I. THE ATTENUATION AND DISPERSION OF SOUND IN A CONDENSING MEDIUM

II. THE FLOW OF A GAS-PARTICLE MIXTURE DOWNSTREAM OF A NORMAL SHOCK IN A NOZZLE

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ABSTRACT

I. The attenuation of sound due to particles suspended in a gas was first calculated by Sewell and later by Epstein in their classical works on the propagation of sound in a two-phase medium. In their work, and in more recent works which include calculations of sound dispersion, the calculations were made for systems in which there was no mass transfer between the two phases. In the present work, mass transfer between phases is included in the calculations.

The attenuation and dispersion of sound in a two-phase condensing medium are calculated as functions of frequency. The medium in which the sound propagates consists of a gaseous phase, a mixture of inert gas and condensible vapor, which contains condensible liquid droplets. The droplets, which interact with the gaseous phase through the interchange of momentum, energy, and mass (through evaporation and condensation), are treated from the continuum viewpoint. Limiting cases, for flow either frozen or in equilibrium with respect to the various exchange processes, help demonstrate the effects of mass transfer between phases. Included in the calculation is the effect of thermal relaxation within droplets. Pressure relaxation between the two phases is examined, but is not included as a contributing factor because it is of interest only at much higher frequencies than the other relaxation processes. The results for a system typical of sodium droplets in sodium vapor are compared to calculations in which there is no mass exchange between phases. It is found that the maximum attenuation is about 25 per cent greater and occurs at about one-half the frequency for the case which includes mass transfer, and that the dispersion at low frequencies is about 35 per cent greater. Results for different values of latent heat are compared.

II. In the flow of a gas-particle mixture through a nozzle, a normal shock may exist in the diverging section of the nozzle. In Marble's calculation for a shock in a constant area duct, the shock was described as a usual gas-dynamic shock followed by a relaxation zone in which the gas and particles return to equilibrium. The thickness of this zone, which is the total shock thickness in the gas-particle mixture, is of the order of the relaxation distance for a particle in the gas. In a nozzle, the area may change significantly over this relaxation zone so that the solution for a constant area duct is no longer adequate to describe the flow. In the present work, an asymptotic solution, which accounts for the area change, is obtained for the flow of a gas-particle mixture downstream of the shock in a nozzle, under the assumption of small slip between the particles and gas. This amounts to the assumption that the shock thickness is small compared with the length of the nozzle. The shock solution, valid in the region near the shock, is matched to the well known small-slip solution, which is valid in the flow downstream of the shock, to obtain a composite solution valid for the entire flow region. The solution is applied to a conical nozzle. A discussion of methods of finding the location of a shock in a nozzle is included.

TABLE OF CONTENTS

Part	Title	Page
	Acknowledgments	ii
	Abstract	iii
	Table of Contents	v
	List of Symbols	vii
I.	THE ATTENUATION AND DISPERSION OF SOUND IN A CONDENSING MEDIUM	1
	1. Introduction	2
	2. Governing Equations	5
	3. The Acoustic Problem	12
	4. Thermal Relaxation within Droplets	19
	5. Limiting Cases for Frozen and Equilibrium Interactions	23
	a. Completely frozen flow	23
	b. Heat and mass transfer frozen	24
	c. Momentum and mass transfer frozen	24
	d. Momentum and heat transfer frozen	25
	e. Droplet temperature frozen	25
	f. Complete equilibrium	26
	6. Effect of Pressure Relaxation	27
	a. Linearized solution	30
	b. The effect of finite liquid conductivity	33
	7. Discussion of Results	37
	Appendices	40
	A. Derivation of the Basic Equations	40
	B. Internal Energy of Droplets	60

Part	Title	Page
	References	62
	Tables and Illustrations	64
II.	THE FLOW OF A GAS-PARTICLE MIXTURE DOWNSTREAM OF A NORMAL SHOCK IN A NOZZLE	73
	1. Introduction	74
	2. Governing Equations	76
	3. Small Slip Solution	78
	4. Shock Solution	84
	5. Matching of the Inner and Outer Solution	95
	6. Composite Expansion	97
	7. Application to a Conical Nozzle	98
	8. Position of Shock in a Nozzle	102
	References	105

LIST OF SYMBOLS

a	speed of sound
A	nozzle area
c	specific heat of solid or liquid material
c_p , c_v	specific heats of gas
$\overline{c}_p$, $\overline{c}_v$	specific heats of gas and particles in equilibrium
e	internal energy of gas, per unit mass
e_p	internal energy of particles per unit mass
f_i	force exerted upon gas by single particle
$F_i = nf_i$	force exerted upon gas by particles, per unit volume
$F(x)$	function defined by equation 2.35
$G(x)$	function defined by equation 2.36
h	enthalpy of gas, per unit mass
h_ℓ	latent heat of vaporization
$K = K_1+iK_2$	complex wave number
k	thermal conductivity of gas
k_ℓ	thermal conductivity of liquid
$K_v^e(T_p)$	equilibrium vapor mass fraction at droplet temperature
K_v	vapor mass fraction
L	length of nozzle
m	mass flow rate in nozzle
M	Mach number
n	number density of particles or droplets
P	thermodynamic pressure
P_ℓ	equilibrium vapor pressure of liquid at liquid temperature

P_v	partial pressure of vapor
Pr	Prandtl number of gas
q	heat transfer rate from a single particle to gas
Q	heat transferred from particles to gas, per unit volume
R	dimensionless radial coordinate
Re	Reynolds number
s	$= c/c_p$
Sc	Schmidt number
T	temperature of gas
T_p	temperature of particles
u	velocity of gas
u_p	velocity of particles
$\alpha = \frac{2K_2a_o}{\omega}$	attenuation coefficient
$\beta = \left(\frac{K_1a_o}{\omega}\right)^2 - 1$	dispersion coefficient
γ	specific heat ratio of gas
$\overline{\gamma}$	specific heat ratio of gas and particles in equilibrium
δ	average spacing between particles
η	$= h_\ell/RT_o$
$\kappa = \rho_p/\rho$	mass fraction of particles or droplets
λ_v	velocity equilibration length of a particle
λ_T	thermal equilibration length of a particle
λ_D	diffusion equilibration length of a particle
μ	viscosity of gas
μ_v	rate of evaporation from droplets, per unit volume

$\xi = x/\lambda_v$	stretched x-coordinate
ρ	density of gaseous phase
ρ_p	density of particulate or liquid phase
σ	diameter of particles or droplets
τ_v	velocity equilibration time of a particle
τ_T	thermal equilibration time of a particle
τ_D	diffusion equilibration time
τ_ℓ	thermal relaxation time for liquid droplet
τ_p	pressure relaxation time
ϕ	complex phase factor defined by equation B. 8
ω	angular frequency

I. THE ATTENUATION AND DISPERSION OF SOUND IN A CONDENSING MEDIUM

1. INTRODUCTION

The attenuation of sound in two-phase media has received considerable attention since Sewell's[1] work in 1910. Sewell calculated the attenuation due to the viscous interaction between an oscillating non-conducting gas and rigid, fixed, spherical, and cylindrical objects. In an appending note he extended his calculation to spherical particles oscillating with some fixed amplitude, small compared to the wavelength of the sound waves. This result is not valid for small particle radii, since as the particle radius tends to zero the particles tend to oscillate with the fluid, and the viscous attenuation tends to zero. In 1941 Epstein[2] calculated the attenuation due to viscous interaction with spheres freely suspended in the fluid. Calculations were made for spheres consisting of a rigid solid, elastic solid, and a viscous liquid. Oscillation of the spheres was restricted to amplitudes small compared to the particle radius. This work was extended by Epstein and Carhart[3] in 1953 to include thermal dissipation due to irreversible heat transfer between the two phases. Their theory was compared to data taken by Knudsen, Wilson, and Anderson[4] for attenuation of sound by water fog in air and was found to give consistently lower values than the experimental data. More recent experiments by Zink and Delsasso[5] using solid particles show good agreement with Epstein and Carhart's theory. In 1964 Chow[6] was able to show that the theory of Epstein and Carhart is valid even for motions of the spherical particles of large amplitude compared to their radius. Chow included surface tension in his analysis and showed that it is significant for gaseous bubbles in a liquid but is of negligible significance for liquid droplets in a gas.

The dispersion of sound in a two-phase medium has received little attention until recently, when the first measurements of sound dispersion by solid particles in a gas were made by Zink and Delsasso[5]. They presented a theory, based upon the alteration of the effective density and specific heat of the gas due to the presence of the particles, which agreed with their measurements. A more complete theory by Chu and Chow[7] shows good agreement with the dispersion measurements of Zink and Delsasso.

The theory developed by Epstein, Epstein and Carhart, and Chow uses the acoustic potential to calculate the scattering of a plane wave by a single spherical particle or droplet. The attenuation is calculated from the increase in entropy due to irreversible transfer of heat and momentum between the particles and the gas. The resulting attenuation due to a single particle is then multiplied by the number of particles to obtain the total attenuation. Temkin[8] and Temkin and Dobbins[9] have calculated attenuation and dispersion by a different method in which the state of the gas and the state of the particles are described by the conservation equations of fluid mechanics. A direct calculation leads to expressions for attenuation and dispersion as functions of the sound frequency. Temkin and Dobbins[10] also report experimental data for oleic acid droplets in nitrogen which agrees very well with their theory, and also agrees with the data taken over a smaller frequency range by Zink and Delsasso. Temkin and Dobbins show that their theory gives attenuations very close to those of Epstein and Carhart, and dispersions close to those calculated by

Chu and Chow. The method of calculation used by Temkin and Dobbins shows clearly the mechanism of attenuation and dispersion due to gas-particle interactions. This approach will be used here to calculate the attenuation and dispersion of sound in a gas-vapor mixture containing condensing liquid droplets.

2. GOVERNING EQUATIONS

A set of general equations governing the dynamics of a condensing medium has been derived by Marble[11]. The system consists of two phases, liquid droplets dispersed in a gaseous phase which is a mixture of vapor and an inert gas. A system in which the gaseous phase consists of pure vapor is just a special case. The droplet cloud is treated as a continuum which exchanges mass, momentum, and energy with the gas phase. The droplet number is assumed conserved and the droplets are assumed to be all of the same size so that nucleation, the initial growth of droplets, and coagulation do not enter the problem. Further, the droplets are assumed sufficiently disperse so that droplet-droplet interaction may be neglected. The liquid phase is assumed to have much greater density than the gaseous phase,and mass per unit volume of the two phases is assumed to be of the same order, so that the volume fraction occupied by the droplets is negligible. The treatment is analogous to that for a gas containing small solid particles[12, 13] except that, in addition to momentum and heat transfer between phases, the exchange of mass due to evaporation and condensation is included.

For the acoustic problem, Marble's general equations are specialized to describe unsteady one-dimensional flow. The gas phase is treated as an ideal gas mixture which is inviscid and nonconducting except in the calculation of the gas-droplet interactions. Further assumptions and properties of the system will be discussed as they arise.

The conservation of mass for the gaseous phase is written

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) = \mu_v \tag{1.1}$$

where μ_v is the rate of mass addition to the gas phase due to evaporation from the droplets. The corresponding equation for the liquid phase is

$$\frac{\partial \rho_p}{\partial t} + \frac{\partial}{\partial x} (\rho_p u_p) = - \mu_v \tag{1.2}$$

where the subscript p refers to the liquid phase. Non-subscripted quantities refer to the gas phase. Though on a microscopic scale, evaporation and condensation occur locally at the droplets, the quantity μ_v is assumed to be a smooth function of x. It is obtained in the same way as the other bulk particulate quantities, by averaging over a unit volume whose dimensions are small compared to the dimensions of the problem but large compared to the inter-particulate distances. For details see Appendix A. Thus, in the continuum description of the flow, local variations on the scale of the particle dimensions do not enter. The small scale properties enter only in the calculation of the interaction between phases.

The conservation of momentum for the gas and liquid phases is written

$$\frac{\partial}{\partial t} (\rho u) + \frac{\partial}{\partial x} (\rho u^2) = - \frac{\partial P}{\partial x} + \mu_v u_p + F \tag{1.3}$$

and

$$\frac{\partial}{\partial t} (\rho_p u_p) + \frac{\partial}{\partial x} (\rho_p u_p^2) = - \mu_v u_p - F \tag{1.4}$$

where F is the force per unit volume exerted upon the gas phase by the droplets. The term $\mu_v u_p$ arises from the addition of momentum to the gas phase by evaporation from the droplets. Since the addition and removal of mass from the gas phase occurs at the droplet surface, it is assumed that the mass is added or removed at the droplet velocity. Thus, this term appears with u_p rather than u or some intermediate velocity.

Conservation of energy for the gas phase is written

$$\frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial x}(\rho u E) = -\frac{\partial}{\partial x}(Pu) + Fu_p + Q + \mu_v\left(e(T_p) + \tfrac{1}{2}u_p^2 + \frac{P}{\rho(T_p)}\right) \tag{1.5}$$

where

$$E = e + \tfrac{1}{2}u^2$$

and where e is the internal energy of the gas at the local bulk temperature of the gas T. The internal energy of the gas at the local droplet temperature is $e(T_p)$. Heat transfer per unit volume from the droplets to the gas is denoted by Q and Fu_p is the work done on the gas by the droplets. The last term in equation 1.5 represents the energy added to the gas phase by evaporation from the droplets. This energy is added at the droplet surface and is assumed to be added at the droplet temperature and velocity. Similarly, the energy equation for the droplets is

$$\frac{\partial}{\partial t}(\rho_p E_p) + \frac{\partial}{\partial x}(\rho_p u_p E_p) = -Fu_p - Q - \mu_v\left(e(T_p) + \tfrac{1}{2}u_p^2 + h_\ell\right) \tag{1.6}$$

where

$$E_p = e_p + \tfrac{1}{2}u_p^2$$

and where e_p is the internal energy of the droplets; h_ℓ is the latent heat associated with the phase change. It is assumed that a molecule condensing upon a droplet releases its latent heat to the droplet rather than to the surrounding gas. Similarly, an evaporating molecule obtains from the droplet the energy or latent heat necessary to go from the liquid to the gaseous phase. Hence, in the phase transition, heat is taken from or added to the droplet and appears only in the energy equation for the liquid phase.

At the liquid surface, the liquid and vapor are assumed to be in equilibrium such that Clausius-Clapeyron holds. It can be expressed as

$$\frac{P_v^e(T_p)}{P_{vo}} = \exp\left(\frac{h_\ell}{RT_o} - \frac{h_\ell}{RT_p}\right)$$

where P_{vo} and T_o are the vapor pressure and temperature at some reference state. The equilibrium vapor pressure at the droplet temperature $P_v^e(T_p)$ can be expressed in terms of the equilibrium vapor mass fraction at the droplet temperature $K_v^e(T_p)$ and the local total gas pressure P. The Clausius-Clapeyron equation can then be written

$$\frac{K_v^e(T_p)}{K_v^e(T_o)} = \frac{P_o}{P} \exp\left[-\frac{h_\ell}{RT_o}\left(\frac{T_o}{T_p} - 1\right)\right]. \tag{1.7}$$

In writing equation 1.7 in this form, the pressure at the droplet surface is assumed equal to the local bulk pressure. That is, the total pressure is assumed not to vary on the microscopic scale of the individual droplets. The conditions under which this assumption is valid are discussed in Section 6. The vapor mass fraction at the droplet let surface $K_v^e(T_p)$ is related to the bulk vapor mass fraction K_v by the diffusion process which determines the mass transfer between phases.

An additional equation is obtained from the conservation of mass of the vapor alone. Written in terms of the vapor mass fraction of the gas, this is

$$\frac{\partial K_v}{\partial t} + u\frac{\partial K_v}{\partial x} = \frac{1}{\rho}(1-K_v)\mu_v \ . \tag{1.8}$$

The gas phase is assumed to obey the ideal gas law

$$P = \rho RT \ . \tag{1.9}$$

The interaction between the two phases F, Q, and μ_v are assumed to obey linear laws and are written

$$F = \rho_p \frac{(u_p-u)}{\tau_v} \tag{1.10}$$

$$Q = \rho_p c \frac{(T_p-T)}{\tau_T} \tag{1.11}$$

$$\mu_v = \rho_p \frac{K_v^e(T_p)-K_v}{\tau_D} \tag{1.12}$$

The droplets are assumed spherical, with the drag given by Stokes law so that

$$\tau_v = \frac{m}{6\pi\sigma\mu} \tag{1.13}$$

where m is the mass of a droplet, σ the droplet radius, and μ is the viscosity of the gaseous phase. τ_v is the velocity equilibration time; it is the time required for the relative velocity of a single droplet moving in the gas phase to be reduced to e^{-1} of its initial value. The thermal equilibration time is

$$\tau_T = \frac{m\,c_p}{4\pi\sigma k} \tag{1.14}$$

where k is the thermal conductivity of the gas phase and c_p its specific heat at constant pressure. The diffusion equilibration time is written

$$\tau_D = \frac{m}{4\pi\sigma\rho D} \tag{1.15}$$

where D is the binary diffusion coefficient of the gas-vapor mixture. In addition to the assumption of spherical droplets of all the same size, two other assumptions are made in writing the force between the two phases as given in equations 1.10 and 1.13. First, the droplet Reynolds number is assumed small so that Stokes drag law is applicable and second, the evaporation and condensation are assumed to have no effect on the drag law. Similar assumptions are made in writing the equations governing heat and mass transfer between phases.

Equations 1.1 - 1.9 are written in terms of the nine dependent variables:

$$\begin{array}{lll} u & u_p & K_v^e(T_p) \\ \rho & \rho_p & K_v \\ T & T_p & \\ P & & \end{array} \tag{1.16}$$

Equations 1.10-1.15 give the interaction quantities F, Q, and μ_v in terms of the dependent variables. In the general problem, the particle radius appears explicitly in the expressions for the equilibration times τ_v, τ_T, and τ_D, and must be included as one of the dependent variables. An additional differential equation for the particle radius can be obtained from the conservation of mass of the liquid phase and the conservation of particle number. This equation is not needed in the acoustic problem, but is written here for completeness.

$$\frac{\partial \Sigma}{\partial t} + u_p \frac{\partial \Sigma}{\partial x} = \frac{2}{3} \frac{K_v - K_v^e(T_p)}{\tau_{D_o}} \tag{1.17}$$

where

$$\Sigma = (\sigma/\sigma_o)^2$$

and σ_o is the particle radius at the reference state.

With equations 1. 10 - 1. 15 giving the interaction terms, equations 1. 1 - 1. 9 and 1. 17 form a general description of the unsteady one-dimensional flow of a condensing medium. This same set of equations is applicable to a gas-particle flow where the solid particles are sublimating. The only difference would be that the latent heat would be the latent heat of sublimation. So the following consideration of the acoustic problem could equally apply to a sublimating particle-gas flow.

A complete derivation of the general governing equations is given in Appendix A, which includes a more thorough discussion of the assumptions made and their implications.

3. ACOUSTIC PROBLEM

The acoustic equations are obtained from the preceding set of equations in the familiar way, by linearization about an equilibrium reference state. We assume that the gas and droplets are initially uniformly at rest and in equilibrium, and that the parameters deviate from the initial values by only a small amount. The perturbation velocities are assumed small compared to the speed of sound in the undisturbed gas. Write the dependent variables as perturbations about the reference state:

$$\begin{aligned} u &= u' & u_p &= u_p' & K_v^e(T_p) &= K_{vo}+K_v^{e}{}'(T_p) \\ \rho &= \rho_o + \rho' & \rho_p &= \rho_{po}+\rho_p' & K_v &= K_{vo} + K_v' \qquad (1.18) \\ T &= T_o + T' & T_p &= T_o + T_p' \\ P &= P_o + P' \end{aligned}$$

In the following, the primes will be dropped with the understanding that variables written with no prime refer to the perturbation quantities.

The linearized interaction quantities, from equations 1.10 - 1.12, are written

$$F = \rho_{po} \frac{u_p - u}{\tau_v} \qquad (1.19)$$

$$Q = \rho_{po} c_p \frac{T_p - T}{\tau_T} \qquad (1.20)$$

$$\mu_v = \rho_{po} \frac{K_v^e(T_p) - K_v}{\tau_D} \qquad (1.21)$$

where the equilibration times τ_v, τ_T, and τ_D are constants evaluated at the equilibrium state. Since only the zeroth order part of the equilibration times enters, the perturbed droplet radius does not explicitly enter the problem. For this reason, equation 1.17 is redundant. However, once the problem is solved, we could go back and calculate the perturbed particle radius from equation 1.17.

The linearized Clausius-Clapeyron equation 1.7 is

$$\frac{K_v^e(T_p)}{K_{vo}} + \frac{P}{P_o} = \frac{h_\ell}{RT_o}\left(\frac{T_p}{T_o}\right) \tag{1.22}$$

and the linearized form of equation 1.8 for the vapor mass fraction is

$$\frac{\partial K_v}{\partial t} = (1-K_{vo})\frac{\mu_v}{\rho_o} \quad . \tag{1.23}$$

The linearized equations for the gas phase, with the interaction terms F and Q substituted from equations 1.19 and 1.20 are

$$\frac{\partial \rho}{\partial t} + \rho_o \frac{\partial u}{\partial x} = \mu_v \tag{1.24}$$

$$\rho_o \frac{\partial u}{\partial t} = -\frac{\partial P}{\partial x} + \frac{\varkappa \rho_o}{\tau_v}(u_p - u) \tag{1.25}$$

$$\rho_o c_p \frac{\partial T}{\partial x} = \frac{\partial P}{\partial t} + \frac{\varkappa \rho_o c_p}{\tau_T}(T_p - T) \tag{1.26}$$

and

$$\frac{P}{P_o} = \frac{\rho}{\rho_o} + \frac{T}{T_o} \tag{1.27}$$

where

$$\varkappa = \frac{\rho_{po}}{\rho_o} \tag{1.28}$$

and a_o is the undisturbed speed of sound in the pure gaseous phase. In writing the energy equation 1.26, the enthalpy of the gas mixture is written as c_pT and c_p assumed constant. If c_p is a function of the vapor mass fraction, this is still correct to the first order since only the zeroth order part of c_p enters into the problem.

The linearized equations for the liquid phase are

$$\frac{\partial \rho_p}{\partial t} + \varkappa \rho_o \frac{\partial u_p}{\partial x} = - \mu_v \qquad (1.29)$$

$$\frac{\partial u_p}{\partial t} = - \frac{1}{\tau_v} (u_p - u) \qquad (1.30)$$

$$\frac{c}{c_p} \frac{\partial T_p}{\partial t} = - \frac{1}{\tau_T} (T_p - T) - \frac{\mu_v}{\rho_o} \frac{T_o}{\varkappa} \left(\frac{\gamma - 1}{\gamma}\right) \frac{h_\ell}{RT_o} \qquad (1.31)$$

The droplet internal energy is written as cT_p, based upon the assumption that a droplet has uniform temperature. This assumption is valid if the liquid thermal conductivity is much greater than that of the gas. This is often true, as in the case of alkali metal droplets in their vapor. However, for some systems such as water droplets in air, the liquid conductivity is of the same order as that of the gas, and the thermal relaxation within droplets may be significant. This effect will be calculated in the next section.

