A GENERALIZED HAUSDORFF DIMENSION FOR FUNCTIONS AND SETS

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ABSTRACT

Let E be a compact subset of the n-dimensional unit cube, I_n, and let C be a collection of convex bodies, all of positive n-dimensional Lebesgue measure, such that C contains bodies with arbitrarily small measure. The dimension of E with respect to the covering class C is defined to be the number

$$d_C(E) = \sup(\beta : H_{\beta,C}(E) > 0) ,$$

where $H_{\beta,C}$ is the outer measure

$$\inf(\sum m(C_i)^{\beta} : \cup C_i \supseteq E, \ C_i \in C) .$$

Only the one and two-dimensional cases are studied. Moreover, the covering classes considered are those consisting of intervals and rectangles, parallel to the coordinate axes, and those closed under translations. A covering class is identified with a set of points in the left-open portion, I_n', of I_n, whose closure intersects $I_n - I_n'$. For $n = 2$, the outer measure $H_{\beta,C}$ is adopted in place of the usual:

$$\inf(\sum (\text{diam. } (C_i))^{\beta} : \cup C_i \supseteq E, \ C_i \in C) ,$$

for the purpose of studying the influence of the shape of the covering sets on the dimension $d_C(E)$.

If E is a closed set in I_1, let $M(E)$ be the class of all non-decreasing functions $\mu(x)$, supported on E with $\mu(x) = 0$, $x \leq 0$ and $\mu(x) = 1$, $x \geq 1$. Define for each $\mu \in M(E)$,

$$d_C(\mu) = \liminf_{c \to 0} \frac{\log \Delta\mu(c)}{\log c}, \qquad (c \in C)$$

where $\Delta\mu(c) = \bigvee_x (\mu(x+c) - \mu(x))$. It is shown that

$$d_C(E) = \sup(d_C(\mu) : \mu \in M(E)) .$$

This notion of dimension is extended to a certain class $\mathfrak{F}$ of subadditive functions, and the problem of studying the behavior of $d_C(E)$ as a function of the covering class C is reduced to the study of $d_C(f)$ where $f \in \mathfrak{F}$. Specifically, the set of points in I_2,

$$(*) \qquad \{(d_B(f), d_C(f)) : f \in \mathfrak{F}\}$$

is characterized by a comparison of the relative positions of the points of B and C. A region of the form (*) is always closed and doubly-starred with respect to the points (0, 0) and (1, 1). Conversely, given any closed region in I_2, doubly-starred with respect to (0, 0) and (1, 1), there are covering classes B and C such that (*) is exactly that region. All of the results are shown to apply to the dimension of closed sets E. Similar results can be obtained when a finite number of covering classes are considered.

In two dimensions, the notion of dimension is extended to the class M, of functions f(x, y), non-decreasing in x and y, supported on I_2 with $f(x, y) = 0$ for $x \cdot y = 0$ and $f(1, 1) = 1$, by the formula

$$d_C(f) = \liminf_{s \cdot t \to 0} \frac{\log \Delta f(s, t)}{\log s \cdot t}, \qquad (s, t) \in C$$

where

$$\Delta f(s,t) = \bigvee_{x,y} \left(f(x+s, y+t) - f(x+s, y) - f(x, y+t) + f(x, t)\right) .$$

A characterization of the equivalence $d_{C_1}(f) = d_{C_2}(f)$ for all $f \in M$, is given by comparison of the gaps in the sets of products $s \cdot t$ and quotients s/t, $(s,t) \in C_i$ $(i = 1, 2)$.

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INTRODUCTION

Let E be a subset of the n-dimensional unit cube In and let S be a collection of convex bodies, all of positive n-dimensional Lebesgue measure, such that S contains bodies with arbitrarily small measure. The dimension of E with respect to the covering class S is defined to be the number

$$d_S(E) = \sup(b : H_b(E) > 0) \quad ,$$

where H_b is the outer measure

$$\inf\left(\sum (m(C_i))^b : \cup C_i \supseteq E, \; C_i \in S\right).$$

The dimension $d_S(E)$ is always a number between 0 and 1. Moreover it is monotone with respect to the covering class in the sense that $S \subseteq T$ implies $d_S(E) \geq d_T(E)$.

The purpose of this dissertation is to study the behavior of $d_S(E)$ as a function of the covering class S. Only the one and two-dimensional cases are considered. In the one-dimensional case convex sets are intervals but the question could arise as to the behavior of $d_S(E)$ when covering classes other than those consisting of intervals are considered. Although this question is for the most part unanswered, some statement can be made.

Theorem. *If S is any covering class of open sets which contains the intervals and their finite unions, then $d_S(E) = 0$ or $d_S(E) = 1$, as E has measure zero or positive outer measure.*

Proof: If E has positive outer measure p, then any covering (C_i) by sets in S has the property that,

$$0 < p \leq \sum m(C_i) \leq \sum m(C_i)^b \qquad (b \leq 1)$$

and so $H_b(E) > 0$, $b \leq 1$, which shows $d_S(E) = 1$. On the other hand, if E has measure zero, there is a sequence of intervals (I_j) such that

$$m(\cup I_j) < \epsilon\,, \quad \sum m(I_j) < \infty\,, \quad \cup I_j \supseteq E\,.$$

For any $r > 0$, there is a seq. N_k such that

$$\sum_{k=1}^{\infty} \left(\sum_{j=N_{k-1}+1}^{N_k} m(I_j) \right)^r < \infty.$$

Thus

$$\sum_{k=1}^{\infty} \left[m\left(\bigcup_{j=N_{k-1}+1}^{N_k} I_j \right) \right]^r < \infty,$$

and so let m be such that

$$\sum_{k=m+1}^{\infty} \left[m\left(\bigcup_{j=N_{k-1}+1}^{N_k} I_j \right) \right]^r < \epsilon^r,$$

then

$$H_r(E) \leq \left(m\left(\bigcup_{j=1}^{N_m} I_j \right) \right)^r + \sum_{k=m+1}^{\infty} \left(m\left(\bigcup_{j=N_{k-1}+1}^{N_k} I_j \right) \right)^r$$
$$< 2\epsilon^r$$

Since ϵ was arbitrary and $r > 0$, $H_r(E) = 0$, and $d_S(E) = 0$.

Since part of the interest of a dimension such as (1) lies in the study of sets of Lebesgue measure zero, it appears that for this purpose the covering classes cannot be too large with respect to the intervals. The fact that the intervals themselves do not form too large a covering class was proved by Hausdorff [11]. The approach taken in this study is motivated by these facts and so covering classes are considered here to be collections of intervals, closed under translation, which contain intervals of arbitrarily small length. Thus a covering class is completely determined by the lengths of its members.

Of great importance to this study is the fact that the study of dimension with respect to covering classes which consist of intervals can be reduced to the study of a dimension of a certain class of increasing functions. The notion of the dimension of a function considerably facilitates the investigations and carries a certain interest for its own sake.

In two dimensions the situation is more interesting because the shape of the covering sets, along with the area, plays a fundamental role. Although the results are fragmentary, certain indications of the role of shape can be made. As a first attempt, covering classes which consist of rectangles are considered and an analysis made of the influence of shape on the dimension.

When S is the collection of all convex bodies, the expression (1) is called the Hausdorff dimension of E. This dimension function on closed sets has been studied in connection with the theory of Trigonometric Series, for example, by Bari [1] and Kahane and Salem [5]. For certain applications to Number Theory and some interesting properties of the Hausdorff dimension see Besocovitch [8] and Eggleston [9]. Finally, for the relationship of the Hausdorff dimension of a closed set to the topological dimension see Hurewitz and Wallman [4].

CHAPTER I

PRELIMINARIES

§1. Sub-additive Functions

A real valued function $f(x)$, defined on $x \geq 0$, is said to be *sub-additive* provided that

$$f(x+y) \leq f(x) + f(y) ,$$

for all $x, y \geq 0$. Only non-negative functions are considered.

Lemma 1. *If* $f(x)$ *is sub-additive and non-decreasing for* $x \geq 0$, *then for all* $t \geq 0$

$$f(tx) \leq 2(t \vee 1) f(x) .$$

Proof: There is a non-negative integer n, such that $n \leq t < n+1$. Thus

$$f(tx) \leq f((n+1)x) \leq (n+1) f(x) \leq (t+1) f(x) \leq 2(t \vee 1) f(x) .$$

In particular, if $t = \frac{1}{x}$, $0 < x \leq 1$,

$$f(1) \leq \frac{2}{x} f(x) , \qquad \text{or}$$

$$f(x) \geq x f(1)/2 . \tag{1.1}$$

A second fact, due to Hille [3], useful in the following, is

Lemma 2. *If* $f(t)/t$ *is non-increasing for* $t > 0$, *then* $f(t)$ *is sub-additive.*

Proof:

$$f(x+y) = x\frac{f(x+y)}{x+y} + y\frac{f(x+y)}{x+y}$$

$$\leq x\frac{f(x)}{x} + y\frac{f(y)}{y} = f(x) + f(y) .$$

A sub-additive function which will be important for later considerations is the function

$$\Delta f(s) = \bigvee_x \left(f(x+s) - f(x)\right) , \tag{1.2}$$

where f is supposed non-decreasing. The fact that Δf is sub-additive can be seen by writing,

$$f(x+s+t) - f(x) \leq \left(f(x+t+s) - f(x+t)\right) + \left(f(x+t) - f(x)\right)$$

$$\leq \Delta f(s) + \Delta f(t) ,$$

and so,

$$\Delta f(s+t) \leq \Delta f(s) + \Delta f(t) .$$

Further, if f itself is sub-additive, then

$$f(x+s) - f(x) \leq f(s) ,$$

and so if $f(0) = 0$, $\Delta f(s) = f(s)$.

§2. The Class M

Let M denote the class of all real-valued, non-decreasing functions $f(x)$, defined on $x \geq 0$ such that $f(0) = 0$, and Δf is bounded on $[0, 1]$.

Define a transformation T on M as follows:

$$(1.3) \qquad T(f)(x) = \begin{cases} x \bigwedge_{0<y\leq x} \dfrac{f(y)}{y} , & \text{if} \quad x > 0 \\ 0 , & x = 0 . \end{cases}$$

Some elementary properties of T are listed below.

Theorem 1.

a) $T(f) \leq f$, $f \in M$.

b) $T(f)(x)/x$ is non-increasing for $x > 0$.

c) $T(M) \subseteq M$.

d) $T(f)$ is continuous at each $x > 0$.

Proof: a) Since $\bigwedge_{y\leq x} f(y)/y \leq f(x)/x$, it follows that

$$T(f)(x) = x \bigwedge_{y\leq x} \frac{f(y)}{y} \leq f(x) ,$$

when $x > 0$. For $x = 0$, the same is true.

b) If $x > 0$, $T(f)(x)/x = \bigwedge_{y\leq x} f(y)/y$, which is evidently non-increasing.

c) $T(f)(0) = 0$ by definition. Let $x, t > 0$, then

$$(x+t) \bigwedge_{y\leq x+t} \frac{f(y)}{y} = (x+t)\left(\bigwedge_{y\leq x} \frac{f(y)}{y} \wedge \bigwedge_{x\leq y\leq x+t} \frac{f(y)}{y}\right)$$

$$\geq x \bigwedge_{y\leq x} \frac{f(y)}{y} \wedge (x+t)\frac{f(x)}{x+t}$$

$$\geq x \bigwedge_{y\leq x} \frac{f(y)}{y} ,$$

which shows that $T(f)$ is non-decreasing. By Lemma 2, § 1 and the remarks following it, parts a) and b), above, imply

$$\Delta T(f) = T(f) \leq f ,$$

so that

$$\Delta T(f)(x) < f(x) \le f(1) , \qquad (0 \le x \le 1) .$$

d) Let $x_0 > 0$ and write $g = T(f)$. If $x > x_0$, then

$$\frac{g(x_0)}{x} \le \frac{g(x)}{x} \le \frac{g(x_0)}{x_0}$$

by c) and so,

$$0 \le g(x) - g(x_0) \le \left(\frac{x}{x_0} - 1\right) g(x_0) .$$

On the other hand, if $x < x_0$,

$$\frac{g(x)}{x_0} \le \frac{g(x_0)}{x_0} \le \frac{g(x)}{x} ,$$

which implies

$$0 \le g(x_0) - g(x) \le \left(\frac{x_0}{x} - 1\right) g(x) \le \left(\frac{x_0}{x} - 1\right) g(x_0) .$$

Thus,

$$|g(x_o) - g(x)| \le g(x_0) \left[\left(\frac{x}{x_0} - 1\right) \vee \left(\frac{x_0}{x} - 1\right)\right]$$

which shows that g is continuous at x_0.

If $f \in M$ and if $f(t)/t$ is non-increasing, it is clear that $T(f) = f$. In the case that $f(x)$ is only sub-additive, there is the following estimation.

<u>Theorem 2.</u> <u>If</u> $f \in M$ <u>is sub-additive, then</u>

$$f(x)/2 \le T(f)(x) \le f(x) .$$

Proof: The fact that $T(f) \leq f$ was just proved above. By Lemma 1,

$$f(x) = f(xy/y) \leq 2(1 \vee x/y)\, f(y) .$$

If $y \leq x$, then $f(x) \leq 2x \frac{f(y)}{y}$, and so

$$f(x) \leq 2x \bigwedge_{y \leq x} \frac{f(y)}{y} = 2T(f)(x) .$$

Denote by $\mathfrak{F}$, the class $T(M)$, then $\mathfrak{F}$ has the following properties:

Theorem 3.

a) If $f \in M$, then $f \in \mathfrak{F}$ if and only if $f(t)/t$ is non-increasing.

b) If $f, g \in \mathfrak{F}$ and $0 \leq \alpha, \beta, \alpha + \beta \leq 1$, then $f^{\alpha} g^{\beta} \in \mathfrak{F}$, where 0^0 is defined to be zero.

c) If $f_a \in \mathfrak{F}$ for all $a \in A$, then

$$\bigwedge_{a \in A} f_a \in \mathfrak{F} \text{ and } \bigvee_{a \in A} f_a \in \mathfrak{F}, \text{ provided these exist.}$$

Proof: a) is proved by Theorem 1, c) and the remark following it.

b) Let $t > x$. Then

$$\frac{f^{a}(t) \cdot g^{\beta}(t)}{t} = \left(\frac{f(t)}{t}\right)^{\alpha} \cdot \left(\frac{g(t)}{t}\right)^{\beta} \cdot t^{\alpha+\beta-1}$$

$$\leq \left(\frac{f(x)}{x}\right)^{\alpha} \cdot \left(\frac{g(x)}{x}\right)^{\beta} t^{\alpha+\beta-1}$$

$$= \frac{f^{\alpha}(x) \cdot g^{\beta}(x)}{x} \left(\frac{t}{x}\right)^{\alpha+\beta-1}$$

$$\leq \frac{f^{\alpha}(x)\, g^{\beta}(x)}{x} .$$

c) If f_a is a non-decreasing for all $a \in A$, then so are $\bigwedge_{a \in A} f_a$ and $\bigvee_{a \in A} f_a$, and both are zero at $x = 0$. By a) it is sufficient to show $\left(\bigwedge_{a \in A} f_a(t)\right)/t$ and $\left(\bigvee_{a \in A} f_a(t)\right)/t$ are non-increasing for $t > 0$. Indeed, since $f_a \in \mathfrak{F}$, for $x < t$,

$$\frac{\left(\bigwedge_{a \in A} f_a(t)\right)}{t} = \bigwedge_{a \in A} \frac{f_a(t)}{t} < \bigwedge_{a \in A} \frac{f_a(x)}{x} = \frac{\left(\bigwedge_{a \in A} f_a(x)\right)}{x}$$

and

$$\frac{\left(\bigvee_{a \in A} f_a(t)\right)}{t} \leq \frac{\left(\bigvee_{a \in A} f_a(x)\right)}{x}$$

for the same reasons.

CHAPTER II

DIMENSION OF FUNCTIONS

§1. Covering Classes, Dimension

A non-empty set S of points in $(0, 1]$ is called a covering class, provided that S has 0 as a limit point.

Definition: Given a function $f \in M$, $f \not\equiv 0$, and a covering class S, the dimension of f with respect to S is defined to be the number,

$$d_S(f) = \liminf_{s \to 0} \frac{\log \Delta f(s)}{\log s} \qquad (s \in S) , \tag{2.1}$$

where Δf is defined by (1. 2).

Remark 1: Observe that if $f \in M$, and $f \not\equiv 0$, then $\Delta f(x) > 0$, if $x > 0$. Indeed, if $f(x_0) > 0$, for some $x_0 > 0$ and $\Delta f(x) = 0$ for some positive x, then let $n \geq 1$ be such that $x_0 \leq nx$. Then

$$f(x_0) \leq f(nx) = \sum_{k=1}^{n} f(kx) - f((k-1)x)$$

$$\leq n \, \Delta f(x) = 0 ,$$

which is a contradiction. Thus $d_S(f)$ is well-defined.

Remark 2: The dimension, $d_S(f)$, is always between 0 and 1. Since $f \in M$, $0 \leq \Delta f(s) \leq C$, for some constant C, when $s \in S$. Thus

$$\frac{\log \Delta f(s)}{\log s} \geq \frac{\log C}{\log s} = o(1) \qquad s \to 0 ,$$

and so $d_S(f) = 0$. On the other hand, since it was observed in § 1, Chapter 1, that Δf is sub-additive,

$$\frac{\log \Delta f(s)}{\log s} \leq \frac{\log s \, \Delta f(1)/2}{\log s} = 1 + \frac{\log \Delta f(1)/2}{\log s} = 1 + o(1) \qquad s \to 0 ,$$

and so by (1. 1) it follows that $d_S(f) \leq 1$. Moreover, x^α is in M, for $0 < \alpha \leq 1$ and $d_S(x^\alpha) = \alpha$. The function x^o defined by

$$x^o = \begin{cases} 1, & x > 0 \\ 0, & x = 0 , \end{cases}$$

satisfies $d_S(x^o) = 0$. Therefore $d_S(f)$ can take any value in $[0, 1]$.

The following can be used as an alternate definition for $d_S(f)$.

<u>Theorem.</u> $d_S(f) = \sup(\beta : s^{-\beta} \Delta f(s) = O(1),\ s \in S)$.

<u>Proof:</u> Let $r(S, f) = \sup(\beta : s^{-\beta} \Delta f(s) = O(1),\ s \in S)$, and suppose that $d_S(f) > 0$. If $0 \leq \beta < d_S(f)$, then for $s \in S$ and s small enough,

$$\frac{\log \Delta f(s)}{\log s} > \beta ,$$

which implies $\Delta f(s) \leq s^\beta$. In this case, $s^{-\beta} \Delta f(s) \leq 1$ when $s \in S$, s small enough so that $\beta \leq r(S, f)$. Since $r(S, f) \geq 0$,

$$d_S(f) \leq r(S, f) .$$

On the other hand, suppose now that $r(S, f) > 0$, and take $0 \leq \beta < r(S, f)$.

Then $$s^{-\beta}\,\Delta f(s) \le M\,, \qquad s \in S\,,$$

for some constant M, and so

$$\frac{\log\,\Delta f(s)}{\log\,s} \ge \frac{\log\,M}{\log\,s} + \beta\,,$$

which implies $d_S(f) \ge \beta$. Thus $d_S(f) \ge r(S,f)$, and equality follows.

