tag{4.51}$$

Within ℓ_1, the particle cloud still retains its upstream temperature T_1. It increases from T_1 to the equilibrium temperature in ℓ_2, and in ℓ_3, the particle cloud approximately has a uniform temperature equal to the equilibrium temperature.

Solutions for various values of Λ_v, Λ_T, and Λ_D are also obtained. They are shown in Figures 4. 7 - 4. 9.

In the case of solid particles, because the vapor pressure is indeed negligible for the kind of solid particles we are considering, phase change is almost non-existent immediately following the compression wave. The curves for particle temperature in Figure 4.10 indicate clearly the regions where the particles exist as solid, coexistence of solid and liquid, and the region where the particles are totally melted. Evaporation of these liquid droplets depends very much on the upstream Mach number, as shown in Fig. 4.11, since the gas temperature downstream of the shock wave increases with increasing Mach number.

As was pointed out before, equilibrium conditions far downstream of the compression wave can be obtained from eqns. 4.34 - 4.39 for liquid droplets, and quite similar equations give those for solid particles. As in the case of non-reacting particles, these conditions are independent of the model describing particle-gas interaction, but they do depend on the size of the particles.

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2. Marble, F. E., "Dynamics of a Gas Containing Small Solid Particles," Proc. 5th AGARD Combustion and Propulsion Colloquium, Braunschweig, April 1962. Pergamon Press, New York (1963), pp. 175-213.

3. Marble, F. E., "Some Effects of Phase Exchange Processes in the Fluid Dynamics of Two Phase Media," International Symposium on Fluid Dynamics of Heterogeneous Multi-Phase Continuous Media, Naples, 3-6 October 1966 (Proceedings to be published.).

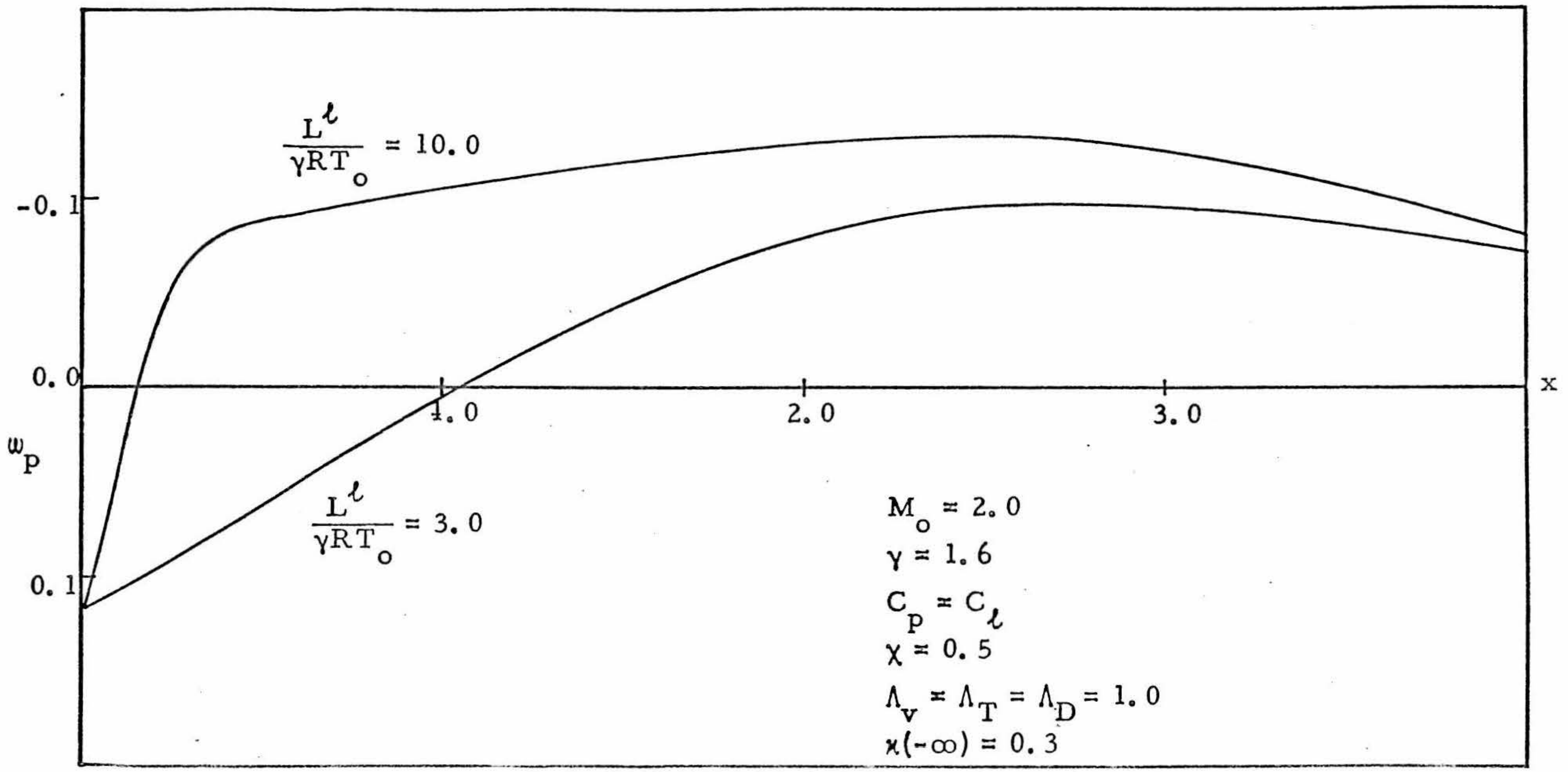


Fig. 4.1. Rate of Mass Transfer Downstream of the Shock.

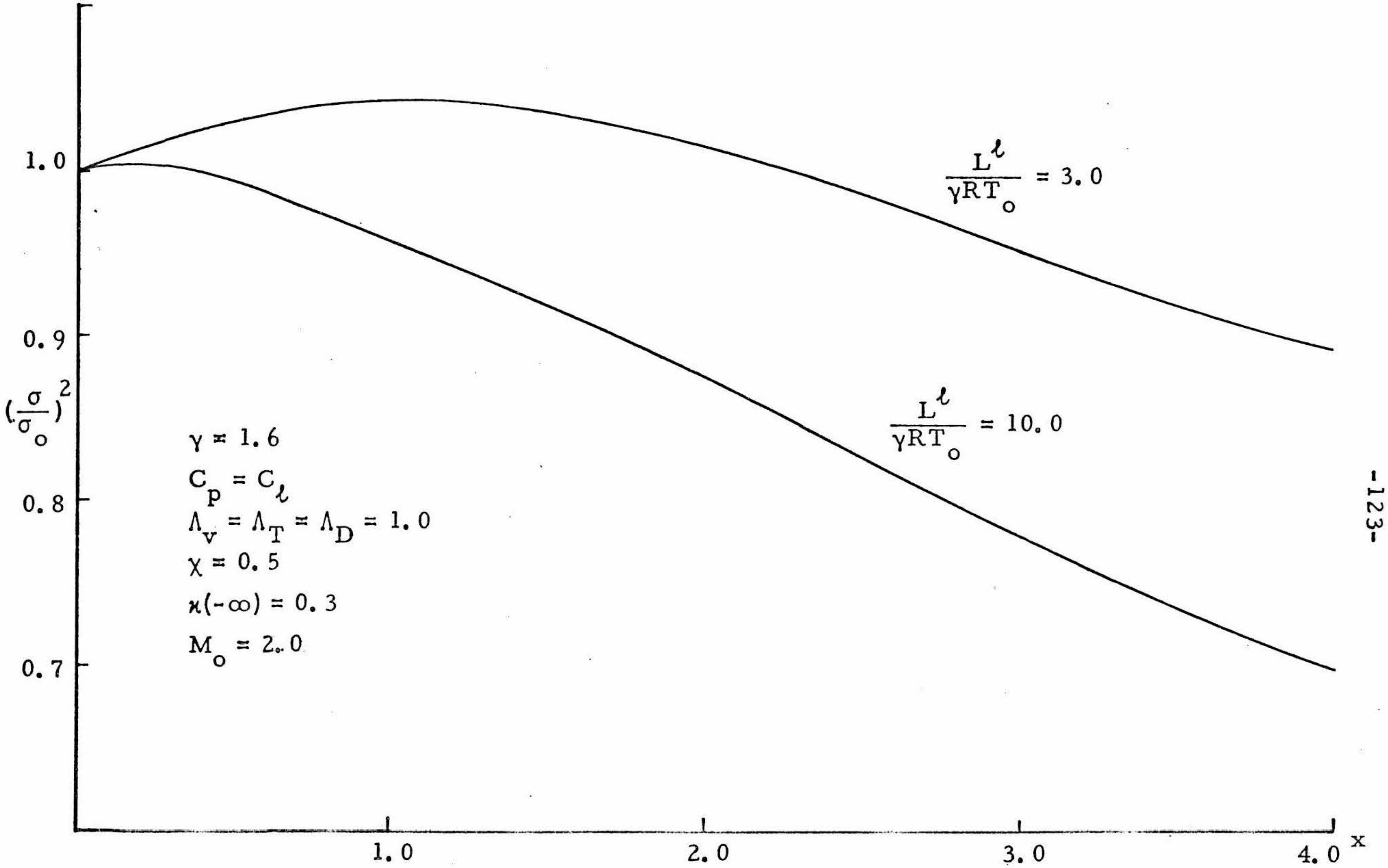
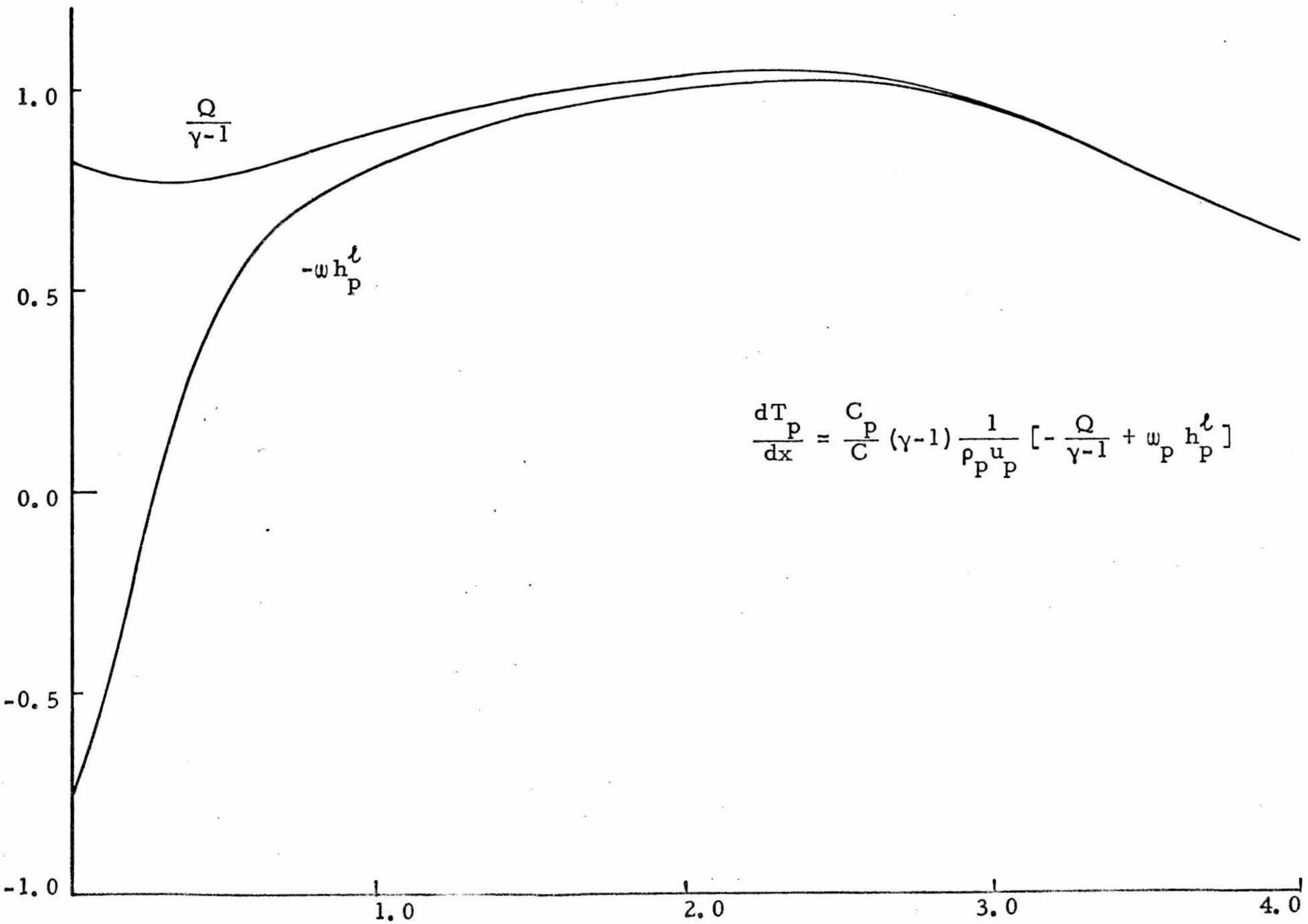


Fig. 4.2. Variation of Particle Radius Downstream of the Shock.



$$\frac{dT_p}{dx} = \frac{C_p}{C} (\gamma-1) \frac{1}{\rho_p u_p} [- \frac{Q}{\gamma-1} + \omega_p h_p^{\ell}]$$

Fig. 4.3. Rate of Heat Transfer to the Particle Cloud.

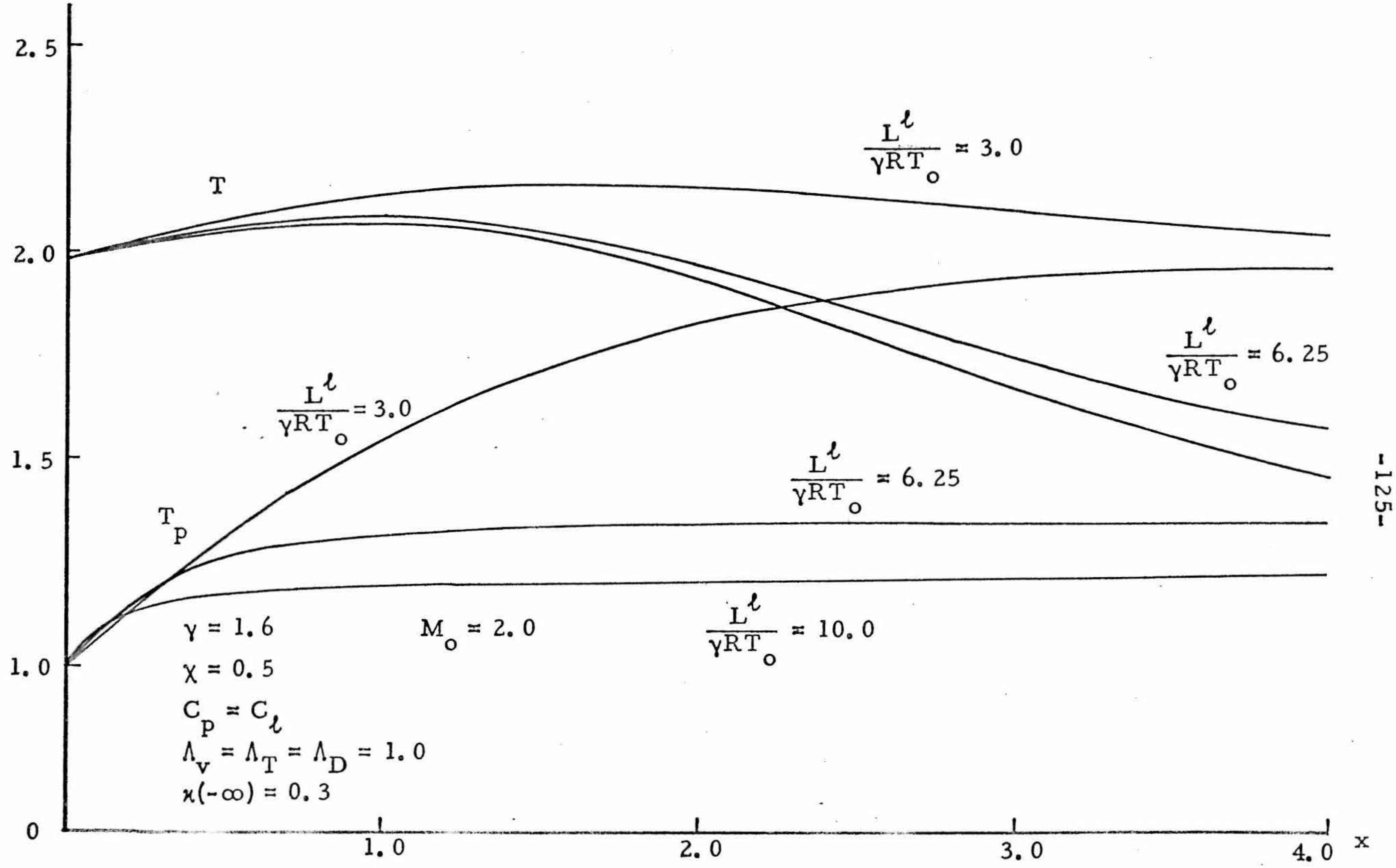


Fig. 4.4. Temperature Variation for Various Latent Heats.

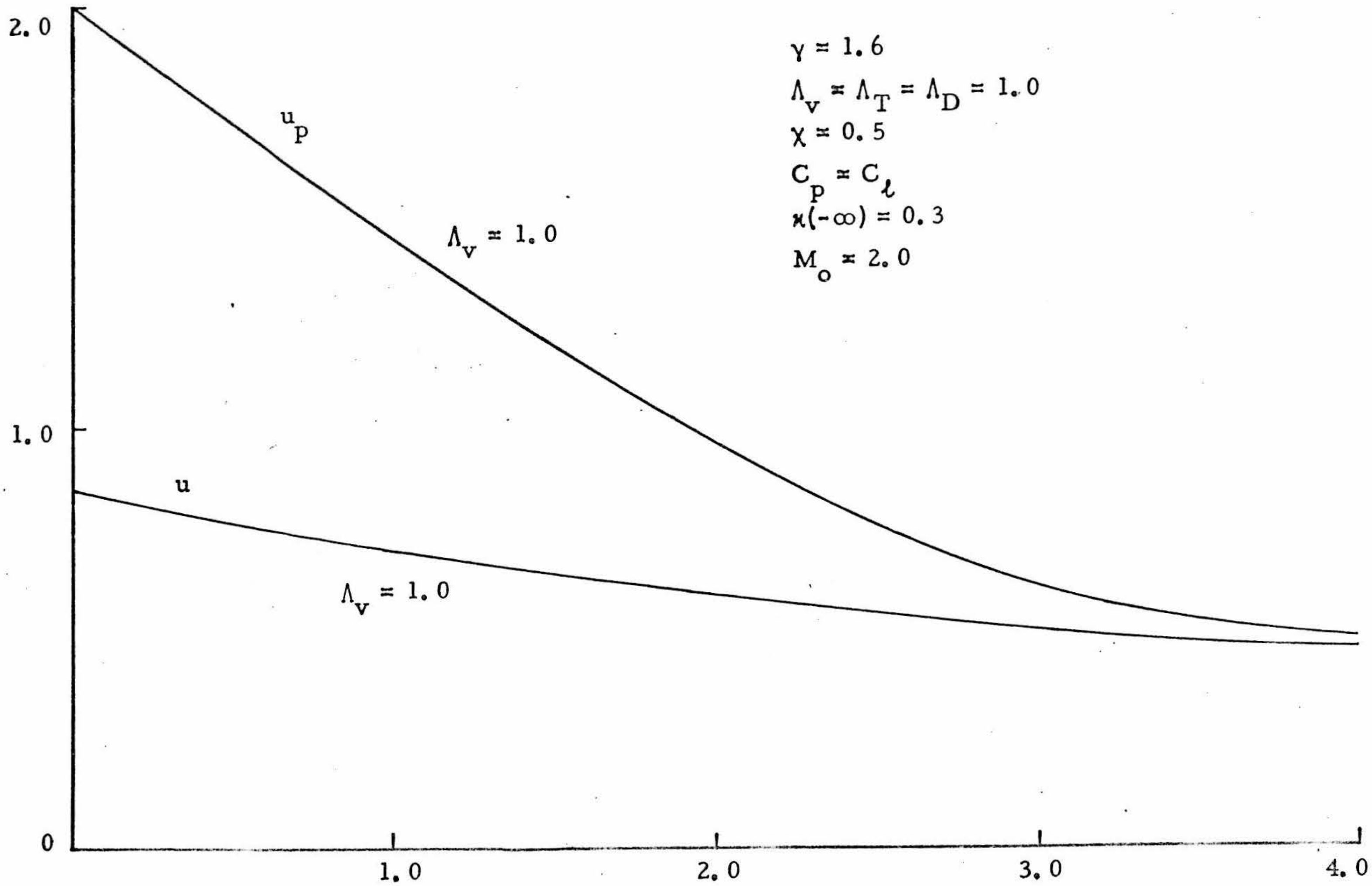


Fig. 4.5. Velocity Variation Downstream of the Shock.