The droplet density only appears in equation 1.29, so it is redundant in the solution. If the equation of state of the gas is used to eliminate the perturbation pressure P, the equations to be solved may be written

$$\frac{K_v^e(T_p)}{K_{vo}} + \frac{\rho}{\rho_o} + \frac{T}{T_o} - \frac{h_\ell}{RT_o} \left(\frac{T_p}{T_o}\right) = 0 \qquad (1.32)$$

$$\frac{\partial K_v}{\partial t} - (1 - K_{vo}) \frac{\mu_v}{\rho_o} = 0 \qquad (1.33)$$

$$\frac{\mu_v}{\rho_o} + \frac{\varkappa}{\tau_D} \left(K_v - K_v^e(T_p)\right) = 0 \qquad (1.34)$$

$$\frac{\partial}{\partial t}\left(\frac{\rho}{\rho_o}\right) + \frac{\partial u}{\partial x} - \frac{\mu_v}{\rho_o} = 0 \tag{1.35}$$

$$\frac{\partial u}{\partial t} + \frac{a_o^2}{\gamma}\frac{\partial}{\partial x}\left(\frac{\rho}{\rho_o} + \frac{T}{T_o}\right) - \frac{\varkappa}{\tau_v}(u_p - u) = 0 \tag{1.36}$$

$$\frac{1}{\gamma}\frac{\partial}{\partial t}\left(\frac{T}{T_o}\right) - \frac{\gamma-1}{\gamma}\frac{\partial}{\partial t}\left(\frac{\rho}{\rho_o}\right) - \frac{\varkappa}{\tau_T}\left(\frac{T_p}{T_o} - \frac{T}{T_o}\right) = 0 \tag{1.37}$$

$$\frac{\partial u_p}{\partial t} + \frac{1}{\tau_v}(u_p - u) = 0 \tag{1.38}$$

and

$$\frac{c}{c_p}\frac{\partial}{\partial t}\left(\frac{T_p}{T_o}\right) + \frac{1}{\tau_T}\left(\frac{T_p}{T_o} - \frac{T}{T_o}\right) + \frac{\mu_v}{\varkappa\rho_o}\left(\frac{\gamma-1}{\gamma}\right)\frac{h_\ell}{RT_o} \ . \tag{1.39}$$

Assume that the dependent variables depend upon x and t through the factor $\exp i(Kx-\omega t)$ for a given angular frequency ω where K is the complex wave number written

$$K = K_1 + iK_2 \ . \tag{1.40}$$

Then from equations 1.33, 1.38, and 1.39, the quantities K_v, u_p, and T_p/T_o can be written

$$K_v = -\frac{(1-K_{vo})}{i\omega}\frac{\mu_v}{\rho_o} \tag{1.41}$$

$$u_p = \frac{u}{1-i\omega\tau_v} \tag{1.42}$$

$$\frac{T_p}{T_o} = \frac{1}{1-i\omega\tau_T s}\left[\frac{T}{T_o} - \frac{\mu_v}{\rho_o}\frac{\tau_T}{\varkappa}\left(\frac{\gamma-1}{\gamma}\right)\left(\frac{h_\ell}{RT_o}\right)\right] \tag{1.43}$$

where $s = c/c_p$.

Using equations 1.41 - 1.43 and 1.32, the variables u_p, T_p/T_o, K_v, and $K_v^e(T_p)$ can be eliminated from equations 1.34 - 1.37. A little manipulation and replacement of the x-derivatives with iKx gives the following equations for u, ρ/ρ_o, T/T_o, and μ_v/ρ_o.

$$i\omega \frac{\rho}{\rho_o} + iKu - \frac{\mu_v}{\rho_o} = 0 \tag{1.44}$$

$$-i\omega \left[1 + \frac{\varkappa}{1-i\omega\tau_v}\right] u + iK \frac{a_o^2}{\gamma} \left(\frac{T}{T_o} + \frac{\rho}{\rho_o}\right) = 0 \tag{1.45}$$

$$-i\omega \left[1 + \frac{\varkappa\gamma s}{1-i\omega\tau_T s}\right] \frac{T}{T_o} + iK(\gamma-1)u - \left[1 - \frac{h_\ell/RT_o}{1-i\omega\tau_T s}\right](\gamma-1)\frac{\mu_v}{\rho_o} = 0 \tag{1.46}$$

$$\left[1 - \frac{h_\ell/RT_o}{1-i\omega\tau_T s}\right]\frac{T}{T_o} + \frac{\rho}{\rho_o} + \left\{\frac{\gamma-1}{\gamma}\frac{(h_\ell/RT_o)^2}{1-i\omega\tau_T s}\left(\frac{\tau_T}{\varkappa}\right) + \left[1 - \frac{\varkappa(1-K_{vo})}{i\omega\tau_D}\left(\frac{\tau_D}{hK_{vo}}\right)\right]\right\}\frac{\mu_v}{\rho_o} = 0 \quad . \tag{1.47}$$

For this set of equations to possess a non-trivial solution, the determinant of the coefficients must be zero. The determinant is

$$\begin{vmatrix} 0 & -i\omega & iK & -1 \\ K\frac{a_o^2}{\gamma} & K\frac{a_o^2}{\gamma} & -i\omega\left[1 + \frac{\varkappa}{1-i\omega\tau_v}\right] & 0 \\ i\omega\left[1 + \frac{\varkappa\gamma s}{1-i\omega\tau_T s}\right] & 0 & -i(\gamma-1)K & (\gamma-1)\left[1 - \frac{h_\ell/RT_o}{1-i\omega\tau_T s}\right] \\ \left[1 - \frac{h_\ell/RT_o}{1-i\omega\tau_T s}\right] & 1 & 0 & \left\{\begin{matrix} \frac{\gamma-1}{\gamma}\frac{\tau_v}{\varkappa}\frac{(h_\ell/RT_o)^2}{1-i\omega\tau_T s} \\ + \frac{\varkappa\tau_D}{K_{vo}}\left[1 - \frac{\varkappa(1-K_{vo})}{i\omega\tau_D}\right] \end{matrix}\right\} \end{vmatrix} \tag{1.48}$$

The upper-left 3×3 matrix is that obtained by Temkin[8] for a gas-particle system with no mass transfer between phases. If τ_D is set equal to zero and K_{vo} to unity, the determinant 1.48 gives the special case in which the gas phase consists of vapor only. Expanding the determinant and setting the result equal to zero gives the following equation for the complex wave number K in terms of the frequency ω.

$$\left(\frac{Ka_o}{\omega}\right)^2 \left\{ \left[1 + \frac{\tau_D}{K_v\tau_T}\left(\frac{\gamma}{\gamma-1}\right)\left(\frac{RT_o}{h_\ell}\right)^2 (1 + \kappa + \frac{\tau_T}{\tau_D}\kappa\,(1-K_{vo})\right] + \right.$$

$$\left. + i\left[\frac{\kappa}{\omega\tau_T s} + \frac{\tau_D}{K_v\tau_T}\left(\frac{\gamma}{\gamma-1}\right)\left(\frac{RT_o}{h_\ell}\right)^2\left(\frac{K(1+\kappa)(1-K_{vo})}{\omega\tau_D} - \omega\tau_T s\right)\right]\right\} =$$

$$\left[1 + \frac{\kappa}{1+\omega^2\tau_v^2} + i\left(\frac{\omega\tau_v}{1+\omega^2\tau_v^2}\right)\right] \times$$

$$\left\{1 + \frac{\gamma}{\gamma-1}\left(\frac{RT_o}{h_\ell}\right)^2\left[\kappa\gamma - \frac{\tau_v}{K_v\tau_T}(1+\kappa\gamma) + \frac{\tau_T}{\tau_D}s\kappa(1-K_{vo})\right] + \right.$$

$$i\left[\frac{\kappa\gamma}{\omega\tau_T}\left(1 - \frac{2RT_o}{h_\ell} + \frac{\gamma}{\gamma-1}(1+\kappa)\left(\frac{RT_o}{h_\ell}\right)^2\right) + \right.$$

$$\left.\left.\frac{\tau_D}{K_v\tau_T}\left(\frac{\gamma}{\gamma-1}\right)\left(\frac{RT_o}{h_\ell}\right)^2\left(\frac{\kappa(1+\kappa\gamma)(1-K_{vo})}{\omega\tau_D} - \omega\tau_T s\right)\right]\right\} . \tag{1.49}$$

K_1 and K_2 can be determined by setting the real and imaginary parts of equation 1.49 equal to zero separately.

Define the attenuation and dispersion coefficients, respectively, as follows

$$\alpha = \frac{2K_2a_o}{\omega} \tag{1.50}$$

and

$$\beta = \left(\frac{K_1a_o}{\omega}\right)^2 - 1 \; . \tag{1.51}$$

α is a dimensionless measure of attenuation and β a measure of dispersion of sound of frequency $\omega/2\pi$.

The dispersion and attenuation coefficients from equations 1.49 - 1.51 are plotted in Figures 1 and 2 for a typical system. The values for a one- and two-component evaporating system are compared to an equivalent system in which there is no phase change. In Figures 3 and 4 the attenuation and dispersion are compared for different values of h_ℓ/RT_o for a single component system. In the single component system the gas phase is just pure vapor.

4. THERMAL RELAXATION WITHIN DROPLETS

In writing the linearized energy equation 1.31 for the liquid phase, the temperature of the droplets was assumed to be uniform over the droplet cross-section. This is not a strictly valid assumption unless the thermal conductivity of the liquid is infinite. Finite liquid conductivity means that there is an additional relaxation process which affects the propagation of sound in the medium. The relaxation processes already considered are the result of finite times required for the transfer of mass, momentum, and heat between the two phases. The additional relaxation process considered here is due to the finite time required to reach thermal equilibrium within a droplet.

The energy contained in a droplet whose surface temperature varies with time like $e^{-i\omega t}$ is derived in Appendix B. The result is

$$\epsilon_p = mcT_s \frac{3}{\phi^2}(1 - \phi\cot\phi) \qquad (1.52)$$

where

$$\phi = \sqrt{i\omega\tau_\ell} \qquad (1.53)$$

and

$$\tau_\ell = \frac{\rho_\ell c\sigma^2}{k_\ell} \quad . \qquad (1.54)$$

The thermal conductivity of the liquid is k_ℓ and the surface temperature of the particle is T_s. As the liquid conductivity approaches infinity, τ_ℓ goes to zero and the droplet internal energy goes to mcT_s, which is the result used in the previous section. The specific energy e_p of the droplets is determined from the relation

$$\rho_p e_p = n e_p$$

where n is the droplet number density. Since nm is just ρ_p, the specific energy of the droplets can be written

$$e_p = c T_s F(\phi) \tag{1.55}$$

where

$$F(\phi) = \frac{3}{\phi^2} [1 - \phi \cot\phi] \ . \tag{1.56}$$

The relaxation time for thermal equilibration of the droplets is τ_ℓ, given by equation 1.54. The previous relaxation times τ_v, τ_T, and τ_D are usually of the same order,as seen from the following relations,

$$\tau_T / \tau_v = \frac{3}{2} \Pr \tag{1.57}$$

and

$$\tau_D / \tau_v = \frac{3}{2} \mathrm{Sc} \tag{1.58}$$

where Pr and Sc are the Prandtl number and Schmidt number of the gas mixture. The Prandtl and Schmidt numbers are of order unity for most gases so τ_v, τ_T, and τ_D are usually about equal. However, τ_ℓ is given by

$$\frac{\tau_\ell}{\tau_v} = \frac{9}{2} \frac{c}{c_p} \Pr \frac{k}{k_\ell} \ . \tag{1.59}$$

Since the specific heats of the liquid and gas are of the same order and the Prandtl number of order unity, the thermal relaxation time of the droplets is significant only if the thermal conductivities of the liquid and the gas-vapor mixture are of the same order of magnitude. The relaxation times are shown in Table II for several systems of interest. For wet sodium and potassium vapors, τ_ℓ is about 10^{-3}

times τ_v, but for water fog in air and wet water vapor at atmospheric pressure, τ_ℓ is of the same order as τ_v. In the latter two systems the thermal equilibration in droplets is of significance in the same frequency range as mass, momentum, and thermal equilibration between the two phases. This effect is incorporated into the preceding calculation as follows.

Equations 1.32 through 1.38 remain unchanged except that the droplet surface temperature T_s replaces the droplet temperature T_p. And instead of equation 1.39, the linearized particle energy equation is written

$$sF(\phi)\frac{\partial}{\partial t}\left(\frac{T_s}{T_o}\right)+\frac{1}{\tau_T}\left(\frac{T_p}{T_o}-\frac{T}{T_o}\right)+\frac{\mu_v}{\varkappa\rho_o}\left(\frac{\gamma-1}{\gamma}\right)\frac{h_\ell}{RT_o}=0. \quad (1.60)$$

A calculation similar to that leading to the determinant 1.48 gives the following determinant which must be set equal to zero for a solution to exist.

$$\begin{vmatrix} 0 & i\omega & iK & -1 \\ K\dfrac{a_o^2}{\omega} & K\dfrac{a_o^2}{\omega} & -i\omega\left[1+\dfrac{\varkappa}{1-i\omega\tau_v}\right] & 0 \\ i\omega\left[1+\dfrac{\varkappa\gamma sF}{1-i\omega\tau_T sF}\right] & 0 & -i(\gamma-1)K & (\gamma-1)\left[1-\dfrac{h_\ell/RT_o}{1-i\omega\tau_T sF}\right] \\ \left[1-\dfrac{h_\ell/RT_o}{1-i\omega\tau_T sF}\right] & 1 & 0 & \left\{\begin{matrix}\dfrac{\gamma-1}{\gamma}\dfrac{\tau_T}{\varkappa}\dfrac{(h_\ell/RT_o)^2}{1-i\omega\tau_T sF}+ \\ \dfrac{\tau_D}{\varkappa K_{vo}}\left[1-\dfrac{\varkappa(1-K_{vo})}{i\omega\tau_D}\right]\end{matrix}\right\} \end{vmatrix} \quad (1.61)$$

The determinant 1.48 can obviously be obtained from 1.61 by setting the complex phase factor F equal to unity. Recall that F approaches unity as the liquid thermal conductivity approaches infinity. The equation obtained by setting this determinant equal to zero is

$$
\begin{aligned}
&\left(\frac{Ka_o}{\omega}\right)^2\left\{1+i\,\frac{\kappa}{\omega\tau_T}+\left(1+\frac{\kappa F s}{1-i\omega\tau_T sF}\right)\times\right.\\
&\qquad\left.\left[(1-i\omega\tau_T sF)\frac{\gamma}{\gamma-1}\left(\frac{RT_o}{h_\ell}\right)^2\frac{\tau_D}{K_{vo}\tau_T}\left(1+\frac{i\kappa(1-K_{vo})}{\omega\tau_D}\right)\right]\right\} =\\
&\left[1+\frac{\kappa}{1-i\omega\tau_v}\right]\left\{1+\frac{i\kappa}{\omega\tau_T}-\frac{2i\kappa\gamma}{\omega\tau_T}\left(\frac{RT_o}{h_\ell}\right)+\right.\\
&\qquad i\,\frac{\kappa\gamma}{\omega\tau_T}\left(\frac{\gamma}{\gamma-1}\right)\left(\frac{RT_o}{h_\ell}\right)^2(1-i\omega\tau_T sF)\left(1+\frac{\kappa sF}{1-i\omega\tau_T sF}\right)\\
&\left.+\left(1+\frac{\kappa\gamma F}{1-i\omega\tau_T sF}\right)\frac{\tau_D}{K_{vo}\tau_T}\left(\frac{\gamma}{\gamma-1}\right)\left(\frac{RT_o}{h_\ell}\right)^2(1-i\omega\tau_T sF)\left(1+\frac{i\kappa(1-K_{vo})}{\omega\tau_D}\right)\right\} \qquad (1.62)
\end{aligned}
$$

The dispersion and attenuation coefficients obtained from equation 1.62 are plotted in Figures 5 and 6 and compared with the previous calculation for no thermal relaxation within the droplets.

5. LIMITING CASES FOR FROZEN AND EQUILIBRIUM INTERACTIONS

The dispersion and attenuation coefficients, calculated from equation 1.62, involve four equilibration times, τ_v, τ_T, τ_D, and τ_ℓ, in a complicated way. The role of each equilibration process is more clearly demonstrated by considering several limiting cases in which certain equilibration times go to either zero or infinity.

If a certain equilibration time τ_i is zero, the flow is in equilibrium with respect to the i^{th} equilibration process. For example, if τ_T is zero, the droplets are in thermal equilibrium with the local gas mixture. That is, we see that, by equation 1.11, for Q to remain finite as τ_T goes to zero we must set $T_p \equiv T$. This is equivalent to saying that the heat transfer berween phases occurs at an infinite rate. On the other hand, if an equilibration time becomes infinite, the flow is frozen with respect to the corresponding equilibration process. For example, if τ_T is infinite, there is no heat transfer due to conduction between the droplets and the gas. This does not mean, however, that the droplet temperature remains constant, since heat is transferred between phases by the evaporation and condensation processes.

Since each of the four equilibration times may take on any of the three values, zero, infinity, or finite and non-zero, there are eighty-one possible combinations. The attenuation and dispersion for some of these limiting cases are of particular interest. They can all be calculated as limiting cases of equation 1.62.

(a) Completely frozen flow is obtained from equation 1.62 by setting $\tau_v = \tau_T = \tau_D = \infty$. The result is

$$\left(\frac{Ka_o}{\omega}\right)^2 = 1 \; . \tag{1.63}$$

This result gives just the frozen speed of sound with no attenuation, independent of the value of τ_ℓ . Since there is no heat or mass transfer to the droplets, these temperature remains constant and τ_ℓ does not enter. In this case, there is no coupling between the two phases, so sound propagates as it would in the gas phase alone.

(b) Heat and mass transfer frozen is obtained by setting $\tau_T = \tau_D = \infty$. If τ_v remains finite we have

$$\left(\frac{Ka_o}{\omega}\right)^2 = 1 + \frac{\varkappa}{1-i\omega\tau_v} \; . \tag{1.64}$$

Again, the thermal relaxation within droplets characterized by τ_ℓ does not enter because there is no heat or mass transfer between phases. If, in addition, $\tau_v = 0$, the droplets and gas are in kinetic equilibrium and equation 1.64 gives for the speed of sound

$$\left(\frac{a_v}{a_o}\right)^2 = \frac{1}{1+\varkappa} \tag{1.65}$$

which is just the speed of sound in a gas of density $\overline{\rho} = \rho(1+\varkappa)$. There is no attenuation with $\tau_v = 0$. This shows how the momentum exchange between phases, characterized by τ_v , increases the effective mass of the gas.

(c) Momentum and mass transfer frozen is a limiting case in which $\tau_v = \tau_D = \infty$. The resulting complex wave number is given by

$$\left(\frac{Ka_o}{\omega}\right)^2 = \left\{ \frac{1 + \varkappa Fs - i\omega\tau_T Fs}{1 + \varkappa\gamma Fs - i\omega\tau_T Fs} \right\} \; . \tag{1.66}$$

If the liquid thermal conductivity is very large compared to that of the gas such that $\tau_\ell \to 0$, then $F \to 1$. Further, if the droplet and gas are in thermal equilibrium, that is, $\tau_T = 0$, then the speed of sound is given by

$$\left(\frac{a_T}{a_o}\right)^2 = \frac{1 + \kappa\gamma s}{1 + \kappa s} \tag{1.67}$$

where the attenuation is zero. This is just the speed of sound in a gas whose specific heat ratio is $\bar{\gamma} = \gamma(1+\kappa s)/(1+\kappa\gamma s)$. Thus, the thermal interaction between phases changes the effective specific heat of the gas.

(d) Momentum and heat transfer frozen is obtained by setting $\tau_v = \tau_T = \infty$. The complex wave number is given by

$$\left(\frac{Ka_o}{\omega}\right)^2 = \frac{K_{vo}\frac{\gamma-1}{\gamma}\eta^2 + \kappa\gamma F s K_{vo} - Fs[i\omega\tau_v - \kappa(1-K_{vo})}{K_{vo}\frac{\gamma-1}{\gamma}\eta^2 - Fs[i\omega\tau_D - \kappa(1-K_{vo})} \,. \tag{1.68}$$

If $\tau_\ell = 0$ such that $F = 1$, and if the gaseous phase is pure vapor so that $K_{vo} = 1$ and $\tau_D = 0$, the speed of sound is given by

$$\left(\frac{a_p}{a_o}\right)^2 = 1 + \frac{\gamma}{\gamma-1}\left(\frac{\kappa\gamma}{\eta^2}\right) \tag{1.69}$$

This is the speed of sound in a wet vapor with only mass transfer and no heat or momentum transfer between phases.

(e) Droplet temperature frozen is the case for $\tau_\ell = \infty$. This may be because either $c \to \infty$ or $k_\ell \to 0$. This implies that the particle temperature remains constant. The complex wave number is given by the relation

$$\left(\frac{Ka_o}{\omega}\right)^2 \left\{\left(1 + \frac{\varkappa}{i\omega\tau_T}\right)\left(\frac{i\omega\tau_v - \varkappa(1-K_{vo})}{K_{vo}}\right)\right\} =$$

$$\left(1 + \frac{\varkappa}{1-i\omega\tau_v}\right)\left\{-\varkappa\gamma\left(1 - \frac{\varkappa}{i\omega\tau_T}\right) + \left(1 - \frac{\varkappa\gamma}{i\omega\tau_T}\right)\left(\frac{i\omega\tau_D - \varkappa(1-K_{vo})}{K_{vo}}\right)\right\} . \qquad (1.70)$$

If we let the mixture be a pure vapor then $K_{vo} = 1$ and $\tau_D = 0$, the speed of sound becomes zero and the attenuation infinite. Thus, sound will not propagate. Since the particle temperature is constant for this case, the vapor pressure at the surface of the droplets is also constant. This places upon the system the constraint that the pressure be constant so that sound cannot propagate.

(f) Complete equilibrium is obtained if all the equilibration times are zero. If the gaseous phase is pure vapor, the equilibrium sound speed is

$$\frac{a^2_{equil}}{a_o^2} = \frac{1}{\gamma(1+\varkappa)} \left[1 - \frac{2}{\eta} + \frac{\gamma}{\gamma-1} \frac{(1+\varkappa s)}{\eta^2}\right]^{-1} \qquad (1.71)$$

and the dispersion is zero.