§2. An Equivalence Relation on M

For the study of the dimension $d_S(f)$ when $f \in M$, there is a natural equivalence relation induced on M which considerably simplifies the work. Namely, two functions f and g in M will be called equivalent, in symbols $f \sim g$, provided that $d_S(f) = d_S(g)$ for all covering classes S. The main result concerning this equivalence is the following:

Theorem 1. Given $f \in M$, $f \not\equiv 0$, there is a function $g \in \mathfrak{F}$, such that $f \sim g$, that is, $d_S(f) = d_S(g)$ for all covering classes S.

Proof: If $f \in M$, define $g = T(\Delta f)$ where T is the transformation defined by (1. 3). Since Δf is sub-additive, Theorem 2 (§2, I), gives

$$T(\Delta f) \le \Delta f \le 2T(\Delta f)\,.$$

By Theorem 1, c) of that section and chapter, $T(\Delta f)(x)/x$ is non-increasing and $T(\Delta f)$ is therefore sub-additive by Lemma 2 (§1, I). Thus the relation,

$$\frac{\log\,2}{\log\,s} + \frac{\log\,T(\Delta f)(s)}{\log\,s} \le \frac{\log\,\Delta f(s)}{\log\,s} \le \frac{\log\,T(\Delta f)(s)}{\log\,s}\,,$$

implies that $d_S(T(\Delta f)) = d_S(f)$, for all covering classes S, or $f \sim T(\Delta f) \in \mathfrak{F}$.

The advantage of being able to replace $f \in M$ by an equivalent g in $\mathfrak{F}$ is that formula (2. 1) simplifies to:

$$d_S(g) = \liminf_{s \to 0} \frac{\log g(s)}{\log s} \qquad (s \in S) .$$

For functions f, g in $\mathfrak{F}$ which are not identically zero, the equivalence $f \sim g$ has an interesting interpretation. Recall that by Remark 1, if $f \in \mathfrak{F}$ and $f \not\equiv 0$, then $f(x) > 0$ when $x > 0$.

Theorem 2. If $f, g \in \mathfrak{F}$, and neither f nor g is identically zero, then $f \sim g$, if and only if,

$$\lim_{x \to 0} \frac{\log \frac{f(x)}{g(x)}}{\log x} = 0 .$$

Proof: Let $a(x) = \frac{\log f(x)}{\log x}$, $b(x) = \frac{\log f(x)}{\log x}$. For every covering class S,

$$\begin{aligned} \limsup_{s \to 0} (a(s)-b(s)) &= -\liminf_{s \to 0} (b(s)-a(s)) \\ &\geq \liminf_{s \to 0} a(s) - \liminf_{s \to 0} b(s) \\ &\geq \liminf_{s \to 0} (a(s)-b(s)), \qquad (s \in S) \end{aligned}$$

since $\liminf (A+B) \geq \liminf A + \liminf B$. If $\lim_{x \to 0} \frac{\log f(x)/g(x)}{\log x} = 0$, it follows that

$$d_S(f) = \liminf a(s) = \liminf b(s) = d_S(g) .$$

Conversely, if $\limsup_{x \to 0} (a(x)-b(x)) > 0$, then there exists S such that

$$\liminf_{s \to 0} (a(s)-b(s)) > 0 , \qquad (s \in S)$$

which implies

$$\liminf_{s \to 0} a(s) > \liminf_{s \to 0} b(s) \qquad (s \in S) .$$

This would contradict $f \sim g$, so that

$$\limsup_{x \to 0} (a(s)-b(x)) \leq 0 .$$

A similar argument shows that $f \sim g$ implies

$$\limsup_{x \to 0} (b(x)-a(x)) \leq 0 ,$$

and so $\lim_{x \to 0} |a(x) - b(x)| = 0$, which completes the proof.

For future use, the following lemma is established.

<u>Lemma 2.</u> <u>If</u> $f \in \mathfrak{F}$ <u>and</u> $0 \leq \alpha \leq 1$, <u>then</u>

$$d_S(f \wedge x^\alpha) = \alpha \vee d_S(f)$$

<u>and</u>

$$d_S(f \vee x^\alpha) = \alpha \wedge d_S(f) .$$

<u>Proof:</u> Since both f and x^α are in $\mathfrak{F}$, Theorem 3 (§2, I) implies that $f \wedge x^\alpha$ and $f \vee x^\alpha$ are in $\mathfrak{F}$. Consequently

$$d_S(f \wedge x^\alpha) = \liminf_{s \to 0} \frac{\log f(s) \wedge s^\alpha}{\log s}$$

and $\qquad (s \in S)$

$$d_S(f \vee x^\alpha) = \liminf_{s \to 0} \frac{\log f(s) \vee s^\alpha}{\log s}$$

Since
$$\frac{\log f(s) \wedge s^{\alpha}}{\log s} = \frac{\log f(s)}{\log s} \vee \alpha$$

and
$$\frac{\log f(s) \vee s^{\alpha}}{\log s} = \frac{\log f(s)}{\log s} \wedge \alpha \ ,$$

it follows that
$$d_S(f \wedge x^{\alpha}) = \alpha \vee d_S(f)$$

and
$$d_S(f \vee x^{\alpha}) = \alpha \wedge d_S(f) \ .$$

§3. Special Functions in $\mathfrak{F}$

Given a covering class S and any point $s \in S$, define
$$s^* = \sup(t : t < s,\ t \in S) \ .$$

If S is a covering class, denote by S^*, the new covering class, $\overline{S} \cap (0, 1]$, where $\overline{S}$ is the closure of S. For $f \in \mathfrak{F}$ define,
$$(f)_S(x) = \bigvee_{s \in S^*} (f(s) \wedge x\, f(s^*)/s^*) \qquad (x \geq 0) \ .$$

The next theorem lists some important properties of $(f)_S$.

Theorem 1. *If $f \in \mathfrak{F}$ and S is a covering class, then:*

a) $(f)_S \in \mathfrak{F}$.

b) $d_S((f)_S) = d_S(f)$.

c) $d_T((f)_S) \leq d_T(f)$, for all covering classes T.

Proof: a) For each $s \in S^*$, the function
$$f(s) \wedge x\, f(s^*)/s^*$$
is in $\mathfrak{F}$. By Theorem 3, c) (§2, I) it follows that $(f)_S \in \mathfrak{F}$.

b) Since $s^* \leq s$ and $f \in \mathfrak{F}$, it follows that $f(s) \leq s\, f(s^*)/s^*$ and so,

$$f(s) = f(s) \wedge s\, f(s^*)/s^* .$$

If $t \in S^*$ and $t > s$, then $t^* \geq s$, so that

$$f(t) \wedge s\, f(t^*)/t^* \leq s\, f(t^*)/t^* \leq f(s) .$$

On the other hand, if $t \in S^*$ and $t \leq s$, then $f(t) \leq f(s)$, so that

$$f(s) = \bigvee_{t \in S^*} \left(f(t) \wedge s\, f(t^*)/t^* \right) = (f)_S(s),$$

holds for $s \in S^*$ and in particular for $s \in S$. This implies that $d_S(f) = d_S((f)_S)$.

c) Finally, observe that

$$f(s) \wedge x\, f(s^*)/s^* \geq f(x) ,$$

whenever $s^* \leq x \leq s$, $s \in S^*$. Since every $x > 0$, satisfies such a relation for some $s \in S^*$, it follows that $(f)_S \geq f$, for $x > 0$, and thus

$$d_T((f)_S) \leq d_T(f) ,$$

for any covering class T.

Remark: Observe that when $s^* \leq x \leq s$, $s \in S^*$, it holds that $(f)_S(x) = f(s) \wedge x\, f(s^*)/s^*$. Indeed, when $t \in S^*$, $t > s$, then $t^* \geq s$ so that $x \leq s$ implies

$$f(t^*) \frac{x}{t^*} \leq f(t^*) \leq f(t) .$$

Moreover, since $t^* \geq s^*$,

$$x\, f(s^*)/s^* \geq x\, f(t^*)/t^* .$$

Again $x \leq s$ implies

$$\frac{f(s)}{x} \geq \frac{f(s)}{x} \geq \frac{f(t^*)}{t^*} ,$$

so that when $t > s$,

$$f(s) \wedge x \frac{f(s^*)}{s^*} \geq f(t) \wedge x \frac{f(t^*)}{t^*} .$$

On the other hand, when $t < s$, $t \in S^*$, then $t \leq s^*$, so that

$$\frac{f(t)}{x} \leq \frac{f(t)}{t} \leq \frac{f(t^*)}{t^*} ,$$

or

$$f(t) \wedge x \frac{f(t^*)}{t^*} = f(t) .$$

Since $f(t) \leq f(s)$ and

$$f(t) \leq f(s^*) \leq \frac{x}{s^*} f(s^*) ,$$

it follows that

$$f(t) \wedge x \frac{f(t^*)}{t^*} \leq f(s) \wedge x \frac{f(s^*)}{s^*}$$

for all $t \in S^*$, which verifys the statement.

Of special interest are the functions $(x^{\alpha})_S$, $0 \leq \alpha \leq 1$. In this case more can be said.

Theorem 2. *If* $f \in \mathfrak{F}$ *such that* $d_S(f) = \alpha$, $0 \leq \alpha < 1$, *then* $d_T(f) \geq d_T((x^{\alpha})_S)$ *for all covering classes* T.

Proof: The proof is based on the following.

Lemma: *If* $g \in \mathfrak{F}$ *and if* S *is a covering class such that* $g(s) \leq s^{\beta}$ *for* $s \in S$, *then* $g \leq (x^{\beta})_S$ *for* $0 < x \leq \delta$ *for some positive* δ.

<u>Proof of the Lemma:</u> Let $t^* \leq x \leq t$ for $t \in S^*$. By Theorem 1, d) (§ 2, I) and the definition of S^*,

$$g(x) \leq g(t) \leq t^\beta$$

and

$$g(x) \leq x \frac{g(t^*)}{t^*} \leq x \; t^{*\beta - 1} .$$

It follows that

$$g(x) \leq t^\beta \wedge x \; t^{*\beta - 1}$$

and so
$$g(x) \leq (x^\beta)_S(x)$$

by the remark made following Theorem 1. This establishes the lemma.

Returning to the proof of Theorem 2, suppose first that $d_S(f) > \alpha$. Then for all $s \in S$, s sufficiently small, it follows that

$$\frac{\log f(s)}{\log s} > \alpha ,$$

or

$$f(s) < s^\alpha .$$

By the lemma, $f(x) \leq (x^\alpha)_S(x)$ when x is sufficiently small, so that

$$d_T(f) \geq d_T((x^\alpha)_S) ,$$

for any covering class T. Now suppose that $d_S(f) = \alpha$ and consider the function $f^{1-t} \cdot x^t$, $0 \leq t \leq 1$. This function is in $\mathfrak{F}$, by Theorem 3 (§ 2, I), and

$$d_S(f^{1-t} \cdot x^t) = \liminf_{s \to 0} \frac{\log f^{1-t}(s) \cdot s^t}{\log s}$$

$$= (1-t) \liminf_{s \to 0} \frac{\log f(s)}{\log s} + t$$

$$= (1-t)\, d_S(f) + t = (1-t)\,\alpha + t\ .$$

Similarly,

$$d_T(f^{1-t} \cdot s^t) = (1-t)\, d_T(f) + t\ .$$

Since $(1-t)\,\alpha + t > \alpha$ when $0 < t$, it follows that

$$d_T((x^\alpha)_S) \leq (1-t)\, d_T(f) + t\ .$$

The right-hand side of this last inequality approaches $d_T(f)$ as t approaches zero, and so $d_T((x^\alpha)_S) \leq d_T(f)$, as desired.

In the case $\alpha = 1$, the situation is not so simple because while $d_T((x)_S) = 1$, there can be f such that $d_S(f) = 1$ but $d_T(f) < 1$. The following theorem makes this clear.

Theorem 3. If $f \in \mathfrak{F}$ and $d_S(f) = 1$, then

$$d_T(f) \geq \sup_{\alpha < 1} d_T((x^\alpha)_S)\ ,$$

and there exists a function $g \in \mathfrak{F}$ such that $d_S(g) = 1$ and $d_T(g) = \sup_{\alpha<1} d_T((x^\alpha)_S)$.

Proof: Suppose that $f \in \mathfrak{F}$ and $d_S(f) = 1$. If $d_T(f) < \sup_{\alpha<1} d_T((x^\alpha)_S)$, then there exists $\gamma < 1$ such that $d_T(f) < d_T((x^\gamma)_S)$. Consider now the function $f(x) \vee x^\gamma$. Then

$$d_S(f \vee x^\gamma) = \gamma \wedge d_S(f) = \gamma$$

and

$$d_T(f \vee x^\gamma) = \gamma \wedge d_T(f) \leq d_T(f) < d_T((x^\gamma)_S) ,$$

but these two statements are in contradiction to Theorem 2. Thus

$$d_T(f) \geq \sup_{\alpha < 1} d_T((x^\alpha)_S) .$$

To prove the second part, suppose first that $\sup_{\alpha<1} d_T((x^\alpha)_S) = 1$. In this case there is nothing to prove, since $d_T(x) = 1$. Now let

$$\sup_{\alpha < 1} d_T((x^\alpha)_S) = \delta < 1 .$$

Select an increasing sequence (α_n) such that

$$\lim_{n \to \infty} \alpha_n = 1$$

and

$$\alpha_n > \delta \qquad (n = 1, 2, \ldots).$$

Since $d_T((x^\alpha)_S) \leq \delta$ for all $\alpha < 1$, there is a decreasing sequence (s_k) of points in S which have the following properties: For each k, there exists $t = t_k \in T$ such that $s_k^* \leq t_k \leq s_k$ and $(x^{\alpha_k})_S (t_k) \geq t_k^{\delta + 1/k}$; and the points s_k satisfy:

$$s_{k+1} \leq (s_k^*)^{\frac{1-\alpha_k}{1-\alpha_{k+1}}} , \qquad (k = 1, 2, \ldots) .$$

Define

$$g(x) = \bigvee_k \left(s_k^{\alpha_k} \wedge x\, s_k^{*\,\alpha_k - 1} \right) .$$

It will now be shown that $d_S(g) = 1$ and $d_T(g) = \delta$. First, it is clear that $g \in \mathfrak{F}$, since each of the functions $s_k^{\alpha_k} \wedge x\, s_k^{*\,\alpha_k - 1}$ $(k = 1, 2, \ldots)$, is in $\mathfrak{F}$. Moreover, if $s \in S$ and $s_{k+1} \leq s \leq s_k^*$, then

$$g(s) = s_{k+1}^{\alpha_{k+1}} \vee s \cdot s_k^{*\,\alpha_k - 1} .$$

Indeed, if $t > k+1$, then

$$s_t^{\alpha_t} \wedge s\ s_t^{*\,\alpha_t - 1} = s_t^{\alpha_t} \leq s_{k+1}^{\alpha_{k+1}} = s_{k+1}^{\alpha_{k+1}} \wedge s\ s_{k+1}^{*\,\alpha_{k+1} - 1} ,$$

since $s \geq s_{k+1}$. On the other hand when $t < k$,

$$s_t^{\alpha_t} \wedge s\ s_t^{*\,\alpha_t - 1} = s\ s_t^{*\,\alpha_t - 1} \leq s\ s_k^{*\,\alpha_k - 1} = s_k^{\alpha_k} \wedge s \cdot s_k^{*\,\alpha_k - 1} ,$$

since $s \leq s_k^*$ and

$$s_k^* \leq s_k \leq s_{k-1}^{*\,\frac{1-\alpha_{k-1}}{1-\alpha_k}} \leq s_{k-1}^{*\,\frac{1-\alpha_t}{1-\alpha_k}} \leq s_t^{*\,\frac{1-\alpha_t}{1-\alpha_k}} .$$

Thus

$$\frac{\log g(s)}{\log s} = \alpha_{k+1} \frac{\log s_{k+1}}{\log s} \wedge 1 + (\alpha_k - 1) \frac{\log s_k^*}{\log s} .$$

$$\geq \alpha_{k+1} \wedge \left(1 - (1-\alpha_k)\right) \frac{\log s_k^*}{\log s}$$

$$\geq \alpha_{k+1} \wedge \alpha_k = \alpha_k .$$

Since α_k approaches 1 as k approaches ∞, it follows that $d_S(g) = 1$. Consider the sequence (t_k) of points in T such that $s_k^* \leq t_k \leq s_k$ and

$$\left(x^{\alpha_k}\right)_S (t_k) \geq t_k^{\delta + \frac{1}{k}} .$$

Since,

$$g(t_k) \geq s_k^{\alpha_k} \wedge t_k\ s_k^{*\,\alpha_k - 1} = \left(x^{\alpha_k}\right)_S (t_k) \geq t_k^{\delta + \frac{1}{k}} ,$$

it follows that $d_T(g) \leq \delta$, and since $g \in \mathfrak{F}$ and by the first part of the theorem, $d_T(g) = \delta$.

Remark: It is indeed possible that $\sup_{\alpha<1} d_T((x^\alpha)_S) < 1$. In fact, there exist covering classes S and T for which $\sup_{\alpha<1} d_T((x^\alpha)_S) = 0$. Let S be the sequence $\left(2^{-2k^2}\right)$ $(k = 1, 2, \ldots)$ and $T = (0, 1]$. Then if $\alpha < 1$,

$$d_T((x^\alpha)_S) \leq \lim_{k\to\infty} \frac{\alpha 2^{k^2}}{\alpha 2^{k^2} - \alpha 2^{(k+1)^2} + 2^{(k+1)^2}}$$

$$= \lim_{k\to\infty} \frac{\alpha}{\alpha - \alpha 2^{2k+1} + 2^{2k+1}}$$

$$= \lim_{k\to\infty} \frac{\alpha}{\alpha + 2^{2k+1}(1-\alpha)} = 0 .$$

§4. A Partial Ordering and Equivalence Relation for Covering Classes

As a first step in the study of $d_S(f)$ as a function of the covering class, it is natural to consider the following partial ordering on the collection of covering classes: a covering class S is said to be less than, or equal to, the covering class T, in symbols $S \leq T$, provided that $d_S(f) \leq d_T(f)$ for all functions f in $\mathfrak{F}$. It is clear that if $T \subseteq S$, then $S \leq T$. Indeed, if for a given f in $\mathfrak{F}$, $s^{-\beta}f(s)$ is bounded for all s in S, then the same is true for $t \in T$. By the Theorem of §1, II, it follows that $d_S(f) \leq d_T(f)$ and thus $S \leq T$. With the help of this remark, the following lemma can be proved.

Lemma 1. If A, B are covering classes and $f \in \mathfrak{F}$, then

$$d_{A\cup B}(f) = d_A(f) \wedge d_B(f) .$$

<u>Proof:</u> Since $A \cup B$ contains both the sets A and B, the inequality,

$$d_{A \cup B}(f) \leq d_A(f) \wedge d_B(f)$$

is immediate by the remarks made above. To show the reverse, let $f \in \mathfrak{F}$, and suppose that $d_A(f) \leq d_B(f)$. If $d_A(f) = 0$, then

$$d_{A \cup B}(f) \geq d_A(f) \wedge d_B(f) \;,$$

automatically. Thus assume $d_A(f) > 0$, and let α be such that $0 \leq \alpha < d_A(f)$. Then

$$a^{-\alpha} f(a) \leq M_A \qquad (a \subset A) \;.$$

Since it is assumed that $d_A(f) \leq d_B(f)$, it follows that

$$b^{-\beta} f(b) \leq M_B \qquad (b \in B) \;.$$

If $M = \max(M_A, M_B)$, then

$$c^{-\alpha} f(c) \leq M \qquad (c \in A \cup B) \;,$$

and so $\alpha \leq d_{A \cup B}(f)$. It then follows that

$$d_A(f) \wedge d_B(f) \leq d_{A \cup B}(f) \;,$$

which proves the lemma.