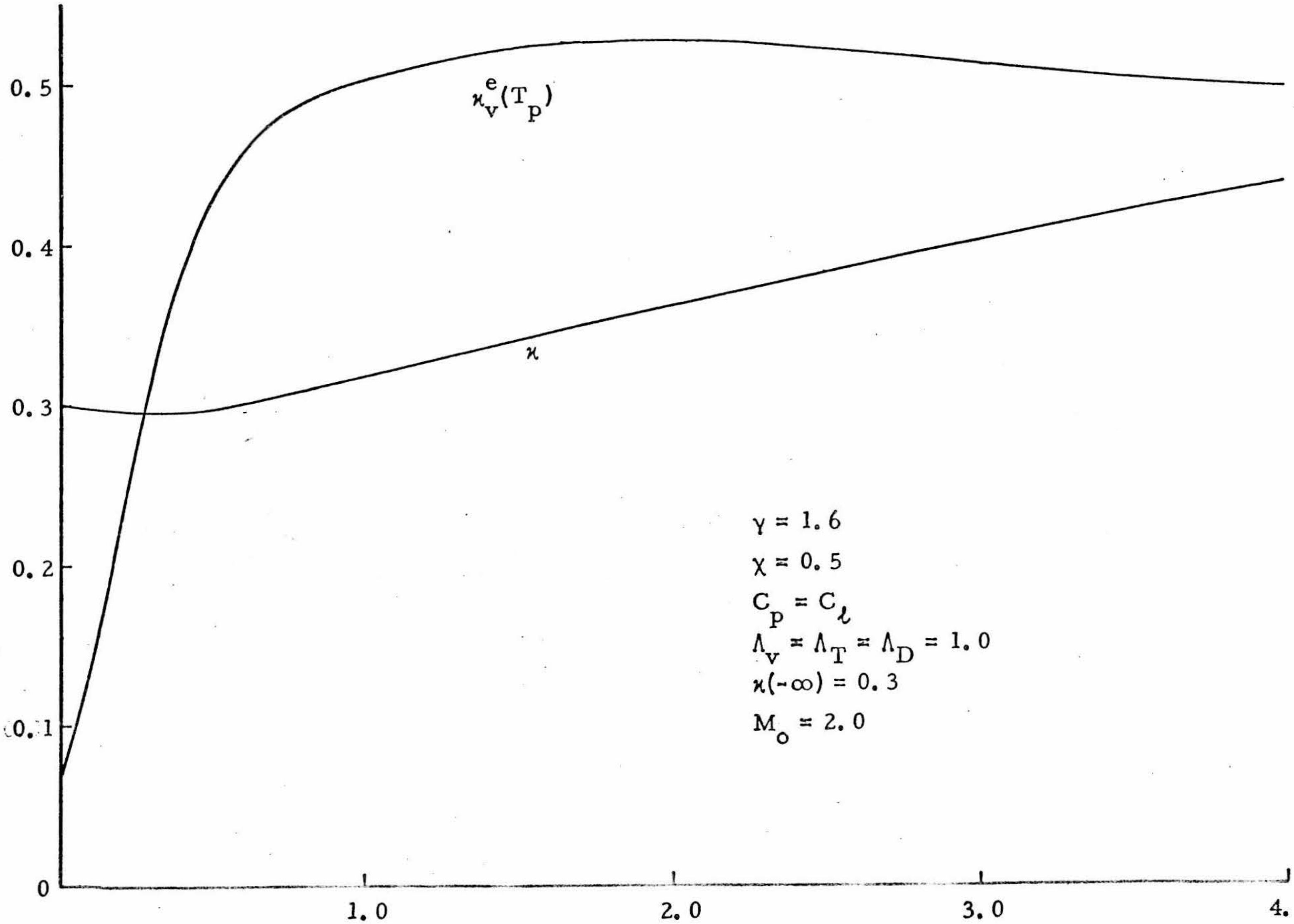


Fig. 4.6. Variation of the Vapor Concentration and the Equilibrium Vapor Concentration at the Particle Temperature.

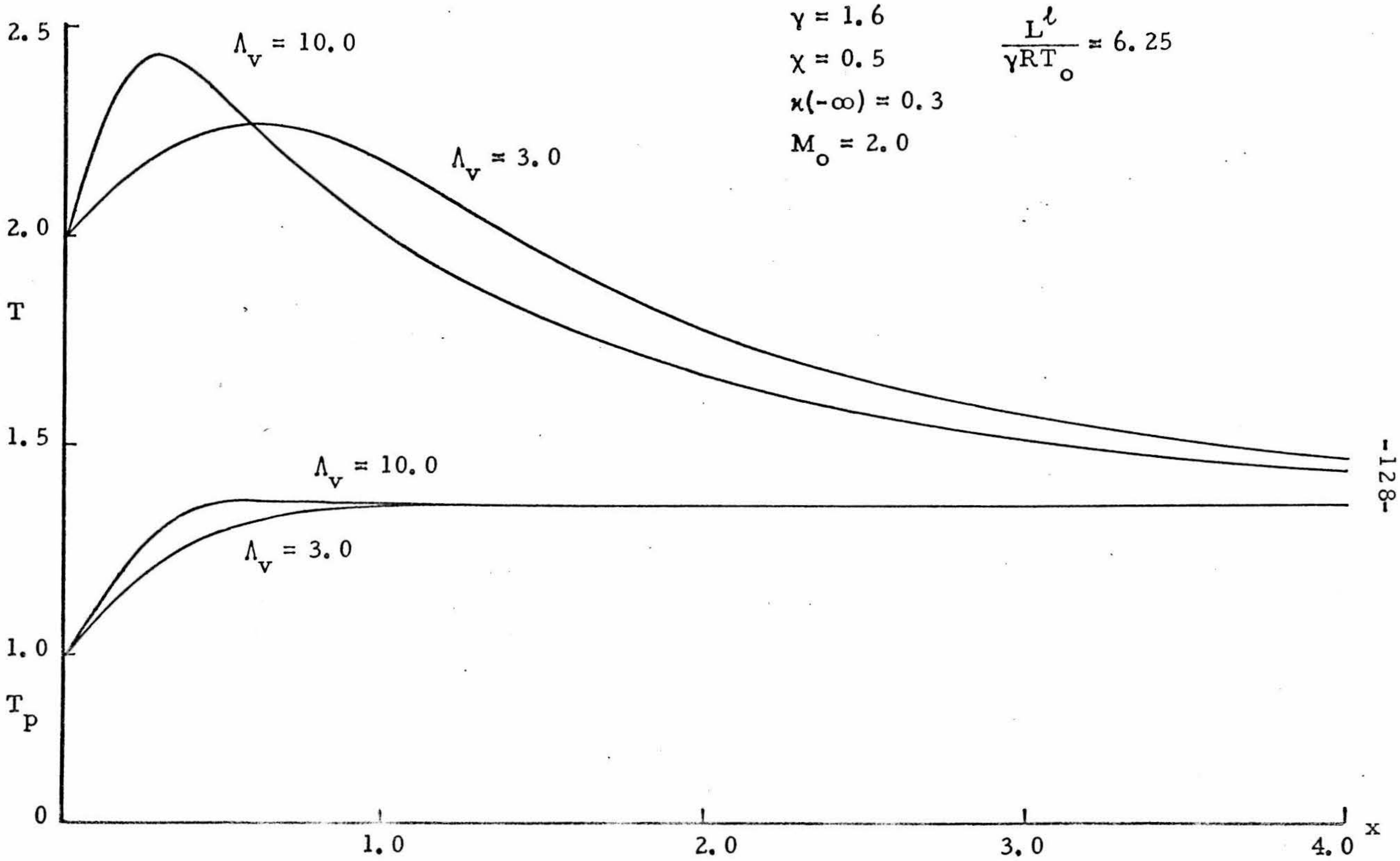


Fig. 4.7. Temperature Variation for Different Values of Λ_v.

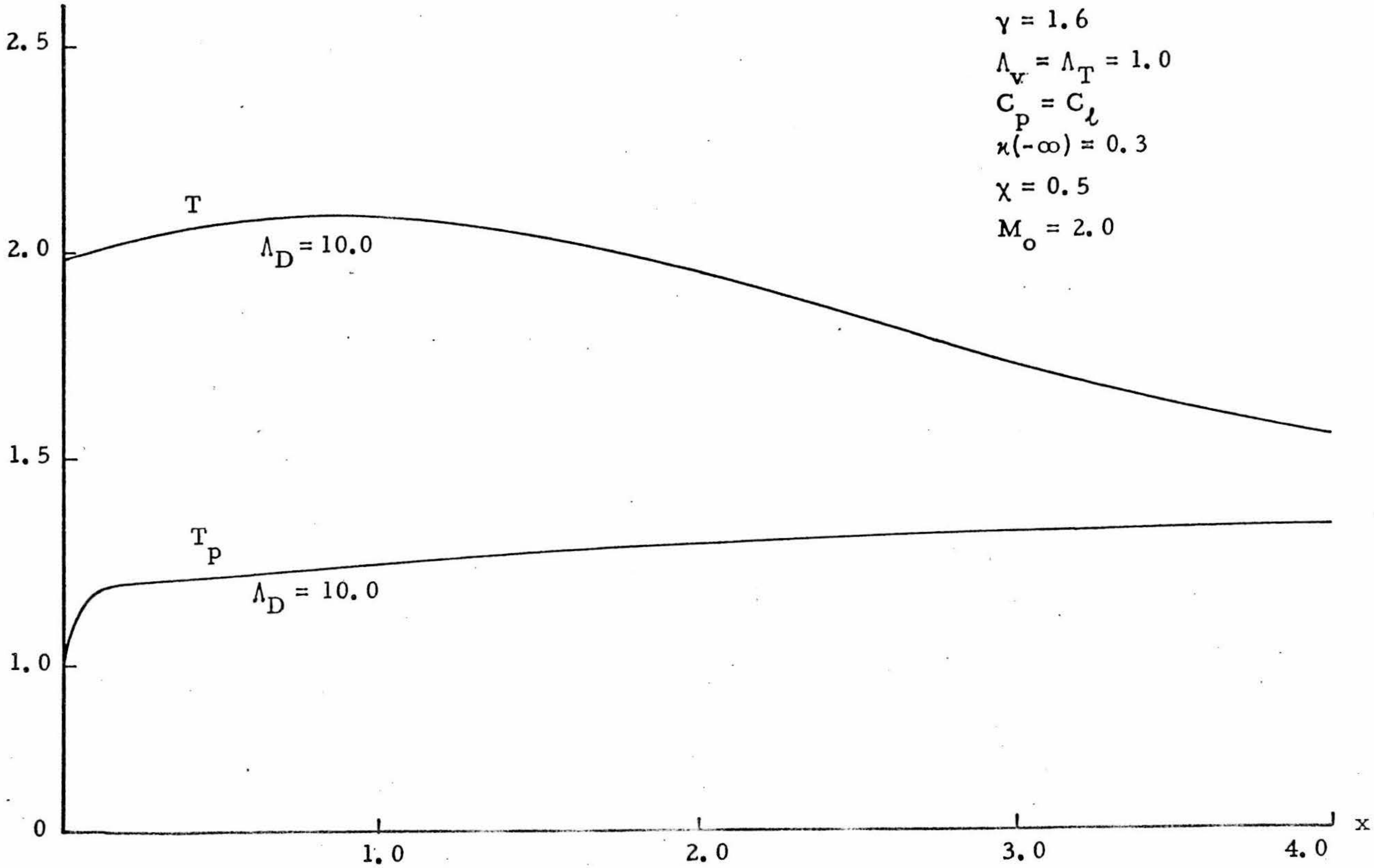


Fig. 4.8. Temperature Variation for $\Lambda_D = 10.0$.

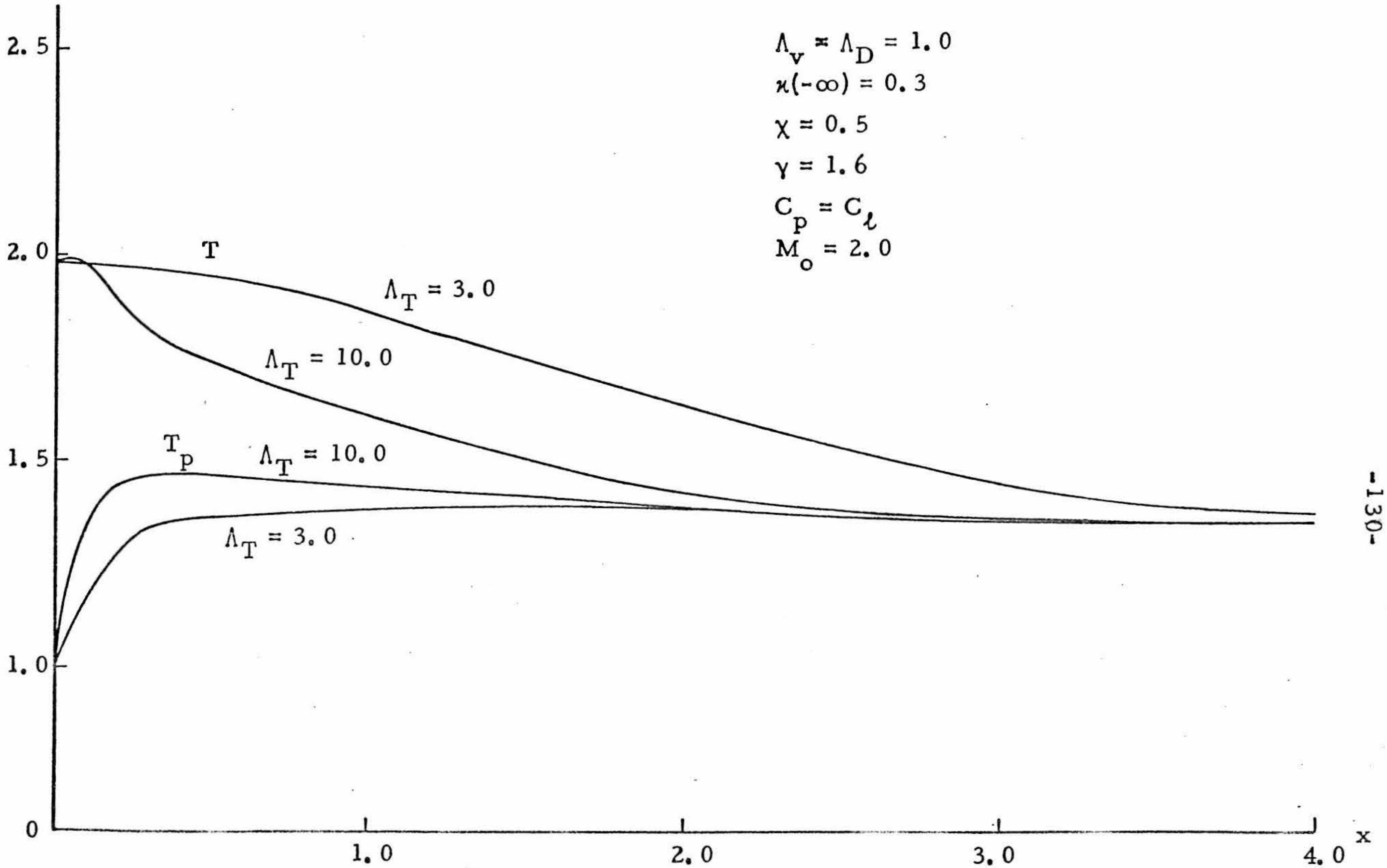


Fig. 4.9. Temperature Variation for Various Values of Λ_T.

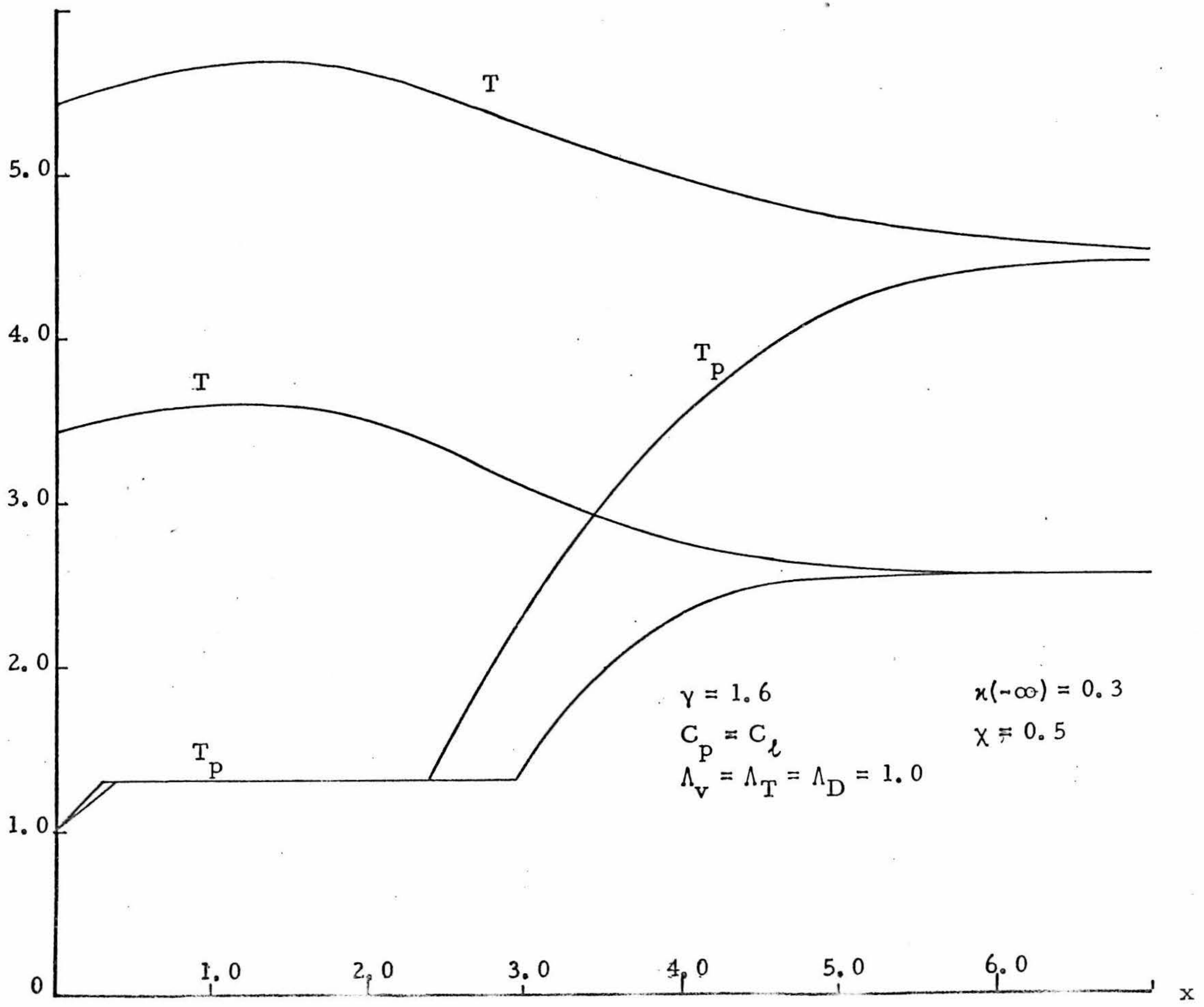


Fig. 4.10. Temperature Variation for Solid Particles.