The other limiting cases may be readily obtained from equation 1.62. They will not be discussed here, however, since they do not add much to the clarification of the roles played by the equilibration processes. If the flow is in equilibrium and/or frozen with respect to all the equilibration processes, the attenuation is zero and the effect of the droplets is to alter the sound speed from that value it would have in the absence of the droplets.

6. EFFECT OF PRESSURE RELAXATION

In writing the general equations, the pressure was assumed locally uniform. That is, the pressure at a droplet surface was assumed equal to the local pressure in the gaseous bulk. This is a reasonable assumption if the relaxation time for a pressure difference across a droplet surface is sufficiently small compared to the other relaxation times. Such a pressure difference arises, for example, if the droplets in an elemental volume are heated or cooled by conduction. This causes a change in the vapor pressure at the droplet's surface. The pressure disturbance propagates into the gaseous bulk which alters the state of the gas. This interaction is in general a continuous process taking place over a finite relaxation time. In the following, we will consider the relaxation of a pressure differential across a liquid surface for the one-dimensional case. We will assume the initial condition that the equilibrium vapor pressure of the liquid at the liquid surface is unequal to the pressure of the vapor just adjacent to the liquid surface. The governing equations will be written down and the pressure relaxation time computed from a linearized solution. The effect of finite liquid conductivity is considered in the subsequent section.

Consider a semi-infinite duct of constant cross-sectional area. The closed end of the duct contains liquid of mass m_o per unit area, and the remainder of the duct is filled with vapor as shown in Figure 7a. The vapor is assumed an ideal gas and the liquid incompressible. The origin of the x-coordinate is at the initial location of the liquid surface. The surface of the liquid and the vapor immediately adjacent to it are assumed to be in equilibrium for time greater than zero. Assume that there is initially a pressure discontinuity at the liquid surface and write the initial conditions as follows.

$$\begin{aligned} P(x, o) &= P_2 && x > 0 \\ &= P_1 && x < 0 \\ T(x, o) &= T_2 && x > 0 \\ &= T_1 && x < 0 \\ u(x, o) &= 0 && \text{all } x \end{aligned} \tag{1.72}$$

Define

$u_i(t)$ = velocity of the liquid interface

$m(t)$ = mass of liquid per unit area

where $m(o) = m_o$.

Equating the rate of increase of the mass of the liquid to the vapor mass flow rate at the liquid surface, we obtain

$$\rho_\ell u_i = \frac{dm}{dt} = -\rho(x_i, t)u(x_i, t) \tag{1.73}$$

where x_i is the coordinate of the interface. The energy balance at the interface is written

$$\frac{d}{dt}(mcT_\ell) = \rho(x_i)u(x_i)\left[\frac{u_i}{u(x_i)} - 1\right]\left[c_p T(x_i) + h_\ell + \tfrac{1}{2}u^2(x_i)\right] \tag{1.74}$$

where the liquid is assumed to have infinite thermal conductivity. Conduction of heat between the two phases is neglected in writing the energy balance 1.74. This assumes that the pressure relaxation takes place before a significant amount of heat is conducted to the surface. It is equivalent to the assumption that the pressure relaxation time is small compared to the thermal relaxation time τ_T . The conditions for this to be true are obtained in the following.

Assume that the vapor pressure at the liquid surface is given by the Clausius-Clapeyron equation

$$\frac{P_\ell}{P_1} = e \frac{h_\ell}{RT_1} \left(1 - \frac{T_1}{T_\ell}\right) \qquad (1.75)$$

where P_ℓ is the equilibrium vapor pressure of the liquid at the liquid temperature T_ℓ. The equations governing the vapor phase are those of conventional one-dimensional gas dynamics

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \qquad (1.76)$$

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} = - \frac{\partial P}{\partial x} \qquad (1.77)$$

$$\frac{\partial}{\partial t}(c_p T + \tfrac{1}{2}u^2) + u \frac{\partial}{\partial x}(c_p T + \tfrac{1}{2}u^2) = \frac{1}{\rho}\frac{\partial P}{\partial t} \qquad (1.78)$$

and

$$P = \rho RT \quad . \qquad (1.79)$$

Assume that the liquid density is much greater than the vapor density so that u_i can be neglected with respect to u and x_i can be set equal to zero. Equations 1.73 and 1.74 can then be written

$$\frac{dm}{dt} = - \rho(o, t)u(o, t) \qquad (1.80)$$

and

$$cT_\ell \frac{dm}{dt} + mc \frac{dT_\ell}{dt} = - \rho u[c_p T + h_\ell + \tfrac{1}{2}u^2]\Big|_{x=0} \quad . \qquad (1.81)$$

In addition to the equations 1.75 - 1.81 in terms of the dependent variables u, ρ, P, T, m, P_ℓ, and T_ℓ, equilibrium at the liquid surface requires that

$$P(o, t) = P_\ell(o, t) \quad , \qquad t > 0 \ . \qquad (1.82)$$

If $P_2 > P_1$, and expansion wave propagates into the vapor as shown in Figure 7b. Vapor flows toward the liquid surface and condenses. This raises the temperature of the liquid which in turn increases the vapor pressure P_ℓ so that compression waves propagate into the vapor, following the expansion. This continues until the vapor pressure reaches P_2. The final liquid temperature is not necessarily equal to T_2 since the initial vapor state may not lie on the saturation curve. The question of accommodation will not be considered here[14, 15]. Rather, it is assumed that all the vapor impinging upon the liquid surface condenses. In a finite system, the pressure equilibration would be complicated by reflected waves and the final vapor pressure would be less than P_2. However, the time required for pressure equilibration in the finite system would be less than that in the present case.

If $P_2 < P_1$ a similar equilibration process occurs, except that the leading wave is a compression which is followed by a series of expansion waves as the liquid is cooled by evaporation. The vapor flow in this case is non-isentropic and, in general, more difficult to calculate. However, we will limit this discussion to the linearized flow in which the cases for $P_1 < P_2$ and $P_1 > P_2$ are both covered by the calculation.

A calculation very similar to this has been carried out by Porter.[16] He treated the reflection of sound from evaporating and adsorbing surfaces.

(a) <u>Linearized solution.</u> Assume that the initial pressure discontinuity at the liquid surface is small. Then a perturbation solution to equations 1.75 - 1.81 can be obtained. Assume a perturbation about the initial state 1.72 which is valid for

$$|(P_2-P_1)/P_1| \ll 1 \ .$$

Write the liquid vapor pressure and temperature as

$$\frac{P_\ell}{P_1} = \frac{P_1 + P'_\ell}{P_1} \tag{1.83}$$

and

$$\frac{T_\ell}{T_1} = \frac{T_1 + T'_\ell}{T_1} \tag{1.84}$$

where P'_ℓ and T'_ℓ are perturbation quantities. For the pressure, temperature, and density of the vapor write

$$\frac{P}{P_2} = \frac{P_2+P'}{P_2} \tag{1.85}$$

$$\frac{T}{T_2} = \frac{T_2+T'}{T_2} \tag{1.86}$$

and

$$\frac{\rho}{\rho_2} = \frac{\rho_2+\rho'}{\rho_2} \quad . \tag{1.87}$$

From the linearized vapor equations 1.76 - 1.79 we obtain the familiar expression for the perturbation velocity

$$\frac{u}{a_2} = \frac{1}{\gamma}\frac{P'}{P_2} \tag{1.88}$$

where u is the vapor velocity and a_2 is the speed of sound in the undisturbed vapor. We will be concerned with the vapor quantities only at the liquid surface, so in the following let all primed vapor quantities refer to values at $x = 0$.

With the aid of equation 1.80, the first order part of equation (1.81) can be written

$$\frac{d}{dt}\left(\frac{T'_\ell}{T_1}\right) = -\frac{\rho_2}{m_o}\, u\left[\frac{c_p T_2 - c T_1}{T_1} + \frac{h_\ell}{RT_1}\right] \quad . \tag{1.89}$$

The term $c_p T_2 - cT_1$ represents the excess non-latent heat carried into the liquid surface by the condensing vapor. Since c and c_p are of the same order, this term will be neglected in comparison with h_ℓ / RT_1 unless T_2 is much greater than T_1. In the linearized solution we assume T_1 and T_2 nearly equal. With the aid of equation 1.88, write equation 1.89 as

$$\frac{d}{dt}\left(\frac{T'_\ell}{T_1}\right) = -\frac{\rho_2 a_2}{m_o \gamma} \frac{c_p}{c} \frac{h_\ell}{c_p T_1} \frac{P'}{P_2} \quad . \tag{1.90}$$

From the linearized Clausius-Clapeyron equation

$$\frac{P'_\ell}{P_1} = \frac{h_\ell}{RT_1} \frac{T'_\ell}{T_1} \tag{1.91}$$

and from equation 1.82,

$$\frac{P_1 + P'_\ell}{P_1} = \frac{P_2 + P'}{P_2} \qquad \text{(1.92}$$

equation 1.90 can be written

$$\frac{d}{dt}\left(\frac{P_\ell}{P_1}\right) = -\frac{\rho_2 a_2}{m_o} \frac{\gamma - 1}{\gamma^2} \left(\frac{c_p}{c}\right)\left(\frac{h_\ell}{RT_1}\right)^2 \frac{P_\ell}{P_1} \quad . \tag{1.93}$$

This can be integrated directly to give

$$\frac{P_2 - P_\ell}{P_2 - P_1} = e^{-t/\tau_p} \tag{1.94}$$

where

$$\tau_p = \frac{c}{c_p} \frac{m_o}{\rho_2 a_o} \frac{\gamma^2}{\gamma - 1} \Big/ \left(\frac{h_\ell}{RT_1}\right)^2 \tag{1.95}$$

is the pressure relaxation time. For a vapor-droplet system, the pressure relaxation time is

$$\tau_p = \frac{c}{c_p} \frac{\rho_\ell}{\rho} \frac{\sigma}{a_o} (\gamma - 1) \Big/ \left(\frac{h_\ell}{RT_1}\right)^2 \quad . \tag{1.96}$$

The pressure relaxation time compared to the velocity relaxation time gives

$$\frac{\tau_p}{\tau_v} = \frac{9}{2}(\gamma-1)\frac{c}{c_p}\frac{u}{a_o} \Big/ \mathrm{Re}\left(\frac{h_\ell}{RT_o}\right)^2 \qquad (1.97)$$

where Re is the Reynolds number of the droplets based upon the vapor velocity u. Since by assumption $\mathrm{Re} \sim 1$ and $c/c_p \sim 1$, τ_p/τ_v is proportional to the perturbation quantity u/a_o divided by $(h_\ell/RT_o)^2$. The ratio (h_ℓ/RT_o) is of order 10 for most systems of interest (see Table I). For the systems tabulated in Table II, τ_p/τ_v is of order 10^{-2} to 10^{-4}. So if $\tau_p << \tau_v$, we are justified in neglecting the pressure relaxation in relation to the other relaxation processes.

(b) The effect of finite liquid conductivity upon pressure relaxation. In the previous calculation, the liquid thermal conductivity was assumed infinite. During the pressure relaxation of a droplet of finite conductivity, the liquid surface is hotter or colder than the interior as condensation or evaporation takes place. If the pressure equilibration takes place rapidly compared to the time for heat to be conducted from the droplet surface to the interior, the heat is effectively confined to a thin layer near the surface. This would effectively decrease the mass entering into the equilibration process, thereby decreasing the pressure equilibration time τ_p. This is examined analytically in the following.

Consider the same one-dimensional system as before, shown in Figure 7a, subject to the initial conditions 1.72. Assume that the wall at $x = -\ell$ is insulated. Then the liquid temperature satisfies the heat equation

$$\rho_\ell c \frac{\partial T'_\ell}{\partial t} = k_\ell \frac{\partial^2 T'_\ell}{\partial x^2} \qquad (1.98)$$

subject to the end conditions

$$\frac{\partial T'_\ell}{\partial x}(-\ell, t) = 0 , \qquad t > 0 \qquad (1.99)$$

and

$$k_\ell \frac{\partial T'_\ell}{\partial x}(0, t) = -\rho_2 u' k_\ell , \qquad t > 0 \qquad (1.100)$$

where the primes stand for perturbation quantities as before. The right hand side of equation 1.100 is the linearized heat flux into the fluid surface due to condensation. From equation 1.88, the condition 1.100 can be written

$$k_\ell \frac{\partial T'_\ell}{\partial x} = -\rho_2 a_2 \frac{h_\ell}{\gamma} \frac{P'}{P_2} , \qquad x = 0 , \ t > 0 .$$

Using equation 1.91 this becomes

$$\frac{\partial}{\partial x}\left(\frac{T'_\ell}{T_1}(0, t)\right) = -\frac{\rho_2 a_2}{h_\ell} \frac{c_p}{\gamma - 1} \left(\frac{h_\ell}{c_p T_1}\right)^2 \frac{T'_\ell}{T_1}(0, t) . \qquad (1.101)$$

Equation 1.98, subject to the conditions 1.99 and 1.101, and the initial condition

$$T'_\ell(x, 0) = 0 \qquad -\ell \le x < 0 \qquad (1.102)$$

can be solved formally in terms of Fourier series[17]. However, this solution does not give the characteristic time explicitly. The following approximate solution by an integral method gives an explicit approximation for the characteristic pressure relaxation time.

Assume that at any time greater than zero the total heat carried to the liquid surface by condensation is contained in a layer of constant thickness δ. From the differential equation 1.98 we write

$$\rho_\ell c \frac{\partial}{\partial t} \int_{-\delta}^{0} T'_\ell dx = k_\ell \frac{\partial T'_\ell}{\partial x}(0,t) \qquad (1.103)$$

where T'_ℓ is assumed zero at $x = -\delta$. Then, from equation 1.101,

$$\rho_\ell c \frac{\partial}{\partial t} \int_{-\delta}^{0} T'_\ell dx = -\rho_2 a_2 \frac{c_p}{\gamma-1} \left(\frac{h_\ell}{c_p T_1}\right)^2 T'_\ell(0,t) \qquad (1.104)$$

Assume the following expansion for $T'_\ell(x,t)$:

$$T'_\ell(x,t) = T'_\ell(0,t) + \frac{\partial T'_\ell}{\partial x}(0,t)\, x + \dots \quad . \qquad (1.105)$$

Substitute equation 1.105 into 1.104 and use 1.101 to obtain

$$\frac{\partial T'_\ell}{\partial t}(0,t) = -\frac{T'_\ell(0,t)}{\tau_p^*} \qquad (1.106)$$

where

$$\tau_p^* = (\gamma-1)\frac{\rho_\ell}{\rho_2}\frac{c}{c_p}\frac{\delta}{a_2}\left[1 + \frac{\delta}{2}\frac{\rho_2 a_2}{k_\ell}\frac{c_p}{\gamma-1}\left(\frac{h_\ell}{c_p T_1}\right)^2\right] \Big/ \left(\frac{h_\ell}{c_p T_1}\right)^2 \quad . \qquad (1.107)$$

τ_p^* is the pressure relaxation time for finite liquid conductivity. Compare $\tau_p{}^*$ with the pressure equilibration time for infinite thermal conductivity of the liquid τ_p.

$$\frac{\tau_p^*}{\tau_p} = \frac{\delta}{\sigma}\left[1 + \frac{\delta}{2\Delta}\right] \qquad (1.108)$$

where

$$\Delta = \frac{k_\ell}{\rho_2 a_2}\frac{\gamma-1}{c_p} \Big/ \left(\frac{h_\ell}{c_p T_1}\right)^2 \qquad (1.109)$$

The thermal layer thickness δ is of the same order or smaller than the particle radius σ, and can always be taken of the order of Δ, such that τ_p^* is less than τ_p. From equations 1.101 and 1.105, we see that to the first order Δ is the depth at which $T'_\ell(0,t) \sim 0$. That is, Δ is a measure of the thickness of the layer at the surface in which the heat due to condensation is contained. Therefore, by definition, $\delta \sim \Delta$.

7. DISCUSSION OF RESULTS

Temkin and Dobbins[9] have shown that their theory, which does not include mass transfer, gives values of attenuation very close to the theory of Epstein and Carhart[3], and values of dispersion very close to the calculations by Chu and Chow[7]. Measurements, reported by Temkin and Dobbins[10], of the attenuation and dispersion of sound in a system consisting of oleic acid in a nitrogen carrier gas were in close agreement with their theory. A comparison between their results for a system with no mass-transfer between phases and the present calculations, which include mass transfer, is shown in Figures 1 and 2. At low frequencies, ($\omega\tau_v < 0.5$), the dispersion coefficient for the case which includes mass transfer is up to 40 per cent higher than that in which there is no mass transfer. The effect of mass transfer is to increase the attenuation by about 20 to 30 per cent in the middle frequencies, ($\omega\tau_v = 0.1$ to 1.0), and to shift the maximum attenuation from a value of $\omega\tau_v = 1$ to about $\omega\tau_v \sim 0.5$. Consequently, in some systems, mass transfer has significant effects upon the attenuation and dispersion coefficients.

The effects of different values of the latent heat of vaporization, h_ℓ, upon the attenuation and dispersion is shown in Figures 3 and 4. Generally, higher values of h_ℓ/RT_o give larger values for the dispersion coefficient at lower frequencies, and higher attenuation in the middle frequencies, as can be seen from the curves. In Figures 5 and 6, the effect of finite droplet thermal conductivity upon the dispersion and attenuation is shown for a particular system. For this system, there is some change in dispersion in the middle frequencies due to particle conductivity effects, and less change in attenuation. For this system, finite particle conductivity does not have a very significant effect.

The attenuation and dispersion have been plotted as functions of $\omega\tau_v$. The variation of $\omega\tau_v$ may be thought of in two ways. It may be considered with respect to a variation in the angular frequency ω for a given system, that is τ_v fixed, or it may be considered as a variation of τ_v for a fixed frequency. Since τ_v is a function of the droplet radius, the latter interpretation may be used, for example, to determine which particle radius, other conditions being the same, gives the maximum attenuation of sound of a given frequency.

In the first order theory, considered here, the problem of droplet collisions, or coagulation, was avoided by assuming all droplets to be of the same size. In a second order theory, however, coagulation must be considered, even for droplets of the same size. The reason for this is that droplets at different locations along the path of the sound wave have different velocities at any given time. The order of magnitude of this velocity difference, Δu_p, is given by the relation

$$\Delta u_p \sim \frac{\partial u_p}{\partial x}\,\delta \tag{1.110}$$

where δ is a measure of the spacing between droplets. Alternately, this relation may be expressed as

$$\frac{\Delta u_p}{a} \sim \frac{u}{a}\cdot\frac{\delta}{\lambda} \tag{1.111}$$

where λ is the wavelength of the sound. The gas velocity u is small compared to the speed of sound, a, in the acoustic approximation, and the quantity δ/λ is assumed to be, at most, of the same order. Hence, the velocity difference Δu_p , which leads to the possibility of coagulation, should be included in a second order theory. Second order coagulation effects may be an appreciable factor in experimental work where measurements are made over time periods which include many cycles, that is, over time periods large compared to $1/\omega$.

APPENDIX A. DERIVATION OF THE BASIC EQUATIONS

Consider a system consisting of liquid droplets dispersed in a mixture of condensible vapor and inert gas. Equations governing the flow of this system are derived in a treatment which largely parallels the work of Marble[11]. Viscous stresses and heat conduction in the gaseous phase are included for generality. The gaseous species are assumed thoroughly mixed and assumed to move together in thermodynamic equilibrium so that differential diffusion rates may be neglected on a scale large compared to the droplet dimensions and so that one may consider the transport properties of the mixture. The liquid droplets are assumed finely divided so that any volume element we may choose contains many droplets; then the droplets may be treated from the continuum viewpoint which clearly does not hold on a scale comparable to the droplet dimensions. A necessary condition for this type of treatment is that the scale of the problem be large compared to the droplet dimensions.

On a scale comparable to the droplet dimensions, there are local variations in velocity, temperature, and vapor concentration. It is a major assumption that these microscopic fields depend only upon the local bulk state of the matter and not upon the relatively large scale properties of the gross motion. Further, the droplets are assumed sufficiently separated that the local fields of one droplet do not interact with the fields of another droplet. As a consequence of these assumptions, it is admissible to employ knowledge of the behavior of isolated droplets in uniform gas fields to calculate the interaction between phases. The liquid density is assumed much greater than the gas density and the mass fraction of the liquid phase is assumed to be at most of order unity so that the volume fraction occupied by the liquid may be neglected. The ensuing calculations are carried out assuming that the droplet and gas are two continua, coexisting in space, and interacting with each other through the transfer of heat, mass, and momentum. Further details and consequences of the assumptions will be treated as they arise in the development of the governing equations.

Separate laws of mass, momentum, and energy conservation are written for the liquid phase and the gaseous mixture. Actually, separate conservation laws may be written for the two gaseous species, but since the gases are thoroughly mixed, this is redundant. In the derivation of the conservation equations, we consider a collection of matter bounded by a surface that always consists of the same elements of matter under consideration. We will frequently calculate the time rate of change of some property integrated over a volume which moves with the substance under consideration. Any tensor quantity $A_{k\ell}$ in the volume V may be differentiated by time through writing

$$\int_V \frac{\partial A_{k\ell}}{\partial t}\, dV + \int_S A_{k\ell} u_i n_i \, dS \qquad \text{(A. 1)}$$

where the first term represents the local time derivative and the second term is the amount convected through the surface S of the volume. If $A_{k\ell}$ is a sufficiently smooth function of x_i, the integral transformation gives

$$\int_V \left\{ \frac{\partial A_{k\ell}}{\partial t} + \frac{\partial}{\partial x_i} (A_{k\ell} u_i) \right\} dV \ . \qquad \text{(A. 2)}$$

This is the familiar form for the time rate of change of the quantity $A_{k\ell}$ in a volume moving with the fluid.

The conservation of mass for the gaseous phase requires that the time rate of change of mass in the volume V be equal to the rate at which vapor is added to the volume by evaporation from the liquid droplets. This may be expressed as

$$\int_V \left\{ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho u_i) \right\} dV = \int_V \mu_v dV \qquad \text{(A. 3)}$$

where μ_v is the mass of vapor per unit volume of gaseous mixture evaporating from the droplets and where ρ and u_i refer to the gaseous mixture. A net condensation would mean that μ_v is negative. In keeping with the earlier assumptions, μ_v is considered to be a continuous smooth function of x_i in spite of the fact that, from a microscopic point of view, the phase change is restricted to a negligible fraction of the volume. Since equation A. 3 holds over an arbitrary volume, by the familiar argument, it must hold point by point. Hence we obtain the differential equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho u_i) = \mu_v \qquad \text{(A. 4)}$$

which is the continuity equation for the gaseous phase. This reduces to the usual continuity equation for gas dynamics if the vapor mass production rate μ_v vanishes.