The following theorem gives a characterization of the relation $S \leq T$ in terms of a comparison of the gaps in the covering classes S and T.

Theorem. A necessary and sufficient condition that $S \leq T$, is that there exist a function g, $g: T \to S$, with the property

$$\lim_{t \to 0} \frac{\log g(t)}{\log t} = 1 \qquad (t \in T) .$$

Proof:

Sufficiency: Let $g: T \to S$ be such that

$$\lim_{t \to 0} \frac{\log g(t)}{\log t} = 1 \qquad (t \in T) .$$

By Lemma 1, (§1, I), for $f \in \mathfrak{F}$,

$$f(t) \leq 2\left(1 \vee \frac{t}{g(t)}\right) f(g(t)) ,$$

or

$$\frac{1}{2}\left(1 \wedge \frac{g(t)}{t}\right) f(t) \leq f(g(t)) .$$

Thus it follows that

$$\frac{\log f(g(t))}{\log g(t)} \leq \frac{\log \frac{1}{2}}{\log g(t)} + \frac{\log\left(1 \wedge \frac{g(t)}{t}\right)}{\log g(t)} + \frac{\log f(t)}{\log t} \cdot \frac{\log t}{\log g(t)} .$$

Since

$$0 \leq \frac{\log\left(1 \wedge \frac{g(t)}{t}\right)}{\log g(t)} = \left|1 - \frac{\log g(t)}{\log t}\right| = o(1) \qquad (t \to 0) ,$$

and

$$0 \leq \frac{\log t}{\log g(t)} \leq \left|1 - \frac{\log (t)}{\log g(t)}\right| + 1 = o(1) + 1 \qquad (t \to 0) ,$$

it follows that

$$\frac{\log f(g(t))}{\log g(t)} \leq o(1)\left(1 + \frac{\log f(t)}{\log t}\right) + \frac{\log f(t)}{\log t} \qquad (t \to 0) .$$

By (1. 1) and the fact that $f \in \mathfrak{F}$, $1 + \frac{\log f(t)}{\log t}$ is bounded for $t \in T$, so that

$$\frac{\log f(g(t))}{\log g(t)} \leq o(1) + \frac{\log f(t)}{\log t} \qquad (t \to 0) ,$$

which implies that

$$d_{g(T)}(f) = \liminf_{t \to 0} \frac{\log f(g(t))}{\log g(t)} \leq \liminf_{t \to 0} \frac{\log f(t)}{\log t} = d_T(f) .$$

Since $g(T) \subseteq S$, it follows that

$$d_S(f) \leq d_T(f) ,$$

and thus $S \leq T$.

Necessity: The following lemma is needed.

Lemma 2. $S \leq S^*$.

Proof of the Lemma: By what has been proved above, it is sufficient to exhibit a mapping, $h: S^* \to S$, such that

$$\lim_{t \to 0} \frac{\log h(t)}{\log t} = 1 \qquad (t \in S^*) .$$

If $t \in S^*$, there is $r \in S$, such that, either $r \leq t$ and $1 \leq \frac{\log r}{\log t} \leq 1 + t$, or $t \leq r$ and $1 - t \leq \frac{\log r}{\log t} \leq 1$. Choose any such r in S and write $h(t) = r$. Then it is clear that

$$\lim_{t \to 0} \frac{\log h(t)}{\log t} = 1 \qquad (t \in S^*) ,$$

and the lemma is proved.

Now define a function $g: T \to S^*$ as follows: If $s^* \leq t \leq s$,

$s \in S^*$, write

$$g(t) = \begin{cases} s\ , & \text{if } \dfrac{\log t}{\log s} \leq \dfrac{\log s^*}{\log t} \\ s^*, & \text{if } \dfrac{\log s^*}{\log t} < \dfrac{\log t}{\log s} \end{cases}$$

Then g is defined on all of T, and since it is now assumed that $d_S(f) \leq d_T(f)$ for all $f \in \mathfrak{F}$, it follows, in particular, that

$$\alpha = d_S((x^\alpha)_S) \leq d_T((x^\alpha)_S) ,$$

for $0 \leq \alpha \leq 1$. Fix α such that $0 < \alpha$, and consider $0 < \epsilon < \alpha$. Then

$$\alpha - \epsilon < d_T()x^\alpha)_S) ,$$

and so

$$(x^\alpha)_S(t) \leq t^{\alpha - \epsilon} .$$

for $t \in T$, $t < \delta$, for some positive δ. From the definition of $(x^\alpha)_S$, it follows that

$$s^\alpha \wedge t\, s*^{\alpha - 1} \leq t^{\alpha - \epsilon}$$

for $s \in S^*$ and $t < \delta$, and so, either

(2.2) $$s^\alpha \leq t^{\alpha - \epsilon}$$

or

(2.3) $$s*^{\alpha - 1} \leq t^{\alpha - \epsilon - 1} .$$

Thus given $t \in T$, $t < \delta$, choose $s \in S^*$ such that $s^* \leq t \leq s$. If (2.2) holds, then

$$s^{\frac{\alpha}{\alpha-\epsilon}} \leq t \leq s \;,$$

and so

$$\frac{\alpha-\epsilon}{\alpha} \leq \frac{\log s}{\log t} \leq 1 \leq \frac{\log t}{\log s} \leq \frac{\alpha}{\alpha-\epsilon} \;.$$

By the definition of g(t), either

$$\frac{\log s}{\log t} = \frac{\log g(t)}{\log t}$$

or

$$\frac{\log t}{\log s} \geq \frac{\log g(t)}{\log t} = \frac{\log s^*}{\log t} \geq 1 \;.$$

Thus in any event,

$$\frac{\alpha-\epsilon}{\alpha} \leq \frac{\log g(t)}{\log t} \leq \frac{\alpha}{\alpha-\epsilon} \;.$$

Similiarly, if (2. 3) holds, then

$$s^* \leq t \leq s^{*\frac{1-\alpha}{1-\alpha+\epsilon}} \;,$$

and so

$$\frac{1-\alpha}{1-\alpha-\epsilon} \leq \frac{\log t}{\log s^*} \leq 1 \leq \frac{\log s^*}{\log t} \leq \frac{1-\alpha-\epsilon}{1-\alpha} \;.$$

Again, since either

$$\frac{\log s^*}{\log t} = \frac{\log g(t)}{\log t}$$

or

$$\frac{\log t}{\log s^*} \leq \frac{\log g(t)}{\log t} = \frac{\log s}{\log t} \leq 1 \;,$$

it follows that

$$\frac{1-\alpha}{1-\alpha+\epsilon} \leq \frac{\log g(t)}{\log t} \leq \frac{1-\alpha+\epsilon}{1-\alpha} \;.$$

Since $\epsilon > 0$ was arbitrary, these considerations show that

$$\lim_{t \to 0} \frac{\log g(t)}{\log t} = 1 \qquad (t \in T) .$$

Finally, consider the composition h(g), where h is the function defined in Lemma 1. Then $h(g): T \to S$ in such a way that

$$\lim_{t \to 0} \frac{\log h(g(t))}{\log t} = 1 \qquad (t \in T) ,$$

and this completes the proof of the theorem.

Remark: The partial ordering $S \leq T$, described above for covering classes, leads immediately to the equivalence $S \equiv T$, when both $S \leq T$ and $T \leq S$. In Lemma 2 it was shown that $S \leq S^*$. Since $S \subseteq S^*$, it follows that $S^* \leq S$, so that $S \equiv S^*$. For this reason it will be assumed henceforth that any covering class in question is closed in the left-open unit interval, unless specifically stated otherwise. Further, since it is clear that the addition or deletion of a finite set of points to or from a covering class does not alter the dimension function, such additions or deletions will be assumed without specific mention whenever convenience dictates. Thus, for example, in later chapters the point, 1, will be assumed to belong to any covering class under consideration, whenever convenient.

CHAPTER III

COMPARISON OF COVERING CLASSES

§1. Doubly-starred Sets

Given any set A in l_2, define the transpose of A, tr A, by

$$\text{tr } A = \{(x, y) : (y, x) \in A\} .$$

If (a, b) is a point of l_2 and $a \geq b$, write

$$(a, b)^s = \{(x, y) : x \geq y, \ ay \geq bx, \ (t-b)(t-x) \geq (1-a)(1-y)\}$$

and if $b \geq a$,

$$(a, b)^s = \text{tr } (b, a)^s .$$

Given any set $A \subseteq l_2$, write

$$A^s = \cup\{(a, b)^s : (a, b) \in A\} .$$

A set A will be called doubly-starred with respect to the points (0, 0) and (1, 1), or more simply, doubly-starred in l_2, provided that $A = A^s$. The next few lemmas describe some properties of doubly-starred sets.

Lemma 1. If $A = \overline{A}$, then $A^s = \overline{A^s}$.

Proof: Suppose $(x_n, y_n) \in A^s$ with $x_n \geq y_n$ $(n = 1, 2, \ldots)$ and $x_n \to x$, $y_n \to y$. Then $x \geq y$. Let $a_n \geq b_n$ be such that $(x_n, y_n) \in (a_n, b_n)^s$, $(a_n, b_n) \in A$. Since $A = \overline{A}$, there is a subsequence $(a_j, b_j) \to (a, b) \in A$.

Since

$$a_j y_j \geq b_j x_j \ ,$$

$$(1-b_j)(1-x_j) \geq (1-a_j)(1-y_j) \ ,$$

it follows that $ay \geq bx$ and $(1-b)(1-x) \geq (1-a)(1-y)$. Thus $(x, y) \in (a, b)^s \subseteq A^s$, which shows $A^s = \overline{A^s}$.

If A is a set in l_2, then the sets tA and $A + s$ for real s and t are defined by

$$tA = \{(ta, tb):(a, b) \in A\} \ ,$$

$$A + s = \{(a+s, b+s):(a, b) \in A\} \ .$$

Lemma 2. $A = A^s$, _if and only if_

$$tA \cup (tA + 1-t) \subseteq A \ ;$$

for all $0 \leq t \leq 1$.

Proof: If $A = A^s$, $0 \leq t \leq 1$, $(a, b) \in A$ with $a \geq b$, then

$$ta \geq tb \ ,$$

$$a(tb) \geq b(ta) \ ,$$

$$\begin{aligned}(1-b)(1-ta) &= (1-b)(1-a) + a(1-b)(1-t) \\ &\geq (1-b)(1-a) + b(1-a)(1-t) = (1-a)(1-tb) \ .\end{aligned}$$

This shows $(ta, tb) \in (a, b)^s$. Moreover, since

$$ta + 1-t \geq tb + 1-t \ ,$$

$$a(tb+1-t) \geq b(ta+1-t) \ ,$$

$$(1-b)(1-(ta+1-t)) = (1-b)t(1-a) = (1-a)(1-(tb+1-t)) \ ,$$

it follows that $(ta+1-t,\ tb+1-t) \in (a,b)^s$. Thus

$$tA \cup (tA+1-t) \subseteq A^s = A .$$

On the other hand, let $(x,y) \in (a,b)^s$ with $(a,b) \in A$. Without loss of generality, suppose $a \geq b$. If $x = y$, then $(1,1) \in A$ implies $(x,y) \in xA \subseteq A$, by hypothesis. Thus suppose $x > y$, and so $a > b$. Write

$$t = \frac{(x-y)}{(x-y) + (by-ax)}$$

and

$$s = \frac{(x-y) + (by-ax)}{a-b} .$$

Since $by \geq ax$, it follows that $0 \leq t \leq 1$ and $s \geq 0$. Writing

$$(x-y) + (by-ax) = (1-a)(1-y) - (1-b)(1-x) + a - b ,$$

shows that $s \leq 1$. Since

$$s(ta+1-t) = s\left(\frac{x(a-b)}{x-y+by-ax}\right) = x$$

and

$$s(tb+1-t) = s\left(\frac{y(a-b)}{x-y+by-ax}\right) = y ,$$

it follows that $(x,y) \in s(tA+1-t)$. Since $s(tA+1-t) \subseteq A$, it follows that $(x,y) \in A$. Thus $(a,b)^s \subseteq A$ which shows that $A = A^s$.

Lemma 3. $(A^s)^s = A^s$.

Proof: Suppose $(a,b) \in A$ with $a \geq b$ and assume $(x,y) \in (a,b)^s$. Since, for $0 \leq t \leq 1$,

$$tx \geq ty ,$$

$$aty \geq btx ,$$

$$\begin{aligned}(1-b)(1-tx) &= (1-b)(1-x) + x(1-t)(1-b) \\ &\geq (1-a)(1-y) + y(1-t)(1-a) = (1-a)(1-ty) ,\end{aligned}$$

it follows that $(tx, ty) \in (a,b)^s$. Moreover,

$$tx + 1-t \geq ty + 1-t ,$$

$$a(ty+1-t) = aty + a(1-t) \geq btx + b(1-t) = b(tx+1-t) ,$$

$$\begin{aligned}(1-b)(1-(tx+1-t)) &= (1-b)(1-x)t \\ &\geq (1-a)(1-y)t = (1-a)(1-(ty+1-t)) ,\end{aligned}$$

and so $(tx+1-t,\ ty+1-t) \in (a,b)^s$. It follows that

$$t(a,b)^s \cup (t(a,b)^s + 1-t) \subseteq (a,b)^s \qquad ((a,b) \in A)$$

for all t, $0 \leq t \leq 1$. This implies

$$tA^s \cup (tA^s + 1-t) \subseteq A^s ,$$

for $0 \leq t \leq 1$; and so by Lemma 2, $(A^s)^s = A^s$.

The next lemma asserts that the transpose of a doubly-starred set is again doubly-starred.

Lemma 4. If $A = A^s$, then $\operatorname{tr} A = (\operatorname{tr} A)^s$.

Proof: $$\begin{aligned}\operatorname{tr}(A^s) &= \operatorname{tr}(\cup\{(a,b)^s : (a,b) \in A\}) \\ &= \cup\{\operatorname{tr}(a,b)^s : (a,b) \in A\} \\ &= \cup\{(b,a)^s : (a,b) \in A\}\end{aligned}$$

$$= \cup\{(b,a)^s : (b,a) \in \operatorname{tr} A\}$$
$$= (\operatorname{tr} A)^s \ .$$

Since $A = A^s$, it follows that $\operatorname{tr} A = (\operatorname{tr} A)^s$.

Under the same assumption made in Lemma 4, it can be shown that $\overline{A} = (\overline{A})^s$.

Lemma 5. $A = A^s$ implies $\overline{A} = (\overline{A})^s$.

Proof: By Lemma 2, $tA \subseteq A$. Since $t\overline{A} = t\overline{A}$, it follows that $t\overline{A} \subseteq \overline{A}$. Similarly

$$t\overline{A} + 1-t = (\overline{tA+1-t}) \subseteq \overline{A} \ ,$$

so that by Lemma 2, $\overline{A} = (\overline{A})^s$.

Given any set $A \subseteq 1_2$, and t, $0 \leq t \leq 1$, define a function A(t) by

$$A(t) = \inf(y:(t,y) \in \overline{A}) \ ,$$

where A(T) is defined to be equal 1, if $(y:(t,y) \in A) = \phi$.

Lemma 6. If $A = A^s$, then the functions A(t) and trA(t) are continuous on (0, 1), non-decreasing on [0, 1], and

(3.1) $A(t)/t$, $\operatorname{tr} A(t)/t$, $(1-A(t))/(1-t)$, $(1-\operatorname{tr} A(t))/(1-t)$,

are all non-decreasing on (0,1).

Proof: Given $0 < t < 1$, it follows that $(t, A(t)) \in \overline{A}$. By Lemma 5, $\overline{A} = (\overline{A})^s$, so that for $0 < \alpha \leq 1$, $(\alpha t, \alpha A(t)) \in \overline{A}$ and

$(\alpha t+1-\alpha,\ \alpha A(t)+1-\alpha) \in \overline{A}$. Then

$$A(\alpha t) \leq \alpha A(t)$$

and

$$A(\alpha t+1-\alpha) \leq \alpha A(t) + 1-\alpha \ .$$

These inequalities imply

$$\frac{A(\alpha t)}{\alpha t} \leq \frac{A(t)}{t}$$

and

$$\frac{1 - A(\alpha t+1-\alpha)}{1 - (\alpha t+1-\alpha)} \leq \frac{1 - A(t)}{1 - t} \ ,$$

which shows that $A(t)/t$ and $(1-A(t))/(1-t)$ are non-decreasing on $(0,1)$. Since $\operatorname{tr} A = (\operatorname{tr} A)^s$, it follows that $\operatorname{tr} A(t)/t$ and $(1-\operatorname{tr} A(t))/(1-t)$ are non-decreasing on $(0,1)$. If $0 < x \leq y$, then

$$A(x) \leq \frac{x}{y} A(y) \leq A(y) \ ,$$

so that $A(t)$ is non-decreasing. The same is true for $\operatorname{tr} A(t)$. Since $A = A^s$ implies $(0,0) \in A$ and $(x,x) \in A$, it follows that $A(0) = 0$ and $A(x) \leq x$, which shows $A(x)$ is continuous at $x = 0$. If $t \geq 0$ and $0 < x < 1$,

$$\frac{1 - A(x+t)}{1 - (x+t)} \geq \frac{1 - A(x)}{1 - x} \ ,$$

so that

$$A(x) \leq A(x+t) \leq \frac{t}{1-x} + \left(1 - \frac{t}{1-x}\right) A(x) \ .$$

By the same reasoning,

$$A(x) \geq A(x-t) \geq \left(1 + \frac{t}{1-x}\right) A(x) - \frac{t}{1-x} \ .$$

These relations show that $A(x)$ is continuous at each $0 < x < 1$, and the lemma is proved.

§2. Main Results

In Chapter II, the partial ordering on covering classes: $d_S(f) \leq d_T(f)$ for all $f \in \mathfrak{F}$, was introduced and studied. The question arises as to what can be said in general about the behavior of $d_S(f)$ and $d_T(f)$ when the covering classes S and T are not necessarily comparable in this ordering. For this purpose the set

$$R(S, T) = \{(d_S(f), d_T(f)) : f \in \mathfrak{F}\} ,$$

is now studied.

The main results concerning the dimension of functions state that the set $R(S, T)$ is closed and doubly-starred in l_2; and conversely, any closed, doubly-starred set in l_2 is of the form $R(S, T)$ for some covering classes S and T. Apart from the inherent interest in these results, they take on significance when, in Chapter IV it is shown that they have direct application to the study of the dimension of closed sets.

Theorem 1. $R(S, T) = (R(S, T))^s$.

Proof: Suppose $(\alpha, \beta) \in R(S, T)$. Then there is $f \in \mathfrak{F}$ with $d_S(f) = \alpha$ and $d_T(f) = \beta$. If $0 \leq t \leq 1$, then both $f(x)^t$ and $f(x)^t \cdot x^{1-t}$ are in $\mathfrak{F}$ by Theorem 3 (§2, I). Since

$$d_A(f^t) = td_A(f)$$

and

$$d_A(f^t \cdot x^{1-t}) = td_A(f) + 1-t ,$$

for every covering class A, it follows that $(t\alpha, t\beta) \in R(S, T)$ and $(t\alpha + 1-t, t\beta + 1-t) \in R(S, T)$. By Lemma 2, §1, it follows that $R(S, T) = R(S, T)^s$.