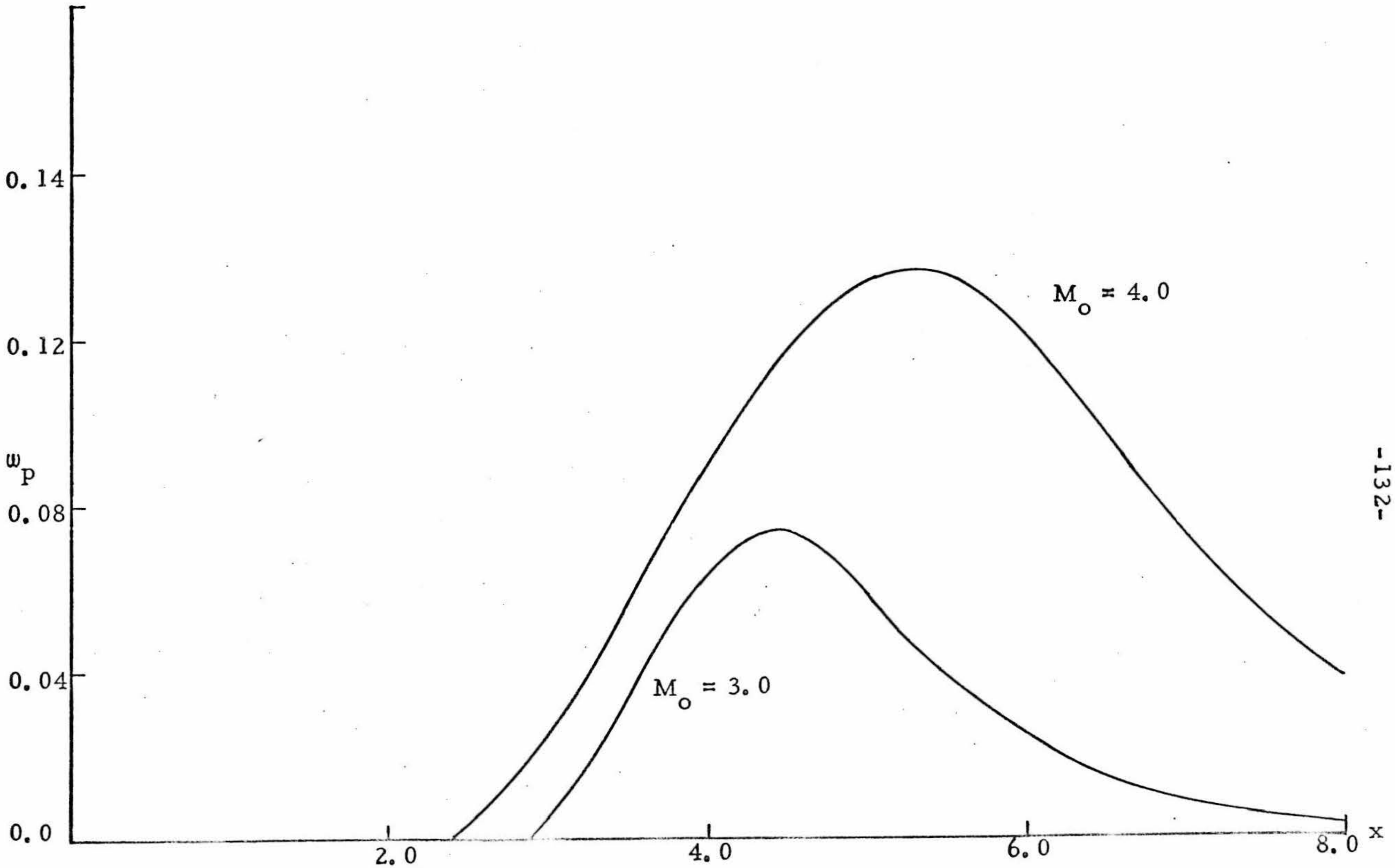


Fig. 4.11. Rate of Mass Transfer for $M_o = 3.0$ and $M_o = 4.0$.

PART V.

SOME FLOW PROBLEMS IN A GAS-PARTICLE MIXTURE WITH PARTICLE COLLISION

1. INTRODUCTION

The effects due to particle collision are very often suppressed in the investigation of the motion of clouds of small solid particles through gases. This approximation is valid when the particle number density is small or when the particles are very nearly of the same size. In the case when both of these two conditions are not satisfied, the effects due to particle-particle collision have to be taken into consideration. Marble(1) first investigated the mechanism of particle collisions by considering the case when a gas contains particles of two distinct sizes which behave as smooth elastic spheres. Collisions take place between the two groups of particles as they move with their own appropriate slip velocities with respect to the gas. The theory developed by Marble is applied to two different problems here. The propagation of a shock wave through a gas-particle mixture and the flow of a gas-particle mixture through a nozzle are investigated.

2. NORMAL SHOCK PROBLEM

The propagation of a shock wave through a homogeneous mixture of a gas and small solid particles was first investigated by Carrier(2). Since the particle size is much bigger than the molecular mean free path, the particle cloud, after traversing through the conventional gas dynamic shock, still retains its upstream velocity and temperature. If it is assumed that the presence of particles does not disturb the one-dimensional configuration of the problem, the change in gas velocity and temperature follows the usual shock relations. Thus, immediately following the gas dynamic shock, the particle cloud and the gas phase have quite different velocities and temperatures. By extending Carrier's analysis, both Marble(3) and Kriebel(4) obtained the shock structure of the mixture as consisting of the conventional gas dynamic shock followed by a relaxation region in which the particle cloud and the gas attain momentum and thermal equilibrium.

For the case when the particles are of two distinct sizes of radius σ_1 and σ_2, the continuity equations read as follows:

$$\rho u = \dot{m} \tag{5.1}$$

$$\rho_{p_1} u_{p_1} = \chi_1 \dot{m} \tag{5.2}$$

$$\rho_{p_2} u_{p_2} = \chi_2 \dot{m} \tag{5.3}$$

where ρ and u denote the gas density and velocity, and those with subscript p_1 pertain to those particles with radius σ_1 and similarly p_2 for the particles of radius σ_2. $\dot{m}$ is the rate of gas mass flow and χ_1, χ_2 are the mass ratio of particles with radius σ_1 and σ_2 with respect to the gas mass at some reference state.

The continuity equation for the mixture is

$$\rho u + \rho_{p1} u_{p1} + \rho_{p2} u_{p2} = (1 + \chi_1 + \chi_2)\dot{m} \tag{5.4}$$

The equation of motion for the gas is

$$\rho u \frac{du}{dx} + \frac{dp}{dx} = F_1 + F_2 \tag{5.5}$$

F_1 and F_2 are the forces exerted on a unit volume of gas by particles with radius σ_1 and σ_2, respectively, and p is the local gas pressure. Using Stokes drag law, a first order approximation to describe the gas-particle interaction, F_1 and F_2 take the following forms:

$$F_1 = n_1 6\pi\mu\sigma_1 (u_{p1} - u) = \rho_{p1}(u_{p1} - u)\frac{a}{\lambda_{v1}} \tag{5.6}$$

$$F_2 = n_2 6\pi\mu\sigma_2 (u_{p2} - u) = \rho_{p2}(u_{p2} - u)\frac{a}{\lambda_{v2}} \tag{5.7}$$

where

$$\lambda_{v1} = \frac{m_1 a}{6\pi\mu\sigma_1} = a\,\tau_{v1} \tag{5.8}$$

$$\lambda_{v2} = \frac{m_2 a}{6\pi\mu\sigma_2} = a\,\tau_{v2} \tag{5.9}$$

n_1 and n_2 are the particle number density with respect to particles of radius σ_1 and σ_2 and a is the local velocity of sound. If we assume that $\mu \sim T^{1/2}$, both λ_{v1} and λ_{v2} are constants. λ_{v1} and λ_{v2} or τ_{v1} and τ_{v2} are the characteristic length or time in which the particle-gas velocity difference is reduced to e^{-1} of its initial value. The equations of motion for the two particle clouds are

$$\rho_{p_1} u_{p_1} \frac{du_{p_1}}{dx} = - F_1 + \mathcal{F} \tag{5.10}$$

$$\rho_{p_2} u_{p_2} \frac{du_{p_2}}{dx} = - F_2 + \mathcal{F} \tag{5.11}$$

$\mathcal{F}$ is the force resulting from particle-particle collisions.

To evaluate $\mathcal{F}$, let us first consider the time between two successive collisions. Using the classical model and denoting τ_{c_1} as the time between collisions for particles of radius σ_1, τ_{c_1} is

$$\tau_{c_1} = (n_2 \pi (\sigma_1 + \sigma_2)^2 | u_{p_2} - u_{p_1} |)^{-1} \tag{5.12}$$

Similarly, τ_{c_2} is

$$\tau_{c_2} = (n_1 \pi (\sigma_1 + \sigma_2)^2 | u_{p_2} - u_{p_1} |)^{-1} \tag{5.13}$$

If both

$$\frac{1}{6} \frac{\rho_{p_2}}{\rho} \frac{\sigma_1^2 (\sigma_1 + \sigma_2)}{\sigma_1^3} \quad \text{and} \quad \frac{1}{6} \frac{\rho_{p_1}}{\rho} \frac{\sigma_2^2 (\sigma_1 + \sigma_2)}{\sigma_1^3}$$

are smaller than unity, we obtain the following condition

$$\frac{\tau_{v_1}}{\tau_{c_1}} << 1 \tag{5.14}$$

$$\frac{\tau_{v_2}}{\tau_{c_2}} << 1 \tag{5.15}$$

Equations 5.14 and 5.15 imply that random velocities introduced by particle collisions are damped out before the next collision takes place. The particle velocities do not decay toward their average velocities u_{p_1} and u_{p_2} but rather towards their collisionless velocities $u_{p_1}^*$ and $u_{p_2}^*$, where $u_{p_1}^*$ and $u_{p_2}^*$ are governed by

$$u_{p_1}^* \frac{d}{dx} u_{p_1}^* = (u - u_{p_1}^*) a / \lambda_{v_1} \tag{5.16}$$

$$u_{p2}^{*}\frac{d}{dx}u_{p2}^{*} = (u - u_{p2}^{*})\, a/\lambda_{v_2} \tag{5.17}$$

If we also assume that

$$\tau_{v_1} |u_{p1}^{*} - u_{p2}^{*}|/\sigma_1 \gg 1 \tag{5.18}$$

$$\tau_{v_2} |u_{p2}^{*} - u_{p1}^{*}|/\sigma_2 \gg 1 \tag{5.19}$$

so that modification of flow fields as two particles approach each other produces negligible effect, the force introduced by particle collision using the billiard ball model is

$$\begin{aligned} \mathcal{F} &= \pi\, n_1 n_2 (\sigma_1 + \sigma_2)^2 \frac{m_1 m_2}{m_1 + m_2} |u_{p1}^2 - u_{p2}| (u_{p2}^{*} - u_{p1}^{*}) \\ &= \frac{\rho_{p1}\rho_{p2}}{\ell} \frac{1}{\rho_0} |u_{p1} - u_{p2}| (u_{p2}^{*} - u_{p1}^{*}) \end{aligned} \tag{5.20}$$

where

$$\ell = (m_1 + m_2)/\rho_0 \pi (\sigma_1 + \sigma_2)^2 \tag{5.21}$$

and ρ_0 is some reference density for the gas. The general momentum integral for the mixture using eqns. 5.5, 5.10, and 5.11 can be written as

$$\rho u^2 + \rho_{p1} u_{p1}^2 + \rho_{p2} u_{p2}^2 + P = (1 + \chi_1 + \chi_2)\rho_0 u_0^2 + P_0 \tag{5.22}$$

u_0 is the upstream velocity; ρ_0 and P_0 are the gas density and pressure upstream of the shock.

For a calorically perfect gas, the first law of thermodynamics can be written as

$$\rho u C_p \frac{dT}{dx} - u\frac{dP}{dx} = Q_1 + Q_2 + \Phi_p \tag{5.23}$$

Q and Q_2 are the rate of heat transfer between a unit volume of gas and the particles with radius σ_1 and σ_2, respectively. To the same approximation as Stokes drag law, Q and Q_2 can be written as

$$Q_1 = n_1 \frac{k}{\sigma_1} 4\pi\sigma_1^2 (T_{p_1} - T) = \rho_{p_1} C_p (T_{p_1} - T)\frac{1}{\lambda_{T_1}} \tag{5.24}$$

$$Q_2 = n_2 \frac{k}{\sigma_2} 4\pi\sigma_2^2 (T_{p_2} - T) = \rho_{p_2} C_p (T_p - T)\frac{1}{\lambda_{T_2}} \tag{5.25}$$

and

$$\lambda_{T_1} = C_p m_1 a / 4\pi\sigma_1 k = \frac{3}{2} P_r \lambda_{v_1} \tag{5.26}$$

$$\lambda_{T_2} = C_p m_2 a / 4\pi\sigma_2 k = \frac{3}{2} P_r \lambda_{v_2} \tag{5.27}$$

T_{p_1} and T_{p_2} are the mean temperature for the particles with radius σ_1 and σ_2. Φ_p is the viscous dissipation corresponding to the work done by the particles and is equal to

$$\overline{\Phi}_p = (u_{p_1} - u)F_1 + (u_{p_2} - u)F_2 + (u_{p_2} - u_{p_1})\mathcal{F}_r \tag{5.28}$$

The energy equations for the particle cloud are

$$\rho_{p_1} u_{p_1} C_1 \frac{dT_{p_1}}{dx} = -Q_1 \tag{5.29}$$

$$\rho_{p_2} u_{p_2} C_2 \frac{dT_{p_2}}{dx} = -Q_2 \tag{5.30}$$

C_1 and C_2 are the constant specific heats for the solid particles. The energy integral for the mixture can be written as

$$\begin{aligned}(c_pT+\tfrac{1}{2}u^2)+\chi_1(c_1T_{p1}+\tfrac{1}{2}u_{p1}^2)+\chi_2(c_2T_{p2}+\tfrac{1}{2}u_{p2}^2)\\ =(c_p+\chi_1c_1+\chi_2c_2)T_0+(1+\chi_1+\chi_2)\tfrac{1}{2}u_0^2\end{aligned} \tag{5.31}$$

T_0 is the upstream temperature.

The equation of state for the gas is

$$p=\rho RT=\rho(1+\chi_1+\chi_2)\frac{R}{1+\chi_1+\chi_2}T=\rho(1+\chi_1+\chi_2)\bar{R}T \tag{5.32}$$

Denoting conditions far downstream of the shock where the gas and particle cloud have attained momentum and thermal equilibrium by subscript ∞, the integrals 5.4, 5.22, and 5.31 can be written in a more familiar form

$$\bar{\rho}_\infty u_\infty=\bar{\rho}_0u_0 \tag{5.33}$$

$$\bar{\rho}_\infty u_\infty^2+p_\infty=\bar{\rho}_0u_0^2+p_0 \tag{5.34}$$

$$\bar{c}_pT_\infty+\tfrac{1}{2}u_\infty^2=\bar{c}_pT_0+\tfrac{1}{2}u_0^2 \tag{5.35}$$

where $\bar{\rho}_\infty=(1+\chi_1+\chi_2)\rho_\infty$, $\bar{\rho}_0=(1+\chi_1+\chi_2)\rho_0$, $\bar{c}_p=(c_p+\chi_1c_1+\chi_2c_2)/(1+\chi_1+\chi_2)$
By defining $\bar{\gamma}=(c_p+\chi_1c_1+\chi_2c_2)/(c_v+\chi_1c_1+\chi_2c_2)$ we obtain the usual shock relations

$$\frac{u_\infty}{u_0}=\frac{(\bar{\gamma}-1)M_0^2+2}{(\bar{\gamma}+1)M_0^2} \tag{5.36}$$

$$\frac{T_\infty}{T_0}=1+\frac{2(\bar{\gamma}-1)}{(\bar{\gamma}+1)^2}\,\frac{\bar{\gamma}M_0^2+1}{M_0^2}(M_0^2-1) \tag{5.37}$$

$$a_\infty^2=\bar{\gamma}\bar{R}T_\infty \tag{5.38}$$

Thus, the equilibrium conditions far downstream of the shock wave can be calculated using the shock relations for a pure gas provided that the parameters γ and R are replaced by $\bar{\gamma}$ and $\bar{R}$.

The structure of the relaxation region can be easily found by numerical integration. Using continuity equations 5. 1, 5. 2, 5. 3; momentum integral 5. 22; energy integral 5. 31; the equation of state 5.32; six of the differential equations 5. 10, 5. 11 for u_{p_1} , u_{p_2} ; 5. 15 and 5. 16 for $u^*_{p_1}$ and $u^*_{p_2}$; 5. 29 and 5. 30 for T_{p_1} and T_{p_2} ; the geometric structure of the equilibrium zone is found for various values of ℓ and λ_{v_1} .

3. FLOW OF A GAS-PARTICLE MIXTURE IN A NOZZLE

Small solid particles are usually present in the exhaust of rocket motors with metallized propellants. The presence of these particles in an optimum gas nozzle results in performance degradations. Based on certain approximations, the losses have been calculated(5, 6) numerically for certain nozzle contours, and Rannie(7) has developed a perturbation scheme for the one-dimensional heterogeneous flow in rocket nozzles. By extending Rannie's analysis, Marble(8) has found an optimum nozzle contour for gas-particle mixtures. Based on Marble's optimum contour, the first order perturbation quantities for a gas-particle mixture, with particle-particle collision taken into consideration, are acquired here.