A similar argument is used to obtain the continuity equation for the droplet phase. Denote by ρ_p the density of the droplet phase, defined as the mass of liquid per unit volume of mixture, and by u_{pi} the velocity vector of the mass motion of the droplet phase. Then, the continuity equation for the droplet cloud is

$$\frac{\partial \rho_p}{\partial t} + \frac{\partial}{\partial x_i} (\rho_p u_{pi}) = - \mu_v \; . \qquad \text{(A. 5)}$$

Since μ_v is the mass evaporation rate from the droplets, $-\mu_v$ is the mass condensation rate because matter is neither created nor destroyed, but just transferred from one phase to the other by evaporation and condensation. The conservation of total mass may be illustrated by adding equations A. 4 and A. 5 to obtain

$$\frac{\partial}{\partial t} (\rho+\rho_p) + \frac{\partial}{\partial x_i} (\rho u_i+\rho_p u_{pi}) = 0 \; . \qquad \text{(A. 6)}$$

This represents the continuity equation for the composite system in which there is no mass production term.

The principle of momentum conservation and its change due to external forces is also applied to the separate phases of the multi-component flow. The momentum vector per unit volume of the gas mixture is ρu_i , and the time rate of change in the total momentum of the moving region is

$$\int_V \left\{ \frac{\partial}{\partial t} (\rho u_i) + \frac{\partial}{\partial x_j} (\rho u_i u_j) \right\} dV \; . \qquad \text{(A. 7)}$$

The gaseous phase is acted upon by two distinct sets of forces: (1) those imposed by the motion or presence of the droplets; and (2) those stresses set up within the gas mixture itself. We explicitly neglect external body forces due to gravitational, electromagnetic, or other fields. The forces per unit volume acting upon the gas due to the droplets will be denoted by F_i. Their nature and origin will be examined later. The internal stresses set up within the gas mixture are assumed to be of the same form as those which would exist in the absence of the liquid phase, since the volume fraction occupied by the liquid phase is assumed negligible. Under these conditions, the stresses in the gaseous medium may be written

$$\sigma_{ij} = -\delta_{ij}\left(P + \frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\right) + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) . \tag{A. 8}$$

The pressure P is the local scalar pressure corresponding to the local temperature and density through the equation of state. Thus, the thermodynamic pressure is defined so long as the gas is in a state where the other thermodynamic quantities are defined. The viscosity coefficient μ is assumed to be that for the gaseous mixture, and due to the small volume fraction occupied by the droplets, the familiar Einstein correction is neglected. For convenience, define the viscous part of the stress tensor as

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\delta_{ij} \tag{A. 9}$$

so that the complete stress tensor is written

$$\sigma_{ij} = -P\,\delta_{ij} + \tau_{ij} . \tag{A. 10}$$

Thus, the forces acting upon the volume of matter under consideration are

$$\int_V F_i dV + \int_S \sigma_{ij} n_j dS . \tag{A. 11}$$

Under the conditions of normal fluid mechanics, the time rate of change of momentum, equation A. 6, would be equated to the external forces, equation A. 11. Here, however, there is an additional source of momentum due to the creation of gaseous medium by evaporation of the droplets. In accounting for this process, we assume that, since the phase transition takes place at the surface of a droplet, the mass is added to or subtracted from the gaseous phase at the droplet velocity. Therefore, the momentum added to our region due to vaporization is

$$\int_V \mu_v u_{pi} \, dV \ . \qquad \text{(A. 12)}$$

Thus, the complete relationship for the change of momentum of the gaseous phase is

$$\int_V \left\{ \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) \right\} dV = \int_V F_i dV + \int_S \sigma_{ij} n_j dS + \int_V \mu_v u_{pi} dV \ . \qquad \text{(A. 13)}$$

Use equation A. 10 for σ_{ij}, and convert the surface integral into a volume integral. Then we make use of the familiar argument to obtain the equation of motion of the gas phase

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = F_i - \frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \mu_v u_{pi} \quad . \qquad \text{(A. 14)}$$

A similar argument may be applied to the change in momentum of the corresponding collection of liquid droplets. However, since the droplets do not form a continuum in the usual sense, there is no surface stress associated with the surface bounding the collection of droplets. The only forces which tend to alter the momentum of the droplet collection are those exerted upon the individual droplets by the gas flow over them; call this force F_i per unit volume. The momentum of the droplet cloud is also modified by the phase transition taking place at the droplet surfaces. Corresponding to the assumption made for the similar term occurring in the momentum consideration of the gas phase, we assume that the condensing mass carries with it the velocity of the droplet as it condenses and that the acceleration or retardation of vapor to the droplet velocity makes itself felt through the drag of the droplet. Utilizing these assumptions, an argument similar to that for the gas phase leads to the droplet momentum equation,

$$\frac{\partial}{\partial t}(\rho_p u_{pi}) + \frac{\partial}{\partial x_j}(\rho_p u_{pi} u_{pj}) = F_{pi} - \mu_v u_{pi} \quad . \qquad \text{(A. 15)}$$

The forces F_i and F_{pi} on the gas and particle collection, respectively, are not independent since they result from the interaction between the two phases at the droplet surface. Thus, clearly,

$$F_i + F_{pi} = 0 \quad , \qquad \text{(A. 16)}$$

which just states that action and reaction are equal. It is interesting to note that the sum of the two momentum equations, equations A. 14 and A. 15, are particularly simple. We find that

$$\frac{\partial}{\partial t}(\rho_{ui} + \rho_p u_{pi}) + \frac{\partial}{\partial x_j}(\rho u_i u_j + \rho_p u_{pi} u_{pj}) = - \frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \quad , \qquad \text{(A. 17)}$$

where the interaction terms do not appear. This total equation represents the momentum conservation relation for the composite gas-droplet mixture.

To complete the statement of the conservation laws, we now apply the conservation of energy to the two phases. We use this principle to state that the rate at which energy (internal and kinetic) of the system is increasing is equal to the sum of (1) the rate at which work is done on the system by external forces (2) the rate at which heat is transferred to the system by outside sources, and (3) the gain in energy from vapor mass increase due to phase change. Denote by e the internal energy per unit mass of the gaseous mixture. The change of energy with respect to time of the gas vapor mixture in a volume V at time t is

$$\int_V \left\{ \frac{\partial}{\partial t}[\rho(e+\tfrac{1}{2}u_i u_i)] + \frac{\partial}{\partial x_j}[\rho(e+\tfrac{1}{2}u_i u_i)u_j] \right\} dV \ . \tag{A.18}$$

Work is done on the gas phase by surface stresses represented by the stress tensor σ_{ij}, and by the force F_i exerted by the droplets and treated here as a distributed volume force. The work done on the volume V by external forces is then

$$\int_S \sigma_{ij} n_j u_i dS + \int_V F_i u_{pi} dV = \int_V \left\{ \frac{\partial}{\partial x_j}(\sigma_{ij} u_i) + F_i u_{pi} \right\} dV \ . \tag{A.19}$$

The heat flux vector q_i defines the rate at which heat is transported out of a unit area normal to the orientation of the vector. Then the heat flowing out of the surface bounding the sample of matter in question is

$$\int_S q_i n_i dS = \int_V \frac{\partial q_i}{\partial x_i} dV \ . \tag{A.20}$$

Finally, there is the energy gained or lost from our sample of matter due to phase transition. This change also takes place very close to the liquid surface. Hence, for consistency with the previous assumptions, the kinetic energy associated with the vapor which is condensed or evaporated must correspond to the velocity of the droplet. Similarly, the internal energy of the vapor which is gained or lost from the local vapor-gas mixture must correspond to the vapor internal energy at the droplet temperature, since that is the gas temperature immediately adjacent to the droplet. The latent heat of phase change requires further consideration. If evaporation is taking place, a molecule escaping the liquid must have left the liquid with sufficient energy to achieve the vapor state, that is, to escape its local liquid bonds. Therefore, the latent heat is extracted from the liquid. Conversely, if the vapor molecule strikes the liquid and remains a part of the liquid structure, then this latent heat again shows up in the liquid droplet. Therefore, the latent heat associated with phase transitions is absorbed from or given to the liquid state. Then if $e(T_p)$ denotes the internal energy of the gas-vapor mixture at the local droplet temperature, the energy added to our sample per unit time due to vapor production is given by

$$\int_V \mu_v \left\{ e(T_p) + \tfrac{1}{2} u_{pi} u_{pi} + \frac{P}{\rho(T_p)} \right\} dV \quad . \qquad \text{(A. 21)}$$

In addition, there is the heat added to the gas-vapor mixture due to heat conduction from the droplets. If Q represents the rate of heat addition to the gas-vapor mixture by condensation from the droplets, the energy added to our sample by this process is given by

$$\int_V Q \, dV \quad . \qquad \text{(A. 22)}$$

Utilizing equations A. 18-22, we now write the appropriate energy equation for the gas-vapor phase.

$$\frac{\partial}{\partial t}\{\rho(e+\tfrac{1}{2}u_iu_i)\}+\frac{\partial}{\partial x_j}\{\rho u_j(e+\tfrac{1}{2}u_iu_i)\}=\frac{\partial}{\partial x_j}(\sigma_{ij}u_i)+F_iu_{pi}-\frac{\partial q_i}{\partial x_i}+$$

$$+\mu_v\{e(T_p)+\frac{P}{\rho(T_p)}+\tfrac{1}{2}u_{pi}u_{pi}\}+Q\ . \quad (A.\ 23)$$

This equation may be transformed into a form that corresponds more clearly to the usual energy equation for a single component. Using the continuity equation A. 4, the momentum equation can be written

$$\rho\frac{\partial u_i}{\partial t}+\rho u_j\frac{\partial u_i}{\partial x_j} = F_i+\mu_v(u_{pi}-u_i)+\frac{\partial\sigma_{ij}}{\partial x_j}\ . \quad (A.\ 24)$$

Arranged in this way, it appears that the droplet cloud exerts an additional trhust on the gas mixture due to change of phase. This is the counterpart of the familiar thrust associated with source flow in an ideal fluid. If this equation is multiplied by u_i and rearranged, an equation for the rate of change of kinetic energy is obtained.

$$\rho\frac{\partial}{\partial t}(\tfrac{1}{2}u_iu_i)+\rho u_i\frac{\partial}{\partial x_j}(\tfrac{1}{2}u_iu_i) = F_iu_i+\mu_v(u_{pi}-u_i)u_i+u_i\frac{\partial}{\partial x_j}(\sigma_{ij})\ . \quad (A.\ 25)$$

With the aid of the continuity equation A. 4, and by using equation A. 25 to subtract out the kinetic energy terms, the energy equation of the gas mixture may be written as

$$\rho\frac{\partial e}{\partial t}+\rho u_j\frac{\partial e}{\partial x_j} = \sigma_{ij}\frac{\partial u_i}{\partial x_j}+F_i(u_{pi}-u_i)-\frac{\partial q_i}{\partial x_i}+Q+$$

$$+\mu_v\{\tfrac{1}{2}u_{pi}u_{pi}-\tfrac{1}{2}u_iu_i-u_i(u_{pi}-u_i)\}+\mu_v\{e(T_p)+\frac{P}{\rho(T_p)}-e\}\ . \quad (A.\ 26)$$

If the term $\sigma_{ij}\frac{\partial u_i}{\partial x_j}$ is expanded using equation A. 8, one can write

$$\sigma_{ij}\frac{\partial u_i}{\partial x_j} = \Phi - P\frac{\partial u_i}{\partial x_i} \quad (A.\ 27)$$

where

$$\Phi \equiv \frac{4}{3}\mu\left(\frac{\partial u_i}{\partial x_i}\right)^2 + \frac{1}{2}\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2_{i\neq j} \qquad (A.\ 28)$$

is the familiar viscous dissipation function. Using this in equation A. 26, one obtains as the energy equation for the gaseous phase

$$\rho\frac{\partial e}{\partial t} + \rho u_j\frac{\partial e}{\partial x_j} + P\frac{\partial u_i}{\partial x_i} = \Phi + (u_{pi}-u_i)(F_i-\mu_v u_i) + Q - \frac{\partial q_i}{\partial x_i}$$

$$+\mu_v\{e(T_p) + \frac{P}{\rho(T_p)} - e + \tfrac{1}{2}u_{pi}u_{pi} - \tfrac{1}{2}u_i u_i\} \ . \qquad (A.\ 29)$$

This equation is now in the conventional form for the first law of thermodynamics.

The energy equation for the droplet cloud may be derived in the same manner. It is

$$\frac{\partial}{\partial t}\{\rho_p(e_p+\tfrac{1}{2}u_{pi}u_{pi})\} + \frac{\partial}{\partial x_j}\{\rho_p(e_p+\tfrac{1}{2}u_{pi}u_{pi})\} =$$

$$u_{pi}F_{pi} - Q - \mu_v\{e_p+h_\ell+\tfrac{1}{2}u_{pi}u_{pi}\} \ . \qquad (A.\ 30)$$

The quantity e_p represents the thermodynamic internal energy of the liquid, and h_ℓ represents the latent heat of phase transition which is defined by the relation

$$h_p + h_\ell = h(T_p) \qquad (A.\ 31)$$

where h_p denotes the enthalpy of the liquid per unit mass and $h(T_p)$ is the enthalpy of the gaseous phase per unit mass at the local droplet temperature.

The sum of the energy equations, equations A. 23 and A. 30, with the aid of equations A. 15 and A. 31 gives

$$\frac{\partial}{\partial t}\{\rho(e+\tfrac{1}{2}u_i u_i)+\rho_p(e_p+\tfrac{1}{2}u_{pi}u_{pi})\}+\frac{\partial}{\partial x_j}\{\rho(e+\tfrac{1}{2}u_i u_i)+\rho_p(e_p+\tfrac{1}{2}u_{pi}u_{pi})\} = \frac{\partial}{\partial x_j}(\sigma_{ij}u_i)-\frac{\partial q_i}{\partial x_i} \qquad \text{(A. 32)}$$

which is the energy equation for the composite system. This equation has the form of the energy equation for a single-phase system and shows that there is no net production or destruction of energy.

To complete the formulation, it is necessary to develop the appropriate expressions for the forces and heat and mass transfer that occur between the liquid and gas phases. In the course of examining this, it will prove necessary to extend the number of transport equations that have been developed. We will, from the beginning, make the assumptions that the droplets are spherical and all of the same size or radius. In addition, the interaction between a single droplet and the gas mixture is summed over the number of droplets to give the total interaction. Further assumptions will arise in the following discussion.

Assume that the force exerted upon a single droplet is given by Stokes law. Then the force f_i exerted upon the gas by a single droplet is

$$f_i = 6\pi\sigma\mu(u_{pi}-u_i) \qquad \text{(A. 33)}$$

where σ is the droplet radius, and μ is the viscosity coefficient of the gas-vapor mixture. If n is the number of droplets per unit volume locally, the total force F_i is simply nf_i. Further, if m denotes the mass of a single droplet, and since all the droplets are assumed to be of the same radius, the local density of the condensed matter is $\rho_p \equiv nm$, and consequently

$$F_i = \frac{6\pi\sigma\mu}{m}\rho_p(u_{pi}-u_i) \quad . \tag{A. 34}$$

The quantity $m/6\pi\sigma\mu$ has the dimension of a time and, in the case of a single free particle of constant radius, it is the time required for the velocity of a droplet relative to the gas to decay to e^{-1} of its initial value. Hence, define the velocity relaxation time to be

$$\tau_v \equiv \frac{m}{6\pi\sigma\mu} \quad . \tag{A. 35}$$

Note that for a given system, τ_v varies as the square of the droplet radius. As a consequence of these assumptions, the volumetric distribution of body force may be written as

$$F_i = \rho_p \frac{u_{pi}-u_i}{\tau_v} \tag{A. 36}$$

so that the only additional quantity which the volumetric forces bring into the problem is the characteristic time τ_v .

Two assumptions, implicit in the above analysis, must be kept in mind. First, the Reynolds number of the droplet motion through the gas is assumed low enough that the Stokes approximation is valid. If this condition is not fulfilled, the characteristic time τ_v becomes a function of Reynolds number; this variation may be carried along numerically in any problem, but will only be mentioned here. The second assumption has to do with the effect of mass transfer upon the drag law. Strictly speaking, within the Stokes approximation, there is no effect and, to higher degrees of approximation, the effect is of the order $\mu_v(u_{pi}-u_i)$. Therefore, to neglect this phenomenon, as we shall do, leads to an inconsistency in the order of terms retained.

Those that have been retained have been done so in order to achieve a certain internal consistency among the equations themselves regardless of the form for the internal forces. In the acoustic problem these terms are of higher order and do not enter in any of the equations, but in other problems where the term $\mu_v(u_{pi}-u_i)$ might be of the same order as other force and inertial terms, an error of this order must be expected if the Stokes law is used.

The processes of heat and mass transfer between the two phases are treated in much the same manner as the interphase forces. The heat transfer rate from a single particle, q, is taken to be

$$q = 4\pi\sigma^2 \frac{k}{\sigma}(T_p - T) \tag{A. 37}$$

where k is the coefficient of thermal conductivity for the gas-vapor mixture. As before, utilizing the number n of droplets locally per unit volume, the total rate of heat transfer from droplets to the gas-vapor mixture is $Q = nq$, which may be written

$$Q = \rho_p c_p \frac{T_p - T}{\tau_v} \tag{A. 38}$$

where the thermal relaxation time τ_T is given by

$$\tau_T = \frac{mc_p}{4\pi\sigma k} \tag{A. 39}$$

and where c_p is the specific heat at constant pressure of the gas-vapor mixture. As in the case of the forces exerted between phases, the effect of mass transfer upon the heat transfer between phases has been neglected in writing equation A. 38. In the acoustic equations, however, these terms do not enter, and will not be discussed further.

The mass transfer between phases μ_v, as in the cases of the other transfer quantities, is assumed continuously distributed throughout the volume. In detail, there are three processes which enter into the rate at which vapor condenses on or vaporizes from a droplet. These are the rate of heat exchange, the rate of vapor transfer, and the kinetic rate of condensation. We shall assume that the condensation rate is infinitely fast in comparison with the other two and consequently is not rate controlling. The approximation is, in principle, that the vapor transfer rate to the liquid surface is slow in comparison with the mass rate of collision of vapor molecules upon the droplet surface. This fact, that the collision rate is very rapid, implies that very close to the droplet surface, the vapor and droplet surface must be in thermodynamic equilibrium. This region is so small that the temperature and composition variation over it are negligible. Thus, at the droplet surface, the partial pressure of vapor must be that corresponding to the surface temperature of the droplet. For the rest of this derivation, we shall consider the droplet bulk to be the same temperature as its surface so the partial pressure of vapor at the droplet surface is the saturation pressure corresponding to the droplet temperature.

From the above observations, it appears that droplet condensation depends only upon the diffusion of vapor to the droplet surface and the conduction away of the resultant heat of condensation. Denote by ρ_v the local density of condensable vapor and by u_D the diffusion velocity of vapor relative to inert gas. Then the mass rate of vapor production from a single droplet is just

$$4\pi\sigma^2 \rho_v u_D$$

where ρ_v and u_D are evaluated at the droplet surface. If there are n droplets per unit volume, the total vapor mass production rate is just

$$\mu_v = n \cdot 4\pi\sigma^2 \rho_v u_D \quad . \tag{A. 40}$$

It is convenient to introduce the mass fraction of vapor K_v based upon the total mass of gaseous species. That is, define

$$K_v \equiv \rho_v / \rho \quad . \tag{A. 41}$$

Then, according to the theory of binary diffusion, the diffusion velocity u_D may be written

$$u_D = - D \frac{1}{K_v} \frac{\partial K_v}{\partial r} \tag{A. 42}$$

where D is the binary diffusion coefficient and $\partial K_v / \partial r$ is the radial vapor concentration evaluated at the droplet surface. We may then write

$$\rho_v u_D = - \rho D \frac{\partial K_v}{\partial r} \quad . \tag{A. 43}$$

This gives for the mass vaporization rate

$$\mu_v = -n\rho D \, 4\pi\sigma^2 \left. \frac{\partial K_v}{\partial r} \right|_\sigma \quad . \tag{A. 44}$$

Under the assumption that the relative motion of the droplets, or the mass flux from the surface, does not appreciably affect the diffusion problem, the surface concentration gradient may be written rigorously as

$$\left. \frac{\partial K_v}{\partial r} \right|_\sigma = \frac{K_v(\infty) - K_v(\sigma)}{\sigma} \tag{A. 45}$$

where $K_v(\infty)$ is the vapor mass fraction far from the droplet, that is, in the gas vapor bulk. Thus, $K_v(\infty) \equiv K_v$, where K_v is understood to be the vapor mass fraction in the medium considered as a continuum. The vapor mass fraction at the droplet is the equilibrium vapor concentration at the ambient mixture pressure and at the droplet temperature. Thus,

$$K_v(\sigma) = K_v^e(T_p) \qquad \text{(A. 46)}$$

where the superscript denotes equilibrium and the argument indicates that it is evaluated at the particle temperature. Utilizing these results, the vapor mass production rate can be written as

$$\mu_v = -n\rho D 4\pi\sigma \{K_v - K_v^e(T_p)\} \quad . \qquad \text{(A. 47)}$$

It is advantageous to treat the vapor production rate in a way analogous to the heat and momentum transfer rates. Define the characteristic concentration equilibration time as

$$\tau_D \equiv \frac{m}{4\pi\sigma\rho D} \quad . \qquad \text{(A. 48)}$$

Then the vapor production rate per unit volume is

$$\mu_v = -\rho_p \frac{K_v - K_v^e(T_p)}{\tau_D} \quad . \qquad \text{(A. 49)}$$

The temperature of the droplets, and consequently the heat transfer between the droplets and the gaseous phase, enters the vapor production rate problem in setting the value of the equilibrium vapor concentration at the droplet surface.

When the vapor is considered as an ideal gas, as it will be here, the equilibrium partial pressure of the vapor is given by the well-known Chausius-Clapeyron relation,

$$\frac{P_v^e(T_p)}{P_{vo}} = \exp\left[-\frac{h_\ell}{RT_o}\left(\frac{T_o}{T_p} - 1\right)\right] \qquad (A.50)$$

where P_{vo} and T_o are the vapor pressure and temperature at some known reference state. However, since the system is in thermodynamic equilibrium, and consequently both vapor and inert gas have the same temperature, then the local equilibrium vapor concentration is

$$K_v^e(T_p) \equiv \frac{P_v^e(T_p)}{P} = K_v^e(T_o)\frac{P_o}{P}\exp\left[-\frac{h_\ell}{RT_o}\left(\frac{T_o}{T_p} - 1\right)\right] \qquad (A.51)$$

where $K_v^e(T_o)$ is the equilibrium vapor mass fraction at the reference temperature and P_o is the total pressure at the reference state. Two new variables have been introduced in the calculation of μ_v, the local mass fraction of vapor K_v, and the vapor mass fraction at the liquid surface $K_v^e(T_p)$. The latter is related to the other dependent variables through equation A. 51. An additional differential equation to account for the local vapor mass fraction can be obtained from the continuity equation for the vapor constituent alone. It is easily shown that

$$\frac{\partial \rho_v}{\partial t} + \frac{\partial}{\partial x_i}(\rho_v u_i) = \mu_v \ . \qquad (A.52)$$

By the definition of the mass fraction, and utilizing the continuity equation of the gas phase, equation A. 4, it follows that

$$\frac{\partial K_v}{\partial t} + u_i\frac{\partial K_v}{\partial x_i} = \frac{1}{\rho}(1-K_v)\mu_v \ . \qquad (A.53)$$

This is the required equation for the local mass fraction of vapor.