The set $R(S, T)$ can be described in terms of the special functions introduced in §3, Chapter II.

Lemma. $R(S, T) = \{(\alpha, \beta): \alpha \geq \beta \geq f(\alpha)$ or $\beta \geq \alpha \geq g(\alpha)\}$, where $(\alpha, \beta) \in 1_2$ and,

$$(3.2) \qquad f(\alpha) = d_T((x^\alpha)_S), \quad 0 \leq \alpha < 1 ; \quad f(1) = \sup_{\alpha<1} f(\alpha)$$

and

$$(3.3) \qquad g(\alpha) = d_S((x^\alpha)_T), \quad 0 \leq \alpha < 1 ; \quad g(1) = \sup_{\alpha<1} g(\alpha) .$$

Proof: If $f \in \mathfrak{F}$ and $d_S(f) = \alpha$, $\alpha < 1$, then

$$d_T(f) \geq d_T((x^\alpha)_S) ,$$

by Theorem 2 (§3, II). Further, by Theorem 3 (§3, II), $f \in \mathfrak{F}$ and $d_S(f) = 1$ implies $d_T(f) \geq \sup_{\alpha<1} d_T((x^\alpha)_S)$, for any covering classes S and T. Consequently

$$R(S, T) \subseteq \{(\alpha, \beta): \alpha \geq \beta \geq f(\alpha) \quad \text{or} \quad \beta \geq \alpha \geq g(\beta)\} .$$

On the other hand, suppose

$$1 > \alpha \geq \beta \geq d_T((x^\alpha)_S) .$$

Then put $f = (x^\alpha)_S \wedge x^\beta$. By Lemma 2 (§2, II)

$$d_S(f) = d_S((x^\alpha)_S) \vee \beta = \alpha \vee \beta = \alpha$$

and

$$d_T(f) = d_T((x^\alpha)_S) \vee \beta = \beta .$$

Now, if $\alpha = 1$ and $1 \geq \beta \geq \sup_{\alpha < 1} d_T((x^\alpha)_S)$, let g be the function defined in the proof of Theorem 3, (§3, II) such that $d_S(g) = 1$ and $d_T(g) = \sup d_T((x^\alpha)_S)$. Write $h = g \wedge x^\beta$ and, as before, it follows that

$$d_S(h) = 1$$
$$d_T(h) = \beta .$$

This shows that

$$\{(\alpha, \beta) : \alpha \geq \beta \geq f(\alpha) \quad \text{or} \quad \beta \geq \alpha \geq g(\alpha)\} \subseteq R(S, T)$$

and the proof is complete.

Theorem 2. $R(S, T)$ is closed in l_2.

Proof: Writing $R = R(S, T)$, the lemma above implies that

$$R(t) = f(t)$$
$$trR(t) = g(t) ,$$

where f and g are defined by (3. 2) and (3. 3). By Lemma 6, §1, f and g are continuous on $[0, 1]$ so that $R(S, T)$ is closed.

The values that $(d_S(f), d_T(F))$ may take, in general, are quite unrestricted as the following theorem demonstrates.

Theorem 3. If R is any closed, doubly-starred set in l_2, there are covering classes S and T such that $R = R(S, T)$.

Proof: For each positive integer k, define

$$f_k(x) = \frac{x}{2k} \vee R(x)$$

and

$$g_k(x) = \frac{x}{2k} \vee trR(x) .$$

Then the functions f_k and g_k decrease monotonically to R(t) and trR(t) respectively; and they are continuous on [0, 1], since R(t) and trR(t) are continuous. Further, since

$$\frac{f_k(x)}{x} = \frac{1}{2k} \vee \frac{R(x)}{x} ,$$

$$\frac{g_k(x)}{x} = \frac{1}{2k} \vee \frac{trR(x)}{x} ,$$

$$\frac{1 - f_k(x)}{1 - x} = \frac{1 - x/2k}{1 - x} \wedge \frac{1 - R(x)}{1 - x} ,$$

and

$$\frac{1 - g_k(x)}{1 - x} = \frac{1 - x/2k}{1 - x} \wedge \frac{1 - trR(x)}{1 - x} ,$$

the functions $\frac{1 - x/2k}{1 - x}$ being non-decreasing in x, it is clear that $f_k(x)$ and $g_k(x)$ satisfy condition (3. 1) for each k.

Define functions

$$v_k(x) , \quad w_k(x) , \quad u_k(x) , \quad z_k(x)$$

as follows:

$$v_k(x) = \frac{x}{f_k(x)} \left(\frac{1 - f_k(x)}{1 - x} \right)$$

$$w_k(x) = \frac{x}{g_k(x)} \left(\frac{1 - g_k(x)}{1 - x} \right)$$

$$u_k(x) = \frac{x}{f_k(x)}$$

$$z_k(x) = \frac{x}{g_k(x)} .$$

Let

$$r_1, r_2, \ldots, r_n, \ldots$$

be an enumeration of the rational numbers in $(0, 1)$ and write

$$v_k = v_k(r_k), \quad w_k = w_k(r_k), \quad u_k = u_k(r_k), \quad z_k = z_k(r_k) .$$

Now define sequences p_k and q_k as follows:

$$p_1 = \frac{1}{2}, \quad p_{2k} = p_{2k-1}^{v_k}, \quad p_{2k+1} = p_{2k}^{w_k}$$

$$q_{2k-1} = p_{2k-1}^{u_k}, \quad q_{2k} = p_{2k}^{z_k}, \quad (k = 1, 2, \ldots) .$$

Since $v_k \geq u_k \geq 1$ and $w_k \geq z_k \geq 1$, it follows that

$$p_{2k} \leq q_{2k-1} \leq p_{2k-1}$$

and

$$p_{2k+1} \leq q_{2k} \leq p_{2k} .$$

If for some $k \geq 1$, both $v_k(x) \equiv 1$ and $w_k(x) \equiv 1$, then it follows that

$$\frac{f_k(x)}{x} = \frac{1 - f_k(x)}{1 - x}$$

or

$$f_k(x)\left(\frac{1}{x} + \frac{1}{1-x}\right) = \frac{1}{1-x} ,$$

which implies $f_k(x) = x$. Similarly, $w_k(x) \equiv 1$ implies $g_k(x) = x$.

These relations in turn imply that $R(x) = x$ and $\operatorname{tr}R(x) = x$ since

$$\frac{x}{2k} < x \qquad (k = 1, 2, \ldots) .$$

Thus taking any covering classes S, T with $S \equiv T$, would give

$$R(S, T) = R .$$

Therefore, without loss of generality assume that for some $k \geq 1$, there is a point x_0, $0 < x_0 < 1$, such that, $v_k(x_0) \geq s > 1$. There is then a neighborhood, U, of x_0 on which $v_k(y) \geq 1 + \frac{s-1}{2} > 1$. Since $j \geq k$ implies $f_j(x) \leq f_k(x)$, it follows that

$$v_j(y) = \frac{y}{f_j(y)}\left(\frac{1 - f_j(y)}{1 - y}\right) = \frac{y}{1 - y}\left(\frac{1 - f_j(y)}{f_j(y)}\right)$$

$$\geq \frac{y}{1 - y}\left(\frac{1 - f_k(y)}{f_k(y)}\right) = v_k(y) ,$$

and so,

$$v_j(y) \geq 1 + \frac{s-1}{2} > 1 ,$$

for y in U and $j \geq k$. Since r_j is in U for infinitely many indices j, it follows that there are infinitely many j for which

$$v_j \geq 1 + \frac{s-1}{2} > 1 ,$$

and thus the product $v_1 \ldots v_j$ approaches infinity as j approaches infinity. Since

$$p_{2k+1} = \left(\frac{1}{2}\right)^{(w_1 \ldots w_k)(v_1 \ldots v_k)} \leq \left(\frac{1}{2}\right)^{v_1 \ldots v_k} ,$$

and since the sequence (p_k) is non-increasing, it follows that p_k tends to zero as k tends to infinity. Thus covering classes S and T can be defined by writing:

$$S = (p_k) \cup (q_{2k}) \qquad (k = 1, 2, \ldots)$$

$$T = (p_k) \cup (q_{2k-1}) \qquad (k = 1, 2, \ldots) .$$

The next step is to show that for $0 < \alpha < 1$

$$d_T((x^\alpha)_S) = R(\alpha)$$

and

$$d_S((x^\alpha)_T) = trR(\alpha) ,$$

and this will complete the proof. First consider the function $(x^\alpha)_S$. At the points p_k, $(k = 1, 2, \ldots)$

$$(x^\alpha)_S(p_k) = p_k^\alpha ,$$

since $p_k \in S$. It remains to determine the values $(x^\alpha)_S(q_{2k-1})$ $(k = 1, 2, \ldots)$. Since $p_{2k} \leq q_{2k-1} \leq p_{2k-1}$, it follows that

$$(x^\alpha)_S(q_{2k-1}) = p_{2k-1}^\alpha \wedge q_{2k-1}\, p_{2k}^{\alpha-1} ,$$

by the remark made following Theorem 1 (§3, II). By definition,

$$p_{2k} = p_{2k-1}^{v_k} \quad \text{and} \quad q_{2k-1} = p_{2k-1}^{u_k} ,$$

so that

$$(x^\alpha)_S(q_{2k-1}) = q_{2k-1}^{\frac{\alpha}{u_k}} \wedge q_{2k-1}^{1-(1-\alpha)\frac{v_k}{u_k}}$$

$$= q_{2k-1}^{\frac{\alpha}{u_k} \vee \left(1-(1-\alpha)\frac{v_k}{u_k}\right)} .$$

Using the definitions of v_k and u_k and substituting, yields

$$(x^\alpha)_S(q_{2k-1}) = q_{2k-1}^{\frac{\alpha f_k(r_k)}{r_k} \vee \left(1 - \frac{(1-\alpha)(1-f_k(r_k))}{1-r_k}\right)} .$$

Thus, for $t \in T$,

$$\frac{\log (x^\alpha)_S(t)}{\log t} = \begin{cases} \alpha , & \text{if } t=p_k \quad (k=1,2,\dots) \\ \frac{\alpha f_k(r_k)}{r_k} \vee \left(1 - \frac{(1-\alpha)(1-f_k(r_k))}{1-r_k}\right) , & \text{if } t=q_{2k-1} \quad (k=1,2,\dots) \end{cases}$$

Since $\quad \alpha \geq \alpha \frac{f_k(r_k)}{r_k} \quad$ and $\quad \alpha \geq 1 - \frac{(1-\alpha)(1-f_k(r_k))}{(1-r_k)}$,

$$d_T((x^\alpha)_S) = \liminf_{k \to \infty} \left(\frac{\alpha f_k(r_k)}{r_k} \vee \left(1 - \frac{(1-\alpha)(1-f_k(r_k))}{1-r_k}\right)\right) .$$

Since the rational numbers are everywhere dense in (0, 1), it follows that

$$d_T((x^\alpha)_S) = \inf_k \left(\frac{\alpha f_k(r_k)}{r_k} \vee \left(1 - \frac{(1-\alpha)(1-f_k(r_k))}{1-r_k}\right)\right) .$$

Now if, $r_k \geq \alpha$, then $f_k(r_k) \leq r_k$ implies

$$\alpha(f_k(r_k) - r_k) \geq r_k(f_k(r_k) - r_k) ,$$

which in turn implies

$$\alpha f_k(r_k) - \alpha r_k - \alpha f_k(r_k) r_k$$
$$\geq r_k f_k(r_k) - r_k^2 - \alpha f_k(r_k) r_k + (r_k - r_k)$$

or

$$\alpha(f_k(r_k))\,(1-r_k) \geq r_k(1-r_k) - r_k(1-\alpha)\,(1-f_k(r_k)) \;,$$

and thus

$$\frac{\alpha f_k(r_k)}{r_k} \geq 1 - \frac{(1-\alpha)(1-f_k(r_k))}{1-r_k} \;.$$

Similarly, if $r_k \leq \alpha$, then

$$\frac{\alpha f_k(r_k)}{r_k} \leq 1 - \frac{(1-\alpha)(1-f_k(r_k))}{1-r_k} \;.$$

Hence

$$d_T((x^\alpha)_S) = \inf_{r_k \geq \alpha}\left(\frac{\alpha f_k(r_k)}{r_k}\right) \wedge \inf_{r_k \leq \alpha}\left(1 - \frac{(1-\alpha)(1-f_k(r_k))}{1-r_k}\right) \;.$$

It is now easy to verify that

$$\inf_{r_k \geq \alpha}\left(\frac{\alpha f_k(r_k)}{r_k}\right) = R(\alpha)$$

and

$$\inf_{r_k \leq \alpha}\left(1 - \frac{(1-\alpha)(1-f_k(r_k))}{1-r_k}\right) \geq R(\alpha)$$

when $0 < \alpha < 1$. Indeed, when $r_k \geq \alpha$,

$$\frac{\alpha f_k(r_k)}{r_k} \geq f_k(\alpha) \geq R(\alpha),$$

so that

$$\inf_{r_k \geq \alpha}\left(\frac{\alpha f_k(r_k)}{r_k}\right) \geq R(\alpha)$$

is immediate.

On the other hand, given $\epsilon > 0$, there is $\delta > 0$, such that $\alpha \leq r_k \leq \alpha + \delta$ implies

$$\frac{\alpha f_k(r_k)}{r_k} \leq f_k(\alpha) + \frac{\epsilon}{2} .$$

Since $\alpha \leq r_k \leq \alpha + \delta$, for arbitrarily large k and since $f_k(\alpha)$ tends to $R(\alpha)$ as k tends to infinity, it follows that there is k, with $\alpha \leq r_k$ and,

$$\frac{\alpha f_k(r_k)}{r_k} \leq R(\alpha) + \epsilon .$$

and this shows that

$$\inf_{r_k \geq \alpha} \frac{\alpha f_k(r_k)}{r_k} = R(\alpha) .$$

Similarly, when $r_k \leq \alpha$,

$$1 - \frac{(1-\alpha)(1-f_k(r_k))}{1-r_k} \geq f_k(\alpha) \geq R(\alpha)$$

so again

$$\inf_{r_k \geq \alpha} \left(1 - \frac{(1-\alpha)(1-f_k(r_k))}{1-r_k}\right) \geq R(\alpha) .$$

An analogous argument shows that $d_S((x^\alpha)_T) = \mathrm{tr}R(\alpha)$ when $\alpha < 1$. Since R is closed, the assertion of the theorem follows by the Lemma above.

§3. A Characterization of R(S, T)

Let S and T be given covering classes. For each point $t \in T$, there is associated a unique point $(a(t), b(t))$ in the unit square I_2. This association is made as follows: Given t, there are points

s_0, s_1 in S, defined by

$$s_0 = \sup (s : s \leq t,\ s \in S)$$
$$s_1 = \inf (s : s \geq t,\ s \in S) ,$$

where s_1 is defined to be 1, if there are no $s \in S$ with $s \geq t$. Write $s_0 = t^{\alpha}$ and $s_1 = t^{\beta}$, where $\alpha \geq 1 \geq \beta$. Observe that if either $\alpha = 1$ or $\beta = 1$, then both are equal 1.

In this case define

$$a(t) = b(t) = 1/2 ,$$

when $\alpha > 1 > \beta$, the equations in $a(t), b(t)$,

$$b(t) = \beta\, a(t) ,$$
$$1 - b(t) = \alpha(1-a(t)) ,$$

have the solution

$$a(t) = \frac{\alpha - 1}{\alpha - \beta} , \quad b(t) = \frac{\beta(\alpha - 1)}{\alpha - \beta} .$$

Altogether then,

$$(3.4) \qquad \begin{aligned} a(t) &= \tfrac{1}{2}, \quad \alpha = 1; \quad a(t) = \frac{\alpha-1}{\alpha-\beta}, \quad \alpha > 1 \\ b(t) &= \tfrac{1}{2}, \quad \beta = 1; \quad b(t) = \frac{\beta(a-1)}{\alpha-\beta}, \quad \beta < 1 . \end{aligned}$$

Note that $a(t) > 0$ for all $t \in T$ and that $b(t) > 0$ for $t \in T$ sufficiently small. Also, if $\beta < 1$ then both

$$\frac{\beta(\alpha - 1)}{\alpha - \beta} < 1$$

and

$$\frac{\alpha - 1}{\alpha - \beta} < 1 .$$

Thus, in any event $a(t) < 1$ and $b(t) < 1$, with $a(t) \geq b(t)$, for all $t \in T$. It follows that

$$s_1 = t^{b(t)/a(t)}$$

and

$$s_0 = t^{(1-b(t))/(1-a(t))} .$$

Define

$$A_n(S:T) = \left\{(a(t), b(t)) : t \in T,\ t \leq \frac{1}{n}\right\} .$$

and finally write,

$$A(S:T) = \bigcap_{n=1}^{\infty} \overline{A_n(S:T)} , \tag{3.5}$$

where $\overline{A}$ denotes the closure of A.

In a similar manner, there is associated with each point $s \in S$, a point $(c(s), d(s))$ in l_2, such that

$$t_0 = s^{\frac{1-c(s)}{1-d(s)}}, \quad t_1 = s^{\frac{c(s)}{d(s)}} ,$$

where t_0, t_1 are the analogues of s_0, s_1 above. Then the set

$$A(T:S) = \bigcap_{n=1}^{\infty} \overline{A_n(T:S)} \tag{3.6}$$

can be defined, where

$$A_n(T:S) = \left\{(c(s), d(s)) : s \in S,\ s \leq \frac{1}{n}\right\} .$$

Since $A_n(S:T) \supseteq A_{n+1}(S:T)$ and each of these sets is non-empty, it follows that the sets $A(T:S)$ and $A(S:T)$ are non-empty. The interest in the sets $A(S:T)$ and $A(T:S)$ is that while they are defined solely in terms of the points of S and T, the smallest doubly-starred set containing $A(S:T) \cup A(T:S)$ is precisely $R(S,T)$, as the following results make clear.

<u>Theorem 1.</u> $(A(S:T) \cup A(T:S))^{s} \subseteq R(S,T)$.