Denoting A as the cross-sectional area of the nozzle, the continuity equation for the gas is

$$\rho u A = \dot{m} \tag{5.39}$$

and u is the gas velocity along the nozzle axis. Because the difference between the particle quantities and the gas quantities or the slip quantities is small, it is convenient to introduce the following variables:

$$\begin{aligned} u_{s_1} &= u - u_{p_1} & u_{s_1}^* &= u - u_{p_1}^* \\ u_{s_2} &= u - u_{p_2} & u_{s_2}^* &= u - u_{p_2}^* \\ T_{s_1} &= T - T_{p_1} & \rho_{s_1} &= 1 - \rho_{p_1}/\rho\chi_1 \\ T_{s_2} &= T - T_{p_2} & \rho_{s_2} &= 1 - \rho_{p_2}/\rho\chi_2 \end{aligned} \tag{5.40}$$

Expressed in terms of the slip quantities 5.34, the continuity equations for the particle cloud read as

$$\beta_{s_1} u + U_{s_1} = \beta_{s_1} u_{s_1} \tag{5.41}$$

$$\beta_{s_2} u + u_{s_2} = \beta_{s_2} U_{s_2} \tag{5.42}$$

The continuity equation 5.4 for the mixture in terms of the slip quantities is

$$\begin{aligned}(1+\chi_1+\chi_2)\rho u A &= (1+\chi_1+\chi_2)\dot{m} + \chi_1 \rho A (U_{s_1} + \beta_{s_1} u - \beta_{s_1} u_{s_1}) \\ &\quad + \chi_2 \rho A (U_{s_2} + \beta_{s_2} u - \beta_{s_2} u_{s_2})\end{aligned} \tag{5.43}$$

The energy integral 5.31 can be expressed in terms of $\bar{C}_p$ and the temperature slip variables, and yields the following relationship:

$$\begin{aligned}\bar{C}_p (T - T_c) + \frac{1}{2} u^2 &= \frac{\chi_1}{1+\chi_1+\chi_2}\left[C_1 T_{s_1} + u U_{s_1} - U_{s_1}^2\right] \\ &\quad + \frac{\chi_2}{1+\chi_1+\chi_2}\left[C_2 T_{s_2} + u U_{s_2} - u_{s_2}^2\right]\end{aligned} \tag{5.44}$$

u, U_s, T_s vanish inside the rocket chamber whose temperature is T_c. Following Rannie's$^{(7)}$ analysis of employing gas pressure p as the independent variable, which has the advantage of avoiding the transonic singularity, the energy equations for the particle clouds, eqns. 5.29 and 5.30, assume the following form:

$$u \frac{dT}{dp} - \frac{\bar{C}_p}{C_1} \frac{a T_{s_1} L}{\lambda_{T_1}} \left(\frac{dp}{d\xi}\right)^{-1} = - \beta_{s_1} \frac{\bar{C}_p}{C_1} \frac{a}{\lambda_{T_1}} T_{s_1} \left(\frac{dp}{d\xi}\right)^{-1} + u \frac{dU_{s_1}}{dp} \tag{5.45}$$

$$u \frac{dT}{dp} - \frac{\bar{C}_p}{C_2} \frac{a T_{s_2} L}{\lambda_{T_2}} \left(\frac{dp}{d\xi}\right)^{-1} = - \beta_{s_2} \frac{\bar{C}_p}{C} \frac{a}{\lambda_{T_2}} T_{s_2} \left(\frac{dp}{d\xi}\right)^{-1} + u \frac{dU_{s_2}}{dp} \tag{5.46}$$

The dimensionless length $\xi = \frac{x}{L}$ where L is the nozzle length.

$\bar{\lambda}_{T_1}$ and $\bar{\lambda}_{T_2}$ are defined as

$$\bar{\lambda}_{T_1} = \bar{c}_p m_1 a / 4\pi \sigma_1 k \qquad (5.47)$$

$$\bar{\lambda}_{T_2} = \bar{c}_p m_2 a / 4\pi \sigma_2 k \qquad (5.48)$$

and a is the equilibrium sound speed, equal to $(\bar{\gamma}\bar{R}T)^{1/2}$. Using the energy integral 5.44 and the momentum equations, a relation corresponding to the isentropic integral for a pure gas can be obtained.

$$\left(\frac{T}{T_c}\right)\left(\frac{p_c}{p}\right)^{\frac{\bar{\gamma}-1}{\bar{\gamma}}} = \exp\left\{\frac{\chi_1}{1+\chi_1+\chi_2}\int_{p_c}^{p}\frac{1}{\bar{c}_p T}\left[c_1\frac{dT_{s1}}{dp} + u_{s1}\frac{du}{dp} - u_{s1}\frac{du_{s1}}{dp}\right]dp\right.$$

$$\left. + \frac{\chi_2}{1+\chi_1+\chi_2}\int_{p_c}^{p}\frac{1}{\bar{c}_p T}\left[c_2\frac{dT_{s2}}{dp} + u_{s2}\frac{du}{dp} - u_{s2}\frac{du_{s2}}{dp}\right]dp\right\} \qquad (5.49)$$

p_c is the rocket chamber pressure and $\bar{\gamma}$, as defined before, is equal to $\bar{\gamma} = (c_p + \chi_1 c_1 + \chi_2 c_2)/(c_v + \chi_1 c_1 + \chi_2 c_2)$. The momentum equations for the particle cloud with p as the independent variable are:

$$u\frac{du}{dp} - \frac{L}{\bar{\lambda}_{v_1}} a u_{s1}\left(\frac{dp}{d\xi}\right)^{-1} - \left[\frac{L}{\gamma}\chi_2\frac{p}{p_0}(1-\rho_{s1})(1-\rho_{s2})\left(\frac{dp}{d\xi}\right)^{-1}\right.$$

$$\left.(u_{s1}-u_{s2})(u_{s1}^{*}-u_{s2}^{*})\right] = -\frac{L}{\bar{\lambda}_{v_1}} a \rho_{s1} u_{s1}\left(\frac{dp}{d\xi}\right)^{-1} + u\frac{du_{s1}}{dp} \qquad (5.50)$$

$$u\frac{du}{dp} - \frac{L}{\bar{\lambda}_{v_2}} a u_{s2}\left(\frac{dp}{d\xi}\right)^{-1} + \left[\frac{L}{\gamma}\chi_1\frac{p}{p_0}(1-\rho_{s1})(1-\rho_{s2})\left(\frac{dp}{d\xi}\right)^{-1}\right.$$

$$\left.(u_{s1}-u_{s2})(u_{s1}^{*}-u_{s2}^{*})\right] = -\frac{L}{\bar{\lambda}_{v_2}} a \rho_{s2} u_{s2}\left(\frac{dp}{d\xi}\right)^{-1} + u\frac{du_{s2}}{dp} \qquad (5.51)$$

$\bar{\lambda}_{v_1}$ and $\bar{\lambda}_{v_2}$ are the velocity equilibrium lengths based on the equilibrium sound speed $(\bar{\gamma} \bar{R} T)^{1/2}$. The equations for the collisionless particle velocities are

$$u \frac{du}{dp} - a \frac{L}{\bar{\lambda}_{v_1}} u_{s1}^* \left(\frac{dp}{dj}\right)^{-1} = u \frac{du_{s1}^*}{dp} + u_{s1}^* \frac{d}{dp} (u - u_{s1}^*) \tag{5.52}$$

$$u \frac{du}{dp} - a \frac{L}{\bar{\lambda}_{v_2}} u_{s2}^* \left(\frac{dp}{dj}\right)^{-1} = u \frac{du_{s2}^*}{dp} + u_{s2}^* \frac{d}{dp} (u - u_{s2}^*) \tag{5.53}$$

The equation of state for the perfect gas is written as

$$p = (1 + \chi_1 + \chi_2) \rho \bar{R} T \tag{5.54}$$

Denoting λ as a typical equilibrium length, it has been discussed(1, 3) that the slip quantities are of order $\frac{\lambda}{L}$. For problems in which $\frac{\lambda}{L} << 1$, a perturbation solution can be found by expressing the flow variables in the following form:

$$\rho(p) = \overset{(0)}{\rho}(p) + \frac{\lambda}{L} \overset{(1)}{\rho}(p) + \left(\frac{\lambda}{L}\right)^2 \overset{(2)}{\rho}(p) + \cdots$$

$$u(p) = \overset{(0)}{u}(p) + \frac{\lambda}{L} \overset{(1)}{u}(p) + \left(\frac{\lambda}{L}\right)^2 \overset{(2)}{u}(p) + \cdots$$

$$T(p) = \overset{(0)}{T}(p) + \frac{\lambda}{L} \overset{(1)}{T}(p) + \left(\frac{\lambda}{L}\right)^2 \overset{(2)}{T}(p) + \cdots$$

$$\rho_{s1}(p) = \overset{(0)}{\rho}_{s1}(p) + \frac{\lambda}{L} \overset{(1)}{\rho}_{s1}(p) + \left(\frac{\lambda}{L}\right)^2 \overset{(2)}{\rho}_{s1}(p) + \cdots$$

$$\rho_{s2}(p) = \overset{(0)}{\rho}_{s2}(p) + \frac{\lambda}{L} \overset{(1)}{\rho}_{s2}(p) + \left(\frac{\lambda}{L}\right)^2 \overset{(2)}{\rho}_{s2}(p) + \cdots$$

$$u_{s1}(p) = \frac{\lambda}{L} \overset{(1)}{u}_{s1}(p) + \left(\frac{\lambda}{L}\right)^2 \overset{(2)}{u}_{s1}(p) + \cdots$$

$$\mathcal{U}_{s2}(p) = \frac{\lambda}{L}\,\mathcal{U}_{s2}^{(1)}(p) + \left(\frac{\lambda}{L}\right)^2 \mathcal{U}_{s2}^{(2)}(p) + \cdots\cdot$$

$$\mathcal{U}_{s1}^{*}(p) = \frac{\lambda}{L}\,\mathcal{U}_{s1}^{*(1)}(p) + \left(\frac{\lambda}{L}\right)^2 \mathcal{U}_{s1}^{*(2)}(p) + \cdots\cdot$$

$$\mathcal{U}_{s2}^{*}(p) = \frac{\lambda}{L}\,\mathcal{U}_{s2}^{*(1)}(p) + \left(\frac{\lambda}{L}\right)^2 \mathcal{U}_{s2}^{*(2)}(p) + \cdots\cdot$$

$$T_{s1}(p) = \frac{\lambda}{L}\,T_{s1}^{(1)}(p) + \left(\frac{\lambda}{L}\right)^2 T_{s1}^{(2)}(p) + \cdots\cdot$$

$$T_{s2}(p) = \frac{\lambda}{L}\,T_{s2}^{(1)}(p) + \left(\frac{\lambda}{L}\right)^2 T_{s2}^{(2)}(p) + \cdots\cdot$$

The zeroth order solution can be easily found. From equation 5.49, we have

$$\frac{T^{(0)}}{T_c} = \left(\frac{p}{p_c}\right)^{\frac{\bar{\gamma}-1}{\bar{\gamma}}} \tag{5.55}$$

and using the equation of state, we obtain

$$\frac{\rho^{(0)}}{\rho_c} = \left(\frac{p}{p_c}\right)^{\frac{1}{\bar{\gamma}}} \tag{5.56}$$

From the energy integral 5.44

$$\mathcal{U}^{(0)} = \left\{2\,\bar{C}_p\,T_c\left[1-\left(\frac{p}{p_c}\right)^{\frac{\bar{\gamma}-1}{\bar{\gamma}}}\right]\right\}^{1/2} \tag{5.57}$$

As discussed in reference 1, the case when the momentum transfer due to particle collisions is of the same order as viscous drag is when $\left(\frac{\lambda}{L}\right)^2 \sim \frac{\ell}{L}$. Under such circumstances, eqns. 5.52 and 5.53 give

$$\mathcal{U}_{s1}^{*(1)} = \frac{\bar{\lambda}_{r_1}}{\lambda}\,a^{(0)}\frac{1}{\rho}\frac{dp}{d\xi} \tag{5.58}$$

$$u_{s2}^{*(1)} = \frac{\bar{\lambda}_{v_2}}{\lambda} a^{(0)} \frac{1}{\rho} \frac{d\rho}{dy} \tag{5.59}$$

From eqns. 5.50 and 5.51, using 5.58 and 5.59, $u_{s1}^{(1)}$ and $u_{s2}^{(1)}$ can be solved algebraically.

$$u_{s1}^{(1)} = -a^{(0)} \frac{\bar{\lambda}_{v_1}}{\lambda} \frac{1}{\bar{r}} \frac{1}{\rho} \frac{d\rho}{dy} \left\{ 1 - \frac{\frac{\lambda^2}{eL} \chi_2 \left(\frac{\bar{\lambda}_{v_1}}{\lambda} - \frac{\bar{\lambda}_{v_2}}{\lambda} \right) \frac{\rho^{(0)}}{\rho_0} \left| \frac{1}{\bar{r}} \frac{1}{\rho} \frac{d\rho}{dy} \right| \left| \frac{\bar{\lambda}_{v_1}}{\lambda} - \frac{\bar{\lambda}_{v_2}}{\lambda} \right|}{1 + \frac{\lambda^2}{eL} \frac{\rho^{(0)}}{\rho_0} \left| \frac{1}{\bar{r}} \frac{1}{\rho} \frac{d\rho}{dy} \right| \left| \frac{\bar{\lambda}_{v_1}}{\lambda} - \frac{\bar{\lambda}_{v_2}}{\lambda} \right| \left(\chi_2 \frac{\bar{\lambda}_{v_1}}{\lambda} + \chi_1 \frac{\bar{\lambda}_{v_2}}{\lambda} \right)} \right\} \tag{5.60}$$

$$u_{s2}^{(1)} = -a^{(0)} \frac{\bar{\lambda}_{v_2}}{\lambda} \frac{1}{\bar{r}} \frac{1}{\rho} \frac{d\rho}{dy} \left\{ 1 - \frac{\frac{\lambda^2}{eL} \chi_1 \left(\frac{\bar{\lambda}_{v_2}}{\lambda} - \frac{\bar{\lambda}_{v_1}}{\lambda} \right) \frac{\rho^{(0)}}{\rho_0} \left| \frac{1}{\bar{r}} \frac{1}{\rho} \frac{d\rho}{dy} \right| \left| \frac{\bar{\lambda}_{v_1}}{\lambda} - \frac{\bar{\lambda}_{v_2}}{\lambda} \right|}{1 + \frac{\lambda^2}{eL} \frac{\rho^{(0)}}{\rho_0} \left| \frac{1}{\bar{r}} \frac{1}{\rho} \frac{d\rho}{dy} \right| \left| \frac{\bar{\lambda}_{v_1}}{\lambda} - \frac{\bar{\lambda}_{v_2}}{\lambda} \right| \left(\chi_2 \frac{\bar{\lambda}_{v_1}}{\lambda} + \chi_1 \frac{\bar{\lambda}_{v_2}}{\lambda} \right)} \right\} \tag{5.61}$$

The corresponding first order $T_{s1}^{(1)}$ and $T_{s2}^{(1)}$ can be obtained from eqns. 5.45 and 5.48

$$T_{s1}^{(1)} = \frac{1}{\bar{C}_p} \frac{C_1}{\bar{C}_p} \left(\frac{\lambda_{T_1}}{\lambda} \right) \frac{a^{(0)}}{\bar{r}} u^{(0)} \frac{1}{\rho} \frac{d\rho}{dy} \tag{5.62}$$

$$T_{s2}^{(1)} = \frac{1}{\bar{C}_p} \frac{C_2}{\bar{C}_p} \left(\frac{\lambda_{T_2}}{\lambda} \right) \frac{a^{(0)}}{\bar{r}} u^{(0)} \frac{1}{\rho} \frac{d\rho}{dy} \tag{5.63}$$

From the continuity equations 5.41 and 5.42, we obtain

$$\rho_{s1}^{(1)} = -\frac{u_{s1}^{(1)}}{u^{(0)}} \tag{5.64}$$

$$\rho_{s2}^{(1)} = -\frac{u_{s2}^{(1)}}{u^{(0)}} \tag{5.65}$$

The remaining first order perturbation quantities for the gas can also be found easily. Equating terms of the first order, 5.49 gives

$$\begin{aligned}\frac{T^{(1)}}{T^{(0)}} &= \frac{\chi_1}{1+\chi_1+\chi_2}\int_{p_c}^{p}\frac{1}{\bar{c}_p T^{(0)}}\left[C_1\frac{dT_{s1}^{(1)}}{dp} + u_{s1}^{(1)}\frac{du^{(0)}}{dp}\right]dp \\ &\quad + \frac{\chi_2}{1+\chi_1+\chi_2}\int_{p_c}^{p}\frac{1}{\bar{c}_p T^{(0)}}\left[C_2\frac{dT_{s2}^{(1)}}{dp} + u_{s2}^{(1)}\frac{du^{(0)}}{dp}\right]dp \\ &= G\left(\frac{p}{p_c}\right)\end{aligned} \tag{5.66}$$

From the equation of state 5.54,

$$\frac{\rho^{(1)}}{\rho^{(0)}} = -\frac{T^{(1)}}{T^{(0)}} = -G\left(\frac{p}{p_c}\right) \tag{5.67}$$

and from the energy integral 5.44, after some simplification, we have

$$\begin{aligned}(\bar{\gamma}-1)M^{2(0)}\frac{u^{(1)}}{u^{(0)}} &= -\frac{T^{(1)}}{T^{(0)}} + \frac{\chi_1}{1+\chi_1+\chi_2}\left[\frac{C_1}{\bar{c}_p}\frac{T_{s1}^{(1)}}{T^{(0)}} + (\bar{\gamma}-1)M^{2(0)}\frac{u_{s1}^{(1)}}{u^{(0)}}\right] \\ &\quad + \frac{\chi_2}{1+\chi_1+\chi_2}\left[\frac{C_2}{\bar{c}_p}\frac{T_{s2}^{(1)}}{T^{(0)}} + (\bar{\gamma}-1)M^{2(0)}\frac{u_{s2}^{(1)}}{u^{(0)}}\right] \\ &= -G\left(\frac{p}{p_c}\right) + F\left(\frac{p}{p_c}\right)\end{aligned}$$

or

$$\frac{u^{(1)}}{u^{(0)}} = \frac{F\left(\frac{p}{p_c}\right) - G\left(\frac{p}{p_c}\right)}{(\bar{\gamma}-1)M^{2(0)}} \tag{5.68}$$

The perturbation quantities are calculated based on the optimum nozzle$^{(8)}$ for gas-particle mixtures consisting only of particles of radius σ_1 giving the pressure distribution

$$\frac{1}{p}\frac{dp}{dy} = -\left[\frac{M_0}{1+\frac{\bar{\lambda}_{\Pi}}{\lambda}\left(\frac{c_L}{c_p}\right)^2(\bar{\gamma}-1)M^{(0)}}\right]^{1/2}\int_{p_e/p_c}^{1}\left[\frac{1+\frac{\bar{\lambda}_{\Pi}}{\lambda}\left(\frac{c_L}{c_p}\right)^2(\bar{\gamma}-1)M^{2(0)}}{M^{(0)}}\right]^{1/2}\frac{dp}{p} \qquad (5.69)$$

where p_e is the nozzle exit pressure.