In dealing with the relations governing the interaction between phases, we have introduced the three characteristic times τ_v, τ_T, and τ_D. Each of these depends explicitly on the square of the droplet radius. Thus, the droplet radius must be accounted for point by point. This is fairly straightforward in the present case because we have assumed that no droplet nucleation, coagulation, or fractionation takes place in the flow field. Then the continuity equation for the liquid phase, since $\rho_p = nm$, may be written

$$\frac{\partial}{\partial t}(nm) + \frac{\partial}{\partial x_i}(nmu_{pi}) = -\mu_v , \tag{A. 54}$$

and since the number of particles is conserved,

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x_i}(nu_{pi}) = 0 . \tag{A. 55}$$

Thus, if m_o is the particle mass corresponding to the reference radius, equation A. 54 may be written

$$nm_o\left\{\frac{\partial}{\partial t}\left(\frac{\sigma}{\sigma_o}\right)^3 + u_{pi}\frac{\partial}{\partial x_i}\left(\frac{\sigma}{\sigma_o}\right)^3\right\} = -\mu_v . \tag{A. 56}$$

By utilizing equation A. 49 to give an explicit expression for the vapor mass production rate we may write

$$\frac{\partial \Sigma}{\partial t} + u_{pi}\frac{\partial \Sigma}{\partial x_i} = \frac{2}{3}\,\frac{K_v - K_v^e(T_p)}{\tau_{Do}} \tag{A. 57}$$

where

$$\Sigma = (\sigma/\sigma_o)^2 \tag{A. 58}$$

and τ_{Do} is the diffusion equliibration time at the reference state.

This equation completes the relations required for the treatment of heterogeneous gas dynamics with phase change. The transport properties of the gaseous phase, of course, depend upon the thermodynamic state of the gas-vapor mixture. If this dependence is known, then the above equations can be used to solve problems involving heterogeneous flow with mass transfer.

Marble[11] has introduced characteristic lengths in place of the characteristic times τ_v, τ_T, and τ_D, and has shown how to account for the variation in the transport properties of the gas mixture with temperature in a simple way. Define the characteristic lengths

$$\begin{aligned} \lambda_v &= a\,\tau_v \\ \lambda_T &= a\,\tau_T \\ \lambda_D &= a\,\tau_D \end{aligned} \tag{A. 59}$$

If the viscosity of the gas mixture is taken to vary as $\sqrt{T}$, then the velocity characteristic length may be written, since $a/\mu = a_o/\mu_o$,

$$\lambda_v = \frac{ma}{6\pi\sigma\mu} = \frac{m_o a_o}{6\pi\sigma_o\mu_o}\left(\frac{\sigma}{\sigma_o}\right)^2 = \lambda_{vo}\Sigma \; . \tag{A. 60}$$

Similarly, if it is assumed that k , the thermal conductivity of the gas mixture, and ρD are assumed to vary as $\sqrt{T}$, then the other two characteristic lengths may be written as

$$\lambda_T = \lambda_{To}\Sigma \tag{A. 61}$$

and

$$\lambda_D = \lambda_{Do}\Sigma \; . \tag{A. 62}$$

Though in the acoustic problem the characteristic times are used, it is convenient in some cases to use the characteristic lengths to describe the interaction parameters.

APPENDIX B. INTERNAL ENERGY OF DROPLETS

Consider a spherical droplet of radius σ whose surface temperature is a function of time $T_s(t)$. Assume that the density, specific heat, and thermal conductivity are constant. The temperature within the droplet then satisfies the heat conduction equation

$$\frac{\partial T}{\partial t} = \frac{k_\ell}{\rho_\ell c r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right), \quad 0 \le r < \sigma, \ t > 0 \tag{B. 1}$$

subject to the conditions

$$T(\sigma, t) = T_s(t), \qquad t \ge 0 . \tag{B. 2}$$

Define the dimensionless quantities

$$\tau_\ell = \frac{\rho_\ell \sigma^2 c}{k_\ell} \tag{B. 3}$$

and

$$R = r/\sigma . \tag{B. 4}$$

Then equation B. 1 can be written

$$\tau_\ell \frac{\partial T}{\partial t} = \frac{1}{R^2} \frac{\partial}{\partial R}\left(R^2 \frac{\partial T}{\partial R}\right) . \tag{B. 5}$$

Assume that the temperature depends upon time only through the factor $\exp(-i\omega t)$. Then equation (B. 5) is reduced to the ordinary differential equation

$$R^2 \frac{d^2T}{dR^2} + 2R \frac{dT}{dR} + i\omega\tau_\ell R^2 T = 0 . \tag{B. 6}$$

The solution, subject to the condition B. 2, is

$$T(R) = T_s \frac{1}{R} \frac{\sin\phi R}{\sin\phi} \tag{B. 7}$$

where

$$\phi = \sqrt{i\omega\tau_\ell} . \tag{B. 8}$$

The energy in the droplet is given by

$$\epsilon_p = \rho_\ell \int_0^R cT(r)4\pi r^2 dr \quad . \tag{B.9}$$

When integrated using equation B. 7, this gives

$$\epsilon_p = mcT_s \frac{3}{\phi^2} [1-\phi\cot\phi] \quad . \tag{b. 10}$$

Equation B. 10 gives the internal energy of a droplet of mass m whose surface temperature T_s varies with time like $\exp(-i\omega t)$. The parameter ϕ is a complex phase factor. The assumption that T_s is a function of time only is valid if the acoustic wavelength is large compared to the droplet radius. This is certainly true for problems of interest here.

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TABLE I

Some Properties of Potassium and Sodium Droplets in Their Respective Vapors, and of Water Fog in Air

	Units	Potassium in Potassium Vapor		Sodium in Sodium Vapor		Water in Air
T		$1500^{\circ}R$	$2500^{\circ}R$	$1500^{\circ}R$	$2500^{\circ}R$	$60^{\circ}F$
P_V	psia	1.50	1.65	0.23	80	0.26
ρ_ℓ/ρ		1.20×10^4	1.38×10^2	1.44×10^5	5.38×10	8.16×10^2
C	$BTU/lb^{\circ}R$	0.182	0.210	0.300	0.326	1.00
C_p	$BTU/lb^{\circ}R$	0.127	0.126	0.214	0.213	0.240
γ		1.64	1.57	1.62	1.53	1.40
k_ℓ/k		3.0×10^3	1.4×10^3	2.8×10^3	1.7×10^3	2.3×10
Pr		0.75	0.73	0.73	0.73	0.71
h_ℓ/RT		11.8	6.7	14.7	8.5	17.8
Sc		-	-	-	-	0.585

TABLE II

Equilibration Times for Potassium and Sodium Droplets in Their Respective Vapors and for Water Fog in Air

Droplet Diameter		Potassium in Potassium Vapor		Sodium in Sodium Vapor		Water in Air
		1500°R	2500°R	1500°R	2500°R	60°F
10 μ	τ_V(sec)	1.0×10^{-3}	0.63×10^{-3}	1.0×10^{-3}	0.73×10^{-3}	1.2×10^{-3}
	τ_T/τ_V	1.1	1.1	1.1	1.1	1.0
	τ_D/τ_V	-	-	-	-	0.88
	τ_ℓ/τ_V	1.1×10^{-3}	2.3×10^{-3}	1.2×10^{-3}	1.9×10^{-3}	0.54
	τ_P/τ_V	9.5×10^{-3}	5.4×10^{-4}	5.6×10^{-2}	8.7×10^{-4}	1.3×10^{-3}
1 μ	τ_V(sec)	1.0×10^{-5}	0.63×10^{-5}	1.0×10^{-5}	0.73×10^{-5}	1.2×10^{-5}
	τ_T/τ_V	1.1	1.1	1.1	1.1	1.0
	τ_D/τ_V	-	-	-	-	0.88
	τ_ℓ/τ_V	1.1×10^{-3}	2.3×10^{-3}	1.2×10^{-3}	1.9×10^{-3}	0.54
	τ_P/τ_V	9.5×10^{-2}	5.4×10^{-3}	5.6×10^{-1}	8.7×10^{-3}	1.3×10^{-2}