<u>Proof:</u> Since $R(S,T)$ is doubly-starred, it is sufficient to show that

$$A(S:T) \cup A(T:S) \subseteq R(S,T) .$$

The proof demonstrates only that

$$A(S:T) \subseteq R(S,T) ,$$

the proof that $A(T:S) \subseteq R(S,T)$ being similar. Hence let $(a,b) \in \bigcap_{n=1}^{\infty} \overline{A_n(S:T)}$, and $0 < a < 1$. Then there is a decreasing sequence $(t_n) \subseteq T$, such that $t_n \to 0$ and

$$\left.\begin{aligned} a_n &= a(t_n) \to a \\ b_n &= b(t_n) \to b \end{aligned}\right. \qquad (n \to \infty) .$$

Write $s_0 \leq t_n \leq s_1$ and put $x_n = s_1^a s_0^{1-a}$, then for t_n satisfying $x_n \leq t_n \leq s_1$,

$$(x^a)_S(t_n) = s_1^a = t_n^{a\, b_n/a_n}$$

and for $s_0 \leq t_n \leq x_n$,

$$\begin{aligned} (x^a)_S(t_n) &= t_0^{a-1}\, t_n \\ &= t_n^{(a-1)\frac{1-b_n}{1-a_n} + 1} . \end{aligned}$$

Thus in any case

$$(x^a)_S(t_n) \geq t_n^{\frac{a\,b_n}{a_n}} \wedge t_n^{1+(a-1)\frac{(1-b_n)}{(1-a_n)}}$$

$$= t_n^{\frac{a\,b_n}{a_n} \vee 1+(a-1)\frac{(1-b_n)}{(1-a_n)}}$$

and so

$$\frac{\log (x^a)_S(t_n)}{\log t_n} \leq \frac{a\,b_n}{a_n} \vee 1 + (a-1)\frac{1-b_n}{1-a_n} \quad .$$

Since $b(t_n) \to b$ and $a(t_n) \to a$ as $n \to \infty$, it follows that

$$d_T((x^a)_S) \leq b \ ,$$

which shows $(a, b) \in R(S, T)$. Now, if $a = 0$ and $(a, b) \in A(S, T)$, then $b = 0$, and $(0, 0)$ is clearly in $R(S, T)$. If $a = 1$ and $b = 1$, then (a,b) is clearly in $R(S, T)$. The only remaining case is $a = 1$, $b < 1$. Again let $a(t_n) \to a$, $b(t_n) \to b$ $t_n \in T$, $t_n \to 0$. Since $b(t_n) < 1$ for n sufficiently large, $a(t_n) < 1$ for the same n. Let $\alpha < 1$ and consider $(x^\alpha)_S(t_n)$. As above

$$(x^\alpha)_S(t_n) \geq t_n^{\frac{\alpha\,b_n}{a_n} \vee 1-(1-\alpha)\left(\frac{1-b_n}{1-a_n}\right)},$$

and

$$\frac{\log (x^\alpha)_S(t_n)}{\log t_n} \leq \frac{\alpha\,b_n}{a_n} \vee 1-(1-\alpha)\left(\frac{1-b_n}{1-a_n}\right) \quad .$$

Since $a(t_n) \to 1$, it follows that

$$\frac{\log (x^{\alpha})_S(t_n)}{\log t_n} \leq \frac{\alpha b_n}{a_n} \to \alpha b , \qquad n \to \infty ,$$

and therefore $d_T((x^{\alpha})_S) \leq \alpha b$, for all $\alpha < 1$. This implies

$$\sup_{\alpha < 1} d_T((x^{\alpha})_S) \leq b .$$

By the lemma of §2 it follows that $(a, b) \in R(S, T)$.

Theorem 2. $R(S, T) \subseteq (A(S:T) \cup A(T:S))^s$.

Proof: Let $f \in \mathfrak{F}$ with $d_S(f) \geq d_T(f)$. If actually $d_S(f) = d_T(f)$ then $(d_S(f), d_T(f)) \in (A(S:T) \cup A(T:S))^s$ automatically. Thus assume $d_S(f) > d_T(f)$, and let α, β be such that

$$d_T(f) < \beta < \alpha < d_S(f) .$$

By Lemma 1, (§1, I) for any $t \in T$, $s \in S$,

$$f(t) \leq \left(1 \vee \frac{t}{s}\right) f(s) ,$$

and so,

$$t^{-\beta} f(t) \leq t^{-\beta} s^{\alpha} \left(1 \vee \frac{t}{s}\right) s^{-\alpha} f(s) .$$

Since $\alpha < d_S(f)$, it follows by the Theorem of §1, Chapter II, that

$$t^{-\beta} f(t) \leq t^{-\beta} s^{\alpha} \left(1 \vee \frac{t}{s}\right) M_{\alpha} \qquad (t \in T)(s \in S) .$$

Since $\beta > d_T(f)$ there is a set $T_0 \subseteq T$, T_0 a covering class, such that $t \in T_0$, implies

$$t^{-\beta} f(t) \geq M_\alpha .$$

Thus for $s \in S$ and $t \in T_0$,

$$1 \leq t^{-\beta} s^{\alpha} \left(1 \vee \frac{t}{s}\right) .$$

Choosing $t \in T_0$, and $s_0(t) \leq t \leq s_1(t)$, $(s_i(t) \in S)$ $(i = 0, 1)$, it follows that

$$1 \leq t^{-\beta} s_1(t)^{\alpha} \qquad (t \in T_0)$$

and

$$1 \leq t^{1-\beta} s_0(t)^{\alpha - 1} . \qquad (t \in T_0)$$

Since $s_1(t) = t^{b(t)/a(t)}$ and $s_0(t) = t^{1-b(t)/1-a(t)}$, these equalities imply that

$$1 \leq t^{-\beta + \alpha \frac{b(t)}{a(t)}} \qquad (t \in T_0)$$

and

$$1 \leq t^{1-\beta + (\alpha - 1)\left(\frac{1-b(t)}{1-a(t)}\right)} . \qquad (t \in T_0)$$

Since $t \leq 1$, it follows that

$$\frac{b(t)}{a(t)} \leq \frac{\beta}{\alpha} \quad \text{and} \quad \frac{1-\beta}{1-\alpha} \leq \frac{1-b(t)}{1-a(t)} \qquad (t \in T_0)$$

Let $(a, b) \in A(S, T_0) \subseteq A(S, T)$ and let t_n be a sequence in T_0 such that

$$a(t_n) \to a , \quad b(t_n) \to b .$$

Then it follows that

$$\frac{b}{a} < \frac{\beta}{\alpha} \quad \text{and} \quad \frac{1-\beta}{1-\alpha} \leq \frac{1-b}{1-a}$$

which implies that $(\alpha, \beta) \in A(S, T_0)^s \subseteq A(S:T)^s \subseteq (A(S:T) \cup A(T:S))^s$. Since (α, β) can be chosen arbitrarily close to $(d_S(f), d_T(f))$ it follows that $(d_S(f), d_T(f)) \in (A(S:T) \cup A(T:S))^s$, this latter set being closed by Lemma 1, §1. The case $d_S(f) < d_T(f)$ is treated similarly, and so the proof is complete.

Theorems 1 and 2 combine to give

Theorem 3. $R(S, T) = (A(S:T) \cup A(T:S))^s$.

CHAPTER IV

APPLICATIONS TO CLOSED SETS

§1. Introduction and Preliminaries

Hausdorff [11] introduced the following outer measure on the subsets of the real line. If $0 \leq p \leq 1$,

$$H_p(E) = \sup_{\varepsilon > 0} \left(\inf \left(\sum \ell(1j)^p : \ell(1_j) < \varepsilon \right) \right)$$

where (1_j) is any countable collection of open intervals containing E, and $\ell(1_j)$ denotes the length of 1_j. The set function $H_p(E)$ is called the p-th dimensional Hausdorff measure of E. For an elementary discussion of the properties of $H_p(E)$ see Halmos [2] or Munroe [7].

The interest in the outer measure H_p for this work lies in the fact that given any closed set E, there is precisely one value p, $0 \leq p \leq 1$ for which $r < p$ implies $H_r(E) = \infty$, and $r > p$ implies $H_r(E) = 0$. The value of $H_p(E)$ may be any non-negative real number or ∞ (for a proof of these facts, see Hurewitz [4]). This unique value p is called the Hausdorff dimension of E and is denoted $d_H(E)$. A slightly different, but more useful description of the Hausdorff dimension of a closed set E is given by the following:

Lemma 1. For each $0 \leq p \leq 1$, define

$$\lambda_p(E) = \inf \left\{ \sum \ell(1_j)^p : \cup 1_j \supseteq E \right\}.$$

Then $H_p(E) = 0$ if and only if $\lambda_p(E) = 0$.

Proof: If $\lambda_p(E) > 0$, then since $\lambda_p(E) \leq H_p(E)$ it follows that $H_p(E) > 0$. On the other hand, if $H_p(E) > 0$, then there are $\delta > 0$, $b > 0$, such that if (J_i) is any sequence of intervals containing E such that $\ell(J_i) \leq \delta$, then

$$\Sigma\, \ell(J_j)^p \geq b > 0 \ .$$

For any other covering (1_k) of E by intervals,

$$\Sigma\, \ell(1_k)^p \geq \delta^p > 0 \ .$$

Thus $\lambda_p(E) > 0$.

Hence the dimension $d_H(E)$ can be defined by

$$d_H(E) = \sup\,(p{:}\lambda_p(E) > 0) \ .$$

This notion, of course, depends upon the particular kind of covering class used to cover the set E. A study is now made of this dependence when the intervals belong to a given class C. Further, this class C is assumed to be closed under translations, that is, $1 \in C \Rightarrow 1+t \in C$ for all real t. Thus whether or not a given interval 1 belongs to C depends only on $\ell(1)$. Moreover it will be assumed that C contains intervals of arbitrarily small length, and that $\ell(1) \leq 1$ if $1 \in C$. Such a class C will be called a covering class. Observe that the set

$$S(C) = S = \{\ell(1){:}1 \in C\}$$

is a covering class in the sense of §1, II. Then given C new set functions,

$$\lambda_{p,C}(E) = \inf\left(\sum \ell(1_j)^p : \cup 1_j \supseteq E,\ 1_j \in C\right) ,$$

can be defined. If $H_{p,C}(E)$ is defined by

$$H_{p,C}(E) = \sup_{\varepsilon > 0} \left(\inf\left(\sum (\ell(1_j))^p : \cup 1_j \supseteq E,\ \ell(1_j) < \varepsilon,\ 1_j \in C\right)\right) ,$$

then the proof of Lemma 1, shows that:

Lemma 2. $H_{p,C}(E) = 0$ *if and only if* $\lambda_{p,C}(E) = 0$.

Since $\lambda_{p,C}(E)$ is clearly a non-increasing function of p it follows that there is precisely one $0 \leq q \leq 1$ such that $r > q$ implies $H_{r,C}(e) = 0$ and $r < q$ implies $H_{r,C}(E) = \infty$. This value q will be called the Hausdorff dimension of E with respect to the covering class C (or S(C)) and is denoted $d_C(E)$. Thus it follows that

$$d_C(E) = \sup (p : \lambda_{p,C}(E) > 0) .$$

Remark: To avoid confusion, the notation $d_S(E)$, $\lambda_{p,S}(E)$ will be used where S refers to a covering class in the sense of §1, Chapter II.

A final lemma is needed for later considerations.

Lemma 3. *If* E_1 *and* E_2 *are any closed sets in* $[0, 1]$ *and* S *any covering class, then*

$$d_S(E_1 \cup E_2) = d_S(E_1) \vee d_S(E_2) .$$

Proof: For each p, the set function $\lambda_{p,S}(E)$ is monotone, so that

$$\lambda_{p,S}(E_1 \cup E_2) \geq \lambda_{p,S}(E_i) \qquad (i = 1, 2) .$$

It follows then that

$$d_S(E_1 \cup E_2) \geq d_S(E_1) \vee d_S(E_2) .$$

On the other hand, assume $0 < d_S(E_1 \cup E_2)$, and take $0 < p < d_S(E_1 \cup E_2)$. Thus

$$\lambda_{p,S}(E_1 \cup E_2) > 0 .$$

Since $\lambda_{p,S}$ is sub-additive, either $\lambda_{p,S}(E_1) > 0$ or $\lambda_{p,S}(E_2) > 0$. In any event,

$$p \leq d_S(E_1) \vee d_S(E_2) ,$$

which shows

$$d_S(E_1 \cup E_2) \leq d_S(E_1) \vee d_S(E_2) .$$

If $d_S(E_1 \cup E_2) = 0$, there is nothing to prove.

§2. Frostman's Theorem

Frostman [10] proved that the Hausdorff dimension of a closed set $E \leq [0,1]$, has the property that if $0 < \alpha < d_H(E)$, there exists a function $\mu(x)$ defined on $(-\infty, \infty)$ which satisfies the following conditions: $\mu(x)$ is non-decreasing, $\mu(x) = 0$ when $x \leq 0$, $\mu(x) = 1$ when $x \geq 1$, and if (a, b) is any open interval not intersecting E, then $\mu(a) = \mu(b)$. Further $\mu(x)$ satisfies a Lipschitz condition of order α at each point of $[0, 1]$. In this section, a generalization of this result to dimension with respect to a covering class is given.

If μ is a non-decreasing function defined on $(-\infty, \infty)$, the support of μ is defined to be the closed set,

$$\mathcal{S}_\mu = (\cup\{(a,b):\mu(a) = \mu(b)\})^c ,$$

where x^c denotes the compliment of x with respect to $(-\infty,\infty)$. Let M(E) denote the collection of all non-decreasing functions μ defined on $(-\infty,\infty)$ with $\mathcal{S}_\mu \leq E$ and $\mu(x) = 0$ for $x \leq 0$, $\mu(x) = 1$ for $x \geq 1$. Observe that if $\mu \in M(E)$ for some closed set E, then $\mu \in M$, defined in §2 of Chapter I. Indeed, the requirements $\mu(0) = 0$, μ non-decreasing and $\Delta\mu$ bounded on $[0,1]$ are all satisfied. With this observation the following fundamental theorem can be stated.

Theorem. Given any closed set E, and any covering class S,

$$d_S(E) = \sup (d_S(\mu):\mu \in M(E)) .$$

Proof: First suppose $\mu \in M(E)$, and consider $d_S(\mu)$. By the Theorem of § 1, Chapter II,

$$d_S(\mu) = \sup (\beta: s^{-\beta}\Delta\mu(s) = O(1),\ s \in S) .$$

Let $\beta < d_S(\mu)$. Then there is a finite constant M, such that $\Delta\mu(s) \leq s^\beta M$, for all $s \in S$. If $\{1_k\}$ is a covering of E by intervals such that $\ell(1_k) \in S$, then

$$\Sigma\, \ell(1_k)^\beta \geq \frac{1}{M} \Sigma\, \Delta\mu(\ell(1_k)) .$$

Since E is compact, a finite number of the intervals 1_k, $(k = 1, 2, \ldots)$ cover E, say (a_j, b_j) $(j = 1, 2, \ldots, n)$. If $1 = (a,b)$, then $\Delta\mu(\ell(1)) \geq \mu(b) - \mu(a)$. Since $\mathcal{S}_\mu \subseteq E$ and $\mu(1) = 1$, it follows that

$$\Sigma \ell(1_k)^\beta \geq \frac{1}{M} \sum_{j=1}^{n} \mu(b_j) - \mu(a_j) \geq 1 .$$

Consequently $\lambda_{\beta,S}(E) > 0$, and it follows that

$$d_S(E) \geq \beta .$$

Since β was arbitrary $< d_S(\mu)$, this shows that $d_S(\mu) \leq d_S(E)$ or that

$$\sup (d_S(\mu)): \mu \in M(E)) \leq d_S(E) .$$

To show the reverse inequality, suppose that $d_S(E) > 0$. If $d_S(E) = 0$, there would be nothing to prove. Let $0 < \beta < d_S(E)$. Then $0 < \lambda_{\beta,S}(E) < \infty$, and it can be assumed that $\lambda_{\beta,S}(E) = 1$, multiplying by an appropriate constant, if necessary. Put

$$\mu(x) = \lambda_{\beta,S}(E \cap [0,x]) , \quad \text{if} \quad x \geq 0$$

$$\mu(x) = 0 \quad \text{if} \quad x < 0 .$$

Then $\mu(x)$ is non-decreasing since $\lambda_{\beta,S}$ is monotone. Since $\lambda_{\beta,S}$ is a sub-additive set function and since $d_S(A) = 0$, if A is a finite set, it follows that $\mu(0) = 0$ and for $x \geq 1$

$$\mu(x) = \lambda_{\beta,S}(E \cap [0,x] \leq \lambda_{\beta,S}(E) + \lambda_{\beta,S}(E \cap [1,x]$$

$$= \lambda_{\beta,S}(E) = 1 .$$

The fact that μ is non-decreasing implies $\mu(x) = 1$ when $x \geq 1$. Moreover, if (a,b) is any open interval not intersecting E, then $E \cap [a,b]$ consists of at most two points so that

$$\mu(b) = \lambda_{\beta,S}(E \cap [0,b] \leq \lambda_{\beta,S}(E \cap [0,a]) + \lambda_{\beta,S}(E \cap [a,b])$$

$$= \mu(a) .$$

This implies that $S_\mu \subseteq E$ and hence $\mu \in M(E)$. Now,

$$\mu(x+s) = \lambda_{\beta,S}\Big(E \cap [0,x+s]\Big)$$

$$\leq \lambda_{\beta,S}\Big(E \cap [0,x]\Big) + \lambda_{\beta,S}\Big((E \cap [x,x+s]\Big) .$$

Since $\lambda_{\beta,S}\Big(E \cap [x,x+s]\Big) \leq s^\beta$, it follows that for every x,

$$\mu(x+s) - \mu(x) \leq s^\beta ,$$

and so $\Delta\mu(s) \leq s^\beta$. This implies that

$$d_S(\mu) \geq \beta ,$$

and so

$$\sup\,(d_S(\mu) : \mu \in M(E)) \geq \beta .$$

Since β was arbitrary $< \alpha$, it follows that

$$\sup\,(d_S(\mu) : \mu \in M(E)) \geq d_S(E) ,$$

and the proof of the theorem is complete.

Remark 1: To obtain Frostman's result from the above theorem, take $S = (0,1]$. Then given E with $d_S(E) > 0$ and $0 < \beta < d_S(E)$, there is $\mu \in M(E)$ such that $d_S(\mu) > \beta$. By the Theorem of §1, Chapter II, $s^{-\beta}\Delta\mu(s) = O(1)$ for $s \in (0,1]$. It follows that μ satisfies a Lipschitz condition of order β at each point x.

Remark 2: It follows immediately from this theorem that $S \leq T$ implies $d_S(E) \leq d_T(E)$ for every closed set, where "$\leq$" is the partial ordering introduced in §4 of Chapter II. Indeed, since $S \leq T$

implies $d_S(f) < d_T(f)$ for every $f \in \mathfrak{F}$, the same is true for every $g \in M$ and thus for every $\mu \in M(E)$. It follows that $d_S(E) \leq d_T(E)$. The converse is also true, that is, if $d_S(E) \leq d_T(E)$ for every closed set E in l_1, then $S \leq T$. This fact will follow from the results in §4 of the present chapter.

§3. Generalized Cantor Sets

It will be useful to distinguish a certain class of closed sets in l_1. Let (n_k) be a given sequence of positive integers, whose k-th partial products $N_k = n_1, \ldots, n_k$ satisfy, $\sum_k N_k^{-1} < \infty$. Define

$$\Omega = \underset{k}{X}\{0, 1, \ldots, n_k - 1\},$$

where the set $\{0, 1, \ldots, n_{k-1}\}$ has the discrete topology and Ω the topology of pointwise convergence. (For definitions and notations see Kelley [6].) In this topology Ω is compact and satisfies the first axiom of countability. The space Ω is totally ordered by the lexicographic ordering: If $a = (a_i)$ and $b = (b_i)$ are distinct points in Ω, then $a < b$ provided that if p is the smallest integer for which $a_p \neq b_p$, then $a_p < b_p$.