Using the first order perturbation quantities, the specific impulse losses can be calculated. If the reference value is taken to be

$$I^{(0)} = \frac{u_e^{(0)}}{g} \qquad (5.70)$$

$$\frac{L}{\lambda}\frac{I^{(0)}-I}{I^{(0)}} = -\left[\frac{u^{(1)}}{u^{(0)}} - \frac{1}{1+\chi_1+\chi_2}\left(\chi_1\frac{u_{s_1}^{(1)}}{u^{(0)}} + \chi_2\frac{u_{s_2}^{(1)}}{u^{(0)}}\right)\right] \qquad (5.71)$$

4. CONCLUSION

The effect upon particle velocities when particle collisions are taken into consideration is very clearly demonstrated by the relatively simple, normal shock wave problem. The transfer of momentum due to collisions of course tends to cause the two particle velocities to be more nearly equal as compared with particle velocities when particle collisions are being suppressed. The change is rather significant, as shown in Figure 5.1 for $\ell/\lambda v_1 = 10.0$. The same change, of course, also appears in the flow of a particle-gas mixture through a nozzle. Here, the first order slip velocities are calculated, and the effect of particle collisions certainly tends to decrease the difference of the two slip velocities. As shown in Figure 5.4, the maximum difference between the two slip velocities for $\frac{\lambda^2}{\ell L} = 5.0$ is reduced to about half of that when particle collisions are not taken into consideration. In the normal shock problem, where the velocities are decreasing downstream of the shock, the effect of particle collisions tends to increase the gas velocity, while in the flow through a nozzle where the velocities are increasing along the nozzle, the effect of particle collisions tends to decrease the gas velocity. The first order perturbation gas velocity has been calculated for the nozzle flow. With $\frac{\lambda^2}{\ell L} = 5.0$, the maximum of the perturbation gas velocity is reduced to one third of that when particle collisions are neglected.

The difference in temperature as compared with the collisionless case is quite negligible. This is true both in the normal shock wave problem and the flow through a nozzle. In Figure 5.3 we observe that the dissipation work associated with the force due to particle collisions is indeed negligible as compared with the rate of heat transfer between the particles and the gas. The fractional loss of specific impulse is shown in Figure 5. 8. The loss is in fact smaller when particle collisions are taken into consideration using the optimum nozzle for collisionless particles. The difference is only a few per cent.

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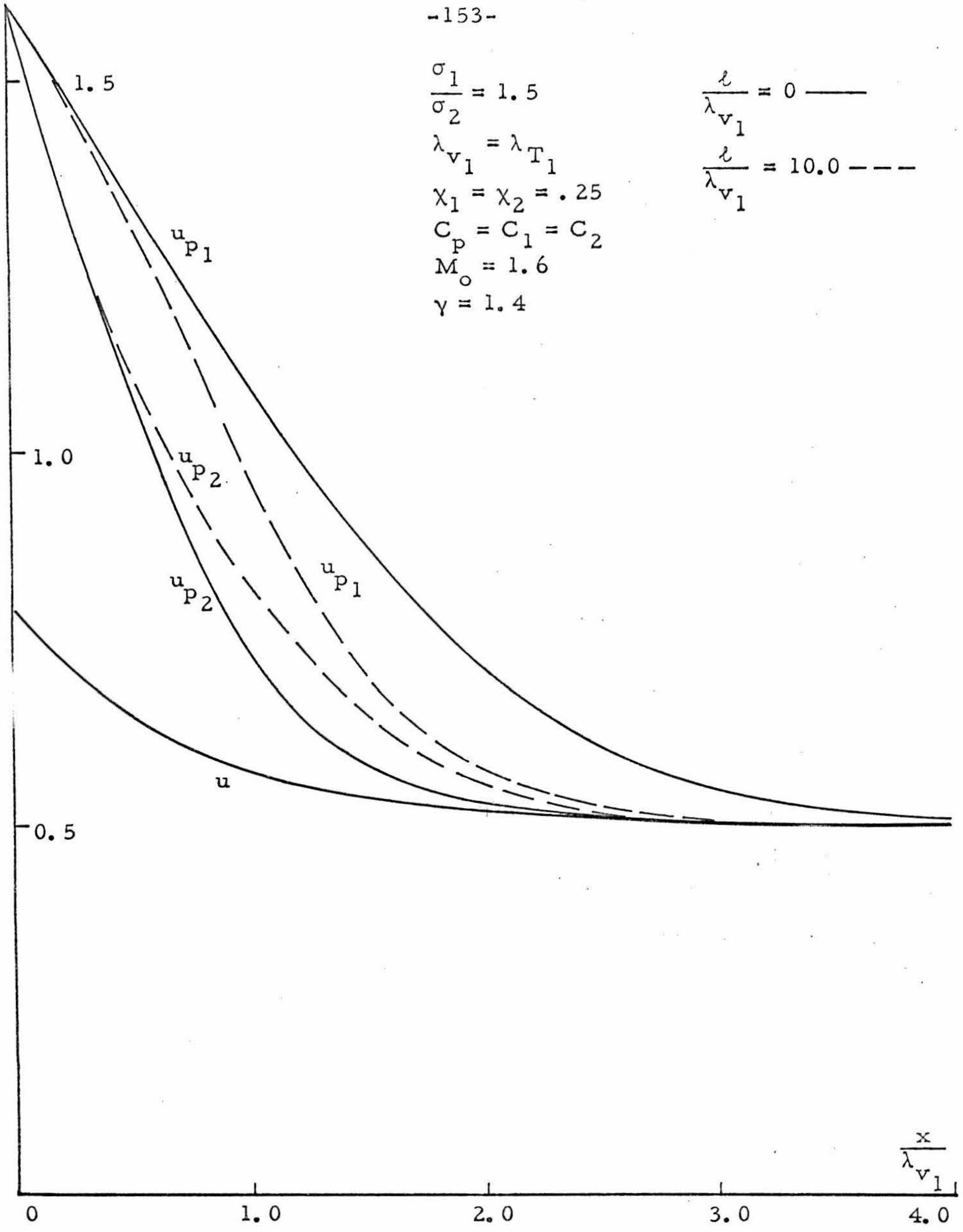

Fig. 5.1. Velocity Variation Downstream of the Gas-Dynamic Shock.

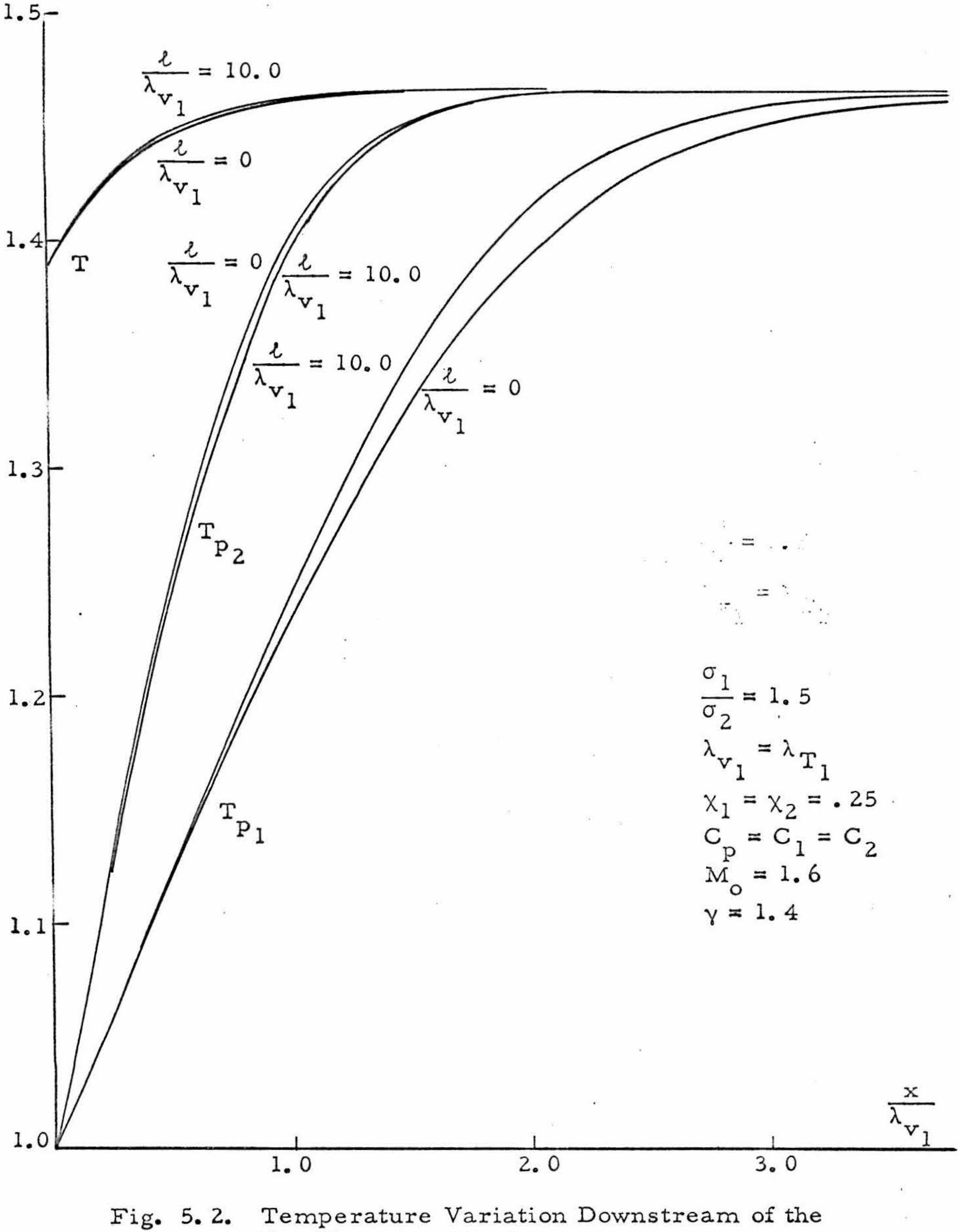

Fig. 5.2. Temperature Variation Downstream of the Gas-Dynamic Shock.

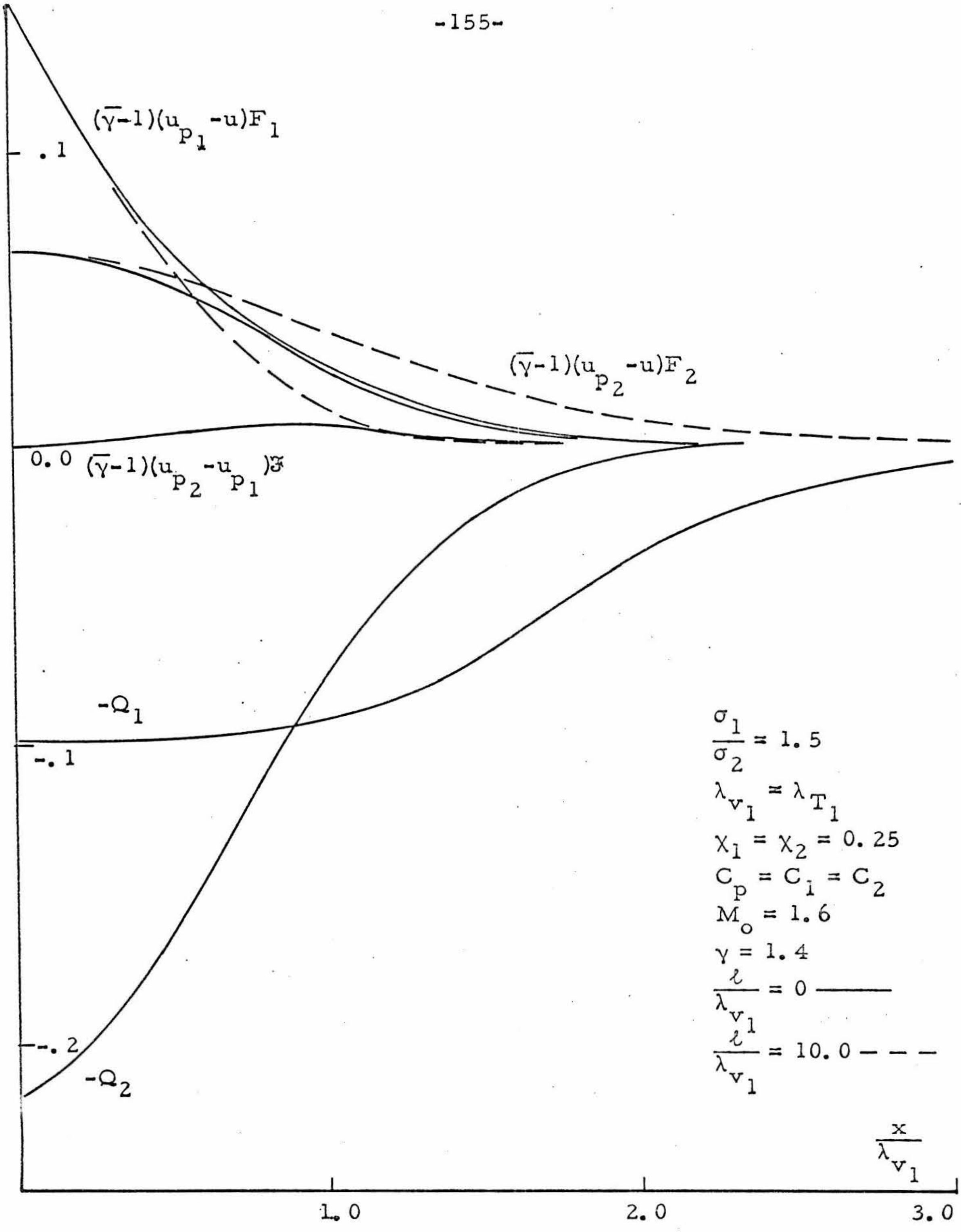

Fig. 5.3. Rate of Energy Transfer to the Particle Cloud.

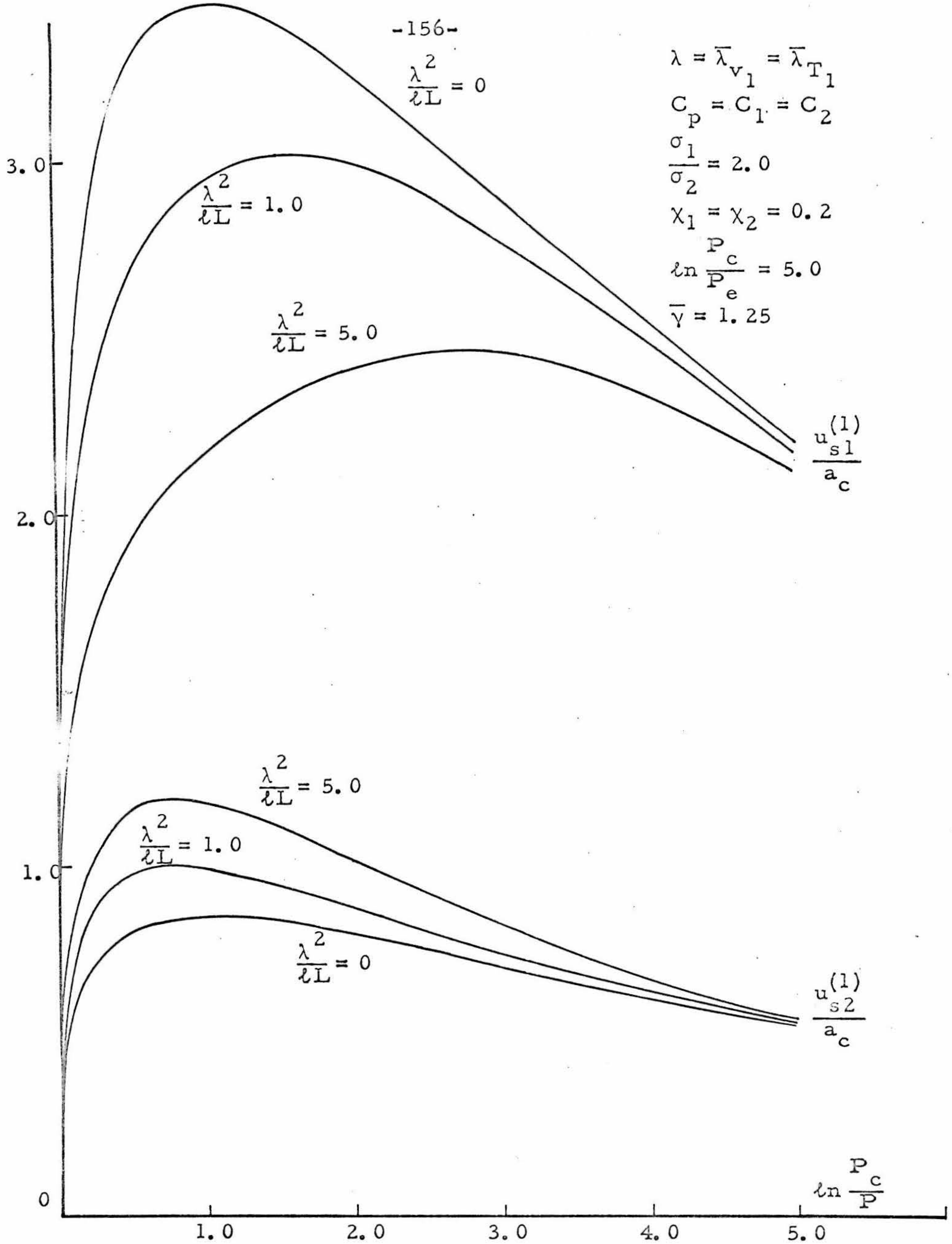

Fig. 5.4. Slip Velocity as a Function of Pressure Along a Nozzle.

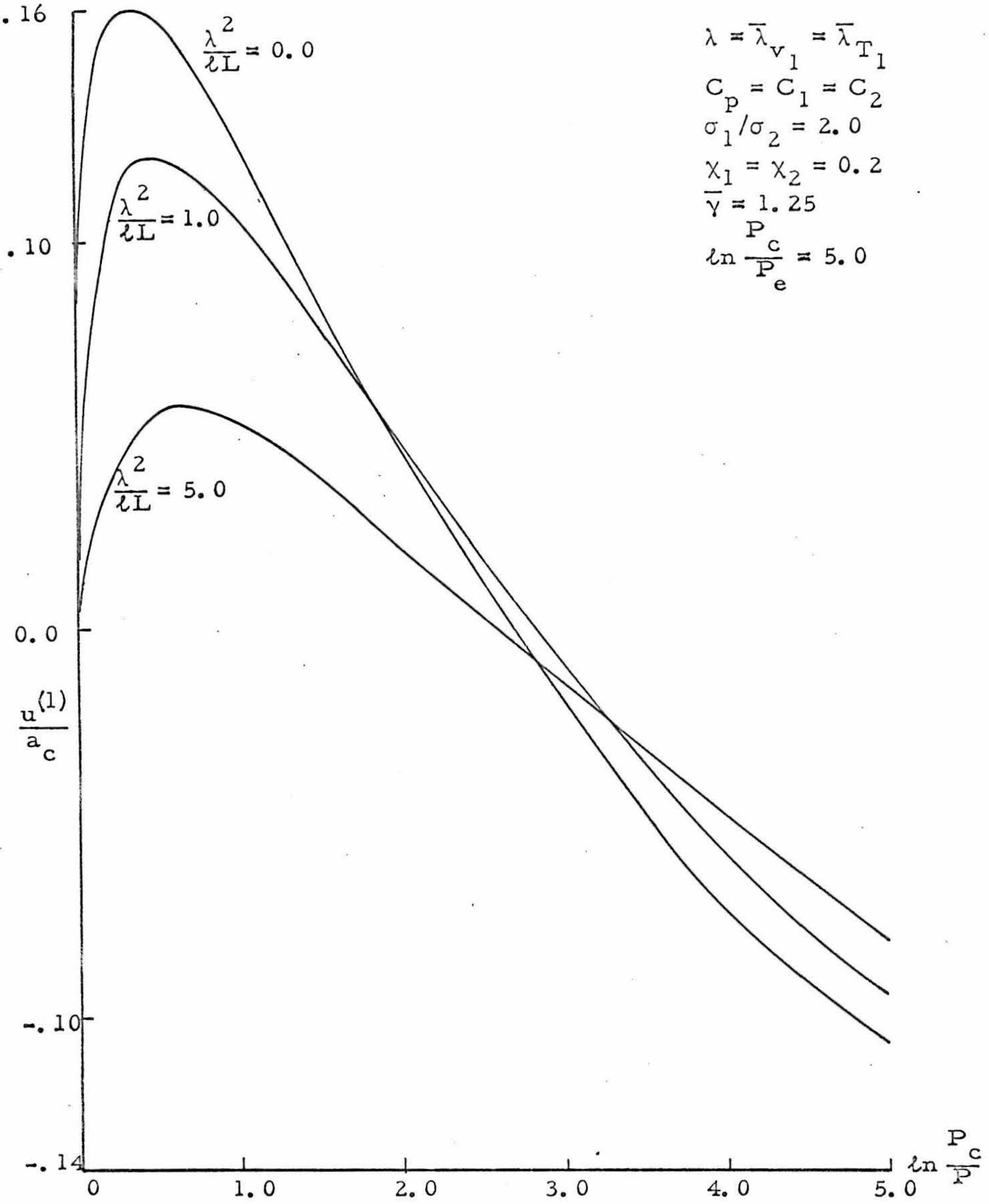

Fig. 5.5. First Order Gas Velocity as a Function of Pressure Along a Nozzle.