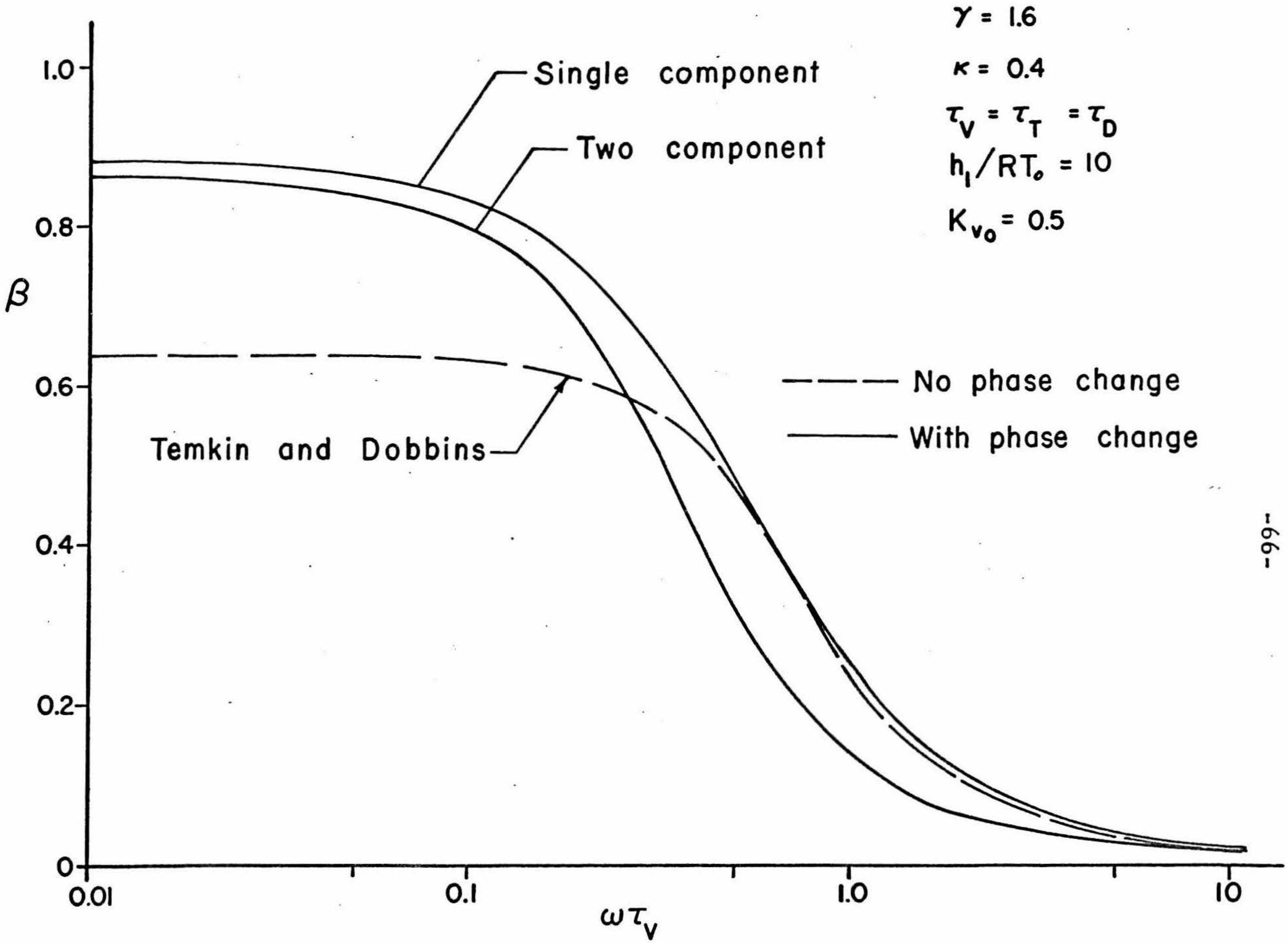


FIGURE I. DISPERSION FOR ONE- AND TWO-COMPONENT SYSTEMS WITH PHASE CHANGE COMPARED TO SYSTEM WITHOUT PHASE CHANGE

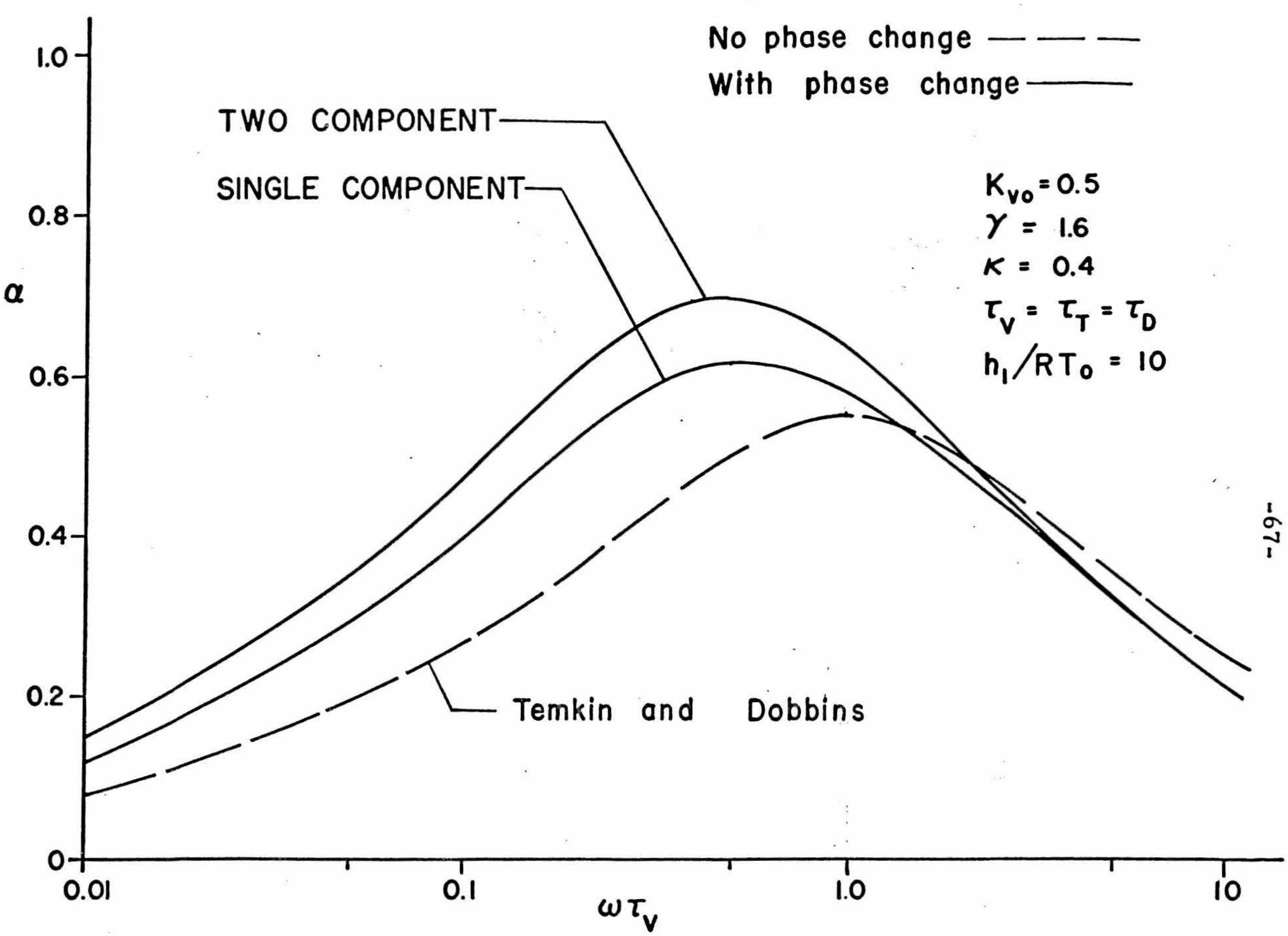


FIGURE 2. ATTENUATION FOR ONE- AND TWO-COMPONENT SYSTEMS WITH PHASE CHANGE COMPARED TO SYSTEM WITHOUT PHASE CHANGE

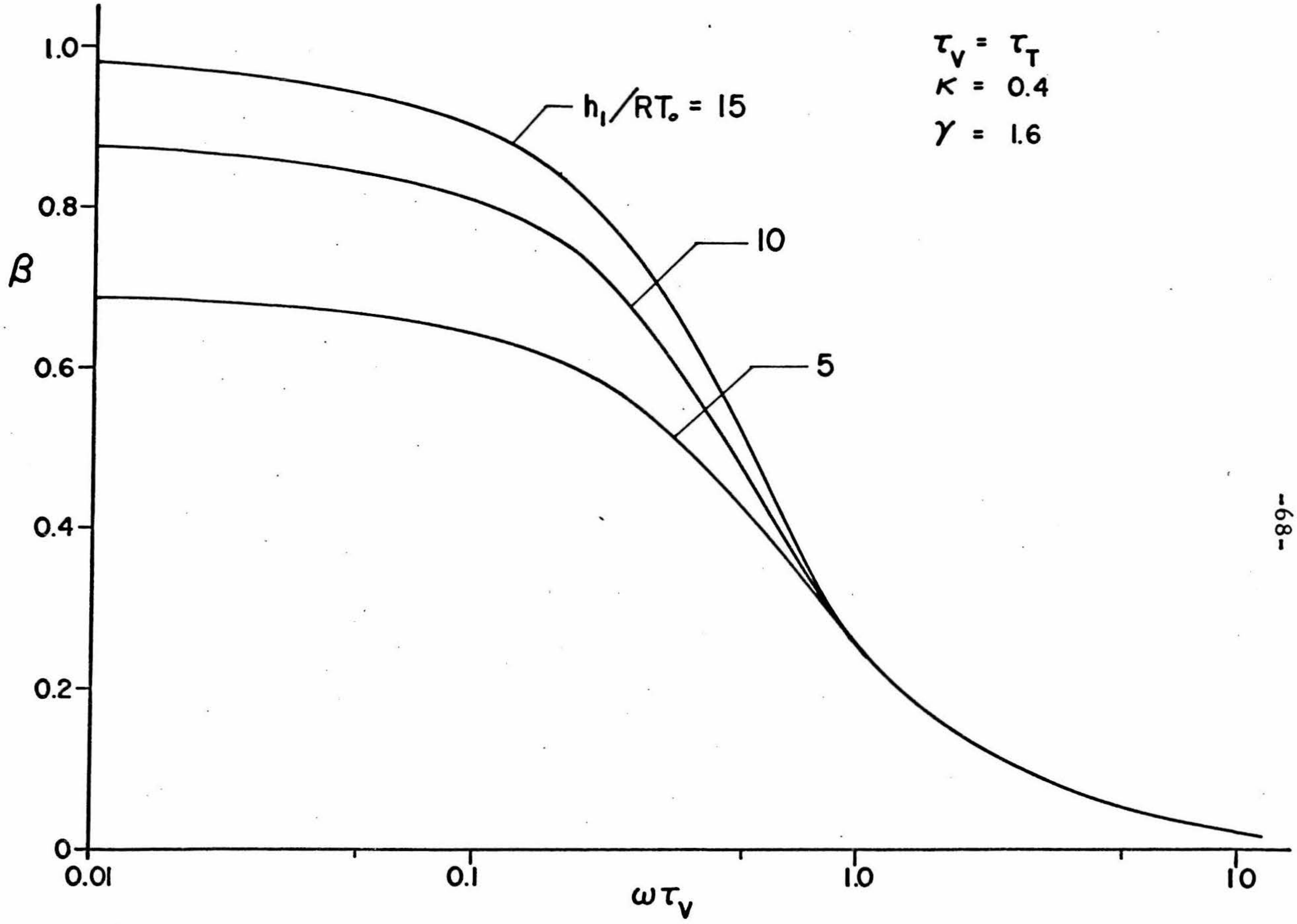


FIGURE 3. DISPERSION COEFFICIENTS FOR SINGLE COMPONENT SYSTEMS

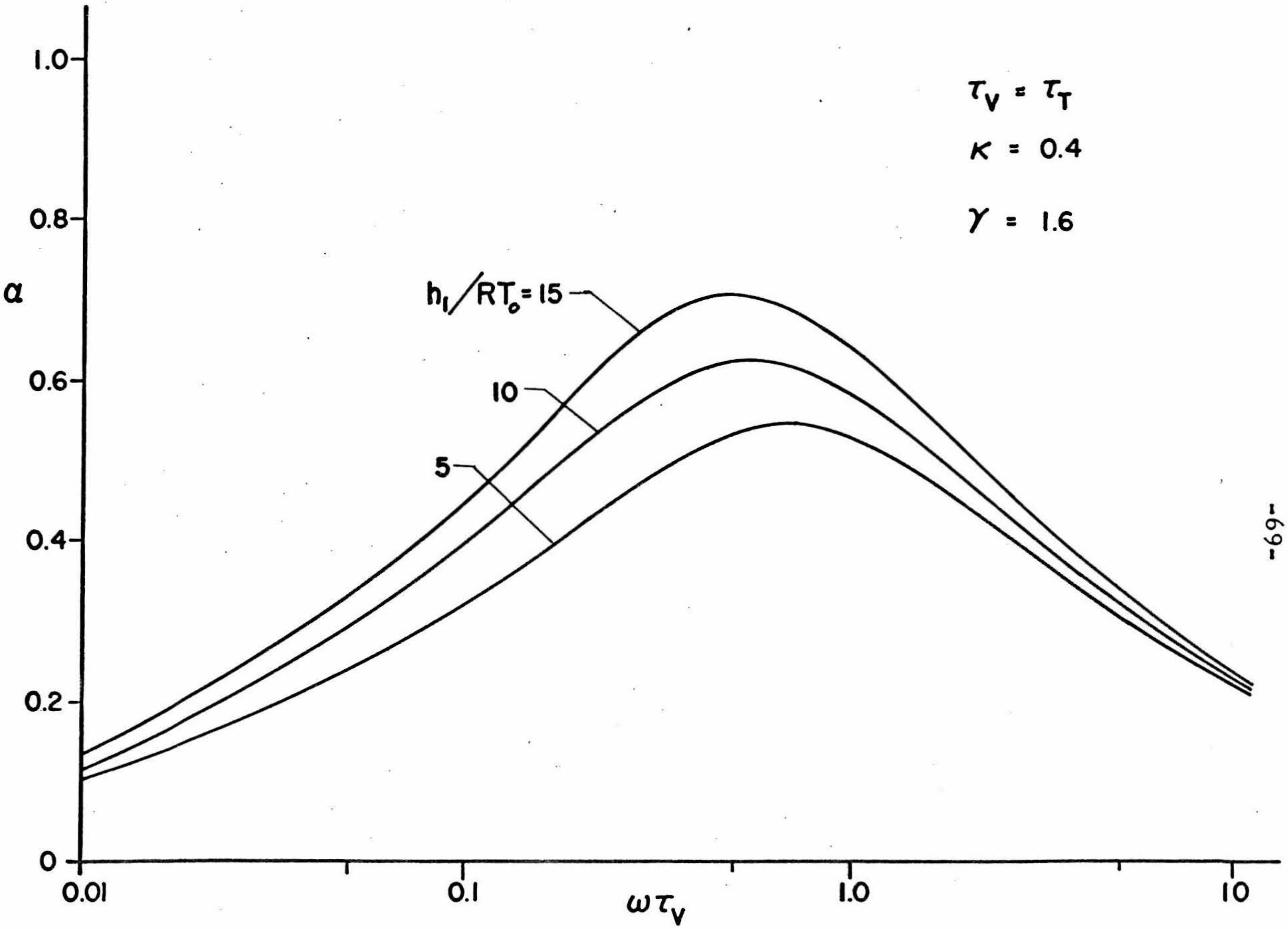


FIGURE 4. ATTENUATION COEFFICIENTS FOR SINGLE COMPONENT SYSTEMS

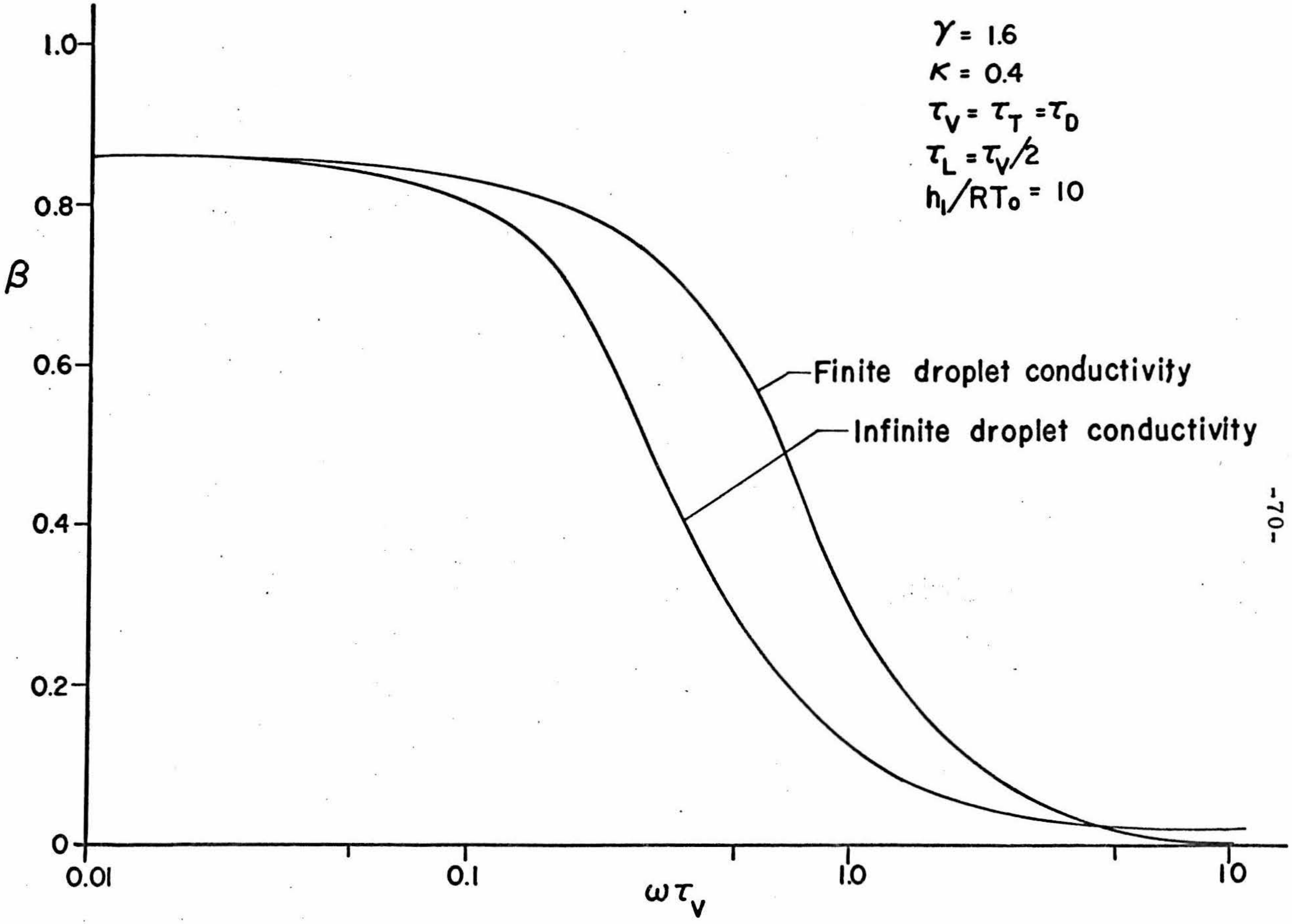


FIGURE 5. DISPERSION FOR TWO-COMPONENT SYSTEM SHOWING THE EFFECT OF THERMAL RELAXATION IN DROPLETS

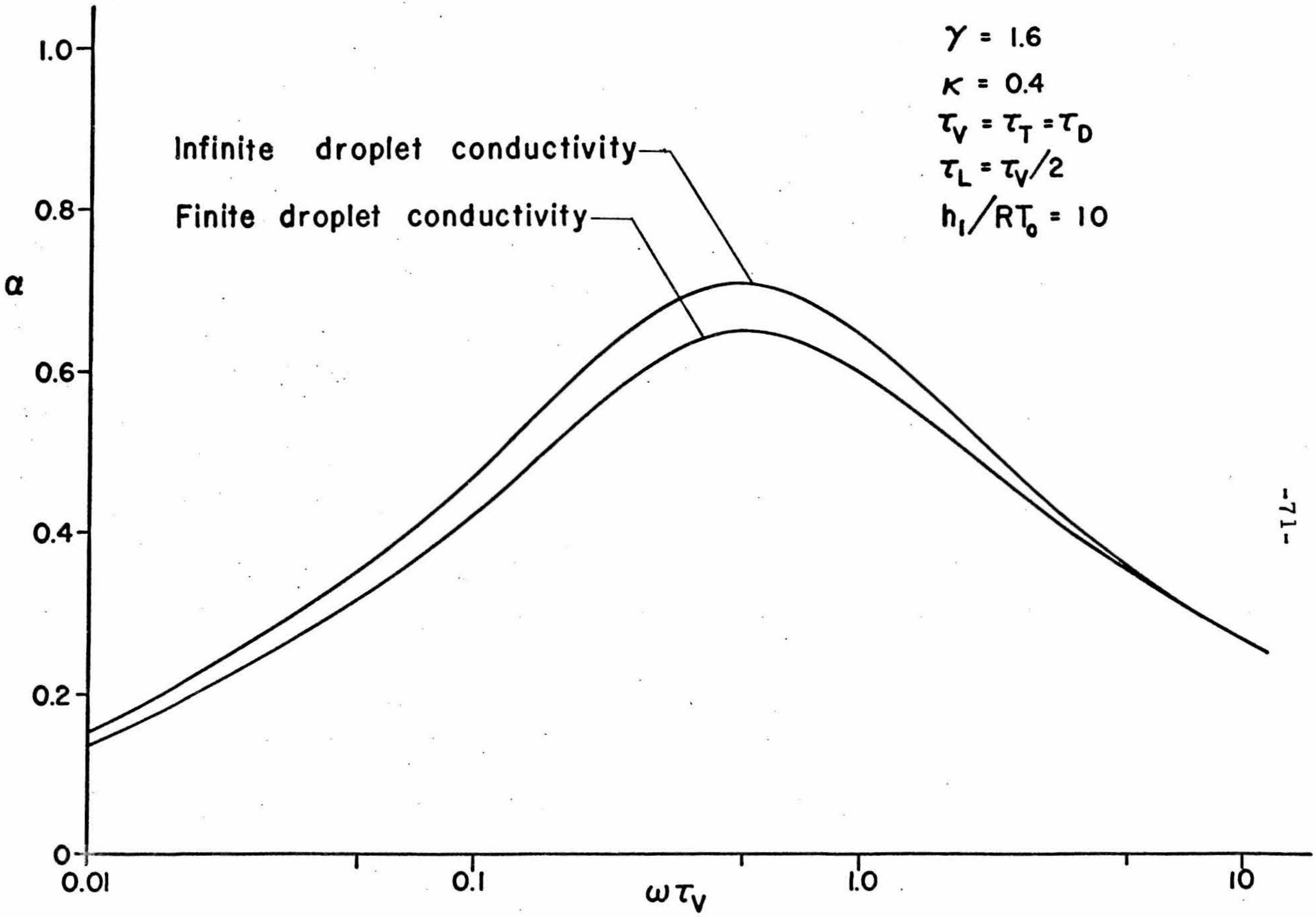
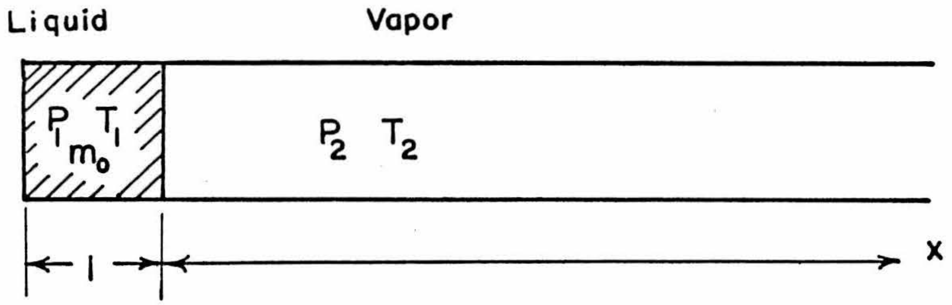
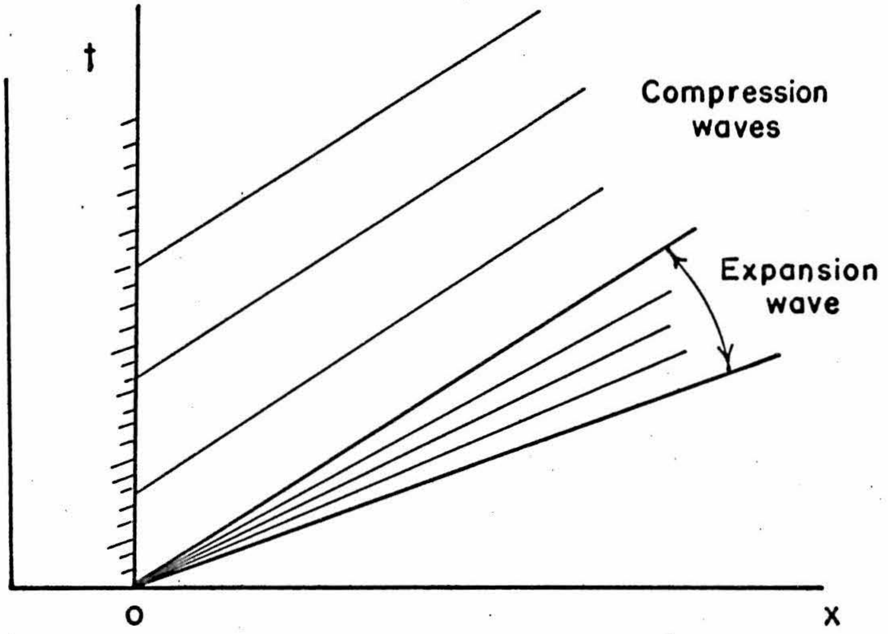


FIGURE 6. ATTENUATION FOR TWO-COMPONENT SYSTEMS SHOWING THE EFFECT OF THERMAL RELAXATION IN DROPLETS



a) INITIAL CONFIGURATION OF CHANNEL (t = 0)



b) WAVE STRUCTURE FOR P_2 P_1. LIQUID SURFACE AT X = 0 ASSUMED STATIONARY

FIGURE 7. FLOW IN ONE-DIMENSIONAL CHANNEL

II. THE FLOW OF A GAS-PARTICLE MIXTURE DOWNSTREAM OF A NORMAL SHOCK IN A NOZZLE

1. INTRODUCTION

The linearized problem of one-dimensional gas-particle flow in a nozzle has been treated by Marble[1, 2] and by Rannie[3]. The linearization was based upon the assumption of small velocity and temperature lag between the gas and particles. The unperturbed state was taken as equilibrium flow where the gas and particles have equal temperatures and velocity. Equilibrium flow was shown to be equivalent to the conventional one-dimensional gas dynamic flow with modified gas properties.

Under the proper conditions, a normal shock may occur in the nozzle, as in the conventional gas dynamic problem. Immediately downstream of the shock the assumption of small lag between the gas and particles is no longer valid,and in this region the above linearized treatment is not applicable.

The normal shock problem for gas-particle flow was originally treated by Carrier[4] and extended by Marble[1]. Solutions were obtained which were applicable to a shock in a constant area duct. The problem was treated under the assumption that the particle thermal and momentum ranges were large compared to the thickness of a gas dynamic shock. Thus, the shock wave in the mixture was treated as a conventional gas dynamic shock followed by a relaxation zone in which the gas and particles come to equilibrium. The thickness of the relaxation zone is of the order of the thermal and momentum ranges of the particles.

If the nozzle area changes appreciably over the relaxation zone downstream of the shock in a divergent nozzle, the shock solution for a constant area duct is no longer adequate, and a solution must be obtained which accounts for the area change. The problem here is to find a solution for a shock in a divergent nozzle and to match this with the small slip solution which is assumed valid far downstream of the shock. The problem is linearized by assuming that the particle thermal and momentum ranges λ_T and λ_v are small compared to the length of the nozzle, which is the length over which the gas properties change appreciably. It is shown that this linearization leads to the appropriate shock solution which can be asymptotically matched with the small slip solution.

The problem is posed as follows. Given a nozzle with a divergent section of length L in which there is a shock, obtain a solution for the gas-particle flow which describes the flow downstream of the shock for λ_v and λ_T small compared to L . Assume that immediately upstream of the shock the gas and particles are flowing in equilibrium and that the flow is known.

2. GOVERNING EQUATIONS

One is referred to the paper by Marble[1] for the formulation of the equations governing one-dimensional gas-particle flow. The equations for the two phases are

$$\rho u A = m \tag{2.1}$$

$$\rho_p u_p A = km \tag{2.2}$$

$$\rho u \frac{du}{dx} + \frac{dP}{dx} = \rho_p a \frac{(u_p - u)}{\lambda_v} \tag{2.3}$$

$$\rho_p u_p \frac{du_p}{dx} = - \rho_p a \frac{(u_p - u)}{\lambda_v} \tag{2.4}$$

$$\rho u c_p \frac{dT}{dx} - u \frac{dP}{dx} = \rho_p c_p a \frac{(T_p - T)}{\lambda_T} + \rho_p a \frac{(u_p - u)^2}{\lambda_v} \tag{2.5}$$

$$\rho_p u_p c \frac{dT_p}{dx} = - \rho_p c_p a \frac{(T_p - T)}{\lambda_T} \tag{2.6}$$

$$P = \rho R T \tag{2.7}$$

The thermal and momentum ranges are λ_T and λ_v, respectively. The subscript p refers to particle quantities except for c_p, the gas specific heat at constant pressure, and c, the specific heat of the solid material of the particles. Both c_p and c are considered constant. The nozzle area A(x), the net gas mass flow rate m, and the mass fractional flow rate of the particles k, are assumed known.

It is convenient to replace equations 2.3 and 2.4 with equations in which interactions between the two phases are absent. Equations 2.3 and 2.4 may be combined to give

$$\rho u \frac{du}{dx} + \rho_p u_p \frac{du_p}{dx} + \frac{dP}{dx} = 0 \tag{2.8}$$

and equations 2.3-2.6 may be combined to give a conventional form of the energy equation

$$\rho u\left[c_p \frac{dT}{dx} + \frac{d}{dx}\left(\frac{u^2}{2}\right)\right] + \rho_p u_p\left[c\,\frac{dT_p}{dx} + \frac{d}{dx}\left(\frac{u_p^2}{2}\right)\right] = 0 \tag{2.9}$$

With the aid of the continuity equations the above equation may be integrated. Thus,

$$\left(c_p T + \frac{u^2}{2}\right) + \left(c T_p + \frac{u_p^2}{2}\right) = \text{const.} \tag{2.10}$$

A set of governing equations equivalent to 2. 1-2. 7 is obtained by replacing 2. 3 and 2. 5 with 2. 8 and 2. 10, respectively.

Assume that λ_v , λ_T , and x are non-dimensionalized in terms of L , which is equivalent to setting L equal to unity. Let the origin of the x-coordinate be at the shock wave.

3. SMALL SLIP SOLUTION

A straightforward perturbation solution of the governing equations, for small λ_v, λ_T leads directly to the linearized problem treated by Marble and Rannie. Since, in general, $\lambda_v \sim \lambda_T$, we may use λ_v as the perturbation parameter, realizing that $\lambda_T/\lambda_v = O(1)$ as $\lambda_v \to 0$. The zeroth order equations are those obtained by setting λ_v equal to zero in the governing equations 2. 1, 2. 2, 2. 4, 2. 6, 2. 7, 2. 8, and 2. 10. From equations 2. 4 and 2. 6, we see that (u_p-u) and (T_p-T) are of order λ_v as $\lambda_v \to 0$. The small parameter λ_v effectively multiplies the derivatives of u_p and T_p, which is the classical warning of the singular behavior of boundary-layer theory. As expected, this restricts the initial values that can be satisfied by this solution.

The small slip or small lag solution is designated as the outer solution. It will turn out to be valid far downstream of the shock. By "far" it is meant a distance large compared to λ_v.

To carry out the solution it is convenient to replace the particle quantities in the governing equations by the slip quantities

$$\begin{aligned} u_s &\equiv u-u_p \\ T_s &\equiv T-T_p \\ \rho_s &\equiv 1-\rho_p/k\rho \end{aligned} \tag{2.11}$$

If modified gas constant and specific heats are defined as follows

$$\begin{aligned} \bar{c}_p &= \frac{c_p+kc}{1+k} \\ \bar{c}_v &= \frac{c_v+kc}{1+k} \end{aligned} \tag{2.12}$$

$$\overline{R} = \overline{c}_p - \overline{c}_v = \frac{R}{1+k}$$

$$\overline{\gamma} = \frac{\overline{c}_p}{\overline{c}_v}$$

four of the appropriate equations for the system are

$$(1+k)\rho u \frac{du}{dx} + \frac{dP}{dx} = k\rho u \frac{du_s}{dx} \tag{2.13}$$

$$\overline{c}_p(T-T_o) + \tfrac{1}{2}(u^2-u_o^2) = \frac{k}{1+k}\left[cT_s + uu_s - \tfrac{1}{2}u_s^2\right] \tag{2.14}$$

$$\rho u A = m \tag{2.15}$$

$$P = (1+k)\rho\overline{R}T \tag{2.16}$$

Equation 2.13 follows from 2.8 and equation 2.14 from 2.10. The reference state u_o, T_o, P_o can be taken as the reservoir condition for flow in a nozzle in which there is no shock. Here, they are determined to all orders by the shock solution, which will be called the inner solution. For the present, consider u_o, T_o, P_o as a yet unspecified reference state.

Equations 2.13-2.16 may be used to write an isentropic integral.

$$\frac{T}{T_o}\left(\frac{P_o}{P}\right)^{\frac{\overline{\gamma}-1}{\overline{\gamma}}} = \exp\left\{\frac{k}{1+k}\int \frac{1}{\overline{c}_p T}\left[c\frac{dT_s}{dx} + u_s\left(\frac{du}{dx} - \frac{du_s}{dx}\right)\right]dx\right\}. \tag{2.17}$$

It will be convenient to take equations 2.14-2.17 as part of the system to be solved.

By substituting the slip quantities in the particle equations 2.2, 2.4, and 2.6, the additional equations required for the solution of the problem are obtained.

$$\left(1 - \frac{\rho_p}{k\rho}\right)u - \left(1 - \frac{\rho_p}{k\rho}\right)u_s + u_s = 0 \tag{2.18}$$

$$u \frac{du}{dx} = \frac{a}{\lambda_v} u_s + u \frac{du_s}{dx} + u_s \frac{du}{dx} - u_s \frac{du_s}{dx} \tag{2.19}$$

$$u \frac{dT}{dx} = \frac{c_p}{c} a \frac{T_s}{\lambda_T} + u \frac{dT_s}{dx} + u_s \frac{dT}{dx} - u_s \frac{dT_s}{dx} \tag{2.20}$$

Equations 2.14-2.20 are equivalent to the original set of equations 2.1-2.7 and are formulated in terms of the gas and slip quantities rather than the gas and particle quantities. The convenience of this is due to the vanishing of zeroth order slip quantities, which can be easily shown from equations 2.18-2.20.

To carry out the perturbation solution, assume that the gas quantities can be expressed as an asymptotic series as follows.

$$\begin{aligned} u(x;\lambda_v) &\sim \tilde{u}^{(0)}(x) + \lambda_v \tilde{u}^{(1)}(x) + \lambda_v^2 \tilde{u}^{(2)}(x) + \dots \\ \rho(x;\lambda_v) &\sim \tilde{\rho}^{(0)}(x) + \lambda_v \tilde{\rho}^{(1)}(x) + \dots \\ P(x;\lambda_v) &\sim \tilde{P}^{(0)}(x) + \lambda_v \tilde{P}^{(1)}(x) + \dots \\ T(x;\lambda_v) &\sim \tilde{T}^{(0)}(x) + \lambda_v \tilde{T}^{(1)}(x) + \dots \end{aligned} \tag{2.21}$$

In addition, assume that the slip quantities can be written

$$\begin{aligned} u_s(x;\lambda_v) &\sim \lambda_v \tilde{u}_s^{(1)}(x) + \lambda_v^2 \tilde{u}_s^{(2)}(x) + \dots \\ \rho_s(x;\lambda_v) &\sim \lambda_v \tilde{\rho}_s^{(1)}(x) + \dots \\ T_s(x;\lambda_v) &\sim \lambda_v \tilde{T}_s^{(1)}(x) + \dots \end{aligned} \tag{2.22}$$

To determine the various order terms, these expansions are substituted in equations 2.14-2.20 which are then separated in like powers of λ_v. Since the shock solution can influence the reference state u_o, T_o, P_o, to all orders of λ_v, it is appropriate to write these quantities as expansions of the form 2.21 and 2.22. However, to simplify the calculations, it is sufficient to keep only the zeroth order of u_o, T_o, P_o. Then the solutions obtained for higher order gas and slip quantities will be correct except for an additive constant. The additive constant is determined by matching with the shock, so there is no benefit in carrying higher order terms of the reference state through the calculations.