Define a mapping $\varphi: \Omega \to [0, 1]$ by

(4.1) $$\varphi(a) = \sum_i a_i N_i^{-1} .$$

Then φ has the following properties:

Theorem 1. a) *If* $a, b, \in \Omega$, *and* $a < b$ or $a = b$, *then* $\varphi(a) \leq \varphi(b)$.

b) *For all* $a \in \Omega$, $0 \leq \varphi(a) \leq 1$.

c) φ is onto.

d) φ is continuous.

Proof: a) Let $a, b \in \Omega$. If $a = b$, then clearly $\varphi(a) = \varphi(b)$, hence suppose that $a < b$. Then there exists k such that $a_i = b_i$, $i=1,\ldots,k-1$, and $a_k < b_k$. In this case,

$$\begin{aligned}
\varphi(b) - \varphi(a) &= \Sigma\, b_i N_i^{-1} - \Sigma\, a_i N_i^{-1} \\
&= (b_k - a_k) N_k^{-1} + \sum_{j>k} (b_j - a_j) N_j^{-1} \\
&\geq N_k^{-1} - \sum_{j>k} n_j N_j^{-1} + \sum_{j>k} N_j^{-1} \\
&= N_k^{-1} - \sum_{j\geq k} N_j^{-1} + \sum_{j>k} N_j^{-1} = 0 ,
\end{aligned}$$

and so

$$\varphi(a) \leq \varphi(b) .$$

b) Clearly $\varphi(a) \geq 0$ for all $a \in \Omega$. Further, since $a \in \Omega$, implies $a \leq w = (n_i - 1)$, it follows from part a) that

$$\begin{aligned}
\varphi(a) \leq \varphi(w) &= \Sigma\, (n_i - 1) N_i^{-1} \\
&= \frac{1}{N_0} = 1 .
\end{aligned}$$

c) Now let $0 \leq x \leq 1$, and define $a = (a_i)$ in Ω as follows. Let a_1 be the greatest integer n, satisfying:

$$0 \leq n \leq n_1 - 1$$

and

$$a_1 N_1^{-1} \leq x .$$

If $a_1, a_2, \ldots, a_{k-1}$ have already been defined, choose a_k to be the greatest integer n satisfying:

$$0 \leq n \leq n_k - 1$$

and

$$\sum_{r=1}^{k-1} a_r N_r^{-1} + n N_k^{-1} \leq x .$$

The point $a = (a_i)$ so defined clearly satisfies

$$\varphi(a) \leq x .$$

Suppose that $\varphi(a) < x$, then two cases arise. If for infinitely many integers j, $a_j < n_j - 1$, then choose j large enough so that

$$N_j^{-1} < \frac{x - \varphi(a)}{2} ,$$

and put $a_j' = a_j + 1$. Then,

$$\sum_{r=1}^{j-1} a_r N_r^{-1} + a_j' N_j^{-1} \leq \varphi(a) + N_j^{-1}$$

$$\leq \varphi(a) + \frac{x - \varphi(a)}{2}$$

$$\leq x ,$$

which would contradict the choice of a_j made above. On the other hand, suppose there exists an integer r such that for all $k \geq r$,

$$a_k = n_k - 1 .$$

Let j be the least such integer r. Since $\varphi(a) < x \leq 1$, it follows that $j > 1$. Moreover $a_{j-1} < n_{j-1} - 1$. Thus

$$\varphi(a) = \sum_{r=1}^{j-2} a_r N_r^{-1} + a_{j-1} N_j^{-1} + \sum_{r \geq j} (n_r - 1) N_r^{-1}$$

$$= \sum_{r=1}^{j-2} a_r N_r^{-1} + a_{j-1} N_j^{-1} + N_j^{-1}$$

$$= \sum_{r=1}^{j-2} a_r N_r^{-1} + (a_{j-1} + 1) N_j < x ,$$

which contradicts the choice of a_{j-1}. Thus the only possibility is $\varphi(a) = x$ which was to be proved.

d) Suppose $a(n)$ is a sequence in Ω such that $a(n)$ converges to $b \in \Omega$. Then for each integer $i=1, 2, \dots,$ there exists M_i such that $k \geq M_i$ implies $a_i(k) = b_i$. Let N be such that $k \geq N$ implies $\sum_{r \geq k} (n_r - 1) N_r^{-1} < \varepsilon$ and put

$$M = \max_{i \leq i \leq N} (M_i) .$$

If $n \geq M$, then $a_i(n) = b_i$, $i=1, \dots, N$ and so

$$\varphi(a(n)) = \sum a_i(n) N_i^{-1} = \sum_{i=1}^{N} b_i N_i^{-1} + \sum_{i>N} a_i(n) N_i^{-1} .$$

Thus

$$\sum_{i=1}^{N} b_i N_i^{-1} \leq \varphi(a(n)) \leq \sum_{i=1}^{N} b_i N_i^{-1} + \varepsilon ,$$

and since

$$\varphi(b) - \varepsilon \leq \sum_{i=1}^{N} b_i N_i^{-1} \leq \varphi(b) ,$$

it follows that

$$\varphi(b) - \varepsilon \le \varphi(a(n)) \le \varphi(b) + \varepsilon .$$

Since Ω is first countable, this shows that φ is continuous on Ω.

Assume that the sequence of positive integers (n_k) is fixed. Given positive sequences (σ_k) and (ξ_k), the triple (n_k, σ_k, ξ_k) is said to be admissible, provided that

$$\text{(4.2)} \qquad \begin{aligned} &\sigma_k < \xi_k && (k = 1, 2, \dots) , \quad \text{and} \\ &n_1\xi_1 \le 1, \quad n_k\xi_k \le \sigma_{k-1} && (k = 2, 3, \dots) . \end{aligned}$$

Given an admissible triple (n_k, σ_k, ξ_k) and a point $a \in \Omega$, consider the sum $\sum_{j=1}^{n} a_j\xi_j$. By (4.2) and the fact that $a_i \le n_{i-1}$,

$$\begin{aligned} \sum_{j=1}^{n} a_j\xi_j \le \sum_{j=1}^{n} (n_j - 1)\xi_j &= \sum_{j=1}^{n} n_j\xi_j - \sum_{j=1}^{n} \xi_j \\ &\le \sum_{j=1}^{n} \sigma_{j-1} - \sum_{j=1}^{n} \sigma_j \\ &= 1 - \sigma_n \le 1 . \end{aligned}$$

Thus the series $\sum a_i\xi_i$ is convergent, and defines a number $\psi(a)$. The function $\psi : \Omega \to [0, 1]$ so defined is called the derived mapping of (n_k, σ_k, ξ_k). Such a function ψ satisfies:

Theorem 2. a) $a < b$, *if and only if*, $\psi(a) < \psi(b)$.

b) $0 \le \psi(a) \le 1$, *for all* $a \in \Omega$.

c) ψ *is continuous.*

Proof: a) If $a, b \in \Omega$ and $a < b$, let k be such that $a_i = b_i$, $i=1,2,\ldots,k-1$, and $a_k < b_k$. Then

$$\begin{aligned}
\psi(b) - \psi(a) &= \Sigma\, b_i \xi_i - \Sigma\, a_i \xi_i \\
&= \sum_{j \geq k} (b_j - a_j)\, \xi_j \\
&\geq \xi_k - \sum_{j \geq k+1} (n_j - 1)\, \xi_j \\
&\geq \xi_k - \sum_{j \geq k+1} \sigma_{k-1} + \sum_{j \geq k+1} \sigma_j \\
&= \xi_k - \sigma_k > 0 .
\end{aligned}$$

Therefore $\psi(b) > \psi(a)$. It follows that $\psi(b) > \psi(a)$ implies $b > a$.

b) It is clear that $\psi(a) \geq 0$. The remarks made preceeding the Theorem established that $\psi(a) \leq 1$.

c) Since $\Sigma\,(n_i - 1)\, \xi_i$ is convergent, the same argument as that used to show the continuity of φ suffices.

A set E is called the Generalized Cantor Set of type (n_k, σ_k, ξ_k), or more briefly, the GCS of type (n_k, σ_k, ξ_k), provided that (n_k, σ_k, ξ_k) is admissible and that $E = \psi(\Omega)$, where ψ is the derived mapping of (n_k, σ_k, ξ_k). Since ψ is continuous and Ω compact, the following theorem is immediate.

Theorem 3. If E is a GCF of type (n_k, σ_k, ξ_k), then E is a closed subset of $[0, 1]$.

Given an admissible triple (n_k, σ_k, ξ_k), define a function $\mu(x)$ on $(-\infty, \infty)$ by writing:

$$\mu(x) = 0 , \quad \text{if} \quad x < 0 ,$$

$$\mu(x) = \sup (\varphi(b) : \psi(b) \leq x) , \quad \text{if} \quad x \geq 0 ,$$

where φ is defined by (4. 1) and ψ is the derived mapping of (n_k, σ_k, ξ_k). The function μ is called the Generalized Cantor Function of type (n_k, σ_k, ξ_k), or the GCF of type (n_k, σ_k, ξ_k).

The following lemma will be useful for the study of Generalized Cantor Functions.

Lemma 1. If $x = \psi(a)$, then $\mu(x) = \varphi(a)$.

Proof: If $x = \psi(a)$, then by definition $\mu(x) \geq \varphi(a)$. Since $\varphi(b) > \varphi(a)$ only if $b > a$, and $\mu(x) > \varphi(a)$ implies the existence of $b \in \Omega$ such that $\varphi(b) > \varphi(a)$ and $\psi(b) \leq x$, it follows from Theorem 2, a) that $\mu(x) \leq \varphi(a)$.

The first general statement about Generalized Cantor functions is:

Theorem 4. If μ is a GCF of type (n_k, σ_k, ξ_k), then $\mu \in M(\psi(\Omega))$.

Proof: Suppose $x \leq y$. Then $\psi(b) \leq x$ implies $\psi(b) \leq y$, and it follows that $\mu(x) \leq \mu(y)$. Since $a \leq w$ for $a \in \Omega$, $\mu(x) \leq \varphi(w) = 1$. The fact that

$$\psi(w) = \Sigma (n_i - 1) \xi_i \leq 1 ,$$

and $\varphi(w) = 1$, implies that $\mu(x) = 1$ when $x \geq 1$. Now write $E = \psi(\Omega)$ and suppose that the open interval (x, y) is disjoint from E. If $y \notin E$, then it is clear that $(x, y) \cap E = \phi$ implies

$$\mu(x) = \sup(\varphi(c):\psi(c) \leq x) = \sup(\varphi(c):\psi(c) \leq y) = \mu(y) .$$

Suppose then that $y \in E$. If also $x \in E$, then there exist points $a, b \in \Omega$ such that $x = \psi(a)$, $y = \psi(b)$. Then by Lemma 1, $\mu(x) = \varphi(a)$, $\mu(y) = \varphi(b)$. If $\varphi(a) < \varphi(b)$ then there would be $c \in \Omega$, such that

$$\varphi(a) < \varphi(c) < \varphi(b) ,$$

since φ is onto. This would mean $a < c < b$, and so $\psi(a) < \psi(c) < \psi(b)$, which would contradict $E \cap (x, y) = \phi$. Thus in the case $x, y \in E$, $\mu(x) = \mu(y)$. Finally, if $x \notin E$ and $x \leq 0$, then $y \leq 0$ and

$$\mu(x) = \mu(0) = \mu(y) ,$$

since $0 \in E$. If $x > 0$, then

$$x \geq \sup(t: t \in E,\ t \leq x) = t_1 \in E$$

since E is closed. Further $(t_1, y) \cap E = \phi$ and so

$$\mu(y) = \mu(t_1) \leq \mu(x) ,$$

and so $\mu(x) = \mu(y)$, which shows that $\mu \in M(\psi(\Omega))$.

The next theorem gives an alternate definition of $\mu(x)$.

<u>Theorem 5.</u> <u>For all</u> x,

$$\mu(x) = \inf(\varphi(c):\psi(c) \geq x) , \qquad (4.3)$$

<u>where the infimum is defined to be</u> 1 <u>in the case that there is no</u> $c \in \Omega$ <u>such that</u> $\psi(c) \geq x$.

Proof: If x is such that $x > \psi(c)$ for all c in Ω, then $\mu(x) = 1$, which agrees with (4. 3). If $x < 0$, then

$$\inf(\varphi(c) : \psi(c) \geq x) = 0 ,$$

so that $\mu(x) = \inf(\varphi(c) : \psi(c) \geq x)$ in this case. Thus suppose that $0 \leq x$ and that there is $c \in \Omega$ for which $\psi(c) \geq x$. Again write $E = \psi(\Omega)$ and let

$$t_0 = \sup(t : t \in E,\ t \leq x)$$

$$t_1 = \inf(t : t \in E,\ t \geq x) .$$

Since E is closed, t_0, t_1 belong to E. Then there are points $a, b \in \Omega$ such that

$$\psi(a) = t_0 \leq x$$

$$\psi(b) = t_1 \geq x .$$

If $\mu(x) < \varphi(b)$, then $\varphi(a) \leq \mu(x) < \varphi(b)$, and so there would be $c \in \Omega$ with $\varphi(a) < \varphi(c) < \varphi(b)$ or $a < c < b$. This would imply $\psi(a) < \psi(c) < \psi(b)$ which contradicts the choice of t_0, t_1. Thus $\mu(x) = \varphi(b)$ which implies $\mu(x) \geq \inf(\varphi(c) : \psi(c) \geq x)$. The fact that $\mu(x) \leq \inf(\varphi(c) : \psi(c) \geq x)$ follows from the definition of μ.

The foregoing proof also establishes:

Lemma 2. Given $0 \leq x$ and $\mu(x) < 1$, there is $b \in \Omega$ such that $\psi(b) \geq x$ and $\mu(x) = \varphi(b)$.

The following lemma will be instrumental in the proof of Theorem 6 below.

Lemma 3. Given a and b in Ω, there is a sequence (ϵ_k) where $\epsilon_k = 0$ or $\epsilon_k = 1$ $(k=1,2,\ldots)$, such that the point $c = (c_i)$ defined by

$$c_i = a_i + b_i + \epsilon_{i+1} - \epsilon_i n_i \qquad (i = 1, 2, \ldots),$$

belongs to Ω.

Proof: $a, b \in \Omega$. The existance of the required (ϵ_k) is proved as follows. Define

$$\Lambda = \mathop{\mathrm{X}}_{k=1}^{\infty} \{0, 1\},$$

where $\{0,1\}$ has the discrete topology and Λ, the topology of pointwise convergence relative to the compact spaces $\{0,1\}$. For each positive integer k define

$$A_k = \{(\epsilon_j): 0 \leq a_j + b_j + \epsilon_{j+1} - \epsilon_j n_j \leq n_j - 1; \qquad j = 1, \ldots, k\}.$$

Thus A_k is closed in Λ for each k, since if $\left(\epsilon_k^{(n)}\right)$ is a sequence in A_k converting to (δ_k) then there exists M such that $n \geq M$ implies $\epsilon_1^{(n)} = \delta_1, \ldots, \epsilon_{k+1}^{(n)} = \delta_{k+1}$, and so (δ_k) is in A_k. Further it is obvious that $A_k \supseteq A_{k+1}$ for all k, and that A_k is non-empty for each k. Since Λ is compact and the sets A_k have the finite intersection property, it follows that $\bigcap_{k=1}^{\infty} A_k$ is non-empty. Hence there is a sequence (ϵ_k) which satisfies the required conditions.

Remark: Let a, b be points of Ω and c a point determined by the lemma above. Then

$$\varphi(a) + \varphi(b) = \varphi(c) + \epsilon_1 .$$

Moreover, if ψ is the derived mapping of the admissible triple (n_k, σ_k, ξ_k), then

$$\psi(c) \geq \psi(a) + \psi(b) - \epsilon_1 .$$

Indeed, if there exists (ϵ_k) such that $c = (c_i)$ defined by

$$c_i = a_i + b_i + \epsilon_{i+1} - \epsilon_i n_i \qquad (i = 1, 2, \ldots)$$

is in Ω, then

$$\begin{aligned} \varphi(c) &= \sum N_i^{-1} (a_i + b_i + \epsilon_{i+1} - \epsilon_i n_i) \\ &= \varphi(a) + \varphi(b) + \sum N_i^{-1} \epsilon_{i+1} - \sum \epsilon_i n_i N_i^{-1} \\ &= \varphi(a) + \varphi(b) - \epsilon_1 . \end{aligned}$$

Moreover,

$$\begin{aligned} \psi(c) = \sum c_i \xi_i &= \sum a_i \xi_i + \sum b_i \xi_i + \sum \epsilon_{i+1} \xi_i - \sum \epsilon_i n_i \xi_i \\ &\geq \psi(a) + \psi(b) + \sum \epsilon_{i+1} \sigma_i - \sum \epsilon_i \sigma_{i-1} \\ &= \psi(a) + \psi(b) - \epsilon_1 . \end{aligned}$$

Theorem 6. If $\mu(x)$ is a GCF of type (n_k, σ_k, ξ_k), then $\mu(x)$ is sub-additive.

Proof: If either $x < 0$ or $y < 0$, then $x+y \leq x \vee y$, so that $\mu(x+y) \leq \mu(x \vee y) \leq \mu(x) + \mu(y)$. If either $\mu(x) = 1$ or $\mu(y) = 1$, then clearly

$$\mu(x+y) \leq \mu(x) + \mu(y) .$$

Thus suppose that $x \geq 0$, $y \geq 0$, $\mu(x) < 1$ and $\mu(y) < 1$. By Lemma 2, there are points $a, b \in \Omega$ such that

$$\psi(a) \geq x, \qquad \psi(b) \geq y$$

and

$$\mu(x) = \varphi(a), \qquad \mu(y) = \varphi(b) .$$

According to Lemma 3, and the remark following it, there is $c \in \Omega$ such that

$$\varphi(c) = \varphi(a) + \varphi(b) - \epsilon_1$$

and $\qquad\qquad (\epsilon_1 = 0 \text{ or } 1)$.

$$\psi(c) \geq \psi(a) + \psi(b) - \epsilon_1$$

If $\epsilon_1 = 0$, then $\psi(c) \geq x+y$. By Theorem 5, this implies

$$\mu(x+y) \leq \varphi(c) = \varphi(a) + \varphi(b) = \mu(x) + \mu(y) .$$

On the other hand, if $\epsilon_1 = 1$, then

$$\mu(x+y) \leq \mu(\psi(a) + \psi(b)) \leq \mu(\psi(c) + 1)$$
$$\leq 1 + \mu(\psi(c)) .$$

Since $\mu(\psi(c)) = \varphi(c) = \varphi(a) + \varphi(b) - 1$, by Lemma 1, it follows that

$$\mu(x+y) \leq \varphi(a) + \varphi(b) = \mu(x) + \mu(y) .$$

Hence in all cases $\mu(x+y) \leq \mu(x) + \mu(y)$, which proves the theorem.

It will now be shown that a GCS, E, of type (n_k, σ_k, ξ_k) and its GCF, μ have the same dimension with respect to any covering class, but first some useful lemmas.