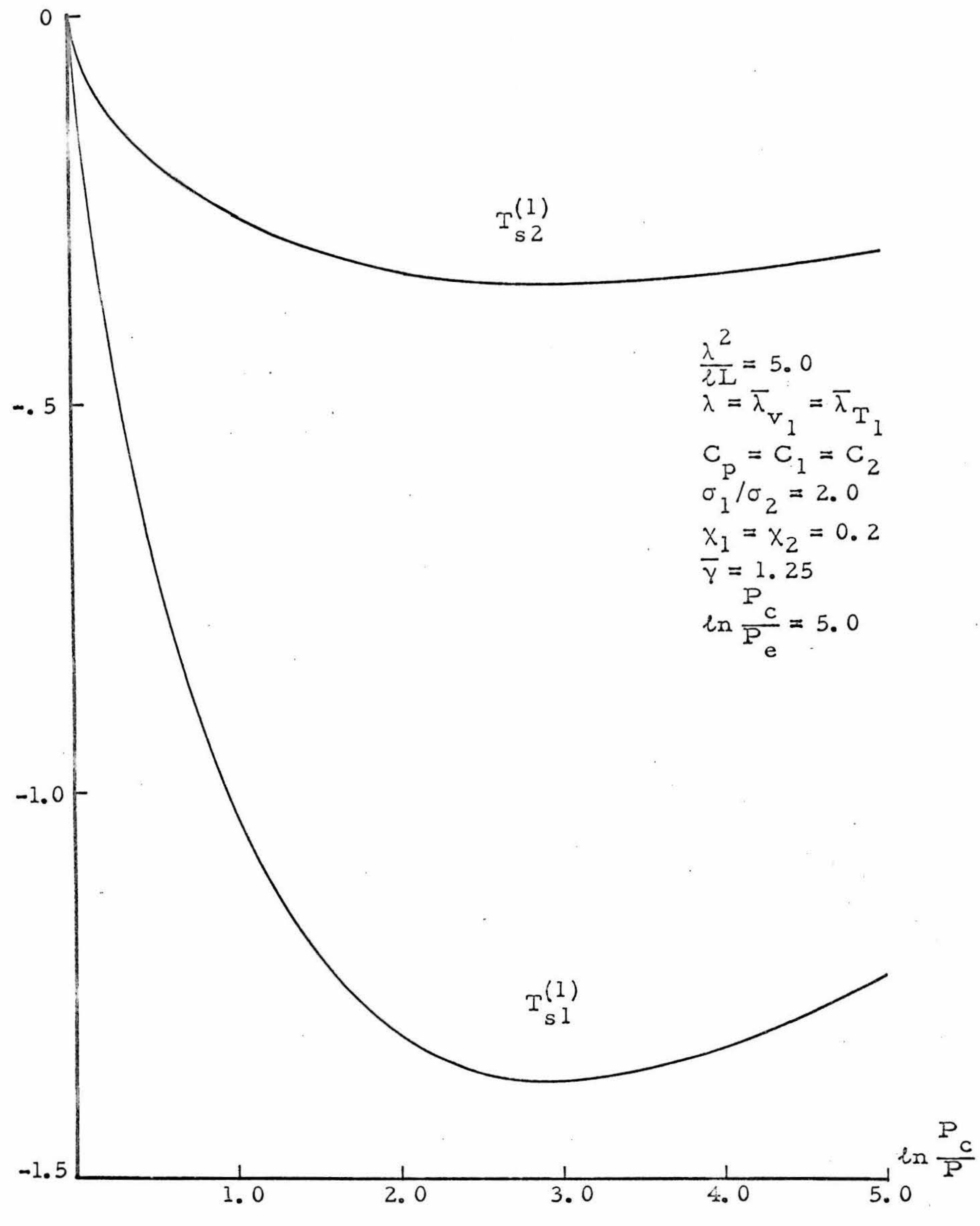

Fig. 5.6. Particle Temperature Lag as a Function of Pressure Along a Nozzle.

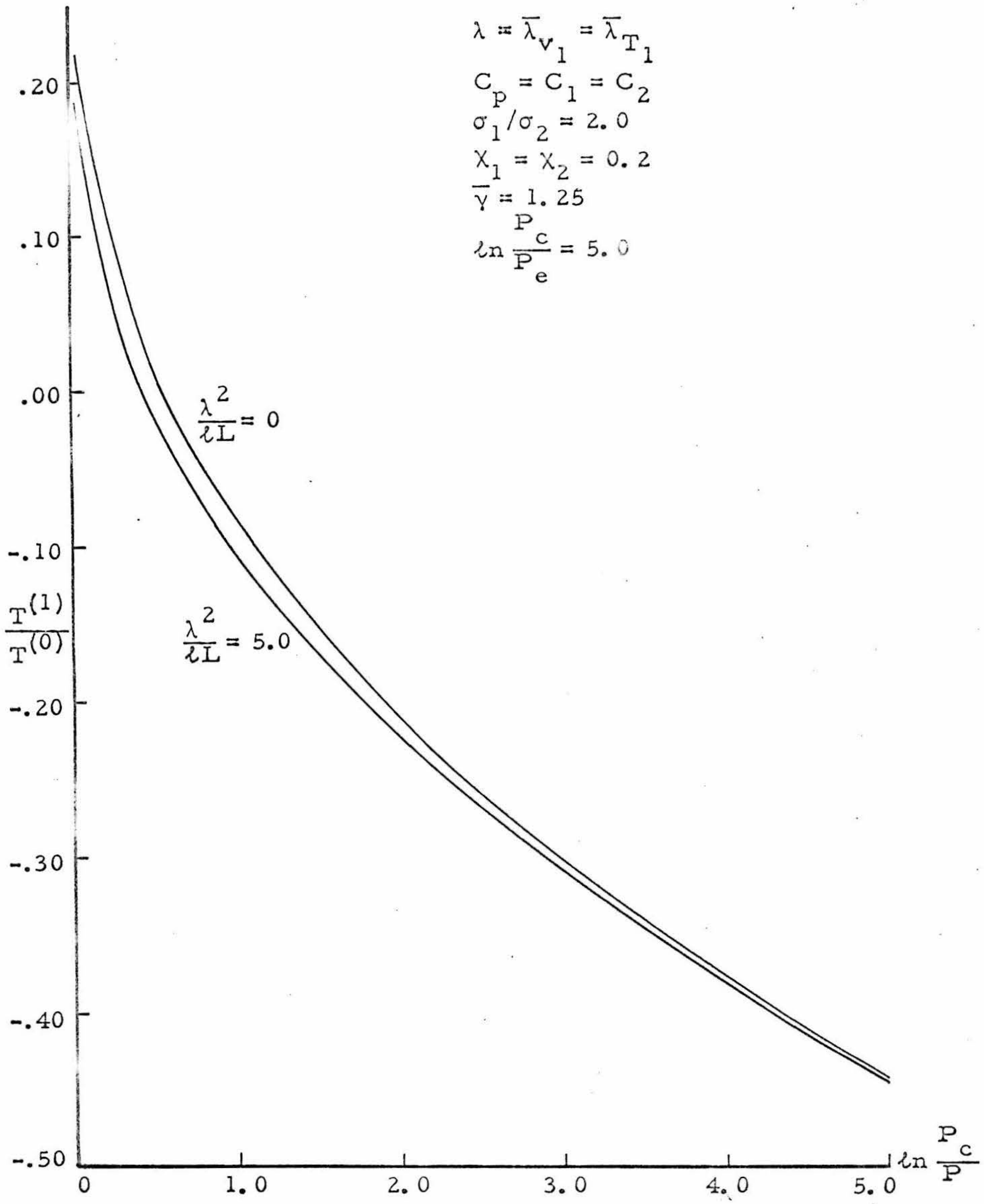

Fig. 5.7. First Order Temperature Variation as a Function of Pressure Along a Nozzle.

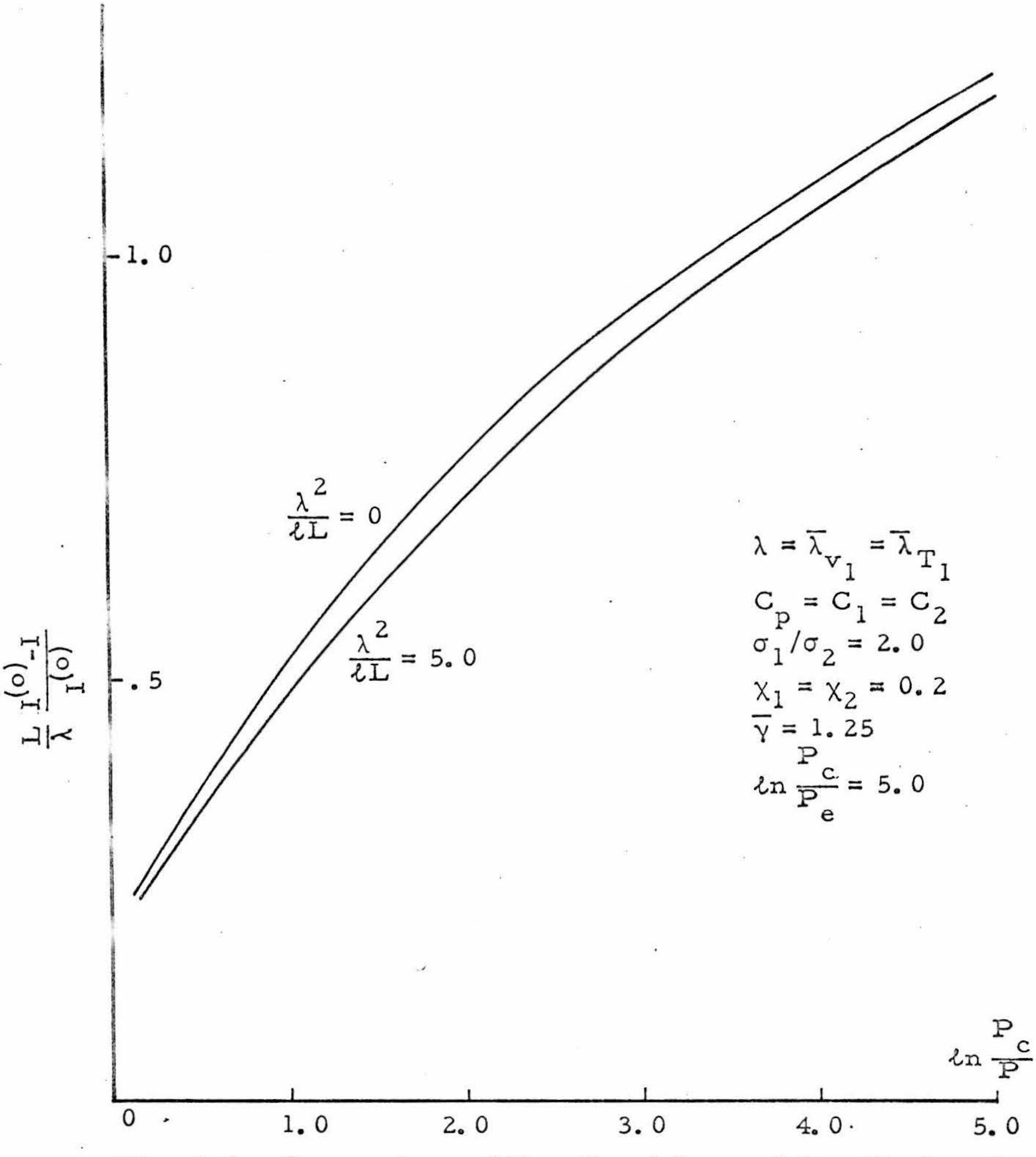

Fig. 5.8. Comparison of Fractional Loss of Specific Impulse With and Without Particle Collision Taken Into Consideration.

APPENDIX A

Equations for Reacting Particles

1. Introduction

The terminology "aerothermochemistry", as first introduced by T. von Kármán(1), deals mainly with problems of gasdynamics in which chemical reactions take place. But in many flow phenomena involving chemical reactions, the presence of a particle cloud plays a dominant role, as in condensation and the combustion of small solid particles and liquid droplets. These problems can probably be considered as aerothermochemistry problems of two phase flow.

It is only relatively recently that a two phase approach has been used to treat some of the combustion problems of liquid droplets(2), and Marble(3) applied the two phase concept to some of the condensation problems and rather interesting and new results were obtained.

The basic equations governing these flow problems are analogous to two phase flow problems without chemical reactions. They are the equations of continuity, momentum, and energy for the two phases of substances present and the equation of state. The presence of different gas species requires continuity equations describing individual species, and the rate of mass production and heat of chemical reactions modify both the equations of motion and the energy equations. In addition, a knowledge of the reaction mechanism is required, and for the problems being considered here, a diffusion relaxation time connected with the rate of mass production is introduced.

$$\tau_D = m/(4\pi\sigma\rho D) \tag{A-1}$$

The relationships between τ_D and τ_v, τ_T are

$$\tau_D = \frac{3}{2} S_c \tau_v \tag{A-2}$$

$$\tau_D = Le\, \tau_T \tag{A-3}$$

where Le is the Lewis number.

2. Basic Equations

Consider a cloud of small solid particles suspended in a gas mixture consisting of N gas species. The particles can exist in either a liquid or solid phase. The total number of particles is assumed to be constant. Thus, in combustion problems, particle breakup is neglected, and spontaneous nucleation is considered to be not a dominant feature in the process of condensation. Oftentimes, as in saturated flows over converging-diverging nozzles, nucleation is usually the most important factor, but in some problems of relatively high vapor concentration in the presence of liquid droplets, the flow may be governed more by the change of size of particles and nucleation can be assumed to be negligible as a first approximation.

Continuity Equation. Consider a volume element fixed in space. The continuity equations for the gas species are

$$\frac{\partial \rho_\alpha}{\partial t} + \frac{\partial}{\partial x_i}(\rho_\alpha u_{\alpha i}) = \omega_\alpha \tag{A-4}$$

ρ_α, $u_{\alpha i}$, ω_α are the individual density, velocity components, and the rate of mass production of the gas species α. The total density

ρ is given by

$$\rho = \sum_{\alpha=1}^{N} \rho_\alpha \tag{A-5}$$

and the mass concentration K_α is defined by

$$K_\alpha = \frac{\rho_\alpha}{\rho} \tag{A-6}$$

The mean velocity of the gas mixture u_i is defined as

$$\rho u_i = \sum_{\alpha=1}^{N} \rho_\alpha u_{\alpha i}$$

or

$$u_i = \sum_{\alpha=1}^{N} K_\alpha u_{\alpha i} \tag{A-7}$$

The diffusion velocity or peculiar velocity of the gas component α is its velocity relative to the mean, and is given by

$$u'_{\alpha i} = u_{\alpha i} - u_i \tag{A-8}$$

Using definitions A-5 and A-7, the continuity equation for the gas mixture, by summing eqn. A-4 over N species, can be written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = \omega \tag{A-9}$$

ω is the total rate of mass production for the gas mixture and is given by

$$\omega = \sum_{\alpha=1}^{N} \omega_\alpha \tag{A-10}$$

The continuity equation for the particle cloud is

$$\frac{\partial \rho_p}{\partial t} + \frac{\partial}{\partial x_i}(\rho_p u_{pi}) = \omega_p \tag{A-11}$$

As before, subscript p is used to denote quantities pertaining to the particle cloud. Since mass is concerned, we have

$$\omega + \omega_p = 0 \tag{A-12}$$

Momentum Equation. The rate of mass production ω_α introduces an additional term in the momentum transfer. If $m_{\alpha i}$ is the average momentum component per unit mass associated with ω_α, the transfer of momentum due to the chemical reaction is $\omega_\alpha m_{\alpha i}$ By considering a volume element fixed in space, the momentum equations for species α are:

$$\frac{\partial}{\partial t}(\rho K_\alpha U_{\alpha i}) + \frac{\partial}{\partial x_i}(\rho K_\alpha U_{\alpha i} U_{\alpha j}) = F_{\alpha i} + \frac{\partial}{\partial x_i}\sigma_{\alpha ij} + \omega_\alpha m_{\alpha i} + f_{\alpha i} \tag{A-13}$$

$F_{\alpha i}$ is the particle-gas momentum interaction on species α. $\sigma_{\alpha ij}$ is the "partial stress" following the terminology given by Truesdell and Toupin(4). The momentum of the gas species α need not be balanced by itself as there is transfer of momentum between gas components. Truesdell(4) calls this additional term $f_{\alpha i}$ "supply of momentum." Since momentum is conserved for the whole system, we have

$$\sum_{\alpha=1}^{N} f_{\alpha i} = 0 \tag{A-14}$$

Summing eqn. A-13 over all gas species, the momentum equation for the gas mixture is

$$\frac{\partial}{\partial t}\rho U_i + \frac{\partial}{\partial x_j}(\rho U_i U_j) = F_i + \frac{\partial}{\partial x_j}\sigma_{ij} + \omega m_i \tag{A-15}$$

where

$$F_i = \sum_{\alpha=1}^{N} F_{\alpha i} \tag{A-16}$$

$$\omega m_i = \sum_{\alpha=1}^{N} \omega_\alpha m_{\alpha i} \qquad \text{(A-17)}$$

and

$$\begin{aligned} \sigma_{ij} &= \sum_{\alpha=1}^{N} (\sigma_{\alpha ij} - \rho_\alpha u'_{\alpha i} u'_{\alpha j}) \\ &= \sum_{\alpha=1}^{N} (P_\alpha \delta_{ij} + \tau_{\alpha ij} - \rho_\alpha u'_{\alpha i} u'_{\alpha j}) \end{aligned} \qquad \text{(A-18)}$$

$\rho_\alpha u'_{\alpha i} u'_{\alpha j}$ are the apparent stresses due to diffusion, and many authors refer to these as "diffusion stresses." In this context, the total stresses acting on the gas mixture are the sum of partial stresses $\sum_{\alpha=1}^{N} \tau_{\alpha ij}$ and the diffusion stresses $-\sum_\alpha \rho_\alpha u'_{\alpha i} u'_{\alpha j}$. Von Kármán$^{(5)}$ defined σ_{ij} as

$$\begin{aligned} \sigma_{ij} &= -p\delta_{ij} + \mu\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right] \\ &= -p\delta_{ij} + \tau_{ij} \end{aligned} \qquad \text{(A-19)}$$

where μ^* is defined in such a way that it includes the apparent stresses caused by diffusion.