Marble[1] has carried out the calculations in some detail. Only the results are given here. The zeroth order part of equations 2. 14-2. 17 determine completely the zeroth order gas quantities. The equations are

$$\bar{c}_p(\tilde{T}^{(0)} - \tilde{T}_o) + \tfrac{1}{2}(\tilde{u}^{(0)2} - \tilde{u}_o^2) = 0$$

or

$$\tilde{T}^{(0)}\left(1 - \frac{\bar{\gamma}-1}{2}\bar{M}^{(0)\,2}\right) = \tilde{T}_o\left(1 - \frac{\bar{\gamma}-1}{2}\bar{M}_o^2\right) \qquad (2.\,23)$$

$$\tilde{\rho}^{(0)}\tilde{u}^{(0)}A = m \qquad (2.\,24)$$

$$\tilde{P}^{(0)} = (1+k)\,\tilde{\rho}^{(0)}\bar{R}\,\tilde{T}^{(0)} \qquad (2.\,25)$$

$$\left(\frac{\tilde{T}^{(0)}}{\tilde{T}_o}\right)\left(\frac{\tilde{P}_o}{\tilde{P}^{(0)}}\right)^{(\bar{\gamma}-1)/\bar{\gamma}} = 0 \qquad (2.\,26)$$

where

$$\bar{M}^{(0)\,2} = \left(\frac{\tilde{u}^{(0)}}{\bar{a}^{(0)}}\right)^2 \qquad (2.\,27)$$

$$\bar{a}^{(0)\,2} = \bar{\gamma}\,\bar{R}\,\tilde{T}^{(0)} \qquad (2.\,28)$$

Clearly, the equations 2. 23-2. 26 represent the conventional "isentropic" flow of a gas of density $(1+k)\tilde{\rho}^{(0)}$ with modified gas properties. It represents equilibrium flow where there is no thermal or velocity slip between the gas and particles.

The first order slip quantities can be obtained directly from the zeroth order parts of equations 2. 18-2. 20 using the known zeroth order gas solution. The results of carrying through the calculation are

$$\tilde{u}_s^{(1)} = \frac{\tilde{u}^{(0)}}{\tilde{a}^{(0)}} \frac{d\tilde{u}^{(0)}}{dx} \tag{2.29}$$

$$\tilde{T}_s^{(1)} = -\frac{1}{\bar{c}_p}\left(\frac{c}{\bar{c}_p}\right)\frac{\lambda_T}{\lambda_v}\,\tilde{u}^{(0)}\,\tilde{u}_s^{(1)} \tag{2.30}$$

$$\tilde{\rho}_s^{(1)} = \frac{-1}{\tilde{a}^{(0)}}\frac{d\tilde{u}^{(0)}}{dx} \tag{2.31}$$

The first order gas quantities are calculated from the first order part of equations 2.13-2.16 using the known first order slip quantities. The first order gas quantities are written in terms of the known zeroth order solution as

$$\frac{\tilde{u}^{(1)}}{\tilde{u}^{(0)}} = -\frac{\tilde{\rho}^{(1)}}{\tilde{\rho}^{(0)}} = -\frac{k}{1+k}\left(\frac{1}{1-\bar{M}^{(0)\,2}}\right)\frac{1}{\bar{\gamma}-1}\left[F(x)-\bar{\gamma}G(x)\right] + \frac{\tilde{u}_1}{\tilde{u}^{(0)}} \tag{2.32}$$

$$\frac{\tilde{T}^{(1)}}{\tilde{T}^{(0)}} = \frac{k}{1+k}\left(\frac{1}{1-\bar{M}^{(0)\,2}}\right)\left[F(x)-\bar{\gamma}\bar{M}^{(0)\,2}G(x)\right] + \frac{\tilde{T}_1}{\tilde{T}^{(0)}} \tag{2.33}$$

$$\frac{\tilde{P}^{(1)}}{\tilde{P}^{(0)}} = \frac{k}{1+k}\left(\frac{\bar{\gamma}}{\bar{\gamma}-1}\right)\frac{1}{1-\bar{M}^{(0)\,2}}\left\{F(x)-\left[1+(\bar{\gamma}-1)\bar{M}^{(0)\,2}\right]G(x)\right\} + \frac{\tilde{P}_1}{\tilde{P}^{(0)}} \tag{2.34}$$

where

$$F(x) = (\bar{\gamma}-1)\bar{M}^{(0)\,2}\left[1-\left(\frac{c}{\bar{c}_p}\right)\frac{\lambda_T}{\lambda_v}\right]\frac{1}{\tilde{a}^{(0)}}\frac{d\tilde{u}^{(0)}}{dx} \tag{2.35}$$

$$G(x) = \int_0^x \frac{1}{\bar{c}_p\tilde{T}^{(0)}}\left\{\left[1-\left(\frac{c}{\bar{c}_p}\right)\frac{\lambda_T}{\lambda_v}\right]\frac{d}{dx}\left(\frac{\tilde{u}^{(0)\,2}}{\tilde{a}^{(0)}}\frac{d\tilde{u}^{(0)}}{dx}\right) - \tilde{u}^{(0)}\frac{d}{dx}\left(\frac{\tilde{u}^{(0)}}{\tilde{a}^{(0)}}\frac{d\tilde{u}^{(0)}}{dx}\right)\right\}dx \tag{2.36}$$

The terms $\tilde{u}_1$, $\tilde{T}_1$, and $\tilde{P}_1$ are constants that will be determined by matching this solution with the shock solution.

The singularity at $\overline{M}^{(0)} = 1$ will present no difficulty here since our region of interest is not near a throat section.

The solution for the gas quantities can clearly satisfy arbitrary initial conditions which determine $\tilde{u}_o$, P_o , $\tilde{T}_o$ and $\tilde{u}_1$, $\tilde{P}_1$, $\tilde{T}_1$. However, the slip quantities cannot independently satisfy initial conditions since, from equations 2. 29-2. 31, they depend entirely upon the zeroth order gas solution.

4. SHOCK SOLUTION

The small slip solution is not valid near the shock since the initial conditions at the shock (x = 0) cannot in general be satisfied. This nonuniformity occurs because the line of reasoning which leads to slip quantities of order λ_v as $\lambda_v \to 0$, is no longer correct. In the region near the shock, gas and particle quantities change very rapidly, so that we can no longer say that terms which appear on the left-hand side of equations 2.3-2.6 are small when multiplied by λ_v. That is, near the shock, $(x \sim \lambda_v)$, $\lambda_v (du_p/dx)$ is no longer arbitrarily small as $\lambda_v \to 0$. The region of nonuniformity of the small slip solution is of thickness of order λ_v. This suggests that a magnified variable be introduced in order to obtain a solution valid at the shock.

Define the magnified coordinate by

$$\xi \equiv x/\lambda_v . \tag{2.37}$$

The governing equations 2.1, 2.2, 2.4, 2.6, 2.7, 2.8, and 2.10 can be written in terms of ξ as follows.

$$\rho u A(\lambda_v \xi) = m \tag{2.38}$$

$$\rho_p u_p A(\lambda_v \xi) = km \tag{2.39}$$

$$\rho u \frac{du}{d\xi} + \rho_p u_p \frac{du_p}{d\xi} + \frac{dP}{d\xi} = 0 \tag{2.40}$$

$$[\bar{c}_p T + \tfrac{1}{2}u^2] + k[cT_p + \tfrac{1}{2}u_p^2] = \text{const.} \tag{2.41}$$

$$u_p \frac{du_p}{d\xi} = -a(u_p - u) \tag{2.42}$$

$$u_p \frac{dT_p}{d\xi} = -a \frac{\lambda_v}{\lambda_T}(T_p - T) \tag{2.43}$$

The area A(x) is written in the continuity equations in terms of the stretched coordinate. We are looking for solutions for small λ_v with ξ of order unity, so it is reasonable to expect that the area $A(\lambda\xi)$ can be expanded as a Maclaurin series

$$A(x) = A^{(0)} + \lambda_v \xi A^{(1)} + (\lambda_v \xi)^2 A^{(2)} + \dots \qquad (2.44)$$

where $A^{(i)}$ $(i = 0, 1, 2, \dots)$, the usual Maclaurin coefficients, are the constants

$$\begin{aligned} A^{(0)} &= A(o) \\ A^{(1)} &= \frac{dA(o)}{dx} \qquad\qquad (2.45) \\ A^{(2)} &= \frac{1}{2!}\,\frac{d^2A(o)}{dx^2} \\ &\dots \end{aligned}$$

Assume the following expansions for the flow quantities:

$$\begin{aligned} u(\xi;\lambda) &\sim u^{(0)}(\xi) + \lambda_v u^{(1)}(\xi) + \lambda_v^2 u^{(2)}(\xi) + \dots \\ \rho(\xi;\lambda) &\sim \rho^{(0)}(\xi) + \lambda_v \rho^{(1)}(\xi) + \dots \\ P(\xi;\lambda) &\sim P^{(0)}(\xi) + \lambda_v P^{(1)}(\xi) + \dots \\ T(\xi;\lambda) &\sim T^{(0)}(\xi) + \lambda_v T^{(1)}(\xi) + \dots \end{aligned}$$

and (2.46)

$$\begin{aligned} u_p(\xi;\lambda) &\sim u_p^{(0)}(\xi) + \lambda_v u_p^{(1)}(\xi) + \dots \\ \rho_p(\xi;\lambda) &\sim \rho_p^{(0)}(\xi) + \lambda_v \rho_p^{(1)}(\xi) + \dots \\ T_p(\xi;\lambda_v) &\sim T_p^{(0)}(\xi) + \lambda_v T_p^{(1)}(\xi) + \dots \end{aligned}$$

Substitution of equations 2.45 and 2.46 into equations 2.38-2.43 and the ideal gas equation yields a set of equations for the various order terms. The zeroth order continuity equations are

$$\rho^{(0)} u^{(0)} A^{(0)} = m \, , \qquad (2.47)$$

$$\rho_p^{(0)} u_p^{(0)} A^{(0)} = km \, . \qquad (2,48)$$

The zeroth order momentum equation can be integrated directly, since the area is constant to the zeroth order. Thus, from equation 2.40, using 2.47 and 2.48,

$$mu^{(0)} + kmu_p^{(0)} + P^{(0)} = (1+k)mu_1 + P_1 \ . \qquad (2.49)$$

The remaining equations are

$$c_pT^{(0)} + \tfrac{1}{2}u^{(0)\,2} + k(cT_p^{(0)} + \tfrac{1}{2}u_p^{(0)\,2}) = (c_p+kc)T_1 + \frac{(1+k)}{2}u_1^2 \qquad (2.50)$$

$$u_p^{(0)}\frac{du_p^{(0)}}{d\xi} = a^{(0)}(u^{(0)} - u_p^{(0)}) \qquad (2.51)$$

$$u_p^{(0)}\frac{dT_p^{(0)}}{d\xi} = a^{(0)}\frac{\lambda_v}{\lambda_T}(T^{(0)} - T_p^{(0)}) \qquad (2.52)$$

$$P^{(0)} = \rho^{(0)}RT^{(0)} \qquad (2.53)$$

The state u_1, P_1, T_1 is the equilibrium state to which the flow would approach far downstream of the shock if it were in a constant area channel. The initial conditions at the shock are calculated from the known upstream conditions using the conventional gas dynamic shock relations. Since the shock thickness is negligible in comparison to λ_v and λ_T, the initial conditions on the particles are just the upstream state. Thus the initial conditions for equations 2.47-2.53 are

$$u^{(0)}(0) = u_u\frac{2}{(\gamma+1)M_u^2}\left(1 + \frac{\gamma+1}{2}M_u^2\right)$$

$$T^{(0)}(0) = T_u\left(1 + \frac{2\gamma}{\gamma+1}(M_u^2-1)\right)\left(1 + \frac{2}{\gamma+1}\,\frac{M_u^2-1}{M_u^2}\right)$$

$$P^{(0)}(0) = 1 + \frac{2\gamma}{\gamma+1}(M_u^2-1) \qquad (2.54)$$

$$u_p^{(0)}(0) = u_u$$

$$T_p^{(0)}(0) = T_u$$

$$\rho_p^{(0)}(0) = k\rho_u$$

where the subscript u refers to the equilibrium state upstream of the shock; M_u is the Mach number of the gas just ahead of the shock.

The quantities u_1, P_1, T_1 can be calculated directly from the upstream state using the momentum and energy conservation equations 2.49 and 2.50.

The numerical solution of equations 2.47-2.53 has been given by Marble[1]. It can be shown that the zeroth order flow approaches the u_1, P_1, T_1 state exponentially.

The first order inner problem is obtained from the first order part of equations 2.38-2.43 and the ideal gas equation.

$$\frac{P^{(1)}}{P^{(0)}} = \frac{\rho^{(1)}}{\rho^{(0)}} + \frac{T^{(1)}}{T^{(0)}} \tag{2.55}$$

$$\frac{\rho^{(1)}}{\rho^{(0)}} + \frac{u^{(1)}}{u^{(0)}} + \frac{A^{(1)}}{A^{(0)}}\xi = 0 \tag{2.56}$$

$$\frac{\rho_p^{(1)}}{\rho_p^{(0)}} + \frac{u_p^{(1)}}{u_p^{(0)}} + \frac{A^{(1)}}{A^{(0)}}\xi = 0 \tag{2.57}$$

$$\begin{aligned}\rho^{(0)}u^{(0)}\frac{du^{(1)}}{d\xi} + \rho_p^{(0)}u_p^{(0)}\frac{du_p^{(1)}}{d\xi} + \frac{dP^{(1)}}{d\xi} + {} \\ + (\rho^{(0)}u^{(1)} + \rho^{(1)}u^{(0)})\frac{du^{(0)}}{d\xi} + (\rho_p^{(0)}u_p^{(1)} + \rho_p^{(1)}u_p^{(0)})\frac{du_p^{(0)}}{d\xi} = 0\end{aligned} \tag{2.58}$$

$$(c_pT^{(1)} + u^{(0)}u^{(1)}) + k(cT_p^{(1)} + u_p^{(0)}u_p^{(1)}) = 0 \tag{2.59}$$

$$u_p^{(0)}\frac{du_p^{(1)}}{d\xi} + u_p^{(1)}\frac{du_p^{(0)}}{d\xi} = a^{(0)}(u^{(1)} - u_p^{(1)}) + a^{(1)}(u^{(0)} - u_p^{(0)}) \tag{2.60}$$

$$u_p^{(0)} \frac{dT_p^{(1)}}{d\xi} + u_p^{(1)} \frac{dT_p^{(0)}}{d\xi} = \frac{\lambda_v}{\lambda_T} [a^{(0)} (T^{(1)} - T_p^{(1)}) + a^{(1)} (T^{(0)} - T_p^{(0)})] \tag{2.61}$$

The initial conditions for equations 2.55-2.61 will be that the first order quantities are zero at the shock (x = 0). That is

$$u^{(1)} = \rho^{(1)} = P^{(1)} = T^{(1)} = u_p^{(1)} = T_p^{(1)} = \rho_v^{(1)} = 0$$

$$\text{at } \xi = 0 \tag{2.63}$$

Using the zeroth order and first order continuity equations, the first order momentum equation 2.58 can be written

$$\frac{du^{(1)}}{d\xi} + k \frac{du_p^{(1)}}{d\xi} + \frac{A(o)}{m} \frac{dP^{(1)}}{d\xi} = \frac{A^{(0)}}{A^{(1)}} \xi \left[\frac{du^{(0)}}{d\xi} + k \frac{du_p^{(0)}}{d\xi} \right] .$$

Eliminate $P^{(1)}$ from this using equations 2.55 and 2.56 to obtain

$$\left(1 - \frac{1}{\gamma M^{(0)\,2}}\right) u^{(1)} + k u_p^{(1)} + \frac{u^{(0)}}{\gamma M^{(0)\,2}} \frac{T^{(1)}}{T^{(0)}} = H(\xi)$$

where, with the aid of the zeroth order momentum equation,

$$H(\xi) = \frac{A^{(1)}}{m} \int_0^{\xi} P^{(0)} \, d\xi \ . \tag{2.64}$$

Now equations 2.59-2.61 and 2.63 can be written in terms of the four quantities $u^{(1)}$, $u_p^{(1)}$, $T^{(1)}$, and $T_p^{(1)}$ as follows.

$$(c_p T^{(1)} + u^{(0)} u^{(1)}) + k (c T_p^{(1)} + u_p^{(0)} u_p^{(1)}) = 0 \tag{2.65}$$

$$\left(1 - \frac{1}{\gamma M^{(0)\,2}}\right) u^{(1)} + k u_p^{(1)} + \frac{u^{(0)}}{\gamma M^{(0)\,2}} \left(\frac{T^{(1)}}{T^{(0)}}\right) = H(\xi) \tag{2.66}$$

$$u_p^{(0)} \frac{du_p^{(1)}}{d\xi} + u_p^{(1)} \frac{du_p^{(0)}}{d\xi} = a^{(0)} (u^{(1)} - u_p^{(1)}) + a^{(1)} (u^{(0)} - u_p^{(0)}) \tag{2.67}$$

$$u_p^{(0)} \frac{dT_p^{(1)}}{d\xi} + u_p^{(1)} \frac{dT_p^{(0)}}{d\xi} = \frac{\lambda_v}{\lambda_T} [a^{(0)} (T^{(1)} - T_p^{(1)}) + a^{(1)} (T^{(0)} - T_p^{(0)})] \tag{2.68}$$

This set of equations may be reduced to a set of two coupled first-order differential equations for $u^{(1)}$ and $u_p^{(1)}$.

Assume that $\lambda_v = \lambda_T$ in the following. Then

$$\begin{aligned} \frac{du_p^{(1)}}{d\xi} &= A_1 u^{(1)} + A_2 u_p^{(1)} + A_3 \\ \frac{du^{(1)}}{d\xi} &= B_1 u^{(1)} + B_2 u_p^{(1)} + B_3 \end{aligned} \tag{2.69}$$

where

$$A_1 = \frac{a^{(0)}}{u_p^{(0)}} \left\{ 1 - \frac{(\gamma M^{(0)\,2} - 1)}{2} \left(1 - \frac{u_p^{(0)}}{u^{(0)}} \right) \right\}$$

$$A_2 = - \frac{a^{(0)}}{u_p^{(0)}} \left\{ 1 + \frac{k\gamma M^{(0)\,2}}{2} \left(1 - \frac{u_p^{(0)}}{u^{(0)}} \right) + \frac{1}{a^{(0)}} \frac{du_p^{(0)}}{d\xi} \right\}$$

$$A_3 = \frac{a^{(0)}}{u_p^{(0)}} \left\{ \frac{\gamma M^{(0)\,2}}{2} \left(1 - \frac{u_p^{(0)}}{u^{(0)}} \right) H(\xi) \right\}$$

$$\begin{aligned} B_1 = - \frac{M^{(0)\,2}}{M^{(0)\,2} - 1} \Bigg\{ & \frac{kc}{c_v} \frac{a^{(0)}}{u_p^{(0)}} \left(1 - \frac{1}{\gamma M^{(0)\,2}} \right) \left[1 + \tfrac{1}{2} \left(1 - \frac{T_p^{(0)}}{T^{(0)}} \right) \right] + \\ & \left(\frac{M^{(0)\,2} - 1}{M^{(0)\,2}} \right) \left(\frac{a^{(0)}}{u_p^{(0)}} + \frac{1}{u^{(0)}} \frac{du^{(0)}}{d\xi} \right) + \frac{d}{d\xi} \left(\frac{M^{(0)\,2} - 1}{M^{(0)\,2}} \right) + \\ & k\gamma \left(1 - \frac{\gamma - 1}{\gamma} \frac{u_p^{(0)}}{u^{(0)}} \right) \left(\frac{a^{(0)}}{u_p^{(0)}} \right) \left[1 - \frac{(\gamma M^{(0)\,2} - 1)}{2} \left(1 - \frac{u_p^{(0)}}{u^{(0)}} \right) \right] \Bigg\} \end{aligned} \tag{2.70}$$

$$\begin{aligned}B_2 = -k\,\frac{M^{(0)\,2}}{M^{(0)\,2}-1}\Bigg\{&\frac{kc}{c_v}\,\frac{a^{(0)}}{u_p^{(0)}}\left[1+\tfrac{1}{2}\left(1-\frac{T_p^{(0)}}{T^{(0)}}\right)\right] + \\ &+\gamma\left(\frac{a^{(0)}}{u_p^{(0)}}+\frac{1}{u^{(0)}}\,\frac{du^{(0)}}{d\xi}\right)\left(1-\frac{\gamma-1}{\gamma}\,\frac{u_p^{(0)}}{u^{(0)}}\right) - \\ -\gamma\left(1-\frac{\gamma-1}{\gamma}\,\frac{u_p^{(0)}}{u^{(0)}}\right)\left(\frac{a^{(0)}}{u_p^{(0)}}\right)&\left[\frac{k\gamma M^{(0)\,2}}{2}\left(1-\frac{u_p^{(0)}}{u^{(0)}}\right)+1+\frac{1}{a^{(0)}}\,\frac{du_p^{(0)}}{d\xi}\right] + \\ &+(\gamma-1)\left[\frac{c}{u^{(0)}\,u_p^{(0)}}\,\frac{dT_p^{(0)}}{d\xi}+\frac{d}{d\xi}\left(\frac{u_p^{(0)}}{u^{(0)}}\right)\right]\Bigg\}\end{aligned}$$

$$\begin{aligned}B_3 = \frac{M^{(0)\,2}}{M^{(0)\,2}-1}\,H(\xi)\Bigg\{&\frac{kc}{c_v}\,\frac{a^{(0)}}{u_p^{(0)}}\left[1+\tfrac{1}{2}\left(1-\frac{T_p^{(0)}}{T^{(0)}}\right)\right] + \\ +\gamma\left[\frac{a^{(0)}}{u_p^{(0)}}+\frac{1}{u^{(0)}}\,\frac{du^{(0)}}{d\xi}\right] - k\gamma&\left(1-\frac{\gamma-1}{\gamma}\,\frac{u_p^{(0)}}{u^{(0)}}\right)\left(\frac{a^{(0)}}{u_p^{(0)}}\right)\left(\frac{\gamma M^{(0)\,2}}{2}\right)\left(1-\frac{u_p^{(0)}}{u^{(0)}}\right)\Bigg\} \\ &+\frac{M^{(0)\,2}}{M^{(0)\,2}-1}\,\frac{dH(\xi)}{d\xi}\end{aligned}$$

Equation 2.69, subject to the initial conditions 2.62, can be integrated straightforwardly by numerical methods, using the A_i and B_i, $(i = 1, 2, 3)$ which are written above in terms of the known zeroth order quantities. In order to asymptotically match this solution with the outer solution, it is only necessary to calculate the asymptotic solution of equation 2.69 for large values of ξ. To accomplish this, it is convenient to reduce the first order system 2.69 to a second order equation for, say, $u_p^{(1)}$. The result is

$$\frac{d^2u_p^{(1)}}{d\xi^2} + p(\xi)\,\frac{du_p^{(1)}}{d\xi} + q(\xi)\,u_p^{(1)} = r(\xi) \qquad (2.71)$$

where

$$p(\xi) = -\left(B_1 + A_2 + \frac{1}{A_1}\frac{dA_1}{d\xi}\right)$$

$$q(\xi) = -\left(A_1B_2 - A_2B_1 - \frac{A_2}{A_1}\frac{dA_1}{d\xi} + \frac{dA_2}{d\xi}\right) \qquad (2.72)$$

and

$$r(\xi) = -\left(A_3B_1 - A_1B_3 + \frac{A_3}{A_1}\frac{dA_1}{d\xi} - \frac{dA_3}{d\xi}\right)$$

The initial conditions for equation 2.71, obtained from 2.62 and 2.69, are

$$u_p^{(1)} = \frac{du_p^{(1)}}{d\xi} = 0 \qquad \text{at } \xi = 0 \; . \qquad (2.73)$$

To obtain the asymptotic form of the solution of equation 2.71, it is necessary to make use of the fact that the zeroth order solution ($u^{(0)}$, $P^{(0)}$, $T^{(0)}$, $u_p^{(0)}$, $T_p^{(0)}$) exponentially approaches the equilibrium values (u_1, P_1, T_1) downstream of the shock. This may be written

$$u^{(0)} = u_1 + \exp$$
$$T^{(0)} = T_1 + \exp$$
$$P^{(0)} = P_1 + \exp$$

and (2.74)

$$u_p^{(0)} = u_1 + \exp$$
$$T_p^{(0)} = T_1 + \exp$$

where "exp" stands for terms that are exponentially small as $\xi \to \infty$.