For each positive integer k, define a set B_k by

$$B_k = \{b : b \in \Omega,\ b_j = 0,\ j > k\} .$$

Then B_k has exactly N_k points and has the following property:

Lemma 4. *If (n_k, σ_k, ξ_k) is admissible and ψ its derived mapping, then*

$$\psi(\Omega) \subseteq \bigcup_{t \in \psi(B_k)} \left[t, t + n_{k+1}\xi_{k+1}\right] \qquad (k = 1, 2, \ldots).$$

Proof: Let $c \in \Omega$. Since B_k is finite, let

$$b(c) = \max_{b \in B_k} (b : b \leq c) ,$$

this maximum being taken with respect to the lexicographic ordering on Ω. Then

$$\psi(c) \geq \psi(b(c)) .$$

If $\psi(c) > \psi(b(c)) + n_{k+1}\xi_{k+1}$, then

$$\begin{aligned} \psi(c) &> \sum_{j=1}^{k} b_j(c)\xi_j + n_{k+1}\xi_{k+1} \\ &\geq \sum_{j=1}^{k} b_j(c)\xi_j + \sum_{j>k} (n_j - 1)\, \xi_j , \end{aligned}$$

since $\sum_{j>k} (n_j - 1)\xi_j \leq n_{k+1}\xi_{k+1}$. If $b_j(c) = n_j - 1$ for $1 \leq j \leq k$, then it would follow that $\psi(c) > 1$, which is false. Thus there is a largest integer $r \leq k$ such that $b_r(c) < n_r - 1$, and so

$$\psi(c) > \sum_{j=1}^{r-1} b_j(c)\xi_j + b_r(c)\xi_r + \sum_{j>r} (n_j - 1)\xi_j$$

$$\geq \sum_{j=1}^{r-1} b_j(c)\xi_j + (b_r(c) + 1)\xi_r \ .$$

Then there would be a point b' in B_k with $b_i' = b_i(c)$ $(i=1,\ldots,r-1)$, $b_r' = b_r(c) + 1$, and

$$b(c) < b' < c \ ,$$

which contradicts the choice of b(c). Consequently

$$\psi(b(c)) \leq \psi(c) \leq \psi(b(c)) + n_k\xi_k$$

and the lemma is established.

With the help of Lemma 4, it is easy to verify

<u>Lemma 5</u>. $\mu(\xi_k) = \mu(\sigma_k) = N_k^{-1}$.

<u>Proof</u>: Let $\delta_{jk} = 0$, if $j \neq k$, $= 1$ if $j = k$. Then $(\delta_{jk}) \in \Omega$ and $\xi_k = \psi((\delta_{jk}))$. By Lemma 1,

$$\mu(\xi_k) = \varphi((\delta_{jk})) = N_k^{-1} \ .$$

To show $\mu(\xi_k) = \mu(\sigma_k)$, observe that $\sigma_k \geq n_{k+1}\xi_{k+1}$ implies that

$$\psi(\Omega) \subseteq \bigcup_{t \in B_k} [t, t + \sigma_k] \ ,$$

by Lemma 4. Since μ is non-decreasing and $\mu(1) = 1$, it follows that

$$1 \leq \sum_{t \in B_k} \mu(t + \sigma_k) - \mu(t) \ ,$$

and so for some $t \in B_k$, $\mu(t + \sigma_k) - \mu(t) \geq N_k^{-1}$. Since μ is sub-additive, and $\sigma_k < \xi_k$,

$$N_k^{-1} = \mu(\xi_k) \geq \mu(\sigma_k) \geq \mu(t+\sigma_k) - \mu(t) \geq N_k^{-1} ,$$

and so $\mu(\xi_k) = \mu(\sigma_k)$.

Lemma 6. For all $s \geq 0$, $\mu(s) \leq 2\left(1 \vee \frac{s}{\xi_k}\right) N_k^{-1}$.

Proof: By Lemma 1 (§1, I), Theorem 6, and Lemma 5,

$$\mu(s) = \mu\left(\frac{s}{\xi_k}, \xi_k\right) \leq 2\left(1 \vee \frac{s}{\xi_k}\right) \mu(\xi_k) = 2\left(1 \vee \frac{s}{\xi_k}\right) N_k^{-1} .$$

The next theorem is a partial answer to the question: For what sets E is it true that there is $\mu \in M(E)$ such that $d_S(\mu) = d_S(E)$ for all S?

Theorem 7. If μ is the GCF of type (n_k, σ_k, ξ_k) and E the GCS of the same type, then

$$d_S(\mu) = d_S(E) ,$$

for every covering class S.

Proof: If ψ is the derived mapping of (n_k, σ_k, ξ_k), then $E = \psi(\Omega)$ by definition and $\mu \in M(E)$ by Theorem 4. From the Theorem of §2, it follows that $d_S(\mu) \leq d_S(E)$. On the other hand, if $\lambda \in M(E)$ and $s \in S$ with $\sigma_k \leq s \leq \sigma_{k-1}$, then

$$S_\lambda \subseteq \psi(\Omega) \subseteq \bigcup_{t \in \psi(B_k)} [t, t+n_{k+1}\xi_{k+1}]$$

$$\subseteq \bigcup_{t \in \psi(B_k)} [t, t+s] ,$$

by Lemma 4. Then

$$1 \leq \sum_{t \in \psi(B_k)} \lambda(t+s) - \lambda(t) ,$$

and it follows that

$$\Delta\lambda(s) \geq N_k^{-1} .$$

If $n_k \xi_k \leq s \leq \sigma_{k-1}$, then since

$$(4.4) \qquad \psi(\Omega) \subseteq \bigcup_{t \in \psi(B_{k-1})} [t, t+n_k\xi_k] \subseteq \bigcup_{t \in \psi(B_{k-1})} [t, t+s] ,$$

it follows that

$$\Delta\lambda(s) \geq N_{k-1}^{-1} = \mu(\sigma_{k-1}) \geq \mu(s) .$$

On the other hand, if $\sigma_k \leq s \leq n_k\xi_k$, then (4.4) and Lemma 1 (§1, I) imply

$$N_{k-1}^{-1} \leq \Delta\lambda(n_k\xi_k) \leq \frac{2\, n_k\xi_k}{s} \Delta\lambda(s) ,$$

from which it follows that

$$\Delta\lambda(s) \geq \frac{1}{2} \frac{s}{\xi_k} N_k^{-1} .$$

Altogether then, for $\sigma_k \leq s \leq \sigma_{k-1}$

$$\begin{aligned}
\Delta\lambda(s) &\geq \left(\frac{1}{2} \frac{s}{\xi_k} N_k^{-1} \wedge \mu(s)\right) \vee N_k^{-1} \\
&\geq \frac{1}{2}\left(\frac{s}{\xi_k} N_k^{-1} \vee N_k^{-1}\right) \wedge \left(\mu(s) \vee N_k^{-1}\right) \\
&\geq \frac{1}{4}\mu(s) \wedge \mu(s) = \frac{1}{4}\mu(s),
\end{aligned}$$

by Lemma 6 and the fact that $\mu(s) \geq \mu(\sigma_k) = N_k^{-1}$. It follows that for all $s \in S$,

$$\frac{\log \Delta\lambda(s)}{\log s} \leq \frac{\log 1/4}{\log s} + \frac{\log \mu(s)}{\log s},$$

which implies $d_S(\lambda) \leq d_S(\mu)$. Hence $d_S(\mu) = d_S(E)$ as desired.

The next result is concerned with the computation of $d_S(\mu)$, where μ is a GCF of type $(n_k, \sigma_k \xi_k)$, in terms of some of the concepts introduced in §3 of Chapter III. Given a covering class S and a positive sequence (σ_k) decreasing to 0, recall the definition of $a(\sigma_k)$ and $b(\sigma_k)$ relative to S, given by (3.4).

Theorem 8. If μ is a GCF of type (n_k, σ_k, ξ_k) where $n_k \xi_k = \sigma_{k-1}$, and if S is any covering class, then

$$(4.5) \qquad d_S(\mu) = \liminf_{k\to\infty}\left(\frac{1}{u_k}\frac{\log \mu(\sigma_k)}{\log \sigma_k}\right) \wedge \liminf_{k\to\infty}\left(\frac{1}{v_k}\frac{\log \mu(\sigma_k)}{\log \sigma_k} + 1 - \frac{1}{v_k}\right),$$

where $u_k = b(\sigma_k)/a(\sigma_k)$, $v_k = (1-b(\sigma_k))/(1-a(\sigma_k))$.

Proof: Since $\left(\sigma_k^{u_k}\right)$ and $\left(\sigma_k^{v_k}\right)$ are contained in S and are covering classes, it follows from Theorem 1 (§2, II) and Lemma 1 (§4, II), that

$$d_S(\mu) \leq d_{\left(\sigma_k^{u_k}\right)}(\mu) \wedge d_{\left(\sigma_k^{v_k}\right)}(\mu).$$

Since $\sigma_k^{u_k} \geq \sigma_k$,

$$\frac{\log \mu\left(\sigma_k^{u_k}\right)}{\log \sigma_k^{u_k}} \leq \frac{1}{u_k}\frac{\log \mu(\sigma_k)}{\log \sigma_k},$$

and thus

$$d_{\left(\sigma_k^{u_k}\right)}(\mu) \leq \liminf_{k \to \infty} \left(\frac{1}{u_k} \frac{\log \mu(\sigma_k)}{\log \sigma_k} \right) .$$

Moreover, the fact that $\sigma_k^{v_k} \leq \sigma_k$ and the sub-additivity of μ imply that

$$\mu(\sigma_k) \leq 2\left(1 \vee \sigma_k^{1-v_k}\right) \mu\left(\sigma_k^{v_k}\right) = 2\,\sigma_k^{1-v_k}\, \mu\left(\sigma_k^{v_k}\right) ,$$

by Lemma 1, (§1, I). Thus

$$\frac{\log \mu\left(\sigma_k^{v_k}\right)}{\log \sigma_k^{v_k}} \leq \frac{\log 1/2}{\log \sigma_k^{v_k}} + \frac{v_k - 1}{v_k} + \frac{1}{v_k} \frac{\log \mu(\sigma_k)}{\log \sigma_k} ,$$

which implies that

$$d_{\left(\sigma_k^{v_k}\right)}(\mu) \leq \liminf_{k \to \infty} \left(\frac{1}{v_k} \frac{\log \mu(\sigma_k)}{\log \sigma_k} + 1 - \frac{1}{v_k} \right) .$$

It follows that,

$$d_S(\mu) \leq \liminf_{k \to \infty} \left(\frac{1}{u_k} \frac{\log \mu(\sigma_k)}{\log \sigma_k} \right) \wedge \liminf_{k \to \infty} \left(\frac{1}{v_k} \frac{\log \mu(\sigma_k)}{\log \sigma_k} + 1 - \frac{1}{v_k} \right) .$$

By Lemma 6 and the fact that $n_k \xi_k = \sigma_{k-1}$,

$$\begin{aligned} \mu(s) &\leq 2\left(1 \vee \frac{s}{\xi_k}\right) N_k^{-1} \\ &= 2\left(N_k^{-1} \vee \frac{s}{\sigma_{k-1}} N_{k-1}^{-1}\right) . \end{aligned} \tag{4. 6}$$

Suppose now that $s \in S$ and $\sigma_k \leq s \leq \sigma_{k-1}$. Then necessarily

$\sigma_k^{u_k} \leq s \leq \sigma_{k-1}^{v_{k-1}}$, and this with (4.6) implies

$$\frac{\log \mu(s)}{\log s} \geq \frac{\log 2}{\log s} + \left(\frac{\log N_k^{-1}}{\log s} \wedge \frac{\log \left(s N_{k-1}^{-1} / \sigma_{k-1}\right)}{\log s}\right)$$

$$\geq \frac{\log 2}{\log \sigma_k} + \left(\frac{1}{u_k} \frac{\log \mu(\sigma_k)}{\log \sigma_k} \wedge \left(1 - \frac{1}{v_{k-1}} \frac{\log N_{k-1}\sigma_{k-1}}{\log \sigma_{k-1}}\right)\right) \tag{4.7}$$

$$= o(1) + \left(\frac{1}{u_k} \frac{\log \mu(\sigma_k)}{\log \sigma_k} \wedge \left(\frac{1}{v_{k-1}} \frac{\log \mu(\sigma_{k-1})}{\log \sigma_{k-1}} + 1 - \frac{1}{v_{k-1}}\right)\right),$$

since $\mu(\sigma_k) = N_k^{-1}$ and $N_k\sigma_k \leq 1$. If $s \in S$ and $k(s)$ denotes the integer k such that $\sigma_k < s \leq \sigma_{k-1}$, then $k(s) \to \infty$ as $s \to 0$. Thus (4.7) implies that

$$d_S(\mu) \geq \liminf_{k \to \infty} \left\{\frac{1}{u_k} \frac{\log \mu(\sigma_k)}{\log \sigma_k} \wedge \left(\frac{1}{v_{k-1}} \frac{\log \mu(\sigma_{k-1})}{\log \sigma_{k-1}} + 1 - \frac{1}{v_{k-1}}\right)\right\}$$

Since $\liminf (A_k \wedge B_k) = \liminf A_k \wedge \liminf B_k$, equality in (4.5) is proved.

The last theorem of this section asserts essentially that if E is a GCS of type (n_k, σ_k, ξ_k), $n_k\xi_k = \sigma_k - 1$, then

$$d_H(E) = \liminf_{k \to \infty} \frac{\log N_k^{-1}}{\log \sigma_k}.$$

<u>Theorem 9.</u> <u>If</u> μ <u>is a GCF of type</u> $(n_k, \sigma_k \xi_k)$, <u>with</u> $n_k\xi_k = \sigma_{k-1}$, <u>then</u> $d_S(\mu) \geq d_{(\sigma_k)}(\mu)$ <u>for every covering class</u> S.

Proof: By Lemma 6,

$$\mu(s) \leq 2\left(1 \vee \frac{s}{\xi_k}\right) N_k^{-1} .$$

If $\sigma_k \leq s \leq \sigma_{k-1}$, then for every $0 \leq \beta < d_{(\sigma_k)}(\mu)$,

$$\begin{aligned} s^{-\beta}\mu(s) &\leq \frac{2s^{-\beta}}{N_k} \vee \frac{2s^{1-\beta}}{\xi_k N_k} = \frac{2s^{-\beta}}{N_k} \vee \frac{2s^{1-\beta}}{\sigma_{k-1}N_{k-1}} \\ &\leq 2\sigma^{-\beta}\ \mu(\sigma_k) \vee 2\sigma_{k-1}^{-\beta}\ \mu(\sigma_{k-1}) \\ &\leq 2C_\beta , \end{aligned}$$

where C_β is a constant independent of k. By the Theorem of §1, Chapter II, it follows that $\beta \leq d_S(\mu)$, and so $d_{(\sigma_k)}(\mu) \leq d_S(\mu)$. Note that if $d_{(\sigma_k)}(\mu) = 0$, the result is trivial.

Remark: Theorem 9 can also be proved as a consequence of Theorem 8.

§4. Main Results

The purpose of this section is to show that the results of §2, Chapter III apply to the dimension of closed sets with respect to a covering class. More precisely, let C be the collection of closed subsets of [0, 1] and

$$M(S, T) = \{(d_S(E), d_T(E)) : E \in C\} ,$$

where S and T are any covering classes. It will be shown that M(S, T) is identical with R(S, T). To show $M(S, T) \subseteq R(S, T)$ is relatively easy:

<u>Theorem 1.</u> $M(S, T) \subseteq R(S, T)$.

<u>Proof:</u> Let E be a closed set such that

$$d_S(E) > d_T(E) .$$

There exists $\mu \in M(E)$ such that

$$d_S(E) \geq d_S(\mu) > d_T(E) ,$$

by the Theorem of §2. Since $\mu \in M$, there is $f \in \mathfrak{F}$ such that $f \sim \mu$, that is $d_A(f) = d_A(\mu)$ for all A, by Theorem 1 (§2, II). Now $f \wedge x^{d_T(E)} \in \mathfrak{F}$, and the lemma of §2, Chapter II implies that

$$d_S\left(f \wedge x^{d_T(E)}\right) = d_S(\mu) \vee d_T(E) = d_S(\mu)$$

and

$$d_T\left(f \wedge x^{d_T(E)}\right) = d_T(\mu) \vee d_T(E) = d_T(E) ,$$

since $\mu \in M(E)$. It follows that

$$(d_S(\mu), d_T(E)) \in R(S, T) ;$$

and, since $d_S(\mu)$ can be chosen arbitrarily close to $d_S(E)$, and since $R(S, T)$ is closed, this implies

$$(d_S(E), d_T(E)) \in R(S, T) .$$

The case $d_S(E) < d_T(E)$ is treated similarly. If $d_S(E) = d_T(E)$, then the fact that $R(S, T)$ is doubly-starred shows that $(d_S(E), d_T(E)) \in R(S, T)$, and this completes the proof.

Theorem 2. Given β such that $0 < \beta < 1$, and any covering class S, there is a GCF, μ of type (n_k, σ_k, ξ_k), with $n_k \xi_k = \sigma_{k-1}$, $(\sigma_k) \subseteq S$ and

$$d_{(\sigma_k)}(\mu) = \beta = d_S(\mu) .$$

Proof: Let $(\sigma_k) \subseteq S$ be a decreasing sequence satisfying:

$$(\sigma_k/\sigma_{k+1})^{1-\beta} \geq 2, \quad \sigma_0 = 1, \quad \sigma_1(\sigma_1^{-\beta} + 1) < 1, \qquad (k = 0, 1, 2, \ldots) .$$

Write

$$n_1 = \left[\sigma_1^{-\beta}\right] + 1 ,$$
$$n_k = \left[\sigma_k^{-\beta}/N_{k-1}\right] + 1 , \qquad (k = 2, 3, \ldots) ,$$

where $[x]$ denotes the greatest integer less than or equal x. Further, put $\xi_k = \sigma_{k-1}/n_k$. In order to show that $\sigma_k < \xi_k$, it suffices to show $n_k \sigma_k < \sigma_{k-1}$. Indeed, for $k = 1$,

$$n_1 \sigma_1 \leq (\sigma_1^{-\beta} + 1) < 1 = \sigma_0 .$$

For $k \geq 2$, $N_{k-1} = N_{k-2}\, n_{k-1} \geq N_{k-2}\, \sigma_{k-1}^{-\beta}/N_{k-2} = \sigma_{k-1}^{-\beta}$, and it follows that

$$n_k \sigma_k \leq \left(\sigma_k^{-\beta}/N_{k-1} + 1\right) \sigma_k \leq \left(\sigma_k^{-\beta}/\sigma_{k-1}^{-\beta} + 1\right) \sigma_k$$
$$\leq \left(2\, \sigma_k^{1-\beta}/\sigma_{k-1}^{1-\beta}\right) \sigma_{k-1} \leq \sigma_{k-1} .$$

Since $N_k \geq \sigma_k^{-\beta}$ and $\Sigma\, \sigma_k^{\beta}$ is a convergent series, it follows that $\Sigma\, N_k^{-1}$ is convergent. It follows that there exists a function μ which is a GCF of type (n_k, σ_k, ξ_k). Since

$$d_{(\sigma_k)}(\mu) = \liminf_{k \to \infty} \frac{\log N_k^{-1}}{\log \sigma_k} ,$$

and $N_k \geq \sigma_k^{-\beta}$, it follows that

$$d_{(\sigma_k)}(\mu) \geq \beta .$$

On the other hand, given k sufficiently large, there are integers $j \leq k$ for which $n_j \geq 2$. Let j_k be the largest of these integers. Then

$$n_{j_k} \leq 1 + \sigma_{j_k}^{-\beta} / N_{j_k - 1} \leq 2 \sigma_{j_k}^{-\beta} / N_{j_k - 1} ,$$

or

$$N_k = N_{j_k} \leq 2 \sigma_{j_k}^{-\beta} \leq 2 \sigma_k^{-\beta} ,$$

since $j_k \leq k$. Thus

$$\frac{\log N_k^{-1}}{\log \sigma_k} \leq \beta + \frac{\log 2^{-1}}{\log \sigma_k} ,$$

which implies $d_{(\sigma_k)}(\mu) \leq \beta$, or $d_{(\sigma_k)}(\mu) = \beta$. Since $(\sigma_k) \subseteq S$, $d_S(\mu) \leq \beta$. But by Theorem 9, §3, $d_S(\mu) \geq d_{(\sigma_k)}(\mu) = \beta$, which proves the theorem.