It has been pointed out by Culick$^{(7)}$ that it is more convenient sometimes to consider just the total stresses σ_{ij} instead of introducing partial stresses $\sigma_{\alpha ij}$ and diffusion stresses $-\sum \rho_\alpha u'_{\alpha i} u'_{\alpha j}$. By considering a volume element moving with the mean velocity u_i, the momentum equations for the gas mixture, by summing over individual species, take the following form:

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[*] μ for binary and multicomponent mixtures is given in reference 6, eqns. (8-2-21) and (8-2-25).

$$\Sigma \frac{\partial}{\partial t}(\rho_\alpha u_{\alpha i}) + \Sigma \frac{\partial}{\partial x_j}(\rho_\alpha u_{\alpha i} u_j) = F_i + \frac{\partial}{\partial x_j}\sigma_{ij} + \omega u_i$$

or

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_j}\rho u_i u_j = F_i + \frac{\partial \sigma_{ij}}{\partial x_j} + \omega u_i \qquad \text{(A-20)}$$

Note that σ_{ij} in eqn. A-20 is the sum of partial stresses while considering a volume element moving with u_i, and diffusion stresses in the form $-\Sigma \rho_\alpha u'_{\alpha i} u'_{\alpha j}$ do not appear at all.

The momentum equation for the particle cloud is

$$\frac{\partial}{\partial t}(\rho_p u_{pi}) + \frac{\partial}{\partial x_j}(\rho_p u_{pi} u_{pj}) = F_{pi} + \omega_{pi} m_{pi} \qquad \text{(A-21)}$$

Since momentum is conserved, we note that

$$F_i + F_{pi} + \omega m_i + \omega_p m_{pi} = 0$$

Adding eqns. A-21 and A-15, the momentum equation for the whole system is

$$\frac{\partial}{\partial t}(\rho u_i + \rho_{pi} u_{pi}) + \frac{\partial}{\partial x_j}(\rho u_i u_j + \rho_p u_{pi} u_{pi}) = \frac{\partial \sigma_{ij}}{\partial x_j} \qquad \text{(A-22)}$$

Using continuity equations A-9 and A-11, the momentum equations can be written in the following form:

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = F_i + \frac{\partial \sigma_{ij}}{\partial x_j} + \omega(m_i - u_i) \qquad \text{(A-23)}$$

$$\rho_p \frac{\partial u_{pi}}{\partial t} + \rho_p u_{pj} \frac{\partial u_{pi}}{\partial x_j} = F_{pi} + \omega_p(m_{pi} - u_{pi}) \qquad \text{(A-24)}$$

Energy Equation. In the momentum equation , we have the term $\omega_\alpha m_{\alpha i}$ which denotes the rate of momentum transfer due to mass production. The corresponding term in the energy equation is clearly equal to $\omega_\alpha [h_\alpha(T_R, P_R) + \frac{1}{2} m_{\alpha i} m_{\alpha i}]$. T_R and P_R are the temperature and pressure at which reaction takes place. h_α is the enthalpy per unit mass for the species α , and is defined as

$$h_\alpha = \int_0^T C_{p\alpha}\, dT + h_\alpha^\circ \qquad \text{(A-25)}$$

h_α° is the enthalpy of formation. The enthalpy of the gas mixture can be defined as

$$h = \Sigma K_\alpha h_\alpha = \Sigma K_\alpha \int_0^T C_{p\alpha}\, dT + \Sigma h_\alpha^\circ K_\alpha \qquad \text{(A-26)}$$

In addition, we have to consider the heat of reaction. Consider a one-step reaction

$$\sum_{\alpha=1}^{N} \nu_\alpha' C_\alpha + \nu_p' C_p \rightleftharpoons \sum_{\alpha=1}^{N} \nu_\alpha'' C_\alpha + \nu_p'' C_p \qquad \text{(A-27)}$$

ν_α', ν_α'' and ν_p' and ν_p'' are the stochiometric coefficients for the α species of the gas mixture and for the particle cloud. C_α and C_p are their respective chemical symbols. The heat of reaction according to eqn. A-27 is

$$-\Sigma(\nu_\alpha'' - \nu_\alpha')\mathcal{M}_\alpha [h_\alpha(T_R, P_R) + \tfrac{1}{2} m_{\alpha i} m_{\alpha i}] - (\nu_p'' - \nu_p')\mathcal{M}_p [h_p(T_R, P_R) + \tfrac{1}{2} m_{p i} m_{p i}]$$

where $\mathcal{M}_\alpha$ and $\mathcal{M}_p$ are their respective atomic weights. The heat of reaction per unit volume is

$$\mathcal{H} = -\Sigma \omega_\alpha [h_\alpha(T_R, P_R) + \tfrac{1}{2} m_{\alpha i} m_{\alpha i}] - \omega_p [h_p(T_R, P_R) + \tfrac{1}{2} m_{p i} m_{p i}] \qquad \text{(A-28)}$$

For liquid droplets, heat of reaction is slightly different. If L denotes the latent heat of vaporization, the heat of reaction per unit volume is

$$\mathcal{H} = -\sum_{\alpha=1}^{N} \omega_\alpha [h_\alpha(T_R, P_R) + \tfrac{1}{2} u_{\alpha i} u_{\alpha i}] - \omega_p [h_p(T_R, P_R) + L(T_R, P_R) + \tfrac{1}{2} u_{pi}^2] \tag{A-29}$$

$L(T_R, P_R)$ is defined as the difference of enthalpies of the vapor and liquid phases. Note also that h_p in eqn. A-28 refers to the enthalpy of the solid particles, and h_p in eqn. A-29 is the enthalpy of the liquid. We assume the heat of reaction is totally absorbed by the gas mixture. Denoting $\mathcal{H}_\alpha$ as the heat of reaction absorbed by species α, we have

$$\sum_{\alpha=1}^{N} \mathcal{H}_\alpha = \mathcal{H} \tag{A-30}$$

Since particle cloud and gas species are moving at different velocities, there is an additional dissipation term $\Phi_{p\alpha}$. $\Phi_{p\alpha}$ is clearly equal to

$$\Phi_{p\alpha} = | F_{\alpha i} (u_{pi} - u_{\alpha i}) | \tag{A-31}$$

We assume this dissipation term is also totally absorbed by the gas species.

The particle cloud and the gas mixture also have different temperatures. Denoting Q as the transfer of thermal energy between the two phases, and Q_α as the fraction absorbed by species α, we have

$$\Sigma Q_\alpha = Q \tag{A-32}$$

q_i is the heat flux vector. If we define $q_{\alpha i}$ as the partial heat flux vector, the energy equation for the species α, by considering a volume element fixed in space, is:

$$\frac{\partial}{\partial t}\left[\rho K_\alpha\left(e_\alpha + \frac{1}{2} u_{\alpha i} u_{\alpha i}\right)\right] + \frac{\partial}{\partial x_j}\left[\rho K_\alpha u_{\alpha j}\left(e_\alpha + \frac{1}{2} u_{\alpha i} u_{\alpha i}\right)\right]$$

$$= \frac{\partial}{\partial x_j}\left(u_{\alpha i} \sigma_{\alpha ij}\right) + F_{\alpha i} u_{\alpha i} + Q_\alpha - \frac{\partial q_{\alpha i}}{\partial x_i} + H_\alpha + \Phi_{p\alpha}$$

$$+ \omega_\alpha\left[h_\alpha(T_R, P_R) + \frac{1}{2} m_{\alpha i} m_{\alpha i}\right] + f_{\alpha i} u_{\alpha i} + \mathcal{E}_\alpha \qquad \text{(A-33)}$$

$\mathcal{E}_\alpha$ is introduced since there is also transfer of energy between the gas species. Analogous with $f_{\alpha i}$, Truesdell calls $\mathcal{E}_\alpha$ the "sppply of energy." Since energy is conserved for the system, we have

$$\sum_{\alpha=1}^{N}\left(f_{\alpha i} u_{\alpha i} + \mathcal{E}_\alpha\right) = 0$$

Summing eqn. A-32 over α, the energy equation for the gas mixture takes the following form:

$$\frac{\partial}{\partial t}\left[\rho\left(h + \frac{1}{2}\Sigma K_\alpha u'_{\alpha i} u'_{\alpha i}\right) + \frac{1}{2}\rho u_i u_i\right] + \frac{\partial}{\partial x_j}\left[\rho u_j\left(h + \frac{1}{2}\Sigma K_\alpha u'_{\alpha i} u'_{\alpha i}\right)\right.$$

$$\left. + \frac{1}{2}\rho u_j u_i u_i\right] = \frac{\partial}{\partial x_j}\left[-\left(q_j - \Sigma u'_{\alpha i} \tau_{\alpha ij}\right) - \Sigma \rho K_\alpha u'_{\alpha j}\left(h_\alpha + \frac{1}{2} u'_{\alpha i} u'_{\alpha i}\right)\right]$$

$$+ \frac{\partial}{\partial x_j} u_i \tau_{ij} + \frac{\partial P}{\partial t} + F_i u_i + F_i\left(u_{pi} - u_i\right) + Q - \omega_p\left[h_p(T_R, P_R)\right.$$

$$\left. + \frac{1}{2} m_{pi} m_{pi}\right] \qquad \text{(A-34)}$$

The energy equation for a mixture, as given by A-34, differs from that of a pure gas in the sense that the total internal energy is the sum of partial internal energies e_α plus the kinetic energies of diffusion $\sum_{\alpha=1}^{N} K_\alpha\left(e_\alpha + \frac{1}{2} u'_{\alpha i} u'_{\alpha i}\right)$ and the total flux of energy consists, in addition to non-mechanical energy flux q_j, of the rates of working of partial stresses against diffusion $-\Sigma u'_{\alpha i} \tau_{\alpha ij}$ and the energy flux due to diffusion $\Sigma \rho K_\alpha u'_{\alpha i}\left(h_\alpha + \frac{1}{2} u'_{\alpha i} u'_{\alpha i}\right)$

Energy Flux Vector $= q_j - \Sigma U'_{\alpha i}\tau_{\alpha ij} + \Sigma \rho K_\alpha U'_{\alpha j}(h_\alpha + \frac{1}{2}U'_{\alpha i}U'_{\alpha i})$* (A-35)

With the definition of total internal energy as given above, eqn. A-34 can also be obtained by considering a volume element moving with mean velocity u_i and using the energy flux vector given by eqn. A-35.

Analogous to what von Kármán did in defining viscous stresses for a mixture (i. e., eqn. A-19), we define

$$q_j - \Sigma U'_{\alpha i}\tau_{\alpha ij} = -k\frac{\partial T}{\partial x_j} \qquad \text{(A-36)}$$

k is the coefficient of thermal conductivity and is defined to include the work done due to diffusion velocity against partial stresses.

The energy equation for the particle cloud is

$$\frac{\partial}{\partial t}[\rho_p(e_p + \frac{1}{2}u_{pi}u_{pi})] + \frac{\partial}{\partial x_j}[\rho_p u_{pj}(e_p + \frac{1}{2}u_{pi}u_{pi})]$$

$$= F_{pi}u_{pi} - Q + \omega_p[h_p(T_R, P_R) + \frac{1}{2}m_{pi}m_{pi}] \qquad \text{(A-37)}$$

The energy equation for the whole system, by adding eqns. A-34 and A-37, is

$$\frac{\partial}{\partial t}[\rho(h + \frac{1}{2}\Sigma K_\alpha U'_{\alpha i}U'_{\alpha i}) + \rho u_i u_i \frac{1}{2} + e_p\rho_p + \frac{1}{2}\rho_p u_{pi}u_{pi}]$$

$$+ \frac{\partial}{\partial x_j}[\rho u_j(h + \frac{1}{2}\Sigma K_\alpha U'_{\alpha i}U'_{\alpha i}) + \rho u_j(\frac{1}{2}u_i u_i) + \rho_p u_{pj}(e_p + \frac{1}{2}u_{pi}u_{pi})]$$

$$= \frac{\partial p}{\partial t} + \frac{\partial}{\partial x_j}(u_i\tau_{ij}) + \frac{\partial}{\partial x_j}[-(q_j - \Sigma U'_{\alpha i}\tau_{\alpha ij}) - \Sigma\rho K_\alpha U'_{\alpha j}(h_\alpha + \frac{1}{2}U'_{\alpha i}U'_{\alpha i})] \qquad \text{(A-38)}$$

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* In accordance with kinetic theory, eqn. A-35 is to be identified with eqn. (7.2-26) of reference 6.

Using momentum equations A-15 and A-21, the energy equations for the gas mixture and the particle cloud can be expressed as

$$\begin{aligned}&\rho\frac{\partial}{\partial t}\Big[h+\frac{1}{2}\sum_{\alpha=1}^{N}u'_{\alpha i}\,u'_{\alpha i}\Big]+\rho u_j\frac{\partial}{\partial x_j}\Big[h+\frac{1}{2}\sum_{\alpha=1}^{N}u'_{\alpha i}\,u'_{\alpha i}\Big]=\frac{\partial p}{\partial t}+u_i\frac{\partial p}{\partial x_i}\\&+\tau_{ij}\frac{\partial u_i}{\partial x_j}+\frac{\partial}{\partial x_j}\Big[-\Big(q_j-\sum_{\alpha=1}^{N}u'_{\alpha i}\,\tau_{\alpha ij}\Big)-\sum\rho K_\alpha u'_{\alpha j}\Big(h_\alpha+\frac{1}{2}u'_{\alpha i}\,u'_{\alpha i}\Big)\Big]+Q\\&+F_i(u_{pi}-u_i)-\omega_p\Big[h_p(T_R,P_R)-h(T,P)+\frac{1}{2}(m_{pi}m_{pi}-u_iu_i)-m_iu_i+u_iu_i\Big]\end{aligned} \tag{A-39}$$

$$\rho_p\frac{\partial e_p}{\partial t}+\rho_p u_{pi}\frac{\partial}{\partial x_i}e_p=-Q+\omega_p\Big[h_p(T_R,P_R)-e(T_p)+\frac{1}{2}m_{pi}m_{pi}-\frac{1}{2}m_{pi}\psi_{pi}\Big] \tag{A-40}$$

Equations A-39 can be expressed in terms of temperature. According to eqn. A-25, we have

$$\begin{aligned}\int_0^T c_{p\alpha}\,dT&=h_\alpha-h_\alpha^o\\\int_0^T c\,dT&=h_p-h_p^o\end{aligned} \tag{A-41}$$

Expressed in terms of $\int_0^T c_{p\alpha}\,dT$ or $h_\alpha-h_\alpha^o$

$$\begin{aligned}&\rho\frac{\partial}{\partial t}\Big[\sum\big(K_\alpha(h_\alpha-h_\alpha^o)+\frac{1}{2}u_{\alpha i}\,u_{\alpha i}\big)\Big]+\rho u_j\frac{\partial}{\partial x_j}\Big[\sum\big(K_\alpha(h_\alpha-h_\alpha^o)+\frac{1}{2}u'_{\alpha i}\,u'_{\alpha i}\big)\Big]\\&=\tau_{ij}\frac{\partial u_i}{\partial x_j}+\frac{\partial p}{\partial t}+u_i\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_j}\Big[-\Big(q_j-\sum_{\alpha=1}^{N}u'_{\alpha i}\,\tau_{\alpha ij}\Big)-\sum\rho K_\alpha u'_{\alpha j}\Big(h_\alpha\\&+\frac{1}{2}u'_{\alpha i}\,u'_{\alpha i}\Big)\Big]+F_i(u_{pi}-u_i)+Q-\omega_p\Big[h_p(T_R,P_R)-h_p^o\\&-\sum K_\alpha(h_\alpha-h_\alpha^o)+\frac{1}{2}(m_{pi}m_{pi}-u_iu_i)+u_iu_i-m_iu_i-\big(\sum\omega_\alpha h_\alpha^o-\omega_p h_p^o\big)\Big]\end{aligned} \tag{A-42}$$

Due to chemical reactions, the particle radius is no longer a constant. Since the particle number density is assumed to be constant

$$\frac{\partial}{\partial t}\left(\frac{\sigma}{\sigma_0}\right)^3 + u_{pi}\frac{\partial}{\partial x_i}\left(\frac{\sigma}{\sigma_0}\right)^3 = \frac{1}{n m_0}\omega_p \qquad \text{(A-43)}$$

The equation of state for a mixture of ideal gases is

$$p = \rho \bar{R} T \qquad \text{(A-44)}$$

where $\bar{R}$ is given as $\Sigma \frac{R}{\mathcal{M}_\alpha} K_\alpha$ with R being the universal gas constant.

In the case of condensation and evaporation, the heat of reaction as given by eqn. A-28 or A-29 is identically equal to zero. Thus, the above equations are valid for condensation processes by letting $\mathcal{H} = 0$. In the case of sublimation, heat of fusion as well as heat of vaporization should be properly included.

3. Reaction Model for a Binary Mixture

We will only consider problems of a binary mixture in which the particle cloud reacts with one of the gas species. The flow medium consists of: (i) a particle cloud which is composed of either liquid droplets or small solid particles; (ii) a gas which reacts with the particle cloud; (iii) an inert gas which can be the reaction product of the particle cloud and the reacting gas in the case of combustion. The reacting gas can be some oxidizing agent when burning of the particle cloud is being investigated. In problems of condensation and evaporation, this gas is the vapor phase of the particle cloud. Subscript $\alpha = 1$ will be used for the reacting gas and $\alpha = 2$ for the inert gas. Since particle clouds can exist either in liquid or solid form, it is sometimes convenient to introduce additional superscripts ℓ and s to distinguish some of the thermodynamic quantities pertaining to the liquid particles or solid particles. Thus, h_p^{ℓ} denotes the enthalpy of the particle cloud of liquid droplets and h_p^{s} is the enthalpy of the solid particles.