It is easy to show from the differential equations that

$$\frac{du^{(0)}}{d\xi} = \exp \qquad \frac{du_p^{(0)}}{d\xi} = \exp$$

$$(2.75)$$

$$\frac{dT^{(0)}}{d\xi} = \exp \qquad \frac{dT_p^{(0)}}{d\xi} = \exp$$

as $\xi \to \infty$.

With the aid of equations 2.74 and 2.75, the coefficients A_i and B_i (i = 1, 2, 3) may be written for large ξ

$$
\begin{aligned}
A_1 &= \frac{a_1}{u_1} + \exp \\
A_2 &= -\frac{a_1}{u_1} + \exp \\
A_3 &= \exp \\
B_1 &= -\frac{a_1}{u_1}\left(\frac{1}{M_1^2-1}\right)\left\{M_1^2\left(\frac{kc}{c_v}+1+k\right)-\left(1+\frac{kc}{c_p}\right)\right\} + \exp \\
B_2 &= -\frac{k^2c}{c_v}\left(\frac{a_1}{u_1}\right)\frac{M_1^2}{M_1^2-1} + \exp \\
B_3 &= \frac{M_1^2}{M_1^2-1}\left\{H(\xi)\left(\frac{a_1}{u_1}\right)\left(\frac{kc}{c_v}+\gamma\right)+\gamma\frac{dH}{d\xi}\right\} + \exp
\end{aligned}
\tag{2.76}
$$

The coefficients $p(\xi)$, $q(\xi)$, and $r(\xi)$, then have the limiting form

$$
\begin{aligned}
p(\xi) &= \frac{a_1}{u_1}\frac{1}{M_1^2-1}\left\{M_1^2\left(\frac{kc}{c_v}+2+k\right)-\left(2+\frac{kc}{c_p}\right)\right\} + \exp \\
q(\xi) &= \left(\frac{a_1}{u_1}\right)^2\left(\frac{1}{M_1^2-1}\right)\left\{M_1^2\left(\frac{kc}{c_v}+1\right)(1+k)-\left(1+\frac{kc}{c_p}\right)\right\} + \exp \\
r(\xi) &= \left(\frac{a_1}{u_1}\right)\left\{H(\xi)\left(\frac{a_1}{u_1}\right)\left(\frac{kc}{c_v}+\gamma\right)+\gamma\frac{dH}{d\xi}\right\}\frac{M_1^2}{M_1^2-1} + \exp
\end{aligned}
\tag{2.77}
$$

The expression for $r(\xi)$ can be further simplified if $H(\xi)$ is written in its limiting form for large ξ.

Instead of the equation 2.74, $P^{o}(\xi)$ may be written

$$P^{(o)}(\xi) = P_1 + P_m(\xi) \tag{2.78}$$

where

$$P_m(\xi) = O(e^{-\xi}) \qquad \text{as} \qquad \xi \to \infty .$$

Then $H(\xi)$, given by equation 2.64, can be written

$$H(\xi) = \frac{A^{(1)}}{m} P_1 \xi + \frac{A^{(1)}}{m} \int_0^{\xi} P_m d\xi \ . \tag{2.79}$$

Using equation 2.77, $r(\xi)$ may be written in the simpler form

$$r(\xi) = C_1 + C_2 \xi + \exp \tag{2.80}$$

where

$$\begin{aligned} C_1 &= \frac{M_1^2}{M_1^2 - 1} \frac{A^{(1)}}{m} \left\{ \left(\frac{a_1}{u_1}\right)^2 \left(\frac{kc}{c_v} + \gamma\right) \int_0^{\infty} P_m d\xi + \gamma \frac{a_1}{u_1} P_1 \right\} \\ C_2 &= \frac{M_1^2}{M_1^2 - 1} \left(\frac{a_1}{u_1}\right)^2 \left(\frac{kc}{c_v} + \gamma\right) \left(\frac{A^{(1)}}{m} P_1\right) \end{aligned} \tag{2.81}$$

The solution of equation 2.71, subject to the conditions 2.73, is simply the particular solution given by

$$u_1^{(1)*} = \left(\frac{C_1}{q_1} + p_1 \frac{C_2}{q_1^2}\right) + \frac{2}{q_1} \xi \qquad \text{as} \quad \xi \to \infty \ , \tag{2.82}$$

where p_1 and q_1 are just the parts of p and q in equations 2.77 which are of order unity as $\xi \to \infty$.

It is easy to show that $u_p^{(1)*}$ is an asymptotic solution for $u_p^{(1)}$ as $\xi \to \infty$.

The coefficient of ξ may be written in a form that is particularly convenient for matching with the outer solution. If the substitutions are carried out, the result is

$$\frac{C_2}{q_1} = \frac{u_1}{\overline{M}_1^2 - 1} \frac{A^{(1)}}{A^{(0)}} \tag{2.83}$$

where

$$\overline{M}_1^2 = u_1^2 / (\bar{\gamma} \bar{R} T_1) \ . \tag{2.84}$$

The asymptotic form for $u^{(1)}(\xi)$ may be obtained in a similar way. It is given by

$$u^{(1)*} = \left(\frac{C_1'}{q_1} + p_1 \frac{C_2}{q_1^2}\right) + \frac{C_2}{q_1}\xi \ . \tag{2.85}$$

The only difference between this and the expression 2.82 for $u_p^{(1)*}$ is in the term C_1', which is given by

$$C_1' = \frac{a_1}{u_1}\left(\frac{M_1^2}{M_1^2-1}\right)\left(\frac{A^{(1)}P_1}{m}\right)\left\{\frac{a_1}{u_1}\left(\gamma+\frac{kc}{c_v}\right)\int_0^\infty \frac{P_m}{P_1}\,d\xi + \left(2\gamma+\frac{kc}{c_v}\right)\right\} \tag{2.86}$$

The asymptotic form for the slip can be calculated from 2.82 and 2.85,

$$u_s^{(1)*} = \left(u^{(1)*} - u_p^{(1)*}\right) = \frac{C_1'-C_1}{q_1} \ . \tag{2.87}$$

Carrying out the calculation gives the asymptotic form for the slip as

$$u_s^{(1)*} = \frac{u_1^2}{a_1}\left(\frac{A^{(1)}}{A^{(0)}}\right)\left(\frac{1}{1-M_1^2}\right) \qquad \text{as } \xi \to \infty \ . \tag{2.88}$$

5. MATCHING OF THE INNER AND OUTER SOLUTIONS

Asymptotic matching of the inner or shock solution with the outer or small slip solution follows the methods of Van Dyke[5]. Each flow quantity in the inner solution must be asymptotically matched with the corresponding quantity in the outer solution. To illustrate the procedure, use the gaseous velocity u . The two-term outer expansion for the gas velocity is

$$u \sim \tilde{u}^{(0)}(x) + \lambda \tilde{u}^{(1)}(x) \tag{2.89}$$

and the two-term inner solution is

$$u \sim u^{(0)}(\xi) + \lambda\, u^{(1)}(\xi) \; . \tag{2.90}$$

Matching requires that

$$\tilde{u}^{(0)}(0) = u^{(0)}(\infty) \tag{2.91}$$

and

$$\xi \frac{d\tilde{u}^{(0)}(0)}{dx} + \tilde{u}^{(1)}(0) = u^{(1)}(\xi) \qquad \text{as } \xi \to \infty \; . \tag{2.92}$$

Recall that $\tilde{u}_o$ is the zeroth order reference state for the outer or small slip solution. Then, using equations 2.74, equation 2.91 requires that

$$\tilde{u}_o = u_1 \tag{2.93}$$

Using equations 2.85 and 2.83, the condition 2.92 requires both that

$$\tilde{u}^{(1)}(0) = \left(\frac{C_1'}{q_1} + p_1 \frac{C_2}{q_1^2}\right) \tag{2.94}$$

and

$$\frac{d\tilde{u}^{(0)}(0)}{dx} = \frac{C_2}{q_1} = \frac{A^{(1)}}{A^{(0)}} \frac{u_1}{\bar{M}_1^2 - 1} \; . \tag{2.95}$$

Thus, the additive constant $\tilde{u}_1$ in equation 2.35 is determined by the matching condition 2.94. Similarly, the other additive constants appearing in the first order outer solutions 2.33 and 2.34 can be determined by matching with the corresponding inner solution.

Application of the well known relations of one-dimensional gas dynamics to the zeroth order outer flow, described by equations 2.23-2.26, gives the following

$$\frac{d\tilde{u}^{(0)}}{dx} = \frac{\tilde{u}^{(0)}}{A} \frac{-1}{1-\overline{M}^{(0)\,2}} \frac{dA}{dx}$$

or, at x = 0 ,

$$\frac{d\tilde{u}^{(0)}(0)}{dx} = \frac{-\tilde{u}_o}{A^{(0)}} \frac{A^{(1)}}{1-\overline{M}_o^2} = \frac{-A^{(1)}}{A^{(0)}} \frac{u_1}{1-\overline{M}_1^2} \tag{2.96}$$

where equation 2.93 has been used. Thus the condition required by 2.95 is satisfied. Condition 2.95 does not involve a matching which determines some unknown constant. Rather, it is a requirement that must come out of the calculations if the asymptotic matching of solutions is to be valid.

6. COMPOSITE EXPANSION

The composite solution is constructed from the inner and outer expansions in such a way as to be a uniformly valid solution for small λ_v . One convenient representation for the composite expansion is

$$u(x) \sim [\tilde{u}^{(0)}(x) - \tilde{u}^{(0)}(0) + u^{(0)}(\xi)] +$$

$$+ \lambda_v [\tilde{u}^{(1)}(x) + u^{(1)}(\xi) - \tilde{u}^{(1)}(0) - \xi \frac{d\tilde{u}^{(0)}(0)}{dx}] \, . \quad (2.97)$$

7. APPLICATION TO A CONICAL NOZZLE

Assume that a shock exists at a known location in a conical nozzle whose area ratio is given by

$$\frac{A}{A_o} = \left(1 + \frac{\alpha}{2} x\right)^2 \tag{2.98}$$

where A_o is the area of the nozzle at the shock. The parameter α is assumed to be of order unity. The gas-particle flow just upstream of the shock is assumed to be a known equilibrium flow. The problem is to calculate the flow far downstream of the shock, using the assumption that λ_v is small compared to the nozzle length.

First, the inner, or shock, solution is calculated since it is completely determined by the known upstream conditions. The upstream variables at the shock are denoted by u_u, P_u, T_u. Then the values of $u(o)$, $P(o)$, $T(o)$ are easily calculated from the conventional shock relations, equations 2.54, and the particle values are taken as $u_p(o) = u_u$ and $T_p(o) = T_u$. These quantities serve as the initial conditions on the inner solution.

The area is expanded in a power series as in equations 2.44 and 2.45.

$$A(x) = A_o\left(1 + \alpha x + \frac{\alpha^2}{4} x^2 + \ldots\right)$$

or

$$A(\lambda\xi) = A_o\left(1 + \alpha\lambda\xi + \frac{\alpha^2\lambda^2}{4} \xi^2 + \ldots\right) . \tag{2.99}$$

The coefficients defined in equation 2.45 are

$$\begin{aligned} A^{(0)} &= A_o \\ A^{(1)} &= A_o\alpha \\ A^{(2)} &= A_o\alpha^2/4 \end{aligned} \tag{2.100}$$

The zeroth order shock solution was carried out by Marble[1], using the equations 2.49-2.53. The zeroth order quantities approach an equilibrium state u_1, T_1, P_1, exponentially. The values of u_1, T_1, P_1 can be calculated directly[1] from the upstream conditions, using the conservation of energy and momentum. A little manipulation of equations 2.49-2.53 then yields the hodograph relation

$$\frac{d(u^{(0)}-u_1)}{d(u_p^{(0)}-u_1)} = F(u^{(0)}, u_p^{(0)}) \qquad (2.101)$$

where

$$F = -(u^{(0)}-u_p)\left[\frac{\bar{\gamma}}{kc(\bar{\gamma}-1)}\left(u^{(0)}-\frac{1}{\gamma M_1^2}u_1\right)-\frac{1}{2kc}(u^{(0)}+u_1)\right]$$

$$-(u_p^{(0)}-u_1)\left[\frac{\bar{\gamma}}{c(\bar{\gamma}-1)}u^{(0)}+\frac{1}{2c}(u_p^{(0)}+u_1)\right]- \qquad (2.102)$$

$$-\frac{\frac{1}{c}[(u^{(0)}-u_1)-(u_p^{(0)}-u_1)][\frac{\gamma}{\gamma-1}(u^{(0)}-u_1)+(u_p^{(0)}-u_p)+\frac{1}{\gamma-1}u_1]}{\frac{1}{kc}[(u^{(0)}-u_1)-(u_p^{(0)}-u_1)][\frac{2\gamma-1}{\gamma-1}(u^{(0)}-u_1)+\frac{\gamma M_1^2-2}{2M_1^2(\gamma-1)}+\frac{k\gamma}{\gamma-1}(u_p^{(0)}-u_1)]}\ .$$

Equation 2.101 may then be numerically integrated in the hodograph plane. To obtain the dependence upon the physical coordinate ξ, use is made of equation 2.51 to write

$$\frac{u_p^{(0)}\,du_p^{(0)}}{a^{(0)}(u^{(0)}-u_p^{(0)})} = d\xi \qquad (2.103)$$

The zeroth order speed of sound can be calculated in terms of $u^{(0)}$ and $u_p^{(0)}$ using equations 2.49 and 2.53. The result in terms of $T^{(0)}$ is

$$T^{(0)} - T_1 = -\frac{u^{(0)}}{R}\left[(u^{(0)} - u_p) + k(u_p^{(0)} - u_1)\right] + \frac{T_1}{u_1}(u^{(0)} - u_1) \ . \tag{2.104}$$

Equation 2.103 can then be integrated numerically to get the zeroth order inner solution.

The first order inner solution can be obtained from a numerical integration of equations 2.69. In practice, however, the details of the first order inner solution would most likely not be required. If only the first order correction factors for the small slip or outer solution reference state are required, then it is only necessary to calculate the asymptotic form of the first order inner solution. Thus, directly from equations 2.32, $\tilde{u}^{(1)}(0)$ may be written

$$\tilde{u}^{(1)}(0) = -\frac{k}{1+k}\left(\frac{M_1^2}{1-M_1^2}\right)\left(1-\frac{c}{c_p}\right)\frac{u_1}{a_1}\frac{du^{(0)}(0)}{d\xi} + \tilde{u}_1$$

and by the matching condition 2.94

$$\tilde{u}_1 - \frac{k}{1+k}\left(\frac{M_1^2}{1-M_1^2}\right)\left(1-\frac{c}{c_p}\right)\frac{u_1}{a_1}\frac{du^{(0)}(0)}{d\xi} = \frac{C_1'}{q_1} + p_1\frac{C_2}{q_1^2} \ . \tag{2.105}$$

Using the familiar gas dynamic equations, the term $\frac{du^{(0)}(0)}{d\xi}$ may be written in terms of the area

$$\frac{d\tilde{u}^{(0)}(0)}{d\xi} = -\frac{u_1}{1-\bar{M}_1^2}\frac{1}{A(0)}\frac{dA(0)}{d\xi} = -\frac{u_1}{1-\bar{M}_1^2}\left(\frac{\alpha}{A_o}\right) \ . \tag{2.106}$$

Then $\tilde{u}_1$ is given by

$$\tilde{u}_1 = \left(\frac{C_1'}{q_1} + \frac{p_1C_2}{q_1^2}\right) - \frac{k}{1+k}\left(\frac{M_1^2}{1-M_1^2}\right)\left(1-\frac{c}{c_p}\right)\left(\frac{M_1^2}{1-\bar{M}_1^2}\right)\frac{\alpha a_1}{A_o} \tag{2.107}$$

where

$$\frac{C_1'}{q_1} = \frac{a_1\alpha}{A_o}\left(\frac{M_1^2}{1-\bar{M}_1^2}\right)\left[\frac{\gamma}{\bar{\gamma}}\left(\frac{a_1}{u_1} + I_1\right) + \frac{c_p}{(1+k)\bar{c}_p}\right]$$

$$\frac{p_1 C_2}{q_1^2} = \frac{a_1\alpha}{A_o}\left(\frac{M_1^2}{(1-\bar{M}_1^2)^2}\right)\left[\frac{\bar{M}_1^2}{1+k}\left(1 + \frac{c_v}{\bar{c}_v}\right) - \left(1 + \frac{c_p}{(1+k)\bar{c}_p}\right)\right] \qquad (2.108)$$

and

$$I_1 = \int_0^\infty \frac{P_m}{P_1}\, d\xi = \int_0^\infty \left[\frac{P^{(o)}(\xi)}{P_1} - 1\right] d\xi$$

Similarly, the other first order corrections to the small slip solution $\tilde{T}_1$ and $\tilde{P}_1$ (see equations 2. 33 and 2. 34) may be calculated.

All the terms in 2. 107 except I_1 are written directly from the easily calculated u_1 , P_1 , T_1 state. The integral I_1 contains the zeroth order inner solution implicitly. Thus, directly from the zeroth order inner solution, the zeroth order reference state $\tilde{u}_o$, $\tilde{P}_o$, $\tilde{T}_o = (u_1 , P_1 , T_1)$ and the first order corrections u_1 , P_1 , T_1 for the outer solution may be calculated. It is not necessary to calculate the detailed first order inner solution unless second order terms of the outer solution are required.

8. POSITION OF SHOCK IN A NOZZLE

In the case of conventional gas dynamics, it is a straightforward calculation to determine the location of a shock in a nozzle. The flow is assumed to be isentropic, except across the shock, and the shock is positioned so that the conditions at the nozzle exit are satisfied. This problem is considered in detail by Crocco[6], and will not be repeated here, except in connection with the corresponding problem for two-phase flow.

In the small slip approximation, that is, the asymptotic solution as $\lambda_v \rightarrow 0$, the location of the shock in an ideal nozzle can be calculated to various orders of λ_v. Assume for the present discussion that the coordinate x is measured from the nozzle throat, increasing toward the nozzle exit, and let x_s denote the location of the normal shock. Assume that x_s can be expanded in powers of λ_v as follows.

$$x_s = x_s^{(0)} + \lambda_v x_s^{(1)} + \lambda_v^2 x_s^{(2)} + \ldots \tag{2.109}$$

where, as before, λ_v has been normalized with respect to the nozzle length. Then $x_s^{(0)}$ is the position of the shock for $\lambda_v = 0$, or for equilibrium flow. The zeroth order or equilibrium flow, as mentioned earlier, is analogous to the conventional isentropic flow of a gas of density $(1+k)\rho$, with modified thermal properties. This flow is described by the zeroth order equations, equations 2.23-2.28. Clearly, $x_s^{(0)}$ is determined in the same way as the shock location problem for conventional gas dynamics. Once the shock location is known to the zeroth order, then the first order slip quantities can be directly calculated from equations 2.29-2.31.

The next step is to calculate the first order correction, $x_s^{(1)}$, to the shock location. The first order correction does not follow directly from the previous calculations without a closer look at the flow conditions. The stagnation conditions upstream of the nozzle are assumed fixed so the flow, for a given system, is completely specified for an ideal nozzle by giving the exit pressure, P_e. The specification of P_e, then, determines the equilibrium flow and the zeroth order location of the shock. Now, if we calculate the first order correction to the gaseous pressure from equation 2. 34, assuming that the shock location is fixed, the exit pressure will be changed by a first order term, say $P_e^{(1)}$. By assumption, however, the exit pressure of the nozzle is fixed, so a correction must be made, which amounts to moving the location of the shock by a first order amount. This may be done in two ways which are actually equivalent.

One way is to modify the zeroth order flow to allow for the first order pressure correction brought about by the slip between phases. This is done by calculating the zeroth order flow for a new exit pressure $P_e' = P_e - \lambda_v P_e^{(1)}$, which changes the shock location by an amount of order λ_v. The corresponding change in the first order gas calculation is of order λ_v^2, so the exit pressure, calculated with the shock at its corrected location will be $P_e + O(\lambda_v^2)$. Higher order corrections may be made by repeating this process but, strictly speaking, terms involving the second order gas flow must be included.

An alternative method is to fix the zeroth order flow, then to move the shock location in such a way that the first order pressure correction, calculated from equation 2.34, gives a zero correction at the nozzle exit. Then, again, the exit pressure will differ from P_e by, at most, an amount of order λ_v^2.

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