Given covering classes S and T, recall the definitions of A(S: T) and A(T:S) given by (3. 5) and (3. 6) of §3, Chapter III.

Theorem 3. Given any $(a, b) \in A(S:T) \cup A(T:S)$, and any t, $0 \leq t \leq 1$, there are closed sets E and F, such that

$$(d_S(E), d_T(E)) = (ta, tb)$$

and

$$(d_S(F), d_T(F)) = (ta + 1 - t, tb + 1 - t) .$$

Proof: Since any finite set E has $d_S(E) = 0$, and any closed interval F of positive length has $d_S(F) = 1$, for all S, only the values t, $0 < t \leq 1$ need to be considered. Assume $(a, b) \in A(S:T)$, then there is a sequence (t_r), $t_0 = 1$, $(t_r) \subseteq T$, (t_r) decreasing to zero such that $\lim_{r \to \infty} a(t_r) = a$ and $\lim_{r \to \infty} b(t_r) = b$. Four cases are considered.

Case 1. If $a \geq b > 0$, $b < 1$, and $0 < t \leq 1$, then there is a GCF, μ, of type (n_k, σ_k, ξ_k) with $n_k \xi_k = \sigma_k - 1$, $(\sigma_k) \subseteq (t_r) \subseteq T$, and

$$d_T(\mu) = d_{(\sigma_k)}(\mu) = tb ,$$

by Theorem 9, §3, and Theorem 2 above. Since $(\sigma_k) \subseteq (t_r)$, it follows that $a_k \to a$, $b_k \to b$, $(k \to \infty)$, where $a_k = a(\sigma_k)$, $b_k = (\sigma_k)$, and thus Theorem 8, §3 implies that

$$d_S(\mu) = ta \wedge \left(\frac{1-a}{1-b} ta + 1 - \frac{1-a}{1-b}\right) .$$

Since $ta \leq 1$ and $\frac{1-a}{1-b} \leq 1$, it follows that

$$\frac{1-a}{1-b} ta + 1 - \frac{1-a}{1-b} \geq ta ,$$

so that

$$d_S(\mu) = ta .$$

By the same reasoning, since $0 < t < 1$ implies $0 < tb + 1 - t < 1$, there is a GCF, λ, of type $(n'_k, \sigma'_i, \xi'_k)$ with $n'_k \xi'_k = \sigma'_{k-1}$, $(\sigma'_k) \subseteq (t_r)$, and $d_T(\lambda) = tb + 1 - t$. Applying Theorem 8, §3, and reducing, implies

$$d_S(\lambda) = \frac{a}{b}(tb+1-t) \wedge (ta+1-t)$$
$$= ta+1-t \ ,$$

since $a \geq b$. Thus taking E to be the GCS of type (n_k, σ_k, ξ_k) and F the GCS of type $(n'_k, \sigma'_k, \xi'_k)$ and applying Theorem 7, §3, proves the theorem in this case.

Case II. If $a = b = 1$, and $0 < t < 1$, then there is a GCF, μ; of type (n_k, σ_k, ξ_k), $n_k \xi_k = \sigma_{k-1}$, and $(\sigma_k) \subseteq (t_r)$ with $d_T(\mu) = d_{(\sigma_k)}(\mu) = t$. Then by Theorem 9, §3, $d_S(\mu) \geq t$. On the other hand, by Theorem 8, §3,

$$d_S(\mu) \leq \liminf_{k \to \infty} \frac{a_k}{b_k} \frac{\log \mu(\sigma_k)}{\log \sigma_k} = d_{(\sigma_k)}(\mu) = t \ ,$$

and so $d_S(\mu) = t$. In this case take E to be the GCS of type (n_k, σ_k, ξ_k) and F any closed interval of positive length.

Case III. If $a = b = 0$ and $0 < t < 1$, then let μ be the GCF described in Case II, with t replaced by $1-t$. Then

$$d_S(\mu) \leq \liminf_{k \to \infty} \left(\frac{1-a_k}{1-b_k} \frac{\log \mu(\sigma_k)}{\log \sigma_k} + 1 - \frac{1-a_k}{1-b_k} \right)$$

$$= d_{\sigma_k}(\mu) = 1-t \ .$$

It follows that $d_S(\mu) = 1-t = d_T(\mu)$, and in this case take E to be a finite set and F the GCS of type (n_k, σ_k, ξ_k).

Case IV. The only remaining case is $a > b = 0$. If $0 < t < 1$, then $tb + 1 - t > 0$, and there is a GCF, μ, of type (n_k, σ_k, ξ_k), $n_k \xi_k = \sigma_{k-1}$ such that

$$d_T(\mu) = d_{(\sigma_k)}(\mu) = tb + 1 - t = 1 - t > 0 .$$

Since $b_k \to 0$, $a_k \to a > 0$ and $d_{(\sigma_k)}(\mu) > 0$, it follows from Theorem 8, §3, that

$$d_S(\mu) = \liminf_{k \to \infty} \left(\frac{1-a_k}{1-b_k} \frac{\log \mu(\sigma_k)}{\log \sigma_k} + 1 - \frac{1-a_k}{1-b_k} \right)$$

$$= (1-a)\, d_{(\sigma_k)}(\mu) + 1 - (1-a)$$

$$= ta + 1 - t .$$

Thus let F be the GCS of type (n_k, σ_k, ξ_k). The existance of a closed set E such that

$$d_S(E) = ta , \quad d_T(E) = 0 \qquad (t > 0) .$$

requires special consideration. Without loss of generality, it can be assumed that the sequence (t_r) is chosen so that $b(t_r)$ decreases to $b = 0$, and that between t_r and t_{r-1} there is a point of S, for every r. This implies that

$$t_r^{\,b(t_r)/a(t_r)} \leq t_{r-1} ,$$

or

$$t_r^{\,-tb(t_r)} \geq t_{r-1}^{\,ta(t_r)} .$$

Since $a(t_r) \to a > 0$ and $t > 0$, it follows that

(4. 8)
$$t_r^{\,-tb(t_r)} \to \infty \qquad r \to \infty .$$

Further, $a(t_r) \to a > 0$ and $b(t_r) \to 0$ implies that

$$a(t_r)\left(\frac{1}{b(t_r)} - t\right) \to \infty \qquad (r \to \infty) \,. \tag{4.9}$$

On the basis of statements (4. 8) and (4. 9), there is a subsequence, (σ_k) of (t_r) which has the following properties:

$$b_1 \leq \frac{1}{2}, \quad \sigma_1 \leq \frac{1}{4}, \quad \sigma_0 = 1 \,.$$

(4. 10)

$$\sigma_k^{-tb_k} \geq \sum_{r=1}^{k-1} \sigma_r^{-tb_r} + 1 \qquad (b_k = b(\sigma_k))$$

$$\sum \sigma_k^{tb_k} < \infty$$

$$a_k\left(\frac{1}{b_k} - t\right) \geq 2 \,.$$

Given such a sequence $(\sigma_k) \subseteq T$, define

$$n_1 = \left[\sigma_1^{-tb_1}\right] + 1$$

$$n_k = \left[\sigma_k^{-tb_k}/N_{k-1}\right] + 1 \qquad (k = 2, 3, \ldots)$$

where $N_r = n_1, \ldots, n_r$ and $[x]$ denotes the greatest integer in x. Further, put

$$\xi_k = \sigma_{k-1}/n_k \,.$$

In order to find a GCF of type (n_k, σ_k, ξ_k), the sequences (n_k), (σ_k), (ξ_k) must satisfy:

$$\sum N_k^{-1} < \infty \,,$$

$$n_k \xi_k \leq \sigma_{k-1} \qquad (k = 1, 2, \ldots)$$

$$\sigma_k < \xi_k \,.$$

Since

$$N_k = n_k \cdot N_{k-1} \geq \frac{\sigma_k^{-tb_k}}{N_{k-1}} \cdot N_{k-1} = \sigma_k^{-tb_k},$$

it follows that

$$\Sigma\, N_k^{-1} \leq \Sigma\, \sigma_k^{tb_k},$$

this latter series being convergent by (4. 10). The fact that $n_k \xi_k \leq \sigma_{k-1}$ follows from the relation $\xi_k = \sigma_{k-1}/n_k$. This same relation implies that $\sigma_k < \xi_k$, whenever $n_k\sigma_k < \sigma_{k-1}$. Now

$$\begin{aligned} n_1\sigma_1 &\leq \left(\sigma_1^{-tb_1} + 1\right)\sigma_1 \leq \left(\sigma_1^{-1/2} + 1\right)\sigma_1 \\ &= \sigma_1^{1/2} + \sigma_1 \leq \frac{3}{4} \leq 1 = \sigma_0, \end{aligned}$$

by (4. 10). To verify $n_k\sigma_k < \sigma_{k-1}$ for $k > 1$, observe that

$$N_k \leq \sigma_{k+1}^{-tb_{k+1}}.$$

Indeed, $N_1 = n_1 \leq \sigma_1^{-tb_1} + 1 \leq \sigma_2^{-tb_2}$ by (4. 10). Suppose that

$$N_{k-1} \leq \sum_{r=1}^{k-1} \sigma_r^{-tb_r} + 1.$$

Then

$$\begin{aligned} N_k = n_k N_{k-1} &\leq \left(\frac{\sigma_k^{-tb_k}}{N_{k-1}} + 1\right) N_{k-1} \\ &= \sigma_k^{-tb_k} + N_{k-1} \leq \sum_{r=1}^{k} \sigma_r^{-tb_r} + 1. \end{aligned}$$

The choice of σ_{k+1} described by (4. 10) implies that $N_k \leq \sigma_{k+1}^{-tb_{k+1}}$. This fact implies that

$$n_k\sigma_k \leq \left(\frac{\sigma_k^{-tb_k}}{N_{k-1}} + 1\right)\sigma_k \leq \frac{2\sigma_k^{-tb_k}}{N_{k-1}} \quad .$$

Since $\sigma_k^{b_k/a_k} \leq \sigma_{k-1}$, $1 - tb_k \geq 0$ and $N_{k-1} \geq \sigma_{k-1}^{-tb_{k-1}}$, it follows that

$$n_k\sigma_k \leq 2\,\sigma_{k-1}^{tb_{k-1} + a_k\left(\frac{1}{b_k} - t\right)}$$

$$\leq 2\,\sigma_{k-1}^{a_k\left(\frac{1}{b_k} - t\right)}$$

$$\leq 2\,\sigma_{k-1}^{2} \leq \sigma_{k-1} \, ,$$

by (4. 10).

Thus let μ be the GCF of type (n_k, σ_k, ξ_k). As has already been observed, $N_k \geq \sigma_k^{-tb_k}$. It is also true that $N_k \leq 2\,\sigma_k^{-tb_k}$. Indeed, since $N_{k-1} \leq \sigma_k^{-tb_k}$, it follows that

$$n_k \leq 1 + \frac{\sigma_k^{-tb_k}}{N_{k-1}} \leq \frac{2\sigma_k^{-tb_k}}{N_{k-1}} \quad ,$$

and multiplying both sides of this inequality by N_{k-1} gives

$$N_k \leq 2\,\sigma_k^{-tb_k} \, .$$

It then follows, since $\mu(\sigma_k) = N_k^{-1}$, that

$$tb_k \leq \frac{\log\,\mu(\sigma_k)}{\log\,\sigma_k} \leq \frac{\log\,1/2}{\log\,\sigma_k} + tb_k \tag{4. 11}$$

and, $b_k \to 0$ implies

$$d_{(\sigma_k)}(\mu) = 0 .$$

Since $(\sigma_k) \subseteq T$, $d_T(\mu) = 0$ as well, By Theorem 8, §3,

$$d_S(\mu) = \liminf_{k \to \infty} \left(\frac{a_k}{b_k} \frac{\log \mu(\sigma_k)}{\log \sigma_k} \right) \wedge a$$

The estimation (4. 11), implies

$$ta_k \leq \frac{a_k}{b_k} \cdot \frac{\log \mu(\sigma_k)}{\log \sigma_k} \leq \frac{a_k}{b_k} \frac{\log 1/2}{\log \sigma_k} + ta_k .$$

Since

$$\sigma_k^{b_k/a_k} \leq \sigma_{k-1} \to 0 \qquad (k \to \infty) ,$$

it follows that $d_S(\mu) = ta \wedge a = ta$. Hence take E to be the GCS of type (n_k, σ_k, ξ_k), and in this case $d_S(E) = ta$, $d_T(E) = 0$.

Thus cases I–IV show the theorem true for (a, b) in $A(S:T)$. A symmetric argument establishes the result when $(a, b) \in A(T:S)$. Therefore the theorem is proved.

The principal theorem concerning the dimension of closed sets is the following:

Theorem 4. $M(S, T) = R(S, T)$.

Proof: By Theorem 1 and Theorem 3 (§3, III), it suffices to show that $(A(S:T) \cup A(T:S))^s \subseteq M(S, T)$. Let $(a, b) \in A(S:T)$, $(x, y) \in (a, b)^s$, and $(c, d) \in A(T:S)$. If $x = ta$, $0 \leq t \leq 1$, let E_1 be the closed set with

$$d_S(E_1) = ta \quad \text{and} \quad d_T(E_1) = tb .$$

There exists $0 \leq s \leq 1$ such that either $y = sd$ or $y = sd + 1-s$, and a closed set E_2 with $d_T(E_2) = y$. In either case, $d_S(E_2) \leq y$. Since $ay \geq bx$, it follows that $y \geq tb$. By Lemma 3, §1, $d_A(E_1 \cup E_2) = d_A(E_1) \vee d_A(E_2)$ for any A, so that

$$d_S(E_1 \cup E_2) = ta \vee d_S(E_2) = ta = x$$

and

$$d_T(E_1 \cup E_2) = tb \vee y = y .$$

If $x = ta + 1 - t$ for $0 \leq t \leq 1$, then let E_3 be the closed set such that

$$d_S(E_3) = ta + 1 - t \quad \text{and} \quad d_T(E_3) = tb + 1 - t .$$

If $(x, y) \in (a, b)^s$, then $(1-a)(1-y) \leq (1-b)(1-x)$ which implies that $(1-y) \leq t(1-b)$. It follows that $y \geq tb + 1 - t$. Consider the set $E_3 \cup E_2$, E_2 defined above. Then

$$d_S(E_3 \cup E_2) = (ta + 1 - t) \vee d_S(E_2) = x \vee d_S(E_2) = x$$

and

$$d_T(E_3 \cup E_2) = (tb + 1 - t) \vee y = y .$$

It follows that $(a, b)^s \subseteq M(S, T)$ if $(a, b) \subseteq A(S:T)$. A symmetric argument shows that $(a, b)^s \subseteq M(S, T)$ when $(a, b) \in A(T:S)$. These facts prove that

$$(A(S:T) \cup A(T:S))^s \subseteq M(S, T) ,$$

and hence $R(S, T) \subseteq M(S, T)$, which proves the theorem.

§5. Conclusions and Generalizations

Theorem 4 of §3 shows that the results obtained in Chapter III apply to the dimension of closed sets. Thus, for example, a knowledge of the relative positions of the covering classes S and T, that is a knowledge of the sets A(S:T) and A(T:S) introduced in §3 of Chapter III, gives a complete description of the values of $(d_S(E), d_T(E))$ as E varies over the collection of closed sets in $[0, 1]$. Moreover, it is clear from the material of §4, that questions of this sort may be dealt with entirely by considering finite unions of Generalized Cantor Sets.

In general, when a finite number of covering classes are considered, a similar analysis can be made using essentially the same methods.

The question arises as to what can be said for closed sets in the unit square. If C is any collection of open rectangles, whose sides are parallel to the coordinate axes, and if C is closed under translations and contains rectangles of arbitrarily small area, the dimension of a closed set E with respect to C is defined to be the number

$$d_C(E) = \sup(\beta:\lambda_{\beta,C}(E) > 0) ,$$

where

$$\lambda_{\beta,C}(E) = \inf\left(\sum (m(R_k))^{\beta}:\cup R_k \supseteq E,\ R_k \in C\right) .$$

The covering class C can be associated with a set of points in the left-open portion, $1_2'$, of the unit square 1_2, whose closure intersects $1_2 - 1_2'$.

If M denotes the class of functions f(x, y), defined on $x \geq 0$, $y \geq 0$, non-decreasing in x and in y such that f(x, y) = 0 if

$x \cdot y = 0$, and such that

$$\Delta f(a,b) = \bigvee_{(x,y) \in R^+} \Big(f(x+a, y+b) - f(x+a, y) - f(x, y+b) + f(x,y) \Big)$$

is bounded in l_2, then for $f \not\equiv 0$, the dimension of f with respect to the covering class S is defined to be the number

$$d_S(f) = \liminf_{st \to 0} \frac{\log \Delta f(s,t)}{\log st} \qquad (s,t) \in S .$$

Since $\Delta f(a,b)$ is sub-additive in each variable, it follows that

$$\Delta f(ta, rb) \leq 4(t \vee 1)(r \vee 1) \; \Delta f(a,b)$$

for $r, t \geq 0$. In particular, for $(x,y) \in l_2$, this implies

$$C \geq \Delta f(x,y) \geq x \cdot y \; \Delta f(1,1)/4 \ ,$$

where C is some positive constant. It follows that $0 \leq d_S(f) \leq 1$ for $f \in M$. On the other hand, the functions $(xy)^\alpha$, $0 \leq \alpha \leq 1$ (with the convention $0^\circ = 0$) are sub-additive in each variable and so for $\alpha > 0$

$$\begin{aligned} \Delta f(a,b) &= \bigvee_{x,y} \left((x+a)^\alpha - x^\alpha)(y+b)^\alpha - y^\alpha \right) \\ &\leq (ab)^\alpha = f(a,b) \ , \end{aligned}$$

while for $\alpha = 0$, $\Delta f(a,b) = 1$. This implies that $d_S((xy)^\alpha) = \alpha$, so that the dimension can take any value between 0 and 1. The interest in studying the dimension of functions in M with respect to various covering classes lies in the study of the influence of the shape, along with the area, of the members of the covering class. In the case of rectangles the shape of the members of a covering class, S, is given by the set of quotients s/t where $(s,t) \in S$. The following theorem is a first indication of the role of shape on the dimension of functions in M.

Theorem. A necessary and sufficient condition that $d_S(f) \le d_T(f)$ for all $f \in M$, is that there exist a function g, $g: T \to S$ with the properties:

$$\lim_{t_1 t_2 \to 0} \frac{\log g(t_1) g(t_2)}{\log t_1 t_2} = 1 \qquad (t_1, t_2) \in T$$

and

$$\lim_{t_1 t_2 \to 0} \frac{\log \left(\frac{t_1}{t_2} / \frac{g(t_1)}{g(t_2)} \right)}{\log t_1 t_2} = 0 .$$

The proof is carried out in a similar manner to that of the Theorem of §4, Chapter II.

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