The detailed mechanism of mass transfer, as pointed out by Marble$^{(3)}$, depends on: (i) the rate of heat exchange, (ii) the rate with which the vapor diffuses towards the particles, and (iii) the kinetic rate of condensation. We shall assume that the kinetic rate is infinitely fast as compared with the other two and thus is not rate controlling. In principle, this approximation is that the rate of condensation is still slow in comparison with the mass rate of collision of gas molecules upon the particle surface. This implies that very close to the particle surface, the gas mixture and the particle surface must be locally in thermodynamic equilibrium. The region is so small that the temperature and concentration variations over this region are negligible. Thus,

$$T_R = T_p \tag{A-45}$$

$$m_{wj} = u_{pj} \tag{A-46}$$

$$p_R = p \tag{A-47}$$

and some of the thermodynamic relations can be used to describe some of the quantities on the surface. At the particle surface, the partial pressure of the vapor must be that corresponding to the surface temperature of the particles. The transfer rate is also sufficiently low that there is no significant difference between the temperature of the surface and its bulk. Thus, the partial vapor pressure at the particle surface is the saturation pressure corresponding to the particle temperature. Using this as the boundary condition for the vapor concentration on the particle surface and the vapor concentration in the flow field as the boundary condition far away from the particle, the equation governing the vapor concentration can be solved. Since the Reynolds number based on the particle radius is much smaller than unity, the convective terms are neglected as in Stokes flow. The continuity equation for the vapor concentrations under these simplifications is

$$\frac{d}{dr}\left[r^2 \rho D_{12} \frac{dK_1^*}{dr}\right] = 0 \tag{A-48}$$

Here, Fick's law is used to describe the diffusion velocity and D_{12} is the binary diffusion coefficient for the two gases in the mixture. K_1^*, according to eqn. A-48, is equal to

$$K_1^* = [K_{1e}(T_p) - K_1(T)]\frac{\sigma}{r} + K_1(T) \tag{A-49}$$

The net mass flux of vapor diffusing towards a particle is

$$\begin{aligned} 4\pi r^2 \rho K_1^* U_D &= -4\pi r^2 \rho D_{12} \frac{dK_1^*}{dr} \\ &= -4\pi\sigma\rho D_{12}[K_1(T) - K_{1e}(T_p)] \end{aligned} \tag{A-50}$$

and

$$\begin{aligned} \omega_p &= n\, 4\pi\sigma\rho D_{12}[K_1(T) - K_{1e}(T_p)] \\ &= \frac{1}{\tau_D}\rho_p[K_1(T) - K_{1e}(T_p)] \end{aligned} \tag{A-51}$$

with

$$\tau_D = m/(4\pi\sigma\rho D_{12}) \tag{A-52}$$

When the vapor is considered to be a perfect gas, and the volume of the substance in liquid state or solid state is negligible compared with the volume in the vapor state, the equilibrium partial pressure is given by the following Clausius-Clapeyron relation

$$P_{1e}(T_p) = P_1(T_0) \exp - \left[\int_{T_0}^{T_p} \left(h_1(T_p) - h_p(T_p) \right) d \frac{1}{T_p} \right] \frac{1}{R} \tag{A-53}$$

where $P_1(T_0)$ is the vapor pressure at some reference temperature T_0. The equilibrium vapor concentration is simply

$$K_{1e}(T_p) = \frac{P_{1e}(T_p)}{P} = K_e(T_0) \frac{P_0}{P} \exp - \left[\int_{T_0}^{T_p} \frac{h_1(T_p) - h_p(T_p)}{R} d\left(\frac{1}{T_p}\right) \right] \tag{A-54}$$

To distinguish the liquid droplets and solid particles, eqn. A-54 is

$$K_{1e}^{s}(T_p) = K_{1e}^{s}(T_0) \frac{P_0}{P} \exp - \int_{T_0}^{T_p} \frac{h_1 - h_p^{s}(T_p)}{R} d\left(\frac{1}{T_p}\right) \tag{A-55}$$

$$K_{1e}^{l}(T_p) = K_{1e}^{l}(T_0) \frac{P_0}{P} \exp - \int_{T_0}^{T_p} \frac{h_1(T_p) - h_p^{l}(T_p)}{R} d\left(\frac{1}{T_p}\right) \tag{A-56}$$

Consider the burning of solid particles. Here, reaction takes place on the surface of the particles. With infinite kinetic reaction rate, K_1^* is

$$K_1^* = K_1 \left(1 - \frac{\sigma}{r}\right) \tag{A. 57}$$

and the net mass flux of the reacting gas diffusing towards a particle is $4\pi\rho D_{12} K_1$.

$$\begin{aligned} \omega_1 &= - n 4\pi \sigma \rho D_{12} K_1 \\ &= - \frac{\rho_p K_1}{\tau_D} \end{aligned} \tag{A-58}$$

The one-step reaction of eqn. A-27 is simply

$$\nu_1 C_1 + \nu_p C_p = \nu_2 C_2 \qquad (A-59)$$

The rate of the particle cloud being consumed is

$$\omega_p = - \frac{\nu_p \mathcal{M}_p}{\nu_1 \mathcal{M}_1} \frac{\rho_p K_1}{\tau_D}$$

$$= - \gamma_p \frac{\rho_p K_1}{\tau_D} \qquad (A-60)$$

with

$$\gamma_p = \frac{\nu_p \mathcal{M}_p}{\nu_1 \mathcal{M}_1} \qquad (A-61)$$

For the burning of liquid droplets, evaporation of the fuel plays a dominant role. The vapor fuel constantly diffuses outwards through the inert gas and the oxidizing agent diffuses toward the particle. Since reaction rate is infinitely fast, combustion takes place on a spherical surface at a distance r^* away from the center of the droplet. Using similar simplifications to those previously outlined, the continuity equation, both for the fuel in vapor form and the oxidizing agent, has the simple form of eq. A-45. Denoting K_p^* as the concentration of the fuel in vapor form, the boundary conditions are

$$K_p^*(\sigma) = K_{pe}(T_p) \qquad (A-62)$$

$$K_p^*(r^*) = 0 \qquad (A-63)$$

$K_{pe}(T_p)$ is the fuel vapor concentration corresponding to the equilibrium vapor pressure as in the case of condensation. The boundary conditions for the oxidizer within this later are

$$K_1^*(r^*) = 0 \qquad (A-64)$$

$$K_1^*(\infty) = K_1(T) \qquad (A-65)$$

and

$$\kappa_p^* = \kappa_{pe}(T_p)\frac{\frac{r^*}{r}-1}{\frac{r^*}{\sigma}-1} \qquad r < r^* \qquad \text{(A-66)}$$

$$\kappa_1^* = \kappa_1\left(1-\frac{r^*}{r}\right) \qquad r > r^* \qquad \text{(A-67)}$$

r^* can be found by equating the mass flux of the vapor fuel and the oxidizer according to the one-step reaction of eqn. A-59.

$$\left|\frac{1}{\nu_p \mathcal{M}_p} 4\pi r^{*2} \rho D_{p2} \frac{d\kappa_p^*}{dr}\Big)_{r=r^*}\right| = \left|\frac{1}{\nu_1 \mathcal{M}_1} 4\pi r^{*2} \rho D_{12} \frac{d\kappa_1^*}{dr}\Big)_{r=r^*}\right|$$

D_{p2} is the binary diffusion coefficient of vapor fuel and the inert gas.

$$\frac{r^*}{\sigma} = 1 + \frac{\nu_1 \mathcal{M}_1}{\nu_p \mathcal{M}_p}\frac{D_{p2}}{D_{12}}\frac{\kappa_p^*(T_p)}{\kappa_1} \qquad \text{(A-68)}$$

$$\omega_p = -\gamma_p \frac{\rho_p \kappa_1}{\tau_D}\left[1 + \frac{1}{\gamma_p}\frac{D_{p2}}{D_{12}}\frac{\kappa_{pe}(T_p)}{\kappa_1}\right] \qquad \text{(A-69)}$$

To summarize, ω_p can be expressed in the following table.

	Condensation	Combustion
(solid particles)	$\frac{\rho_p}{\tau_D}[\kappa_1 - \kappa_{1e}^s(T_p)]$	$-\gamma_p\frac{\rho_p \kappa_1}{\tau_D}$
(liquid particles)	$\frac{\rho_p}{\tau_D}[\kappa_1 - \kappa_{1e}^l(T_p)]$	$-\gamma_p\frac{\rho_p \kappa_1}{\tau_D}\left[1+\frac{1}{\gamma_p}\frac{D_{p2}}{D_{12}}\frac{\kappa_{pe}(T_p)}{\kappa_1}\right]$

The momentum and heat transfer between the particle cloud and the gas mixture are still to have the form

$$F_i = -F_{pi} = \frac{\rho_p}{\tau_v}(u_{pi} - u_i) \qquad \text{(A-70)}$$

$$Q = \frac{\rho_p C_p}{\tau_T}(T_p - T) \qquad \text{(A-71)}$$

These expressions for burning particles can conceivably be in error. Williams$^{(2)}$ pointed out that there might be a thermal jet-like effect due to combustion which can change the drag law; and combustion and condensation appearing on the surface or close to the surface of the particles may introduce a non-uniform temperature within the particle.

In the derivation of ω_p , we have introduced a relaxation time of diffusion τ_D . Although τ_D does not quite have the same physical interpretation as τ_v and τ_D , they are all very simply related through Lewis number and Schmidt number as shown in eqn. A-2 and eqn. A-3. It is convenient to express both τ_D and τ_T in terms of

$$\tau_D = \frac{3}{2} S_c \tau_v \qquad \text{(A-72)}$$

$$\tau_T = \frac{3}{2} L_e \tau_v \qquad \text{(A-73)}$$

Using Stokes drag law, τ_v is

$$\tau_v = \frac{m}{4\pi\mu\sigma} = \frac{m_o}{4\pi\mu_o\sigma_o}\frac{\mu_o}{\mu}\left(\frac{\sigma}{\sigma_o}\right)^2 = \tau_{v_o}\left(\frac{\sigma}{\sigma_o}\right)^2\frac{\mu_o}{\mu} \qquad \text{(A-74)}$$

m_o and σ_o are the mass and the radius of the particles before reaction and μ_o is the viscosity at some reference state. If both Schmidt number S_c and Prandtl number P_r are assumed to be constants,

$$\tau_T = \tau_{T_0}\frac{\mu_0}{\mu}\left(\frac{\sigma}{\sigma_0}\right)^2 \qquad (A-75)$$

$$\tau_D = \tau_{D_0}\frac{\mu_0}{\mu}\left(\frac{\sigma}{\sigma_0}\right)^2 \qquad (A-76)$$

Using eqns. A-53 - A-55, we have

$$F_i = \frac{\rho_p}{\tau_{V_0}}(u_{p_i} - u_i)\left(\frac{\mu}{\mu_0}\right)\left(\frac{\sigma_0}{\sigma}\right)^2 \qquad (A-77)$$

$$Q = \frac{\rho_p}{\tau_{T_0}}(T_p - T)\left(\frac{\mu}{\mu_0}\right)\left(\frac{\sigma_0}{\sigma}\right)^2 \qquad (A-78)$$

	Condensation	Combustion
(solid particles)	$\frac{\rho_p}{\tau_{D_0}}K_1\frac{\mu}{\mu_0}\left(\frac{\sigma_0}{\sigma}\right)^2\left(1 - \frac{K_{10}^{S}(T_p)}{K_1}\right)$	$-r_p\frac{\rho_p}{\tau_{D_0}}K_1\frac{\mu}{\mu_0}\left(\frac{\sigma_0}{\sigma}\right)^2$
(liquid particles)	$\frac{\rho_p}{\tau_{D_0}}K_1\frac{\mu}{\mu_0}\left(\frac{\sigma_0}{\sigma}\right)^2\left(1 - \frac{K_{1e}^{l}(T_p)}{K_1}\right)$	$-r_p\frac{\rho_p}{\tau_{D_0}}K_1\frac{\mu}{\mu_0}\left(\frac{\sigma_0}{\sigma}\right)^2\left[1 + \frac{1}{r_p}\frac{D_{p1}}{D_{12}}\frac{K_{pe}(T_p)}{K_1}\right]$

(A-79)

4. Summary of Basic Equations

The continuity equations for the gas mixture and the particle cloud are

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = -\,\omega_p \qquad (A-80)$$

$$\frac{\partial \rho_p}{\partial t} + \frac{\partial}{\partial x_i}(\rho_p u_{p_i}) = \omega_p \qquad (A-81)$$

Using Fick's law to describe the diffusion velocities for a binary mixture, the continuity equation for the oxidizing agent can be written as

$$\rho\frac{\partial K_1}{\partial t} + \rho u_i\frac{\partial K_1}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\rho D_{12}\frac{\partial K_1}{\partial x_i}\right) + \left(K_1 + \frac{1}{r_p}\right)\omega_p \qquad (A-82)$$

and $\gamma_p = -1$ for the case of condensation.

The momentum equations for the gas mixture and the particle cloud, with $m_j = u_{pj}$, are

$$\rho\frac{\partial u_i}{\partial t} + \rho u_j\frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\tau_{ij} + \frac{\rho_p}{\tau_{v_0}}(u_{pi} - u_i)\left(\frac{\mu}{\mu_0}\right)\left(\frac{\sigma_0}{\sigma}\right)^2$$
$$- \omega_p[u_{pi} - u_i] \qquad \text{(A-83)}$$

$$\rho\frac{\partial u_{pi}}{\partial t} + \rho_p u_{pj}\frac{\partial u_{pi}}{\partial x_j} = -\frac{\rho_p}{\tau_{v_0}}(u_{pi} - u_i)\left(\frac{\mu}{\mu_0}\right)\left(\frac{\sigma_0}{\sigma}\right)^2 \qquad \text{(A-84)}$$

Since the particle cloud exists either in liquid or solid phase, we assume that

$$h_p(T_p) = e_p(T_p) \qquad \text{(A-85)}$$

The energy equations as given by eqns. A-42 and A-40 are

$$\rho\frac{\partial}{\partial t}\left[\Sigma\left(K_\alpha(h_\alpha - h_\alpha^\circ) + \tfrac{1}{2}u'_{\alpha i}u'_{\alpha i}\right)\right] + \rho u_j\frac{\partial}{\partial x_j}\left[\Sigma\left(K_\alpha(h_\alpha - h_\alpha^\circ) + \tfrac{1}{2}u'_{\alpha i}u'_{\alpha i}\right]\right.$$

$$= \tau_{ij}\frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial t} + u_i\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[-\left(q_j - \sum_{\alpha=1}^{N}u'_{\alpha i}\tau_{\alpha ij}\right) - \Sigma\rho K_\alpha u'_{\alpha j}\left(h_\alpha + \tfrac{1}{2}u'_{\alpha i}u'_{\alpha i}\right)\right]$$

$$- \omega_p\left[(h_p - h_p^\circ) - \Sigma K_\alpha(h_\alpha - h_\alpha^\circ) + \tfrac{1}{2}(m_{pi}m_{pi} - u_iu_i) + u_iu_i - m_iu_i\right]$$

$$+ F_i(u_{pi} - u_i) + Q - \left[\Sigma\omega_\alpha h_\alpha^\circ + \omega_p h_p^\circ\right] \qquad \text{(A-86)}$$

$$\rho_p\frac{\partial}{\partial t}[h_p - h_p^\circ] + \rho_p u_{pi}\frac{\partial}{\partial x_i}[h_p - h_p^\circ] = -Q \qquad \text{(A-87)}$$

For calorically perfect gas with the specific heat of gas species equal, $C_{p1} = C_{p2}$, we have $h_\alpha - h_\alpha^\circ = C_p T$ and $h_p - h_p^\circ = c\,T_p$.

Equations A-86 and A-87 are reduced to

$$\rho\frac{\partial}{\partial t}[C_pT+\Sigma\tfrac{1}{2}u'_{\alpha i}u'_{\alpha i}]+\rho u_j\frac{\partial}{\partial x_j}[C_pT+\Sigma\tfrac{1}{2}u'_{\alpha i}u'_{\alpha i}]=\frac{\partial P}{\partial t}$$
$$+u_j\frac{\partial P}{\partial x_j}+\tau_{ij}\frac{\partial u_i}{\partial x_j}+F_i(u_{pi}-u_i)+Q+\frac{\partial}{\partial x_j}[-(q_j-\Sigma u'_{\alpha i}\tau_{\alpha ij})$$
$$-\Sigma\rho K_\alpha u'_{\alpha j}(\tfrac{1}{2}u'_{\alpha i}u'_{\alpha i})]-\omega_p[CT_p-C_pT+\tfrac{1}{2}(m_{pi}m_{pi}-u_iu_i$$
$$+u_iu_i-m_iu_i]-[\Sigma\omega_\alpha h^\circ_\alpha+\omega_p h^\circ_p] \qquad \text{(A-88)}$$

$$\rho_p C\frac{\partial T_p}{\partial t}+\rho_p u_{pi}C\frac{\partial T_p}{\partial x_i}=-Q \qquad \text{(A-89)}$$

For liquid droplets, latent heat of vaporization should be included, and eqns. A-88 and A-89 are

$$\rho\frac{\partial}{\partial t}[C_pT+\Sigma\tfrac{1}{2}u'_{\alpha i}u'_{\alpha i}]+\rho u_j\frac{\partial}{\partial x_j}[C_pT+\Sigma\tfrac{1}{2}u'_{\alpha i}u'_{\alpha i}]=\frac{\partial P}{\partial t}$$
$$+u_j\frac{\partial P}{\partial x_j}+\tau_{ij}\frac{\partial u_i}{\partial x_j}+F_i(u_{pi}-u_i)+Q+\frac{\partial}{\partial x_j}[-(q_j-\Sigma u'_{\alpha i}\tau_{\alpha ij})$$
$$-\Sigma\rho K_\alpha u'_{\alpha j}(\tfrac{1}{2}u'_{\alpha i}u'_{\alpha i})]-\omega_p[CT_p+L(T_p)-C_pT+\tfrac{1}{2}(u_{pi}u_{pi}$$
$$+u_iu_i-u_{pi}u_i]-[\Sigma\omega_\alpha h^\circ_\alpha+\omega_p h^\circ_p] \qquad \text{(A-90)}$$

$$\rho C\frac{\partial T_p}{\partial t}+\rho_p u_{pj}C\frac{\partial T_p}{\partial x_j}=-Q+\omega_p L(T_p) \qquad \text{(A-91)}$$

In the problem of condensation of liquid droplets, eqn. A-90 is modified by letting $\mathcal{H}=0$. The energy equation for the gas mixture is

$$\rho\frac{\partial}{\partial t}\left[C_pT+\Sigma\tfrac{1}{2}u'_{\alpha i}\,u'_{\alpha i}\right]+\rho u_j\frac{\partial}{\partial x_j}\left[C_pT+\Sigma\tfrac{1}{2}u'_{\alpha i}\,u'_{\alpha i}\right]=\frac{\partial P}{\partial t}$$

$$+\,u_i\frac{\partial P}{\partial x_i}+\tau_{ij}\frac{\partial u_i}{\partial x_j}+F_i(u_{pi}-u_i)+Q+\frac{\partial}{\partial x_j}\Big[-(q_j-\Sigma u'_{\alpha i}\tau_{\alpha ij})$$

$$-\,\Sigma\rho K_\alpha u'_{\alpha j}(\tfrac{1}{2}u'_{\alpha i}\,u'_{\alpha i})\Big]-\omega_p\left[C_p(T_p-T)\right]-\omega_p\Big[u_iu_i$$

$$-\,u_iu_{pi}+\tfrac{1}{2}(u_{pi}u_{pi}-u_iu_i)\Big] \qquad \text{(A-92)}$$

and eqn. A-91 is still valid for the particle cloud.

When there is change of phase between solid particles and the vapor state, $L(T_p)$ in eqn. A-91 includes the heat of fusion as well as the heat of vaporization.

The remaining governing equations are

$$P=\rho\bar{R}T \qquad \text{(A-93)}$$

and

$$\frac{\partial}{\partial t}\left(\frac{\sigma}{\sigma_0}\right)^3+u_{pi}\frac{\partial}{\partial x_i}\left(\frac{\sigma}{\sigma_0}\right)^3=\frac{1}{n m_0}\,\omega_p \qquad \text{(A-94)}$$

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